Uniformly Perfect Boundaries of Gromov Hyperbolic Spaces

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ABSTRACT. For a Gromov hyperbolic space X there exists a boundary at infinity $\partial_\infty X$. This boundary is equipped in a natural way with a quasi-metric with respect to a base point $o \in X$.

Uniformly perfectness is a weaker condition than connectedness, but the two properties belong together.

Let X be a geodesic, Gromov hyperbolic Space. In this thesis we show that there exists a quasi-isometric invariant criterion for the uniformly perfectness of $\partial_\infty X$ that can be applied to X.

In the second part we proof that the property for a space to be uniformly perfect is invariant under a generalized involution.

ZUSAMMENFASSUNG. Zu einem Gromov-Hyperbolischen Raum X existiert der Rand im Unendlichen $\partial_\infty X$. Diesem Rand können wir auf natürliche Art eine Quasimetrik in Abhängigkeit von einem Fusspunkt $o \in X$ zuordnen.

Uniform perfekt zu sein ist schwächer als Zusammenhang, aber die beiden Eigenschaften gehören zusammen.

Sei X ein geodätischer Gromov-hyperbolischer Raum. In dieser Arbeit wird zuerst gezeigt, dass es ein quasi-Isometrie invariantes Kriterium für X gibt, an welchem wir erkennen, ob $\partial_\infty X$ uniform perfekt ist oder nicht.

Im zweiten Teil wird bewiesen, dass die Eigenschaft für einen Raum uniform perfekt zu sein invariant ist unter einer verallgemeinerten Involution.

Contents

1 Introduction

This thesis investigates some relations between Gromov hyperbolic spaces X and their boundaries at infinity $\partial_\infty X$. A geodesic Gromov hyperbolic space is - roughly spoken - a space where triangles are thin. The boundary at infinity is defined over equivalence classes of sequences.

The attempt to understand the process from the inner to the boundary of a Gromov hyperbolic space leads to numerous observations, many of which are collected in the book [BS07] by Sergei Buyalo and Viktor Schroeder. There they state that $\partial_\infty X$ is equipped in a natural way with a quasi-metric with respect to a base point $o \in X$. For a space to be quasi-metric, it means that in every triangle the two larger sides have the same length up to a factor K. More precisely:

Definition 2.3 A *quasi-metric space* is a set Z, consisting of at least two points, equipped with a function $\rho : Z \times Z \to [0, \infty]$, which satisfies the conditions

(1) $\rho(P, Q) \geq 0$ for every P, $Q \in Z$ and $\rho(P, Q) = 0$ if and only if $P = Q$;

(2) $\rho(P, Q) = \rho(Q, P)$ for every P, $Q \in Z$;

(3) $\rho(P, R) \leq K \max\{\rho(P, Q), \rho(Q, R)\}$ for every P, Q, $R \in Z$ and some fixed $\infty > K \geq 1$.

(4) There exists at most one infinitely remote point $\omega \in Z$. Thus, there exists a set $Z_\infty \subset Z$ with cardinality $|Z_\infty| \leq 1$, such that ρ restricted to $Z \setminus Z_\infty \times Z \setminus Z_\infty$ is finite.

Further [BS07] states that one can find a metric (visual metric, see Definition 3.11) on $\partial_\infty X$ which behaves like the quasi-metric up to some multiplicative constants. It is also of great interest to observe what happens if two Gromov hyperbolic spaces have a map between their boundaries. The main result is the following

Theorem 5.9 *Let X be a visual and X' a geodesic hyperbolic space. Assume that there is a bilipschitz embedding $f : (\partial_\infty X, d) \to (\partial_\infty X', d')$ where d, d' are visual metrics with respect to base points $o \in X$, $o' \in X'$ and the same parameter a. Then there exists a roughly isometric map $F : X \to X'$ such that $f = \partial_\infty F$.*

In some places in [BS07] they are forced to claim that a space is uniformly perfect, which is rather surprising at first sight. Uniformly perfectness is a weaker condition than (pathwise) connectedness but, in many contexts, to be uniformly perfect is strong enough to maintain some specific structures.

Definition 4.1 A quasi-metric space is called *uniformly perfect*, if there is a constant $\mu \in (0,1)$, such that for each $P \in Z \backslash Z_\infty$ and every $r > 0$ for which the set $Z \backslash B_r(P)$ is nonempty, we have that $B_r(P) \backslash B_{\mu\, r}(P)$ is nonempty.

A connected space is uniformly perfect, a space with isolated points is not uniformly perfect.

Buyalo and Schroeder introduce a procedure to construct a Gromov hyperbolic space for a given metric space Z, the hyperbolic approximation (Definition 3.1). This space is a Gromov hyperbolic, connected graph. In [BS07] we see that the boundary of an approximation of a complete space Z and Z are pointwise identical. If Z is bounded, we obtain a graph with a infinitely long ray which contains no information about the space. We can cut this off and obtain a truncated hyperbolic approximation. It turns out that the boundary of such a approximation is bounded and complete.

In this context three questions arise:

- Consider a graph of a hyperbolic approximation. Is it possible to find out whether the boundary of the graph is uniformly perfect or not?
- Is there a quasi-metric invariant criterion for the uniformly perfectness of the boundary that can be applied to geodesic, Gromov hyperbolic spaces?
- Is the property for a space to be uniformly perfect invariant under involutions?

This thesis...

...answers all these questions with “yes”. The key to answer the first question is the observation that a geodesic ray in the graph of the hyperbolic approximation without any branching leads to an isolated point in the boundary. Thus, the following lemma contains the criterion we are looking for:

Lemma 4.2 *The boundary $\partial_\infty X$ of a truncated hyperbolic approximation is uniformly perfect if and only if there exists a $N \in \mathbb{N}$ such that in any closed interval of length N on any equivalence class of geodesic rays $[\xi]$ in X lies at least one fork.*

A fork (Definitions 3.26, 3.27) is the spot, where two geodesic rays or equivalence classes of rays separate.

The wish for a more general approach forces us to leave the geodesic rays in a graph and to find a more flexible tool: for a graph to have no arbitrary long, forkless pieces on geodesic rays is equivalent to the claim that there exist large equilateral triangles in every large ball. This is expressible in terms of the Gromov product (Definition 2.1) and leads to the following Definition where the transfer from graphs to general metric spaces is achieved:

Definition 5.1 Let X be a unbounded metric space. We call X *uniformly equilateral,* if there exist two numbers $S_0 > 0$, $\lambda > 0$, such that for every $w \in X$ and every $S \geq S_0$ the ball $B_S(w) \subset X$ contains three points x, y, z with

$$(x|y)_z, (y|z)_x, (x|z)_y \geq \lambda S.$$

To be uniformly equilateral is a quasi-isometric invariant for geodesic spaces (Theorem 5.5). Further, to be uniformly equilateral is the property a Gromov hyperbolic space must fulfill if the boundary is uniformly perfect:

Theorem 5.11 *Let X be a geodesic, visual, Gromov hyperbolic space. Then:*

$$X \textit{ is uniformly equilateral} \Leftrightarrow \partial_\infty X \textit{ is uniformly perfect.}$$

In classical terms, a involution at $0 \in \mathbb{R}^n$ is a map $I : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ where $I(x) = \frac{x}{\|x\|^2} = \frac{x}{|0x|\,|0x|}$. Now consider the metric ρ_0 on $\mathbb{R}^n$ which makes I an isometriy: $\rho_0(x, y) := \frac{|xy|}{|0x|\,|0y|}$.

Answering the third question is equivalent to proofing Theorem 6.1., which states:

Theorem 7.1 *Let (Z, ρ) be a quasi-metric, uniformly perfect space and let $z \in Z$. Then the space (Z, ρ_z) is also quasi-metric and uniformly perfect.*

Overall, this thesis brings some light in the role of the uniformly perfectness in the geometry of Gromov hyperbolic spaces and their boundary. Finally, it follows that uniformly perfectness belongs to Möbius geometry as well as to Gromov hyperbolic geometry.

Danksagung

Allen die zum Gelingen dieser Dissertation beigetragen haben, möchte ich herzlich danken. Besonderen Dank gilt Professor Viktor Schroeder für das Thema und die Unterstützung während meiner Arbeit und Professor Sergei Buyalo für das Begutachten. Auch Julian Jordi und Anna Mätzener möchte ich für viele hilfreiche Diskussionen und das Gegenlesen danken.

2 Preliminaries

2.1 Gromov Hyperbolic Spaces

We work with Gromov hyperbolic spaces. This is a familiy of negativ curved spaces. For geodesic spaces, one can define such a space using a condition for every triangle or, in general, using a certain behavior of Gromov products.

Definition 2.1 (Gromov product)**.** Let X be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ set $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality, and it is called the *Gromov product* of x, x' with regard to the base point o.

Definition 2.2 (Gromov hyperbolic space, δ-triple)**.** A metric space X is called *Gromov hyperbolic* if it satisfies the δ-inequality:

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta \quad \forall\, x, y, z, o \in X.$$

In other words, $(x|y)_o$, $(x|z)_o$ and $(z|y)_o$ form a *δ-triple*, i.e. the two smaller of the three numbers have a difference smaller than δ.

2.2 Quasi-Metric Spaces

In this subsection we give an introduction to K-quasi-metric spaces.

Definition 2.3 (Quasi-metric space)**.** A *quasi-metric space* is a set Z, consisting of at least two points equipped with a function $\rho : Z \times Z \to [0, \infty]$, which satisfies the conditions

(1) $\rho(P, Q) \geq 0$ for every P, $Q \in Z$ and $\rho(P, Q) = 0$ if and only if $P = Q$;

(2) $\rho(P, Q) = \rho(Q, P)$ for every P, $Q \in Z$;

(3) $\rho(P, R) \leq K \max\{\rho(P, Q), \rho(Q, R)\}$ for every P, Q, $R \in Z$ and some fixed $\infty > K \geq 1$.

(4) There exists at most one infinitely remote point $\omega \in Z$. Thus there exists a set $Z_\infty \subset Z$ with cardinality $|Z_\infty| \leq 1$ such that ρ restricted to $Z \setminus Z_\infty \times Z \setminus Z_\infty$ is finite.

For a given value K, we call such a space a K-quasi-metric space. The function ρ is in that case called a *quasi-metric*, or, more specifically, a K-quasi-metric. Three positive reals L_1, L_2, L_3 are called a *multiplicative K-triple* if $L_i \leq K \max\{L_j, L_k\}$ where i, j, $k \in \{1, 2, 3\}$ and $i \neq j \neq k$.

If $K = 1$ we have an *ultra-metric*, and a metric space is 2-quasi-metric. The following two remarks are important:

Remark 2.4. Let $\{a, b, c\}$ be a multiplicative K-triple. If $Ka < b$ (or $Ka < c$), then a is the smallest number of $\{a, b, c\}$, since $b \leq K \max\{a, c\}$ holds, and we obtain $c > a$ (analogously for $Ka < c$).

Remark 2.5. Let $\{a, b, c\}$ be a multiplicative K-triple and a its smallest number. Then we obtain: $b \leq K\,c$, $c \leq K\,b \Rightarrow \frac{1}{K} \leq \frac{b}{c} \leq K$ and analogously $\frac{1}{K} \leq \frac{c}{b} \leq K$.

Frink showed in [FR37] that every K-quasi-metric space is metrizable if $K \leq 2$. The following Proposition 2.2.6 from [BS07] states, that the metrization can become independent of K:

Proposition 2.6. *Let ρ be a K-quasi-metric on a set Z. Then, there exists $\epsilon_0 > 0$ only depending on K, such that ρ^ϵ is bilipschitz equivalent to a metric for each $0 < \epsilon \leq \epsilon_0$. More precisely, there exists a metric d_ϵ on Z such that*

$$\frac{1}{2K^\epsilon}\rho^\epsilon(z, z') \leq d_\epsilon(z, z') \leq \rho^\epsilon(z, z').$$

Now, by taking a metric d_ϵ, Z is a topological space with a metric topology.

3 Hyperbolic Approximation of Metric Spaces

We introduce a procedure from [BS07] to construct a Gromov hyperbolic space for a given metric space Z. If you see from far this constructed space $HA(Z)$ seems to be a tree. But there are some loops along geodesic rays. So it is more precisely only a graph.

3.1 Construction of a Hyperbolic Approximation

In this subsection, i recall some definitions from [BS07]. A subset V of a metric space Z is called *a-separated*, $a > 0$, if $d(v, v') \geq a$ for each distinct pair v, $v' \in V$. Note that if V is maximal with this property, then the union $\cup_{v \in V} B_a(v)$ covers Z.

A *hyperbolic approximation* of a metric space Z is a graph X which is defined as follows. We fix a positive $r \leq 1/6$ which is called *parameter of* X. For every $k \in \mathbb{Z}$, let $V_k \subset Z$ be a maximal r^k-separated set. We associate with every $v \in V_k$ the ball $B(v) \subset Z$ of radius $r(v) := 2r^k$ centered at v. We consider the set $V = \cup_{k \in \mathbb{Z}} V_k$, or better the set of balls $B(v)$, $v \in V$, as the vertex set of a graph X. Vertices v, $v' \in V$ are connected by an edge if and only if they belong to the same level V_k, and the closed balls $\overline{B}(v) = \{z | \rho(z, v) \leq r\}$, $\overline{B}(v') = \{z | \rho(z, v') \leq r\}$ intersect, $\overline{B}(v) \cap \overline{B}(v') \neq \emptyset$, or they lie on neighboring levels V_k, V_{k+1} and the ball of the upper level, V_{k+1} is contained in the ball of the lower level, V_k.

Definition 3.1 (Level function). The function $\ell : V \to \mathbb{Z}$, $\ell(v) :=$ level of v, is called the *level function* of the hyperbolic approximation.

Remark 3.2. It follows directly from the definition that the set of vertices V connected by the edges from the graph of a hyperbolic approximation X is metric ($|xy|$ is the smallest number of edges between two vertices x and y), unbounded and pathwise connected. *From now on we denote by X the set of vertices connected by the edges equipped with the mentioned metric.*

Definition 3.3 (Geodesic ray). An edge $\overline{xy}$ in X is called *horizontal* if $\ell(x) = \ell(y)$. Other edges are called *radial.* A geodesic consisting of only radial edges is called a *radial geodesic.* For a radial geodesic ξ, ξ_m denotes the unique vertex of ξ on level m. A geodesic η which starts from a vertex η_n and which has a strictly to infinity growing level function along η is called a *geodesic ray.*

We state Lemma 6.2.6 and 6.2.10 from [BS07]:

Lemma 3.4. *Any vertices v, $v' \in V$ can be connected in X by a geodesic which contains at most one horizontal edge. If there is such an edge, then it lies on the lowest level of the geodesic.* □

Lemma 3.5. *A hyperbolic approximation of any metric space is a geodesic 2δ-hyperbolic space with $2\delta = 3$.* □

3.2 The Boundary at Infinity of a Hyperbolic Approximation

Definition 3.6 (Boundary at infinity $\partial_\infty X$)**.** Let X be a Gromov hyperbolic space and $o \in X$ a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity*, if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

This property is independent of the choice of o (Chapter 2.2. in [BS07]). Two sequences $\{x_i\}$ and $\{y_i\}$ that converge to infinity are *equivalent*, if

$$\lim_{i\to\infty} (x_i|y_i)_o = \infty.$$

This defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity. Hence by $[x] \in \partial_\infty X$ we denote a set of equivalent sequences $\{x_i\} \in X$.

Lemma 3.7. *Let $\{x_i\}$ be a sequence in X which converges to infinity and $\{x'_i\} \subset \{x_i\}$ a subsequence. Then, $\{x_i\}$ is equivalent to $\{x'_i\}$.*

Proof. $(x_i|x_j) \overset{i,j\to\infty}{\longrightarrow} \infty$ i.e. we can find a J for every C such that $(x_i|x_j) > C \,\forall\, i,j > J$. In particular $(x_i|x'_i) > C$ since $x'_i = x_{k(i)}$, $k(i) \geq i$. Hence $(x_i|x'_i) \overset{i\to\infty}{\longrightarrow} \infty$. □

Definition 3.8 (Gromov product on $\partial_\infty X$)**.** Fix a base point $o \in X$. For points $[x]$, $[x'] \in \partial_\infty X$, we define their *Gromov product on* $\partial_\infty X$ by

$$([x]|[x'])_o = \inf \liminf_{i\to\infty} (x_i|x'_i)_o$$

where the infimum is taken over all sequences $(x_i) \in [x]$, $(x'_i) \in [x']$. Note that $([x]|[x'])_o$ takes values in $[0, \infty]$ and that $([x]|[x'])_o = \infty$ if and only if $[x] = [x']$.

We state some facts from the sections 2.2.2 and 2.2.3 from [BS07]:

Lemma 3.9. *Let X be a Gromov hyperbolic space with the hyperbolicity constant δ. Fix $a > 1$ and consider the function $\rho : \partial_\infty X \times \partial_\infty X \to \mathbb{R}$, $\rho([x], [x']) = a^{-([x]|[x'])_o}$. Then, ρ is a K-quasi-metric on $\partial_\infty X$ with $K = a^\delta$.* □

Definition 3.10 (Quasi-metric boundary $\partial_\infty X$)**.** From now on, we denote by $\partial_\infty X$ the quasi-metric space $(\partial_\infty X, \rho)$.

Definition 3.11 (Visual metric)**.** A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X, a > 1$ and positive constants c_1, c_2, such that

$$c_1 a^{([x]|[x'])_o} \leq d([x], [x']) \leq c_2 a^{([x]|[x'])_o}$$

for all $[x]$, $[x'] \in \partial_\infty X$. We say that d is a visual metric with regard to the base point o and the parameter a.

We state Theorem 2.2.7. and some facts from sections 2.2.3 and 6.4.4 from [BS07]:

Theorem 3.12. *Let X be a hyperbolic space. Then for any $o \in X$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a metric d on $\partial_\infty X$, which is visual with regard to o and a.* □

We define the topology at infinity $\partial_\infty X$ for a hyperbolic space X as the metric topology for some visual metric on $\partial_\infty X$. This topology is independent of the choice of a visual metric.

Lemma 3.13. *The quasi-metric space $\partial_\infty X$ and $\partial_\infty X$ with a visual metric is bounded.*

Proof. Let $[x]$ and $[y]$ be two arbitrary points in $\partial_\infty X$. Then $([x]|[y])_o = \inf \liminf_{i \to \infty} (x_i|y_i)_o = \inf \liminf_{i \to \infty} \frac{1}{2}(|ox_i| + |oy_i| - |x_i y_i|) \geq 0$ since X is metric, i.e. the triangle inequality holds. It follows that $a^{-(x|y)_o} \leq 1$. With Definition 3.11 we obtain that $\partial_\infty X$ is also bounded with a visual metric. □

Now, we show that $\partial_\infty X$ is complete.

Lemma 3.14. *Let X be a Gromov hyperbolic space. Then $\partial_\infty X$ is complete with a visual metric.*

Proof. Let (ξ_n) be a Cauchy sequence in $\partial_\infty X$. We construct a limit of (ξ_n) trough a diagonal procedure.

Let $(\xi_n) \subset \partial_\infty X$ be a Cauchy sequence. By taking a subsequence if necessary, we may assume that for $i < j$: $(\xi_i|\xi_j) > i$: we find an i, such that $(\xi_i|\xi_j) > 1\ \forall j > i$. Rename ξ_i to ξ_1, define new indices according to ξ_1 and omit the beginning of the sequence. Now find i' such that $(\xi_{i'}|\xi_j) > 2\ \forall j > i'$. Rename $\xi_{i'}$ to ξ_2, define new indices indices according to ξ_2 and omit the elements between ξ_1 and ξ_2. By repeating this we obtain $(\xi_i|\xi_j) > i$ for $i < j$.

Then, since $(\xi_1|\xi_2) > 1$, we can find an m such that $(x_n^1|x_n^2) \geq 1\, \forall n \geq m$, where $(x_i^1|x_i^2)$ are any representatives of ξ_1, ξ_2. Define new representatives for ξ_1, ξ_2 by throwing away the terms $x_2^1, ..., x_{m-1}^1$ from ξ_1 and $x_2^2, ..., x_{m-1}^2$ from ξ_2, such that x_m^1 becomes x_2^1 and x_m^2 becomes x_2^2.

We then have:

$$(x_2^1|x_2^2) \geq 1 \text{ and } (x_n^1|x_m^1) \geq \min\{n, m\}, (x_n^2|x_m^2) \geq \min\{n, m\} \text{ is preserved.}$$

Then, proceed similarly to find representatives for ξ_1, ξ_2, ξ_3, such that $(x_3^1|x_3^3) \geq 1$, $(x_3^2|x_3^3) \geq 2$, while preserving $(x_j^i|x_k^i) \geq \min\{j, k\}$, $i = 1, 2, 3$. Iteratively, we find representatives for all ξ_i with $(x_\ell^k|x_\ell^\ell) \geq k$ if $k \leq \ell$ and $(x_j^i|x_k^i) \geq \min\{j, k\}$ and we define $\xi := (x_i^i)_i$.

This indeed represents a point in $\partial_\infty X$, since (without loss of generality $i \leq j$) $\{(x_i^i|x_j^j), (x_i^i|x_j^j), (x_j^i|x_j^j)\}$ is a δ-triple, and $(x_i^i|x_j^i) \xrightarrow{i,j} \infty$ and $(x_j^i|x_j^j) \xrightarrow{i,j} \infty$ hence $(x_i^i|x_j^j) \xrightarrow{i,j} \infty$, so $\xi \in \partial_\infty X$.

Also, $\xi_n \to \xi$ since $(x_k^n|x_k^k) \geq n \Rightarrow (\xi_n|\xi) \geq n$, i.e. $d(\xi_n, \xi) \xrightarrow{n} 0 \Rightarrow \partial_\infty X$ is complete with a visual metric. □

Definition 3.15 (Geodesic boundary)**.** Two geodesic rays γ, γ' in a geodesic space X are called *asymptotic* if $|\gamma(t)\gamma'(t)| \leq C < \infty$ for some constant C and all $t \geq a$. To be asymptotic is an equivalence relation on the set of rays in X, and the set of classes of asymptotic rays is sometimes called the *geodesic boundary* of X, $\partial^g X$.

Definition 3.16 (Truncated hyperbolic approximation)**.** Assume now that the metric space Z is bounded, diam $Z < \infty$, and nontrivial, i.e. it contains at least two points. Then the largest integer k with diam $Z < r^k$ exists, and we denote it by $k_0 = k_0($ diam $Z, r)$. Note that if $r < \min\{\text{diam } Z, 1/\text{diam } Z\}$ then $k_0 = 0$ (the case diam $Z < 1$) or $k_0 = -1$ (the case diam $Z \geq 1$).

Note that for every $k \leq k_0$ the vertex set V_k consists of one point, and therefore contains no essential information about Z. Thus we modify the graph X by setting $V_k = \emptyset$ for every $k < k_0$, and call the modified graph the *truncated hyperbolic approximation of Z*. Clearly, all properties of the hyperbolic approximation as discussed above hold as well for the truncated hyperbolic approximation.

Theorem 6.4.1 and Proposition 6.4.3 of [BS07] state:

Theorem 3.17. *Let X be a truncated hyperbolic approximation of a complete, bounded metric space Z. Then there is the canonical identification $\partial_\infty X = Z$ under which the metric d of Z is a visual metric on $\partial_\infty X$ with respect to the base point o of X and the parameter $a = \frac{1}{r}$.* □

Proposition 3.18. *Let X be a truncated hyperbolic approximation of a complete bounded space Z. Then $\partial^g X = \partial_\infty X$, and for any two radial rays $\gamma = o...v_k..., \gamma' = o...v'_k...(v_k, v'_k \in V_k)$ in X, representing the same point in $\partial_\infty X$, we have $|v_k v'_k| \leq 1$ for all $k \geq k_o$.* □

Remark 3.19. From now on, whenever we make a truncated hyperbolic approximation of a complete bounded space, we consider the equivalence classes of the geodesic rays as $\partial_\infty X$. In this case we denote a point from the border with $[\xi]$, where ξ is a geodesic ray in X.

Definition 3.20 (Gromov product for geodesic rays)**.** Let ξ, η be two geodesic rays in X. Then

$$(\xi|\eta)_o := \liminf_k (\xi_k|\eta_k)_o.$$

Remark 3.21. Together with Definition 3.8 we can now define in the case of a truncated hyperbolic approximation of a complete bounded space the Gromov Product on the boundary in the following way:

$$([\xi]|[\eta])_o := \inf_{\xi \in [\xi], \eta \in [\eta]} (\xi|\eta)_o$$

where $[\xi]$, $[\eta] \in \partial_\infty X$. Note that in general $(\xi|\eta)_o \neq ([\xi]|[\eta])_o$. We make a more precise statement in Lemma 3.28.

3.3 Properties of the Hyperbolic Approximation

In this subsection we describe what it looks like when two geodesic rays separate. This leads to the definition of the fork. Then we show the connection between a fork and the Gromov product of its geodesic rays.

Definition 3.22 (Fork for rays)**.** Let ξ, η be two geodesic rays. The *fork of* ξ *and* η, denoted by $\varphi(\xi, \eta)$, is the minimal level k where ξ and η hold $|\xi_k \eta_k| > 1$. If $\varphi(\xi, \eta) = \infty$, there is no fork between ξ and η.

Remark 3.23. A fork is not a vertex of the graph, it is a level.

Lemma 3.24. *There exist two different types of forks (see Figure 1).*

Proof. If k is a fork for ξ and η, we know that $|\xi_k \eta_k| > 1$ and $|\xi_m \eta_m| \leq 1$ for $m < k$. In particular, $|\xi_{k-1} \eta_{k-1}| \leq 1$, i.e. (i) $\xi_{k-1} = \eta_{k-1}$ or (ii) the vertices ξ_{k-1} and η_{k-1} are connected by a horizontal edge. □

Remark 3.25. If k is a fork for ξ and η of type (i), then $|\xi_k \eta_k| = 2$, otherwise $|\xi_k \eta_k| = 3$.

Lemma 3.26. *Let* ξ*,* η *be two radial geodesics in* X*, the graph of a hyperbolic approximation of a metric space* Z*. Then the following equivalence holds:*

$$\varphi(\xi, \eta) < \infty \Leftrightarrow \xi \textit{ is not equivalent to } \eta.$$

Proof. For the first direction assume that ξ is equivalent to η and hence $\lim_k \xi_k = z = \lim_k \eta_k$ and then $z \in \bar{B}(\xi_k) \cap \bar{B}(\eta_k)$ for all k.

For the other direction, we assume that there is no fork between η and μ. Without loss of generality we fix an arbitrary vertex on η and consider it as the base point o. Then it follows: $2\,(\xi_n|\eta_n)_o = |\xi_n o| + |\eta_n o| - |\xi_n \eta_n| > |\xi_n o| + |\eta_n o| - 2$. We obtain: $\lim_n (\xi_n|\eta_n)_o \to \infty$, which is the equivalence of ξ and η. □

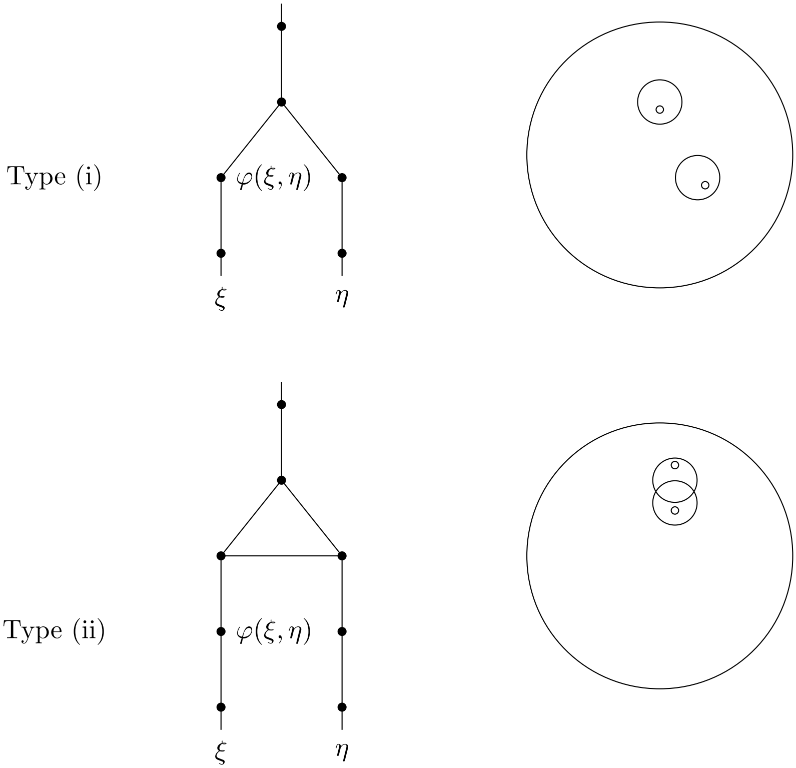

Figure 1: Two examples of forks of type (i) and (ii) between geodesic rays and the associated situations in the hyperbolic approximation.

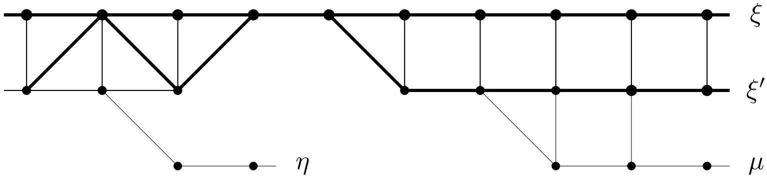

Figure 2: A sequence of a equivalence class $[\xi] = [\xi']$ of geodesic rays with two separating rays η and μ. Note that $\varphi(\xi, \mu) \neq \varphi(\xi', \mu)$ and $\varphi([\xi], [\mu]) = \varphi(\xi, \mu)$.

Now we define forks for equivalence classes of geodesic rays:

Definition 3.27 (Fork for boundary points)**.** Let $\partial^g X$ be the geodesic boundary of a Gromov hyperbolic space and $[\xi]$, $[\eta] \in \partial^g X$ two points of it. In this case the equivalence classes $[\xi]$ and $[\eta]$ contain geodesic rays. Then

$$\varphi([\xi], [\eta]) := \inf_{\xi \in [\xi], \eta \in [\eta]} \varphi(\xi, \eta)$$

is called the *fork for the boundary points* $[\xi]$ *and* $[\eta]$. Again, if $\varphi([\xi], [\eta]) = \infty$, there is no fork between $[\xi]$ and $[\eta]$. By a closed interval determined by two levels $m < n$ on $[\xi]$ we mean the set of intervals $\{[\xi_m ... \xi_n] | \xi \in [\xi]\}$. See Figure 2.

Now we investigate the connection between Gromov products and forks:

Lemma 3.28. *Let ξ, η be two geodesic rays in the graph of the hyperbolic approximation X and o a vertex on a ray in $[\xi]$ such that $\ell(o) < \varphi([\xi]|[\eta])_o$. Assume that $\varphi(\xi, \eta) = \varphi([\xi], [\eta]) =: F$. Then*

$$F - \ell(o) - \frac{3}{2} \leq ([\xi]|[\eta])_o \leq F - \ell(o) \quad \textit{and}$$

$$F - \ell(o) - \frac{3}{2} \leq (\xi|\eta)_o \leq F - \ell(o).$$

Proof. Let $i > \ell(o)$. Note that $\min_{\xi \in [\xi]} |o\,\xi_i| \geq i - \ell(o)$ and $\max_{\xi \in [\xi]} |o\,\xi_i| \leq i - \ell(o) + 1$. Further, since $\ell(o) < F$, i.e. $|\xi_j \eta_j| \leq 1$ for $j \leq F$ we get that $\min_{\eta \in [\eta]} |o\,\eta_i| \geq i - \ell(o)$ and $\max_{\eta \in [\eta]} |o\,\eta_i| \leq i - \ell(o) + 1$.

Consider $([\xi]|[\eta])_o = \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i \frac{1}{2}(|o\xi_i| + |o\eta_i| - |\xi_i\eta_i|)\,(*)$. We obtain

$$\begin{aligned}
(*) \quad &\leq \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i \frac{1}{2}(i - \ell(o) + 1 + i - \ell(o) + 1 - (2(i-F)+2)) && (1)\\
&= \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i \frac{1}{2}(2i - 2\ell(o) + 2 - 2i + 2F - 2) && (2)\\
&= \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i (F - \ell(o)) = F - \ell(o), && (3)
\end{aligned}$$

and similarly

$$\begin{aligned}
(*) \quad &\geq \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i \frac{1}{2}(i - \ell(o) + i - \ell(o) - (2(i-F)+3)) && (4)\\
&= \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i \frac{1}{2}(2i - 2\ell(o) - 2i + 2F - 3) && (5)\\
&= \inf_{\xi\in[\xi],\eta\in[\eta]} \liminf_i (F - \ell(o) - \frac{3}{2}) = F - \ell(o) - \frac{3}{2}. && (6)
\end{aligned}$$

Hence $([\xi]|[\eta])_o$ is, up to the additive constant $\frac{3}{2}$ and the fork, independent of the representants. Thus, the second claim is also true, since we have $\varphi(\xi, \eta) = \varphi([\xi], [\eta])$. □

4 Uniformly Perfect Spaces

In this section we discuss some relations between a hyperbolic space and its boundary at infinity.

4.1 Uniformly perfect spaces

In this subsection, we denote by X the graph of a hyperbolic approximation of the metric space Z. We proof that $\partial_\infty X$ is μ-uniformly perfect if and only if Z is μ-uniformly perfect.

Definition 4.1 (Uniformly perfect space). A quasi-metric space Z is called *uniformly perfect*, if there is a constant $\mu \in (0,1)$, such that for each $P \in Z \backslash Z_\infty$ and every $r > 0$ for which the set $Z \backslash B_r(P)$ is nonempty, we have that $B_r(P) \backslash B_{\mu\, r}(P)$ is nonempty.

For a given value μ, we call such a space a μ-uniformly perfect space.

Uniformly perfectness is a weaker condition than connectedness. Connected spaces are uniformly perfect, those with isolated points are not. Many disconnected fractals such as the Cantor ternary Set (Figure 4) are uniformly perfect.

Now, we want to determine, how one can decide whether $\partial_\infty X$ is uniformly perfect or not simply by applying the following Lemma to the the graph of a truncated hyperbolic approximation X.

Lemma 4.2. *The boundary $\partial_\infty X$ of a truncated hyperbolic approximation is uniformly perfect if and only if there exists a $N \in \mathbb{N}$ such that in any closed interval of length N on any equivalence class of geodesic rays $[\xi]$ in X lies at least one fork.*

Proof. For the first implication we show that the length of the longest interval without a fork in equivalence classes in X depends on μ, the constant of the μ-uniformly perfect space $\partial_\infty X$. It follows immidiately that, if we don't have a maximal N, μ converges to 0, and hence, $\partial_\infty X$ is not μ-uniformly perfect.

Since $\partial_\infty X$ is bounded, we set X as the graph of the truncated hyperbolic approximation, and the root as base point o.

Choose a equivalence class $[\xi] \subset X$ and a forkless interval with lenght $M > 0$ on $[\xi]$ between two forks $m := \varphi([\xi],[\eta])$ and $n := \varphi([\xi],[\nu])$. Without loss of generality we say that $m < n$. The interval starts on level $m+1$. Define $r := e^{-(\xi|\eta)_o} \in [0,1]$ and choose $\tilde{\mu} \in (0,1)$ maximal, such that the set $B_r([\xi]) \backslash B_{\tilde{\mu}\, r}([\xi])$ in $\partial_\infty X$ is nonempty. The equivalence class $[\nu]$ has the

Figure 3: This figure shows a uniformly perfect subset of $\mathbb{R}$.

property that $\varphi([\xi],[\nu]) = \varphi([\xi],[\eta]) + M + 2$. Consider $[\nu]$, $[\eta]$ and $[\xi]$ as points in $\partial_\infty X$, then $[\nu]$ is the next closest point after $[\eta]$ to $[\xi]$. It follows

$$\tilde{\mu}\, r \leq e^{-([\xi]|[\nu])_o}.$$

With Lemma 3.28 and $\varphi([\xi],[\nu]) = \varphi([\xi],[\eta]) + M + 2$, we obtain

$$\varphi([\xi],[\eta]) + M + 2 - \ell(o) - \frac{3}{2} \leq ([\xi]|[\nu])_o \leq \varphi([\xi],[\eta]) + M + 2 - \ell(o).$$

If there exists no maximal N, we can find a sequence of forkless intervals whose lenghts go to infinity, hence, since $\varphi([\xi],[\eta]) \geq \ell(o)$ (in particular $\varphi([\xi],[\eta])$ does not converge to $-\infty$), $([\xi]|[\nu])_o \to \infty$ thus $e^{-([\xi]|[\nu])_o} \to 0$ and we get that $\tilde{\mu} \to 0$.

On the other hand, assume there exists an N. Given is $[\xi] \in \partial_\infty X$ and a radius $r \in [0,1]$ such that $\partial_\infty X \setminus B_r([\xi]) \neq \emptyset$. Take a (existing) closest point $[\eta] \in \partial_\infty X$ to $[\xi]$ such that $e^{-([\xi]|[\eta])_o} \geq r$. By assumption, the next closest point after $[\eta]$ to $[\xi]$, $[\nu]$ has its fork with $[\xi]$ in the next N levels, i.e. $\varphi(\xi,\eta) + N \geq \varphi(\xi,\nu)$. So we have the following three facts:

$$r \leq e^{-(\xi|\eta)_o} \tag{1}$$

$$e^{-(\xi|\nu)_o} \leq r \tag{2}$$

$$(\xi|\eta)_o + N \geq (\xi|\nu)_o. \tag{3}$$

Set $\mu = e^{-N}$ and we obtain

$$\mu\, r = e^{-N}\, r \overset{(1)}{\leq} e^{-N}\, e^{-(\xi|\eta)_o} = e^{-((\xi|\eta)_o + N)} \overset{(3)}{\leq} e^{-(\xi|\nu)_o} \overset{(2)}{\leq} r.$$

In other words: $\mu r \leq \rho(\xi,\nu) \leq r$, and further $B_r([\xi]) \setminus B_{\mu r}([\xi])$ is nonempty. $\square$

Lemma 4.3. *Let Z be a complete bounded metric space, X the graph of its truncated hyperbolic approximation, and $\partial_\infty X$ the boundary of X at infinity. Then*

$$\partial_\infty X \text{ is uniformly perfect} \Leftrightarrow Z \text{ is uniformly perfect.}$$

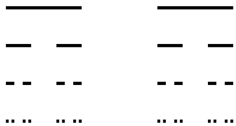

Figure 4: The first iteration steps of the origination of the Cantor ternary set.

Proof. For the first direction, we assume that Z is not uniformly perfect, i.e. for every μ we can find a point $P \in Z$ and a radius s such that $B_s(P) \backslash B_{\mu\, s}(P) = \emptyset$ where $Z \backslash B_s(P)$ is nonempty.

To apply Lemma 4.2, we show that there exist arbitrary long forkless intervals on equivalence classes of the graph X. A geodesic ray $\xi \in X$ has no fork on a certain level m, if the ball in Z belonging to the vertex ξ_{m-1} contains only one ball of level m (i.e. there exists no fork of type (i)) and if there exists no other ball of level $m-1$ with a nonempty intersection with the ball which belongs to ξ_{m-1} (i.e. there exists no fork of type (ii)).

Let r be the constant of the hyperbolic approximation. Choose an integer $N > 0$ arbitrary and μ smaller than $\frac{r^{N+1}}{2}$. Choose $P \in Z$ and $s \in \mathbb{R}$ such that $B_s(P) \backslash B_{\mu\, s}(P) = \emptyset$ and an integer k minimal such that more than one point of the r^k-separated net of the hyperbolic approximation lie in $B_{\mu\, s}(P)$.

By the choice of μ we have $2\, r^{-N-1} \mu s \leq s$ $(*)$ and by the choice of k: $r^k \leq 2\, \mu\, s$ $(**)$, otherwise there could not lie two points of the r^k-separated net in $B_{\mu s}(P)$.

For every $m \in \mathbb{N}$ such that $r^{k-1-m} < s$ we observe, that the ball from the hyperbolic approximation in Z in which P lies, contains only one ball of level $k-1-m+1$ and it does not intersect another ball of level $k-1-m$, since the r^{k-1-m}-separated net has only one point in $B_{\mu\, s}(P)$ and the next closest point from the net lies outside $B_s(P)$. Consider the following:

$$r^{k-1-N} = r^k\, r^{-1-N} \overset{(**)}{\leq} r^{-1-N}\, 2\, \mu\, s = 2\, r^{-N-1}\, \mu\, s \overset{(*)}{\leq} s,$$

i.e. for every level $i \in \{k-1-N, ..., k-2\}$ there is no fork on the sequence of a geodesic ray which we obtain with the balls containing P and the equivalence class of this ray consists in this section only of one geodesic. Since we may choose N arbitrary large, the claim follows with Lemma 4.2.

For the other direction, we assume that Z is μ-uniformly perfect. Let r be the parameter of the hyperbolic approximation. Take an arbitrary ball of an arbitrary level $B_{r^k}(P)$ of the construction of the hyperbolic approximation. We know by assumption that there is at least one point Q in $B_{r^k}(P)\backslash B_{\mu\, r^k}(P)$. Consider the sequence of balls containing P: How many levels further must we go until P and Q are no longer in balls which have a nonempty intersection? In other words: how many levels further must we go until we can be sure that a fork appeared? The distance between P and Q is at least $\mu\, r^k$. We are looking for an N such that

$$\mu\, r^k > 4\, r^{k+N},$$

$$\frac{\log \frac{\mu}{4}}{\log(r)} < N.$$

The left side is constant, therefore we can choose such an N and we know that after N steps a fork must appear. Again, the claim follows with Lemma 4.2.

□

4.2 Morphisms

Definition 4.4 (Quasi-isometric map and spaces)**.** A map $f : X \to Y$ between metric spaces is said to be *quasi-isometric* if there are $a \geq 1$, $b \geq 0$ such that

$$\frac{1}{a}|xx'| - b \leq |f(x)f(x')| \leq a|xx'| + b$$

for all x, $x' \in X$. Here, $|xx'|$, respectively $|f(x)f(x')|$, denote the distance in the corresponding space. We define two spaces X and X' to be *quasi-isometric* if there are quasi-isometric maps $f : X \to X'$, $f' : X' \to X$ and a constant $d \geq 0$ such that $|f' \circ f(x)x| \leq d$ and $|f \circ f'(x')x'| \leq d$ for all $x \in X$ and all $x' \in X'$.

Definition 4.5 (Roughly Isometric map)**.** A map $f : X \to X'$ between metric spaces is said to be *roughly isometric,* if there exists a $b \geq 0$, such that for all x, $y \in X$:

$$|xy| - b \leq |f(x)f(y)| \leq |xy| + b.$$

If such an f exists, we call X and X' roughly isometric to each other, and this relation is obviously an equivalence relation.

We state a definition from chapter 4.1 and one from chapter 4.2 from [BS07]:

Definition 4.6 (Classical cross-difference)**.** The additive version of the classical cross-ratio is the *classical cross-difference*

$$\langle x, y, z, u\rangle = \frac{1}{2}(|xz| + |yu| - |xy| - |zu|).$$

Definition 4.7 (Cross-difference triple)**.** For a quadruple $Q = (x, y, z, u)$ of points in a metric space X, we form the triple $A = \{(x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o\}$ and call it the *cross-difference triple* of Q. We define the *cross-difference* of Q

$$\mathrm{cd}(Q) = \max_{a,a' \in A}(a - a').$$

The cross-difference is independent of the choice of a base point $o \in X$ and it is an invariant of the unordered Q.

Definition 4.1.1. from [BS07] states:

Definition 4.8 (Strongly PQ-isometric map)**.** We say that a map $f : X \to X'$ between metric spaces is *strongly PQ-isometric*, if there are constants $c \geq 1, d \geq 0$ such that for all quadruples $(x, y, z, u) \in X^4$ with $\langle x, y, z, u\rangle \geq 0$

$$\frac{1}{c}\langle x, y, z, u\rangle - d \leq \langle x', y', z', u'\rangle \leq c\langle x, y, z, u\rangle + d,$$

where x', y', z' and u' are the images of x, y, z and u under f.

5 Uniformly Equilateral Spaces

With Lemma 4.2 we have a criterion to decide wheter $\partial_\infty X$ is uniformly perfect or not. We want to express this criterion in a quasi-isometric invariant way. To have no arbitrary long forkless pieces on geodesic rays is equivalent to the claim that there exist large equilateral triangles in every large ball . This, again, is expressible in terms of the Gromov product:

Definition 5.1 (Uniformly equilateral space)**.** Let X be a unbounded metric space. We call X *uniformly equilateral,* if there exist two numbers $S_0 > 0$, $\lambda > 0$, such that for every $w \in X$ and every $S \geq S_0$ the Ball $B_S(w) \subset X$ contains three points x, y, z with

$$(x|y)_z, (y|z)_x, (x|z)_y \geq \lambda S.$$

Now, the aim is to show that uniformly equilaterality is a quasi-isometric invariant for geodesic spaces.

Lemma 5.2. *If $Q = (x, y, z, z)$ and the base point is z, then $cd(Q) = (x|y)_z = \langle x, y, z, z\rangle$.*

Proof. $A = ((x|y)_z + (z|z)_z, (x|z)_z + (y|z)_z, (x|z)_z + (y|z)_z) = ((x|y)_z, 0, 0)$. We obtain that $\mathrm{cd}(Q) = (x|y)_z = \frac{1}{2}(|xz| + |yz| - |xy|) = \frac{1}{2}(|xz| + |yz| - |xy| - |zz|) = \langle x, y, z, z\rangle$. □

We state Theorem 4.4.1. and Corollary 4.2.3. from [BS07]:

Theorem 5.3. *Let $f : X \to X'$ be a (c, b)-quasi-isometric map of hyperbolic geodesic spaces. Then there is a constant $d \geq 0$ depending only on c, b and the hyperbolicity constants δ, δ' of X, X', such that f is strongly $(c, d) - PQ$-isometric and, in particular, $(c, d) - PQ$-isometric.* □

Lemma 5.4. *Let $f : X \to Y$ be a strongly $(c, d) - PQ$- isometric map. Then*

$$\frac{1}{c} cd(Q) - d \leq cd(Q') \leq c\, cd(Q) + d$$

for every quadruple $Q \subset X$, where $Q' = f(Q)$. □

Now we show that the property to be quasi equilateral is a quasi-isometric invariant for geodesic spaces:

Theorem 5.5. *Let X and X' be geodesic, quasi-isometric, Gromov hyperbolic spaces. Then*

$$X \text{ is uniformly equilateral } \Leftrightarrow X' \text{ is uniformly equilateral.}$$

Proof. We only show one direction. We construct the values λ' and S_0' and show, that X' is uniformly equilateral with these numbers.

We can apply Theorem 5.3 to X and X' and obtain that the (a, b)-quasi-isometry f is a strongly (a, d)-PQ-isometric map. Let S_0 and λ be the constants of the uniformly equilateral space X. Later, we need that

$$S_0 > \frac{ad}{\lambda}. \quad (*)$$

Hence we might be forced to enlarge S_0. But X is also uniformly equilateral with the new S_0.

First, we set some interdependent variables S and S': $S' := aS + b$. Thus, we define $S_0' := aS_0 + b$ and $S = \frac{S'-b}{a}$. Now we check the conditions for λ': later in the proof λ' needs to hold the following inequality: $\lambda' S' \leq \frac{\lambda}{a} S - d$ $(**)$. By replacing S we get $\lambda' S' \leq \frac{\lambda(S'-b)-a^2 d}{a^2}$ and furthermore $a^2\lambda' S' \leq \lambda S' - \lambda b - a^2 d$. On each side we have a linear function in S'. The inequality is fulfilled for all $S' \geq S_0'$, if the following conditions are fulfilled:

- $a^2\lambda' \leq \lambda$, i.e. $\lambda' \leq \frac{\lambda}{a^2}$
- $a^2\lambda' S_0' \leq \lambda S_0' - \lambda b - a^2 d$, i.e. $\lambda' \leq \frac{\lambda S_0' - \lambda b - a^2 d}{a^2 S_0'}$. The right side is greater than 0, if $\lambda(aS_0 + b) > \lambda b + a^2 d$, i.e. $S_0 > \frac{ad}{\lambda}$, and this is guaranteed by $(*)$.

We set $\lambda' := \min\{\frac{\lambda}{a^2}, \frac{\lambda S_0' - \lambda b - a^2 d}{a^2 S_0'}\}$. Now we show that X' is uniformly equilateral with the constants λ' and S_0': given are $S' \geq S_0'$ and $w \in X'$ arbitrary. Consider the ball $B_S(f^{-1}(w))$ and let B' be the smallest ball in X', which contains the set $f(B_S(f^{-1}(w))$. We observe that the radius of B' is smaller or equal to $a\, S + b = S'$.

Since X is uniformly equilateral, we have three points x, y, $z \in B_S(f^{-1}(w))$, such that the corresponding Gromov products are greater or equal than $\lambda\, S$.

Consider without loss of generality:

$$(f(x)|f(y))_{f(z)} \geq \frac{1}{a}(x|y)_z - d \geq \frac{\lambda}{a} S - d \geq \lambda'\, S'.$$

The first inequality follows with Theorem 5.3, Lemma 5.2 and Lemma 5.4, the second follows with the uniformly equilaterality of X and the third by $(**)$. □

Corollary 5.6. *Let X and X' be Gromov hyperbolic spaces and f a strongly PQ-isometric map between X and X'. Then*

$$X \text{ is uniformly equilateral } \Rightarrow X' \text{ is uniformly equilateral.}$$

Proof. The proof is the same as for Theorem 5.5, with the only difference that there is no need for Theorem 5.3. □

The following Lemma states a connection between Definition 5.1 and Lemma 4.2:

Lemma 5.7. *If X is the graph of a truncated hyperbolic approximation of a complete bounded metric space, then*

$$X \text{ is uniformly equilateral} \Leftrightarrow \partial_\infty X \text{ is uniformly perfect.}$$

Proof. Assume that $\partial_\infty X$ is not uniformly perfect, i.e. for every $N \in \mathbb{N}$ we can find a equivalence class $[\xi]$ containing a forkless interval, where the interval is longer than N (this is equivalent to "$\partial_\infty X$ is not μ-uniformly perfect" by Lemma 4.2).

Choose R_0 arbitrarily large and search a forkless intervall on a equivalence class $[\xi]$ with length $> 2\,R_0$. Find a Point P in this interval with the property that the ball $B_{R_0}(P) \subset X$ contains no fork (the interval and P exist).

Now we need to consider the following for u and $v \in B_{R_0}(P)$ (note that $|uv| \geq |\ell(u) - \ell(v)|$): If $u,\, v \in \xi$, then $|uv| = |\ell(u) - \ell(v)|$. If not, we assume without loss of generality that $u \in \xi,\ v \notin \xi$. Since there is no fork on ξ in $B_{R_0}(P)$, we know that $|\xi_{\ell(v)}v| = 1$. It follows that $|uv| \leq |\ell(u) - \ell(v)| + 1$. Overall, we obtain: $|\ell(u) - \ell(v)| \leq |uv| \leq |\ell(u) - \ell(v)| + 1$.

Now choose $x, y, z \in B_{R_0}(P)$ and assume without loss of generality that $\ell(x) \geq \ell(y) \geq \ell(z)$. We obtain: $(x|z)_y \leq \frac{1}{2}(|\ell(x) - \ell(y)| + 1 + |\ell(z) - \ell(y)| + 1 - |\ell(x) - \ell(z)|) = \frac{1}{2}(\ell(x) - \ell(y) + 1 + \ell(y) - \ell(z) + 1 - \ell(x) + \ell(z)) = 1$. In other words, one of the three Gromov products is allways 0 or 1, and hence X is not uniformly equilateral, since X is unbounded.

Assume now that $\partial_\infty X$ is uniformly perfect, i.e. there exists $N \in \mathbb{N}$, such that for every point $P \in X$ the ball $B_{\frac{N}{2}}(P)$ contains at least one fork. Choose

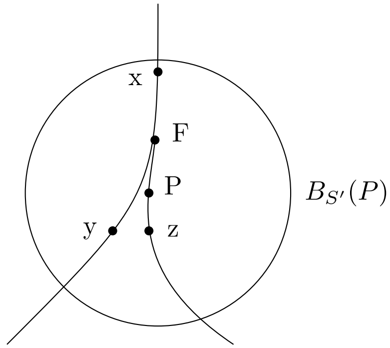

Figure 5: Situation in the proof of Lemma 5.7.

an arbitrary point P and let F be the fork closest to P and $S_0 := 2\,(N+2)$. Choose $S \geq S_0$ and $S' := S$ if S is even, otherwise set $S' := S-1$. For the following see Figure 5. In $B_{S'}(P)$ exist two points y, z, such that $\ell(y) = F + \frac{S'}{2}$ and $\ell(z) = F + \frac{S'}{2}$ and the distance between y and z is $2\,\frac{S'}{2} + 2$ or $2\,\frac{S'}{2} + 3$ (this depends on the type of the fork). F is a fork between two geodesic rays. Follow one of these in the direction of the starting point. Because $S' \geq 2\,(N+2)$ we can find a point $x \in B_{S'}(P)$ with $\ell(x) = F - \frac{S'}{2} - 2$. Without loss of generality we obtain $|xy| = S' + 2$ and $|xz| = S' + 2$ or $|xz| = S' + 3$ (this depends on the type of the fork). So we have found three points x, y, $z \in B_{S'}(P) \subset B_S(P)$ such that

$$(x|y)_z = \frac{1}{2}\,(|xz| + |yz| - |xy|) \geq \frac{1}{2}\,((S'+2) + (S'+1) - (S'+2)) = \frac{S'+1}{2} \geq \frac{S}{2},$$

$$(x|z)_y = \frac{1}{2}\,(|xy| + |zy| - |xz|) \geq \frac{1}{2}\,((S'+2) + (S'+2) - (S'+3)) = \frac{S'+1}{2} \geq \frac{S}{2},$$

$$(y|z)_x = \frac{1}{2}\,(|yx| + |xz| - |yz|) \geq \frac{1}{2}\,((S'+2) + (S'+2) - (S'+3)) = \frac{S'+1}{2} \geq \frac{S}{2}.$$

Thus, we see that X with $\lambda = \frac{1}{2}$ and S_0 satisfies the conditions of a uniformly equilateral space. $\square$

The next goal is to proof Theorem 5.11 which is a more general version of Lemma 5.7. The difference is that we leave the graph and consider a Gromov hyperbolic space.

Definition 5.8 (Visual space)**.** A hyperbolic space Y is said to be *visual,* if for some base point $o \in Y$ there is a positive constant D, such that for every $y \in Y$ there is $\xi \in \partial_\infty Y$ with $|oy| \leq (y|\xi)_o + D$ (one easily sees that this property is independent of the choice of o). For hyperbolic geodesic spaces this property is a rough version of the property that every segment $oy \subset Y$ can be extended to a geodesic ray beyond the end point y.

Its necessary to recall Theorem 7.1.2 from [BS07]:

Theorem 5.9. *Let X be a visual and X' a geodesic hyperbolic space. Assume that there is a bilipschitz embedding $f : (\partial_\infty X, d) \to (\partial_\infty X', d')$ where d, d' are visual metrics with respect to base points $o \in X$, $o' \in X'$ and the same parameter a. Then there exists a roughly isometric map $F : X \to X'$ such that $f = \partial_\infty F$.* □

In particular, there exists a roughly isometric map $G : X' \to X$.

Lemma 5.10. *Every visual, hyperbolic space X is roughly isometric to any hyperbolic approximation of its boundary with parameter $\frac{1}{a}$ at infinity equipped with a visual metric $d \sim a^{-(\,|\,)_o}$ with respect to a base point $o \in X$.*

Proof. Let $\partial^d_\infty X = (\partial_\infty X, d)$ be the boundary at infinity of X, equipped with the visual metric d. This space is bounded (Lemma 3.13) and by Lemma 3.14 complete. Consider the graph of a truncated hyperbolic approximation $HA_r(\partial^d_\infty X)$ with parameter $r \leq \frac{1}{6}$. With Theorem 3.17 we obtain that $\partial_\infty X = \partial_\infty(HA_r(\partial^d_\infty X))$ and d is also a visual metric on $\partial_\infty(HA_r(\partial^d_\infty X))$ with respect to a base point $o \in X$ and the parameter $\frac{1}{r}$, i.e.

$$\partial^d_\infty X = \partial^d_\infty(HA_r(\partial^d_\infty X)).$$

It obviously follows that the identity $id : \partial^d_\infty X \to \partial^d_\infty(HA_r(\partial^d_\infty X))$ is a bilipschitz embedding. Now we can apply Theorem 5.9 and we obtain that there exists a roughly isometric map $F : X \to HA_r(\partial^d_\infty X)$ and a roughly isometric map $G : HA_r(\partial^d_\infty X) \to X$. □

Theorem 5.11. *Let X be a geodesic, visual, Gromov hyperbolic space. Then:*

$$X \text{ is uniformly equilateral} \Leftrightarrow \partial_\infty X \text{ is uniformly perfect.}$$

Proof. Construct the graph X' of a truncated hyperbolic approximation of $\partial_\infty X$ equipped with a bounded visual metric d (Lemma 3.13) with respect to a base point $o \in X$. Note that $(\partial_\infty X, d)$ is complete by Lemma 3.14 and X' is visual and geodesic and by Lemma 5.10, we know that X and X' are roughly isometric, i.e. in particular quasi isometric. Now consider:

$\partial_\infty X$ is uniformly perfect $\Leftrightarrow$ $\partial_\infty X'$ is uniformly perfect $\Leftrightarrow$ X' is uniformly equilateral $\Leftrightarrow$ X is uniformly equilateral.

The first equivalence follows by Lemma 4.3, the second by Lemma 5.7 and the third by Theorem 5.5. □

6 Quasi-Metric, Uniformly Perfect Spaces

In this Chapter we give some preparations for the proof of Theorem 7.1 which states that if (Z, ρ) is a quasi-metric, uniformly perfect space and $z \in Z$, then the involution of Z is also quasi-metric and uniformly perfect. In classical terms, a involution at $0 \in \mathbb{R}^n$ is a map $I : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ where $I(x) = \frac{x}{\|x\|^2} = \frac{x}{|0x|\,|0x|}$. Consider now the metric ρ_0 on $\mathbb{R}^n$ which makes I to an isometriy: $\rho_0(x, y) := \frac{|xy|}{|0x|\,|0y|}$. This motivates a definition of an abstract version of a function which measures the distance to a point, the admissible function. And thus we obtain a generalized involution.

Definition 6.1 (Admissible function)**.** Let (Z, ρ) be a K-quasi-metric space. We call a function $\lambda : Z \to [0, \infty]$ *admissible* , if $Z_\infty = \lambda^{-1}(\infty)$ and $\{\rho(P, Q)$, $\lambda(P)$, $\lambda(Q)\}$ is a multiplicative K-triple. We denote by Δ the infimum of λ.

Remark 6.2. The following shows that Δ is smaller than ∞: Assume that $\lambda(A) = \infty$ for every point $A \in Z$. By definition $\lambda^{-1}(\infty) \overset{\text{Def.}}{=} Z_\infty$. Since $|Z_\infty| \leq 1$, it follows that Z consists of only one point. Hence $\Delta < \infty$.

The following Lemma states two examples of admissible functions (for the proof see [S06]):

Lemma 6.3. *Let $o \in Z \backslash \{\infty\}$ be any point, then the functions λ, $\bar{\lambda} : Z \to [0, \infty]$,*

$$\lambda(z) = \rho(o, z)$$

$$\bar{\lambda}(z) = \sqrt{\rho^2(z, o) + 1}/\sqrt{2}$$

are admissible.

The involution is a special case of the following, more general construction.

Definition 6.4 (Generalized involution ρ_λ)**.** Given a K-quasi-metric space Z with an admissible function λ, we define $\rho_\lambda : Z \times Z \to [0, \infty)$ by

$$\rho_\lambda(P, Q) = \frac{\rho(P, Q)}{\lambda(P)\,\lambda(Q)},$$

where we set (in the case that $Z_\infty = \{\omega\}$) $\rho_\lambda(\omega, P) = \rho_\lambda(P, \omega) = \frac{1}{\lambda(P)}$, in particular, $\rho_\lambda(\omega, \omega) = 0$. In the case that $\lambda(O) = 0$ for some point $O \in Z$, we have $\rho_\lambda(O, P) = \infty$ for all $P \neq O$.

We want to investigate the K-quasi-metric space (Z, ρ) after applying the generalized involution, i.e. the space (Z, ρ_λ). It follows directly from the properties of an admissible function that (Z, ρ_λ) is a K^2-quasi-metric space:

Proposition 6.5. *Let (Z, ρ) be a K-quasi-metric space. Then for every admissible function λ on Z, the space (Z, ρ_λ) is K^2-quasi-metric.* □

Next, we define sequences which converge towards one point:

Definition 6.6 (η-sequence)**.** For $\eta \in (0, 1)$ and a Point $P \in Z \backslash Z_\infty$, we call a sequence of points (W_i) in a μ-uniformly perfect space Z an *η-sequence for* P, if the following holds:

$$\mu\, \eta\, \rho(W_i, P) \leq \rho(W_{i+1}, P) \leq \eta\, \rho(W_i, P).$$

Proposition 6.7. *Given $P \in Z \backslash Z_\infty$ and $\eta \in (0, 1)$, there exists an η-sequence (W_i) for P in Z.*

Proof. Choose an arbitrary point as starting point W_0. If we have the point W_i, then $\rho(W_i, P)$ is known. Because of the uniformly perfectness of Z, we can find a point Q, such that $\mu\, \eta\, \rho(W_i, P) \leq \rho(Q, P) \leq \eta\, \rho(W_i, P)$. Now, $W_{i+1} := Q$. □

The factor η is needed for the following proposition:

Proposition 6.8. *Every η-sequence for P converges to P.*

Proof. Consider the following sequence, which follows from the definition of the η-sequence:

$$\eta\, \rho(W_0, P) \geq \rho(W_1, P) \geq \frac{1}{\eta}\, \rho(W_2, P) \geq \frac{1}{\eta^2}\, \rho(W_3, P) \geq ...$$

or in general: $\eta^{1-i}\, \rho(W_i, P) \geq \eta^{-i}\, \rho(W_{i+1}, P)$ for $i \in \mathbb{N}_0$. This sequence has an upper bound $\eta\, \rho(W_0, P)$. Therefore, we see from $\frac{1}{\eta^i} \longrightarrow \infty$ that $\rho(W_i, P) \longrightarrow 0$ and thus, the sequence converges to P. □

The space Z itself can be bounded or not. In the bounded case, we need some terms to express its size:

Definition 6.9. For a fixed point A, the value R_A is defined as $\sup\{\rho(A, B)| B \in Z\}$.
The largest R_A for all A, $\sup\{R_A | A \in Z\}$, is denoted by R. A specific value for Z is $R_\Delta := \max\{R, \Delta\}$. It holds $0 < R_A \le R \le R_\Delta$.

Remark 6.10. The largest admissible radius r for the point A to find a point N in the μ-uniformly perfect space Z such that $\mu r \le \rho(z, N) < r$ needs to hold $r < R_A$.

If Z is unbounded, it certainly holds $R_A = R = R_\Delta = \infty$ for all $A \in Z$.

Now, we show in particular that λ is bounded, if Z is bounded:

Proposition 6.11. *Let Z be a K-quasi-metric space, then $\lambda(B) < 2\,K^2\,R_\Delta$, $\forall B \in Z$.*

Proof. We can find a point $P \in Z$, such that $2\,K\,\Delta > \lambda(P)$. Now assume that there exists a point $Q \in Z$: $\lambda(Q) \ge 2\,K^2\,R_\Delta$. Since $\rho(P, Q) < 2\,K\,R_\Delta$, we have $K\,\rho(P, Q) < 2\,K^2\,R_\Delta \le \lambda(Q)$. This means that by Remark 2.4, $\rho(P, Q)$ is the smallest value in the multplicative K-triple $\{\lambda(P), \lambda(Q), \rho(P, Q)\}$.
Since $2\,K\,\Delta > \lambda(P)$ we obtain

$$2\,K^2\,\Delta > K\,\lambda(P) \ge \lambda(Q) \ge 2\,K^2\,R_\Delta.$$

And thus the contradiction: $\Delta > R_\Delta$. □

Remark 6.12. The factor 2 in the claim of this Proposition appears to catch the case $K = 1$. The value 2 is not specific. It is sufficient to have any factor greater than 1.

6.1 The Infimum of the Admissible Function is 0

If $\Delta = 0$, one can imagine λ as a distance-function to a point of the space. Indeed, in Lemma 6.17 we show that this is the right idea. First, we show that a point, where λ is 0, is unique:

Proposition 6.13. *In a K-quasi-metric space, there exists at most one point where λ is* 0.

Proof. If we have two points o and o' with $\lambda(o) = \lambda(o') = 0$, then $\{\lambda(o), \lambda(o'), \rho(o, o')\}$ is only a multiplicative K-triple if $\rho(o, o') = 0$. □

If $\inf(\lambda) = 0$, we can find a sequence (W_i) with $\lambda(W_i) \to 0$. But a point $o \in Z$, where $\lambda(o) = 0$, does not have to exist. But in some situations, it can be useful to add one point O with this property. Roughly spoken, $O := \lim(W_i)$, where (W_i) is the sequence from above. More precisely:

Definition 6.14. Assume that $\Delta = 0$, but $\lambda(P) > 0$ for all $P \in Z$. Then, we add a point O with the property $\lambda(O) = 0$. By Proposition 6.13, O is unique. The distance from an arbitrary point A to O is defined as $\rho(A, O) := \lambda(A)$. Set $Z_O := Z \cup \{O\}$.

We need to show that Z_O is also a K-quasi-metric, μ-uniformly perfect space:

Lemma 6.15. *Let Z be a K-quasi-metric, μ-uniformly perfect space with an admissible function λ. Then the space Z_O is a K-quasi-metric, $\frac{\mu}{K^2}$-uniformly perfect space with the admissible function λ.*

Proof. (i) First of all, we show that λ is admissible on Z_O: we need to show that $\{\rho(O, A), \lambda(A), \lambda(O)\} = \{\lambda(A), \lambda(A), \lambda(O)\}$ is a multiplicative K-triple for every $A \in Z$ and this is certainly true. Since $\lambda(O) = 0$, the condition $Z_\infty = \lambda^{-1}(\infty)$ is not affected and thus still true.

(ii) Z_O is K-quasi-metric. We verify the properties of Definition 2.3 for the point O. Let P and Q be two arbitrary points of Z.

(1) $\rho(P, O) = \lambda(P) \geq 0$ and $\lambda(P) = 0 \Leftrightarrow P = O$ (uniqueness of O).

(2) $\rho(P, O) = \lambda(P) = \rho(O, P)$.

(3) We have to show that $\{\rho(P, Q), \rho(O, Q), \rho(O, P)\} = \{\rho(P, Q), \lambda(Q), \lambda(P)\}$ is a multiplicative K-triple, and this is true by Definition 6.1.

(4) The point O should not lie in Z_∞. By Definition 6.1: $O \notin \lambda^{-1}(\infty) = Z_\infty$ holds, since $\lambda(O) = 0$.

(iii) Z_O is uniformly perfect. We have to show the uniformly perfect property for the point O, given $r < R_O$. We can find a point A, such that $\frac{\mu\, r}{K^2} \leq \rho(A, O) < r$. This is equvalent to Definition 4.1. Choose a sequence (V_i), where $\lambda(V_i) \to 0$ and fix a $N \in \mathbb{N}$ such that $K\,\lambda(V_N) < \mu \frac{r}{K}$. We can find a point A, such that $\mu \frac{r}{K} \leq \rho(V_N, A) < \frac{r}{K}$. Thus: $K\,\lambda(V_N) < \rho(V_N, A)$ and by Remark 2.4 we get that $\lambda(V_N)$ is the smallest value in $\{\rho(V_N, A), \lambda(A), \lambda(V_N)\}$. Overall we achieve the following facts:

(a) $K\,\rho(V_N, A) < r$;

(b) $\frac{\mu\, r}{K^2} \leq \frac{\rho(V_N,A)}{K}$;

(c) $\frac{1}{K} \leq \frac{\lambda(A)}{\rho(V_N,A)} \leq K \Rightarrow \frac{\rho(V_N,A)}{K} \leq \lambda(A)$;

(d) $\frac{1}{K} \leq \frac{\rho(V_N,A)}{\lambda(A)} \leq K \Rightarrow \lambda(A) \leq K\,\rho(V_N, A)$.

Overall, we obtain:

$$\frac{\mu\, r}{K^2} \overset{(b)}{\leq} \frac{\rho(V_N, A)}{K} \overset{(c)}{\leq} \lambda(A) \overset{(d)}{\leq} K\,\rho(V_N, A) \overset{(a)}{<} r.$$

This means that Z_O is $\frac{\mu}{K^2}$-uniformly perfect, since $\lambda(A) = \rho(A, O)$. □

Definition 6.16. From now on, we denote by Z a K-quasi-metric, μ-uniformly perfect space with an admissible function λ including the (eventually added) base point O and the (eventually modified) uniformly perfect-constant μ.

Note that, since the set $\{\rho(O, A), \lambda(A), \lambda(O)\}$ is a multiplicative K-triple and $\lambda(O) = 0$, we have, in the case that $A \neq O$,

$$\frac{1}{K} \leq \frac{\lambda(A)}{\rho(A, O)} \leq K.$$

This means, that, if the infimum of $\lambda = 0$, the functions $\rho(A, O)$ and $\lambda(A)$, behave similar, even if we did not add O:

Lemma 6.17. *Let (Z, ρ) be a K-quasi-metric, μ-uniformly perfect space with $\Delta = 0$. Then, for an arbitrary point $A \neq O$, the function λ holds:*

$$\frac{1}{K} \leq \frac{\lambda(A)}{\rho(A, O)} \leq K \text{ and } \frac{1}{K} \leq \frac{\rho(A, O)}{\lambda(A)} \leq K.$$

□

If Z is a uniformly perfect space, we can find a property for the function λ which reminds us of Definition 4.1 (μ-uniformly perfect space).

Lemma 6.18. *Let (Z,ρ) be a K-quasi-metric, μ-uniformly perfect space with $\Delta = 0$, then for a given $r < R_O$, we can find a point B, such that:*

$$\frac{\mu\, r}{K} \leq \lambda(B) < K\, r.$$

Proof. If the base point O has been added, the Lemma is trivial by Definition 6.14 and the fact that Z_O is a μ-uniformly perfect space.

For a given $r \geq 0$, we know that there exists a point A, such that

$$\mu\, r \leq \rho(A,O) < r.$$

Together with Lemma 6.17 this provides $\frac{\mu\, r}{K} \leq \frac{\rho(A,O)}{K} \overset{6.17}{\leq} \lambda(A) \overset{6.17}{\leq} K\,\rho(A,O) < r\,K \Rightarrow$

$$\frac{\mu\, r}{K} \leq \lambda(A) < r\, K.$$

□

Remark 6.19. With some modifications of the proof, one can express this lemma in the following way: if $\Delta = 0$, for a given $\frac{r}{K} < R_O$, we can find a point B, such that: $\frac{\mu\, r}{K^2} \leq \lambda(B) < r$.

Proposition 6.20. *If $\Delta = 0$, the space (Z,ρ_λ) is unbounded.*

Proof. Choose a sequence V_i converging to O and an arbitrary point $A \neq O$. Observe now $\frac{\rho(A,V_i)}{\lambda(A)\,\lambda(V_i)}$. It goes to infinity, since $\{\lambda(A), \lambda(V_i), \rho(A,V_i)\}$ is a multiplicative K-triple and $\lambda(V_i)$ goes to 0. □

Since we know that the functions $\rho(A,O)$ and $\lambda(A)$ behave similar, it should be possible to find something similar to an η-sequence for O expressed in λ:

Definition 6.21 (Λ-sequence)**.** Let $\Delta = 0$. For $\Lambda \in (0,1)$, we call a sequence of points (W_i) in a μ-uniformly perfect space a Λ-*sequence*, if it holds :

$$\mu\frac{\Lambda\,\lambda(W_i)}{K^2} \leq \lambda(W_{i+1}) \leq \Lambda\,\lambda(W_i).$$

Proposition 6.22. *Given $\Lambda \in (0,1)$, there exists an Λ-sequence (W_i) in Z.*

Proof. If we have the point W_i, then $\lambda(W_i)$ is known. As we have seen in Lemma 6.18, we can find a point Q, such that $\frac{\mu\,\Lambda\,\lambda(W_i)}{K^2} \leq \lambda(Q) \leq \frac{K\,\Lambda\,\lambda(W_i)}{K}$ (consider $\frac{\Lambda\,\lambda(W_i)}{K}$ as r in Lemma 6.18 , so we need to choose the starting point W_0, such that $\frac{\Lambda\,\lambda(W_0)}{K} < R_0$). Now, $W_{i+1} := Q$. □

The factor Λ is necessary for the following proposition:

Proposition 6.23. *Every Λ-sequence converges to O.*

Proof. With the same argument as in Lemma 6.8 (replace $\rho(W_i, P)$ by $\lambda(W_i)$), we can conclude that $\lambda(W_i) \to 0$ and thus with Lemma 6.17 that $\rho(W_i, O) \to 0$. This is equivalent to the convergence of the Λ-sequence. □

6.2 The Infimum of the Admissible Function is Larger than 0

As remarked, the infimum of λ can be greater than 0. In this situation, some things become more complicated. Having a point P with $\lambda(P) = \Delta$ does not help, since this point is probably not unique. What we have instead of a base point O, is a zone where λ is small. The size of this zone depends on the size of Δ (if $\lambda(P) = \lambda(P') = \Delta$, then by the K-quasi-metric: $\rho(P, P') \leq K\,\Delta$). Close to this zone, or within it, we cannot control the behavior of λ. If we have a sequence (V_i), such that $\lambda(V_i) \to \Delta$, it is for instance not possible to conclude that (V_i) converges and this was an important tool so far. The convergence of η-sequences is still possible, but the Λ-sequence is no longer realizable. One may interprete the function λ as a distance-function to a point which lies *above* the space and does not belong to Z.

Compare the following Lemma to Lemma 6.18 and Remark 6.19.

Lemma 6.24. *Let (Z, ρ) be a unbounded K-quasi-metric, μ-uniformly perfect space with an admissible function λ and $\Delta > 0$. For all $s > \Delta$ we can find a point $N \in Z$, such that the following is true:*

$$\frac{\mu}{2\,K^3}\,s < \lambda(N) < s.$$

Proof. Consider two cases:

(i) $\mu \frac{s}{K} > \Delta$

(ii) $\mu \frac{s}{K} \leq \Delta$

Case (i): We don't have a base point O. But we need an alternative: we can find a point P, such that

$$\mu \frac{s}{K} > \lambda(P) \geq \Delta \text{ and } 2\,K\,\Delta > \lambda(P). \tag{1}$$

Since (Z,ρ) is μ-uniformly perfect and by the assumption that (Z,ρ) is unbounded, we can find a point N, such that

$$\mu \frac{s}{K} \leq \rho(P,N) < \frac{s}{K}. \tag{2}$$

$\overset{(2)(1)}{\Longrightarrow} \rho(P,N) > \lambda(P)$, i.e. $\rho(P,N)$ is not the smallest value in $\{\rho(P,N), \lambda(P), \lambda(N)\}$, i.e.

$$K\,\rho(P,N) \geq \lambda(N). \tag{3}$$

By $\lambda(N) \geq \Delta$ and the right-hand side of (1) we obtain that $2\,K\,\lambda(N) > \lambda(P)$, i.e. $2\,K\,\lambda(N)$ is bigger than the second largest value in $\{\rho(P,N), \lambda(P), \lambda(N)\}$. We obtain

$$2\,K^2\,\lambda(N) > \rho(P,N).$$

By inverting this inequality, multiplying with $\lambda(N)$ and applying (3), we get

$$\frac{1}{2\,K^2} < \frac{\lambda(N)}{\rho(P,N)} \overset{(3)}{\leq} K. \tag{4}$$

On the other hand, by inverting (2) and then multiplying with $\lambda(N)$, we get

$$\frac{K\,\lambda(N)}{s} < \frac{\lambda(N)}{\rho(P,N)} \leq \frac{K\,\lambda(N)}{\mu\,s}. \tag{5}$$

Overall:

$$\frac{1}{2\,K^2} \overset{(4)(5)}{<} \frac{K\,\lambda(N)}{\mu\,s} \Rightarrow \frac{\mu}{2\,K^2} < \frac{K\,\lambda(N)}{s},$$

and

$$\frac{K\,\lambda(N)}{s} \overset{(5)(4)}{<} K.$$

The two last inequalities yield

$$\frac{\mu}{2\,K^2} < \frac{K\,\lambda(N)}{s} < K,$$

multiplying with $\frac{s}{K}$:

$$\frac{\mu}{2\,K^3}\,s < \lambda(N) < s,$$

and this is what we need.

Case (ii): The second case is easier: since $s > \Delta$, it follows that there exists a point P, such that $\lambda(P) < s$. On the other hand, by assumption: $\mu\frac{s}{K} < \Delta \leq \lambda(P) \Rightarrow$

$$\frac{\mu}{K}\,s < \lambda(P) < s$$

and certainly

$$\frac{\mu}{2\,K^3}\,s < \lambda(P) < s.$$

□

Remark 6.25. This Lemma could be prooved in less generality for a bounded space (Z, ρ). But then, one has to consider, how large s could be. Basically, for s to hold (2) we need $\frac{s}{K} < R_P$. But in general, this is not enough. It would make sense to set $s := K\,R_\Delta$, since the upper bound for λ is $2\,K^2\,R_\Delta$ (Lemma 6.11). But then, $\frac{s}{K} < R_P$ is not satisfied. This means, even if we proof this Lemma for bounded spaces as strong as we can, there are admissible values s, such that we cannot find the point M.

In the claim of Lemma 6.24, we multiply K^3 with 2. This factor 2 shows up to catch the case $K = 1$ in (1). The value 2 is not specific. It is sufficient to have a factor greater 1.

Now, we want to understand an important consequence for (Z, ρ_λ), if the infimum of λ is greater than 0.

Lemma 6.26. *Assume, that $\Delta > 0$, then the K-quasi-metric space (Z, ρ_λ) is bounded by $\frac{K}{\Delta}$.*

Proof. If we have an arbitrary point $A \in Z$, then for any point $B \in Z$ one of the two following conditions is true:

(i) $K\,\lambda(A) < \rho(A,B) \Rightarrow \frac{1}{K} \le \frac{\rho(A,B)}{\lambda(B)} \le K \Rightarrow \frac{\rho(A,B)}{\lambda(A)\,\lambda(B)} \le \frac{K}{\lambda(A)} \le \frac{K}{\Delta} < \infty.$

(ii) $K\,\lambda(A) \ge \rho(A,B) \Rightarrow K \ge \frac{\rho(A,B)}{\lambda(A)} \Rightarrow \frac{\rho(A,B)}{\lambda(A)\,\lambda(B)} \le \frac{K}{\lambda(B)} \le \frac{K}{\Delta} < \infty.$

Hence, we see that the space (Z, ρ_λ) is bounded, i.e. the distance between two points is maximally $\frac{K}{\Delta}$. □

7 Invariance of Uniformly Perfectness

In this section, we show that uniformly perfectness is invariant under the generalized involution as introduced above.

Theorem 7.1. *Let (Z, ρ) be a K-quasi-metric, μ-uniformly perfect space with an admissible function λ. There exists a $\bar{\mu} \leq \mu$, such that the K^2-quasi-metric, uniformly perfect space (Z, ρ_λ) is $\bar{\mu}$-uniformly perfect.*

Proof. Given K, μ, Δ, a point $A \in Z$ and $r < R_A$,

we show that there exists a $\bar{\mu} \leq \mu$, such that for all r there exists a point M, such that

$$\bar{\mu}\, r \leq \rho_\lambda(A, M) < r.$$

Basically, we have four situations:

(i) (Z, ρ) and (Z, ρ_λ) are unbounded.

(ii) (Z, ρ) is bounded and (Z, ρ_λ) is unbounded.

(iii) (Z, ρ) is unbounded and (Z, ρ_λ) is bounded.

(iv) (Z, ρ) and (Z, ρ_λ) are bounded.

It is possible to proof (i) and (ii) together, but (iii) and (iv) need to be treated separately. So the proof of this Theorem consists of three parts: in the first part, we assume the existence of O, in the second part, we assume that the infimum of the admissible function λ is greater than 0 and (Z, ρ) is unbounded, in part three, Δ is greater than 0 and (Z, ρ) is bounded.

For *Part 1* assume the existence of O. We consider three cases where we assume that $A \notin Z_\infty$:

Case 1: We assume that a given radius r is small.

Case 2: We assume that a given radius r is large.

Case 3: We assume that a given radius r lies between an upper and a lower bound.

Then, we consider $A \in Z_\infty$.

Some preparations for *Part 1*:

Note that $A \neq O$, since O is the infinitely remote point of (Z, ρ_λ) and this point is not to consider by Definition 4.1.

Later in the proof, we need that

$$\frac{\mu\,\lambda(A)}{K} \overset{!}{<} R_A \quad \text{and} \quad \frac{\mu\,\lambda(A)}{K} \overset{!}{<} R_O.$$

This is obviously true, if Z is unbounded since $\lambda(A) < \infty$, but in the case that Z is a bounded space, we need to apply Lemma 6.17: out of $\frac{\lambda(A)}{K} \leq \rho(A, O)$ we obtain $\frac{\mu\,\lambda(A)}{K} < \rho(A, O) \leq R_A$ and $\frac{\mu\,\lambda(A)}{K} < \rho(A, O) \leq R_O$.

Some preparations for *Case 1* :

We need a point *close* to A, but as far away from A *as possible*. Because (Z, ρ) is a μ-uniformly perfect space and $\frac{\mu\,\lambda(A)}{K} < R_A$, we can choose a point N, such that

$$\frac{\mu^2\,\lambda(A)}{K} \leq \rho(A, N) < \frac{\mu\,\lambda(A)}{K}$$

$$\Rightarrow \quad \mu^2\,\lambda(A) \leq K\,\rho(A, N) < \mu\,\lambda(A)$$

$$\Rightarrow \quad K\,\rho(A, N) < \lambda(A).$$

With Remark 2.4, we see that $\rho(A, N)$ is the smallest value in $\{\rho(A, N), \lambda(A), \lambda(N)\}$.

From $\frac{\mu^2\,\lambda(A)}{K} \leq \rho(A, N) < \frac{\mu\,\lambda(A)}{K}$, we further get

$$\frac{\mu^2}{K\,\lambda(N)} \leq \frac{\rho(A, N)}{\lambda(A)\,\lambda(N)} < \frac{\mu}{K\,\lambda(N)}.$$

We set $s := \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)} = \rho_\lambda(A, N)$.

Case 1: $r \leq s$.

Claim: There exists a $\iota \in (0,1)$, such that for all $r \leq s$ there is a point M, such that $\iota\, r \leq \rho_\lambda(A,M) < r$.

Choose an η-sequence (W_j) for A with $\eta \geq \frac{\mu}{K}$ starting at N. We can do this, since $\frac{\mu}{K} \in (0,1)$. The factor μ is only needed to catch the case $K = 1$. Since $\rho_\lambda(A, W_i) \to 0$, we can locate the index i where

$$\frac{\rho(A,W_{i+1})}{\lambda(A)\,\lambda(W_{i+1})} < r \leq \frac{\rho(A,W_i)}{\lambda(A)\,\lambda(W_i)} \Leftrightarrow \frac{\mu^2\,\rho(A,W_{i+1})}{K^3\,\lambda(A)\,\lambda(W_{i+1})} < \frac{\mu^2\, r}{K^3} \leq \frac{\mu^2\,\rho(A,W_i)}{K^3\,\lambda(A)\,\lambda(W_i)}.$$

If we can show that

$$\frac{\mu^2\,\rho(A,W_i)}{K^3\,\lambda(A)\,\lambda(W_i)} \leq \frac{\rho(A,W_{i+1})}{\lambda(A)\,\lambda(W_{i+1})}, \qquad (*)$$

then we have

$$\frac{\mu^2\, r}{K^3} \leq \frac{\rho(A,W_{i+1})}{\lambda(A)\,\lambda(W_{i+1})} < r$$

and W_{i+1} is the point we are looking for. Let us verify $(*)$: $K\,\rho(A,N) < \lambda(A) \Rightarrow K\,\rho(A,W_i) < \lambda(A)$, i.e. $\lambda(A)$ and $\lambda(W_i)$ are large in $\{\lambda(A), \lambda(W_i), \rho(A,W_i)\}$. It follows:

$$\frac{1}{K} \leq \frac{\lambda(W_i)}{\lambda(A)} \leq K \Leftrightarrow \frac{\lambda(A)}{K} \leq \lambda(W_i) \leq K\,\lambda(A)$$

and if we do the same for W_{i+1}:

$$\frac{1}{K} \leq \frac{\lambda(W_{i+1})}{\lambda(A)} \leq K \Leftrightarrow \frac{\lambda(A)}{K} \leq \lambda(W_{i+1}) \leq K\,\lambda(A).$$

Now, we replace $\lambda(W_i)$ and $\lambda(W_{i+1})$ in $(*)$ with the "worst-case-terms" from the last inequalities:

$$\frac{\mu^2\,\rho(A,W_i)}{K^3\lambda(A)\,\frac{\lambda(A)}{K}} \leq \frac{\rho(A,W_{i+1})}{\lambda(A)\,K\,\lambda(A)} \Leftrightarrow \frac{\mu^2}{K}\,\rho(A,W_i) \leq \rho(A,W_{i+1}).$$

And this is true by definition of the η-sequence (Definition 6.8) and since we have chosen $\eta \geq \frac{\mu}{K}$:

$$\frac{\mu^2}{K}\,\rho(W_i,A) \leq \mu\,\eta\,\rho(W_i,A) \overset{\text{Def.}}{\leq} \rho(W_{i+1},A).$$

Choose $M := W_{i+1}$ and $\iota := \frac{\mu^2}{K^3}$ and the proof for $(*)$ and also for the *Case 1* is finished. Note that so far we did not use that λ converges to 0.

Some preparations for *Case 2*:
Contrary to *Case 1*, in *Case 2* , we need a point *close*, but not *too close* to O. From Lemma 6.18 and the fact that $\frac{\lambda(A)\,\mu}{K} < R_O$, we know that for $\frac{\lambda(A)\,\mu}{K^2}$ we can find a point $\tilde{N}$, such that

$$\frac{\mu\,\frac{\lambda(A)\,\mu}{K^2}}{K} \leq \lambda(\tilde{N}) < K\,\frac{\lambda(A)\,\mu}{K^2}$$

$$\Leftrightarrow \frac{\mu^2\,\lambda(A)}{K^3} \leq \lambda(\tilde{N}) < \frac{\lambda(A)\,\mu}{K}$$

$$\Rightarrow \frac{\mu^2\,\lambda(A)}{K^2} \leq K\,\lambda(\tilde{N}) < \lambda(A)\,\mu \Rightarrow K\,\lambda(\tilde{N}) < \lambda(A).$$

With Remark 2.4, we see that $\lambda(\tilde{N})$ is the smallest number in $\{\lambda(A),\lambda(\tilde{N}),\rho(A,\tilde{N})\}$.
Set $S := \frac{\rho(A,\tilde{N})}{\lambda(A)\,\lambda(\tilde{N})} = \rho_\lambda(A,\tilde{N})$.

Case 2: $r > S$.

Claim: There exists a $\iota' \in (0,1)$, such that for all $r > S$ there is a point M, such that $\iota'\,r \leq \rho_\lambda(A,M) < r$.

Choose a Λ-sequence (V_j) with $\Lambda > \frac{\mu}{K}$, starting at $\tilde{N}$. Clearly, $\frac{\rho(A,V_i)}{\lambda(A)\lambda(V_i)}$ converges to ∞. Locate the index i where

$$\frac{\rho(A,V_i)}{\lambda(A)\,\lambda(V_i)} < r \leq \frac{\rho(A,V_{i+1})}{\lambda(A)\,\lambda(V_{i+1})} \Leftrightarrow \frac{\mu^2\,\rho(A,V_i)}{K^5\,\lambda(A)\,\lambda(V_i)} < \frac{\mu^2\,r}{K^5} \leq \frac{\mu^2\,\rho(A,V_{i+1})}{K^5\,\lambda(A)\,\lambda(V_{i+1})}.$$

If we can show that

$$\frac{\mu^2\,\rho(A,V_{i+1})}{K^5\,\lambda(A)\,\lambda(V_{i+1})} \leq \frac{\rho(A,V_i)}{\lambda(A)\,\lambda(V_i)}, \quad (**)$$

then we have

$$\frac{\mu^2\,r}{K^5} \leq \frac{\rho(A,V_i)}{\lambda(A)\,\lambda(V_i)} < r.$$

We now show $(**)$: $K\,\lambda(\tilde{N}) < \lambda(A) \Rightarrow K\,\lambda(V_i) < \lambda(A)$ because $\lambda(\tilde{N}) = \lambda(V_0)$, it folllows that $\lambda(V_i)$ is the smallest value in $\{\rho(A,V_i), \lambda(A), \lambda(V_i)\}$. We get

$$\frac{1}{K} \leq \frac{\rho(A,V_i)}{\lambda(A)} \leq K \Leftrightarrow \frac{\lambda(A)}{K} \leq \rho(A,V_i) \leq K\,\lambda(A)$$

and

$$\frac{1}{K} \leq \frac{\rho(A,V_{i+1})}{\lambda(A)} \leq K \Leftrightarrow \frac{\lambda(A)}{K} \leq \rho(A,V_{i+1}) \leq K\,\lambda(A).$$

Now we replace $\rho(A,V_i)$ and $\rho(A,V_{i+1})$ in $(**)$ with the "worst-case-terms" from the last inequalities:

$$\frac{\mu^2\,K\,\lambda(A)}{K^5\lambda(A)\,\lambda(V_{i+1})} \overset{!}{\leq} \frac{\frac{\lambda(A)}{K}}{\lambda(A)\,\lambda(V_i)} \Leftrightarrow \frac{\mu^2}{K^3}\lambda(V_i) \overset{!}{\leq} \lambda(V_{i+1})$$

And this is by the definition of the Λ-sequence (6.21) and the choice of Λ true:

$$\frac{\mu^2}{K^3}\,\lambda(V_i) \leq \frac{\mu\,\Lambda}{K^2}\,\lambda(V_i) \overset{\text{Def.}}{\leq} \lambda(V_{i+1}).$$

Choose $M := V_i$, $\iota' := \frac{\mu^2}{K^5}$ and *Case 2* is finished. Note that s and S depend on A (more precisely on $\lambda(A)$).

Case 3 connects s and S, and therefore *Part 1* becomes independent of A.

Case 3: $s < r \leq S$.

Claim: There exists a $\iota'' \in (0,1)$, such that for all $s < r \leq S$ there is a point M, such that $\iota''\,r \leq \rho_\lambda(A,M) < r$.

We use the points N, $\tilde{N}$ which we constructed in *Case 1* and *Case 2* . We show that we can find a constant ι'', such that ι'' times the largest possible r is smaller than $\rho_\lambda(A,N)$. This means that N is the point we are looking for. More precisely:

$$\iota''\, \rho_\lambda(A,\tilde{N}) \leq \rho_\lambda(A,N) \Leftrightarrow \iota''\, S \leq s,$$

which is certainly equivalent to

$$\iota'' \leq \frac{s}{S},$$

where ι'' should be a constant $\in (0,1)$ which only depends on K and μ and not on A. From the proof of *Case 1*, we know that

$$s \geq \frac{\mu^2}{K\,\lambda(N)}.$$

From the proof of *Case 2*, we know that

$$S = \frac{\rho(A,\tilde{N})}{\lambda(A)\,\lambda(\tilde{N})} \leq \frac{\rho(A,\tilde{N})}{\lambda(A)\,\frac{\mu^2\lambda(A)}{K^3}} = \frac{K^3\,\rho(A,\tilde{N})}{\mu^2\lambda(A)^2}.$$

Overall:

$$\frac{s}{S} \geq \frac{\frac{\mu^2}{K\,\lambda(N)}}{\frac{K^3\,\rho(A,\tilde{N})}{\mu^2\lambda(A)^2}} = \frac{\mu^4\,\lambda(A)^2}{K^4\,\lambda(N)\,\rho(A,\tilde{N})}. \quad (*)$$

Since $\rho(A,N)$ is the smallest value in $\{\rho(A,N),\lambda(A),\lambda(N)\}$ and $\lambda(\tilde{N})$ is the smallest value in $\{\lambda(A),\, \lambda(\tilde{N}),\rho(A,\tilde{N})\}$, we have with Remark 2.5:

$$\frac{\lambda(A)}{\lambda(N)} \geq \frac{1}{K} \quad \text{and} \quad \frac{\lambda(A)}{\rho(A,\tilde{N})} \geq \frac{1}{K}.$$

In $(*)$ we can insert the last two inequalities and we get

$$\frac{s}{S} \geq \frac{\mu^4\,\lambda(A)^2}{K^4\,\lambda(N)\,\rho(A,\tilde{N})} \geq \frac{\mu^4}{K^6}.$$

This is the constant independent of A we are looking for.

$$\Rightarrow \frac{\mu^4}{K^6}\, r \leq \rho_\lambda(A,N) < r.$$

Now we set $M := N$ and $\iota'' := \frac{\mu^4}{K^6}$.

We now verify the case $A \in Z_\infty$. Since $R_O = \infty$, we can apply Lemma 6.18 for any r and we can find a point M, such that

$$\frac{\mu}{K}\,\frac{K^2}{r\,\mu} \leq \lambda(M) < \frac{K^3}{r\,\mu}.$$

By inverting this inequality, we obtain

$$\frac{r\,\mu}{K^3} < \frac{1}{\lambda(M)} \leq \frac{r}{K} < r \text{ and further } \frac{\mu}{K^3}\,r < \frac{1}{\lambda(M)} < r.$$

Since $\frac{1}{\lambda(M)} = \frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} = \rho_\lambda(A, M)$, we have what we need by Defintion 6.4 .

We now undertake *Part 2* of the proof: (Z, ρ) is not bounded but (Z, ρ_λ) is bounded, i.e. $\Delta > 0$. Again, we have three cases, where we can adopt *Case 1* from *Part 1*. Some preparations:
Given Δ, a point $A \in Z$ and a radius r. With Lemma 6.26, we know that $r \leq \frac{K}{\Delta}$. In order to apply Lemma 6.24, μ needs to hold two conditions. Therefore it can be necessary to modify it. Set $\mu' := \frac{\mu}{2\,K^5}$. Now it holds:

(a) $\mu'\,K^2 = \frac{\mu}{2\,K^3}$;

(b) $\mu' < \frac{1}{K^2}$;

(c) $\frac{1}{K\,\mu'\,r} > \Delta$,

since $\frac{1}{K\,\mu'\,r} \geq \frac{1}{K\,\mu'\,\frac{K}{\Delta}} = \frac{\Delta}{K^2\,\mu'} \overset{(b)}{>} \frac{\Delta}{K^2\,\frac{1}{K^2}} = \Delta$.

We consider three cases, assuming $A \notin Z_\infty$:

Case 1': We assume that a given radius r is small.

Case 2': We assume that a given radius r is large.

Case 3': We assume that a given radius r lies between an upper and a lower bound.

Then, we consider $A \in Z_\infty$.

Construct N as in *Case 1,* but with the value μ' instead of μ and set $s := \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$.

Case 1': $r \leq s$.

Claim: There exists a $\iota_1 \in (0,1)$, such that for all $r \leq s$ there is a point M, such that $\iota_1 \, r \leq \rho_\lambda(A,M) < r$.

We find the point M with the η-sequence from *Case 1* with $\iota_1 := \frac{\mu'^2}{K^3}$.

Case 2': $r \geq \frac{1}{\lambda(A)\,\mu'}$.

Claim: There exists a $\iota_2 \in (0,1)$, such that for all $r \geq \frac{1}{\lambda(A)\,\mu'}$ there is a point M, such that $\iota_2 \, r \leq \rho_\lambda(A,M) < r$.

First of all, note that

$$\frac{1}{\lambda(A)\,\mu'} \leq r \Leftrightarrow \frac{\lambda(A)}{K} \geq \frac{1}{K\,\mu'\,r}. \tag{1}$$

Since $\frac{1}{K\,\mu'\,r} > \Delta$ we can use Lemma 6.24 and it follows that we can find a point M, such that

$$\frac{\mu}{2\,K^3}\,\frac{1}{K\,\mu'\,r} = \frac{K}{r} < \lambda(M) < \frac{1}{K\,\mu'\,r}.$$

Since (Z,ρ) is originally μ-uniformly perfect, here, μ is strong enough, moreover, with μ we are a bit more precise.
We now show that M is already the point we are looking for. It is crucial that, if r is large enough for *Case 2*, it is not possible that M lies close to the border, because the points there have a maximal distance of $\frac{K}{\lambda(A)}$ to A but $\frac{K}{\lambda(A)} < \frac{1}{\lambda(A)\,\mu'}$, since $\mu' < \frac{1}{K^2}$. We show that M lies close to the zone where λ is very small, small enough that $\frac{1}{K} \leq \frac{\rho(A,M)}{\lambda(A)} \leq K$:

$$\frac{K}{r} < \lambda(M) < \frac{1}{K\,\mu'\,r} \tag{2}$$

$$\Leftrightarrow K\,\mu'\,r < \frac{1}{\lambda(M)} < \frac{r}{K} \tag{3}$$

$$\Leftrightarrow K^2\,\mu'\,r < \frac{K}{\lambda(M)} < r \tag{4}$$

$$\overset{(3)}{\Longrightarrow} \mu'\,r < \frac{1}{K\,\lambda(M)}\,. \tag{5}$$

Consider now (2) together with (1), this yields:

$$\lambda(M) < \frac{\lambda(A)}{K}.$$

This means that $\lambda(M)$ is the smallest value in $\{\rho(A,M), \lambda(A), \lambda(M)\}$. It follows

$$\frac{1}{K\,\lambda(M)} \leq \frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} \leq \frac{K}{\lambda(M)}.$$

Together with (5) and the right-hand side of (4), we obtain:

$$\mu'\,r \overset{(5)}{<} \frac{1}{K\,\lambda(M)} \leq \frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} \leq \frac{K}{\lambda(M)} \overset{(4)}{<} r$$

In other words

$$\mu'\,r < \rho_\lambda(A,M) < r.$$

We obtain $\iota_2 := \mu'$.

Case 3': $s < r < \frac{1}{\lambda(A)\,\mu'}$.

Claim: There exists a $\iota_3 \in (0,1)$, such that for all $s < r < \frac{1}{\lambda(A)\,\mu'}$ there is a point M, such that $\iota_3\,r \leq \rho_\lambda(A,M) < r$.

We search ι_3 by multiplying ι_3 with the largest r that can occur in this case ($\frac{1}{\lambda(A)\,\mu'}$), and this product should be smaller than $\rho_\lambda(A,M)$. We now demonstrate how we obtain M. We have seen in the preparations of *Case 1* that

$$\frac{\mu'^2}{K\,\lambda(N)} \leq \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$$

and $\rho(A,N)$ is the smallest value in $\{\rho(A,N), \lambda(A), \lambda(N)\}$. It follows that $K\,\lambda(A) \geq \lambda(N)$ and so we obtain

$$\frac{\mu'^2}{K^2\,\lambda(A)} \leq \frac{\mu'^2}{K\,\lambda(N)} \leq \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)},$$

i.e. $\frac{\mu'^2}{K^2\,\lambda(A)} \leq s$. Therefore, we can apply *Case 1'* with $\frac{\mu'^2}{K^2\,\lambda(A)}$ and we get M, such that

$$\iota_1 \frac{\mu'^2}{K^2\,\lambda(A)} \leq \frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} < \frac{\mu'^2}{K^2\,\lambda(A)}.$$

The largest admissible r for *Case 3'* is $\frac{1}{\lambda(A)\,\mu}$. Now, we set

$$\iota_3 \frac{1}{\lambda(A)\,\mu'} \overset{!}{\leq} \iota_1 \frac{\mu'^2}{K^2\,\lambda(A)} = \frac{\mu'^4}{K^5\,\lambda(A)},$$

and we get

$$\iota_3 \leq \frac{\mu'^5}{K^5}.$$

Setting $\iota_3 := \frac{\mu'^5}{K^5}$, we obtain:

$$\iota_3\, r = \frac{\mu'^5}{K^5}\, r \overset{\text{asspt.}}{\leq} \frac{\mu'^5}{K^5}\, \frac{1}{\lambda(A)\,\mu'} = \iota_1 \frac{\mu'^2}{K^2\,\lambda(A)} \overset{\textit{Case 1'}}{\leq}$$

$$\frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} \overset{\textit{Case 1'}}{<} \frac{\mu'^2}{K^2\,\lambda(A)} \leq s \overset{\text{asspt.}}{<} r$$

$$\Rightarrow \iota_3\, r \leq \rho_\lambda(A,M) < r.$$

We now verify the case $A \in Z_\infty$. Since (Z,ρ) is unbounded and by our preparations in the beginning of *Part 2* (i.e. $\frac{2\,K^3}{\mu'\,r} > \frac{1}{K\,\mu'\,r} > \Delta$), we can apply Lemma 6.24 and we can find a point M, such that

$$\frac{\mu'}{2\,K^3}\,\frac{2\,K^3}{\mu'\,r} < \lambda(M) < \frac{2\,K^3}{\mu'\,r}$$

$$\Leftrightarrow \frac{1}{r} < \lambda(M) < \frac{2\,K^3}{\mu'\,r}$$

$$\Leftrightarrow \frac{\mu'}{2\,K^3}\, r < \frac{1}{\lambda(M)} < r.$$

Since $\frac{1}{\lambda(M)} = \frac{\rho(A,M)}{\lambda(A)\,\lambda(M)} = \rho_\lambda(A,M)$, we have what we need by Defintion 6.4.

We now undertake *Part 3* : (Z,ρ) and (Z,ρ_λ) are bounded i.e. $\Delta > 0$. We have seen that if (Z,ρ) is bounded, we cannot apply Lemma 6.24. The radius of the ball for a point A, where the space is locally uniformly perfect, is determined by $\frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$.

We have seen in *Case 1* that

$$\frac{\mu^2}{K\,\lambda(N)} \leq \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$$

for the point N. For small r (i.e. $r \leq \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$), we can simply adopt *Case 1*. For larger r we show that, multiplying r with a certain $\tilde{\imath}$, the point N already fulfills the condition.

Assume that $r > \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}$. Now observe that $\frac{\mu^2}{K\,\lambda(N)}$ has a lower bound, since λ has a upper bound (Lemma 6.11). It follows:

$$\frac{\mu^2}{2\,K^3\,R_\Delta} < \frac{\mu^2}{K\,\lambda(N)} \leq \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)}.$$

Now set $\tilde{\imath} := \frac{\mu^2\,\Delta}{2\,K^4\,R_\Delta}$. We obtain

$$\tilde{\imath}\, r \leq \tilde{\imath}\,\frac{K}{\Delta} = \frac{\mu^2}{2\,K^3\,R_\Delta} < \frac{\rho(A,N)}{\lambda(A)\,\lambda(N)} < r,$$

and *Part 3* is finished. □

Remark 7.2. This proof is not of a quantitative nature. With more cases one could find more precise uniformly perfect constants. In spite of this, i give an overview of the values we found in this proof (see page 43):

Situation (i) and (ii) i.e. (Z, ρ_λ) is unbounded: $\bar{\mu} = \frac{\mu^4}{K^6}$.

Situation (iii) i.e. (Z, ρ) is unbounded, (Z, ρ_λ) is bounded: $\bar{\mu} = \frac{\mu^5}{32\,K^{30}}$.

Situation (iv) i.e. (Z, ρ) and (Z, ρ_λ) is bounded: $\bar{\mu} = \frac{\mu^2\,\Delta}{2\,K^4\,R_\Delta}$.

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