Topology and Geometry of Stable Map Spaces

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To Mª Carmen, Jesús,
Penélope, and Mar.

Abstract

Algebraic stacks were introduced as a solution to the construction of moduli spaces in situations where the theories of varieties and schemes were not good enough. This thesis tackles three different projects concerning the topology and geometry of moduli stacks.

In the first chapter of this thesis, we study the enumerative significance of genus zero Gromov-Witten invariants with a Grassmannian target in terms of rational curves in the Grassmannian.

The second of the projects in this thesis is the computation of the Betti numbers of the spaces $\overline{M}_{0,0}(G(k,n),d)$ for $d = 2,3$, with $G(k,n)$ the Grassmannian parametrizing k-dimensional vector spaces in $V \cong \mathbb{C}^n$.

The last project in this thesis compares the virtual fundamental classes (introduced by Behrend and Fatenchi in [BF97]) of the stack of (g,β,μ)-stable ramified maps $\overline{\mathfrak{U}}_{g,\mu}(X,\beta)$ ([KKO]) and of the stack of (g,β,μ)-log stable ramified maps $\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)$, constructed using the machinery described in [Kim].

Zusammenfassung

Algebraische Stacks sind als Lösung für die Konstruktion von Modulräumen eingeführt worden, wo Varietäten und Schemata nicht ausreichten. Diese Arbeit besteht aus drei verschiedenen Projekten zur Topologie und Geometrie von algebraischen Stacks.

Im ersten Kapitel der Arbeit geht es um Gromov-Witten-Invarianten von Kurven des Geschlechts Null in der Grassmannschen. Wir analisieren ihre enumerative Bedeutung bezüglich rationaler Kurven in der Grassmannschen.

Der zweite Teil handelt von der Berechnung der Betti-Zahlen der Räume $\overline{M}_{0,0}(G(k,n),d)$ für $d = 2,3$, wobei $G(k,n)$ die Grassmannsche ist, welche die k-dimensionale Untervektorräume in $V \cong \mathbb{C}^n$ parametrisiert.

Schliesslich vergleichen wir die virtuellen Fundamentalklassen (eingeführt von Behrend und Fantechi in [BF97]) des Stacks von (g,β,μ)-stabilen verzweigten Abbildungen $\overline{\mathfrak{U}}_{g,\mu}(X,\beta)$ ([KKO]) und des Stacks von (g,β,μ)-log-stabilen verzweigten Abbildungen $\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)$, die wie in [Kim] beschrieben konstruiert worden sind.

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Tal com’ as nubes
que impele o vento,
i agora asombran, i agora alegran
os espaços inmensos do ceo,
así as ideas
loucas que eu teño,
as imaxes de múltiples formas,
de estranas feituras, de cores incertos,
agora asombran,
agora acraran,
o fondo sin fondo do meu pensamento.

Rosalía de Castro
In Follas Novas, 1880

Contents

Introduction

Stacks were introduced by Grothendieck to provide a general framework for studying local-global phenomena in mathematics. In this sense, the theory of stacks generalizes sheaf theory of Serre and Cartan. The early stages of the development of the theory can be traced in the Ph.D. thesis of Grothendieck's student Giraud on non-abelian cohomology [Gir71]. Not long after their introduction, stacks were used by Deligne and Mumford in a different context, the study of moduli of algebraic curves [DM69]. In their paper they gave two different approaches to the proof of irreducibility of M_g: one involving standard techniques and another based on the use of a category larger than the category of schemes. They established the foundations of the theory of stacks, introducing what are today called *Deligne-Mumford stacks* together with general theorems. Later, Artin [Art74] generalized the work of Deligne and Mumford by presenting *Artin stacks*, which have ever since proved to be a vital tool in algebraic geometry, especially in the study of moduli problems and quotient spaces.

For example, stacks are used to parametrize geometric objects varying in families. The understanding of these families, also called moduli, is a central topic in algebraic geometry. The authors above introduced the language of stacks to study moduli problems of objects with symmetries. Other spaces used in geometry, such as varieties or schemes, are not good enough to parametrize geometric objects which are self-similar. Stacks not only encode the way these objects vary in families, but also remember the symmetries of each object in the family.

This thesis is divided into three chapters, in which different aspects of the topology and geometry of moduli stacks are studied. Each of these chapters can be (in principle) read independently from the others. Some of them may have some definitions, but the reader is often referred to other sources for foundational material.

The purpose of the first chapter of this thesis is to rephrase the enumerative interpretation for the Gromov-Witten invariant given by Fulton and Pandharipande in [FP97] to one in terms of rational curves on a homogeneous space X satisfying incidence conditions. For a large class of enumerative conditions, we are able to exclude that due to reparametrizations of the source curve a map f contributes multiply to the Gromov-Witten invariant. Then the Gromov-Witten invariant simply counts rational curves in a given curve class incident to general translates of the given subvarieties. Theorem 1.4 asserts this to be the case when X is a Grassmannian variety and the subvarieties are Schubert varieties, up to a correction factor of the degree of the curve class for each Schubert variety of codimension one.

The goal of the second chapter is to compute the Betti numbers of the spaces of stable maps $\overline{M}_{0,0}(G(k,n),d)$, for $d = 2$ and 3. For that, we consider a $\mathbb{C}^*$-action on the moduli stack with isolated fixed points. The method described in this chapter is different to those used by other authors in the study of these numbers or the presentation of the cohomology rings of other spaces. These numbers are computed as a first step towards understanding a presentation and the structure of the cohomology ring of the space of stable maps to $G(k,n)$.

Finally, in the third chapter of the thesis, we focus our attention on two new moduli spaces, constructed by Kim, Kresch, and Oh [KKO], and Kim [Kim], respectively. Behrend and Fantechi introduced, as a solution to computations in spaces with components exceeding the expected dimension, the notion of *virtual fundamental class*. The aim of this last chapter is to compare the two virtual fundamental classes carried by the two spaces above.

The results from this thesis can be found in [LM11], [LM10a], and [LM10b].

Chapter 1

Gromov-Witten invariants and rational curves on Grassmannians

A typical question in enumerative geometry asks about the number of solutions of a given geometric problem expressed in terms of some type of geometric objects: *how many geometric structures of a given type satisfy a given collection of geometric conditions?* Since their appearance in the algebraic context, Gromov--Witten invariants have proven to be an indispensable tool for enumerative geometry. The problem of determining the number N_d of rational curves of degree d passing through $3d-1$ points in general position in the complex projective plane $\mathbb{P}^2$ was solved, by means of Gromov-Witten theory, by Kontsevich (cf. [KM94]), yielding the well-known recursive formula

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right),$$

with seed value N_1: the number of lines passing through two points in the projective plane.

Gromov-Witten invariants arose as enumerative invariants of stable maps, which had been previously introduced independently by Ruan and Tian [RT95] in the symplectic case, and by Kontsevich and Manin [KM94] in the algebraic case.

The purpose of this work is to study the enumerative significance of genus zero Gromov-Witten invariants with a Grassmannian target. While Gromov--Witten theory in the algebraic context in general requires the sophisticated machinery of virtual fundamental classes [BF97, LT98] and the invariants have no clear enumerative significance in general, in the case of genus zero with target a homogeneous variety $X = G/P$ the moduli stack is smooth of the expected dimension and it makes sense to ask for the enumerative significance of the Gromov-Witten invariants. Given a tuple of classes of subvarieties of X of suitable dimensions, the Gromov-Witten invariant counts isomorphism classes of $(C \cong \mathbb{P}^1, p_1, \dots, p_s, f)$ such that $f_*[\mathbb{P}^1]$ is equal to a given curve class and f maps p_i into a general translate of the ith subvariety for all i, according to Fulton and Pandharipande [FP97].

1.1 Preliminaries

Let X be a smooth projective variety over $\mathbb{C}$, and let β be a curve class on X. The set of isomorphism classes of pointed maps $(C, p_1, \dots, p_s, f)$, where C is a projective nonsingular curve and f is a morphism from C to X with $f_*([C]) = \beta$, is denoted as $M_{g,s}(X, \beta)$. Its compactification, the moduli space $\overline{M}_{g,s}(X, \beta)$, parametrizes stable maps and was introduced by Kontsevich (see [KM94]), where was used to formulate and predict enumerative results in the geometry of curves. A point in this compactification is a map $(C, p_1, \dots, p_s, f)$ where C is a projective, connected, nodal curve of arithmetic genus g, the markings $p_1, \dots, p_s$ are distinct nonsingular points of C, and f is a morphism from C such that $f_*([C]) = \beta$. We say that a point in C is *special* if it is either a marked point or a point where two components of C meet. The map $(C, p_1, \dots, p_s, f)$ is *stable* if for every irreducible component $E \subset C$ the following conditions hold:

i. If $E \cong \mathbb{P}^1$ and E is mapped to a point by f, then E must contain at least three special points;

ii. If E has arithmetic genus one and E is mapped to a point by f, then E must contain at least one special point.

Let G be a complex simple Lie group of classical type and P a maximal parabolic subgroup. The homogeneous space $X = G/P$ is a Grassmannian variety, a usual Grassmannian of subspaces of some finite-dimensional complex vector space V when G is of type A, or of subspaces isotropic for a given nondegenerate symmetric or skew-symmetric bilinear form on V in the other classical Lie types. Throughout this chapter we assume that $\dim X \geq 2$.

We restrict now our attention to the genus zero case. The moduli space of stable genus zero degree d maps $\overline{M}_{0,s}(X,d)$ is smooth (as a stack) of dimension $\dim X + s - 3 + d \deg c_1(X)$. For each $i = 1, \ldots, s$ there is an *evaluation map*

$$ev_i : \overline{M}_{0,s}(X,\beta) \to X, \quad ev_i\big((C, p_1, \ldots, p_s, f)\big) = f(p_i)$$

which evaluates the map f at the marked point p_i.

Definition 1.1. Let X be a homogeneous variety. If $\alpha_1, \ldots, \alpha_s \in A^*(X)$ are classes in the Chow (or cohomology) group of X whose codimensions sum to $\dim \overline{M}_{0,s}(X,d)$ then the (non-gravitational, genus zero) *Gromov-Witten invariant* is defined as

$$I_d(\alpha_1 \cdots \alpha_s) := \int_{\overline{M}_{0,s}(X,d)} ev_1^*\alpha_1 \cup \cdots \cup ev_s^*\alpha_s.$$

If $\Gamma_1, \ldots, \Gamma_s$ are subvarieties of X with α_i (Poincaré dual to) the fundamental class of Γ_i for each i, then for general $(g_1, \ldots, g_s) \in G^s$ the Gromov-Witten invariant is equal to the number of degree d maps $\mathbb{P}^1 \to X$ sending the ith marked point into $g_i\Gamma_i$ for each i.

Lemma 1.2 ([FP97]). *Let $s \geq 0$. Let $X = G/P$ be a homogeneous variety, and $\Gamma_1, \ldots, \Gamma_s$ be subvarieties of X such that $\sum \operatorname{codim}(\Gamma_i) = \dim \overline{M}_{0,s}(X,d)$. Then, for general elements $g_1, \ldots, g_s \in G$, the scheme-theoretic intersection*

$$ev_1^{-1}(g_1\Gamma_1) \cap \cdots \cap ev_s^{-1}(g_s\Gamma_s)$$

is a finite number of reduced points supported on the locus $M_{0,s}(X,d)$. Furthermore, the number of points in this intersection above equals $I_d(\alpha_1 \cdots \alpha_s)$.

The *quantum Schubert calculus* is a set of combinatorial rules that determine the genus zero 3-point Gromov-Witten invariants of X. Quantum analogues of the classical Pieri and Giambelli formulas are given for the usual Grassmannians by Bertram [Ber97] and for isotropic Grassmannians by Buch, Kresch, and Tamvakis [BKT09, BKTa, BKTb, KT03, KT04]. In type A an explicit combinatorial rule for the invariants, i.e., a quantum Littlewood-Richardson rule, is available, due to Coskun [Cos09].

Let us recall now some definitions. The Schubert varieties on the usual Grassmannian of m-planes in $V \cong \mathbb{C}^n$ are indexed by integer partitions of length at most m and biggest part at most $n - m$. There are analogous descriptions in the other Lie types, based on *k-strict partitions*; the reader is referred to [BKT09] for a detailed description and facts about Schubert varieties in the isotropic Grassmannians. Here k is $n - m$ when the bilinear form on V is skew symmetric and $X = \mathrm{IG}(m, 2n)$ and in the case of a symmetric bilinear form is $n-m$, respectively $n + 1 - m$, when X is $\mathrm{OG}(m, 2n + 1)$, respectively $\mathrm{OG}(m, 2n + 2)$; a partition is k--strict when there are no repeated parts greater than k. We will generally denote by X_λ the Schubert variety in X corresponding to a partition λ. Its codimension is $|\lambda|$, where for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ the weight is $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$.

A rational map of degree d to X is a morphism $f : \mathbb{P}^1 \to X$ such that $f_*[\mathbb{P}^1]$ has degree d. Here degree is understood with respect to the projective embedding of X corresponding to a fundamental representation of G with point stabilizer P. In dimension one the unique Schubert class has degree one and the corresponding Schubert variety is a line under this embedding. Since $\dim X \geq 2$, X contains a projective plane or a nonsingular quadric threefold, and therefore there exist rational curves of every degree $d \geq 1$ on X.

The kernel and span of a rational map as the one described above were introduced and discussed by Buch in [Buc03]. The original formulation, given in that paper, was presented in terms of subvarieties in the Grassmannian of l-dimensional subspaces in a n-dimensional vector space over the complex numbers, $G(l, n)$.

Definition 1.3 ([Buc03])**.** If Y is any subvariety of $G(l, n)$, we define the *span* of Y as the smallest subspace of $G(l, n)$ containing all the l-dimensional spaces given by points of Y. The *kernel* of the subvariety Y is the largest subspace of $G(l, n)$

contained in all the spaces given by points of Y.

We refer the reader to [Buc03] for further results, and stress the importance of Lemma 1 in that paper, which will be used in our proof.

1.2 Main result

We adopt the notation of Section 1.1 and prove the following result.

Theorem 1.4. *If $X = G/P$ where G is a complex simple Lie group of classical type and P is a maximal parabolic subgroup, d and s are positive integers, and $\Gamma_1, \ldots, \Gamma_s$ are Schubert varieties whose codimensions sum to $\dim \overline{M}_{0,s}(X, d)$, then the Gromov--Witten invariant $I_d([\Gamma_1], \ldots, [\Gamma_s])$ is divisible by d^r where $r = \#\{i : \operatorname{codim} \Gamma_i = 1\}$, and the quotient is equal to the number of degree d rational curves on X incident to general translates of the Γ_i.*

We note, if $\Gamma_i = X$ for any i then the Gromov-Witten invariant is zero (fundamental class axiom) and by a dimension argument the indicated set of curves is empty. If some $r \geq 1$ of the Γ_i have codimension 1, then the Gromov-Witten invariant is equal to d^r times the $(s-r)$-point Gromov-Witten invariant with the divisor classes omitted (divisor axiom). This implies the divisibility assertion. We first prove the enumerative claim assuming $\operatorname{codim} \Gamma_i \geq 2$ for all i, then we treat the case when some of the Γ_i are divisors.

Lemma 1.5. *Let Γ be a Schubert variety in X of codimension at least 2 with Schubert cell Γ^0. Fix a point $p_1 \in \mathbb{P}^1$. Then for each $d \geq 1$ there exists a degree d map $f : \mathbb{P}^1 \to X$ such that*

i. *f is an unramified morphism.*

ii. *$f(p_1) \in \Gamma^0$.*

iii. *f maps a nonzero tangent vector at p_1 to a tangent vector at $f(p_1)$ not contained in the tangent space to Γ^0.*

iv. *$f^{-1}(\Gamma) = \{p_1\}$.*

Recall that $\overline{M}_{0,s}(X,d)$ is irreducible [KP01, Tho98]. For a point $x \in X$ and for each i we observe that by the group action there is a birational isomorphism between $ev_i^{-1}(x) \times \mathbb{A}^{\dim X}$ and $\overline{M}_{0,s}(X,d)$, and hence $ev_i^{-1}(x)$ is irreducible as well. In the situation of the Lemma we therefore obtain that $ev_i^{-1}(\Gamma)$ is irreducible, by [DM69, Theorem 4.17].

Proof. It suffices to verify (ii)-(iv) for a point of $\overline{M}_{0,1}(X,d)$, since the combination of these is an open condition in $ev_1^{-1}(\Gamma)$ and (i) is satisfied on a dense open subset of $ev_1^{-1}(\Gamma) \cap M_{0,1}(X,d)$. When $d = 1$ it is clear that (ii)–(iv) may be satisfied. When $d = 2$ and X is an orthogonal Grassmannian, this follows from the description in [BKT09, Lemma 2.1, proof of Thm. 2.3 or Lemma 3.1, proof of Thm. 3.3].

Otherwise, we consider two cases. There is a critical degree d_0, the smallest for which two general points on X are joined by a rational curve of that degree: $d_0 = \min(m, n-m)$ when $X = G(m,n)$, otherwise d_0 is m when $X = IG(m, 2n)$ and m (or $m-1$ when $X = OG(m, 2m)$) rounded up to the next even integer for the orthogonal Grassmannians (divided by two in the case of the maximal orthogonal Grassmannians). The first case we consider is when $d \leq d_0$. Then we have the following set-up from [BKT03, §2.2, 3.2, 4.2], [BKT09, §1.4, 2.4, 3.4]: there is a variety Y_d, an incidence correspondence

$$\begin{array}{ccc} T_d & \longrightarrow & X \\ {\scriptstyle\pi}\big\downarrow & & \\ Y_d & & \end{array}$$

"modified" Schubert varieties $Y_\lambda \subset Y_d$ each defined as the image by π of the preimage $T_\lambda \subset T_d$ of X_λ and a result identifying the three-point genus zero Gromov-Witten invariants in degree d with (in some cases up to a certain power of 2) intersection points of modified Schubert varieties in Y_d. (In types B and D when d is odd there is a codimension 3 subvariety of Y_d denoted Z_d° in [BKT09] to which we need to restrict our attention.) The fiber of π above a general point of Y_λ is a Grassmannian of particular type (e.g., the Lagrangian Grassmannian of a symplectic $2d$-space in type C). Letting λ^+ be the smallest partition containing λ so that according to [BKT03, proof of Cor. 2, 4 or 6] or [BKT09, Lemma 1.3, 2.1 or 3.1] the map $T_{\lambda^+} \to Y_{\lambda^+}$ is generically finite, the inequality $\dim Y_\lambda \geq \dim Y_{\lambda^+} =$

$\dim T_{\lambda^+}$ is enough to guarantee: the intersection of T_λ with the fiber of π above a general point of Y_λ has codimension at least 2 in the fiber and is generically smooth. Choose a general point of the intersection to be $f(p_1)$, then a general rational map of degree d in the fiber of π meets requirements (ii)-(iv), since we have an explicit description of a general rational curve on the fiber of π by [BKT09, Prop. 1.1]. For $d > d_0$ we simply have to take a degree d_0 curve as just described and attach $d - d_0$ copies of a degree 1 tail. If the point of attachment is general with respect to $f(p_1)$ then a general line will for dimension reasons be disjoint from the Schubert variety, and (ii)-(iv) remain valid.

□

Proof of the Theorem. In [FP97, Lemma 14] it is proven that, in the conditions of the Theorem, the intersection $ev_1^{-1}(g_1\Gamma_1)\cap\cdots\cap ev_s^{-1}(g_s\Gamma_s)$ is a finite set of reduced points, each corresponding to an irreducible source curve $\mathbb{P}^1$ with automorphism-free map to the target variety X, for general $(g_1,\ldots,g_s) \in G^s$. The number of these is the Gromov-Witten invariant.

We prove the Theorem first in the case when each of the Γ_i has codimension at least 2. We claim, for general $(g_1,\ldots,g_s) \in G^s$ each of these finitely many intersection points satisfies (i)-(iv) of the Lemma, with $\Gamma = g_1\Gamma_1$. Given this, we may repeat the argument with the other Γ_i in place of Γ_1 and find for general $(g_1,\ldots,g_s) \in G^s$ that for each of the finitely many intersection points $(C \cong \mathbb{P}^1, p_1,\ldots,p_s, f)$ and each i, the point p_i is the unique one on C having image contained in $g_i\Gamma_i$.

To prove the claim, by homogeneity we may fix $g_1 = e$ and verify the conditions for general $(g_2,\ldots,g_s) \in G^{s-1}$. Let $ev_1^{-1}(\Gamma_1^0)^*$ denote the open subset of $ev_1^{-1}(\Gamma_1^0)\cap M_{0,s}(X,d)$ satisfying (i)-(iv) of the Lemma. Now we apply the Kleiman--Bertini theorem to the diagram

$$\begin{array}{ccc} & & ev_1^{-1}(\Gamma_1^0)^* \\ & & \downarrow \\ \Gamma_2^0 \times \cdots \times \Gamma_s^0 & \longrightarrow & X^{s-1} \end{array}$$

with action of G^{s-1}. Then for general $(g_2,\ldots,g_s)$ the intersection is a finite set of reduced points, and there is no contribution to the intersection from points of

$ev_1^{-1}(\Gamma_1^0)$ not in $ev_1^{-1}(\Gamma_1^0)^*$. The claim is verified.

For the case when some of the Γ_i are divisors, we repeat the above argument but taking $ev_1^{-1}(\Gamma_1^0)^*$ to be defined by only conditions (i)-(iii) of the Lemma. Then for each of the surviving intersection points the map $f : \mathbb{P}^1 \to X$ has properties (i)-(iii) at every preimage point of Γ. It follows that there are d distinct choices for a marked point on $\mathbb{P}^1$ mapping to Γ_1. Hence there is a d^r-to-1 correspondence (with r as in the statement of the Theorem) between intersection points in $\overline{M}_{0,s}(X, d)$ and degree d rational curves on X satisfying the incidence conditions.

□

Remark 1.6. The three-point genus zero Gromov-Witten invariants are the ones that arise as structure constants in the small quantum cohomolgy ring; they are the ones given by the quantum Schubert calculus. For these, according to the Theorem, the Gromov-Witten invariant precisely counts rational curves on X except in one case: when $d = 2$ and one of the Γ_i has codimension 1, then the Gromov-Witten invariant is twice the number of conics satisfying the incidence conditions. This is only possible when X is an orthogonal Grassmannian, reflecting the presence of q^2 terms in the quantum Pieri formula for multiplication with the codimension 1 Schubert class (where q is the formal parameter of the quantum cohomology ring).

Chapter 2

Poincaré polynomials of stable map spaces to Grassmannians

As seen in the previous chapter moduli spaces of stable maps have been used to give answers to many problems in enumerative geometry.

At first, techniques were developed which allowed intersection theory to be carried out in top codimension in arbitrary moduli spaces of stable maps using indirect methods such as localization under torus actions [Kon95]. It was acknowledged that presentations for the Chow rings of $\overline{M}_{g,s}(X,\beta)$ would give a much better understanding of their geometry and would allow arbitrary calculation of intersections in these spaces.

The computation of the Betti numbers has proven to be useful in the calculation of Chow or cohomology rings of different spaces. That is the case of the work of Getzler and Pandharipande in [GP06], where they computed Betti numbers in the case of a projective space target. Their results were used later by Behrend and O'Halloran [BO03] in the computation of cohomology rings of the stable map spaces $\overline{M}_{0,0}(\mathbb{P}^n, d)$. In [Beh], Behrend computes the Betti numbers of the space of stable maps of genus zero and degree three to the infinite-dimensional Grassmannian $\overline{M}_{0,0}(G(k,\infty), 3)$ using the methods described in [GP06]. In this chapter, we will consider the particular case when X is the Grassmann variety of k-dimensional planes in $\mathbb{C}^n$, $G(k,n)$. The Betti numbers of the spaces

$\overline{M}_{0,0}(G(k,n),d)$, for $d = 2$ and 3, are presented in the form of their generating functions: the Poincaré polynomials. For that, we use a method different from those mentioned above: we consider a $\mathbb{C}^*$-action on $\overline{M}_{0,0}(G(k,n),d)$ with isolated fixed points. Making use of results by Oprea, see [Opr06], the cohomology (with $\mathbb{Q}$ coefficients) of our moduli stacks of stable maps is determined from the fixed loci in the manner of the usual Białynicki-Birula decomposition of smooth projective varieties, see [BB73] and [BB76]. The method developed here can be, in principle, applied to maps of arbitrary degree, as it relies on the combinatorial description of the fixed points of the torus action on the moduli stack.

It is relevant to mention that Strømme [Str87] worked out the cohomology of quot schemes parametrizing maps from the projective line to a Grassmannian variety using torus actions and the Białynicki-Birula decomposition already mentioned above. Later, Chen generalized the method to the case of hyperquot schemes and computed a generating function for the Poincaré polynomials of these, see [Che01].

2.1 Set-up

In [Kon95, §3.2.], Kontsevich makes a description of $\overline{M}_{g,r}(\mathbb{P}^n,d)^{\mathbf{T}}$, the fixed locus of the moduli space of degree d stable maps under the action of the group $\mathbf{T} \simeq (\mathbb{C}^*)^{n+1}$ of diagonal matrices. To keep track of the combinatorial data arising from this description, the fixed points are written as labelled connected graphs. In our case a similar description can be done. The torus action on $\mathbb{P}^{n-1}$ induces a $(\alpha_1,\dots,\alpha_n)$-weighted action on the Grassmannian $G(k,n)$. We will denote fixed points of the action by k-tuples of natural numbers in the following way: for each $a_1 < \cdots < a_k$, the k-tuple $(a_1 \dots a_k)$ represents the point in $G(k,n)$ corresponding to the matrix H with ones in the entries $(h_{\ell a_\ell})_{\ell=1,\dots,k}$, and zeros elsewhere. Two fixed points are joined by a line if both points have all the coordinates but one equal.

For example, in $G(2,4)$, the point (12) and the line (123) joining (12) and (13), correspond to the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v & 0 \end{pmatrix},\ [u:v] \in \mathbb{P}^1,$$

respectively.

Besides, in [Kon95, §3.2.] complete formulas for the weights of the fixed loci are displayed in terms of combinatorial data.

The following results list the extra weights obtained after considering the inclusion of $G(k,n)$ into $G(k,n+1)$ and $G(k+1,n+1)$, respectively.

Proposition 2.1. *Let V be a subvector space of $\mathbb{C}^{n+1}$ spanned by n coordinate vectors, $V = span(e_1, \ldots, \widehat{e}_m, \ldots, e_{n+1})$. Let us consider $f : C \to G(k,V)$ a torus-invariant genus 0 stable map. Then the inclusion $\iota : G(k,V) \to G(k,\mathbb{C}^{n+1})$ induces an injective map $H^0(C, f^*T_{G(k,V)}) \to H^0(C, f^*\iota^*T_{G(k,\mathbb{C}^{n+1})})$ whose cokernel is given by the following list of weights*

$$[\alpha_m - \alpha_{a_1}, \ldots, \alpha_m - \alpha_{a_{k-1}}, \{\alpha_m - (\frac{s}{d_i}\alpha_{b_u} + \frac{t}{d_i}\alpha_{b_v})\}_{0\le s,t\le d_i, s+t=d_i}]$$

for every irreducible component of C mapping with degree d_i to the line joining $(a_1, \ldots, a_{k-1}, b_u)$ and $(a_1, \ldots, a_{k-1}, b_v)$, with the following weights removed

$$[\alpha_m - \alpha_{a_1}, \ldots, \alpha_m - \alpha_{a_k}]$$

for every node of C mapping to the point $(a_1, \ldots, a_k)$.

Proof. The exact sequence

$$0 \longrightarrow T_{G(k,V)} \longrightarrow \iota^*T_{G(k,\mathbb{C}^{n+1})} \longrightarrow N \longrightarrow 0,$$

where $N = \iota^*T_{G(k,\mathbb{C}^{n+1})}/T_{G(k,V)}$, gives rise to an exact sequence in cohomology. Using the vanishing $H^1(C, f^*T_{G(k,V)}) = 0$, [FP97, Lemma 10], it reduces to the following short exact sequence

$$0 \longrightarrow H^0(C, f^*T_{G(k,V)}) \longrightarrow H^0(C, f^*\iota^*T_{G(k,\mathbb{C}^{n+1})}) \longrightarrow H^0(C, f^*N) \longrightarrow 0.$$

The normal bundle N equals $S^* \otimes \iota^*\widetilde{Q}/Q$, where S is the universal subbundle, and Q and $\widetilde{Q}$ are the quotient subbundles of $G(k,V)$ and $G(k,\mathbb{C}^{n+1})$ respectively.

Let $\phi : \widehat{C} \to C$ be the normalization. So, the following sequence is exact

$$0 \longrightarrow H^0(C, f^*N) \longrightarrow H^0(\widehat{C}, \phi^*f^*N) \longrightarrow \oplus N_{p_i} \longrightarrow H^1(C, f^*N)$$

where p_i denote the images in $G(k,V)$ of the nodes of C.

Since S^* is globally generated, we have $H^1(C, f^*S^*) = 0$. The vanishing of $H^1(C, f^*N)$ follows from that of $H^1(C, f^*S^*)$, since $\iota^*\widetilde{Q}/Q$ is a trivial line bundle.

The list of weights firstly given corresponds to $H^0(\widehat{C}, \phi^* f^*N)$, whereas the term $\oplus N_{p_i}$ gives the list of weights to be removed.

□

Corollary 2.2. *The map $T_{\overline{M}_{0,0}(G(k,V),d),(C,f)} \to T_{\overline{M}_{0,0}(G(k,\mathbb{C}^{n+1}),d),(C,\iota\circ f)}$, induced by the inclusion ι in Proposition 1, is injective and the cokernel is given by exactly the weights listed above, at the point (C, f) in $\overline{M}_{0,0}(G(k, V), d)$.*

Proposition 2.3. *Let $\{e_1, \ldots, e_m, \ldots, e_{n+1}\}$ be a basis for $\mathbb{C}^{n+1}$ and V be the subvector space given by $\mathbb{C}^{n+1}/\{e_m\}$. Let us consider $f : C \to G(k, V)$ a torus-invariant genus 0 stable map. Then the inclusion $\kappa : G(k, V) \to G(k+1, \mathbb{C}^{n+1})$ induces an injective map $H^0(C, f^*T_{G(k,V)}) \to H^0(C, f^*\kappa^*T_{G(k+1,\mathbb{C}^{n+1})})$ whose cokernel is given by the following list of weights*

$$[\alpha_{a'_1} - \alpha_m, \ldots, \alpha_{a'_{n-k-1}} - \alpha_m, \{(\frac{s}{d_i}\alpha_{b_u} + \frac{t}{d_i}\alpha_{b_v}) - \alpha_m\}_{0\leq s,t\leq d_i, s+t=d_i}]$$

for every irreducible component of C mapping with degree d_i to the line joining $(a_1, \ldots, a_{k-1}, b_u)$ and $(a_1, \ldots, a_{k-1}, b_v)$, with the following weights removed

$$[\alpha_{a'_1} - \alpha_m, \ldots, \alpha_{a'_{n-k}} - \alpha_m]$$

for every node of C mapping to the point $(a_1, \ldots, a_k)$, and where $(a'_1, \ldots, a'_{n-k})$ denotes the complement of $(a_1, \ldots, a_k)$.

Proof. Arguing as in the proof of the previous proposition, we start by considering the short exact sequence in cohomology arising from the exact sequence

$$0 \longrightarrow T_{G(k,n)} \longrightarrow \kappa^*T_{G(k+1,n+1)} \longrightarrow N \longrightarrow 0,$$

where $N = \kappa^*T_{G(k+1,n+1)}/T_{G(k,n)}$. In this case, the normal bundle N equals $(\ker(\widetilde{S} \to S))^* \otimes Q$, where Q is the quotient subbundle, and S and $\widetilde{S}$ are the universal subbundles of $G(k, n)$ and $G(k+1, n+1)$ respectively.

An argument similar to that used in the previous proof, gives us the list of weights in the statement: the first of them corresponds to $H^0(\widehat{C}, \phi^* f^*N)$, whereas the term $\oplus N_{p_i}$ gives the list of weights to be removed. □

Corollary 2.4. *The map* $T_{\overline{M}_{0,0}(G(k,V),d),(C,f)} \to T_{\overline{M}_{0,0}(G(k+1,\mathbb{C}^{n+1}),d),(C,\kappa\circ f)}$*, induced by the inclusion* κ *in Proposition 2, is injective and the cokernel is given by exactly the weights listed above, at the point* (C, f) *in* $\overline{M}_{0,0}(G(k, V), d)$*.*

2.2 The Poincaré polynomials

Oprea in [Opr06] obtains a description of a Białynicki-Birula decomposition of the stack of stable maps. Using this result, the Betti numbers of the moduli stack can be read off the weights of the tangent bundle. The use of this technique together with the results presented above allows us to compute the Betti numbers of the spaces of stable maps to arbitrary classical Grassmannians.

2.2.1 The degree 2 stable maps case

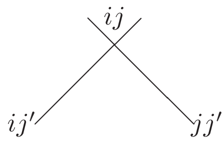
Theorem 2.5. *Let X denote the moduli stack of stable maps* $\overline{M}_{0,0}(G(k, n), 2)$*. Then its Poincaré polynomial is given by*

$$P_X(q) = \frac{\big((1+q^n)(1+q^3) - q(1+q)(q^k+q^{n-k})\big)\prod_{i=k}^{n}(1-q^i)}{(1-q)^2(1-q^2)^2\prod_{i=1}^{n-k-1}(1-q^i)}.$$

Proof. In the case of degree two maps, fixed points of the $\mathbb{C}^*$-action in an arbitrary Grassmannian $G(k, n)$ carry some kind of redundancy and can be described as points in a smaller Grassmannian. For instance, the tree of $\mathbb{P}^1$'s in $G(3, 4)$ fixed by the corresponding action: $\{(1423), (2413); (124)\}$, can actually be seen as $\{(123), (213); (12)\}$ in $G(2, 3)$. Using this observation, in the case of degree two maps, we only need to consider fixed points in $G(1, 2)$, $G(1, 3)$, $G(2, 3)$, and $G(2, 4)$.

The contributions of the different fixed points of the $\mathbb{C}^*$-action on $\overline{M}_{0,0}(G(k, n), 2)$ to the Poincaré polynomial are all computed in a similar way. Here only some of the possible cases will be worked out in detail.

Let us first consider the contribution to the fixed points of $\overline{M}_{0,0}(G(k, n), 2)$ given by



$$i < j < j'$$

Taking $i = 1$, $j = 2$, and $j' = 3$ on $G(2,3)$ we get a list of 5 weights. Of these, 4 are positive, under the convention on weights $\alpha_1 \ll \alpha_2 \ll \alpha_3$.

We consider now the process of extending to $G(k,n)$:

$$S_0 \cup T_0 = \{1, \dots, i-1\}, \qquad S_1 \cup T_1 = \{i+1, \dots, j-1\}$$
$$S_2 \cup T_2 = \{j+1, \dots, j'-1\}, \qquad S_3 \cup T_3 = \{j'+1, \dots, n\}$$

with $i_\nu = \#S_\nu$ and $j_\nu = \#T_\nu$ $(0 \le \nu \le 3)$ related by

$$i_0 + i_1 + i_2 + i_3 = k-2, \qquad i_0 + j_0 = i-1, \qquad i_2 + j_2 = j' - j - 1,$$
$$j_0 + j_1 + j_2 + j_3 = n-k-2, \qquad i_1 + j_1 = j - i - 1, \qquad i_3 + j_3 = n - j'.$$

If we identify

$$G(2,3) = G(2, \mathbb{C}\langle e_i, e_j, e_{j'}\rangle),$$
$$G(2, n-k+2) = G(2, \mathbb{C}\langle e_\tau,\ \tau \in \{i, j, j'\} \cup T_0 \cup T_1 \cup T_2 \cup T_3\rangle)$$

the embedding $G(2,3) \hookrightarrow G(2, n-k+2)$ gives rise to the following added weights:

$$\alpha_t - \alpha_i \qquad \alpha_t - \alpha_j \qquad t \in T_0 \cup T_1 \cup T_2 \cup T_3$$
$$\alpha_t - \alpha_{j'} \qquad \alpha_t - \alpha_{j'}$$

The further embedding in $G(k,n)$ gives rise to the additional weights:

$$\alpha_i - \alpha_s \qquad \alpha_{j'} - \alpha_s \qquad s \in S_0 \cup S_1 \cup S_2 \cup S_3$$
$$\alpha_j - \alpha_s$$
$$\alpha_t - \alpha_s \qquad t \in T_0 \cup T_1 \cup T_2 \cup T_3$$

If we let $n_\nu = \#\{(s,t) \in S_\nu \times T_\nu \mid t > s\}$ then the number of added weights under $G(2,3) \hookrightarrow G(k,n)$ that are positive is

$$n_0 + n_1 + n_2 + n_3 + 3i_0 + 2i_1 + j_1 + 2j_2 + 4j_3$$
$$+ i_0 j_1 + i_0 j_2 + i_0 j_3 + i_1 j_2 + i_1 j_3 + i_2 j_3.$$

Remark 2.6. Sums over partitions of a finite set into the disjoint union of two sets of given cardinality give rise to q-binomialcoefficients:

$$\sum_{\substack{S\cup T=\{1,\ldots,n\}\\ \#S=k,\#T=n-k}} q^{\#\{(s,t)\in S\times T|t>s\}} = \binom{n}{k}_q .$$

Therefore the contribution to the Poincaré polynomial of $\overline{M}_{0,0}(G(k,n),2)$ is

$$\sum q^{4+3i_0+2i_1+i_2+j_1+2j_2+4j_3+i_0j_1+i_0j_2+i_0j_3+i_1j_2+i_1j_3+i_2j_3} \qquad (\star)$$
$$\cdot \binom{i_0+j_0}{i_0}_q \binom{i_1+j_1}{i_1}_q \binom{i_2+j_2}{i_2}_q \binom{i_3+j_3}{i_3}_q ,$$

where the sum is over $i_0+i_1+i_2+i_3 = k-2$, $i_0+j_0 = i-1$, $i_1+j_1 = j-i-1$, $i_2+j_2 = j'-j-1$, $i_3+j_3 = n-j'$.

To compute sums of this kind we make use of the identities on q-binomial coefficients given in Section 2.3 .

For example, applying the corresponding identity to $\sum q^{(i_2+2)j_3}\binom{i_2+j_2}{i_2}_q\binom{i_3+j_3}{i_3}_q$, the sum $(\star)$ equals

$$\sum q^{4+3i_0+2i_1+i_2+j_1+2j_2'+i_0j_1+i_0j_2'+i_1j_2'} \cdot \binom{n-j+1}{i_2+i_3+2}_q \binom{i_0+j_0}{i_0}_q \binom{i_1+j_1}{i_1}_q$$
$$-\sum q^{7+4i_0+3i_1+2i_2+j_1+2j_2'+i_0j_1+i_0j_2'+i_1j_2'} \cdot \binom{n-j}{i_2+i_3+2}_q \binom{i_0+j_0}{i_0}_q \binom{i_1+j_1}{i_1}_q ,$$

where $j_2' = j_2+j_3$, and the sums are over $i_0+i_1+i_2+i_3 = k-2$, $i_0+j_0 = i-1$, $i_1+j_1 = j-i-1$, and $i_2+i_3+j_2' = n-j-1$ and $i_2+i_3+j_2' = n-j-2$ respectively.

We recognize that the sums above are equal to

$$q^4 \sum_{i_0+i_1+i_2+i_3=k-2} q^{3i_0+2i_1+i_2}\binom{n+1}{k+2}_q - q^7 \sum_{i_0+i_1+i_2+i_3=k-2} q^{4i_0+3i_1+2i_2}\binom{n}{k+2}_q .$$

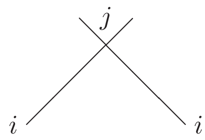
Continuing applying identities on q-binomial coefficients leads to the expression

$$q^4\binom{k+1}{3}_q\binom{n+1}{k+2}_q - q^7\binom{k+2}{4}_q\binom{n}{k+2}_q + q^8\binom{k+1}{4}_q\binom{n}{k+2}_q,$$

and expanding the q-coefficients we get:

$$\left(q^4\frac{(1-q^{n+1})}{(1-q^{k+2})} - q^7\frac{(1-q^{n-k-1})}{(1-q^4)} + q^8\frac{(1-q^{n-k-1})(1-q^{k-2})}{(1-q^{k+2})(1-q^4)}\right)\frac{(1-q^{k-1})}{(1-q^3)}\binom{n}{k+1}_q\binom{k+1}{2}_q .$$

Another degree two map, fixed by the torus action, is that given by the following tree of $\mathbb{P}^1$'s:



$j < i$

Taking $j = 1$ and $i = 2$ on $G(1,2)$ we get a list of 2 weights which are all positive, under the convention on weights $\alpha_1 \ll \alpha_2$.

We consider again the process of extending to $G(k,n)$:

$$S_0 \cup T_0 = \{1,\ldots,j-1\}, \quad S_1 \cup T_1 = \{j+1,\ldots,i-1\}, \quad S_2 \cup T_2 = \{i+1,\ldots,n\}$$

with $i_\nu = \#S_\nu$ and $j_\nu = \#T_\nu$ $(0 \leq \nu \leq 2)$ related by

$$\begin{aligned} &i_0+i_1+i_2 = k-1, && i_0+j_0 = j-1, && i_2+j_2 = n-i, \\ &j_0+j_1+j_2 = n-k-1, && i_1+j_1 = i-j-1. \end{aligned}$$

If we identify

$$\begin{aligned} G(1,2) &= G(1,\mathbb{C}\langle e_j, e_i\rangle) \\ G(1,n-k+1) &= G(1,\mathbb{C}\langle e_\tau,\ \tau \in \{j,i\} \cup T_0 \cup T_1 \cup T_2\rangle), \end{aligned}$$

then the embedding $G(1,2) \hookrightarrow G(1,n-k+1)$ gives rise to the following list of added weights:

$$[\alpha_t - \alpha_i, \alpha_t - \alpha_j, \alpha_t - \alpha_i \;:\; t \in T_0 \cup T_1 \cup T_2],$$

and the further embedding in $G(k,n)$ gives rise to the additional weights:

$$[\alpha_i - \alpha_s, \alpha_j - \alpha_s, \alpha_j - \alpha_s, \alpha_t - \alpha_s \;:\; s \in S_0 \cup S_1 \cup S_2, t \in T_0 \cup T_1 \cup T_2].$$

If again we let $n_\nu = \#\{(s,t) \in S_\nu \times T_\nu \,|\, t > s\}$ then the number of added weights under $G(1,2) \hookrightarrow G(k,n)$ that are positive is

$$n_0 + n_1 + n_2 + 3i_0 + i_1 + j_1 + 3j_2 + i_0 j_1 + i_0 j_2 + i_1 j_2.$$

Using again Remark 2.6, the contribution of this configuration to the Poincaré polynomial of $\overline{M}_{0,0}(G(k,n),2)$ is

$$\sum q^{2+3i_0+i_1+j_1+3j_2+i_0j_1+i_0j_2+i_1j_2} \cdot \binom{i_0+j_0}{i_0}_q \binom{i_1+j_1}{i_1}_q \binom{i_2+j_2}{i_2}_q, \qquad (\star\star)$$

where the sum is over $i_0 + i_1 + i_2 = k - 1$, $i_0 + j_0 = j - 1$, $i_1 + j_1 = i - j - 1$, $i_2 + j_2 = n - i$.

As we did earlier we make use of the identities on q-binomial coefficients in Section 2.3 until arriving to the expression:

$$q^2 \left(\binom{k+2}{3}_q \binom{n+1}{k+2}_q - q^2 \binom{k+1}{3}_q \binom{n+1}{k+2}_q - q^2 \binom{k+3}{4}_q \binom{n}{k+2}_q + q^3 \binom{k+2}{4}_q \binom{n}{k+2}_q \right.$$
$$\left. + q^5 \binom{k+2}{4}_q \binom{n}{k+2}_q - q^6 \binom{k+1}{4}_q \binom{n}{k+2}_q \right).$$

The different configurations for the fixed points give different sums. Added up they give the expression presented as the generating function for the Poincaré polynomial of $\overline{M}_{0,0}(G(k,n),2)$.

□

2.2.2 The degree 3 stable maps case

In this subsection we present the generating function for the Poincaré polynomial in the case of no pointed, genus zero, degree three stable maps to the Grassmannian. The method used to organize the combinatorial data involved and prove Theorem 2.7 is the same as the one described for the case of degree 2 stable maps. The number of different configurations for the labelled connected graphs representing the fixed locus increases rapidly, what makes the follow up a bit more complicated.

Theorem 2.7. *Let X be the moduli stack of stable maps $\overline{M}_{0,0}(G(k,n),3)$, then its Poincaré polynomial $P_X(q)$ is given by the following expression:*

$$\frac{(F_1(q)(1+q^{2n})+(1+q)^2(F_2(q)q^n(1+q^2)-F_3(q)q(1+q^n)(q^k+q^{n-k}))+F_4(q)q^2(q^{2k}+q^{2n-2k}))\prod_{i=k}^{n}(1-q^i)}{(1-q)^2(1-q^2)^3(1-q^3)^2\prod_{i=1}^{n-k-1}(1-q^i)}$$

where

$$F_1(q) = 1 + 2q^2 + 3q^3 + 3q^4 - q^5 + q^6 - 3q^7 - 3q^8 - 2q^9 - q^{11},$$
$$F_2(q) = 1 + 5q^2 + 2q^3 - 2q^4 - 5q^5 - q^7,$$
$$F_3(q) = 2 + 3q^2 + q^3 - q^4 - 3q^5 - 2q^7,$$
$$F_4(q) = 1 + 6q + 3q^2 + 2q^3 - 2q^4 - 3q^5 - 6q^6 - q^7.$$

2.3 Some identities on q-Binomial coefficients

The *q-binomial coefficients* are classical combinatorial numbers with diverse interpretations. They arise in number theory, combinatorics, linear algebra, and finite geometry.

We define for $0 \leq k \leq n$ the *q-binomial coefficient* $\binom{n}{k}_q$, firstly introduced by Gauss, as

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)\cdots(1-q)}.$$

The following basic identities are proven in [Gau73]:

$$\begin{aligned}
\binom{n}{k}_q &= \binom{n}{n-k}_q, \\
\binom{n}{k}_q &= \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q, \\
\binom{n}{k}_q &= q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \\
\sum_{j=0}^{a} q^j \binom{d+j}{j}_q &= \binom{d+a+1}{a}_q.
\end{aligned}$$

One further identity, established using the basic identities in q-binomial coefficients above, is

$$\sum_{i+j=a} q^{(c+1)j}\binom{c+i}{i}_q\binom{d+j}{j}_q = \binom{c+d+a+1}{a}_q.$$

More general identities can be obtained for $\sum_{i+j=a} q^{(c+n)j}\binom{c+i}{i}_q\binom{d+j}{j}_q$, whenever $n > 0$, using the previous identities. As seen, sums of this kind are used in the proof of Theorems 2.5 and 2.7, but they will not be stated explicitly.

Chapter 3

Virtual normalization and virtual fundamental classes

An important aspect of moduli spaces in algebraic geometry is the notion of *virtual fundamental class*, introduced by K. Behrend and B. Fantechi in [BF97].

Inspired by work of Fukaya and Oh in the symplectic case, B. Kim, A. Kresch, and Y.G. Oh [KKO] present a new compactification of the space of maps from pointed nonsingular projective stable curves to a nonsingular complex projective variety with given ramification indices at the marked points.

B. Kim [Kim] constructed the moduli space of log stable maps with given ramification indices. Each space is equipped with a virtual fundamental class. In this chapter we compare the virtual fundamental classes, and show, in particular, that they determine the same numerical invariants.

3.1 Virtual normalization

We will assume the reader is familiar with basic concepts (cf. [Kat89]) from logarithmic geometry, and adopt the notation in [Ols03] unless otherwise stated.

Let S denote a normal locally noetherian base scheme. Throughout the chapter, schemes will be locally of finite type over S, and algebraic stacks will be locally of finite type over S with finite-type though not necessarily separated diagonal. All monoids will be commutative.

Notation: We will use $P \subset P^{sat} \subset P^{gp}$ for an integral monoid P. For P a fine monoid (i.e. integral and finitely generated), $\mathcal{S}_P$ will denote the algebraic stack $[\mathrm{Spec}(S[P])/\mathrm{Spec}(S[P^{gp}])]$.

Lemma 3.1. *For any fine monoid P the natural morphism*

$$\mathcal{S}_{P^{sat}} \to \mathcal{S}_P$$

is a normalization.

Proof. This is immediate from the (standard) fact that $\mathrm{Spec}(S[Q])$ is normal for any saturated torsion-free fine monoid Q. □

Let us introduce some further notation: we will denote by $\mathcal{L}og$ the category of schemes with fine log structures treated in [Ols03], and as in [Ols05] $\mathcal{L}^1$ will denote the stack whose fiber over a scheme T is the category of pairs of fine log structures on T with a morphism between them. Furthermore, $\widehat{\mathcal{L}og}$ will denote the substack of $\mathcal{L}^1$ where the morphism of log structures is saturation.

Proposition 3.2. *The natural forgetful map (forgetting the saturation)*

$$\widehat{\mathcal{L}og} \to \mathcal{L}og$$

is a normalization.

Proof. By the cartesian diagram

$$\begin{array}{ccc} \mathcal{S}_{P^{sat}} & \longrightarrow & \widehat{\mathcal{L}og} \\ \downarrow & & \downarrow \\ \mathcal{S}_P & \longrightarrow & \mathcal{L}og \end{array}$$

the result follows from Lemma 3.1.

The diagram above is clearly commutative, and by Corollary 5.25 of [Ols03] both horizontal arrows are étale. Therefore, to prove it is cartesian we are reduced to checking a bijection between k-points (where k is an algebraically closed field). By Lemma 2.15 of [Kat96] we have this bijection. □

Definition 3.3. Let $\mathcal{S}$ be an algebraic stack, X a scheme and $X \to \mathcal{S}$ a morphism. Let $\widehat{\mathcal{S}} \to \mathcal{S}$ be a normalization. Then the *virtual normalization* of X with respect to $\mathcal{S}$ is the fiber product $X \times_{\mathcal{S}} \widehat{\mathcal{S}}$.

When X has a relative perfect obstruction theory over $\mathcal{S}$, it may be described as virtually smooth over $\mathcal{S}$. Since normalization commutes with smooth base change, the fiber product in the definition makes sense as a virtual normalization. That will be the setting in Section 3.2.

Example 3.4. When X is smooth over $\mathcal{S}$, the normalization coincides with the virtual normalization. The morphism

$$\phi : \mathscr{B}^{\mathrm{bal}}_{g,n}(\mathscr{S}_d) \to \mathscr{A}dm_{g,n,d}$$

of [ACV03, Prop. 4.2.2], which is a normalization, is therefore also a virtual normalization for the morphism from $\mathscr{A}dm_{g,n,d}$ to $\mathcal{L}og$ described in [Moc95, §3B].

In the next result, virtual normalization is again taken with respect to the stack $\mathcal{L}og$. The fiber products in question are described in Lemma 2.15 and Corollary 2.16 of [Kat96], respectively Proposition 2.7 of [Kat89].

Corollary 3.5. *The fiber product of a pair of morphisms in the category of schemes with fs log structures is the virtual normalization of the fiber product of the same morphisms in the category of schemes with fine log structures.*

3.2 Stable ramified and log stable ramified map spaces

Our goal in Section 3.3 will be to compare the virtual fundamental class of the stack of (g, β, μ)-stable ramified maps $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$ with that of the stack of (g, β, μ)--log stable ramified maps $\overline{\mathfrak{U}}^{\log}_{g,\mu}(X, \beta)$, constructed using the machinery described by B. Kim in [Kim]. Let us now recall some definitions.

Let k be an algebraically closed field of characteristic zero. Let X be a nonsingular projective variety over k. In [KKO], B. Kim, A. Kresch, and Y.G. Oh gave a definition for (g, β, μ)-stable ramified maps: n-pointed, genus g stable maps to a Fulton-MacPherson (FM) degeneration space with prescribed ramification indices at the marked points. We refer the reader to [FM94] for foundational material on FM degeneration spaces, to [FP97] for an introduction to stable maps, and to [KKO] for the notation not made explicit here.

Definition 3.6 ([KKO]). Let $\beta \in NE_1(X)$ be an element in the semigroup of effective curve classes. Let $g, n \in \mathbb{Z}_{\geq 0}$, and $\overline{\mu} = (\mu_1, \dots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$.

A triple

$$\big((\pi : C \to S, p_1, \dots, p_n), (\pi_{W/S} : W \to S, \pi_{W/X} : W \to X), f : C \to W\big)$$

is called a *stable map with $\overline{\mu}$-ramification from a n-pointed, genus g curve to a FM degeneration space W of X, representing class β* if:

i. $(C, p_1, \dots, p_n)$ is a n-pointed, genus g, prestable curve over k,

ii. $(\pi_{W/S} : W \to S, \pi_{W/X} : W \to X)$ is a FM degeneration of X over S,

iii. $f : C \to W$ is a morphism over S,

iv. over each geometric point of S the pushforward of the fundamental class of C is β,

v. the following four conditions are satisfied:

(a) *Prescribed Ramification Index Condition:*

- $C^{sm} = f^{-1}(W^{sm})$,
- $f|_{C^{sm}}$ is unramified everywhere possibly at p_i,
- f has ramification index μ_i at p_i.

(b) *Distinct Points Condition:* $f(p_i)$, $i = 1, \dots n$ are pairwise distinct points of W, over each geometric point of S.

(c) *Admissibility Condition:* For any geometric point t of S, if p is a nodal point of C_t, then $\widehat{f}^*$ can be expressed as:

$$\widehat{\mathcal{O}}_{f(p)} \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[z_1, \dots, z_{r+1}]]/(z_1 z_2 - s) \xrightarrow{\widehat{f}^*} \widehat{\mathcal{O}}_p \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[x, y]]/(xy - s')$$

with $\widehat{f}^* z_1 = \alpha_1 x^m$, $\widehat{f}^* z_2 = \alpha_2 y^m$, no restriction on z_i for $i > 2$, α_i being units in $\widehat{\mathcal{O}}_p$ with condition $\alpha_1 \alpha_2 \in \widehat{\mathcal{O}}_{\pi_S(p)}$, $s, s' \in \widehat{\mathcal{O}}_{\pi_S(p)}$, and m being a positive integer.

(d) *Stability Condition:* over each geometric point of S, the following are satisfied:

- For each ruled component W_r of W, there is an image of a marking in W_r ($f(p_i) \in W_r$ for some i), or a non-fiber image $f(D) \subset W_r$ of an irreducible component D of C,
- For each end component $W_e \cong \mathbb{P}^r$ of W, there are either images of two distinct markings in W_e or a non-line image $f(D) \subset W_e$ of an irreducible component D of C.

The stack of (g, β, μ)-stable ramified maps to FM degeneration spaces of X is denoted by $\overline{\mathfrak{U}}_{g,\mu}(X, \beta)$. They proved it to be a Deligne-Mumford stack of finite type carrying a perfect obstruction theory:

$$R\pi_*(f^*T_W^\dagger(-\sum \mu_i p_i))^\vee \to L^\bullet_{\overline{\mathfrak{U}}_{g,\mu}(X,\beta)/\mathfrak{B}}$$

over the stack $\mathfrak{B}$ of n-pointed, genus g, prestable curves, FM spaces (of X, with n-tuples of smooth pairwise distinct points), fine log structures, and pairs of morphisms of log structures:

$$(C \to S, W \to S, N, N^{C/S} \to N, N^{W/S} \to N).$$

By the results of Behrend and Fantechi [BF97], this perfect obstruction theory yields a virtual fundamental class $[\overline{\mathfrak{U}}_{g,\mu}(X, \beta)]^{\mathrm{vir}}$.

Now, let us introduce the definition of the second space of interest to us, as well as the perfect obstruction theory carried by it. In [Kim, 5.2.7.], Kim defines *log stable μ-ramified maps*. From this construction, we now introduce the notion of (g, β, μ)*-log stable ramified map*.

Definition 3.7. Let $\beta \in NE_1(X)$. Let $g, n \in \mathbb{Z}_{\geq 0}$, and $\overline{\mu} = (\mu_1, \dots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$. A pair

$$\big((f : (C, M_C, p_1, \dots, p_n) \to (W, M_W))/(S, N), \pi_{W/X} : W \to X\big)$$

is called a (g, β, μ)*-log stable ramified map* if:

i. $\big((C, M)/(S, N), p_1, \dots, p_n\big)$ is a n-pointed, genus g, minimal log prestable curve,

ii. $(W, M_W)/(S, N)$ is a log twisted FM type space,

iii. $f : (C, M_C) \to (W, M_W)$ is a log morphism over (S, N),

iv. $\underline{f} : C \to W$, the underlying map to f, is a stable (g, β, μ)-ramified map over S.

As mentioned before we use the notation $\overline{\mathfrak{U}}^{\log}_{g,\mu}(X, \beta)$ for the stack of such maps. By Kim's construction, this stack also carries a perfect obstruction theory:

$$(R\pi_* f^* T^{\dagger}_{\mathfrak{X}^+/\mathfrak{X}}(-\sum \mu_i p_i))^{\vee} \to L^{\bullet}_{\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)/\mathfrak{MB}}$$

over $\mathfrak{MB} := \mathfrak{M}^{\log}_{g,n} \times_{\mathcal{L}og} \mathfrak{X}^{+(n),\mathrm{tw}}$, giving rise to a virtual fundamental class $[\overline{\mathfrak{U}}^{\log}_{g,\mu}(X, \beta)]^{\mathrm{vir}}$. Here, following [KKO], $\mathfrak{X}$ denotes the stack of FM degeneration spaces, with universal FM space $\mathfrak{X}^+ \to \mathfrak{X}$, we have the space $\mathfrak{X}^{+(n)}$, open in the n-fold fiber product of $\mathfrak{X}^+$ over $\mathfrak{X}$ of pairwise distinct n-tuples of points, and the tw denotes log twisted FM type space as in [Kim]. (Because all nodes are distinguished, we may use log twisted rather than extended log twisted FM type spaces here.) So $\mathfrak{MB}$ is an open substack of $\mathfrak{B}$.

Proposition 3.8. *Let N' be the log structure on $\overline{\mathfrak{U}}^{\log}_{g,\mu}(X, \beta)$ whose sheaf of monoids is the subsheaf of $N^{C/S}$ generated by $m \log(s')$ for every node of C. Then, the commuting diagram*

$$\begin{array}{ccc} (\overline{\mathfrak{U}}^{\log}_{g,\mu}(X, \beta), N) & \longrightarrow & (\overline{\mathfrak{U}}_{g,\mu}(X, \beta), N^{W/S}) \\ \downarrow & & \downarrow \\ (\overline{\mathfrak{U}}_{g,\mu}(X, \beta), N^{C/S}) & \longrightarrow & (\overline{\mathfrak{U}}_{g,\mu}(X, \beta), N') \end{array}$$

is cartesian in the category of schemes with fs log structures.

Proof. This follows directly from the definition of log prestable map in [Kim, 5.2.2.]. □

3.3 Comparison of virtual fundamental classes

Let us define $\widehat{\mathfrak{B}} = \mathfrak{B} \times_{\mathcal{L}og} \widehat{\mathcal{L}og}$. Then, we have the cartesian diagram:

$$\begin{array}{ccccc}
\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta) & \longrightarrow & \widehat{\mathfrak{B}} & \longrightarrow & \widehat{\mathcal{L}og} \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow & & \Big\downarrow \\
\overline{\mathfrak{U}}_{g,\mu}(X,\beta) & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathcal{L}og
\end{array}$$

where ϕ is the forgetful map. (The outer square is cartesian by Corollary 3.5 and Proposition 3.8.) From $\widehat{\mathfrak{B}}$ to $\mathfrak{B}$ there is another morphism, which forgets the not-necessarily-saturated log structure, and this morphism is relatively Deligne--Mumford and étale by [Ols05, Proposition 2.11]. It follows that $L^{\bullet}_{\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)/\mathfrak{M}\mathfrak{B}} = L^{\bullet}_{\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)/\widehat{\mathfrak{B}}}$.

Theorem 3.9. *We have the equality of virtual fundamental classes*

$$[\overline{\mathfrak{U}}_{g,\mu}(X,\beta)]^{\mathrm{vir}} = \phi_*[\overline{\mathfrak{U}}^{\log}_{g,\mu}(X,\beta)]^{\mathrm{vir}}.$$

Proof. By the above cartesian diagram, this follows from [Cos06, Theorem 5.0.1], as normalizations have generic degree 1 and the compatibility of obstruction theories follows from the equality of cotangent complexes above.

□

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