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Min-max constructions of 2d-minimal surfaces

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Min–Max Constructions of 2d–Minimal Surfaces

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Abstract

In this thesis we will present (following [**CDL03**]) the proof of the existence of closed embedded minimal surfaces in a closed 3-dimensional manifold constructed via min-max arguments and we will prove genus bounds for the produced surfaces. A stronger estimate was announced by Pitts and Rubinstein but to our knowledge its proof has never been published. Our proof follows ideas of Simon and uses an extension of a famous result of Meeks, Simon and Yau on the convergence of minimizing sequences of isotopic surfaces.

Zusammenfassung

Eine Minimalfläche ist eine Fläche im Raum, die lokal minimalen Flächeninhalt hat. In dieser Doktorarbeit studiere ich ähnlichen Probleme nicht in Raum, sondern zum Beispiel in einer 3–dimensionale Sphere. Es wird gezeigt, dass 2–dimensionale minimale Flächen in einer 3–Sphere existieren. Danach analysiere ich deren Geometrie.

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Filippo Pellandini, December 2009.

Contents

Introduction

In this thesis we will collect and present answers to some long standing open questions concerning the existence of minimal surfaces in Riemannian 3–Manifolds (we recall that minimal surfaces are regular surfaces whose mean curvature vanishes identically, i.e. critical points of the area functional). We will give a partial answer, in particular, to the following question:

Do closed 3–dimensional manifolds contain closed embedded minimal surfaces and if yes, of which genus?

Note that, in one dimension less, i.e. on 2-dimensional Riemannian manifolds, "closed 1-dimensional minimal surfaces" are closed simple geodesics. Minimizing the length functional in a given nontrivial homotopy class shows the existence of nontrivial simple closed geodesics in any 2-dimensional manifold which is not diffeomorphic to the sphere. The existence of nontrivial simple closed geodesics on manifolds diffeomorphic to the 2-sphere is one of the most classical and celebrated results of the first half of the 20th century in differential geometry. Its solution started with the pioneering work of Birkhoff (see [**Bir17**]) and culminated in the famous Theorem of Lyusternik and Shnirelman (see [**LS29**]).

The first breakthrough on the question above was achieved by J. Pitts in his monograph [**Pit81**]. The ideas contained in the latter reference have influenced much of the subsequent literature and also this thesis. Further results are due to F. Smith, L. Simon, and Pitts and H. Rubistein (see [**PR86**] and [**PR87**]). All the theorems claimed by these authors build upon variational approaches which we call min-max constructions. The first argument of this type was introduced by Birkhoff to handle the existence of closed geodesics in the work mentioned above.

0.1. Min–max construction of surfaces

Following [**CDL03**] we shortly present the general min-max construction which is central for the purposes of this thesis. In the following M denotes a closed Riemannian 3–dimensional manifold, Diff_0

is the identity component of the diffeomorphism group of M, and $\mathfrak{Is}$ is the set of smooth isotopies. Thus $\mathfrak{Is}$ is the set of maps $\psi \in C^\infty([0,1] \times M, M)$ such that $\psi(0,\cdot)$ is the identity and $\psi(t,\cdot) \in \mathrm{Diff}_0$ for every t.
An n–parameter family of surfaces is a collection of sets $\Sigma_t \subset M$ where the parameter t belongs to $[0,1]^n$, Σ_t is an embedded closed surface except for a small set T of exceptional t's and the map $t \mapsto \Sigma_t$ enjoys some continuity properties. In our case we have $n = 1$ and T is finite.

Definition 0.1. A generalized family of surfaces is a collection $\{\Sigma_t\}_{t\in[0,1]}$ of closed subsets of M satisfying the following properties

1. The collection depends continuously on t, in the sense that
   (c1) $\mathcal{H}^2(\Sigma_t)$ is a continuous function of t;
   (c2) $\Sigma_t \to \Sigma_{t_0}$ in the Hausdorff topology whenever $t \to t_0$.
2. There is a finite exceptional set $T \subset [0,1]$ such that Σ_t is an embedded smooth surface for every $t \notin T$;
3. There is a finite set of points P such that, for $t \in T$, Σ_t is a surface in $M \setminus P$.

Here $\mathcal{H}^2$ denotes the 2–dimensional Hausdorff measure, which for smooth surfaces reduces to the usual surface area. Figure 1 gives (in one dimension less) an example of a generalized 1–parameter family with $T = \{0,1\}$.

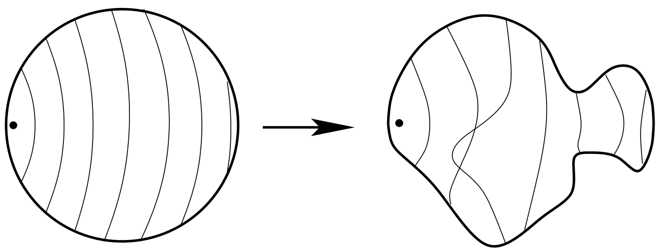

Figure 1. A 1–parameter family of curves on a 2–sphere which induces a map $F : \mathbf{S}^2 \to \mathbf{S}^2$ of degree 1.

With a small abuse of notation, we shall use the word "surface" even for the sets Σ_t with $t \in T$. To avoid confusion, families of surfaces will always be denoted by $\{\Sigma_t\}$. Thus, when referring to a surface a subscript will denote a real parameter, whereas a superscript will denote an integer as in a sequence.

Given a generalized family $\{\Sigma_t\}$ we can generate new generalized families via the following procedure. Take an arbitrary map $\psi \in$

$C^\infty([0,1]\times M, M)$ such that $\psi(t,\cdot)\in \mathrm{Diff}_0$ for each t and define $\{\Sigma'_t\}$ by $\Sigma'_t = \psi(t,\Sigma_t)$. We will say that a set Λ of generalized families is *saturated* if it is closed under this operation.

REMARK 0.2. For technical reasons we require an additional property for any saturated set Λ considered in this work: the existence of some $N = N(\Lambda) < \infty$ such that for any $\{\Sigma_t\}\subset\Lambda$, the set P in Definition 0.1 consists of at most N points. This additional property will play an important role later on.

Given a family $\{\Sigma_t\}\in\Lambda$ we denote by $\mathcal{F}(\{\Sigma_t\})$ the area of its maximal slice and by $m_0(\Lambda)$ the infimum of $\mathcal{F}$ taken over all families of Λ; that is,

$$\mathcal{F}(\{\Sigma_t\}) = \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t) \qquad \text{and} \tag{0.1}$$

$$m_0(\Lambda) = \inf_{\Lambda} \mathcal{F} = \inf_{\{\Sigma_t\}\in\Lambda}\left[\max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t)\right]. \tag{0.2}$$

If $\lim_n \mathcal{F}(\{\Sigma_t\}^n) = m_0(\Lambda)$, then we say that the sequence of generalized families of surfaces $\{\{\Sigma_t\}^n\}\subset\Lambda$ is a *minimizing sequence*. Assume $\{\{\Sigma_t\}^n\}$ is a minimizing sequence and let $\{t_n\}$ be a sequence of parameters. If the areas of the slices $\{\Sigma^n_{t_n}\}$ converge to m_0, i.e. if $\mathcal{H}^2(\Sigma^n_{t_n})\to m_0(\Lambda)$, then we say that $\{\Sigma^n_{t_n}\}$ is a *min–max sequence*.

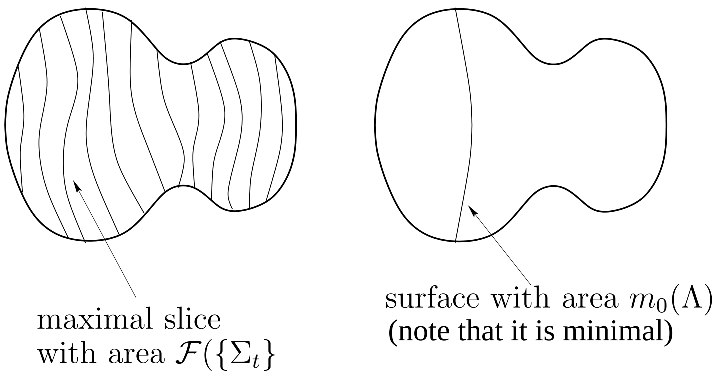

FIGURE 2. $\mathcal{F}(\{\Sigma_t\})$ and $m_0(\Lambda)$.

It is natural to expect that, when $m_0(\Lambda) > 0$, there exists at least a min-max sequence $\{\Sigma^n_{t_n}\}$ which converges to a minimal surface of area $m_0(\Lambda)$ (here we must take into account multiplicities: in principle the $\Sigma^n_{t_n}$ could converge to an N–fold covering of the limiting surface). In one dimension less, this intuitive idea was made rigorous in the pioneering work of Birkhoff (see [**Bir17**]). The following elementary proposition (proved in Section 1.4 following the Appendix of [**CDL03**]) shows that, for any given M, there are many saturated sets Λ for which $m_0(\Lambda) > 0$.

Proposition 0.3. *Let M be a closed 3-manifold with a Riemannian metric and let $\{\Sigma_t\}$ be the level sets of a Morse function. The smallest saturated set Λ containing the family $\{\Sigma_t\}$ has $m_0(\Lambda) > 0$.*

This proposition shows that the assumptions of Definition 0.1 are indeed quite natural. When M is diffeomorphic to S^3, a key example of Λ is the smallest saturated set contanining the obvious generalized family which "sweeps out" M with 2–dimensional spheres (see Remark 1.13).

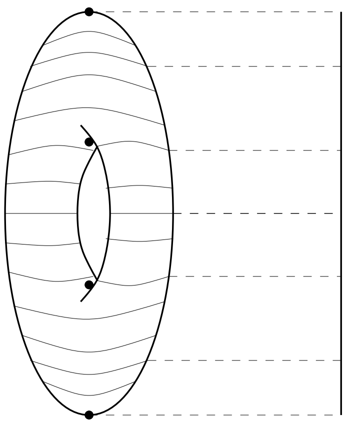

Figure 3. A sweep–out of the torus by level sets of a Morse function. In this case there are four degenerate slices in the 1–parameter family.

The min-max construction used by J. Pitts in his monograph [**Pit81**] is rather different than the one above. In [**Pit81**] he proves that *in any closed Riemannian manifold M of dimension $n \leq 6$, there exists at least one minimal embedded closed (smooth) hypersurface*. Later, in [**SS81**] Schoen and Simon extend Pitts' approach to $n \geq 7$ (producing embedded minimal surfaces with singularity) and a much shorter proof of the Pitts-Schoen-Simon Theorem has been given in [**DT09**].

However, Pitts min-max construction does not allow to control the topology of the resulting minimal surface. The min-max construction used in this thesis was (essentially) introduced in the PhD thesis of F. Smith [**Smi82**] (written under the supervision of Simon). Combining some ideas of Pitts with results of Meeks, Simon and Yau [**MSY82**] and

Almgren and Simon [**AS79**], Smith showed the following remarkable theorem (cp. with [**SU81**] for the "immersed" version, which appeared few years before).

Theorem 0.4 (Smith). *In any Riemannian manifold diffeomorphic to the* 3*–dimensional sphere there exists at least one minimal embedded* 2*–dimensional sphere.*

Later on Pitts and Rubinstein [**PR86**] announced an extension of the work of Smith to more general min-max constructions in 3–dimensional manifolds, dealing with multiparameter families, surfaces of higher genus and bounds on the Morse index. To our knowledge, Pitts and Rubinstein have never published proofs of their claims. Actually, not even Smith's PhD thesis has been published, and it turns out to be very difficult to follow his arguments. Only recentely T. Colding and C. De Lellis, following the ideas introduced by Smith, have written a complete proof of the first portion of the theorems of Smith and Pitts-Rubinstein, which is the following regularity result (see [**CDL03**]).

Theorem 0.5. *Let M be a closed* 3*-manifold with a Riemannian metric. For any saturated Λ, there is a min–max sequence $\Sigma^n_{t_n}$ converging in the sense of varifolds to a smooth embedded minimal surface Σ with area $m_0(\Lambda)$ (multiplicity is allowed).*

We refer to Section 1.1 for the relevant notion of varifolds convergence.

The original part of this thesis is a complete proof of the second portion of the program, i.e. we give here genus bounds for the surface Σ produced in the proof of Theorem 3.1 (this proof will appear on Journal for Pure and Applied Mathematics, see for now [**DP09**]). However, since several parts of the two portions are in common, we also present the proof of Theorem 3.1. Summarizing, Chapters 1 and 3 of this thesis are widely taken from the paper [**CDL03**] of T. Colding and C. De Lellis, while the content of Chapters 2 and 4 is original research of the author jointly with C. De Lellis (see [**DP09**]). This thesis is therefore the first complete reference answering to the question stated at the beginning of the introduction and it is obviously shorter than the sum of the two papers [**CDL03**] and [**DP09**].

0.2. Genus bounds

In chapter 4 we bound the topology of Σ under the assumption that the t–dependence of $\{\Sigma_t\}$ is smoother than just the continuity required in Definition 0.1. This is the content of the next definition.

DEFINITION 0.6. A generalized family $\{\Sigma_t\}$ as in Definition 0.1 is said to be *smooth* if:

(s1) Σ_t varies smoothly in t on $[0,1]\setminus T$;
(s2) For $t\in T$, $\Sigma_\tau \to \Sigma_t$ smoothly in $M\setminus P$.

Here P and T are the sets of requirements 2. and 3. of Definition 0.1. We assume further that Σ_t is orientable for any $t\notin T$.

Note that, if a set Λ consists of smooth generalized families, then the elements of its saturation are still smooth generalized families. Therefore, the saturated set considered in Proposition 0.3 is smooth.

We next introduce some notation which will be consistently used during the proofs in the next chapters. We decompose the surface Σ of Theorem 3.1 as $\sum_{i=1}^N n_i\Gamma^i$, where the Γ^i's are the connected components of Σ, counted without multiplicity, and $n_i\in\mathbb{N}\setminus\{0\}$ for every i. We further divide the components $\{\Gamma^i\}$ into two sets: the orientable ones, denoted by $\mathcal{O}$, and the non–orientable ones, denoted by $\mathcal{N}$. We are now ready to state the second main theorem presented in this thesis.

THEOREM 0.7. *Let Λ be a saturated set of smooth generalized families and Σ and $\Sigma^n_{t_n}$ the surfaces produced in the proof of Theorem 3.1 given in Chapter 3. Then*

$$\sum_{\Gamma^i\in\mathcal{O}}\mathbf{g}(\Gamma^i)+\frac{1}{2}\sum_{\Gamma^i\in\mathcal{N}}(\mathbf{g}(\Gamma^i)-1)\ \le\ \mathbf{g}_0\ :=\ \liminf_{j\uparrow\infty}\liminf_{\tau\to t_j}\mathbf{g}(\Sigma^j_\tau)\,. \tag{0.3}$$

REMARK 0.8. According to our definition, $\Sigma^j_{t_j}$ is not necessarily a smooth submanifold, as t_j could be one of the exceptional parameters of point 3. in Definition 0.1. However, for each fixed j there is an $\eta>0$ such that Σ^j_t is a smooth submanifold for every $t\in]t_j-\eta,t_j[\cup]t_j,t_j+\eta[$. Hence the right hand side of (0.3) makes sense.

Note that Smith's Theorem 0.4 is an obvious corollary of Theorem 0.7 applied to the obvious min-max construction on M using 2-spheres (see Remark 1.13).

In fact the inequality (0.3) holds with $\mathbf{g}_0=\liminf_{j\to\infty}\mathbf{g}(\Sigma^j)$ for every limit Σ of a sequence of surfaces Σ^j's that enjoy certain requirements of variational nature, i.e. that are *almost minimizing in sufficiently small annuli* (see section 1.3). The precise statement will be given in Theorem 4.1, after introducing the suitable concepts.

As usual, when Γ is an orientable 2–dimensional connected surface, its genus $\mathbf{g}(\Gamma)$ is defined as the number of handles that one has to attach to a sphere in order to get a surface homeomorphic to Γ. When Γ is non–orientable and connected, $\mathbf{g}(\Gamma)$ is defined as the number of cross caps that one has to attach to a sphere in order to get a surface homeomorphic to Γ (therefore, if χ is the Euler characteristic of the surface, then

$$\mathbf{g}(\Gamma) = \begin{cases} \frac{1}{2}(2-\chi) & \text{if } \Gamma \in \mathcal{N} \\ 2-\chi & \text{if } \Gamma \in \mathcal{O} \end{cases}$$

see [**Mas91**]). For surfaces with more than one connected component, the genus is simply the sum of the genus of each connected component.

Our genus estimate (0.3) is weaker than the one announced by Pitts and Rubinstein in [**PR86**], which reads as follows (cp. wih Theorem 1 and Theorem 2 in [**PR86**]):

$$\sum_{\Gamma^i \in \mathcal{O}} n_i \mathbf{g}(\Gamma^i) + \frac{1}{2} \sum_{\Gamma^i \in \mathcal{N}} n_i \mathbf{g}(\Gamma^i) \leq \mathbf{g}_0 . \tag{0.4}$$

In Section 4.3 a very elementary example shows that (0.4) is false for sequences of almost minimizing surfaces (in fact even for sequences which are locally strictly minimizing). In this case the correct estimate should be

$$\sum_{\Gamma^i \in \mathcal{O}} n_i \mathbf{g}(\Gamma^i) + \frac{1}{2} \sum_{\Gamma^i \in \mathcal{N}} n_i (\mathbf{g}(\Gamma^i) - 1) \leq \mathbf{g}_0 . \tag{0.5}$$

Therefore, the improved estimate (0.4) can be proved only by exploiting an argument of more global nature, using a more detailed analysis of the min–max construction.

The estimate (0.5) respects the rough intuition that the approximating surfaces Σ^j are, after appropriate surgeries, isotopic to coverings of the surfaces Γ^i. For instance Γ can consist of a single component that is a real projective space, and Σ^j might be the boundary of a tubular neighborhood of Γ of size $\varepsilon_j \downarrow 0$, i.e. a sphere. In this case Σ^j is a double cover of Γ.

Our proof uses the ideas of an unpublished argument of Simon, reported by Smith in [**Smi82**] to show the existence of an embedded minimal 2–sphere when M is a 3–sphere. These ideas do not seem enough to show (0.4): its proof probably requires a much more careful analysis. In Section 4.3 we discuss this issue.

The unpublished argument of Simon has been used also by Grüter and Jost in [**GJ86**]. The core of Simon's argument is reported here with a technical simplification. In Chapter 2 we then give a detailed proof of an auxiliary proposition which plays a fundamental role in the argument. We will therein state a suitable modification of a celebrated result of Meeks, Simon and Yau (see [**MSY82**]) in which we handle minimizing sequences of isotopic surfaces with boundaries (see Proposition 2.2). This part is, to our knowledge, new: neither Smith, nor Grüter and Jost provide a proof of it. Smith suggests that the proposition can be proved by suitably modifying the arguments of [**MSY82**] and [**AS79**]. Though this is indeed the case, the strategy suggested by Smith leads to a difficulty which we overcome with a different approach: see the discussion in Section 2.3. Moreover, [**Smi82**] does not discuss the "convex–hull property" of Section 2.1, which is a basic prerequisite to apply the boundary regularity theory of Allard in [**All75**] (in fact we do not know of any boundary regularity result in the minimal surface theory which does not pass through some kind of convex hull property).

0.3. Why is the proof of Theorem 0.7 so complicated?

The main reason is the very weak notion of convergence under which Theorem 3.1 is proved. Under the varifold convergence the genus is indeed not necessarily lower semicontinuous: we give below a simple example of this fact. On the other hand, the varifold convergence seems to be the only one which allows us to build a succesfull variational theory.

We end this introduction with a brief discussion of how a sequence of closed surfaces Σ^j could converge, in the sense of varifolds, to a smooth surface with higher genus. This example is a model situation which must be ruled out by any proof of a genus bound.

- First take a sphere in $\mathbf{R}^3$ and squeeze it in one direction towards a double copy of a disk (see the first three pictures in Figure 4). Recall that the convergence in the sense of varifolds does not take into account the orientation.
- Next take the disk and wrap it to form a torus in the standard way (see the last five pictures in Figure 4).

With a standard diagonal argument we find a sequence of smooth embedded spheres in $\mathbf{R}^3$ which, in the sense of varifolds, converges to a double copy of an embedded torus.

This example does not occur in min–max sequences for variational reasons. In particular, it follows from the arguments of our proofs that such a sequence does not have the almost minimizing property in (sufficiently small) annuli discussed in Section 1.3.

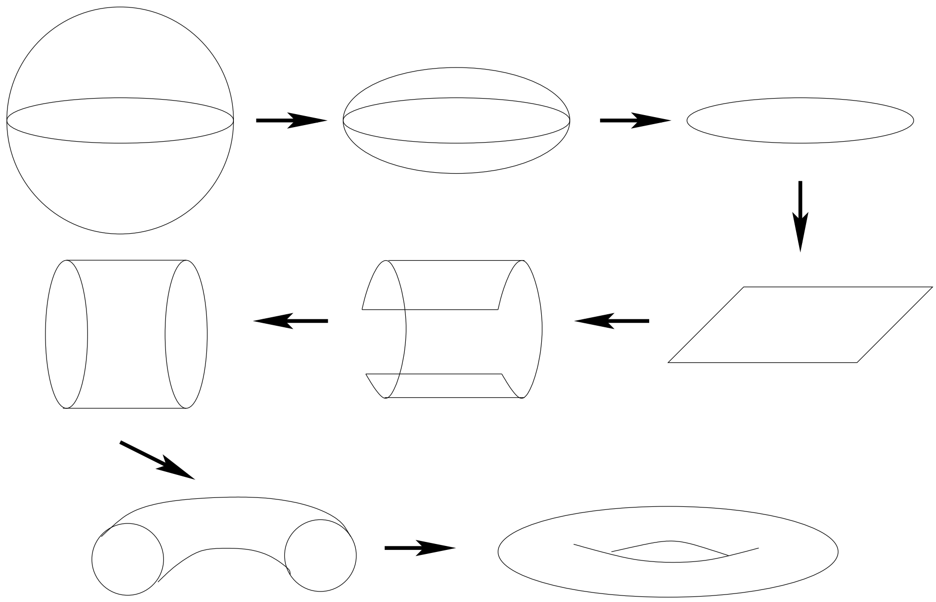

Figure 4. Failure of genus bounds under varifold convergence. A sequence of embedded spheres converges to a double copy of a torus.

CHAPTER 1

Min-max construction

In this Chapter we will firstly introduce a more general setting of min–max surfaces. Section 1.1 will be dedicated to the definition of the measure theoretic concept of varifolds and their properties. In sections 1.2 and 1.3 we will prove the existence, under some variational hypothesis, of converging min–max sequences whose limits are good candidates for being smooth embedded minimal surfaces (as it will be proved in Chapter 3). In Section 1.4 we give some concrete examples of min–max constructions and finally in Section 1.5 we recall an important Theorem of Schoen and Simon for minimal surfaces which will be used plenty throughout this work.

1.1. Stationary varifolds

1.1.1. Varifolds. We recall some basic facts from the theory of varifolds; see for instance chapter 4 and chapter 8 of [**Sim83**] for further information. Varifolds are a convenient way of generalizing surfaces to a category that has good compactness properties. An advantage of varifolds, over other generalizations (like currents), is that they do not allow for cancellation of mass. This last property is fundamental for the min–max construction.

If U is an open subset of M, any finite nonnegative measure on the Grassmannian of unoriented 2–planes on U is said to be a 2*–varifold in* U. The Grassmannian of 2–planes will be denoted by $G^2(U)$ and the vector space of 2–varifolds is denoted by $\mathcal{V}^2(U)$. Throughout we will consider only 2–varifolds; thus we drop the 2.

We endow $\mathcal{V}(U)$ with the topology of the weak convergence in the sense of measures, thus we say that a sequence V^k of varifolds converge to a varifold V if for every function $\varphi \in C_c(G(U))$

$$\lim_{k\to\infty} \int \varphi(x,\pi)\, dV^k(x,\pi) \;=\; \int \varphi(x,\pi)\, dV(x,\pi)\,.$$

Here π denotes a 2–plane of T_xM. If $U' \subset U$ and $V \in \mathcal{V}(U)$, then we denote by $V \llcorner U'$ the restriction of the measure V to $G(U')$. Moreover,

$\|V\|$ will be the unique measure on U satisfying

$$\int_U \varphi(x)\, d\|V\|(x) \;=\; \int_{G(U)} \varphi(x)\, dV(x,\pi) \qquad \forall \varphi \in C_c(U)\,.$$

The support of $\|V\|$, denoted by supp $(\|V\|)$, is the smallest closed set outside which $\|V\|$ vanishes identically. The number $\|V\|(U)$ will be called the *mass of* V *in* U. When U is clear from the context, we say briefly the *mass of* V.

Recall also that a 2–dimensional rectifiable set is a countable union of closed subsets of C^1 surfaces (modulo sets of $\mathcal{H}^2$–measure 0). Thus, if $R \subset U$ is a 2–dimensional rectifiable set and $h : R \to \mathbf{R}^+$ is a Borel function, then we can define a varifold V by

(1.1)
$$\int_{G(U)} \varphi(x,\pi)\, dV(x,\pi) = \int_R h(x)\varphi(x, T_x R)\, d\mathcal{H}^2(x) \quad \forall \varphi \in C_c(G(U))\,.$$

Here $T_x R$ denotes the tangent plane to R in x. If h is integer–valued, then we say that V is an *integer rectifiable varifold*. If $\Sigma = \bigcup n_i \Sigma_i$, then by slight abuse of notation we use Σ for the varifold induced by Σ via (1.1).

1.1.2. Pushforward, first variation, monotonicity formula. If V is a varifold induced by a surface $\Sigma \subset U$ and $\psi : U \to U'$ a diffeomorphism, then we let $\psi_\# V \in \mathcal{V}(U')$ be the varifold induced by the surface $\psi(\Sigma)$. The definition of $\psi_\# V$ can be naturally extended to *any* $V \in \mathcal{V}(U)$ by

$$\int \varphi(y,\sigma)\, d(\psi_\# V)(y,\sigma) \;=\; \int J\psi(x,\pi)\, \varphi(\psi(x), d\psi_x(\pi))\, dV(x,\pi)\,;$$

where $J\psi(x,\pi)$ denotes the Jacobian determinant (i.e. the area element) of the differential $d\psi_x$ restricted to the plane π; cf. equation (39.1) of [**Sim83**].

Given a smooth vector field χ, let ψ be the isotopy generated by χ, i.e. with $\frac{\partial \psi}{\partial t} = \chi(\psi)$. The first variation of V with respect to χ is defined as

$$[\delta V](\chi) \;=\; \frac{d}{dt}(\|\psi(t,\cdot)_\# V\|)\Big|_{t=0}\quad ;$$

cf. sections 16 and 39 of [**Sim83**].

When Σ is a smooth surface we recover the classical definition of first variation of a surface:

$$[\delta\Sigma](\chi) \;=\; \int_\Sigma \mathrm{div}_\Sigma \chi\, d\mathcal{H}^2 \;=\; \frac{d}{dt}(\mathcal{H}^2(\psi(t,\Sigma)))\Big|_{t=0}\,.$$

If $[\delta V](\chi) = 0$ for every $\chi \in C_c^\infty(U, TU)$, then V is said to be *stationary in* U. Thus stationary varifolds are natural generalizations of minimal surfaces.

Stationary varifolds in Euclidean spaces satisfy the monotonicity formula (see sections 17 and 40 of [**Sim83**]):

(1.2)

$$\text{For every } x \text{ the function } f(\rho) = \frac{\|V\|(B_\rho(x))}{\pi\rho^2} \text{ is non–decreasing.}$$

When V is a stationary varifold in a Riemannian manifold a similar formula with an error term holds. Namely, there exists a constant $C(r) \geq 1$ such that

$$f(s) \;\leq\; C(r) f(\rho) \qquad \text{whenever } 0 < s < \rho < r. \tag{1.3}$$

Moreover, the constant $C(r)$ approaches 1 as $r \downarrow 0$. This property allows us to define the *density* of a stationary varifold V at x, by

$$\theta(x, V) \;=\; \lim_{r \downarrow 0} \frac{\|V\|(B_r(x))}{\pi r^2}.$$

Thus $\theta(x, V)$ corresponds to the upper density θ^{*2} of the measure $\|V\|$ as defined in section 3 of [**Sim83**]. The following theorem gives a useful condition for rectifiability in terms of density:

Theorem 1.1. *(Theorem 42.4 of* [**Sim83**]*). If V is a stationary varifold with $\theta(V, x) > 0$ for $\|V\|$–a.e. x, then V is rectifiable.*

1.1.3. Tangent cones, Constancy Theorem. Tangent varifolds are the natural generalization of tangent planes for smooth surfaces. In order to define tangent varifolds in a 3–dimensional manifold we need to recall what a dilation in a manifold is. If $x \in M$ and $\rho < \mathrm{Inj}\,(M)$, then the dilation around x with factor ρ is the map $T_\rho^x : B_\rho(x) \to \mathcal{B}_1$ given by $T_\rho^x(z) = (\exp_x^{-1}(z))/\rho$; thus if $M = \mathbf{R}^3$, then T_ρ^x is the usual dilation $y \to (y - x)/\rho$.

Definition 1.2. If $V \in \mathcal{V}(M)$, then we denote by V_ρ^x the dilated varifold in $\mathcal{V}(\mathcal{B}_1)$ given by $V_\rho^x = (T_\rho^x)_\sharp V$. Any limit $V' \in \mathcal{V}(\mathcal{B}_1)$ of a sequence $V_{s_n}^x$ of dilated varifolds, with $s_n \downarrow 0$, is said to be a *tangent varifold at* x. The set of all tangent varifolds to V at x is denoted by $T(x, V)$.

It is well known that if the varifold V is stationary, then any tangent varifold to V is a stationary *Euclidean* cone (see section 42 of [**Sim83**]); that is a stationary varifold in $\mathbf{R}^3$ which is invariant under the dilations $y \to y/\rho$. If V is also integer rectifiable and the support of V is contained in the union of a finite number of disjoint connected surfaces

Σ_i, i.e. $\mathrm{supp}\,(\|V\|) \subset \bigcup \Sigma_i$, then the Constancy Theorem (see theorem 41.1 of [**Sim83**]) gives that $V = \bigcup m_i \Sigma^i$ for some natural numbers m_i.

1.1.4. Two lemmas about varifolds. We state and prove in this subsection two technical lemmas on varifolds which will be useful later. The first lemma is a weak version (in the varifolds setting) of the classical maximum principle for minimal surfaces.

LEMMA 1.3. *Let U be an open subset of a 3–manifold M and W a stationary 2–varifold in $\mathcal{V}(U)$. If $K \subset\subset U$ is a smooth strictly convex set and $x \in (\mathrm{supp}\,(\|W\|)) \cap \partial K$, then*

$$(B_r(x) \setminus \overline{K}) \cap \mathrm{supp}\,(\|W\|) \neq \emptyset \qquad \textit{for every } r > 0.$$

PROOF. For simplicity assume that $M = \mathbf{R}^3$. The proof can be easily adapted to the general case. Let us argue by contradiction; so assume that there are $x \in \mathrm{supp}\,(\|W\|)$ and $B_r(x)$ such that $(B_r(x) \setminus \overline{K}) \cap \mathrm{supp}\,(\|W\|) = \emptyset$. Given a vector field $\chi \in C^\infty_c(U, \mathbf{R}^3)$ and a 2–plane π we set

$$\mathrm{Tr}\,(D\chi(x), \pi) \;=\; D_{v_1}\chi(x) \cdot v_1 + D_{v_2}\chi(x) \cdot v_2$$

where $\{v_1, v_2\}$ is an orthonormal base for π. Recall that the first variation of W is given by

$$\delta W(\chi) \;=\; \int_{G(U)} \mathrm{Tr}\,(D\chi(x), \pi)\, dW(x, \pi)\,.$$

Take an increasing function $\eta \in C^\infty([0, 1])$ which vanishes on $[3/4, 1]$ and is identically 1 on $[0, 1/4]$. Denote by φ the function given by $\varphi(x) = \eta(|y - x|/r)$ for $y \in B_r(x)$. Take the interior unit normal ν to ∂K in x, and let z_t be the point $x + t\nu$. If we define vector fields ψ_t and χ_t by

$$\psi_t(y) \;=\; -\frac{y - z_t}{|y - z_t|} \qquad \text{and} \qquad \chi_t = \varphi\psi_t\,,$$

then χ_t is supported in $B_r(x)$ and $D\chi_t = \varphi D\psi_t + \nabla\varphi \otimes \psi_t$. Moreover, by the strict convexity of the subset K,

$$\nabla\varphi(y) \cdot \nu > 0 \qquad \text{if } y \in \overline{K} \cap B_r(x) \text{ and } \nabla\varphi(y) \neq 0.$$

Note that ψ_t converges to ν uniformly in $B_r(x)$, as $t \uparrow \infty$. Thus, $\psi_T(y) \cdot \nabla\varphi(y) \geq 0$ for every $y \in \overline{K} \cap B_r(x)$, provided T is sufficiently large. This yields that

$$\mathrm{Tr}\,(\nabla\varphi(y) \otimes \psi_T(y), \pi) \geq 0 \qquad \text{for all } (y, \pi) \in G(B_r(x) \cap \overline{K})\,. \tag{1.4}$$

Note that $\mathrm{Tr}\,(D\psi_t(y),\pi) > 0$ for all $(y,\pi) \in G(B_r(x))$ and all $t > 0$. Thus

$$\begin{aligned}\delta W(\chi_T) &= \int_{G(B_r(x)\cap\overline{K})} \mathrm{Tr}\,(D\chi_T(y),\pi)\,dW(y,\pi) \\ &\overset{(1.4)}{\geq} \int_{G(B_r(x)\cap\overline{K})} \mathrm{Tr}\,(\varphi(y)D\psi_T(y),\pi)\,dW(y,\pi) \\ &\geq \int_{G(B_{r/4}(x)\cap\overline{K})} \mathrm{Tr}\,(D\psi_T(y),\pi)\,dW(y,\pi) \;>\; 0.\end{aligned}$$

This contradicts that W is stationary and completes the proof. □

LEMMA 1.4. *Let $x \in M$ and V be a stationary integer rectifiable varifold in M. Assume T is the subset of the support of $\|V\|$ given by*

$$T \;=\; \{T(y,V) \text{ consists of a plane transversal to } \partial B_{d(x,y)}(x)\}\,.$$

If $\rho < Inj(M)$, then T is dense in $(\mathrm{supp}\,(\|V\|)) \cap B_\rho(x)$.

PROOF. Since V is integer rectifiable, then V is supported on a rectifiable 2–dimensional set R and there exists a Borel function $h : R \to \mathbb{N}$ such that $V = hR$. Assume the lemma is false; then there exists $y \in B_\rho(x) \cap \mathrm{supp}\,(\|V\|)$ and $t > 0$ such that

- the tangent plane to R in z is tangent to $\partial B_{d(z,x)}(x)$, for any $z \in B_t(y)$.

We choose t so that $B_t(y) \subset B_\rho(x)$. Take polar coordinates (r,θ,φ) in $B_\rho(x)$ and let f be a smooth nonnegative function in $C^\infty_c(B_t(y))$ with $f = 1$ on $B_{t/2}(y)$. Denote by χ the vector field $\chi(\theta,\varphi,r) = f(\theta,\varphi,r)\frac{\partial}{\partial r}$. We use the notation of the proof of Lemma 1.3. For every $z \in R\cap B_t(x)$, the plane π tangent to R in z is also tangent to the sphere $\partial B_{d(z,x)}(x)$. Hence, an easy computation yields that $\mathrm{Tr}\,(\chi,\pi)(z) = 2\psi(z)/d(z,x)$. This gives

$$[\delta V](\chi) = \int_{R\cap B_t(y)} \frac{2h(z)\psi(z)}{\mathrm{d}\,(z,x)} d\mathcal{H}^2(z) \;>\; C\|V\|(B_{t/2}(y))\,,$$

for some positive constant C. Since $y \in \mathrm{supp}\,(\|V\|)$, we have

$$\|V\|(B_{t/2}(y)) \;>\; 0\,.$$

This contradicts that V is stationary. □

1.2. Limits of suitable min–max sequences are stationary

In the following we fix a saturated set Λ of generalized 1-parameter families of surfaces and denote by $m_0 = m_0(\Lambda)$ the infimum of the areas of the maximal slices in Λ; cf. (0.1). If $\{\{\Sigma_t\}^k\} \subset \Lambda$ is a minimizing sequence, then it is easy to show the existence of a min–max sequence which converge (after possibly passing to subsequences) to a stationary varifold. However, as Fig. 1 illustrate, a general minimizing sequence $\{\{\Sigma_t\}^k\}$ can have slices $\Sigma^k_{t_k}$ with area converging to m_0 but not "clustering" towards stationary varifolds.

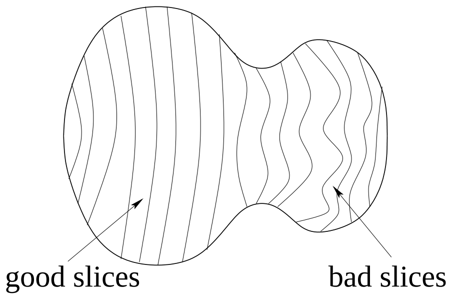

FIGURE 1. Slices with area close to m_0. The good ones are very near to a minimal surface of area m_0, whereas the bad ones are far from any stationary varifold.

In the language introduced above, this means that a given minimizing sequence $\{\{\Sigma_t\}^k\}$ can have min–max sequences which are not clustering to stationary varifolds. This is a source of some technical problems and forces us to show how to choose a "good" minimizing sequence $\{\{\Sigma_t\}^k\}$. This is the content of the following proposition:

PROPOSITION 1.5. *There exists a minimizing sequence* $\{\{\Sigma_t\}^n\} \subset \Lambda$ *such that every min–max sequence* $\{\Sigma^n_{t_n}\}$ *clusters to stationary varifolds.*

A result similar to Proposition 1.5 appeared in [**Pit81**] (see theorem 4.3 of [P]). The proof follows from ideas of [**AJ65**] (cf. 12.5 there). For simplicity we metrize the weak topology on the space of varifolds and restate Proposition 1.5 using this metric.

Denote by X the set of varifolds $V \in \mathcal{V}(M)$ with mass bounded by $4\,m_0$, i.e., with $\|V\|(M) \leq 4m_0$. Endow X with the weak* topology and let $\mathcal{V}_\infty$ be the set of stationary varifolds contained in X. Clearly, $\mathcal{V}_\infty$ is a closed subset of X. Moreover, by standard general topology theorems, X is compact and metrizable. Fix one such metric and denote it by

$\mathfrak{d}$. The ball of radius r and center V in this metric will be denoted by $U_r(V)$.

PROPOSITION 1.6. *There exists a minimizing sequence* $\{\{\Sigma_t\}^n\} \subset \Lambda$ *such that, if* $\{\Sigma^n_{t_n}\}$ *is a min–max sequence, then* $\mathfrak{d}(\Sigma^n_{t_n}, \mathcal{V}_\infty) \to 0$.

PROOF. The key idea of the proof is building a continuous map $\Psi : X \to \mathfrak{Is}$ such that :

- If V is stationary, then Ψ_V is the trivial isotopy;
- If V is not stationary, then Ψ_V decreases the mass of V.

Since each Ψ_V is an isotopy, and thus is itself a map from $[0,1]\times M \to M$, to avoid confusion we use the subscript V to denote the dependence on the varifold V. The map Ψ will be used to deform a minimizing sequence $\{\{\Sigma_t\}^n\} \subset \Lambda$ into another minimizing sequence $\{\{\Gamma_t\}^n\}$ such that :

For every $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\text{if} \quad \left\{ \begin{array}{cc} & n > N \\ \text{and} & \mathcal{H}^2(\Gamma^n_{t_n}) > m_0 - \delta \end{array} \right\}, \quad \text{then} \quad \mathfrak{d}(\Gamma^n_{t_n}, \mathcal{V}_\infty) < \varepsilon. \tag{1.5}$$

Such a $\{\{\Gamma_t\}^n\}$ would satisfy the requirement of the proposition.

The map Ψ_V should be thought of as a natural "shortening process" of varifolds which are not stationary. If the mass (considered as a functional on the space of varifolds) were smoother, then a gradient flow would provide a natural shortening process like Ψ_V. However, this is not the case; even if we start with smooth initial datum, in very short time the motion by mean curvature, i.e. the gradient flow of the area functional on smooth submanifolds, gives surfaces which are not isotopic to the initial one.

Step 1: A map from X to the space of vector fields.

The isotopies Ψ_V will be generated as 1–parameter families of diffeomorphisms satisfying certain ODE's. In this step we associate to any V a suitable vector field, which in Step 2 will be used to construct Ψ_V.

For $k \in \mathbb{Z}$ define the annular neighborhood of $\mathcal{V}_\infty$

$$\mathcal{V}_k = \left\{ V \in X | 2^{-k+1} \geq \mathfrak{d}(V, \mathcal{V}_\infty) \geq 2^{-k-1} \right\}.$$

There exists a positive constant $c(k)$ depending on k such that to every $V \in \mathcal{V}_k$ we can associate a smooth vector field χ_V with

$$\|\chi_V\|_\infty \leq 1 \qquad \text{and} \qquad \delta V(\chi_V) \leq -c(k).$$

Our next task is choosing χ_V with continuous dependence on V. Note that for every V there is some radius r such that $\delta W(\chi_V) \leq -c(k)/2$

for every $W \in U_r(V)$. Hence, for any k we can find balls $\{U_i^k\}_{i=1,\dots,N(k)}$ and vector fields χ_i^k such that :

(1.6) The balls $\tilde{U}_i^k$ concentric to U_i^k with half the radii cover $\mathcal{V}_k$;

(1.7) If $W \in U_i^k$, then $\delta W(\chi) \le -c(k)/2$;

(1.8) The balls U_i^k are disjoint from $\mathcal{V}_j$ if $|j-k| \ge 2$.

Hence, $\{U_i^k\}_{k,i}$ is a locally finite covering of $X \setminus \mathcal{V}_\infty$. To this family we can subordinate a continuous partition of unit φ_i^k. Thus we set $H_V = \sum_{i,k} \varphi_i^k(V)\chi_i^k$. The map $H : X \to C^\infty(M, TM)$ which to every V associates H_V is continuous. Moreover, $\|H_V\|_\infty \le 1$ for every V.

Step 2: A map from X to the space of isotopies.

For $V \in \mathcal{V}_k$ we let $r(V)$ be the radius of the smaller ball $\tilde{U}_i^j$ which contains it. We find that $r(V) > r(k) > 0$, where $r(k)$ only depends on k. Moreover, by (1.7) and (1.8), for every W contained in the ball $U_{r(V)}(V)$ we have that

$$\delta W(H_V) \;\le\; -\frac{1}{2}\min\{c(k-1), c(k), c(k+1)\}.$$

Summarizing there are two continuous functions $g : \mathbf{R}^+ \to \mathbf{R}^+$ and $r : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\text{(1.9)}\quad \delta W(H_V) \;\le\; -g(\mathfrak{d}\,(V, \mathcal{V}_\infty)) \qquad \text{if} \qquad \mathfrak{d}\,(W, V) \le r(\mathfrak{d}\,(V, \mathcal{V}_\infty)).$$

Now for every V construct the 1–parameter family of diffeomorphisms

$$\Phi_V : [0, +\infty) \times M \to M \qquad \text{with} \qquad \frac{\partial \Phi_V(t,x)}{\partial t} \;=\; H_V(\Phi_V(t,x)).$$

For each t and V, we denote by $\Phi_V(t, \cdot)$ the corresponding diffeomorphism of M. We claim that there are continuous functions $T : \mathbf{R}^+ \to [0,1]$ and $G : \mathbf{R}^+ \to \mathbf{R}^+$ such that

- If $\gamma = \mathfrak{d}\,(V, \mathcal{V}_\infty) > 0$ and we transform V into V' via the diffeomorphism $\Phi_V(T(\gamma), \cdot)$, then $\|V'\|(M) \le \|V\|(M) - G(\delta)$;
- $G(s)$ and $T(s)$ both converge to 0 as $s \downarrow 0$.

Indeed fix V. For every $r > 0$ there is a $T > 0$ such that the curve of varifolds

$$\{V(t) = (\Phi_V(t, \cdot))_\sharp V\,, \quad t \in [0, T]\}$$

stays in $U_r(V)$. Thus

$$\begin{aligned} \|V(T)\|(M) - \|V\|(M) \;&=\; \|V(T)\|(M) - \|V(0)\|(M) \\ &\le\; \int_0^T [\delta V(t)](H_V)\, dt\,, \end{aligned}$$

and therefore if we choose $r = r(\mathfrak{d}\,(V, \mathcal{V}_\infty))$ as in (1.9), then we get the bound

$$\|V(T)\|(M) - \|V\|(M) \ \leq\ -Tg(\mathfrak{d}\,(V, \mathcal{V}_\infty)).$$

Using a procedure similar to that of Step 1 we can choose T depending continuously on V. It is then trivial to see that we can in fact choose T so that at the same time it is continuous and depends only on $\mathfrak{d}\,(V, \mathcal{V}_\infty)$.

Step 3: Constructing the competitor and the conclusion.

For each V, set $\gamma = \mathfrak{d}\,(V, \mathcal{V}_\infty)$ and

$$\Psi_V(t, \cdot) \ =\ \Phi_V([T(\gamma)]\, t, \cdot) \qquad \text{for } t \in [0, 1].$$

Ψ_V is a "normalization" of Φ_V. From Step 2 we know that there is a continuous function $L : \mathbf{R} \to \mathbf{R}$ such that

- L is strictly increasing and $L(0) = 0$;
- $\Psi_V(1, \cdot)$ deforms V into a varifold V' with $\|V'\| \leq \|V\| - L(\gamma)$.

Choose a sequence of families $\{\{\Sigma_t\}^n\} \subset \Lambda$ with $\mathcal{F}(\{\Sigma_t\}^n) \leq m_0 + 1/n$ and define $\{\Gamma_t\}^n$ by

$$\Gamma^n_t = \Psi_{\Sigma^n_t}(1, \Sigma^n_t) \qquad \text{for all } t \in [0, 1] \text{ and all } n \in \mathbb{N} \tag{1.10}$$

Thus

$$\mathcal{H}^2(\Gamma^n_t) \leq \mathcal{H}^2(\Sigma^n_t) - L(\mathfrak{d}\,(\Sigma^n_t, \mathcal{V}_\infty)). \tag{1.11}$$

Note that $\{\Gamma_t\}^n$ does not necessarily belong to Λ, since the families of diffeomorphisms $\psi_t(\cdot) = \Psi_{\Sigma^n_t}(1, \cdot)$ may not depend smoothly on t. In order to overcome this technical obstruction fix n and note that $\Psi_t = \Psi_{\Sigma^n_t}$ is the 1–parameter family of isotopies generated by the 1–parameter family of vector fields $h_t = T(\Sigma^n_t) H_{\Sigma^n_t}$. Think of h as a continuous map

$$h : [0, 1] \to C^\infty(M, TM) \quad \text{with the topology of } C^k \text{ seminorms.}$$

Thus h can be approximated by a *smooth* map $\tilde{h} : [0, 1] \to C^\infty(M, TM)$. Consider the *smooth* 1–parameter family of isotopies $\tilde{\Psi}_t$ generated by the vector fields $\tilde{h}_t$ and the family of surfaces $\{\Gamma_t\}^n$ given by $\Gamma^n_t = \tilde{\Psi}_t(1, \Sigma^n_t)$. If $\sup_t \|h_t - \tilde{h}_t\|_{C^1}$ is sufficiently small, then we easily get (by the same calculations of the previous steps)

$$\mathcal{H}^2(\Gamma^n_t) \leq \mathcal{H}^2(\Sigma^n_t) - L(\mathfrak{d}\,(\Sigma^n_t, \mathcal{V}_\infty))/2. \tag{1.12}$$

Moreover, since $\tilde{\Psi}_t(1, \cdot)$ is a smooth map, this new family belongs to Λ.

Clearly $\{\{\Gamma_t\}^n\}$ is a minimizing sequence. We next show that $\{\{\Gamma_t\}^n\}$ satisfies (1.5). Note first that the construction yields a continuous and increasing function $\lambda : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\lambda(0) = 0 \quad \text{and} \quad \mathfrak{d}\,(\Sigma_t^n, \mathcal{V}_\infty) \geq \lambda(\mathfrak{d}\,(\Gamma_t^n, \mathcal{V}_\infty))\,. \tag{1.13}$$

Fix $\varepsilon > 0$ and choose $\delta > 0$, $N \in \mathbb{N}$ such that $L(\lambda(\varepsilon))/2 - \delta > 1/N$. We claim that (1.5) is satisfied with this choice. Suppose not; then there are $n > N$ and t such that $\mathcal{H}^2(\Gamma_t^n) > m_0 - \delta$ and $\mathfrak{d}\,(\Gamma_t^n, \mathcal{V}_\infty) > \varepsilon$. Hence, from (1.12) and (1.13) we get

$$\mathcal{H}^2(\Sigma_t^n) \;\geq\; \mathcal{H}^2(\Gamma_t^n) + \frac{L(\lambda(\varepsilon))}{2} - \delta \;>\; m_0 + \frac{1}{N} \geq m_0 + \frac{1}{n}\,.$$

This contradicts the assumption that $\mathcal{F}(\{\Sigma_t\}^n) \leq m_0 + 1/n$. Thus (1.5) holds and the proof is completed. $\square$

1.3. Almost minimizing min-max sequences

A stationary varifold can be quite far from an embedded minimal surface. The key point for getting regularity for varifolds produced by min–max sequences is the concept of "almost minimizing surfaces" or a.m. surfaces. Roughly speaking a surface Σ is almost minimizing if any path of surfaces $\{\Sigma_t\}_{t\in[0,1]}$ starting at Σ and such that Σ_1 has small area (compared to Σ) must necessarily pass through a surface with large area. That is, there must exist a $\tau \in]0,1[$ such that Σ_τ has large area compared with Σ; see Fig. 2.

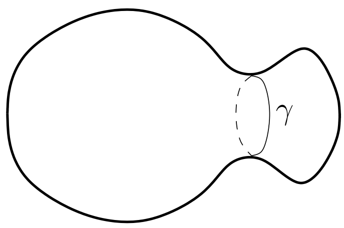

FIGURE 2. Curves near γ are ε–a.m.: It is impossible to deform any such curve isotopically to a much smaller curve without passing through a large curve.

The precise definition of a.m. surfaces is the following:

DEFINITION 1.7. Given $\varepsilon > 0$, an open set $U \subset M^3$, and a surface Σ, we say that Σ is *ε–a.m. in U* if there DOES NOT exist any isotopy ψ supported in U such that

$$\mathcal{H}^2(\psi(t,\Sigma)) \leq \mathcal{H}^2(\Sigma) + \varepsilon/8 \text{ for all } t; \tag{1.14}$$

$$\mathcal{H}^2(\psi(1,\Sigma)) \leq \mathcal{H}^2(\Sigma) - \varepsilon. \tag{1.15}$$

A *sequence* $\{\Sigma^n\}$ is said to be *a.m. in* U if each Σ^n is ε_n–a.m. in U for some $\varepsilon_n \downarrow 0$.

This definition first appeared in Smith's dissertation, [**Smi82**], and was inspired by a similar one of Pitts (see the definition of almost minimizing varifolds in 3.1 of [**Pit81**]). In section 4 of his book, Pitts used combinatorial arguments (some of which were based on ideas of Almgren, [**AJ65**]) to prove a general existence theorem for almost minimizing varifolds. The situation we deal with here is much simpler, due to the fact that we only consider 1–parameter families of surfaces and not general multi–parameter families. In the previous section we showed that there exists a family $\{\Sigma_t\}$ such that every min–max sequence is clustering towards stationary varifolds. Using a version of the combinatorial arguments of Pitts, we will prove in the following proposition that one of these min–max sequences is almost minimizing in sufficiently many annuli:

Proposition 1.8. *There exists a function* $r : M \to \mathbf{R}^+$ *and a min–max sequence* $\{\Sigma^j\}$ *such that:*

(1.16) $\{\Sigma^j\}$ *is a.m. in every* $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$*, for all* $x \in M$;

In every such An*,* Σ^j *is a smooth surface*

(1.17) *when* j *is sufficiently large*;

(1.18) Σ^j *converges to a stationary varifold* V *in* M *as* $j \uparrow \infty$.

The reason why we work with annuli is two fold. The first is that we allow the generalized families to have slices with point–singularities. The second is that even if any family of Λ were made of smooth surfaces, then the combinatorial proof of Proposition 1.8 would give a point $x \in M$ in which we are forced to work with annuli as the reader will later see. For a better understanding of this point consider the following example, due to Almgren ([**AJ65**], p. 15–18; see also [**Pit81**], p. 20–21). The surface M in Fig. 3 is diffeomorphic to $\mathbf{S}^2$ and metrized as a "three–legged starfish". The picture shows a sweep–out with a unique maximal slice, which is a geodesic figure–eight (cf. Fig. 5 of [**Pit81**]). The slices close to the figure–eight are *not* almost minimizing in balls centered at its singular point P. But they are almost minimizing in every sufficiently small annulus centered at P.

If Λ is the saturated set generated by the sweep–out of Fig. 3, then no min–max sequence generated by Λ converges to a simple closed geodesic. However, there are no similar examples of sweep–outs of 3–dimensional manifolds by 2–dimensional objects: The reason for this is that point–singularities of (2–dimensional) minimal surfaces are removable.

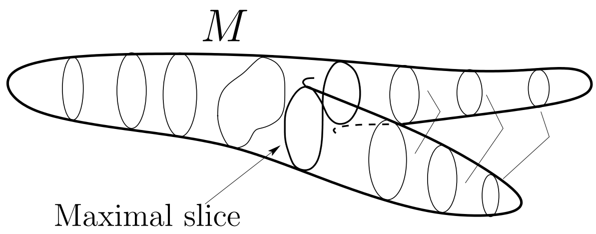

FIGURE 3. A sweep–out of the three–legged starfish, which can be realized as level–sets of a Morse function.

It's useful to introduce the following notation.

DEFINITION 1.9. Given a pair of open sets (U^1, U^2) we say that a surface Σ is *ε–a.m. in* (U^1, U^2) if it is ε–a.m. in at least one of the two open sets. Let A and B be open sets and we consider the following distance d:

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

We denote by $\mathcal{CO}$ the set of pairs (U^1, U^2) of open sets with

$$\mathrm{d}\,(U^1, U^2) \geq 2 \min\{\mathrm{diam}(U^1), \mathrm{diam}(U^2)\}\,.$$

Proposition 1.8 will be an easy corollary of the following:

PROPOSITION 1.10. *There is a min–max sequence* $\{\Sigma^L\} = \{\Sigma^{n(L)}_{t_{n(L)}}\}$ *which converges to a stationary varifold and such that*

$$\text{each } \Sigma^L \text{ is } 1/L\text{–a.m. in every } (U^1, U^2) \in \mathcal{CO}\,. \tag{1.19}$$

Note that the Σ^L's in the previous proposition may be degenerate slices (that is, they may have a finite number of singular points). The key point for proving Proposition 1.10 is the following easy lemma:

LEMMA 1.11. *If* (U^1, U^2) *and* $(V^1, V^2) \in \mathcal{CO}$ *, then there are* $i, j \in \{1, 2\}$ *with* $\mathrm{d}\,(U^i, V^j) > 0$.

PROOF. W.l.o.g., assume that U_1 is, among U_1, U_2, V_1, V_2, the set with smallest diameter. We claim that either $d(U_1, V_1) > 0$ or $d(U_1, V_2) > 0$. If this were false, then there is a point $x \in \overline{U}_1 \cap \overline{V}_1$ and a point $y \in \overline{U}_1 \cap \overline{V}_2$. But then $d(x, y) \leq \mathrm{diam}(U_1) \leq \min\{\mathrm{diam}(V_1), \mathrm{diam}(V_2)\}$, and hence

$$d(V_1, V_2) \leq d(x, y) \leq \min\{\mathrm{diam}(V_1), \mathrm{diam}(V_2)\},$$

contradicting the assumption $(V_1, V_2) \in \mathcal{CO}$. $\square$

Before giving a rigorous proof of Proposition 1.10 we will explain the ideas behind it.

1.3.1. Outline of the proof of Proposition 1.10. First of all note that if a slice $\Sigma^n_{t_0}$ is not ε–a.m. in a given open set U, then we can decrease its area by an isotopy ψ satisfying (1.14) and (1.15). Now fix an open interval I around t_0 and choose a smooth bump function $\varphi \in C^\infty_c(I, [0,1])$ with $\varphi(t_0) = 1$. Define $\{\Gamma_t\}^n$ by

$$\Gamma^n_t \;=\; \psi(\varphi(t), \Sigma^n_t)\,.$$

If the interval I is sufficiently small, then by (1.14), for any $t \in I$, the area of Γ^n_t will not be much larger than the area of Σ^n_t. Moreover, for t very close to t_0 (say, in a smaller interval $J \subset I$) the area of Γ^n_t will be much less than the area of Σ^n_t.

We will show Proposition 1.10 by arguing by contradiction. So suppose that the proposition fails; we will construct a better competitor $\{\{\Gamma_t\}^n\}$. Here the pairs $\mathcal{CO}$ will play a crucial role. Indeed when the area of Σ^n_t is sufficiently large (i.e. close to m_0), we can find *two* disjoint open sets U_1 and U_2 in which Σ^n_t is not almost minimizing. Consider the set $K_n \subset [0,1]$ of slices with sufficiently large area. Using Lemma 1.11 (and some elementary considerations), we find a finite family of intervals I_j, open sets U_j, and isotopies $\psi_j : I_j \times M \to M$ satisfying the following conditions; see Fig. 4:

(1.20) ψ_j is supported in U_j and is the identity at the ends of I_j.

(1.21) If $I_j \cap I_k \neq \emptyset$, then $U_j \cap U_k = \emptyset$.

(1.22) No point of $[0,1]$ belong to more than two I_j's.

(1.23) $\mathcal{H}^2(\psi_j(t, \Sigma^n_t))$ is never much larger than $\mathcal{H}^2(\Sigma^n_t)$.

(1.24) For every $t \in K_n$, there is j s.t. $\mathcal{H}^2(\psi_j(t, \Sigma^n_t))$ is much smaller than $\mathcal{H}^2(\Sigma^n_t)$.

Conditions (1.20) and (1.21) allow us to "glue" the ψ_j's in a unique $\psi \in \mathfrak{Is}$ such that $\psi = \psi_j$ on $I_j \times U_j$. The family $\{\Gamma_t\}^n$ given by $\Gamma^n_t = \psi(t, \Sigma^n_t)$ is our competitor. Indeed for every t, there are at most two ψ_j's which change Σ^n_t. If $t \notin K_n$, then none of them increases the area of Σ^n_t too much. Whereas, if $t \in K_n$, then one ψ_j decreases the area of Σ^n_t a definite amount, and the other increases the area of Σ^n_t a small amount. Thus, the area of the "small–area" slices are not increased much and the area of "large–area" slices are decreased. This yields that $\mathcal{F}(\{\Gamma_t\}^n) - \mathcal{F}(\{\Sigma_t\}^n) < 0$. We will now give a rigorous bound for this (negative) difference.

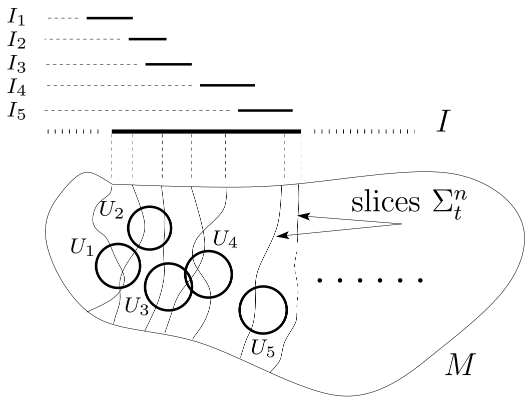

FIGURE 4. The covering I_j and the sets U_j. No point of I is contained in more than two I_j's. The intersection $U_j \cap U_k = \emptyset$ if I_j and I_k overlap.

1.3.2. Proofs of Propositions 1.10 and 1.8.

PROOF OF PROPOSITION 1.10. We choose $\{\{\Sigma_t\}^n\} \subset \Lambda$ such that $\mathcal{F}(\{\Sigma_t\}^n) < m_0 + 1/n$ and satisfying the requirements of Proposition 1.6. Fix $L \in \mathbb{N}$. To prove the proposition we claim there exist $n > L$ and $t_n \in [0,1]$ such that $\Sigma^n = \Sigma^n_{t_n}$ satisfies (1.19) and $\mathcal{H}^2(\Sigma^n) \geq m_0 - 1/L$. We define the sets

$$K_n = \left\{ t \in [0,1] : \mathcal{H}^2(\Sigma^n_t) \geq m_0 - \frac{1}{L} \right\}$$

and argue by contradiction. Suppose not; then for every $t \in K_n$ there exists a pair of open subsets (U^1_t, U^2_t) such that Σ^n_t is not $1/L$–a.m. in either of them. So for every $t \in K_n$ there exists isotopies ψ^i_t such that

(1) ψ^i_t is supported on U^i_t;
(2) $\mathcal{H}^2(\psi^i_t(1, \Sigma^n_t)) \leq \mathcal{H}^2(\Sigma^n_t) - 1/L$;
(3) $\mathcal{H}^2(\psi^i_t(\tau, \Sigma^n_t)) \leq \mathcal{H}^2(\Sigma^n_t) + 1/(8L)$ for every $\tau \in [0,1]$.

In the following we fix n and drop the subscript from K_n. Since $\{\Sigma^n_t\}$ is continuous in t, if $t \in K$ and $|s - t|$ is sufficiently small, then

(2') $\mathcal{H}^2(\psi^i_t(1, \Sigma^n_s)) \leq \mathcal{H}^2(\Sigma^n_s) - 1/(2L)$;
(3') $\mathcal{H}^2(\psi^i_t(\tau, \Sigma^n_s)) \leq \mathcal{H}^2(\Sigma^n_s) + 1/(4L)$ for every $\tau \in [0,1]$.

By compactness we can cover K with a finite number of intervals satisfying (2') and (3'). This covering $\{I_k\}$ can be chosen so that I_k overlaps only with I_{k-1} and I_{k-2}. Summarizing we can find

closed intervals	$I_1, \dots I_r$
pairs of open sets	$(U_1^1, U_1^2), \dots, (U_r^1, U_r^2) \in \mathcal{CO}$
and pairs of isotopies	$(\psi_1^1, \psi_1^2), \dots, (\psi_r^1, \psi_r^2)$

such that

(A) the interiors of I_j cover K and $I_j \cap I_k = \emptyset$ if $|k-j| \geq 2$;
(B) ψ_j^i is supported in U_j^i;
(C) $\mathcal{H}^2(\psi_j^i(1, \Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) - 1/(2L)\ \ \forall s \in I_j$;
(D) $\mathcal{H}^2(\psi_j^i(\tau, \Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) + 1/(4L)\ \ \forall s \in I_j$ and $\tau \in [0,1]$.

In Step 1 we refine this covering. In Step 2 we use the refined covering to construct a competitor $\{\Gamma_t\}^n \in \Lambda$ with

$$\mathcal{F}(\{\Gamma_t\}^n) \leq \mathcal{F}(\{\Sigma_t\}^n) - 1/(2L)\,. \tag{1.25}$$

The arbitrariness of n will give that $\liminf_n \mathcal{F}(\{\Gamma_t\}^n) < m_0$. This is the desired contradiction which yields the proposition.

Step 1: Refinement of the covering.

First we want to find

a covering $\{J_1, \dots, J_R\}$	which is a refinement of $\{I_1, \dots I_r\}$,
open sets $V_1, \dots V_R$	among $\{U_j^i\}$,
and isotopies $\varphi_1, \dots, \varphi_R$	among $\{\psi_j^i\}$,

such that:

(A1) The interiors of J_i cover K and $J_i \cap J_k = \emptyset$ for $|k-i| \geq 2$;
(A2) If $J_i \cap J_k \neq \emptyset$, then $d(V_i, V_k) > 0$;
(B') φ_i is supported in V_i;
(C') $\mathcal{H}^2(\varphi_i(1, \Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) - 1/(2L)\ \ \forall s \in J_i$;
(D') $\mathcal{H}^2(\varphi_i(\tau, \Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) + 1/(4L)\ \ \forall s \in J_i$ and $\tau \in [0,1]$.

We start by setting $J_1 = I_1$ and we distinguish two cases.

- **Case a1**: $I_1 \cap I_2 = \emptyset$; we set $V_1 = U_1^1$, and $\varphi_1 = \psi_1^1$.
- **Case a2**: $I_1 \cap I_2 \neq \emptyset$; by Lemma 1.11 we can choose $i, k \in \{1, 2\}$ such that $\mathrm{d}\,(U_1^i, U_2^k) > 0$ and we set $V_1 = U_1^i$, $\varphi_1 = \psi_1^i$.

We now come to the choice of J_3. If we come from case a1 then:

- **Case b1:** We make our choice as above replacing I_1 and I_2 with I_2 and I_3;

If we come from case a2, then we let i and k be as above and we further distinguish two cases.

- **Case b21**: $I_2 \cap I_3 = \emptyset$; we define $J_2 = I_2$, $V_2 = U_2^k$, $\varphi_2 = \psi_2^k$.

- **Case b22**: $I_2 \cap I_3 \neq \emptyset$; by Lemma 1.11 there exist $l, m \in \{1, 2\}$ such that $d(U_2^l, U_3^m) > 0$. If $l = k$, then we define $J_2 = I_2$, $V_2 = U_2^k$, $\varphi_2 = \psi_2^k$. Otherwise we choose two closed intervals $J_2, J_3 \subset I_2$ such that
  - their interiors cover the interior of I_2,
  - J_2 does not overlap with any I_h for $h \neq 1, 2$,
  - J_3 does not overlap with any I_h for $h \neq 2, 3$.

  Thus we set $V_2 = U_2^k$, $\varphi_2 = \psi_2^k$, and $V_3 = U_2^l$, $\varphi_3 = \psi_2^l$.

An inductive argument using this procedure gives the desired covering. Note that the cardinality of $\{J_1, \ldots, J_R\}$ is at most $2r - 1$.

Step 2: Construction.

Choose C^∞ functions η_i on $\mathbf{R}$ taking values in $[0, 1]$, supported in J_i, and such that for every $s \in K$, there exists η_i with $\eta_i(s) = 1$. Fix $t \in [0, 1]$ and consider the set $\mathrm{Ind}_t \subset \mathbb{N}$ of all i containing t; thus Ind_t consists of at most two elements. Define subsets of M by

$$\Gamma_t^n = \begin{cases} \varphi_i(\eta_i(t), \Sigma_t^n) & \text{in the open sets } V^i,\ i \in \mathrm{Ind}_t\ , \\ \Sigma_t^n & \text{outside.} \end{cases} \tag{1.26}$$

In view of (A1), (A2) and (B'), then $\{\Gamma_t\}^n$ is well defined and belongs to Λ.

Step 3: The contradiction.

We now want to bound the energy $\mathcal{F}(\{\Gamma_t\}^n)$ and hence we have to estimate $\mathcal{H}^2(\Gamma_t^n)$. Note that by (A1) every Ind_t consists of at most two integers. Assume for the sake of argument that Ind_t consists of *exactly* two integers. From the construction, there exist $s_i, s_k \in [0, 1]$ such that Γ_t^n is obtained from Σ_t^n via the diffeomorphisms $\varphi_i(s_i, \cdot)$, $\varphi_k(s_k, \cdot)$. By (A2) these diffeomorphisms are supported on disjoint sets. Thus if $t \notin K$, then (D') gives

$$\mathcal{H}^2(\Gamma_t^n) \ \leq\ \mathcal{H}^2(\Sigma_t^n) + \frac{2}{4L} \ \leq\ m_0 - \frac{1}{2L}.$$

If $t \in K$, then at least one of s_i, s_k is equal to 1. Hence (C) and (D) give

$$\mathcal{H}^2(\Gamma_t^n) \ \leq\ \mathcal{H}^2(\Sigma_t^n) - \frac{1}{L} + \frac{1}{4L} \ \leq\ \mathcal{F}(\{\Sigma_t^n\}) - \frac{3}{4L}\,.$$

Therefore $\mathcal{F}(\{\Gamma_t\}^n) \leq \mathcal{F}(\{\Sigma_t\}^n) - 1/(2L)$. This is the desired bound (1.25). □

We now come to Proposition 1.8.

Proof of Proposition 1.8. We claim that a subsequence of the Σ^k's of Proposition 1.10 satisfies the requirements of Proposition 1.8.

Indeed fix $k \in \mathbb{N}$ and r such that $\text{Inj}\,(M) > 4r > 0$. Since $(B_r(x), M \setminus B_{4r}(x)) \in \mathcal{CO}$. Thus we have that

$$\text{either } \Sigma^k \text{ is } 1/k\text{–a.m. on } B_r(y) \text{ for every } y \tag{1.27}$$

$$\text{or there is } x_r^k \in M \text{ s.t. } \Sigma^k \text{ is } 1/k\text{–a.m. on } M \setminus B_{4r}(x_r^k). \tag{1.28}$$

If for some $r > 0$ there exists a subsequence $\{\Sigma^{k(n)}\}$ satisfying (1.27), then we are done. Otherwise we may assume that there are two sequences of natural numbers $n \uparrow \infty, j \uparrow \infty$ and points x_j^n such that

- For every j, and for n large enough, Σ^n is $1/n$–a.m. in $M \setminus B_{1/j}(x_j^n)$.
- $x_j^n \to x_j$ for $n \uparrow \infty$ and $x_j \to x$ for $j \uparrow \infty$.

Thus for every j, the sequence $\{\Sigma^n\}$ is a.m. in $M \setminus B_{2/j}(x)$. Of course if $U \subset V$ and N is ε–a.m. in V, then N is ε–a.m. in U. This proves that there exists a subsequence $\{\Sigma^j\}$ which satisfies conditions (1.16) and (1.18) for some positive function $r : M \to \mathbf{R}^+$.

It remains to show that an appropriate further subsequence satisfies (1.17). Each Σ^j is smooth except at finitely many points. We denote by P_j the set of singular points of Σ^j. After extracting another subsequence we can assume that P_j is converging, in the Hausdorff topology, to a finite set P. If $x \in P$ and An is any annulus centered at x, then $P_j \cap \text{An} = \emptyset$ for j large enough. If $x \notin P$ and An is any (small) annulus centered at x with outer radius less than $\text{d}\,(x, P)$, then $P_j \cap \text{An} = \emptyset$ for j large enough. Thus, after possibly modifying the function r above, the sequence $\{\Sigma^j\}$ satisfies (1.16), (1.17), and (1.18). □

From now on, in order to simplify our notation, a sequence $\{\Sigma^j\}$ satisfying the conclusions of Proposition 1.8 will be simply called *almost minimizing in sufficiently small annuli.*

1.4. Examples

A key example of saturated sets Λ with $m_0(\Lambda) > 0$ has already been given in the introduction by Proposition 0.3, which we recall here for the reader's convenience.

Proposition 1.12. *Let M be a closed* 3*-manifold with a Riemannian metric and let $\{\Sigma_t\}$ be the level sets of a Morse function. The smallest saturated set Λ containing the family $\{\Sigma_t\}$ has $m_0(\Lambda) > 0$.*

Proof. Let M^3 be a closed Riemannian 3–manifold with a Morse function $f : M \to [0, 1]$. Denote by Σ_t the level set $f^{-1}(\{t\})$ and let Λ

be the saturated set of families

$$\Big\{\{\Gamma_t\}\Big|\ \Gamma_t = \psi(t,\Sigma_t) \text{ for some } \psi\in C^\infty([0,1]\times M, M) \\ \text{with } \psi_t\in \mathrm{Diff}_0 \text{ for every } t\Big\} .$$

To prove Proposition 0.3 we need to show that $m_0(\Lambda)>0$. To do that set $U_t = f^{-1}([0,t[)$ and $V_t=\psi(t,U_t)$. Clearly $\Gamma_t=\partial V_t$ and if we let Vol denote the volume on M, then $\mathrm{Vol}(U_t)$ is a continuous function of t. Since V_0 is a finite set of points and $V_1 = M$, then there exists an s such that $\mathrm{Vol}(V_s)=\mathrm{Vol}(M)/2$. By the isoperimetric inequality there exists a constant $c(M)$ such that

$$\frac{\mathrm{Vol}(M)}{2} \;=\; \mathrm{Vol}(V_s) \;\leq\; c(M)\mathcal{H}^2(\Gamma_s)^{3/2} .$$

Hence,

$$\mathcal{F}(\{\Gamma_t\}) \;=\; \max_{t\in[0,1]}\mathcal{H}^2(\Gamma_t) \;\geq\; \left(\frac{\mathrm{Vol}(M)}{2c(M)}\right)^{\frac{2}{3}} > 0\,, \tag{1.29}$$

and the proposition follows. □

REMARK 1.13. Assume M is diffeomorphic to $\mathrm{S}^3 := \{x\in\mathrm{R}^4 : |x| = 1\}$ and fix a diffeomorphism $\Phi:\mathrm{S}^3\to M$. Then the map

$$\tfrac{1}{2}(x_1(\Phi^{-1})+1) : M \quad\mapsto\quad [0,1]$$

is obviously a Morse function. Moreover, $f^{-1}(\{t\})$ is diffeomorphic to S^2 for every $t\in[0,1]$.

1.5. Schoen–Simon curvature estimates

We end this Chapter recalling some fundamental results due to Schoen and Simon. Consider an orientable $U\subset M$. We look here at closed sets $\Gamma\subset M$ of codimension 1 satisfying the following regularity assumption:

(SS) $\Gamma\cap U$ is a smooth embedded hypersurface outside a closed set S with $\mathcal{H}^{n-2}(S)=0$.

Γ induces an integer rectifiable varifold V. Thus Γ is said to be minimal (resp. stable) in U with respect to the metric g of U if V is stationary (resp. stable). The following compactness theorem, a consequence of the Schoen–Simon curvature estimates (cp. with Theorem 2 of Section 6 in [**SS81**]), is going to be very helpful in many proofs of the following chapters.

THEOREM 1.14. *Let U be an orientable open subset of a manifold and $\{g^k\}$ and $\{\Gamma^k\}$, respectively, sequences of smooth metrics on U and of hypersurfaces $\{\Gamma^k\}$ satisfying (SS). Assume that the metrics g^k converge smoothly to a metric g, that each Γ^k is stable and minimal relative to the metric g^k and that $\sup \mathcal{H}^n(\Gamma^k) < \infty$. Then there are a subsequence of $\{\Gamma^k\}$ (not relabeled), a stable stationary varifold V in U (relative to the metric g) and a closed set S of Hausdorff dimension at most $n-7$ such that*

(a) *V is a smooth embedded hypersurface in $U \setminus S$;*
(b) *$\Gamma^k \to V$ in the sense of varifolds in U;*
(c) *Γ^k converges smoothly to V on every $U' \subset\subset U \setminus S$.*

REMARK 1.15. Since in our work we will always have $n = 3$ parts (a) and (b) of Theorem 1.14 can be rephrased as follows:

$$\begin{aligned} &\text{If } \{\Sigma^n\} \text{ is a sequence of stable minimal surfaces in } U, \\ &\text{then there exists a subsequence converging to} \\ &\text{a stable minimal surface } \Sigma^\infty . \end{aligned} \tag{1.30}$$

CHAPTER 2

Minimizing sequences of isotopic surfaces

In this chapter we will study the regularity of the following problem:

DEFINITION 2.1. Let $\mathcal{I}$ be a class of isotopies of M and $\Sigma \subset M$ a smooth embedded surface. If $\{\varphi^k\} \subset \mathcal{I}$ and

$$\lim_{k\to\infty} \mathcal{H}^2(\varphi^k(1,\Sigma)) \;=\; \inf_{\psi\in\mathcal{I}} \mathcal{H}^2(\psi(1,\Sigma))\,,$$

then we say that $\varphi^k(1,\Sigma)$ is a *minimizing sequence for Problem* $(\Sigma, \mathcal{I})$.

If U is an open set of M, Σ a surface with $\partial\Sigma \subset \partial U$ and $j \in \mathbb{N}$ an integer, then we define

(2.1)

$$\mathfrak{Is}_j(U,\Sigma) \;:=\; \left\{\psi \in \mathfrak{Is}(U) \middle|\, \mathcal{H}^2(\psi(\tau,\Sigma)) \leq \mathcal{H}^2(\Sigma) + 1/(8j) \quad \forall \tau \in [0,1]\right\}\,.$$

We state our main regularity result in the next theorem.

THEOREM 2.2 (Regularity). *Let $U \subset M$ be an open set and consider a smooth embedded surface Σ. Let $\Delta^k := \varphi^k(1,\Sigma)$ be a minimizing sequence for Problem $(\Sigma, \mathfrak{Is}_j(U,\Sigma))$, converging to a stationary varifold V.*

(a) *(Interior regularity) There exists a smooth minimal surface Δ with $\overline{\Delta} \setminus \Delta \subset \partial U$ and $V = \Delta$ in U;*
(b) *(Boundary regularity) If U is an open ball with sufficiently small radius and Σ is such that $\partial\Sigma \subset \partial U$ is also smooth. Then there exists a smooth minimal surface Δ with $\overline{\Delta} \setminus \Delta \subset \partial U$, $V = \Delta$ in U and with smooth boundary $\partial\Delta = \partial\Sigma$;*
(c) *(Connected components) Let U and Σ as in (b). If we form a new sequence $\tilde{\Delta}^k$ by taking an arbitrary union of connected components of Δ^k, it converges, up to subsequences, to the union of some connected components of Δ.*

This theorem has been proved in [**MSY82**] for $U = M$ for surfaces without boundary. Our proof for the case of surfaces with fixed boundary relies on the techniques introduced by Almgren and Simon in [**AS79**] and Meeks, Simon and Yau in [**MSY82**]. Our proof is quite technical and will require deep results from geometric measure theory due to Allard, Almgren and Simon. We are going to split its proof in 5 parts: Section 2.1 discusses the convex–hull properties needed for the boundary regularity. In Section 2.2 we introduce and prove the "squeezing lemmas" which allow to pass from almost–minimizing sequences to minimizing sequences. Section 2.3 discusses the γ–reduction (see section 3 of [**MSY82**])and how one applies it to get the interior regularity. We also point out why the γ–reduction cannot be applied directly to the surfaces of Proposition 2.2. Section 2.4 proves the boundary regularity. Finally, section 2.5 handles the part of Proposition 2.2 involving limits of connected components.

Before we concentrate on the proof of the Theorem, we collect some elementary remarks on minimizing sequences of isotopic surfaces which will be used later in this work.

Remark 2.3. If Σ is $1/j$–a.m. in an open set U and $\tilde{U}$ is an open set contained in U, then Σ is $1/j$–a.m. in $\tilde{U}$.

Remark 2.4. If Σ is $1/j$–a.m. in U and $\psi \in \mathfrak{Is}_j(\Sigma, U)$ is such that $\mathcal{H}^2(\psi(1,\Sigma)) \leq \mathcal{H}^2(\Sigma)$, then $\psi(1,\Sigma)$ is $1/j$–a.m. in U.

2.1. Part I: Convex hull property

2.1.1. Preliminary definitions. Consider an open geodesic ball $U = B_\rho(\xi)$ with sufficiently small radius ρ and a subset $\gamma \subset \partial U$ consisting of finitely many disjoint smooth Jordan curves.

Definition 2.5. We say that an open subset $A \subset U$ meets ∂U in γ transversally if there exists a positive angle θ_0 such that:

(a) $\partial A \cap \partial U \subset \gamma$.

(b) For every $p \in \partial A \cap \partial U$ we choose coordinates (x, y, z) in such a way that the tangent plane T_p of ∂U at p is the xy-plane and $\gamma'(p) = (1, 0, 0)$. Then in this setting every point $q = (q_1, q_2, q_3) \in A$ satisfies $\frac{q_3}{q_2} \geq \tan(\frac{\pi}{2} - \theta_0)$.

Remark 2.6. Condition (b) of the above definition can be stated in the following geometric way: there exixt two halfplanes π_1 and π_2 meeting at the line through p in direction $\gamma'(p)$ such that

- they form an angle θ_0 with T_p;
- the set A is all contained in the wedge formed by π_1 and π_2;

see Figure 1.

In this subsection we will show the following lemma.

Lemma 2.7 (Convex hull property). *Let V and Σ be as in Proposition 2.2. Then, there exists a convex open set $A \subset U$ which intersects U in $\partial\Sigma$ transversally and such that* $\mathrm{supp}\,(\|V\|) \subset \overline{A}$.

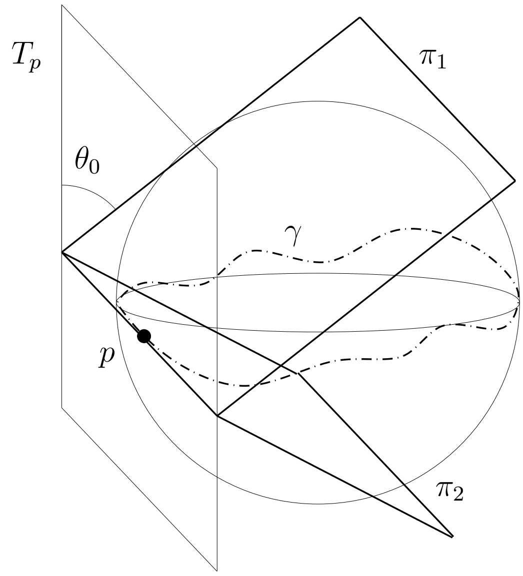

FIGURE 1. For any $p \in A \cap \partial U$, A is contained in a wedge delimited by two halfplanes meeting at p transversally to the plane T_p.

Our starting point is the following elementary fact about convex hulls of smooth curves lying in the euclidean two–sphere.

PROPOSITION 2.8. *If $\beta \subset \partial \mathcal{B}_1 \subset \mathbf{R}^3$ is the union of finitely many C^2−Jordan curves, then its convex hull meets $\mathcal{B}_1$ transversally in β.*

The proof of this proposition follows from the regularity and the compactness of β and from the fact that β is not self-intersecting. We leave its details to the reader.

2.1.2. Proof of Lemma 2.7. From now on, we consider $\gamma = \partial\Sigma$: this is the union of finitely many disjoint smooth Jordan curves contained in ∂U. Recall that U is a geodesic ball $B_\rho(\xi)$. Without loss of generality we assume that ρ is smaller than the injectivity radius.

Step 1 Consider the rescaled exponential coordinates induced by the chart $f : \overline{B}_\rho(\xi) \to \overline{\mathcal{B}}_1$ given by $f(z) = (\exp_\xi^{-1}(z))/\rho$. These coordinates will be denoted by (x_1, x_2, x_3). We apply Proposition 2.8 and consider the convex hull B of $\beta = f(\partial\Sigma)$ in $\mathcal{B}_1$. According to our definition, $f^{-1}(B)$ meets U transversally in γ.

We now let θ_0 be a positive angle such that condition (b) in Definition 2.5 is fulfilled for B. Next we fix a point $x \in f(\gamma)$ and consider the halfplanes π_1 and π_2 delimiting the wedge of condition (b). Without loss of generality, we can assume that the coordinates are chosen so that π_1 is given by

$$\pi_1 = \{(z_1, z_2, z_3) : z_3 \leq a\}$$

for some positive constant a. Condition (b) ensures that $a \leq a_0 < 1$ for some constant a_0 inpendent of the point $x \in f(\gamma)$.

For $t \in]0, \infty[$ denote by C_t the points $C_t := \{(0, 0, -t)\}$ and by $r(t)$ the positive real numbers

$$r(t) \ := \ \sqrt{1 + t^2 + 2at}$$

We finally denote by R_t the closed balls

$$R_t \ := \ \overline{\mathcal{B}}_{r(t)}(C_t)\,.$$

The centers C_t and the radii $r(t)$ are chosen in such a way that the intersection of the sphere ∂R_t and $\partial \mathcal{B}_1$ is always the circle $\pi_1 \cap \partial \mathcal{B}_1$.

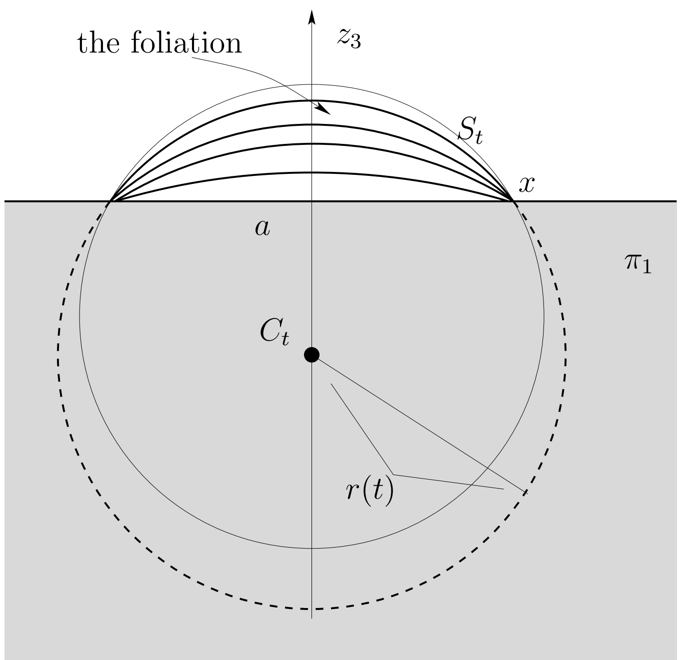

FIGURE 2. A planar cross-section of the foliation $\{S_t : t \in]0, \infty[\}$.

Note, moreover, that for t coverging to $+\infty$, the ball R_t converges towards the region $\{z_3 \leq a\}$. Therefore, the region $\{z_3 > a\} \cap \mathcal{B}_1$ is foliated with the caps

$$S_t \ := \ \partial R_t \cap \mathcal{B}_1 \qquad \text{for } t \in]0, \infty[.$$

$\rho_0 \leq \rho(c_0)$, then the nearest point projection π on $\overline{f(U)}$ is a Lipschitz map with constant 1. Moreover, at every point $P \notin \overline{f(U)}$, $|\nabla\pi(P)| < 1$.

Proof. Let $d_e(y)$ be the euclidean distance of y to $\overline{U}$ and $d(y)$ the geodesic distance of y to $\overline{f(U)}$. The function d_e is C^2 and uniformly convex on the closure of $\mathcal{B}_1 \setminus U$. Therefore, if ε_0 is sufficiently small, the function d is uniformly convex on the closure of $B_\varepsilon(x) \setminus f(U)$. Let now $y_0 \in B_\varepsilon(x) \setminus \overline{f(U)}$. In order to find $\pi(x)$ it suffices to follow the flow line of the ODE $\dot{y} = -\nabla d(y)/|\nabla d(y)|^2$, with initial condition $y(0) = y_0$, until the line hits $\overline{f(U)}$. Thus, the inequality $|\nabla\pi(x)| < 1$ follows from Lemma 1 of [**Ban79**]. On the other hand, $\pi(x) = x$ on $\overline{f(U)}$, and therefore the map is Lipschitz with constant 1. □

Next, it is obvious that π_0 is the identity map and that the map $(t, x) \mapsto \pi_t(x)$ is smooth.

Assume now for a contradiction that V is not supported in $f^{-1}(R_{t_0})$. By Lemma 2.9, the varifold $(\pi_{t_0})_\# V$ has, therefore, strictly less mass than the varifold V.

Next, consider a minimizing sequence Δ^k as in the statement of theorem 2.2. Since $\partial\Delta^k = \partial\Sigma$, the intersection of $\overline{\Delta^k}$ with ∂U is given by $\partial\Sigma$. On the other hand, by construction $\partial\Sigma \subset f^{-1}(R_t)$ and therefore, if we consider $\Delta^k_t := (\pi_t)_\# \Delta^k$ we obtain a (continuous) one-parameter family of currents with the properties that

(i) $\partial\Delta^k_t = \partial\Sigma$;
(ii) $\Delta^k_0 = \Delta_0$;
(iii) The mass of Δ^k_t is less or equal than $\mathcal{H}^2(\Delta^k)$;
(iv) The mass of $\Delta^k_{t_0}$ converges towards the mass of $(\pi_{t_0})_\# V$ and hence, for k large enough, it is strictly smaller than the mass of V.

Therefore, if we fix a sufficiently large number k, we can assume that (iv) holds with a gain in mass of a positive amount $\varepsilon = 1/j$. We can, moreover, assume that $\mathcal{H}^2(\Delta^k) \leq \mathcal{H}^2(\Sigma) + 1/(8j)$. By an approximation procedure, it is possible to replace the family of projections $\{\pi_t\}_{t\in[0,t_0]}$ with a smooth isotopy $\{\psi_t\}_{t\in[0,1]}$ with the following properties:

(v) ψ_0 is the identity map and $\psi_t|_{\partial U}$ is the identity map for every $t \in [0, 1]$;
(vi) $\mathcal{H}^2(\Delta^k) \leq \mathcal{H}^2(\psi_t(\Sigma)) + 1/(8j)$;
(vii) $\mathcal{H}^2(\psi_1(\Delta^k)) \leq \mathbf{M}((\pi_{t_0})_\# V) - 1/j$.

This contradicts the $1/j$–almost minimizing property of Σ.

In showing the existence of the family of isotopies ψ_t, a detail must be taken into account: the map π_t is smooth everywhere on $\overline{U}$ but on the circle $f^{-1}(R_t)\cap\partial U$ (which is the same circle for every t!). We briefly indicate here a procedure to construct ψ_t, skipping the cumbersome details.

We replace the sets $\{R_t\}$ with a new family $\mathcal{R}_t$ which have the following properties:

- $\mathcal{R}_0 = \overline{\mathcal{B}}_1$;
- $\mathcal{R}_{t_0} = R_{t_0}$;
- For $t \in [0, t_0]$ the boundaries $\partial\mathcal{R}_t$ are uniformly convex;
- $\partial\mathcal{R}_t \cap \partial\mathcal{B}_1 = \partial R_t \cap \partial\mathcal{B}_1$;
- The boundaries of $\partial\mathcal{R}_t$ are smooth for $t \in [0, t_0[$ and form a smooth foliation of $\mathcal{B}_1(0) \setminus R_{t_0}$.

The properties of the new sets are illustrated in Figure 4

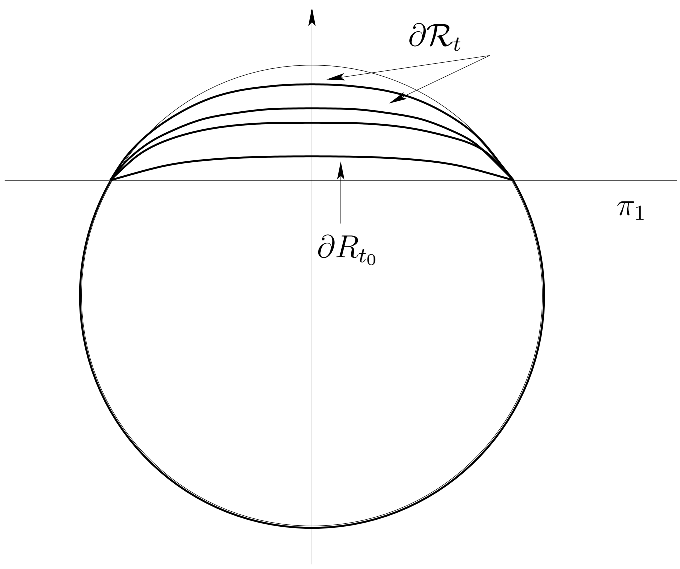

FIGURE 4. A planar cross-section of the new foliation.

Since $\overline{\Delta^k}$ touches ∂U in $\partial\Sigma$ transversally and $\partial\Sigma \subset f^{-1}(\mathcal{R}_t)$ for every t, we conclude the existence of a small δ such that $\Delta^k \subset f^{-1}(\mathcal{R}_{2\delta})$. Moreover, for δ sufficiently small, the nearest point projection $\tilde{\pi}_{t_0-\delta}$ on $f^{-1}(\mathcal{R}_{t_0-\delta})$ is so close to π_{t_0} that

$$\mathbf{M}((\tilde{\pi}_{t_0-\delta})_\# \Delta^k) \;\leq\; \mathbf{M}((\pi_{t_0})_\# \Delta^k) + \varepsilon/4\,.$$

We then construct ψ_t in the following way. We fix a smooth increasing bijective function $\tau : [0, 1] \to [\delta, t_0 - \delta]$,

- ψ_t is the identity on $\overline{U} \setminus \mathcal{R}_\delta$ and on $\mathcal{R}_{\tau(t)}$;
- On $\mathcal{R}_\delta \setminus \mathcal{R}_{\tau(t)}$ it is very close to the projection $\tilde{\pi}_{\tau(t)}$ on $\mathcal{R}_{\tau(t)}$.

In particular, for this last step, we fix for a smooth function $\sigma : [0,1] \times [0,1]$ such that, for each t, $\sigma(t,\cdot)$ is a smooth bijection between $[0,1]$ and $[\delta, \tau(t)]$ very close to the function which is identically $\tau(t)$ on $[0,1]$. Then, for $s \in [0,1]$, we define ψ_t on the surface $\partial\mathcal{R}_{(1-s)\delta+s\tau(t)}$ to be the nearest point projection on the surface $\partial\mathcal{R}_{\sigma(t,s)}$. So, ψ_t fixes the leave $\partial\mathcal{R}_\delta$ but moves most of the leaves between $\partial\mathcal{R}_\delta$ and $\partial\mathcal{R}_{\tau(t)}$ towards $\partial\mathcal{R}_{\tau(t)}$. This completes the proof of Lemma 2.7.

2.2. Part II: Squeezing Lemma

In this section we prove the following Lemma.

LEMMA 2.10 (Squeezing Lemma). *Let $\{\Delta^k\}$ be as in Theorem 2.2, $x \in \overline{U}$ and $\beta > 0$ be given. Then there exists an $\varepsilon_0 > 0$ and a $K \in \mathbb{N}$ with the following property. If $k \geq K$ and $\varphi \in \mathfrak{Is}(B_{\varepsilon_0}(x) \cap U)$ is such that $\mathcal{H}^2(\varphi(1,\Delta^k)) \leq \mathcal{H}^2(\Delta^k)$, then there exists a $\Phi \in \mathfrak{Is}(B_{\varepsilon_0}(x) \cap U)$ such that*

$$\Phi(1,\cdot) \;=\; \varphi(1,\cdot) \tag{2.2}$$

$$\mathcal{H}^2(\Phi(t,\Delta^k)) \;\leq\; \mathcal{H}^2(\Delta^k) + \beta \qquad \text{for every } t \in [0,1]. \tag{2.3}$$

If x is an interior point of U, this lemma reduces to Lemma 7.6 of [**CDL03**]. When x is on the boundary of U, one can argue in a similar way (cp. with Section 7.4 of [**CDL03**]). Indeed, the proof of Lemma 7.6 of [**CDL03**] relies on the fact that, when ε is sufficiently small, the varifold V is close to a cone. For interior points, this follows from the stationarity of the varifold V. For points at the boundary this, thanks to a result of Allard (see [**All75**]), is a consequence of the stationarity of V and of the convex hull property of Lemma 2.7.

2.2.1. Tangent cones. Consider the varifold V of Theorem 2.2. Given a point $x \in \overline{U}$ and a radius $\rho > 0$, consider the chart $f_{x,\rho} : B_\rho(x) \to \mathcal{B}_1$ given by $f_{x,\rho}(y) = \exp_x^{-1}(y)/\rho$. We then consider the varifolds $V_{x,\rho} := (f_{x,\rho})_\# V$. Moreover, if $\lambda > 0$, we will denote by $O_\lambda : \mathbf{R}^3 \to \mathbf{R}^3$ the rescaling $O_\lambda(x) = x/\lambda$.

If $x \in U$, the monotonicity formula and a compactness result (see Theorem 19.3 of [**Sim83**]) imply that, for any $\rho_j \downarrow 0$, there exists a subsequence, not relabeled, such that V_{x,ρ_j} converges to an integer rectifiable varifold W supported in $\mathcal{B}_1$ with the property that $(O_\lambda)_\# W \llcorner B_1(0) = W$ for any $\lambda < 1$. The varifolds W which are limit of subsequences V_{x,ρ_j} are called tangent cones to V at x. The monotonicity formula implies that the mass of each W is a positive constant $\theta(x,V)$ independent of W (see again Theorem 19.3 of [**Sim83**]).

If $x \in \partial U$, we fix coordinates y_1, y_2, y_3 in R^3 in such a way that $f_{x,\rho}(U \cap B_\rho(x))$ converges to the half-ball $\mathcal{B}_1^+ = \mathcal{B}_1 \cap \{y_1 > 0\}$.

Recalling Lemma 2.7, we can infer with the monotonicity formula of Allard for points at the boundary (see 3.4 of [**All75**]) that $V_{x,\rho} = (f_{x,\rho})_\# V$ have equibounded mass. Therefore, if $\rho_j \downarrow 0$, a subsequence of V_{x,ρ_j}, not relabeled, converges to a varifold W.

By Lemma 2.7, there is a positive angle θ_0 such that, after a suitable change of coordinates, W is supported in the set

$$\{|y_2| \leq y_1 \tan\theta_0\}\,.$$

Therefore $\mathrm{supp}\,(W) \cap \{y_1 = 0\} = \{(0,0,t) : t \in [-1,1]\} =: \ell$. Applying the monotonocity formula of 3.4 of [**All75**], we conclude that

$$\|W\|(\ell) \;=\; 0 \tag{2.4}$$

and

$$\|W\|(\mathcal{B}_\rho(0)) \;=\; \pi\theta(\|V\|,x)\rho^2\,, \tag{2.5}$$

where

$$\theta(\|V\|,x) \;=\; \lim_{r\downarrow 0} \frac{\|V\|(B_\rho(x))}{\pi\rho^2}$$

is independent of W. Being W the limit of a sequence V_{x,ρ_j} with $\rho_j \downarrow 0$, we conclude that W is a stationary varifold.

Now, define the reflection map $r : \mathbf{R}^3 \to \mathbf{R}^3$ given by $r(z_1,z_2,z_3) = (-z_1,-z_2,z_3)$. By (2.4), using the reflection principle of 3.2 of [**All75**], the varifold $W' := W + r_\# W$ is a stationary varifold. By (2.5) and Corollary 2 of 5.1 in [**All72**], we conclude that $(O_\lambda)_\# W' \llcorner \mathcal{B}_1^+ = W'$ for every $\lambda < 1$. On the other hand, this implies $(O_\lambda)_\# W \llcorner \mathcal{B}_1^+ = W$. Therefore W is a cone and we will call it *tangent cone to V at x*.

2.2.2. A squeezing homotopy. Since for points in the interior the proof is already given in [**CDL03**], we assume that $x \in \partial U$. Moreover, the proof given here in this case can easily be modified for $x \in U$. Therefore we next fix a small radius $\varepsilon > 0$ and consider an isotopy φ of $U \cap B_\varepsilon(x)$ keeping the boundary fixed.

We start by fixing a small parameter $\delta > 0$ which will be chosen at the end of the proof. Next, we consider a diffeomorphism G_ε between $\mathcal{B}_\varepsilon^+ = \mathcal{B}_\varepsilon \cap \{y_1 > 0\}$ and $B_\varepsilon(x) \cap U$. Consider on $\mathcal{B}_\varepsilon^+$ the standard Euclidean metric and denote the corresponding 2-dimensional Hausdorff measure with $\mathcal{H}_e^2$. If ε is sufficiently small, then G_ε can be chosen so that the Lipschitz constants of G_ε and G_ε^{-1} are both smaller than $1+\varepsilon$. Then, for any surface $\Delta \subset B_\varepsilon(x)\cap U$,

$$(2.6) \qquad (1-C\delta)\mathcal{H}^2(\Delta) \;\le\; \mathcal{H}^2_e(G_\varepsilon(\Delta)) \;\le\; (1+C\delta)\mathcal{H}^2(\Delta)\,,$$

where C is a universal constant.

We want to construct an isotopy $\Lambda \in \mathfrak{Is}(\mathcal{B}^+_\varepsilon)$ such that $\Lambda(1,\cdot) = G_\varepsilon\circ\varphi(1, G_\varepsilon^{-1}(\cdot))$ and (for k large enough)

$$(2.7) \qquad \mathcal{H}^2_e(\Lambda(t, G_\varepsilon(\Delta^k))) \;\le\; \mathcal{H}^2_e(G_\varepsilon(\Delta^k))(1+C\delta)+C\delta \qquad \text{for every } t\in[0,1].$$

After finding Λ, $\Phi(t,\cdot) = G_\varepsilon^{-1}\circ\Lambda(t, G_\varepsilon(\cdot))$ will be the desired map. Indeed Φ is an isotopy of $B_\varepsilon(x)\cap U$ which keeps a neighborhood of $B_\varepsilon(x)\cap U$ fixed. It is easily checked that $\Phi(1,\cdot)=\varphi(1,\cdot)$. Moreover, by (2.6) and (2.7), for k sufficiently large we have

$$(2.8) \qquad \mathcal{H}^2(\Phi(t,\Delta^k)) \;\le\; (1+C\delta)\mathcal{H}^2(\Delta^k)+C\delta \qquad \forall t\in[0,1]\,,$$

for some constant C inpendent of δ and k. Since $\mathcal{H}^2(\Delta^k)$ is bounded by a constant independent of δ and k, by choosing δ sufficiently small, we reach the claim of the Lemma.

Next, we consider on $\mathcal{B}^+_\varepsilon$ a one-parameter family of diffeomorphisms. First of all we consider the continuous piecewise linear map $\alpha : [0,1[\times[0,1[\to[0,1]$ defined in the following way:

- $\alpha(t,s)=s$ for $(t+1)/2\le s\le 1$;
- $\alpha(t,s)=(1-t)s$ for $0\le s\le t$;
- $\alpha(t,s)$ is linear on $t\le s\le (t+1)/2$.

So, each $\alpha(t,\cdot)$ is a biLipschitz homeomorphism of $[0,1]$ keeping a neighborhood of 1 fixed, shrinking a portion of $[0,1]$ and uniformly stretching the rest. For t very close to 1, a large portion of $[0,1]$ is shrinked into a very small neighborhood of 0, whereas a small portion lying close to 1 is stretched to almost the whole interval.

Next, for any given $t\in[0,1[$, let $y_t := ((1-t)\eta\varepsilon,0,0)$ where η is a small parameter which will be fixed later. For any $z\in\mathcal{B}^+_\varepsilon$ we consider the point $\pi_t(z)\in\partial\mathcal{B}^+_\varepsilon$ such that the segment $[y_t,\pi_t(z)]$ containing z. We then define $\Psi(t,z)$ to be the point on the segment $[y_t,\pi_t(z)]$ such that

$$|y_t-\Psi(t,z)| = \alpha\left(t, \frac{|y_t-z|}{|y_t-\pi_t(z)|}\right)|y_t-\pi_t(z)|\,.$$

It turns out that $\Psi(0,\cdot)$ is the identity map and, for fixed t, $\Psi(t,\cdot)$ is a biLipschitz homeomorphism of $\mathcal{B}^+_\varepsilon$ keeping a neighborhood of $\partial\mathcal{B}^+_\varepsilon$ fixed. Moreover, for t close to 1, $\Psi(t,\cdot)$ shrinks a large portion of $\mathcal{B}^+_\varepsilon$ in a neighborhood of y_t and stretches uniformly a layer close to $\partial\mathcal{B}_\varepsilon$. See Figure 5.

In Figure 2, we see a section of this foliation with the plane z_2z_3.

We claim that, for some constant $t_0 > 0$ independent of the choice of the point $x \in f(\gamma)$, the varifold V is supported in $f^{-1}(R_{t_0})$. A symmetric procedure can be followed starting from the plane π_2. In this way we find two off-centered balls and hence a corresponding wedge W_x satisfying condition (b) of Definition 2.5 and containing the support of V; see Picture 3. Our claim that the constant t_0 can be chosen independently of x and the bound $a \leq a_0 < 1$ imply that the the planes delimiting the wedge W_x form an angle larger than some fixed constant with the plane T_x tangent to $\partial\mathcal{B}_1$ at x. Therefore, the intersections of all the wedges W_x, for x varying among the points of γ, yield the desired set A.

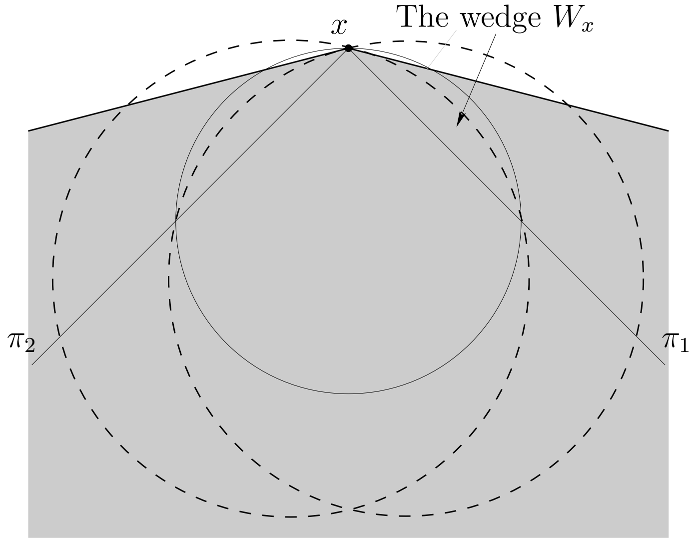

FIGURE 3. A planar cross-section of the wedge W_x.

Step 2 We next want to show that the varifold V is supported in the closed ball $f^{-1}(R_{t_0})$. For any $t \in [0, t_0[$, denote by $\pi_t : \overline{U} \to f^{-1}(R_t)$ the nearest point projection. If the radius ρ_0 of U and the parameter t_0 are both sufficiently small, then π_t is a well defined Lipschitz map (because there exists a unique nearest point). Moreover, the Lipschitz constant of π_t is equal to 1 and, for $t > 0$, $|\nabla \pi_t| < 1$ on $U \setminus f^{-1}(R_t)$. In fact the following lemma holds.

LEMMA 2.9. *Consider in the euclidean ball $\mathcal{B}_1$ a set U that is uniformly convex, with constant c_0. Then there is a $\rho(c_0) > 0$ such that, if*

We next consider the isotopy $\Xi(t,\cdot) := G_\varepsilon^{-1} \circ \Psi(t, G_\varepsilon(\cdot))$. It is easy to check that, if we fix a Δ^k and we let $t \uparrow 1$, then the surfaces $\Psi(1, G_\varepsilon(\Delta^k))$ converge to the cone with center 0 and base $G_\varepsilon(\Delta^k) \cap \partial\mathcal{B}_\varepsilon$.

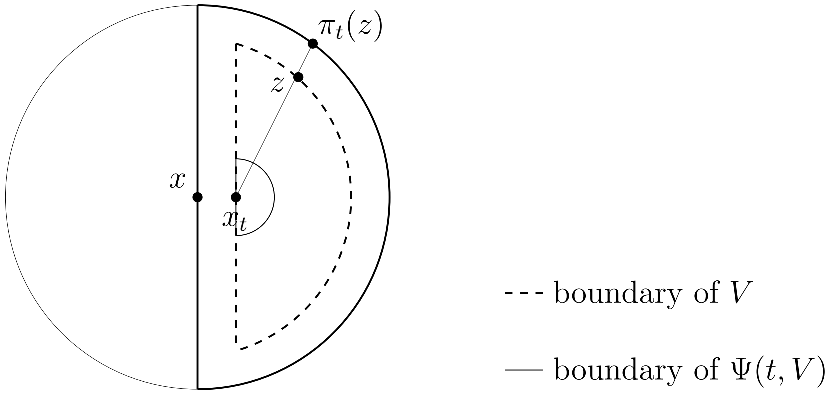

FIGURE 5. For t close to 1 the map $\Psi(t,\cdot)$ shrinks homotethically a large portion of $\mathcal{B}_\varepsilon^+$.

2.2.3. Fixing a tangent cone. By Subsection 2.2.1, we can find a sequence $\rho_l \downarrow 0$ such that V_{x,ρ_l} converges to a tangent cone W. Our choice of the diffeomorphism G_{ρ_l} implies that $(O_{\rho_l} \circ G_{\rho_l})_\# V$ has the same varifold limit as V_{x,ρ_l}.

Since Δ^k converges to V in the sense of varifolds, by a standard diagonal argument, we can find an increasing sequence of integers K_l such that:

(C) $(O_{\rho_l}(G_{\rho_l}(\Delta^{k_l}))$ converges in the varifold sense to W, whenever $k_l \geq K_l$.

(C), the conical property of W and the coarea formula imply the following fact. For ρ_l sufficiently small, and for k sufficiently large, there is an $\varepsilon \in]\rho_l/2, \rho_l[$ such that:

(2.9)
$$\mathcal{H}^2_e\big(\Psi(t, G_\varepsilon(\Delta^k) \cap L)\big) \;\leq\; \mathcal{H}^2_e\big(G_\varepsilon(\Delta^k) \cap L\big) + \delta \qquad \forall t \text{ and all open } L \subset \mathcal{B}^+_\varepsilon,$$

where Ψ is the map constructed in the previous subsection. This estimate holds independently of the small parameter η. Moreover, it fixes the choice of ε_0 and K as in the statement of the Lemma. K depends only on the parameter δ, which will be fixed later. ε might depend on $k \geq K$, but it is always larger than some fixed ρ_l, which will then be the ε_0 of the statement of the Lemma.

2.2.4. Construction of Λ. Consider next the isotopy $\psi = G_\varepsilon \circ \varphi \circ G_\varepsilon^{-1}$. By definition, there exists a compact set K such that $\psi(t,z) = z$ for $z \in \mathcal{B}^+_\varepsilon \setminus K$ and every t. We now choose η so small that $K \subset \{x : x_1 > \eta\varepsilon\}$. Finally, consider $T \in]0,1[$ with T sufficiently close to 1. We build the isotopy Λ in the following way:

- for $t \in [0, 1/3]$ we set $\Lambda(t,\cdot) = \Psi(3tT,\cdot)$;
- for $t \in [1/3, 2/3]$ we set $\Lambda(t,\cdot) = \Psi(3tT, \psi(3t-1,\cdot))$;
- for $t \in [2/3, 1]$ we set $\Lambda(t,\cdot) = \Psi(3(1-t)T, \psi(1,\cdot))$.

If T is sufficiently large, then Λ satisfies (2.7). Indeed, for $t \in [0, 1/3]$, (2.7) follows from (2.9). Next, consider $t \in [1/3, 2/3]$. Since $\psi(t,\cdot)$ moves only points of K, $\Lambda(t,x)$ coincides with $\Psi(T,x)$ except for x in $\Psi(T,K)$. However, $\Psi(T,x)$ is homotethic to K with a very small shrinking factor. Therefore, if T is chosen sufficiently large, $\mathcal{H}^2_e(\Lambda(t, G_\varepsilon(\Delta^k)))$ is arbitrarily close to $\mathcal{H}^2_e(\Lambda(1/3, G_\varepsilon(\Delta^k)))$. Finally, for $t \in [2/3, 1]$, $\Lambda(t,x) = \Psi(3(1-t)T, x)$ for $x \notin \Psi(3(1-t)T, K)$ and it is $\Psi(3(1-t)T, \psi(1,x))$ otherwise. Therefore, $\Lambda(t, G_\varepsilon(\Delta^k))$ differs from $\Psi(3(1-t)T, G_\varepsilon(\Delta^k))$ for a portion which is a rescaled version of $G_\varepsilon(\varphi(1,\Delta^k) \setminus G_\varepsilon(\Delta^k)$. Since by hypothesis $\mathcal{H}^2(\varphi(1,\Delta^k)) \leq \mathcal{H}^2(\Delta^k)$, we actually get

$$\mathcal{H}^2_e\big(G_\varepsilon(\varphi(1,\Delta^k)) \setminus G_\varepsilon(\Delta^k)\big) \;\leq\; (1 + C\delta)\mathcal{H}^2_e\big(G_\varepsilon(\Delta^k) \setminus G_\varepsilon(\varphi(1,\Delta^k))\big)$$

and by the scaling properties of the euclidean Hausdorff measure we conclude (2.7) for $t \in [2/3, 1]$ as well.

Though Λ is only a path of biLipschitz homeomorphisms, it is easy to approximate it with a smooth isotopy: it suffices indeed to smooth $\alpha|_{[0,T]\times[0,1]}$, for instance mollifying it with a standard kernel.

2.3. Part III: γ–reduction

In this section we prove the following

Proposition 2.11. *Let U be an open set. If Λ is an embedded surface with smooth boundary $\partial\Lambda \subset \partial U$ and $\{\Lambda^k\}$ is a minimizing sequence for Problem $(\Lambda, \mathfrak{Is}(U))$ converging to a varifold W, then there exists a stable minimal surface Γ with $\overline{\Gamma} \setminus \Gamma \subset \partial\Lambda$ and $W = \Gamma$ in U.*

Note that this proposition is in fact part (a) of Theorem 2.2. This Proposition has been claimed in [**CDL03**] (cp. with Theorem 7.3 therein) and since nothing on the behavior of W at the boundary is claimed, it follows from a straightforward modification of the theory of γ-reduction of [**MSY82**] (as asserted in [**CDL03**]). This simple modification of the γ–reduction is, as the original γ-reduction, a procedure to reduce through simple surgeries the minimizing sequence Λ^k into a more suitable sequence.

In this section we also wish to explain why this argument cannot be directly applied neither to the surfaces Δ^k of Theorem 2.2 (b) on the *whole* domain U (see Remark 2.16), nor to their intersections with a smaller set U' (see Remark 2.18). In the first case, the obstruction comes from the $1/j$-a.m. property, which is not powerful enough to perform certain surgeries. In the second case this obstruction could be removed by using the squeezing lemma, but an extra difficulty pops out: the intersection $\Delta^k \cap \partial U'$ is, this time, not fixed and the topology of $\Delta^k \cap U'$ is not controlled. These technical problems are responsible for most of the complications in our proof.
Firstly we define here the surgeries permitted in this work and used to perform the γ–reduction.

2.3.1. Surgery. The surgeries that we will use are of two kind: we are allowed to

- remove a small cylinder and replace it by two disks (as in Fig. 6);
- discard a connected component.

We give below the precise definition.

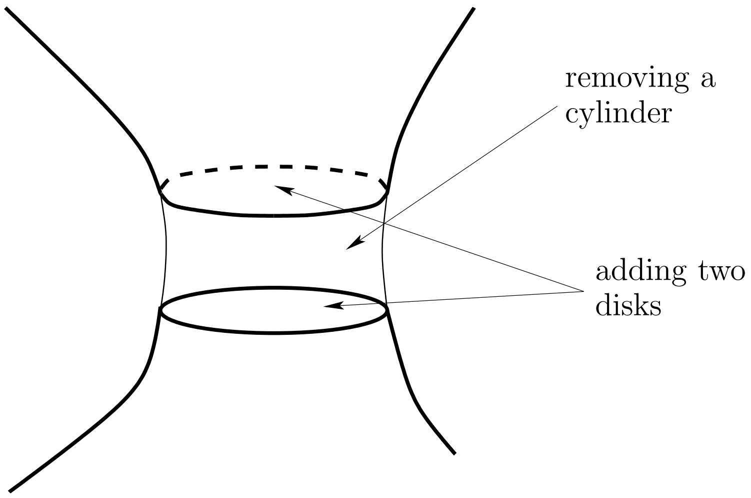

FIGURE 6. Cutting away a neck

DEFINITION 2.12. Let Σ and $\tilde{\Sigma}$ be two closed smooth embedded surfaces. We say that $\tilde{\Sigma}$ is obtained from Σ by cutting away a neck if:

- $\Sigma \setminus \tilde{\Sigma}$ is homeomorphic to $S^1 \times]0, 1[$;
- $\tilde{\Sigma} \setminus \Sigma$ is homeomorphic to the disjoint union of two open disks;
- $\tilde{\Sigma} \Delta \Sigma$ is a contractible sphere.

We say that $\tilde{\Sigma}$ is obtained from Σ through surgery if there is a finite number of surfaces $\Sigma_0 = \Sigma, \Sigma_1, \ldots, \Sigma_N = \tilde{\Sigma}$ such that each Σ_k is

- either isotopic to the union of some connected components of Σ_{k-1};
- or obtained from Σ_{k-1} by cutting away a neck.

2.3.2. Definition of the γ–reduction. In what follows, we assume that an open set $U \subset M$ and a surface Λ in M with $\partial\Lambda \subset \partial U$ are fixed. Moreover, we let $\mathcal{C}$ denote the collection of all compact smooth 2-dimensional surfaces embedded in U with boundary equal to $\partial\Lambda$.

We next fix a positive number δ such that the conclusion of Lemma 1 in [**MSY82**] holds and consider $\gamma < \delta^2/9$. Following [**MSY82**] we define the γ-reduction and the strong γ-reduction.

Definition 2.13. For $\Sigma_1, \Sigma_2 \in \mathcal{C}$ we write

$$\Sigma_2 \overset{(\gamma,U)}{\ll} \Sigma_1$$

and we say that Σ_2 is a (γ, U)–reduction of Σ_1, if the following conditions are satisfied:

(γ1) Σ_2 is obtained from Σ_1 through a surgery as described in Definition 2.12. Therefore:
  - $\overline{\Sigma_1 \setminus \Sigma_2} = A \subset U$ is diffeomorphic to the standard closed annulus $\overline{\mathrm{An}(x, 1/2, 1)}$;
  - $\overline{\Sigma_2 \setminus \Sigma_1} = D_1 \cup D_2 \subset U$ with each D_i diffeomorphic to $\mathcal{D}$;
  - There exists a set Y embedded in U, homeomorphic to $\mathcal{B}_1$ with $\partial Y = A \cup D_1 \cup D_2$ and $(Y \setminus \partial Y) \cap (\Sigma_1 \cup \Sigma_2) = \varnothing$. (See Picture 6).

(γ2) $\mathcal{H}^2(A) + \mathcal{H}^2(D_1) + \mathcal{H}^2(D_2) < 2\gamma$;

(γ3) If Γ is the connected component of $\Sigma_1 \cap \overline{U}$ containing A, then for each component of $\Gamma \setminus A$ we have one of the following possibilities:
  - either it is a disc of area $\geq \delta^2/2$;
  - or it is not simply connected.

Remark 2.14. The previous definition has another interesting consequence that the reader could easily check: $\Sigma \in \mathcal{C}$ is (γ, U)–irreducible if and only if whenever Δ is a disc with $\partial\Delta = \Delta \cap \Sigma$ and $\mathcal{H}^2(\Delta) < \gamma$, then there is a disc $D \subset \Sigma$ with $\partial D = \partial\Delta$ and $\mathcal{H}^2(D) < \delta^2/2$.

A slightly weaker relation than $\overset{(\gamma,U)}{\ll}$ can be defined as follows. We consider $\Sigma_1, \Sigma_2 \in \mathcal{C}$ and we say that Σ_2 is a strong (γ, U)–reduction of Σ_1, written $\Sigma_2 \overset{(\gamma,U)}{<} \Sigma_1$, if there exists an isotopy $\psi \in \mathfrak{Is}(U)$ such that

($s1$) $\Sigma_2 \overset{(\gamma,U)}{\ll} \psi(\Sigma_1)$;
($s2$) $\Sigma_2 \cap (M \setminus U) = \Sigma_1 \cap (M \setminus U)$;
($s3$) $\mathcal{H}^2(\psi(\Sigma_1)\triangle\Sigma_1) < \gamma$.

We say that $\Sigma \in \mathcal{C}$ is strongly $(\gamma, U)-$irreducible if there is no $\tilde{\Sigma} \in \mathcal{C}$ such that $\tilde{\Sigma} \overset{(\gamma,U)}{<} \Sigma$.

Remark 2.15. Arguing as in [**MSY82**] one can prove that, for every $\Lambda' \in \mathcal{C}$, there exist a constant $c \geq 1$ (depending on δ, $\mathbf{g}(\Lambda')$ and $\mathcal{H}^2(\Lambda')$) and a sequence of surfaces Σ_j, $j = 1, ..., k$, such that

$$k \leq c; \tag{2.10}$$

$$\Sigma_j \in \mathcal{C}; \quad j = 1, ..., k; \tag{2.11}$$

$$\Sigma_k \overset{(\gamma,U)}{<} \Sigma_{k-1} \overset{(\gamma,U)}{<} \dots \overset{(\gamma,U)}{<} \Sigma_1 = \Lambda'\,; \tag{2.12}$$

$$\mathcal{H}^2(\Sigma_k \Delta \Lambda') \;\leq\; 3c\gamma\,; \tag{2.13}$$

$$\Sigma_k \text{ is strongly } (\gamma, U)-\text{irreducible.} \tag{2.14}$$

Compare with Section 3 of [**MSY82**] and in particular with (3.3), (3.4), (3.8) and (3.9) therein.

2.3.3. Proof of Proposition 2.11**.** Applying Lemma 2.7, we conclude that a subsequence, not relabeled, of Λ^k converges to a stationary varifold V in $\overline{U}$ such that $\overline{U} \cap \mathrm{supp}\,(V) \subset \partial\Lambda$. Next, arguing as in Subsection 2.2.1, we conclude that $\|V\|(\partial\Lambda) = 0$, and hence that $\|V\|(\partial U) = 0$. Proceeding as in pages 364-365 of [**MSY82**] (see (3.22)–(3.26) therein), we find a $\gamma_0 > 0$ and a sequence of γ_0–strongly irreducible surfaces Σ^k with the following properties:

- Σ^k is obtained from Λ^k through a number of surgeries which can be bounded independently of k;
- Σ^k converges, in the sense of varifolds, to V.

This allows to apply Theorem 2 and Section 5 of [**MSY82**] to the surfaces Σ^k to conclude that $\mathrm{supp}\,(V) \setminus \partial U$ is a smooth embedded stable minimal surface.

Remark 2.16. This procedure *cannot* be applied if the minimality of the sequence Λ^k in $\mathfrak{Is}(U)$ were replaced by the minimality in $\mathfrak{Is}_j(U)$. In fact, the proof of Theorem 2 in [**MSY82**] uses heavily the minimality in $\mathfrak{Is}(U)$ and we do not know how to overcome this issue.

2.3.4. Interior regularity. As a consequence of Proposition 2.11 we have:

LEMMA 2.17 (Interior regularity). *Let V be as in Theorem 2.2. Then $\|V\| = \mathcal{H}^2 \llcorner \Delta$ where Δ is a smooth stable minimal surface in U (multiplicity is allowed).*

The proof of this Lemma follows with the help of the squeezing Lemma and the regularity theory of replacements as described in [**CDL03**] (cp. with Section 7 therein). We will recall this regularity technique for varifolds in Chapter 3, therefore we withhold the proof of the Lemma until then (see Section 3.3).

REMARK 2.18. Note that the arguments of Section 3 of [**MSY82**] *cannot* be applied directly to the sequence Δ^k. It is indeed possible to modify Δ^k in $B_\varepsilon(x) =: U'$ to a strongly γ-irreducible $\tilde{\Delta}^k$. However, the number of surgeries needed is controlled by $\mathcal{H}^2(\Delta^k \cap B_\varepsilon(x))$ and $\mathbf{g}(\Delta^k \cap U')$. Though the first quantity can be bounded independently of k, on the second quantity (i.e. $\mathbf{g}(\Delta^k \cap U')$) we do not have any a priori uniform bound.

2.4. Part IV: Boundary regularity

In this section we conclude the proof of part (b) of Theorem 2.2. More precisely, we show that the surface Δ of Lemma 2.17 is regular up to the boundary and its boundary coincides with $\partial\Sigma$.

LEMMA 2.19 (Boundary regularity). *Let Δ be as in Lemma 2.17. Then Δ has a smooth boundary and $\partial\Delta = \partial\Sigma$.*

As a corollary, we conclude that the multiplicity of Δ is everywhere 1.

COROLLARY 2.20. *There exist finitely many stable embedded connected disjoint minimal surfaces $\Gamma_1, \ldots, \Gamma_N \subset U$ with disjoint smooth boundaries and with multiplicity 1 such that*

$$\Delta = \Gamma_1 \cup \ldots \cup \Gamma_N \qquad \text{and} \qquad \partial\Delta = \partial\Gamma_1 \cup \ldots \cup \partial\Gamma_N . \tag{2.15}$$

PROOF. Lemmas 2.17 and 2.19 imply that Δ is the union of finitely many disjoint connected components $\Gamma_1 \cup \ldots \cup \Gamma_N$ contained in U and that either $\partial\Gamma_i = 0$ or $\partial\Gamma_i$ is the union of some connected components of $\partial\Sigma$. In this last case, the multiplicity of Γ_i is necessarily 1. On the other hand, $\partial\Gamma_i = 0$ cannot occur, otherwise Γ_i would be a smooth embedded minimal surface without boundary contained in a convex ball of a Riemannian manifold, contradicting the classical maximum principle. □

2.4.1. Tangent cones at the boundary. Consider now $x \in \operatorname{supp}\|V\| \cap \partial U$. We follow Subsection 2.2.1 and consider the chart $f_{x,\rho} : B_\rho(x) \to \mathcal{B}_1$ given by $f_{x,\rho}(y) = \exp_x^{-1}(y)/\rho$. We then denote by $V_{x,\rho}$ the varifolds $(f_{x,\rho})_\# V$. Moreover, if $\lambda > 0$, we will denote by $O_\lambda : \mathbf{R}^3 \to \mathbf{R}^3$ the rescaling $O_\lambda(x) = x/\lambda$.

Let next W be the limit of a subsequence V_{x,ρ_j}. Again following the discussion of Subsection 2.2.1, we can choose a system of coordinates (y_1, y_2, y_3) such that:

- W is integer rectifiable and $\operatorname{supp}(W)$ is contained in the wedge
$$\text{Wed} \;:=\; \{(y_1, y_2, y_3) : |y_2| \;\leq\; y_1 \tan\theta_0\} \cap \overline{\mathcal{B}}_1(0)\,.$$
- $\operatorname{supp}(W)$ containes the line $\ell = \{(0,0,t) : t \in [-1,1]\}$, (which is the limit of the curves $f_{x,\rho}(\partial\Sigma \cap B_\rho(x))$).
- If we denote by $r : \mathbf{R}^3 \to \mathbf{R}^3$ the reflection given by $r(z_1, z_2, z_3) = (-z_1, -z_2, z_3)$, then $r_\# W + W$ is a stationary cone.

By the Boundary regularity Theorem of Allard (see Section 4 of [**All75**]), in order to show regularity it suffices to prove that

(TC) Any W as above (i.e. any varifold limit of a subsequence $(f_{x,\rho_n})_\# V$ with $\rho_n \downarrow 0$) is a half–disk of the form

$$P_\theta := \{(y_1, y_2, y_3) : y_1 > 0, y_3 = y_1 \tan\theta\} \cap \mathcal{B}_1(0) \tag{2.16}$$

for some angle $\theta \in]-\pi/2, \pi/2[$.

In the rest of this section we aim, therefore, at proving (TC). As a first step we now show that

$$W \;=\; \sum_{i=1}^{N} k_i P_{\theta_i} \tag{2.17}$$

where $k_i \geq 1$ are integers and θ_i are angles in $[-\theta_0, \theta_0]$. There are two possible ways of seeing this. One way is to use the Classification of stationary integral varifolds proved by Allard and Almgren in [**AA76**].

The second, which is perhaps simpler, is to observe that, on $\mathcal{B}^+$ the varifold W is actually smooth. Indeed, by the interior regularity, V is a smooth minimal surface in $B_\rho(x) \cap V$ and it is stable, therefore, by Schoen's curvature estimates (see Remark 1.15), a subsequence of V_{x,ρ_n} converges smoothly in compact subsets of $\mathcal{B}^+$. It follows that $W^r := W + r_\# W$ coincides with a smooth minimal surface outside on $\mathcal{B}_1(0) \setminus \ell$. On the other hand W^r is a cone and therefore we conclude that $\partial\mathcal{B}_{1/2}(0) \cap W^r \setminus \{(0,0,1/2), (0,0,-1/2)\}$ is a smooth 1-d manifold consisting of arcs of great circles. Since $\operatorname{supp}(W) \subset \text{Wed}$, we conclude that in fact $\partial\mathcal{B}_{1/2}(0) \cap W^r \setminus \{(0,0,1/2), (0,0,-1/2)\}$ consists of finitely many planes (mupltiplicity is allowed) passing through ℓ. This proves (2.17).

2.4.2. Diagonal sequence. We are now left with the task of showing that $N = 1$ and $k_1 = 1$. We will, indeed, assume the contrary and derive a contradiction. In order to do so, we consider a suitable diagonal sequence $f_{x,\rho_n}(\Delta^{k_n})$ converging, in the sense of varifolds, to W. We can select Δ^{k_n} in such a way that the following minimality property holds:

(F) If Λ is any surface isotopic to Δ^{k_n} with an isotopy fixing $\partial(U \cap B_{\rho_n}(x))$, then $\mathcal{H}^2(\Lambda) \geq \mathcal{H}^2(\Delta^{k_n}) - \rho_n^3$.

Indeed, we apply the Squeezing Lemma 2.10 with $\beta = 1/(16j)$ and let n be so large that ρ_n is smaller than the constant ε_0 given by the Lemma. Since Δ^k is $1/j$–a.m. in U, we conclude therefore that, if we set

$$M_{k,n} := \inf\{\Phi(1, \Delta^k) : \Phi \in \mathfrak{Is}(U \cap B_{\rho_n}(x))\}\,,$$

then

$$\lim_{k\uparrow\infty} \mathcal{H}^2(\Delta^k \cap B_{\rho_n}(x)) - M_{n,k} = 0\,.$$

Therefore, having fixed $\rho_n < \varepsilon_0$, we can choose k_n so large that $M_{n,k} \geq \mathcal{H}^2(\Delta^{k_n}) - \rho_n^3$.

Next, it is convenient to introduce a slightly perturbed chart $g_x^{\rho_n}$ which maps $\partial U \cap B_{\rho_n}(x)$ onto $\mathcal{B}_1 \cap \{y_1 = 0\}$ and $\partial\Sigma \cap B_{\rho_n(x)}$ onto ℓ. This can be done in such a way that $f_{x,\rho_n} \circ g^{-1}_{x,\rho_n}$ and $g_{x,\rho_n} \circ f^{-1}_{x,\rho_n}$ converge smoothly to the identity map as $\rho_n \downarrow 0$.

Having set $\Gamma_n = g_{x,\rho_n}(\Delta^{k_n})$, we have that Γ_n converges to W in the sense of varifolds. Moreover, our discussion implies that $\mathcal{H}^2(\Delta^{k_n} \cap B_{\rho_n}(x)) = \rho_n^2 \mathcal{H}^2_e(\Gamma_n) + O(\rho_n^3)$. Therefore we conclude from (F) that

(F') Let m_n be the minimum of $\mathcal{H}^2_e(\Lambda)$ over all surfaces Λ isotopic to Γ_n with an isotopy which fixes $\partial(U \cap \mathcal{B}_1)$. Then $\mathcal{H}^2_e(\Gamma_n) - m_n \downarrow 0$.

We next claim that

$$\liminf_{n\downarrow 0} \mathcal{H}^1_e(\Gamma_n \cap \partial\mathcal{B}_\sigma) \geq \pi\sigma \sum_{i=1}^N k_i \qquad \text{for every } \sigma \in]0,1[. \tag{2.18}$$

Indeed, using the squeezing homotopies of Section 2.2.2 it is easy to see that

$$\mathcal{H}^2_e(\Gamma_n) - m_n \geq \mathcal{H}^2_e(\Gamma_n \cap \mathcal{B}_\sigma) - \sigma\mathcal{H}^1_e(\Gamma_n \cap \partial\mathcal{B}_\sigma)$$

Letting $n \uparrow 0$ and using (2.17) with the convergence of Γ_n to the varifold W we conclude

$$\liminf_{n\uparrow\infty}(\mathcal{H}^2_e(\Gamma_n) - m_n) \;\geq\; \sigma\left(\sigma\pi\sum_i k_i - \liminf_{n\downarrow 0}\mathcal{H}^1_e(\Gamma_n\cap\partial\mathcal{B}_\sigma)\right).$$

Therefore, from (F') we conclude (2.18).

We next claim the existence of a $\sigma \in [1/2, 1[$ and a subsequence $n(j)$ such that $\Gamma_{n(j)} \cap \partial\mathcal{B}_\sigma$ is a smooth 1-dimensional manifold with boundary $(0,0,\sigma) - (0,0,-\sigma)$ and, at the same time,

$$\lim_{j\uparrow\infty}\mathcal{H}^1_e(\Gamma_{n(j)}\cap\partial\mathcal{B}_\sigma) \;=\; \pi\sigma\sum_{i=1}^N k_i \tag{2.19}$$

and

$$\lim_{j\uparrow\infty}\mathcal{H}^1_e(\Gamma_{n(j)}\cap\partial\mathcal{B}_\sigma\cap K) \;=\; 0 \qquad \text{for every compact } K\subset\mathcal{B}_1\setminus\textstyle\bigcup_i P_{\theta_i}. \tag{2.20}$$

In fact, let $\{K_l\}_l$ be an exhaustion of $\mathcal{B}_1 \setminus \bigcup_i P_{\theta_i}$ by compact sets. Observe that, by the convergence of Γ_n to W, we get

$$\lim_{n\uparrow\infty}\left(\mathcal{H}^2_e(\Gamma_n\cap\mathcal{B}_1\setminus\mathcal{B}_{1/2}) + \sum_{l=0}^\infty 2^{-l}\mathcal{H}^2_e(\Gamma_n\cap K_l\cap(\mathcal{B}_1\setminus\mathcal{B}_{1/2}))\right) \;=\; \frac{\pi}{8}\sum_i k_i\,. \tag{2.21}$$

Using the coarea formula, we conclude

$$\int_{1/2}^1 \sigma\pi\sum_i k_i\,d\sigma \;\geq\; \lim_{n\uparrow\infty}\int_{1/2}^1\left(\mathcal{H}^1_e(\Gamma_n\cap\partial\mathcal{B}_\sigma) + \sum_l 2^{-l}\mathcal{H}^1_e(\Gamma_n\cap\partial\mathcal{B}_\sigma\cap K_l)\right)d\sigma\,.$$

Therefore, by Fatou's Lemma, for a.e. $\sigma \in [1/2, 1[$ there is a subsequence $n(j)$ such that

$$\lim_{j\uparrow\infty}\left(\mathcal{H}^1_e(\Gamma_n\cap\partial\mathcal{B}_\sigma) + \sum_l 2^{-l}\mathcal{H}^1_e(\Gamma_n\cap\partial\mathcal{B}_\sigma\cap K_l)\right) \;\leq\; \pi\sigma\sum_i k_i\,. \tag{2.22}$$

Clearly, (2.18) and (2.22) imply (2.19) and (2.20). On the other hand, by Sard's Theorem, for a.e. $\sigma \in [1/2, 1[$ every surface $\partial\mathcal{B}_\sigma \cap \Gamma_n$ is a smooth 1-dimensional submanifold with boundary $(0,0,\sigma)-(0,0,-\sigma)$.

2.4.3. Disks. From now on we fix the radius σ found above and we use Γ_n in place of $\Gamma_{n(i)}$ (i.e. we do not relabel the subsequence). Consider now the Jordan curves $\gamma^n_1, \ldots, \gamma^n_{M(n)}$ forming $\Gamma^n \cap \partial\mathcal{B}^+_\sigma$ (by $\mathcal{B}^+_\sigma$ we understand the half ball $\mathcal{B}_\sigma \cap \{y_1 \geq 0\}$).

Since $\partial\Gamma^n \cap \{y_1 = 0\}$ is given by the segment ℓ, there is one curve, say γ_1^n, which contains the segment ℓ. All the others, i.e. the curves γ_i^n with $i \geq 2$ lie in $\partial\mathcal{B}_\sigma \cap \{y_1 > 0\}$.

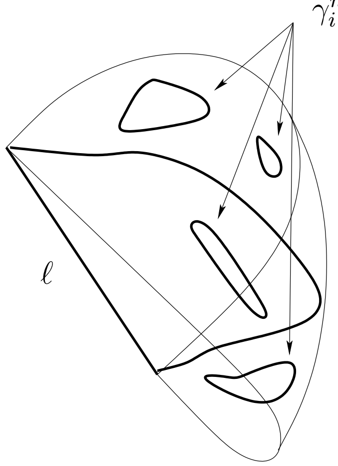

FIGURE 7. The curves γ_i^n.

Next, for every γ_l^n consider the number

$$\kappa_l^n := \inf\{\mathcal{H}_e^2(D) : D \text{ is an embedded smooth disk bounding } \gamma_l^n\}\,. \tag{2.23}$$

We will split our proof into several steps.

(a) In the first step, we combine a simple desingularization procedure with the fundamental result of Almgren and Simon (see [**AS79**]), to show that

there are disjoint embedded smooth disks $D_1^n, \ldots D_{M(n)}^n$ s.t.

$$\sum_{i=1}^{M(n)} \mathcal{H}_e^2(D_i^n) \leq \sum_{i=1}^{M(n)} \kappa_i^n + \frac{1}{n}\,. \tag{2.24}$$

A simple topological observation (see Lemma 2.22 in subsection 2.4.7) shows that, for each fixed n, there exist isotopies Φ_l keeping $\partial\mathcal{B}_\sigma^+$ fixed and such that $\Phi_l(\Gamma_n \cap \mathcal{B}_\sigma)$ converges, in the sense of varifolds, to the union of the disks D_i^n. Combining (F'), (2.24) and the convergence of Γ_n to the varifold W we then conclude

$$\limsup_{n\uparrow\infty} \sum_{i=1}^{M(n)} \kappa_i^n = \frac{\pi\sigma^2}{2} \sum_j k_j\,. \tag{2.25}$$

(b) In the second step we will show the existence of a $\delta > 0$ (independent of n) such that

$$\kappa_i^n \;\leq\; \sigma\left(\frac{1}{2}-\delta\right)\mathcal{H}^1_e(\gamma_i^n) \qquad \text{for every } i \geq 2 \text{ and every } n. \tag{2.26}$$

A simple cone construction shows that

$$\kappa_1^n \;\leq\; \frac{\sigma}{2}\mathcal{H}^1_e(\gamma_1^n)\,. \tag{2.27}$$

So, (2.19), (2.26) and (2.27) imply

$$\lim_{n\uparrow\infty}\sum_{i=2}^{M(n)}\mathcal{H}^1_e(\gamma_i^n) \;=\; 0 \qquad \text{and} \qquad \lim_{n\uparrow\infty}\mathcal{H}^1_e(\gamma_1^n) \;=\; \sigma\sum_j k_j\,, \tag{2.28}$$

which in turn give

$$\lim_{n\uparrow\infty}\kappa_1^n \;=\; \frac{\pi\sigma^2}{2}\sum_j k_j\,. \tag{2.29}$$

(c) We next fix a parameterization $\beta_1^n : \mathbf{S}^1 \to \partial\mathcal{B}^+_\sigma$ of γ_1^n with a multiple of the arc-length and extract a further subsequence, not relabeled such that β_1^n converges to a β^∞. By (2.20), the image of β^∞ is then contained in the union of the curves $P_{\theta_l}\cap\partial\mathcal{B}^+_\sigma$. We will then show that

$$\limsup_{n\downarrow\infty}\kappa_1^n \;=\; \frac{\pi\sigma^2}{2}\,. \tag{2.30}$$

(2.29) and (2.30) finally show that W consists of a single half-disk $P_\theta\cap\mathcal{B}^+_1$, counted once. This will therefore complete the proof.

2.4.4. Proof of (2.24)**.** In this step we fix n and prove the claim (2.24). First of all, note that each γ_i^n with $i \geq 2$ is a smooth Jordan curve lying in $\partial B_\sigma \cap \{y_1 > 0\}$.

We recall the following result of Almgren and Simon (see [**AS79**]).

Theorem 2.21. *For every curve γ_i^n with $i \geq 2$ consider a sequence of smooth disks D^j with $\mathcal{H}^2_e(D^j)$ converging to κ_i^n. Then a subsequence, not relabeled, converges, in the sense of varifolds, to an embedded smooth disk $D_i^n \subset \mathcal{B}^+_\sigma$ bounding γ_i^n and such that $\mathcal{H}^2_e(D_i^n) = \kappa_i^n$. (The disk is smooth also at the boundary).*

For each γ_i^n select therefore a disk D_i^n as in Theorem 2.21. We next claim that these disks are all pairwise disjoint. Fix in fact two such disks. To simplify the notation we call them D^1 and D^2 and assume they bound, respectively, the curves γ_1 and γ_2. Clearly, D^1 divides $\mathcal{B}^+_\sigma$

into two connected components A and B and γ_2 lies in one of them, say A. We will show that D^2 lies in A.

Assume by contradiction that D^2 intersects D^1. By perturbing D^2 a little we modify it to a new disk E^j such that $\mathcal{H}^2_e(E^j) \leq \mathcal{H}^2_e(D^2) + 1/j$ and E^j intersects D^1 transversally in finitely many smooth Jordan curves α_m.

Each α_m bounds a disk F^m in E^j. We call α_m maximal if it is not contained in any F^l. Each maximal α_m bounds also a disk G^m in D^1. By the minimality of D^1, clearly $\mathcal{H}^2_e(G^m) \leq \mathcal{H}^2_e(F^m)$. We therefore consider the new disk H^j given by

$$D^2 \setminus \left(\bigcup_{\alpha_m \text{ maximal}} F^m \right) \cup \bigcup_{\alpha_m \text{ maximal}} G^m .$$

Clearly $\mathcal{H}^2_e(H^j) \leq \mathcal{H}^2_e(E^j) + 1/j$. With a small perturbation we find a nearby smooth embedded disk K^j which lies in A and has $\mathcal{H}^2_e(K^j) \leq \mathcal{H}^2_e(E^j) + 1/(2j)$. By letting $j \uparrow \infty$ and applying Theorem 2.21, a subsequence of K^j converges to a smooth embedded minimal disk D^3 in the sense of varifolds. On the other hand, by choosing K^j sufficiently close to H^j, we conclude that H^j converges as well to the same varifold. But then,

$$D^2 \setminus \left(\bigcup_{\alpha_m \text{ maximal}} F^m \right) \subset D^3$$

and hence $D^2 \cap D^3 \neq \emptyset$. Since D^3 lies on one side of D^2 (i.e. in $\overline{A}$) this violates the maximum principle for minimal surfaces.

Having chosen $D^n_2, \ldots D^n_{M(n)}$ as above, we now choose a smooth disk E^n_1 bounding γ^n_1 and with

$$\mathcal{H}^2_e(E^n_1) \leq \kappa^n_1 + \frac{1}{3n} .$$

In fact we cannot apply directly Theorem 2.21 since in this case the curve γ^n_1 is not smooth but has, in fact, two corners at the points $(0, 0, \sigma)$ and $(0, 0, -\sigma)$.

γ^n_1 lies in one connected component A of $\overline{\mathcal{B}^+_\sigma}$. We now find a new smooth embedded disk D^n_1 with

$$\mathcal{H}^2_e(D^n_1) \leq \kappa^n_1 + \frac{1}{n}$$

and lying in the interior of A This suffices to prove (2.24).

Consider the disks $D'_1, \ldots D'_l$ which, among the D^n_j with $j \geq 2$, bound A. We first perturb E^n_1 to a smooth embedded F^n_1 which intersects all the D'_j. We then inductively modify E^n_1 to a new disk which does not intersect D'_j and looses at most $1/(3ln)$ in area. This is done exactly with the procedure outlined above and since the distance between different D'_j's is always positive, it is clear that while removing the intersections with D'_j we can do it in such a way that we do not add intersections with D'_i for $i < j$.

2.4.5. Proof of (2.26)**.** In this step we show the existence of a positive δ, independent of n and j, such that

$$\kappa^n_j \;\le\; \sigma\left(\frac{1}{2}-\delta\right)\mathcal{H}^1_e(\gamma^n_j) \qquad \forall j\ge 2, \forall n\,. \tag{2.31}$$

Observe that for each γ^n_j we can construct the cone with vertex the origin, which is topologically a disk and achieves area equal to $\frac{\sigma}{2}\mathcal{H}^1_e(\gamma^n_j)$. On the other hand, this cone is clearly not stationary, because γ^n_j is not a circle, and therefore there is a disk diffeomorphic to the cone with area strictly smaller than $\frac{\sigma}{2}\mathcal{H}^1_e(\gamma^n_j)$. A small perturbation of this disk yields a smooth embedded disk D bounding γ^n_j such that

$$\mathcal{H}^2_e(D) \;<\; \frac{\sigma}{2}\mathcal{H}^1_e(\gamma^n_j)\,. \tag{2.32}$$

Therefore, it is clear that it suffices to prove (2.31) when n is large enough.

Next, by the isoperimetric inequality, there is a constant C such that, any curve γ in $\partial\mathcal{B}_\sigma$ bounds, in $\mathcal{B}_\sigma$, a disk D such that

$$\mathcal{H}^2_e(D) \;\le\; C\left(\mathcal{H}^1_e(\gamma)\right)^2\,. \tag{2.33}$$

Therefore, (2.31) is clear for every γ^n_j with $\mathcal{H}^1_e(\gamma^n_j)\le \sigma/4C$.

We conclude that the only way of violating (2.31) is to have a subsequence, not relabeled, of curves $\gamma^n := \gamma^n_{j(n)}$ such that

- $\mathcal{H}^1_e(\gamma^n)$ converges to some constant $c_0 > 0$;
- $\kappa^n := \kappa^n_{j(n)}$ converges to $c_0\sigma/2$.

Consider next the wedge $\mathrm{Wed} = \{|y_2| \le y_1 \tan\theta_0\}$ containing the support of the varifold V. If we enlarge this wedge slightly to

$$\mathrm{Wed}' \;:=\; \{|y_2|\le y_1(\tan\theta_0+1)\}\,,$$

we conclude, by (2.20), that

$$\lim_{n\uparrow\infty}\mathcal{H}^1_e(\gamma^n\setminus \mathrm{Wed}') \;=\; 0\,. \tag{2.34}$$

Perturbing γ^n slightly we find a nearby smooth Jordan curve β^n contained in $\partial\mathcal{B}_\sigma\cap\mathrm{Wed}'$. Consider next

$$\mu^n \;:=\; \min\{\mathcal{H}^2_e(D) : \text{smooth embedded disk } D \text{ bounding } \beta^n\}\,. \tag{2.35}$$

Given a D bounding β^n, it is possible to construct a D' bounding γ^n with

$$\mathcal{H}^2_e(D') \leq \mathcal{H}^2_e(D) + o(1).$$

Therefore, we conclude that

- $\mathcal{H}^1_e(\beta^n)$ converges to $c_0 > 0$;
- μ^n converges to $\sigma c_0/2$;
- β^n is contained in Wed$'$.

Consider next the projection of the curve $\alpha = \text{Wed}' \cap \mathcal{B}_\sigma$ on the plane $\pi = y_1 y_3$. This projection is an ellipse bounding a domain Ω in π. Clearly α is the graph of a function over this ellypse. The function is Lipschitz (actually it is analytic except for the two points $(0, \sigma)$ and $(0, -\sigma)$) and we can therefore find a function $f : \Omega \to \mathrm{R}$ which minimizes the area of its graph. This function is smooth up to the boundary except in the points $(0, \sigma)$ and $(0, -\sigma)$ where, however, it is continuous. Therefore, the graph of f is an embedded disk.

We denote by Λ the graph of f. Λ is in fact the *unique* area-minimizing current spanning α, by a well-known property of area–minimizing graphs. By the classical maximum principle, Λ is contained in the wedge Wed$'$ and does not contain the origin. Consider next the cone C^n having vertex in 0 and β^n as base. Clearly, this cone intersects Λ in a smooth Jordan curve $\tilde{\beta}^n$ and hence there is a disk D^n in Λ bounding this curve. Moreover, we call E^n the cone constructed on $\tilde{\beta}^n$ with vertex 0 (see Figure 8).

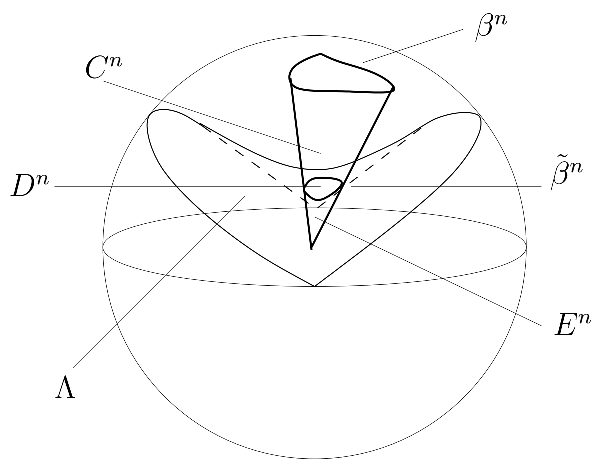

FIGURE 8. The minimal surface Λ, the cones C^n and E^n and the domain D^n.

Clearly,

$$\liminf_{n\uparrow\infty} \mathcal{H}^1_e(\beta^n) \;>\; 0\,. \tag{2.36}$$

Consider next the current given by $D^n \cup (C^n \setminus E^n)$. These coverge, up to subsequences, to some integer rectifiable current. Therefore, the disks D^n converge, in the sense of currents, to a 2–dimensional current D supported in Λ. It is easy to check that D must be the current represented by a domain of Λ, counted with multiplicity 1. Therefore

$$\lim_{n\uparrow\infty} \mathcal{H}^2_e(D^n) \;=\; \mathcal{H}^2_e(D)\,. \tag{2.37}$$

Similarly, E^n converges, up to subsequences, to a current E. By the minimizing property of Λ, $\mathcal{H}^2_e(D) < \mathbf{M}(E)$, unless $\mathcal{H}^2_e(D) = \mathbf{M}(E) = 0$, where $\mathbf{M}(E)$ denotes the mass of E.

So, if $\mathbf{M}(E) > 0$, we then have

$$\liminf_{n\uparrow\infty} \mathcal{H}^2_e(E^n) \;\geq\; \mathbf{M}(E) \;>\; \mathcal{H}^2_e(D) \;=\; \lim_{n\uparrow\infty} \mathcal{H}^2_e(D^n)\,.$$

If $\mathbf{M}(E) = 0$, by (2.36), we conclude

$$\liminf_{n\uparrow\infty} \mathcal{H}^2_e(E^n) \;>\; 0 \;=\; \lim_{n\uparrow\infty} \mathcal{H}^2_e(D^n)\,.$$

In both cases we conclude that the embedded disk $H^n = (C^n \setminus E^n) \cup D^n$ bounds β^n and satisfies

$$\lim_{n\uparrow\infty} \mathcal{H}^2_e(H^n) \;<\; \lim_{n\uparrow\infty} \mathcal{H}^2_e(C^n) \;=\; \frac{\sigma c_0}{2} \;=\; \lim_{n\uparrow\infty} \mu^n\,. \tag{2.38}$$

Therefore, there exists an n such that $\mu^n > \mathcal{H}^2_e(H^n)$. A small perturbation of H^n gives a smooth embedded disk bounding β^n with area strictly smaller than μ^n. This contradicts the minimality of μ^n (see (2.35)) and hence proves our claim.

2.4.6. Proof of (2.30)**.** In this final step we show (2.30). Our arguments are inspired by those of Section 7 in [**AS79**].

Consider the curve γ^n_1. Again applying (2.20) we conclude that, for every compact set

$$K \subset \overline{\mathcal{B}}^+_\sigma \setminus \bigcup_i P_{\theta_i}$$

we have

$$\lim_{n\uparrow\infty} \mathcal{H}^1_e(\gamma^n_1 \cap K) \;=\; 0\,. \tag{2.39}$$

Consider next the solid sector $S := \mathrm{Wed}' \cap \mathcal{B}_\sigma$. Clearly $\mathcal{H}^2_e(\partial S) = (3\pi - \eta)\sigma^2$, where η is a positive constant. Clearly a curve contained in ∂S bounds always a disk with area at most $(\frac{3\pi}{2} - \frac{\eta}{2})\sigma^2$. For large n

we can modify γ_1^n to a new curve $\tilde{\gamma}^n$ contained in ∂S, and hence find a smooth embedded disk bounding $\tilde{\gamma}^n$ with area at most $(\frac{3\pi}{2} - \frac{\eta}{4})\sigma^2$. This and (2.29) implies that

$$\frac{\pi\sigma^2}{2}\sum_i k_i \;=\; \lim_{n\uparrow\infty} \kappa_1^n \;<\; \frac{3\pi}{2}\sigma^2 .$$

Therefore we conclude that $\sum_i k_i \leq 2$.

Extracting a subsequence, not relabeled, we can assume that γ_1^n converges to an integer rectifiable current γ. The intersection of the support of γ with $\partial\mathcal{B}_\sigma \setminus \{(0,0,\sigma),(0,0,-\sigma)$ is then contained in the arcs $\alpha_i := P_{\theta_i} \cap \partial\mathcal{B}_\sigma$. Therefore if we denote by $[[\alpha_i]]$ the current induced by α_i then we have

$$\gamma \llcorner \partial\mathcal{B}_\sigma \;=\; \sum_i h_i[[\alpha_i]]$$

where the h_i are integers.

On the other hand, $\gamma_1^n \llcorner \mathcal{B}_\sigma$ is given by the segment ℓ. Therefore we conclude that

$$\gamma \llcorner \mathcal{B}_\sigma \;=\; [[\ell]] .$$

It turns out that

$$\gamma \;=\; [[\ell]] + \sum_i h_i[[\alpha_i]]$$

and of course $\sum_i |h_i| \leq \sum_i k_i$.

Since $\partial\gamma = 0$, we conclude that

$$0 \;=\; \partial[[l]] + \sum_i h_i \partial[[\alpha_i]] \;=\; \delta_P - \delta_N + \sum_i h_i(\delta_N - \delta_P)$$

where $N = (\sigma, 0, 0)$, $P = (-\sigma, 0, 0)$ and δ_X denotes the Dirac measure in the point X. Hence we conclude

$$\left(1 - \sum_i h_i\right)\delta_P - \left(1 - \sum_i h_i\right)\delta_N \;=\; 0$$

and therefore $\sum_i h_i = 1$. This implies that $\sum_i |h_i|$ is odd. Since $\sum_i |h_i| \leq \sum_i k_i \leq 2$, we conclude $\sum_i |h_i| = 1$.

Therefore, γ consists of the segment ℓ and an arc, say, α_1. Clearly, γ bounds P_{θ_1}, which has area $\pi\sigma^2/2$. Consider next the closed curve β^n made by joining $\gamma_1^n \cap \partial\mathcal{B}_\sigma$ and $-\alpha_1$. These curves might have self–intersections, but they are close. Moreover, they have bounded length and they converge, in the sense of currents, to the tivial current $\alpha_1 - \alpha_1 = 0$.

There are therefore domains $D^n \subset \mathcal{B}_\sigma^+$ such that $\partial D^n = \beta^n$ and $\mathcal{H}_e^2(D^n) \downarrow 0$. It is not difficult to see that the union of the domains

D^n and of P_{θ_1} gives embedded disks E^n bounding γ_1^n and with area converging to $\pi\sigma^2/2$ (see Figure 9). Approximating these disks E^n with smooth embedded ones, we conclude that

$$\lim_{n\uparrow\infty} \mu_n \;\leq\; \frac{\pi}{2}\sigma^2\,.$$

This shows that $\sum_i k_i \leq 1$. Hence the varifold W is either trivial or it consists of at most one half-disk. Since it cannot be trivial by the considerations of Subsections 2.2.1 and 2.4.1, we conclude that W consists in fact of exactly one half-disk.

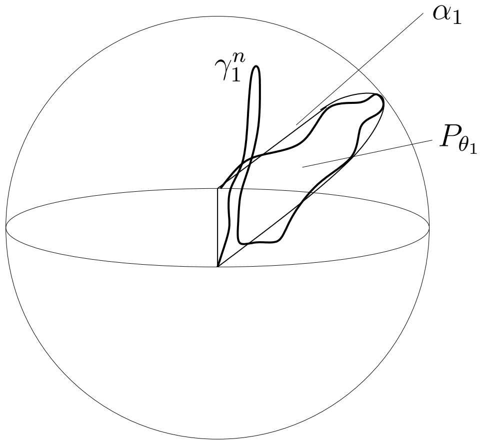

FIGURE 9. The curves γ_1^n and α_1.

2.4.7. A simple topological fact. We summarize the topological fact used in (a) of Section 2.4.3 in the following lemma.

LEMMA 2.22. *Condider a smooth 2–dimensional surface* $\Sigma \subset \mathcal{B}_1$ *with smooth boundary* $\partial\Sigma \subset \partial\mathcal{B}_1$. *Let* $\Gamma \subset \mathcal{B}_1$ *is a smooth surface with* $\partial\Gamma = \partial\Sigma$ *consisting of disjoint embedded disks. Then there exists a smooth map* $\Phi : [0,1[\times\overline{\mathcal{B}}_1 \to \overline{\mathcal{B}}_1$ *such that*

(i) $\Phi(0,\cdot)$ *is the identity and* $\Phi(t,\cdot)$ *is a diffeomorphism for every* t*;*
(ii) *For every* t *there exists a neighborhood* U_t *of* $\partial\mathcal{B}_1$ *such that* $\Phi(t,x) = x$ *for every* $x \in U_t$*;*
(iii) $\Phi(t,\Sigma)$ *converges to* Γ *in the sense of varifolds as* $t \to 1$.

PROOF. The proof consists of two steps. In the first one we show the existence of a surface Γ' and of a map $\Psi : [0,1[\times\overline{\mathcal{B}}_1 \to \overline{\mathcal{B}}_1$ such that

- $\partial\Gamma' = \partial\Sigma$,

- Γ' consists of disjoint embedded disks,
- Ψ satisfies (i) and (ii),
- $\Psi(t,\Sigma) \to \Gamma'$ as $t \to 1$.

In the second we show the existence of a $\tilde{\Psi} : [0,1[\times\overline{\mathcal{B}}_1 \to \overline{\mathcal{B}}_1$ such that (i) and (ii) hold and $\tilde{\Psi}(t,\Gamma') \to \Gamma$ as $t \to 1$.

In order to complete the proof from these two steps, consider the map $\tilde{\Phi}(s,t,x) = \tilde{\Psi}(t,\Psi(s,x))$. Then, for every smooth $g : [0,1[\to [0,1[$ with $g(0) = 0$, the map $\Phi(t,x) = \tilde{\Phi}(g(t),t,x)$ satisfies (i) and (ii) of the Lemma. Next, for any fixed t, if s is sufficiently close to 1, then $\tilde{\Phi}(s,t,\Sigma)$ is close, in the sense of varifolds, to $\tilde{\Psi}(t,\Gamma')$. This allows to find a piecewise constant function $h : [0,1[\to [0,1[$ such that

$$\lim_{t\to 1} \tilde{\Phi}(g(t),t,\Sigma) \;=\; \Gamma \qquad\qquad \text{(in the sense of varifolds)}$$

whenever $g \geq h$ in a neighborhood of 1. If we choose, therefore, a smooth $g : [0,1[\to [0,1[$ with $g(0) = 0$ and $g \geq h$ on $[1/2,1[$, the map $\Phi(t,x) = \tilde{\Phi}(g(t),t,x)$ satisfies all the requirements of the lemma.

We now come to the existence of the maps Ψ and $\tilde{\Psi}$.

Existence of Ψ. Let $\mathcal{G}$ be the set of all surfaces Γ' which can be obtained as $\lim_{t\to 1}\Psi(t,\Sigma)$ for maps Ψ satisfying (i) and (ii). It is easy to see that any Γ' which is obtained from Σ through surgery as in Definition 2.12 is contained in $\mathcal{G}$. Let $\mathbf{g}_0$ be the smallest genus of a surface contained in $\mathcal{G}$. It is then a standard fact that $\mathbf{g}(\Gamma') = \mathbf{g}_0$ if and only if the surface is incompressible. However, if this holds, then the first homotopy group of Γ' is mapped injectively in the homotopy group of $\mathcal{B}_1$ (see for instance [**Jac80**]). Therefore there is a $\Gamma' \in \mathcal{G}$ which consists of disjoint embedded disks and spheres. The embedded spheres can be further removed, yielding a $\Gamma' \in \mathcal{G}$ consisting only of disjoint embedded disks.

Existence of $\tilde{\Psi}$. Note that each connected component of $\mathcal{B}_1 \setminus \Gamma'$ (and of $\mathcal{B}_1 \setminus \Gamma$) is a, piecewise smooth, embedded sphere. Therefore the claim can be easily proved by induction from the case in which Γ and Γ' consist both of a single embedded disk. This is, however, a standard fact (see once again [**Jac80**]). □

2.5. Part V: Convergence of connected components

In this section we complete the proof of Theorem 2.2 and we show part (c) of it. In particular, building on Corollary 2.20, we show the following.

LEMMA 2.23. *Let Σ and Δ^k be as in Theorem 2.2 and consider their varifold limit V. According to Lemma 2.17, Lemma 2.19 and Corollary*

2.20, V is a smooth stable minimal surface with boundary $\partial\Delta = \partial\Sigma$ and with multiplicity 1. *Let $\Gamma_1, \ldots, \Gamma_N$ be the connected components of Δ.*

If $\tilde{\Delta}^k$ is an arbitrary union of connected components of Δ^k which converges, in the sense of varifolds, to a W, then W is given by $\Gamma_{i_1} \cup \ldots \cup \Gamma_{i_l}$ for some $1 \le i_1 < i_2 < \ldots < i_l \le N$.

Proof. This lemma is indeed a simple consequence of some known facts in geometric measure theory. Fix a sequence $\tilde{\Delta}^k$ and a W as in the statement of the lemma. Note that $\partial\tilde{\Delta}^k \subset \partial\Delta^k = \partial\Sigma$.

We can therefore apply the compactness of integer rectifiable currents and, after a further extraction of subsequence, assume that the $\tilde{\Delta}^k$ are converging, as currents, to an integer rectifiable current T with boundary ∂T which is the limit of the boundaries $\partial\tilde{\Delta}^k$. Since these boundaries are all contained in ∂U, we conclude that ∂T is also contained in ∂U. It is a known fact in geometric measure theory that

$$\|T\| \;\le\; \|W\|\,. \tag{2.40}$$

On the other hand,

$$\|W\| \;\le\; \|V\| \;\le\; \sum_i \mathcal{H}^2 \llcorner \Gamma_i\,. \tag{2.41}$$

So T is actually supported in the current given by the union of the currents induced by the Γ_i's, which we denote by $[[\Gamma_i]]$. Since ∂T and $\partial\Gamma_i$ lie both on ∂U, a second standard fact in geometric measure theory imply the existence of integers $h_1, \ldots, h_N$ such that

$$T \;=\; \sum_{i=1}^N h_i[[\Gamma_i]]$$

Therefore,

$$\|T\| \;=\; \sum_i |h_i| \mathcal{H}^2 \llcorner \Gamma_i\,. \tag{2.42}$$

Hence, (2.40), (2.41) and (2.42) give $h_i \in \{-1, 0, 1\}$. On the other hand, since each $\partial\tilde{\Delta}^k$ is the union of connected components of $\partial\Sigma$ (with positive orientation), it turns out that ∂T is the union of the currents induced by some connected components of $\partial\Sigma$, with *positive* orientation. Moreover, since U is a sufficiently small ball, by the maximum principle each surface Γ_i must have nontrivial boundary. Hence, we conclude that $h_i \in \{0, 1\}$.

Arguing in the same way, we conclude that $\Delta^k \setminus \tilde{\Delta}^k$ converge, as currents, to a current T', and, as varifolds, to a varifold W' with the properties that

$$T' \;=\; \sum_{i=1}^{N} h'_i[[\Gamma_i]] \tag{2.43}$$

$$\|T'\| \le \|W'\| \tag{2.44}$$

and $h'_i \in \{0,1\}$. Since $W + W' = V$, (and hence $\|W\| + \|W\|' = \|V\|$), we conclude that $h''_i = h'_i + h_i \in \{0,1\}$. On the other hand, Δ^k converges, in the sense of currents, to $T + T'$, which is given by

$$T + T' \;=\; \sum_i (h_i + h'_i)[[\Gamma_i]]\,. \tag{2.45}$$

Moreover, since $\partial\Delta^k = \partial\Sigma$,

$$[[\partial\Sigma]] \;=\; \partial(T + T') \;=\; \sum_i (h_i + h'_i)[[\partial\Gamma_i]]\,. \tag{2.46}$$

Since the $\partial\Gamma_i$ are all nonzero, disjoint and contained in $\partial\Sigma$, we conclude that $h_i + h'_i = 1$ for every i.

Summarizing, we conclude that $\|V\| = \|W\| + \|W'\| \ge \|T\| + \|T'\| \ge \|T + T'\| = \|V\|$. This implies that $\|W\| + \|W'\| = \|T\| + \|T'\|$ and hence that $\|W\| = \|T\|$. Therefore

$$\|W\| \;=\; \sum_i h_i \mathcal{H}^2 \llcorner \Gamma_i$$

and since $h_i \in \{0,1\}$, this last claim concludes the proof. □

CHAPTER 3

Existence and regularity of min-max surfaces

In this Chapter we will recall the proof of the existence and regularity result for min–max surfaces announced by Pitts and Rubistein in [**PR86**] and proved by T. Colding and C. De Lellis in their paper [**CDL03**]. Thereby the authors gave a complete proof of the following Theorem due to Simon and Smith. Its proof will use the material covered in Chapter 1 and Chapter 2.

THEOREM 3.1. *[Simon–Smith] Let M be a closed 3-manifold with a Riemannian metric. For any saturated Λ, there is a min–max sequence $\Sigma^n_{t_n}$ converging in the sense of varifolds to a smooth embedded minimal surface Σ with area $m_0(\Lambda)$ (multiplicity is allowed).*

3.1. Overview of the proof of Theorem 3.1

In the following we fix a saturated set Λ of generalized 1-parameter families of surfaces and denote by $m_0 = m_0(\Lambda)$ the infimum of the areas of the maximal slices in Λ.

What we need to prove is that the stationary varifold V of Proposition 1.8 is a smooth surface.

In Section 3.2 we will see that if An is an annulus in which $\{\Sigma^j\}$ is a.m., then there exists a stationary varifold V', referred to as a *replacement*, such that

(3.1) V and V' have the same mass and $V = V'$ on $M \setminus \mathrm{An}$.

(3.2) V' is a stable minimal surface inside An.

The stragegy of the proof will be twofold: in Section 3.2 (Lemma 3.5), following [**Smi82**], we will show that this "replacement property" and (1.30) will imply that the stationary varifold V of Proposition 1.8 is integer rectifiable and hence regular. Section 3.4 will be devoted to the construction of such replacements and will follow ideas of Pitts (see [**Pit81**]). More precisely we can split the proof of theorem 3.1 in two steps:

Step 1: Fix an annulus An in M in which Σ^k is ε_k–a.m. In this annulus we deform Σ^k into a further sequence of surfaces $\{\Sigma^{k,l}\}^l$ with the following properties:

- $\Sigma^{k,l}$ is the image of Σ^k under some isotopy ψ which satisfies (1.14) (with $\varepsilon = \varepsilon_k$ and $U = \mathrm{An}$);
- If we denote by $\mathcal{S}^k$ the family of all such isotopies, then

$$\lim_{l\to\infty} \mathcal{H}^2(\Sigma^{k,l}) = \inf_{\psi\in\mathcal{S}^k} \mathcal{H}^2(\psi(1,\Sigma^k))\,. \tag{3.3}$$

After possibly passing to a subsequence, then $\Sigma^{k,l} \to V^k$ and $V^k \to V'$, where V^k is a varifold which is stationary *in* An. By the a.m. property of V, it follows that V' is stationary *in all of* M and satisfies (3.1).

The second step is to prove that V^k is a (smooth) stable minimal surface in An. Thus, (1.30) will give that also V' is a stable minimal surface in An. After checking some details we will show that V meets the technical requirements of Proposition 3.4.

Step 2: It remains to prove that V^k is a stable minimal surface. Stability is a trivial consequence of (3.3). For the regularity we use again Proposition 3.4. Key is going to be the following property which is an immediate consequence of the squeezing Lemma proved in Chapter 2 (see Lemma 2.10):

(P) If $B \subset \mathrm{An}$ is a sufficiently small ball and l is a sufficiently large number, then *any* $\psi \in \mathfrak{Is}(B)$ with $\mathcal{H}^2(\psi(1,\Sigma^{k,l})) \leq \mathcal{H}^2(1,\Sigma^k)$ can be replaced by a $\Psi \in \mathfrak{Is}(\mathrm{An})$ with

$$\Psi(1,\cdot) = \psi(1,\cdot) \quad \text{and} \quad \mathcal{H}^2(\Psi(t,\Sigma^{k,l})) \leq \mathcal{H}^2(\Sigma^{k,l}) + \varepsilon_k/8 \tag{3.4}$$

for all $t \in [0,1]$ and all given ε_k.

We will now discuss how (P) gives the regularity of V^k.

Fix a sufficiently small ball B and a large number l so that the property (P) above holds. Take a sequence of surfaces $\Gamma^j = \Sigma^{k,l,j}$ which are isotopic to $\Sigma^{k,l}$ in B and such that $\mathcal{H}^2(\Gamma^j)$ converges to

$$\inf_{\psi\in\mathfrak{Is}(B)} \mathcal{H}^2(1,\psi(\Sigma^{k,l}))\,.$$

By Theorem 2.2 of Chapter 2, Γ^j converges to a varifold $V^{k,l}$ which is a stable minimal surface in B. Thus, by (1.30), the sequence of varifolds $\{V^{k,l}\}^l$ converges to a varifold W^k which is a stable minimal surface in B. The property (P) is used to show that, for j and l sufficiently large, $\Sigma^{k,l,j}$ is a good competitor with respect to the ε_k–a.m. property of Σ^k. This is then used to show that W^k is a replacement for V^k in B. Again it is only a technical step to check that we can apply Proposition 3.4, and hence get that V^k is a stable minimal surface in An.

3.2. Regularity theory for replacement

We should start defining what we intended as "good replacement" for stationary varifolds.

DEFINITION 3.2. Let $V \in \mathcal{V}(M)$ be stationary and $U \subset M$ be an open subset. A stationary varifold $V' \in \mathcal{V}(M)$ is said to be a *replacement for* V *in* U if (3.5) and (3.6) below hold.

$$V' = V \text{ on } G(M \setminus \overline{U}) \text{ and } \|V'\|(M) = \|V\|(M). \tag{3.5}$$

$$V \llcorner U \text{ is a stable minimal surface } \Sigma \text{ with } \overline{\Sigma} \setminus \Sigma \subset \partial U. \tag{3.6}$$

DEFINITION 3.3. Let V be a stationary varifold and $U \subset M$ be an open subset. We say that V has the *good replacement property* in U if the following conditions hold.

(a) There is a positive function $r : U \to \mathbf{R}$ such that for every annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ there is a replacement V' for V in An.
(b) The replacement V' has a replacement V'' in any $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ and in any $\mathrm{An} \in \mathcal{AN}_{r'(y)}(y)$ (where r' is positive).
(c) V'' has a replacement V''' in any $\mathrm{An} \in \mathcal{AN}_{r''(y)}(y)$ (where $r'' > 0$).

If V and V' are as above, then we will say that V' is a *good replacement* and V'' a *good further replacement*.

We are now able to state the proposition which will give us regularity.

PROPOSITION 3.4. *Let G be an open subset of M. If V has the good replacement property in G, then V is a (smooth) minimal surface in G.*

The two lemmas in subsection 1.1.4 are useful for the proof of this proposition. We will also need the following lemma:

LEMMA 3.5. *Let U be an open subset of M and V a stationary varifold in U. If there exists a positive function r on M such that V has a replacement in any annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$, then V is integer rectifiable. Moreover, $\theta(x, V) \geq 1$ for any $x \in U$ and any tangent cone to V in x is an integer multiple of a plane.*

PROOF. Since V is stationary, the monotonicity formula (1.3) gives a constant C_M such that

$$\frac{\|V\|(B_\sigma(x))}{\sigma^2} \leq C_M \frac{\|V\|(B_\rho(x))}{\rho^2} \qquad \forall \sigma < \rho < \mathrm{Inj}\,(M) \text{ and } \forall x \in M\,. \tag{3.7}$$

Fix $x \in \mathrm{supp}\,(\|V\|)$ and $r < r(x)$ so that $4r$ is smaller than the convexity radius. Replace V with V' in $\mathrm{An}(x, r, 2r)$. We claim that $\|V'\|$ cannot be identically 0 on $\mathcal{AN}(x, r, 2r)$. Assume it was; since

$x \in \operatorname{supp}(\|V'\|)$, there would be a $\rho \le r$ such that V' "touches" ∂B_ρ from the interior. More precisely, there would exist ρ and ε such that $\operatorname{supp}\|V'\| \cap \partial B_\rho(x) \neq \emptyset$ and $\operatorname{supp}\|V'\| \cap \mathcal{AN}(x,\rho,\rho+\varepsilon) = \emptyset$. Since $B_\rho(x)$ is convex this would contradict Lemma 1.3. Thus $V' \llcorner \mathrm{An}(x,r,2r)$ is a non–empty smooth surface and so there is $y \in \mathrm{An}(x,r,2r)$ with $\theta(V',y) \ge 1$. Using (3.7) we get
(3.8)
$$\frac{\|V\|(B_{4r}(x))}{16r^2} = \frac{\|V'\|(B_{4r}(x))}{16r^2} \ge \frac{C_M\|V'\|(B_{2r}(y))}{16r^2} \overset{(3.7)}{\ge} \frac{\pi C_M}{4}.$$
Hence, $\theta(x,V)$ is bounded uniformly from below on $\operatorname{supp}(\|V\|)$ and applying Theorem 1.1 we conclude that V is rectifiable.

We next prove that V is *integer* rectifiable. We use the notation of Definition 1.2. Fix $x \in \operatorname{supp}(\|V\|)$, a stationary cone $C \in TV(x,V)$, and a sequence $\rho_n \downarrow 0$ such that $V^x_{\rho_n} \to C$. Replace V by V'_n in $\mathrm{An}(x,\rho_n/4,3\rho_n/4)$ and set $W'_n = (T^x_{\rho_n})_\# V'_n$. After possibly passing to a subsequence, we can assume that $W'_n \to C'$, where C' is a stationary varifold. The following properties of C' are trivial consequences of the definition of replacements:

$$C' = C \text{ in } \mathcal{B}_{1/4}(x) \cup \mathrm{An}(x,3/4,1), \tag{3.9}$$
$$\|C'\|(\mathcal{B}_\rho) = \|C\|(\mathcal{B}_\rho) \text{ if } \rho \in]0,1/4[\cup]3/4,1[. \tag{3.10}$$

Since C is a cone, in view of (3.10) we have

$$\frac{\|C'\|(\mathcal{B}_\sigma)}{\sigma^2} = \frac{\|C'\|(\mathcal{B}_\rho)}{\rho^2} \qquad \forall \sigma,\rho \in]0,1/4[\cup]3/4,1[. \tag{3.11}$$

Hence, the stationarity of C' and the monotonicity formula imply that C' is a cone. By (1.30), W'_n converge to a stable embedded minimal surface in $\mathrm{An}(x,1/4,3/4)$. This means that $C' \llcorner \mathrm{An}(x,1/4,3/4)$ is an embedded minimal cone in the classical sense and hence it is supported on a disk containing the origin. This forces C' and C to coincide and be an integer multiple of the same plane. □

We are now able to prove the proposition.

Proof of Proposition 3.4. The strategy of the proof is as follows. Fix $x \in M$, a good replacement V' for V in $\mathrm{An}(x,\rho,2\rho)$, and let Σ' be the stable minimal surface given by V' in $\mathrm{An}(x,\rho,2\rho)$. Consider $t \in]\rho,2\rho[$, $s < \rho$ and the replacement V'' of V' in $\mathrm{An}(x,s,t)$, which in this annulus coincides with a smooth minimal surface Σ''. In step 2 we will prove that, for ρ sufficiently small and for an appropriate choice of t, then $\Sigma'' \cup \Sigma'$ is a smooth surface. Letting $s \downarrow 0$ we get a minimal surface $\Sigma \subset B_\rho(x) \setminus \{x\}$ such that every Σ'' constructed as above is a subset of Σ. Loosely speaking, any replacement of V' will coincide with Σ in the annular region where it is smooth.

Now, fix a z which belongs to $\operatorname{supp}(\|V\|)$ and such that V intersects $\partial B_s(x)$ "transversally" in z. If we consider a replacement V'' of V' in $\mathrm{An}(x, s, \rho)$, then z will belong to the closure of the minimal surface $\Sigma'' = V'' \llcorner \mathrm{An}(x, s, \rho)$. The discussion above gives that $z \in \Sigma$. Lemma 1.4 implies that "transversality" to the spheres centered at x is a dense subset of $(\operatorname{supp}(\|V\|)) \cap B_\rho(x)$. Thus in step 3 we conclude that

$$(\operatorname{supp}(\|V\|)) \cap B_\rho(x) \setminus \{x\} \subset \Sigma.$$

Since $\mathcal{H}^2(\Sigma \cap B_\rho(x)) = \|V\|(B_\rho(x))$, then $V = \Sigma$ in $B_\rho(x)$. Step 4 concludes the proof by showing that x is a removable singularity for Σ.

The key fact that Σ'' and Σ' can be "glued" smoothly together is a consequence of the curvature estimates for stable minimal surfaces combined with the characterization of the tangent cones given in Lemma 3.5. These two ingredients will be used to prove that Σ'' is (locally) a Lipschitz graph nearby $\partial B_t(x)$; thus allowing us to apply standard theory of Elliptic PDE.

Step 1: The set up.

Fix x, V, V', V'', Σ', Σ'', ρ, s, and t as above. We require that 2ρ is less than the convexity radius of M and that Σ' intersects $\partial B_t(x)$ transversally. Fix a point $y \in \Sigma' \cap \partial B_t(x)$ and a sufficiently small radius r, so that $\Sigma' \cap B_r(y)$ is a disk and $\gamma = \Sigma' \cap \partial B_t(x) \cap B_r(y)$ is a smooth arc.

Let $\zeta : B_r(y) \to \mathcal{B}_1$ be a diffeomorphism such that

$$\zeta(\partial B_t(x)) \subset \{z_1 = 0\} \qquad \text{and} \qquad \zeta(\Sigma'') \subset \{z_1 > 0\},$$

where z_1, z_2, z_3 are orthonormal coordinates on $\mathcal{B}_1$; see Fig. 1. We will also assume that $\zeta(\gamma) = \{(0, z_2, g'(0, z_2))\}$ and $\zeta(\Sigma') \cap \{z_1 \leq 0\} = \{(z_1, z_2, g'(z_1, z_2))\}$ where g' is smooth.

Note the following elementary facts:

- Any kind of estimates (like curvature or area bounds or monotonicity) for a minimal surface $\Sigma \subset B_r(y)$ translates into similar estimates for the surface $\zeta(\Sigma)$.
- Varifolds in $B_r(y)$ are push–forwarded to varifolds in $\mathcal{B}_1$ and there is a natural correspondence between tangent cones to V in ξ and tangent cones to $\zeta_\# V$ in $\zeta(\xi)$.

By slight abuse of notation, we use the same symbols (e.g. γ, V', Σ') for both the objects of $B_r(y)$ and their images under ζ.

Step 2: Graphicality; gluing Σ' and Σ'' smoothly together.

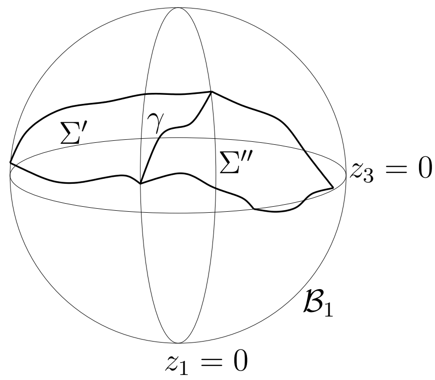

FIGURE 1. The surfaces Σ' and Σ'' and the curve γ in $\mathcal{B}_1$.

The varifold V'' consists of $\Sigma'' \cup \Sigma'$ in $B_r(y)$. Moreover, Lemma 3.5 applied to V'' gives that $TV(z, V'')$ is a family of (multiples of) 2–planes. Fix $\overline{z} \in \gamma$. Since Σ' is regular and transversal to $\{z_1 = 0\}$ in $\overline{z}$, each plane $P \in TV(\overline{z}, V'')$ coincides with the half plane $T_{\overline{z}}\Sigma'$ in $\{z_1 < 0\}$. Hence $TV(\overline{z}, V'') = \{T_{\overline{z}}\Sigma'\}$. Let $\tau(\overline{z})$ be the unit normal to the graph of g'

$$\tau(\overline{z}) \;=\; \frac{(-\partial_1 g'(0, \overline{z}_2), -\partial_2 g'(0, \overline{z}_2), 1)}{\sqrt{1 + |\nabla g' 0, \overline{z}_2)|^2}}$$

and let $R_r^{\overline{z}} : \mathbf{R}^3 \to \mathbf{R}^3$ be the dilatation of 3-space defined by

$$R_r^{\overline{z}}(z) \;=\; \frac{z - \overline{z}}{r}\,.$$

Since $TV(\overline{z}, V'') = \{T_{\overline{z}}\Sigma'\}$, the surfaces $\Sigma_r = R_r^{\overline{z}}(\Sigma'')$ converge to the half plane $HP = \{\tau(\overline{z})\cdot v = 0, v_1 > 0\}$ — half of the plane $\{\tau(\overline{z})\cdot v = 0\}$. This convergence implies that

$$\lim_{z \to \overline{z}, z \in \Sigma''} \frac{|(z - \overline{z}) \cdot \tau(\overline{z})|}{|\overline{z} - z|} = 0\,. \tag{3.12}$$

Indeed assume that (3.12) fails; then there is a sequence $\{z_n\} \subset \Sigma''$ such that $z_n \to \overline{z}$ and $|(z_n - \overline{z}) \cdot \tau(\overline{z})| \geq k|z_n - \overline{z}|$ for some $k > 0$. Set $r_n = |z_n - \overline{z}|$. There exists a constant k_2 such that $\mathcal{B}_{2k_2 r_n}(z_n) \cap HP = \emptyset$. Thus $\mathrm{dist}(HP, \mathcal{B}_{k_2 r_n}(z_n)) \geq k_2 r_n$. Since Σ'' is regular in z_n we get by the monotonicity formula that

$$\|V''\|(\mathcal{B}_{k_2 r_n}(z_n)) \geq C k_2^2 r_n^2 \qquad \text{where } C \text{ depends on } \zeta.$$

This contradicts the fact that HP is the only element of $TV(\overline{z}, V'')$. Note also that the convergence of (3.12) is uniform for $\overline{z}$ in compact subsets of γ. The argument is explained in Fig. 2.

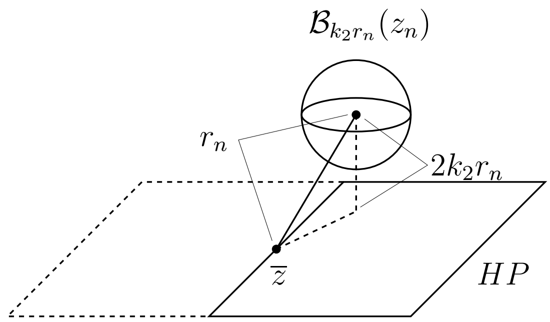

FIGURE 2. If $z_n \in \Sigma''$ is far from the plane HP, the monotonicity formula gives a “good amount” of the varifold V'' which lives far from HP.

Let ν denote the smooth unit vector field to Σ'' such that $\nu \cdot (0,0,1) \geq 0$. We next use the stability of Σ'' to show that

$$\lim_{z \to \overline{z}, z \in \Sigma''} \nu(z) \; = \; \tau(\overline{z})\,. \tag{3.13}$$

Indeed let σ be the plane $\{(0,\alpha,\beta),\, \alpha,\beta \in \mathbf{R}\}$, assume that $z_n \to \overline{z}$ and set $r_n = \operatorname{dist}(z_n,\sigma)$. Define the rescaled surfaces $\Sigma^n = R^{z_n}_{r_n}(\Sigma'' \cap \mathcal{B}_{r_n}(z_n))$. Each Σ^n is a stable minimal surface in $\mathcal{B}_1$, and hence, after possibly passing to a subsequence, Σ^n converges smoothly in $\mathcal{B}_{1/2}$ to a minimal surface Σ^∞ (by (1.30)). By (3.12), we have that Σ^∞ is the disk $T_{\overline{z}}\Sigma' \cap \mathcal{B}_{1/2}$. Thus the normals to Σ^n in 0, which are given by $\nu(z_n)$, converge to $\tau(\overline{z})$; see Fig. 3. It is easy to see that the convergence in (3.13) is uniform on compact subsets of γ.

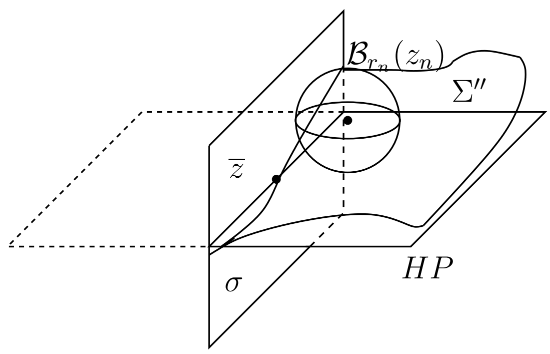

FIGURE 3. If we rescale $\mathcal{B}_{r_n}(z_n)$, then we find a sequence of stable minimal surfaces Σ^n which converge to the half–plane HP.

Hence, for each $\overline{z} \in \gamma$, there exists $r > 0$ and a function $g'' \in C^1(\{z_1 \geq 0\})$ such that

$$\Sigma'' \cap B_r(\overline{z}) \;=\; \{(z_1, z_2, g''(z_1, z_2)), z_1 > 0\}\,,$$
$$g''(0, z_2) = g'(0, z_2)\,, \qquad \text{and} \qquad \nabla g''(0, z_2) = \nabla g'(0, z_2)\,.$$

In the coordinates z_1, z_2, z_3, the minimal surface equation yields a second order uniformly elliptic equation for g' and g''. Thus the classical theory of elliptic PDE gives that g' and g'' are restrictions of a unique smooth function g.

Step 3: Regularity of V in the punctured ball.

Let Σ' and Σ'' be as in the previous step. We will now show that :

(3.14)

If Γ is a connected component of Σ'', then $\overline{\Gamma} \cap \Sigma' \cap \partial B_t(y) \neq \emptyset$.

Indeed assume that for some Γ equation (3.14) fails. Since t is assumed to be less than the convexity radius we have by the maximum principle that $\overline{\Gamma} \cap \partial B_t(x) \neq \emptyset$. Fix z in $\overline{\Gamma} \cap \partial B_t(x)$. If (3.14) were false, then the varifold V'' would "touch" $\partial B_t(x)$ in z from the interior. More precisely, there would be an $r > 0$ such that

$$z \in \operatorname{supp}(\|V''\|) \qquad \text{and} \qquad (B_r(y) \cap \operatorname{supp}(\|V''\|)) \subset \overline{B}_t(x)\,.$$

This contradicts Lemma 1.3; thus (3.14) holds.

Let t, ρ be as in the first paragraph of Step 1. Step 2 and (3.14) imply the following:

if $s < \rho$, then Σ' can be extended to

(3.15) a surface Σ_s in $\mathrm{An}(x, s, 2\rho)$

(3.16) if $s_1 < s_2 < \rho$, then $\Sigma_{s_1} = \Sigma_{s_2}$ in $\mathrm{An}(x, s_2, 2\rho)$.

Thus $\Sigma = \bigcup_s \Sigma_s$ is a stable minimal surface Σ with $\overline{\Sigma} \setminus \Sigma \subset (\partial B_{2\rho}(x) \cup \{x\})$, i.e. Σ' can be continued up to x (which, in principle, could be a singular point).

We will next show that V coincides with Σ in $B_\rho(x) \setminus \{x\}$. Recall that $V = V'$ in $B_\rho(x)$. Fix

$$y \in (\operatorname{supp}(\|V\|)) \cap B_\rho(x) \setminus \{x\} \qquad \text{and set } s = d(y, x).$$

We first prove that if $TV(y, V)$ consists of a (multiple of a) plane π transversal to $\partial B_s(x)$, then y belongs to Σ. Consider the replacement V'' of V' in $\mathrm{An}(x, s, t)$ and split V'' into the three varifolds

$$\begin{array}{lllll} V_1 & = & V'' \llcorner B_s(x) & = & V \llcorner B_s(x)\,, \\ V_2 & = & V'' \llcorner \mathrm{An}(x, s, 2\rho) & = & \Sigma \cap \mathrm{An}(x, s, 2\rho)\,, \\ V_3 & = & V'' - V_1 - V_2\,. & & \end{array}$$

By Lemma 3.5, the set $TV(y, V'')$ consists of planes and since $V_1 = V \llcorner B_s(x)$, all these planes have to be multiples of π. Thus y is in the closure of $(\mathrm{supp}\,(\|V''\|)) \setminus \overline{B}_s(x)$, which implies $y \in \overline{\Sigma_t} \subset \Sigma$.

Let T be the set of points $y \in B_\rho(x)$ such that $TV(y, V)$ consists of a (multiple of) a plane transversal to $\partial B_{d(y,x)}(x)$. Lemma 1.4 gives that T is dense in $\mathrm{supp}\,(\|V\|)$. Thus

$$(\mathrm{supp}\,(\|V\|)) \cap B_\rho(x) \setminus \{x\} \;\subset\; \Sigma\,.$$

Property (3.5) of replacements implies $\mathcal{H}^2(\Sigma \cap B_\rho(x)) = \|V\|(B_\rho(x))$. Hence $V = \Sigma$ on $B_\rho(x) \setminus \{x\}$.

Step 4: Regularity in x.

We will next show that Σ is smooth also in x, i.e. that x is a removable singularity for Σ. If $x \notin \mathrm{supp}\,(\|V\|)$, then we are done. So assume that $x \in \mathrm{supp}\,(\|V\|)$. In the following we will use that, by Lemma 3.5, every $C \in TV(x, V)$ is a multiple of a plane.

Map $B_t(x)$ into $\mathcal{B}_t(0)$ by the exponential map, use the notation of Step 1, and set $\Sigma_r = R^x_r(\Sigma)$. Every convergent subsequence $\{\Sigma_{r_n}\}$ converges to a plane in the sense of varifolds. The curvature estimates for stable minimal surfaces (see (1.30)) gives that this convergence is actually smooth in $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$. Thus, for r sufficiently small, there exist natural numbers $N(\rho)$ and $m_i(\rho)$ such that

$$\Sigma \cap \mathrm{An}(x, \rho/2, \rho) \;=\; \bigcup_{i=1}^{N(\rho)} m_i(\rho)\Sigma^i_\rho\,,$$

where each Σ^i_ρ is a Lipschitz graph over a (planar) annulus. Note also that the Lipschitz constants are uniformly bounded, independently of ρ.

By continuity, the numbers $N(r)$ and $m_i(r)$ do not depend on r. Moreover, if $s \in]\rho/2, \rho[$, then each Σ^i_ρ can be continued through $\mathrm{An}(s, \rho/2, x)$ by a Σ^i_s. Repeating this argument a countable number of times, we get N minimal punctured disks Σ^i with

$$\Sigma \cap B_\rho(x) \setminus \{x\} \;=\; \bigcup_{i=1}^{N} m_i \Sigma^i\,.$$

Note that x is a removable singularity for each Σ^i. Indeed, Σ^i is a stationary varifold in $B_\rho(x)$ and $TV(x, \Sigma^i)$ consists of planes with multiplicity one. This means that

$$\lim_{r\downarrow 0} \frac{\|V\|(B_r(x))}{\pi r^2} \;=\; 1\,.$$

Hence we can apply Allard's regularity theorem (see section 8 of [**All72**]) to conclude that Σ^i is a graph in a sufficiently small ball around x. Standard elliptic PDE theory gives that x is a removable singularity.

Finally, the maximum principle for minimal surfaces implies that N must be 1. This completes the proof. □

3.3. Proof of Lemma 2.17

With the help of the replacements theory developed in the previous Section we are now able to prove the Lemma on interior regularity stated in Subsection 2.3.4.

Proof of Lemma 2.17. Let Δ^k and V be as in Theorem 2.2 and in Lemma 2.17. Let $x \in U$ and consider a $U' = B_\varepsilon(x) \subset U$ as in Lemma 2.10. Applying Lemma 2.10 we can modify Δ^k in $B_\varepsilon(x)$ getting a minimizing sequence $\{\Delta^{k,j}\}_j$ for $\mathfrak{Is}_j(B_\varepsilon(x))$. Applying Proposition 2.11, we can assume that $\Delta^{k,j}$ converges, as $j \uparrow \infty$ to a varifold V'_k which in $B_\varepsilon(x)$ is a stable minimal surface Σ^k. By the curvature estimates for minimal surfaces (cp. also with the Choi-Schoen compactness Theorem), we can assume that Σ^k converges to a stable smooth minimal surface Σ^∞. Extracting a diagonal subsequence $\tilde{\Delta}^k := \Delta^{k,j(k)}$, we can assume that $\tilde{\Delta}^k$ is still minimizing for problem $\mathfrak{Is}_j(U)$ and hence that it converges to a varifold V'. V' coincides with Σ^∞ in $B_\varepsilon(x)$ and with V outside $B_\varepsilon(x)$ and hence it is a replacement according to Definition 3.3. By Proposition 3.4, V coincides with a smooth embedded minimal surface in U. □

3.4. Construction of replacements

In this section we conclude the proof of Theorem 3.1 by showing that the varifold V of Proposition 1.8 is a smooth minimal surface. We need to show that V satisfies the requirements of Proposition 3.4. As outlined is the fist section we will construct the required good replacements and therefore prove the follwing theorem.

Theorem 3.6. *Let $\{\Sigma^j\}$ be a sequence of compact surfaces in M which converge to a stationary varifold V. If there exists a function $r : M \to \mathbf{R}^+$ such that*

- *in every annulus of $\mathcal{AN}_{r(x)}(x)$ and for j large enough Σ^j is a $1/j$–a.m. smooth surface in* An,

then V is a smooth minimal surface.

Key for the proof of Theorem 3.6 is the next Proposition. The first part implies that that we can deform Σ^j into a sequence $\Sigma^{j,k}$ which is minimizing for Problem $(\Sigma^j, \mathfrak{Is}_j(\mathrm{An}))$ (cp. with notation in Chapter 2) and converges, as $k \to \infty$, to a stable minimal surface in An. The second part will give us the desired replacements.

Proposition 3.7. *Let* $\{\Sigma^{j,k}\}^k$ *be a minimizing sequence for Problem* $(\Sigma^j, \mathfrak{Is}_j(\mathrm{An}))$ *and converging to a varifold* V^j. *Then*

(i) V^j *is a stable minimal surface in* An*;*
(ii) *any* V^* *which is the limit of a subsequence of* $\{V^j\}$ *is a replacement for* V.

Proof. (i) is a corollary of Lemma 2.17 and remarks 2.3 and 2.4. (ii) will follow from (i): Without loss of generality, we can assume that the sequence $\{V^j\}$ converges to V. Note that every V^j coincides with V in $M \setminus \overline{\mathrm{An}}$; thus the same is true for V^*. Moreover, $\|V^j\|(M) \geq \mathcal{H}^2(\Sigma^j) - 1/j$ since Σ^j is a.m. This gives that $\|V^*\|(M) = \|V\|(M)$. By (i) and (1.30) we have that V^* is a stable minimal surface in An.

To complete the proof we need to show that V^* is stationary. Since $V = V^*$ in $M \setminus \overline{\mathrm{An}}$, then V^* is stationary in this open set. Hence it suffices to prove that V^* is stationary in an open annulus $\mathrm{An}' \in \mathcal{AN}_r$ containing $\overline{\mathrm{An}}$. Choose such an An' and suppose that V^* is not stationary in An'; we will show that this contradict that $\{\Sigma^j\}$ is a.m. in An'. Namely, suppose that for some vector field χ supported in An' we have $\delta V^*(\chi) \leq -C < 0$. Let ψ be the isotopy given by that $\frac{\partial \psi(t,x)}{\partial t} = \chi(\psi(t,x))$ and set

$$\begin{aligned} V^*(t) &= \psi(t)_\sharp V^* \,, \\ V^j(t) &= \psi(t)_\sharp V^j \,, \\ \Sigma^{j,k}(t) &= \psi(t, \Sigma^{j,k}) \,. \end{aligned}$$

For ε sufficiently small, we have that

$$[\delta V^*(t)](\chi) \leq -\frac{C}{2} \qquad \text{for all } t < \varepsilon.$$

Since $V^j(t) \to V^*(t)$, there exists J such that

$$[\delta V^j(t)](\chi) \leq -\frac{C}{4} \qquad \text{for every } j > J \text{ and every } t < \varepsilon.$$

Moreover, since $\Sigma^{j,k}(t) \to V^j(t)$, for each $j > J$ there exists $K(j)$ with

$$[\delta \Sigma^{j,k}(t)](\chi) \leq -\frac{C}{8} \qquad \text{for all } t < \varepsilon \text{ and all } k > K(j). \tag{3.17}$$

Integrating both sides of (3.17) we get

$$\mathcal{H}^2(\psi(t, \Sigma^{j,k})) - \mathcal{H}^2(\Sigma^{j,k}) \leq -\frac{tC}{8} \tag{3.18}$$

Choose j and k sufficiently large so that $\varepsilon C/8 > 1/j$ and (3.18) holds. Each $\Sigma^{j,k}$ is isotopic to Σ^j via an isotopy $\varphi^{j,k} \in \mathfrak{Is}_j(\mathrm{An})$. By gluing $\varphi^{j,k}$ and ψ smoothly together, we find a smooth isotopy $\Phi : [0, 1+\varepsilon] \times M \to M$ supported on An'. In view of (3.18), Φ satisfies

$$\begin{aligned} \mathcal{H}^2(\Phi(t, \Sigma^j)) &\leq \mathcal{H}^2(\Sigma^j) + 1/(8j) \qquad \forall t \in [0, 1+\varepsilon]\,, \\ \mathcal{H}^2(\Phi(1+\varepsilon, \Sigma^j) &< \mathcal{H}^2(\Sigma^j) - 1/j\,, \end{aligned}$$

which give the desired contradiction and prove the proposition. □

We are now able to prove the main theorem of this section.

Proof of Theorem 3.6. We will apply Proposition 3.4. From Proposition 3.7 we know that in every annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ there is a replacement V^* for V. We still need to show that V satisfies (a), (b), and (c) in Definition 3.3. Consider the family of surfaces $\Sigma^{j,k}$ as in Proposition 3.7. By a diagonal argument we can extract a subsequence $\Sigma^{j,k(j)}$ converging to V^*. Note the following consequence of the way we constructed $\{\Sigma^{j,k(j)}\}^j$. If U is open and

- either $U \cup \mathrm{An}$ is contained in some annulus $\mathcal{AN}_{r(x)}(x)$
- or $U \cap \mathrm{An} = \emptyset$ and U is contained in some annulus of $\mathcal{AN}_{r(y)}(y)$ with $y \neq x$,

then $\Sigma^{j,k(j)}$ is a.m. in U. Thus $\{\Sigma^{j,k(j)}\}$ is still a.m. in

- every annulus of $\mathcal{AN}_{r(x)}(x)$;
- every annulus of $\mathcal{AN}_{\rho(y)}(y)$ for $y \neq x$, provided $\rho(y)$ is sufficiently small.

This shows that (b) in Definition 3.3 holds for V. Similarly, we can show that also condition (c) of that Definition holds. Hence Proposition 3.4 applies and we conclude that V is a smooth surface. □

CHAPTER 4

Genus bounds

In this chapter we will prove a slightly different version of Theorem 0.7. Our genus estimate is valid, in general, for limits of min–max sequences of surfaces which are almost minimizing in sufficiently small annuli.

THEOREM 4.1. *Let $\Sigma^j = \Sigma^j_{t_j}$ be a sequence which is a.m. in sufficiently small annuli. Let $V = \sum_i n_i \Gamma^i$ be the varifold limit of $\{\Sigma^j\}$, where Γ^i are as in Theorem 0.7. Then*

$$\sum_{\Gamma^i \in \mathcal{O}} \mathbf{g}(\Gamma^i) + \frac{1}{2} \sum_{\Gamma^i \in \mathcal{N}} (\mathbf{g}(\Gamma^i) - 1) \;\leq\; \liminf_{j\uparrow\infty} \liminf_{\tau\to t_j} \mathbf{g}(\Sigma^j_\tau)\,. \tag{4.1}$$

4.1. Overview of the proof

In this section we give an overview of the proof of Theorem 4.1. The contents of Chapter 2 will have a prominent role proving the theorem. We fix a min–max sequence $\Sigma^j = \Sigma^j_{t_j}$ as in Theorem 4.1 and we let $\sum_i n_i \Gamma^i$ be its varifold limit. Consider the smooth surface $\Gamma = \cup_i \Gamma^i$ and let $\varepsilon_0 > 0$ be so small that there exists a smooth retraction of the tubular neighborhood $T_{2\varepsilon_0}\Gamma$ onto Γ. This means that, for every $\delta < 2\varepsilon_0$,

- $T_\delta \Gamma^i$ are smooth open sets with pairwise disjoint closures;
- if Γ^i is orientable, then $T_\delta\Gamma^i$ is diffeomorphic to $\Gamma^i \times]-1, 1[$;
- if Γ^i is non–orientable, then the boundary of $T_\delta\Gamma^i$ is an orientable double cover of Γ^i.

4.1.1. Simon's Lifting Lemma. The following Proposition is the core of the genus bounds. Similar statements have been already used in the literature (see for instance [**GJ86**] and [**FH89**]). We recall that the surface Σ^j might not be everywhere regular (see discussion in Introduction), and we denote by P_j its set of singular points (possibly empty).

PROPOSITION 4.2 (Simon's Lifting Lemma). *Let γ be a closed simple curve on Γ^i and let $\varepsilon \leq \varepsilon_0$ be positive. Then, for j large enough,* *there is a positive* $n \le n_i$ *and a closed curve* $\tilde{\gamma}^j$ *on* $\Sigma^j \cap T_\varepsilon \Gamma^i \setminus P_j$ *which is homotopic to* $n\gamma$ *in* $T_\varepsilon \Gamma^i$.

Simon's lifting Lemma implies directly the genus bounds if we use the characterization of homology groups through integer rectifiable currents and some more geometric measure theory. However, we choose to conclude the proof in a more elementary way, using Proposition 4.3 below.

4.1.2. Surgery. The idea is that, for j large enough, one can modify any $\{\Sigma^j_t\}$ sufficiently close to $\Sigma^j = \Sigma^j_{t_j}$ through surgery to a new surface $\tilde{\Sigma}^j_t$ such that

- the new surface lies in a tubular neighborhood of Γ;
- it coincides with the old surface in a yet smaller tubular neighborhood.

In this chapter we will use surgeries of the type defined in Definition 2.12. It is important to note that, if $\tilde{\Sigma}$ is obtained from Σ through surgery, then $\mathbf{g}(\tilde{\Sigma}) \le \mathbf{g}(\Sigma)$.

We are now ready to state our next Proposition.

Proposition 4.3. *Let* $\varepsilon \le \varepsilon_0$ *be positive. For each* j *sufficiently large and for* t *sufficiently close to* t_j*, we can find a surface* $\tilde{\Sigma}^j_t$ *obtained from* Σ^j_t *through surgery and satisfying the following properties:*

- $\tilde{\Sigma}^j_t$ *is contained in* $T_{2\varepsilon}\Gamma$*;*
- $\tilde{\Sigma}^j_t \cap T_\varepsilon \Gamma = \Sigma^j_t \cap T_\varepsilon \Gamma$.

4.1.3. Proof of Proposition 4.3**.** Consider the set $\Omega = T_{2\varepsilon}\Gamma \setminus \overline{T_\varepsilon \Gamma}$. Since Σ^j converges, in the sense of varifolds, to Γ, we have

$$\lim_{j\uparrow\infty} \limsup_{t\to t_j} \mathcal{H}^2(\Sigma^j_t \cap \Omega) \;=\; 0\,. \tag{4.2}$$

Let $\eta > 0$ be a positive number to be fixed later and consider N such that

$$\limsup_{t\to t_j} \mathcal{H}^2(\Sigma^j_t \cap \Omega) \;<\; \eta/2 \qquad \text{for each } j \ge N. \tag{4.3}$$

Fix $j \ge N$ and let $\delta_j > 0$ be such that

$$\mathcal{H}^2(\Sigma^j_t \cap \Omega) \;<\; \eta \qquad \text{if } |t_j - t| < \delta_j. \tag{4.4}$$

For each $\sigma \in]\varepsilon, 2\varepsilon[$ consider $\Delta_\sigma := \partial\,(T_\sigma \Gamma)$, i.e. the boundary of the tubular neighborhood $T_\sigma\Gamma$. The surfaces Δ_σ are a smooth foliation of $\Omega \setminus \Gamma$ and therefore, by the coarea formula

$$\int_\varepsilon^{2\varepsilon} \mathrm{Length}(\Sigma^j_t \cap \Delta_\sigma)\, d\sigma \;\le\; C\mathcal{H}^2(\Sigma^j_t \cap \Omega) \;<\; C\eta \tag{4.5}$$

where C is a constant independent of t and j. Therefore,

$$\text{Length}(\Sigma_t^j \cap \Delta_\sigma) < \frac{2C\eta}{\varepsilon} \tag{4.6}$$

holds for a set of σ's with measure at least $\varepsilon/2$.

By Sard's Lemma we can fix a σ such that (4.5) holds and Σ_t^j intersects Δ_t transversally.

For positive constants λ and C, independent of j and t, the following holds:

(B) For any $s \in]0, 2\varepsilon[$, any simple closed curve γ lying on Δ_s with $\text{Length}(\gamma) \leq \lambda$ bounds an embedded disk $D \subset \Delta_s$ with $\text{diam}(D) \leq C\text{Length}(\gamma)$.

Assume that $2C\eta/\varepsilon < \lambda$. By construction, $\Sigma_t^j \cap \Delta_\sigma$ is a finite collection of simple curves. Consider $\tilde{\Omega} := T_{\sigma+\delta}\Gamma \setminus \overline{T_{\sigma-\delta}\Gamma}$. For δ sufficiently small, $\tilde{\Omega} \cap \Sigma_t^j$ is a finite collection of cylinders, with upper bases lying on $\Delta_{\sigma+\delta}$ and lower bases lying on $\Delta_{\sigma-\delta}$. We "cut away" this finite number of necks by removing $\tilde{\Omega} \cap \Sigma_t^j$ and replacing them with the two disks lying on $\Delta_{\sigma-\delta} \cup \Delta_{\sigma+\delta}$ and enjoying the bound (B). For a suitable choice of η, the union of each neck and of the corresponding two disks has sufficiently small diameter. This surface is therefore a compressible sphere, which implies that the new surface $\hat{\Sigma}_t^j$ is obtained from Σ_t^j through surgery.

We can smooth it a little: the smoothed surface will still be obtained from Σ_t^j through surgery and will not intersect Δ_σ. Therefore $\tilde{\Sigma}_t^j := \hat{\Sigma}_t^j \cap T_\sigma\Gamma$ is a closed surface and is obtained from $\hat{\Sigma}_t^j$ by dropping a finite number of connected components. □

4.1.4. Proof of Theorem 4.1**.** Proposition 4.3 and Proposition 4.2 allow us to conclude the proof of Theorem 4.1. We only need the following standard fact for the first integral homology group of a smooth closed connected surface (see Sections 4.2 and 4.5 of [**Mas91**]).

LEMMA 4.4. *Let* Γ *be a connected closed* 2*–dimensional surface with genus* $\mathbf{g}$*. If* Γ *is orientable, then* $H_1(\Gamma) = \mathbb{Z}^{2\mathbf{g}}$*. If* Γ *is non–orientable, then* $H_1(\Gamma) = \mathbb{Z}^{\mathbf{g}-1} \times \mathbb{Z}_2$*.*

We now come to the proof of Theorem 4.1 then the rest of the Chapter will be dedicated to prove Simon's Lifting Lemma.

PROOF OF THEOREM 4.1. Define $m_i = \mathbf{g}(\Gamma^i)$ if i is orientable and $(\mathbf{g}(\Gamma^i) - 1)/2$ if not. Our aim is to show that

$$\sum_i m_i \leq \liminf_{j\uparrow\infty} \liminf_{t\to t_j} \mathbf{g}(\Sigma_t^j)\,. \tag{4.7}$$

By Lemma 4.4, for each Γ^i there are $2m_i$ curves $\gamma^{i,1}, \ldots, \gamma^{i,2m_i}$ with the following property:

(Hom) If $k_1, \ldots, k_{2m_i}$ are integers such that $k_1\gamma^{i,1} + \ldots + k_{2m_i}\gamma^{i,2m_i}$ is homologically trivial in Γ^i, then $k_l = 0$ for every l.

Since $\varepsilon < \varepsilon_0/2$, $T_{2\varepsilon}\Gamma^i$ can be retracted smoothly on Γ^i. Hence:

(Hom') If $k_1, \ldots, k_{2m_i}$ are integers such that $k_1\gamma^{i,1} + \ldots + k_{2m_i}\gamma^{i,2m_i}$ is homologically trivial in $T_{2\varepsilon}\Gamma^i$, then $k_l = 0$ for every l.

Next, fix $\varepsilon < \varepsilon_0$ and let N be sufficiently large so that, for each $j \geq N$, Simon's Lifting Lemma applies to each curve $\gamma^{i,l}$. We require, moreover, that N is large enough so that Proposition 4.3 applies to every $j > N$.

Choose next any $j > N$ and consider the curves $\tilde{\gamma}^{i,l}$ lying in $T_\varepsilon\Gamma^i \cap \Sigma^j$ given by Simon's Lifting Lemma. Such curves are therefore homotopic to $n_{i,l}\gamma^{i,l}$ in $T_\varepsilon\Gamma^i$, where each $n_{i,l}$ is a positive integer. Moreover, for each t sufficiently close to t_j consider the surface $\tilde{\Sigma}^j_t$ given by Proposition 4.3. The surface $\tilde{\Sigma}^j_t$ decomposes into the finite number of components (not necessarily connected) $\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i$. Each such surface is orientable and

$$\sum_i \mathbf{g}(\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i) \;=\; \mathbf{g}(\tilde{\Sigma}^j_t) \;\leq\; \mathbf{g}(\Sigma^j_t)\,. \tag{4.8}$$

We claim that

$$m_i \;\leq\; \liminf_{t\to t_j} \mathbf{g}(\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i)\,, \tag{4.9}$$

which clearly would conclude the proof.

Since Σ^j_t converges smoothly to Σ^j outside P_j, we conclude that $\tilde{\Sigma}^j_t \cap T_\varepsilon\Gamma^i$ converges smoothly to $\Sigma^j \cap T_\varepsilon\Gamma^i$ outside P_j. Since each $\gamma^{i,l}$ does not intersect P_j, it follows that, for t large enough, there exist curves $\hat{\gamma}^{i,l}$ contained in $\tilde{\Sigma}^j_t \cap T_\varepsilon\Gamma^i$ and homotopic to $\tilde{\gamma}^{i,l}$ in $T_\varepsilon\Gamma^i$.

Summarizing:

(i) Each $\hat{\gamma}^{i,l}$ is homotopic to $n_{i,l}\gamma^{i,l}$ in $T_{2\varepsilon}\Gamma^i$ for some positive integer $n_{i,l}$;
(ii) Each $\hat{\gamma}^{i,l}$ is contained in $\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i$;
(iii) $\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i$ is a closed surface;
(iv) If $c_1\gamma^{i,1} + \ldots + c_{2m_i}\gamma^{i,2m_i}$ is homologically trivial in $T_{2\varepsilon}\Gamma^i$ and the c_l's are integers, then they are all 0.

These statements imply that:

(Hom") If $c_1\hat{\gamma}^{i,1} + \ldots + c_{2m_i}\hat{\gamma}^{i,2m_i}$ is homologically trivial in $\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i$ and the c_l's are integers, then they are all 0.

From Lemma 4.4, we conclude again that $\mathbf{g}(\tilde{\Sigma}^j_t \cap T_{2\varepsilon}\Gamma^i) \geq m_i$. □

4.2. Proof of Proposition 4.2

4.2.1. Two lemmas. We state and prove two useful lemmas. We will use the notation of Definition 2.1.

Lemma 4.5. *Let Σ_j be $1/j$–a.m. in annuli and $r : M \to \mathbf{R}^+$ be the function of Theorem 3.6. Assume U is an open set with closure contained in* $\mathrm{An}(x,\tau,\sigma)$*, where $\sigma < r(x)$. Let $\psi_j \in \mathfrak{Js}_j(\Sigma_j, U)$ be such that $\mathcal{H}^2(\psi_j(1,\Sigma_j)) \leq \mathcal{H}^2(\Sigma)$. Then $\psi_j(1,\Sigma_j)$ is $1/j$–a.m. in sufficiently small annuli.*

Proof. Recall the definition of $1/j$–a.m. in sufficiently small annuli. This means that there is a function $r : M \to \mathbf{R}^+$ such that Σ is $1/j$–a.m. on every annulus centered at y and with outer radius smaller than $r(y)$. Let $\mathrm{An}(x,\tau,\sigma)$ be an annulus on which Σ is $1/j$–a.m. and $U \subset\subset \mathrm{An}(x,\tau,\sigma)$. If $y \notin B_\sigma(x)$, then $\mathrm{dist}(y,U) > 0$. Set $r_1(y) := \min\{r(y), \mathrm{dist}(y,U)\}$. Then $\psi(1,\Sigma) = \Sigma$ on every annulus with center y and radius smaller than $r_1(y)$, and therefore it is $1/j$–a.m. in it. If $y = x$, then the statement is obvious because of Remark 2.4. If $y \in B_\sigma(x) \setminus \{x\}$, then there exists $\rho(y), \tau(y)$ such that $U \cup B_{\rho(y)}(y) \subset \mathrm{An}(x, \tau(y), \sigma)$. By Remarks 2.4 and 2.3, $\psi(1,\Sigma)$ is $1/j$–a.m. on every annulus centered at y and outer radius smaller than $\rho(y)$. □

Lemma 4.6. *Let $\{\Sigma^j\}$ be a sequence as in Theorem 3.6 and U and ψ_j be as in Lemma 4.5. Assume moreover that U is contained in a convex set W. If Σ^j converges to a varifold V, then $\psi_j(1,\Sigma^j)$ converges as well to V.*

Proof. By Theorem 3.6 V is a smooth minimal surface (multiplicity allowed). By Lemma 4.5, $\psi_j(1,\Sigma^j)$ is also $1/j$–a.m. and again by Theorem 3.6 a subsequence (not relabeled) converges to a varifold V' which is a smooth minimal surface. Since $\Sigma^j = \psi_j(1,\Sigma^j)$ outside W, $V = V'$ outside W. Being W convex, it cannot contain any closed minimal surface, and hence by standard unique continuation, $V = V'$ in W as well. □

4.2.2. Step 1. Preliminaries. Let $\{\Sigma^j\}$ be a sequence as in Theorem 4.1. We keep the convention that Γ denotes the union of disjoint closed connected embedded minimal surfaces Γ^i (with multiplicity 1) and that Σ^j converges, in the sense of varifolds, to $V = \sum_i n_i \Gamma^i$. Finally, we fix a curve γ contained in Γ.

Let $r : \Gamma \to \mathbf{R}^+$ be such that the three conclusions of Proposition 1.8 hold. Consider a finite covering $\{B_{\rho_l}(x_l)\}$ of M with $\rho_l < r(x_l)$ and denote by C the set of the centers $\{x_l\}$. Next, up to extraction of subsequences, we assume that the set of singular points $P_j \subset \Sigma^j$ converges in the sense of Hausdorff to a finite set P (recall Remark 0.2) and we denote by E the union of C and P. Recalling Remark 2.3, for each $x \in M \setminus E$ there exists a ball B centered at x such that:

- $\Sigma^j \cap B$ is a smooth surface for j large enough;
- Σ^j is $1/j$–a.m. in B for j large enough.

Deform γ to a smooth curve contained in $\Gamma \setminus E$ and homotopic to γ in Γ. It suffices to prove the claim of the Proposition for the new curve. By abuse of notation we continue to denote it by γ. In what follows, we let ρ_0 be any given positive number so small that:

- $T_{\rho_0}(\Gamma)$ can be retracted on Γ;
- For every $x \in \Gamma$, $B_{\rho_0}(x) \cap \Gamma$ is a disk with diameter smaller than the injectivity radius of Γ.

For any positive $\rho \leq 2\rho_0$ sufficiently small, we can find a finite set of points $x_1, \ldots, x_N$ on γ with the following properties (to avoid cumbersome notation we will use the convention $x_{N+1} = x_1$):

(C1) If we let $[x_k, x_{k+1}]$ be the geodesic segment on Γ connecting x_k and x_{k+1}, then γ is homotopic to $\sum_k [x_k, x_{k+1}]$.
(C2) $B_\rho(x_{k+1}) \cap B_\rho(x_k) = \emptyset$;
(C3) $B_\rho(x_k) \cup B_\rho(x_{k+1})$ is contained in a ball $B^{k,k+1}$ of radius 3ρ;
(C4) In any ball $B^{k,k+1}$, Σ^j is $1/j$–a.m. and smooth provided j is large enough;

see Figure 1. From now on we will consider j so large that (C4) holds for every k. The constant ρ will be chosen (very small, but independent of j) only at the end of the proof. The existence of the points x_k is guaranteed by a simple compactness argument if ρ_0 is a sufficiently small number.

4.2.3. Step 2. Leaves. In every $B_\rho(x_k)$ consider a minimizing sequence $\Sigma^{j,l} := \psi_l(1, \Sigma^j)$ for Problem $(\Sigma^j, \mathfrak{Is}_j(B_\rho(x_k), \Sigma^j))$. Using Proposition 2.2, extract a subsequence converging (in $B_\rho(x_k)$) to a smooth minimal surface $\Gamma^{j,k}$ with boundary $\partial\Gamma^{j,k} = \Sigma^j \cap B_\rho(x_k)$. This is a stable minimal surface, and we claim that, as $j \uparrow \infty$, $\Gamma^{j,k}$ converges smoothly on every ball $B_{(1-\theta)\rho}(x_k)$ (with $\theta < 1$) to V. Indeed, this is a consequence of Schoen's curvature estimates, see Section 1.5.

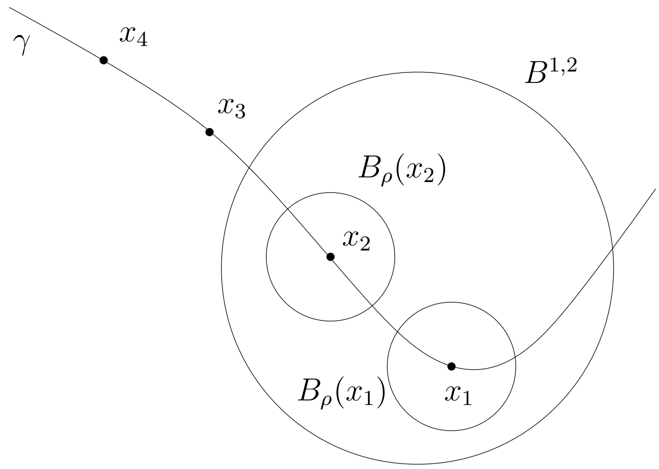

FIGURE 1. The points x_l of (C1)-(C4).

By a diagonal argument, if $\{l_j\}$ grows sufficiently fast, $\Sigma^{j,l_j} \cap B_\rho(x_k)$ has the same limit as $\Gamma^{j,k}$. On the other hand, for $\{l_j\}$ growing sufficiently fast, Lemmas 4.5 and 4.6 apply, giving that Σ^{j,l_j} converges to V.

Therefore, $\Gamma^{j,k}$ converges smoothly to $n_i\Gamma^i \cap B_{(1-\theta)\rho}(x_k)$ in $B_{(1-\theta)\rho}(x_k)$ for every positive $\theta < 1$. Therefore any connected component of $\Gamma^{j,k} \cap B_{(1-\theta)\rho}(x_k)$ is eventually (for large j's) a disk (multiplicity allowed). The area of such a disk is, by the monotonicty formula for minimal surfaces, at least $c(1-\theta)^2\rho^2$, where c is a constant depending only on M. From now on we consider θ fixed, though its choice will be specified later.

Up to extraction of subsequences, we can assume that for each connected component $\hat{\Sigma}^j$ of Σ^j, $\psi_l(1, \hat{\Sigma}^j)$ converges to a finite union of connected components of $\Gamma^{j,k}$. However, in $B_{(1-\theta)\rho}(x_k)$,

- either their limit is zero;
- or the area of $\psi_l(1, \hat{\Sigma}^j)$ in $B_{(1-\theta)\rho}(x_k)$ is larger than $c(1-2\theta)^2\rho^2$ for l large enough.

We repeat this argument for every k. Therefore, for any j sufficiently large, we define the set $\mathcal{L}(j,k)$ whose elements are those connected components $\hat{\Sigma}^j$ of $\Sigma^j \cap B_\rho(x_k)$ such that $\psi_l(1, \hat{\Sigma}^j)$ intersected with $B_{(1-\theta)\rho}(x_k)$ has area at least $c(1-2\theta)^2\rho^2$.

Recall that Σ^j is converging to $n_i\Gamma^i \cap B_\rho(x_k)$ in $B_\rho(x_k)$ in the sense of varifolds. Therefore, the area of Σ^j is very close to $n_i\mathcal{H}^2(\Gamma^i \cap B_\rho(x_k))$. On the other hand, by definition $\mathcal{H}^2(\psi_l(1, \Sigma^j) \cap B_\rho(x_k))$ is not larger. This gives a bound to the cardinality of $\mathcal{L}(j,k)$, independent of j and

k. Moreover, if ρ and θ are sufficiently small. the constants c and ε get so close, respectively, to 1 and 0 that the cardinality of $\mathcal{L}(j,k)$ can be at most n_i.

4.2.4. Step 3. Continuation of the leaves. We claim the following

LEMMA 4.7 (Continuation of the leaves). *If ρ is sufficiently small, then for every j sufficiently large and for every element Λ of $\mathcal{L}(j,k)$ there is an element $\tilde{\Lambda}$ of $\mathcal{L}(j,k+1)$ such that Λ and $\tilde{\Lambda}$ are contained in the same connected component of $\Sigma^j \cap B^{k,k+1}$.*

The lemma is sufficient to conclude the proof of the Theorem. Indeed let $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_k\}$ be the elements of $\mathcal{L}(j,1)$. Choose a point y_1 on Λ_1 and then a point y_2 lying on an element $\tilde{\Lambda}$ of $\mathcal{L}(j,2)$ such that $\Lambda_1 \cup \tilde{\Lambda}$ is contained in a connected component of $\Sigma^j \cap B^{1,2}$. We proceed by induction and after N steps we get a point y_{N+1} in some Λ_k. After repeating at most $n_i + 1$ times this procedure, we find two points y_{lN+1} and y_{rN+1} belonging to the same Λ_s. Without loss of generality we discard the first lN points and renumber the remaining ones so that we start with y_1 and end with $y_{nN+1} = y_1$. Note that $n \leq n_i$. Each pair y_k, y_{k+1} can be joined with a path $\gamma_{k,k+1}$ lying on Σ^j and contained in a ball of radius 3ρ, and the same can be done with a path $\gamma_{nN+1,1}$ joining y_{nN+1} and y_1. Thus, if we let

$$\tilde{\gamma} = \sum_k \gamma_{k,k+1} + \gamma_{nN+1,1}$$

we get a closed curve contained in Σ^j.

It is easy to show that the curve $\tilde{\gamma}$ is homotopic to $n\gamma$ in $\cup_k B^{k,k+1}$. Indeed, for each $sN + r$ fix a path $\eta^{sN+r} : [0,1] \to B_\rho(x_r)$ with $\eta^{sN+r}(0) = y_{sN+r}$ and $\eta^{sN+r}(1) = x_r$. Next fix an homotopy $\zeta^{sN+r} : [0,1] \times [0,1] \to B^{k,k+1}$ with

- $\zeta^{sN+r}(0,\cdot) = \gamma_{sN+r,sN+r+1}$,
- $\zeta^{sN+r}(1,\cdot) = [x_r, x_{r+1}]$,
- $\zeta^{sN+r}(\cdot,0) = \eta^{iN+r}(\cdot)$
- and $\zeta^{sN+r}(\cdot,1) = \eta^{sN+r+1}(\cdot)$.

Joyning the ζ^k's we easily achieve an homotopy between γ and $\tilde{\gamma}$. See Figure 2. If ρ is chosen sufficiently small, then $\cup_k B^{k,k+1}$ is contained in a retractible tubular neighborhood of Γ and does not intersect E.

4.2.5. Step 4. Proof of the Continuation of the Leaves. Let us fix a ρ for which Lemma 4.7 does not hold. Our goal is to show that for ρ sufficiently small, this leads to a contradiction. Clearly, there is

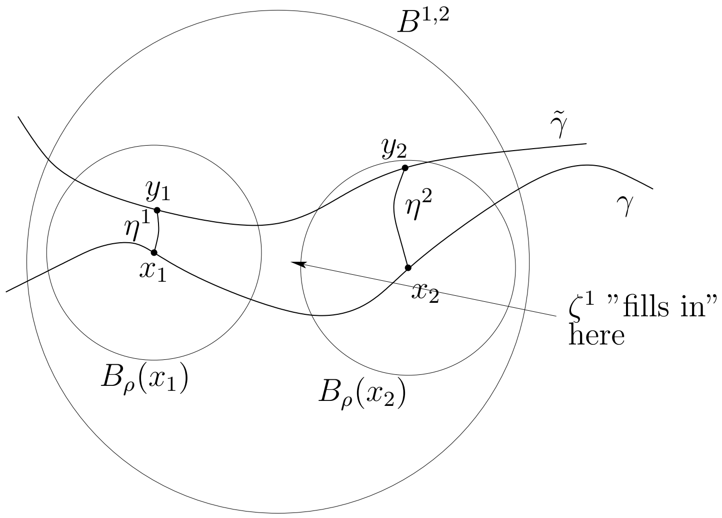

FIGURE 2. The homotopies ζ^{iN+r}.

an integer k and a subsequence $j_l \uparrow \infty$ such that the statement of the Lemma fails. Without loss of generality we can assume $k = 1$ and we set $x = x_1$, $y = x_2$ and $B^{1,2} = B$. Moreover, by a slight abuse of notation we keep labeling Σ^{j_l} as Σ^j.

Consider the minimizing sequence of isotopies $\{\psi_l\}$ for Problem $(\Sigma^j, \mathfrak{Is}_j(B_\rho(x), \Sigma^j))$ and $\{\phi_l\}$ for Problem $(\Sigma^j, \mathfrak{Is}_j(B_\rho(y), \Sigma^j))$ fixed in Step 3. Since $B_\rho(x) \cap B_\rho(y) = \emptyset$ and ψ_l and ϕ_l leave, respectively, $M \setminus B_\rho(y)$ and $M \setminus B_\rho(x)$ fixed, we can combine the two isotopies in

$$\Phi_l(t, z) \;:=\; \begin{cases} \psi_l(2t, z) & \text{for } t \in [0, 1/2] \\ \phi_l(2t-1, z) & \text{for } t \in [1/2, 1]. \end{cases}$$

If we consider $\Sigma^{j,l} = \Phi_l(1, \Sigma^j)$, then $\Sigma^{j,l} \cap B_\rho(x) = \psi_l(1, \Sigma^j) \cap B_\rho(x)$ and $\Sigma^{j,l} \cap B_\rho(y) = \phi_l(1, \Sigma^j) \cap B_\rho(y)$. Moreover for a sufficiently large l, the surface $\Sigma^{j,l}$ by Lemma 4.5 is $1/j$–a.m. in B and in sufficiently small annuli.

Arguing as in Step 2 (i.e. applying Theorem 3.6, Lemma 4.5 and Lemma 4.6), without loss of generality we can assume that:

(i) $\Sigma^{j,l}$ converges, as $l \uparrow \infty$, to smooth minimal surfaces Δ^j and Λ^j respectively in $B_\rho(x)$ and $B_\rho(y)$;
(ii) Δ^j and Λ^j converge, respectively, to $n_i\Gamma^i \cap B_\rho(x)$ and $n_i\Gamma^i \cap B_\rho(y)$;
(iii) For l_j growing sufficiently fast, Σ^{j,l_j} converges to the varifold $V = \sum_i n_i\Gamma^i$.

Let $\hat{\Sigma}^j$ be the connected component of $\Sigma^j \cap B_\rho(x)$ which contradicts Lemma 4.7. Denote by $\tilde{\Sigma}^j$ the connected component of $B \cap \Sigma^j$ containing $\hat{\Sigma}^j$.

Now, by Proposition 2.2, $\Phi_l(1, \tilde{\Sigma}^j) \cap B_\rho(x)$ converges to a stable minimal surface $\tilde{\Delta}^j \subset \Delta^j$ and $\Phi_l(1, \hat{\Sigma}^j)$ converges to a stable minimal surface $\hat{\Delta}^j \subset \tilde{\Delta}^j$. Because of (ii) and of curvature estimates (see Section 1.5), $\hat{\Delta}^j$ converges necessarily to $r\Gamma^i \cap B_\rho(x)$ for some integer $r \geq 0$. Since $\hat{\Sigma}^j \in \mathcal{L}(j, 1)$, it follows that $r \geq 1$. Similarly, $\Phi_l(1, \tilde{\Sigma}^j) \cap B_\rho(y)$ converges to a smooth minimal surface $\tilde{\Lambda}^j$ and $\tilde{\Lambda}^j$ converges to $s\Gamma^i \cap B_\rho(y)$ for some integer $s \geq 0$. Since $\tilde{\Sigma}^j$ does not contain any element of $\mathcal{L}(j, 2)$, it follows necessarily $s = 0$.

Consider now the varifold W which is the limit in B of $\tilde{\Sigma}^{j,l_j} = \Phi_{l_j}(1, \tilde{\Sigma}^j)$. Arguing again as in Step 2 we choose $\{l_j\}$ growing so fast that W, which is the limit of $\tilde{\Sigma}^{j,l_j}$, coincides with the limit of $\tilde{\Delta}^j$ in $B_\rho(x)$ and with the limit of $\tilde{\Lambda}_j$ in $B_\rho(y)$. According to the discussion above, V coincides then with $r\Gamma^i \cap B_\rho(x)$ in $B_\rho(x)$ and vanishes in $B_\rho(y)$. Moreover

$$\|W\| \;\leq\; \|V\| \llcorner B \;=\; n\mathcal{H}^2 \llcorner \Gamma^i \cap B \tag{4.10}$$

in the sense of varifolds. We recall here that $\|W\|$ and $\|V\| \llcorner B$ are nonnegative measures defined in the following way:

$$\int \varphi(x) d\|W\|(x) \;=\; \lim_{j\uparrow\infty} \int_{\tilde{\Sigma}^{j,l_j}} \varphi \tag{4.11}$$

and

$$\int \varphi(x) d\|V\|(x) \;=\; \lim_{j\uparrow\infty} \int_{\Sigma^{j,l_j}} \varphi \tag{4.12}$$

for every $\varphi \in C_c(B)$. Therefore (4.10) must be understood as a standard inequality between measures, which is an effect of (4.11), (4.12) and the inclusion $\tilde{\Sigma}^{j,l_j} \subset \Sigma^{j,l_j} \cap B$. An important consequence of (4.10) is that

$$\|W\|(\partial B_\tau(w)) \;=\; 0 \qquad \text{for every ball } B_\tau(w) \subset B. \tag{4.13}$$

Next, consider the geodesic segment $[x, y]$ joining x and y in Γ^i. For $z \in [x, y]$, $B_{\rho/2}(z) \subset B$. Moreover,

$$\text{the map} \quad z \;\mapsto\; \|W\|(B_{\rho/2}(z)) \quad \text{is continuous in } z, \tag{4.14}$$

because of (4.10) and (4.13).

Since $\|W\|(B_{\rho/2}(x)) \geq \mathcal{H}^2(\Gamma^i \cap B_{\rho/2}(x))$ and $\|W\|(B_{\rho/2}(y)) = 0$, by the continuity of the map in (4.14), there exists $z \in [x, y]$ such that

$$\|W\|(B_{\rho/2}(z)) \;=\; \frac{1}{2}\mathcal{H}^2(\Gamma^i \cap B_{\rho/2}(z))\,.$$

Since $\|W\|(\partial B_{\rho/2}(z)) = 0$, we conclude (see Proposition 1.62(b) of [**AFP00**]) that

$$\lim_{j\uparrow\infty} \mathcal{H}^2(\tilde{\Sigma}^{j,l_j} \cap B_{\rho/2}(z)) \;=\; \frac{1}{2}\mathcal{H}^2(\Gamma^i \cap B_{\rho/2}(z)) \tag{4.15}$$

(see Figure 3).

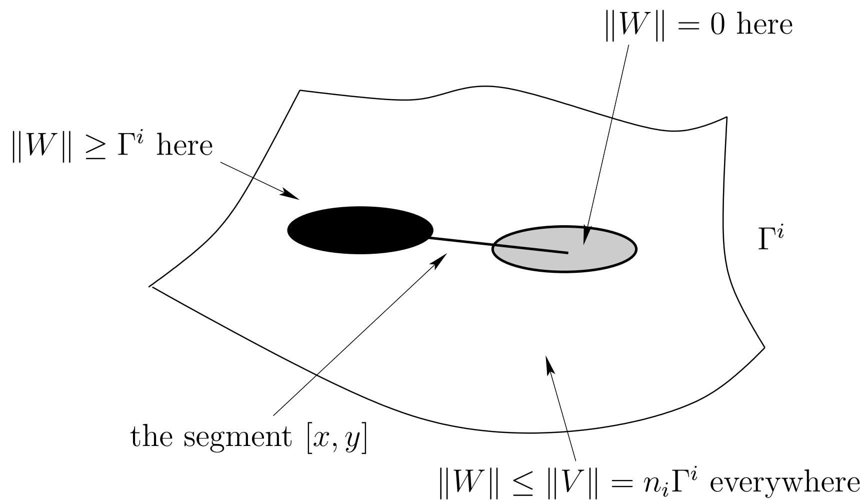

FIGURE 3. The varifold W.

On the other hand, since Σ^{j,l_j} converges to V in the sense of varifolds and $V = n_i\Gamma^i \cap B_{\rho/2}(z)$ in $B_{\rho/2}(z)$, we conclude that

$$\lim_{j\uparrow\infty} \mathcal{H}^2((\Sigma^{j,l_j} \setminus \tilde{\Sigma}^{j,l_j}) \cap B_{\rho/2}(z)) \;=\; \left(n_i - \frac{1}{2}\right)\mathcal{H}^2(\Gamma^i \cap B_{\rho/2}(z))\,. \tag{4.16}$$

If ρ is sufficiently small, $\Gamma^i \cap B_{\rho/2}(z)$ is close to a flat disk and $B_{\rho/2}(z)$ is close to a flat ball.

Using the coarea formula and Sard's lemma, we can find a $\sigma \in]0, \rho/2[$ and a subsequence of $\{\Sigma^{j,l_j}\}$ (not relabeled) with the following properties:

(a) Σ^{j,l_j} intersects $\partial B_\sigma(z)$ transversally;
(b) $\mathrm{Length}(\tilde{\Sigma}^{j,l_j} \cap \partial B_\sigma(z)) \leq 2(1/2 + \varepsilon)\pi\sigma$;
(c) $\mathrm{Length}((\Sigma^{j,l_j} \setminus \tilde{\Sigma}^{j,l_j}) \cap \partial B_\sigma(z)) \leq 2((n_i - 1/2) + \varepsilon)\pi\sigma$;
(d) $\mathcal{H}^2(\Gamma^i \cap B_\sigma(z)) \geq (1 - \varepsilon)\pi\sigma^2$.

Note that the geometric constant ε can be made as close to 0 as we want by choosing ρ sufficiently small.

In order to simplify the notation, set $\Omega^j = \Sigma^{j,l_j}$. Consider a minimizing sequence $\Omega^{j,s} = \varphi_s(1, \Omega^j)$ for Problem $(\Omega^j, \mathfrak{Is}_j(B_\sigma(z), \Omega^j))$. By Proposition 2.2, $\Omega^{j,s} \cap B_\sigma(z)$ converges, up to subsequences, to a minimal surface Ξ^j with boundary $\Omega^j \cap \partial B_\sigma(z)$. Moreover, using Lemma 4.6 and arguing as in the previous steps, we conclude that Ξ^j converges to $n_i \Gamma^i \cap B_\sigma(z)$.

Next, set:

- $\tilde{\Omega}^j = \tilde{\Sigma}^{j,l_j} \cap B_\sigma(z)$, $\tilde{\Omega}^{j,s} = \varphi_s(1, \tilde{\Omega}^j)$;
- $\hat{\Omega}^j = (\Sigma^{j,l_j} \setminus \tilde{\Sigma}^{j,l_j}) \cap B_\sigma(z)$, $\hat{\Omega}^{j,s} = \varphi_s(1, \hat{\Omega}^j)$.

By Proposition 2.2, since $\tilde{\Omega}^j$ and $\hat{\Omega}^j$ are unions of connected components of $\Omega^j \cap B_\sigma(z)$, we can assume that $\tilde{\Omega}^{j,s}$ and $\hat{\Omega}^{j,s}$ converge respectively to stable minimal surfaces $\tilde{\Xi}^j$ and $\hat{\Xi}^j$ with

$$\partial \tilde{\Xi}^j \;=\; \tilde{\Sigma}^{j,l_j} \cap \partial B_\sigma(z) \qquad \partial \hat{\Xi}^j \;=\; (\Sigma^{j,l_j} \setminus \tilde{\Sigma}^{j,l_j}) \cap \partial B_\sigma(z)\,.$$

Hence, by (b) and (c), we have

(4.17)

$$\text{Length}(\partial \tilde{\Xi}^j) \;\leq\; 2\left(\frac{1}{2} + \varepsilon\right)\pi\sigma \qquad \text{Length}(\partial \hat{\Xi}^j) \;\leq\; 2\left(n_i - \frac{1}{2} + \varepsilon\right)\pi\sigma\,.$$

On the other hand, using the standard monotonicity estimate of Lemma 4.8 below, we conclude that

$$\mathcal{H}^2(\hat{\Xi}^j) \;\leq\; \left(n_i - \frac{1}{2} + \eta\right)\pi\sigma^2 \tag{4.18}$$

$$\mathcal{H}^2(\tilde{\Xi}^j) \;\leq\; \left(\frac{1}{2} + \eta\right)\pi\sigma^2\,. \tag{4.19}$$

As the constant ε in (d), η as well can be made arbitrarily small by choosing ρ suitably small. We therefore choose ρ so small that

$$\mathcal{H}^2(\hat{\Xi}^j) \;\leq\; \left(n_i - \frac{3}{8}\right)\pi\sigma^2\,, \tag{4.20}$$

$$\mathcal{H}^2(\tilde{\Xi}^j) \;\leq\; \frac{5}{8}\pi\sigma^2 \tag{4.21}$$

and

$$\mathcal{H}^2(\Gamma^i \cap B_\sigma(z)) \;\geq\; \left(1 - \frac{1}{8n_i}\right)\pi\sigma^2\,. \tag{4.22}$$

Now, by curvature estimates (see Section 1.5), we can assume that the stable minimal surfaces $\tilde{\Xi}^j$ and $\hat{\Xi}^j$, are converging smoothly (on compact subsets of $B_\sigma(z)$) to stable minimal surfaces $\tilde{\Xi}$ and $\hat{\Xi}$. Since $\Xi^j = \tilde{\Xi}^j + \hat{\Xi}^j$ converges to $n_i\Gamma^i \cap B_\sigma(z)$, we conclude that $\tilde{\Xi} = \tilde{n}\Gamma^i \cap B_\sigma(z)$ and $\hat{\Xi} = \hat{n}\Gamma^i \cap B_\sigma(z)$, where $\tilde{n}$ and $\hat{n}$ are nonnegative integers with $\tilde{n} + \hat{n} = n_i$. On the other hand, by (4.20), (4.21) and (4.22), we conclude

$$\tilde{n}\left(1 - \frac{1}{8n_i}\right)\pi\sigma^2 \;\le\; \mathcal{H}^2(\tilde{\Xi}) \;\le\; \liminf_j \mathcal{H}^2(\tilde{\Xi}^j) \;\le\; \frac{5}{8}\pi\sigma^2 \tag{4.23}$$

$$\hat{n}\left(1 - \frac{1}{8n_i}\right)\pi\sigma^2 \;\le\; \mathcal{H}^2(\hat{\Xi}) \;\le\; \liminf_j \mathcal{H}^2(\hat{\Xi}^j) \;\le\; \left(n_i - \frac{3}{8}\right)\pi\sigma^2 \,. \tag{4.24}$$

From (4.23) and (4.24) we conclude, respectively, $\tilde{n} = 0$ and $\hat{n} \le n_i - 1$, which contradicts $\tilde{n} + \hat{n} = n_i$.

4.2.6. A simple estimate. The following lemma is a standard fact in the theory of minimal surfaces.

LEMMA 4.8. *There exist constants C and $r_0 > 0$ (depending only on M) such that*

$$\mathcal{H}^2(\Sigma) \;\le\; \left(\frac{1}{2} + C\sigma\right)\sigma \, \mathit{Length}\,(\partial\Sigma) \tag{4.25}$$

for any $\sigma < r_0$ and for any smooth minimal surface Σ with boundary $\partial\Sigma \subset \partial B_\sigma(z)$.

Indeed, (4.25) follows from the usual computations leading to the monotonicty formula. However, since we have not found a reference for (4.25) in the literature, we will give a proof of it.

PROOF. Let Σ be a smooth minimal surface with $\partial\Sigma \subset \partial B_\sigma(x)$, where $\sigma < r_0$ and r_0 is a positive constant to be chosen later. We recall that, for every vector field $X \in C^1_c(B_\sigma(x))$, we have

$$\int_{B_\sigma(x)} \mathrm{div}_\Sigma X \;=\; 0\,. \tag{4.26}$$

We assume $r_0 < \mathrm{Inj}\,(M)$ (the injectivity radius of M) and we use geodesic coordinates centered at x. For every $y \in B_\sigma(x)$ we denote by $r(y)$ the geodesic distance between y and x. Recall that r is Lipschitz on $B_\sigma(x)$ and C^∞ in $B_\sigma(x) \setminus \{x\}$, and that $|\nabla r| = 1$, where $|\nabla r| = \sqrt{g(\nabla r, \nabla r)}$.

We let $\gamma \in C^1([0,1])$ be a cut-off function, i.e. $\gamma = 0$ in a neighborhood of 1 and $\gamma = 1$ in a neighborhood of 0. We set $X = \gamma(r) r \nabla r =$

$\gamma(r)\nabla\frac{|r|^2}{2}$. Thus, $X \in C^\infty_c(B_\sigma(x))$ and from (4.26) we compute

$$0 \;=\; \int_\Sigma \gamma(r)\,\mathrm{div}_\Sigma\,(r\nabla r) + \int_\Sigma r\,\gamma'(r) \sum_i \partial_{e_i} r\, g(\nabla r, e_i)\,, \tag{4.27}$$

where $\{e_1, e_2\}$ is an orthonormal frame on $T\Sigma$. Clearly

$$\sum_i \partial_{e_i} r\, g(\nabla r, e_i) = \sum_i (\partial_{e_i} r)^2 = |\nabla_\Sigma r|^2 = |\nabla r|^2 - |\nabla^\perp r|^2 = 1 - |\nabla^\perp r|^2\,, \tag{4.28}$$

where $\nabla^\perp r$ denotes the projection of ∇r on the normal bundle to Σ. Moreover, let ∇^e be the euclidean connection in the geodesic coordinates and consider a 2-d plane π in T_yM, for $y \in B_\sigma(x)$. Then

$$\mathrm{div}_\pi\,(r(y)\,\nabla r(y)) - \mathrm{div}^e_\pi\,(|y|\,\nabla^e|y|) = O(|y|) = O(\sigma)\,.$$

Since $\mathrm{div}^e_\pi\,(|y|\,\nabla^e|y|) = 2$, we conclude the existence of a constant C such that

$$\left|\int_\Sigma \gamma(r)\mathrm{div}_\Sigma(r\nabla r) - 2\int_\Sigma \gamma(r)\right| \;\le\; C\|\gamma\|_\infty \sigma \mathcal{H}^2(\Sigma\cap B_\sigma(x))\,. \tag{4.29}$$

Inserting (4.28) and (4.29) in (4.27), we conclude

$$\int_\Sigma 2\,\gamma(r) + \int_\Sigma r\,\gamma'(r) \;=\; \int_\Sigma r\,\gamma'(r)\,|\nabla^\perp r|^2 + \mathrm{Err} \tag{4.30}$$

where, if we test with functions γ taking values in $[0,1]$, we have

$$|\mathrm{Err}| \le C\sigma\mathcal{H}^2(\Sigma\cap B_\sigma(x))\,. \tag{4.31}$$

We test now (4.30) with functions taking values in $[0,1]$ and approximating the characteristic functions of the interval $[0,\sigma]$. Following the computations of pages 83-84 of [**Sim83**], we conclude

$$\frac{d}{d\rho}\left(\rho^{-2}\mathcal{H}^2(\Sigma\cap B_\rho(x))\right)\Big|_{\rho=\sigma} \;=\; \frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho(x)} \frac{|\nabla^\perp r|^2}{r^2}\right)\Big|_{\rho=\sigma} + \sigma^{-3}\mathrm{Err}\,. \tag{4.32}$$

Straightforward computations lead to

$$\mathcal{H}^2(\Sigma\cap B_\sigma(x)) \;=\; \underbrace{\frac{\sigma}{2}\,\frac{d}{d\rho}\left(\mathcal{H}^2(\Sigma\cap B_\rho(x))\right)\Big|_{\rho=\sigma} - \frac{\sigma^3}{2}\,\frac{d}{d\rho}\left(\int_{\Sigma\cap B_\rho(x)} \frac{|\nabla^\perp r|^2}{r^2}\right)\Big|_{\rho=\sigma}}_{=(A)} + \mathrm{Err}\,. \tag{4.33}$$

Moreover, by the coarea formula, we have

$$(A) \;=\; \frac{\sigma}{2}\int_{\partial B_\sigma(x)\cap\Sigma}\frac{1}{|\nabla_\Sigma r|} - \frac{\sigma^3}{2}\int_{\partial B_\sigma(x)\cap\Sigma}\frac{|\nabla^\perp r|^2}{\sigma^2|\nabla_\Sigma r|} \;=\; \frac{\sigma}{2}\int_{\partial\Sigma}\frac{1-|\nabla^\perp r|^2}{|\nabla_\Sigma r|}$$

$$\frac{\sigma}{2}\int_{\partial\Sigma}|\nabla_\Sigma r| \;\leq\; \frac{\sigma}{2}\text{Length}\,(\partial\Sigma)\,. \tag{4.34}$$

Inserting (4.34) into (4.33), we conclude that

$$\mathcal{H}^2(\Sigma\cap B_\sigma(x)) \;\leq\; \frac{\sigma}{2}\text{Length}\,(\partial\Sigma) + |\text{Err}|\,, \tag{4.35}$$

which, taking into account (4.31), becomes

$$(1-C\sigma)\mathcal{H}^2(\Sigma\cap B_\sigma(x)) \;\leq\; \frac{\sigma}{2}\text{Length}\,(\partial\Sigma)\,. \tag{4.36}$$

So, for $r_0 < \min\{\text{Inj}\,(M), (2C)^{-1}\}$ we get (4.25). □

4.3. Considerations on (0.5) and (0.4)

4.3.1. Coverings. In this subsection we discuss why (0.5) seems ultimately the correct estimate. Fix a sequence $\{\Sigma^j_{t_j}\}$ which is $1/j$–a.m. in suffciently small annuli and assume for simplicity that each element is a smooth embedded surface and that the varifold limit is given by

$$\Gamma \;=\; \sum_{\Gamma^i\in\mathcal{O}} n_i\Gamma^i + \sum_{\Gamma^i\in\mathcal{N}} n_i\Gamma^i\,.$$

Then, one expects that, after appropriate surgeries (which can only bring the genus down) $\Sigma^j_{t_j}$ split into three groups.

- The first group consists of
$$m_1 \;=\; \sum_{\Gamma^i\in\mathcal{O}} n_i$$
surfaces, each isotopic to a $\Gamma^i\in\mathcal{O}$;
- The second group consists of
$$m_2 \;=\; \frac{1}{2}\sum_{\Gamma^i\in\mathcal{N}} n_i$$
surfaces, each isotopic to the boundary of a regular tubular neighborhood of $\Gamma^i\in\mathcal{N}$, (which is a double cover of Γ^i);
- The sum of the areas of the the third group vahishes as $j\uparrow\infty$.

As a consequence one would conclude that n_i is even whenever $\Gamma^i\in\mathcal{N}$ and that (0.5) holds.

The type of convergence described above is exactly the one proved by Meeks, Simon and Yau in [**MSY82**] for sequences of surfaces which are minimizing in a given isotopy class. The key ingredients of their proof is the γ–reduction and the techniques set forth by Almgren and Simon in [**AS79**] to discuss sequences of minimizing disks. However, in their situation there is a fundamental advantage: when the sequence $\{\Sigma^j\}$ is minimizing in a given isotopy class, one can perform the γ–reduction "globally", and conclude that, after a finite number of surgeries which do not increase the genus, there is a constant $\sigma > 0$ with the following property:

- For any ball B with radius σ, each curve in $\partial B \cap \Sigma^j$ bounds a small disk in Σ^j.

In the case of min–max sequences, their weak $1/j$–almost minimizing property on subsets of the ambient manifold allows to perform the γ–reduction only to surfaces which are appropriate local modifications of the Σ^j's, see the Squeezing Lemma of Section 2.2 and the modified γ–reduction of Section 2.2. Unfortunately, the size of the open sets where this can be done depends on j. In order to show that the picture above holds, it seems necessary to work directly in open sets of a fixed size.

4.3.2. An example. In this section we show that (0.4) cannot hold for sequences which are $1/j$–a.m.. Consider in particular the manifold $M =]-1,1[\times \mathbf{S}^2$ with the standard product metric. We parameterize $\mathbf{S}^2$ with $\{|\omega| = 1 : \omega \in \mathbf{R}^3\}$. Consider on M the orientation–preserving diffeomorphism $\varphi : (t, \omega) \mapsto (-t, -\omega)$ and the equivalence relation $x \sim y$ if $x = y$ or $x = \varphi(y)$. Let $N = M/_\sim$ be the quotient manifold, which is an oriented Riemannian manifold, and consider the projection $\pi : M \to N$, which is a local isometry. Clearly, $\Gamma := \pi(\{1\} \times \mathbf{S}^2)$ is an embedded 2–dimensional (real) projective plane. Consider a sequence $t_j \downarrow 1$. Then, each $\Lambda^j := \{t_j\} \times \mathbf{S}^2$ is a totally geodesic surface in M and, therefore, $\Sigma^j = \pi(\Lambda_j)$ is totally geodesic as well. Let r be the injectivity radius of N and consider a smooth open set $U \subset N$ with diameter smaller than r such that ∂U intersects Σ^j transversally. Then $\Sigma^j \cap U$ is the unique area–minimizing surface spanning $\partial U \cap \Sigma^j$.

Hence, the sequence of surfaces $\{\Sigma^j\}$ is $1/j$–a.m. in sufficiently small annuli of N. Each Σ^j is a smooth embedded minimal sphere and Σ^j converges, in the sense of varifolds, to 2Γ. Since $\mathbf{g}(\Sigma^j) = 0$ and $\mathbf{g}(\Gamma) = 1$, the inequality

$$\mathbf{g}(\Gamma) \ \leq \ \liminf_{j\uparrow\infty} \mathbf{g}(\Sigma^j)\,,$$

which corresponds to (0.4), fails in this case.

Table of symbols

T_xM	the tangent space of M at x
TM	the tangent bundle of M.
$\mathrm{Inj}\,(M)$	the injectivity radius of M.
$\mathcal{H}^2$	the 2–d Hausdorff measure in the metric space (M,d).
$\mathcal{H}^2_e$	the 2–d Hausdorff measure in the euclidean space $\mathbf{R}^3$.
$B_\rho(x)$	open ball
$\overline{B}_\rho(x)$	closed ball
$\partial B_\rho(x)$	distance sphere of radius ρ in M.
$\mathrm{diam}(G)$	diameter of a subset $G \subset M$.
$d(G_1, G_2)$	the Hausdorff distance between the subsets G_1 and G_2 of M.
$\mathcal{D}$, $\mathcal{D}_\rho$	the unit disk and the disk of radius ρ in $\mathbf{R}^2$.
$\mathcal{B}$, $\mathcal{B}_\rho$	the unit ball and the ball of radius ρ in $\mathbf{R}^3$.
$\exp_x$	the exponential map in M at $x \in M$.
$\mathfrak{Is}(U)$	smooth isotopies which leave $M \setminus U$ fixed.
$G^2(U)$, $G(U)$	grassmannian of (unoriented) 2–planes on $U \subset M$.
$\mathrm{An}(x,\tau,t)$	the open annulus $B_t(x) \setminus \overline{B}_\tau(x)$.
$\mathcal{AN}_r(x)$	the set $\{\mathrm{An}(x,\tau,t)$ where $0 < \tau < t < r\}$.
$C^\infty(X,Y)$	smooth maps from X to Y.
$C^\infty_c(X,Y)$	smooth maps with compact support from X to the vector space Y.

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