Fold-type Solution Singularities and Characteristic Varieties of Nonlinear PDEs

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Michael Johannes Bächtold
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Prof. Dr. Alberto S. Cattaneo (Vorsitz)
Prof. Dr. Alexandre M. Vinogradov

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…You cant always get what you want
but if you try some time you might find
you get what you need

- The Rolling Stones

Synopsis

The concept of generalized (distribution) solutions has been of central importance for the development of the theory of linear PDEs, but it is commonly acknowledged to be inadequate for nonlinear PDEs.

In the 70's A. M. Vinogradov introduced a geometric analog of generalized solutions for (nonlinear) PDEs in the context of the geometric approach to PDEs. These solutions are certain *smooth* sub-manifolds of jet spaces which distinguish themselves from classical solutions in that they are not everywhere transversal to the projections to jets of lower order. The points where this transversality is lost are naturally interpreted as (geometric) singularities of the solution.

Since its introduction there have only been a hand-full of works on geometric generalized solutions and their singularities, most of them by A.M. Vinogradov and V. Lychagin. It was nevertheless already observed that geometric generalized solutions are related to distribution solutions of linear PDE's and several results were obtained indicating that the concept was a good generalization which could even give a refined picture in the linear theory.

In one of the last articles on the subject Vinogradov introduced the so called *singularity equations* associated to a PDE $\mathcal{E}$. These are systems of nonlinear PDEs deduced from the equation $\mathcal{E}$ whose solutions describe the shape which geometric singularities of solutions of $\mathcal{E}$ can have.

In this work we pick up the study of the singularity equations for the case of singularities of fold-type. We obtain general results which describe how the fold-type singularity equations behave under the process of prolongation of the original equation $\mathcal{E}$. We show that from the point of view of the infinite prolongation of the equation these fold-type singularity equations are fully described by the characteristics of the equations and how this allows to construct them for all prolongations of $\mathcal{E}$.

We also observe the existence of an additional structure on the singularity equations which in the case of scalar PDEs in two independent variables gives rise to an ODE which is a contact invariant of the original equation.

One of our two main results states that the characteristics are actually intrinsic to the diffiety and not only to the infinitely prolonged equation. The second central result states that under coverings of diffieties the characteristics grow, and that for finite coverings characteristics coincide. This has several implications like the observation that generic equations of different orders may not cover each other with finite coverings.

Another result we obtain from our intrinsic approach to the characteristic variety of a nonlinear PDE is a necessary condition for the existence of an integrating field of the PDE, which is a generalization of the method of characteristics as it is known for first order scalar equations.

In a final section we describe more explicitly the geometric structure of fold-type singularity equations for all prolongations of hyperbolic and parabolic Monge-Ampere equations and compute them explicitly for the first prolongation of the Monge-Ampere equation.

Some minor results include a local classification of the infinite dimensional manifolds underlying diffieties as well as an intrinsic characterization of distributions which might appear as Cartan distributions of an infinitely prolonged PDE.

Zusammenfassung

Das Konzept der verallgemeinerten (Distributions) Lösungen ist von zentraler Bedeutung in der Theorie linearer partieller Differentialgleichungen (PDEs), aber dessen Limitationen für die Theorie nichtlineare PDEs sind weit anerkannt.

In den 70er Jahren hat A. M. Vinogradov ein geometrisches Analog der verallgemeinerten Lösungen im Kontext der geometrischen Theorie nichtlineare PDEs eingeführt. Diese Lösungen sind bestimmte *glatte* Untermannigfaltigkeiten in Jet Räumen welche sich von den klassischen Lösungen dadurch unterscheiden, dass sie an gewissen Stellen nicht transversal zu den Projektionen auf tiefere Jet Räume sind. Diese Stellen lassen sich in natürlicher Weise als (geometrische) Singularitäten der Lösung interpretieren.

Seit der Einführung des Begriffs sind erst eine kleine Anzahl an Arbeiten erschienen welche sich mit geometrischen verallgemeinerten Lösungen und dessen Singularitäten beschäftigen. Die meisten dieser Arbeiten stammen von A.M. Vinogradov und V. Lychagin. Trotzdem wurde in diesen Arbeiten bereits bemerkt, dass geometrische verallgemeinerte Lösungen mit Distributions Lösungen linearer PDEs verwandt sind. Ausserdem wurden mehrere Resultate erhalten, welche darauf hinweisen, dass der Begriff eine angemessene Verallgemeinerung klassischer Lösungen darstellt und selbst im Fall linearer PDEs, ein verfeinertes Bild ermöglicht.

In einen der letzten Artikel auf dem Gebiet hat A. M. Vinogradov die sogenannten *Singularitäten Gleichungen*, welche einer PDE $\mathcal{E}$ zugeordnet werden, eingeführt. Diese sind Systeme nichtlinearer PDEs, dessen Lösungen die geometrische Form der Singularitäten der verallgemeinerten Lösungen von $\mathcal{E}$ beschreiben.

In dieser Arbeit greifen wir das Studium der Singularitäten Gleichung für den Fall von Falt-Singularitäten auf. Wir erhalten allgemeine Aussagen, welche diese Singularitäten Gleichungen für die Verlängerungen der Gleichung $\mathcal{E}$ beschreiben. Wir zeigen, dass aus Sicht der unendlichen Verlängerung der Gleichung, diese Falt-Singularitäten-Gleichungen vollkommen durch die Charakteristiken der Gleichung $\mathcal{E}$ bestimmt werden.

Wir beobachten desweiteren, die Existenz einer zusätzlichen Struktur auf den Singularitäten Gleichungen, welche im Fall skalarer Gleichungen in zwei Veränderlichen einer gewöhnlichen Differenzialgleichung entsprechen, die eine Invariante der PDE unter Kontakt Transformationen darstellt.

Eines der zwei zentralen Resultate sagt aus, dass die Charakteristiken tatsächlich intrinsisch der Diffiety zugeordnet sind und nicht nur der unendlichen Verlängerung der Gleichung. Das zweite wichtige Resultat sagt, dass unter Überdeckungen von PDEs die Charakteristiken mehr werden und im Falle endlicher Überdeckung gleich bleiben. Dies hat mehrere Konsequenzen. Eine dieser ist, dass es zwischen zwei generischen PDEs verschiedener Ordnung, keine endliche Überdeckung geben kann.

Ein weiteres Resultat, welches wir aus unserem intrinsischen Zugang zu den Charakteristiken erhalten, gibt eine notwendige Bedingung für die Existenz eines integrierenden Vektorfleds der PDE, was eine Verallgemeinerung der Methode der Charakteristiken für scalare PDEs erster Ordnung darstellt.

In einem letzten Abschnitt beschreiben wir expliziter die geometrische Struktur der Singularitäten Gleichungen für alle Verlängerungen von hyperbolischer und parabolischer Monge-Ampere Gleichungen und berechnen diese explizit für die erste Verlängerung.

Einige kleinere Resultate sind eine lokale Klassifizierung der unendlich dimensionalen Mannigfaltigkeiten welche Diffieties unterliegen, als auch eine intrinsische Charakterisierung von Distributionen, welche als Cartan Distributionen einer unendlich verlängerten PDE auftreten können.

Sinossi

Il concetto di soluzione generalizzata (nel senso delle distribuzioni) è stato di grande importanza nella teoria delle equazioni lineari alle derivate parziali (PDE), pur incontrando delle ben note limitazioni nel caso delle equazioni non lineari.

Negli anni '70 A. M. Vinogradov introdusse il concetto analogo di una soluzione generalizzata nel contesto della teoria geometrica delle PDE non lineari. Queste soluzioni vanno intese come quelle particolari sottovarietà lisce nei spazi di getti, che si distinguono delle soluzioni classiche per non essere in tutti i punti trasversali alle proiezioni sugli spazi di getti di ordine inferiore. I punti in cui la varietà che rappresenta una soluzione non soddisfa la suddetta condizione di trasversalità si interpretano naturalmente come le singolarità di quella soluzione.

Successivamente all'introduzione delle soluzioni geometriche e delle loro singolarità, solo pochi lavori sono apparsi intorno all'argomento. Ciò non ostante, è stato notato che le soluzioni geometriche generalizzate sono legate con le soluzioni, nel senso delle distribuzioni, di equazioni lineari. Sono stati anche trovati vari risultati che indicano che il concetto geometrico ed sua una appropriata generalizzazione permettono di trovare nuovi aspetti della teoria lineare.

In uno degli ultimi articoli in questo campo, Vingradov ha introdotto le equazioni delle singolaritá associate a una PDE $\mathcal{E}$. Si tratta di sistemi di PDE non lineari le cui soluzioni descrivono la forma geometrica delle singolarità delle soluzioni generalizzate di $\mathcal{E}$.

In questa tesi approfondiremo lo studio delle equazioni delle singolarità nel caso in cui le singolarità siano del cosiddetto tipo piega. Verranno presentati dei risultati generali che permettono di descrivere queste equazioni associate a tutti i prolungamenti di une certa equazione $\mathcal{E}$. Si proverà inoltre che, dal punto di vista del prolungamento infinito, queste equazioni sono completamente determinate delle caratteristiche della equazione $\mathcal{E}$.

Accidentalmente, verrà scoperta l'esistenza di una struttura addizionale sulle equazioni delle singolarità, la quale, nel caso di equazioni scalari in due variabili indipendenti, corrisponde ad una equazione ordinaria associata. Quest'ultima fornisce un nuovo invariante dell'equazione sotto trasformazioni di contatto.

Uno dei due risultati centrali stabilisce che le caratteristiche sono intrinsecamente associate ad una qualsiasi diffiety, e non solo al prolungamento infinito di una equazione. Il secondo risultato stabilisce che, passando ai ricoprimenti di una diffiety, le caratteristiche, in generale, aumentano, pur rimanendo le stesse nel caso di ricoprimenti finiti. Da ciò scaturiscono varie conseguenze. La prima è che per due equazioni generiche di ordine differente non può esistere un ricoprimento finito.

Un altro risultato che segue dall'approccio intrinseco alle caratteristiche è una condizione necessaria per l'esistenza di un campo vettoriale integrante per l'equazione. Questo fornisce una generalizzazione del metodo delle caratteristiche, ben noto nel caso di equazioni scalari del primo ordine.

In un'ultima sezione verrà descritta più dettagliatamente la struttura geometrica delle equazioni delle singolarità per le equazioni iperboliche e paraboliche di Monge-Ampère, che saranno calcolate esplicitamente per il primo prolungamento di tali equazioni.

Fra i risultati minori si trovano una classificazione locale delle varietà infinito-dimensionali soggiagenti alle diffiety, ed una caratterizzazione intrinseca delle distribuzioni che possono apparire come distribuzioni di Cartan sui prolungamenti infiniti delle PDE.

FOLD-TYPE SOLUTION SINGULARITIES AND CHARACTERISTIC VARIETIES OF NON-LINEAR PDES

MICHAEL JOHANNES BÄCHTOLD

Notations and conventions:

- For a vector space V over a field $\mathbb{K}$ we use the following notations:
  - $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.
  - $\langle v, \alpha \rangle$ for the canonical pairing between a vector $v \in V$ and a co-vector $\alpha \in V^*$: $\langle v, \alpha \rangle = \alpha(v)$.
  - $W^\circ \subset V^*$ for the annihilator of a subspace $W \subset V$, i.e.
$$W^\circ = \{\alpha \in V^* \mid \alpha|_W = 0\}$$
  - $\langle U \rangle$ for the span of a subset $U \subset V$.
  - $S^k V$ for the k-th symmetric tensor product of V.
  - $\mathrm{Grass}(V, m)$ for the Grassmannian of m-dimensional subspaces of V.
- All manifolds and maps are assumed to be real and C^∞ unless otherwise stated.
- We regularly make use of the duality between differential geometry and commutative algebra (see [24]) based on the equivalences:
  - manifold $M \leftrightarrow$ commutative $\mathbb{R}$-algebra $C^\infty(M)$
  - points of $M \leftrightarrow$ algebra morhism $C^\infty(M) \to \mathbb{R}$
  - vector bundles on $M \leftrightarrow$ finitely generated projective $C^\infty(M)$-modules
  - vector fields on $M \leftrightarrow$ derivations on $C^\infty(M)$
  - etc...
- Given a fiber bundle $\pi : N \to M$, the fiber over a point $x \in M$ is denoted with
$$N_x := \pi^{-1}(x).$$
- The pullback of a fiber bundle $\pi : E \to M$ along a map $\phi : N \to M$ is denoted with $\phi^*\pi : \phi^*E \to N$.
- To reduce the load of notation, the same symbol is used to denote the total space of a vector bundle $\pi : P \to M$ and its module of sections, i.e. $P = \Gamma(\pi)$. To avoid confusion, the expression $Y \in P$ will always mean that Y is a section and not an element of the total space. To deal with single elements of the total space we write $\xi \in P_x$, where P_x is the fiber of the vector bundle over $x \in M$. Similarly for a section $Y \in P$ we write Y_x to denote the value of Y at x, i.e. $Y_x \in P_x$.
  Recall that in algebraic terms one has $P_x = P/\mu_x P$ where $\mu_x \subset C^\infty(M)$ is the vanishing ideal of the point x.
- The only exception to the previous rule is made with tangent and cotangent bundles where we write TM and T^*M for the tangent and co-tangent space and $D(M)$ resp. $\Lambda^1(M)$ for vector fields and one forms.
- Given two $C^\infty(M)$-modules P, Q , their tensor product $P \otimes Q$ is always understood over $C^\infty(M)$ unless otherwise stated. Similarly for symmetric products, wedge products and duals $P^* = \mathrm{Hom}_{C^\infty(M)}(P, C^\infty(M))$.

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Contents

Introduction

The mathematical context of this thesis is the geometric theory of nonlinear partial differential equations (PDEs) which studies these from the point of view of jet spaces and the geometrical structures they posess like the Cartan distribution [2, 10]. Since the theory is not yet a part of the mathematical mainstream there is no consensus on the terminology and even the subject as a whole appears under different guises like "exterior differential systems", the "formal theory of PDEs", or in its linear version as the theory of D-modules, algebraic analysis etc.

Despite this status it has proven to be a powerful unifying language to describe many phenomena related with PDEs and given several new insights. In one of its conceptually most evolved form it deals with so called diffieties and the "secondary" calculus on them [30]. It is this approach which serves as background for this thesis, though we do not assume the reader to be familiar with it. If the reader wishes to get further motivations for this approach he may consult the first chapter in [30] and references therein.

The more specific subject of this thesis started in [26], where A. M. Vinogradov introduced a geometric concept of generalized solutions for nonlinear PDEs which is very natural from the point of view of the geometric approach to nonlinear PDEs. It is based on the observation that smooth (or regular) solutions to PDEs correspond to smooth integral submanifolds of the Cartan distribution on jet spaces, which are moreover everywhere transversal to the projections to lower order jets. But several well known examples of PDEs posess multivalued solutions with singularities, like caustics in the case of Hamilton-Jacobi equations or branching points in the case of Cauchy-Riemann equations (further examples can be found in [21]). Seen from the point of view of integral submanifolds of jet spaces, these solututions exhibit no real singularities but simply points where the transversality condition fails. Hence one is lead to generalize the concept of solutions of PDEs by allowing *smooth* integral submanifolds of the Cartan distribution which are not necessarily everywhere transversal to the projection onto the base.

After the introdiction of the concept a series of fundamental works by Krishchenko, Lychagin and Vinogradov on geometric generalized solutions followed [12, 13, 18, 19, 20, 21] which ended with the two review articles [22] and [28] in 1986. Since then only [17], which is of "phenomenological" nature and [23, 3] which may be seen as an application to compute differential invariants of Monge-Ampère equations appeared. Nevertheless in these works some very satisfying results about geometric generalized solutions were obtained. First of all some first order classification results on the type of singularities were obtained in [13, 22] which revealed that they posses interesting internal structures. In [22, 28] it was shown how these generalized solutions are related with the more familiar concept of generalized solution of linear PDEs in the sense of Schwartz. In [20] it was shown how singularities of multivalued generalized solutions give rise to characteristic classes similarly as in Maslov's global analysis of the quasi-classical approximation to quantum mechanics. Some further conjectures about the role of these and related singularities of solutions in the problem of quantization were made in [29]. In one of the last articles [28] Vinogradov introduced the so called singularity equations of a non-linear PDE. Roughly speaking the solutions of these associated PDEs correspond to the geometrical shapes which the singularities of generalized solutions of the original PDE may assume. One motivation for studying these equations is the conjecture [29] that they might be useful in describing the motion of vortices in turbulent fluids.

In this thesis we pick up the subject of these singularity equations, mainly for the case of fold-type singularities (i.e. singularities which occur along submanifolds of co-dimension one). The main contributions show how these concepts are seen from the point of view of the diffiety and how they are actually intrinsic to the category of diffieties. This category is basically the category with objects nonlinear PDEs and morphism nonlinear partial differential operators sending solutions of one equation to solutions of the other [30].

Detailed outline of the thesis. Since the starting point of the geometrical approach to non-linear PDEs are jet spaces and their geometry we review some basic facts in section 1. In particular we recall the main geometric structures on jet spaces which are R-planes and the Cartan distribution. We explain how submanifolds of the starting manifold (or sections of the starting fiber bundle) are identified with certain integral submanifolds of the Cartan distribution on jet spaces and how this leads naturally to the concept of "generalized" manifolds (resp. sections) and their singularities. These will play the role of geometric generalized solutions of nonlinear PDEs

In section 2 we recall how non-linear PDEs are seen from the point of view of the geometry of jet spaces. We avoid defining a PDE as lying in a single jet space of fixed degree and instead pass immediately to consider all its prolongations together as the representative of the PDE. This is not only for aesthetic reasons but unavoidable if one wants to consider non-linear differential operators as morphisms between PDEs and prove the result on coverings which we later do. This leads us to review what we call "co-filtered" manifolds (which are infinite towers of fiber bundles) and differential calculus on them. We make a distinction between "co-filtered" manifolds and "pro-finite" manifolds, the latter roughly being the inverse limit of the tower of projections without the preferred co-filtration. With this at hand we also define what a diffiety is and what morphism between them are. As a result we give an intrinsic characterization of the distributions on co-filtered manifolds which appear as Cartan distribution of infinitely prolonged PDEs. As a corollary we give an a posteriori justification of why morphism of diffieties can only have a bounded shift. Finally we give the definitions of geometric generalized solutions and the associated singularity equations of a PDE as introduced by Vinogradov in [28].

A fundamental structure to study singularities of generalized solutions and the singularity equations are the symbols of a non-linear PDE which we review in section 3. To do this we first recall the affine structures on fibers of projections between consecutive jet spaces and show that the symbols of a PDE form what is called a symbolic system. We also recall that symbolic systems are just the dual notion of graded modules over polynomial algebras, intimately linking the subject of geometric singularities with commutative algebra (in the sense of algebraic geometry). Finally we explain what characteristics of a PDE are and how they are described via the annihilator of the symbolic module dual to the symbolic system.

Section 4 deals with some aspects of the structure and classification of involutive subspaces of the Cartan distribution. This is of necessity since tangent planes to singular points of generalized solutions are involutive.

Finally in section 5 we link the fold singularity equations with the characteristics of the symbolic module and show that they are basically equivalent.

Section 6 contains in a sense the most fundamental results which show that characteristics are actually invariantly associated to the diffiety (and not only to the particular infinitely prolonged PDE). To do this we first define an intrinsic object called the *characteristic ideal* of a diffiety using a standard construction from the theory of D-modules applied to the left $\mathcal{C}$Diff-module of Cartan forms $\mathcal{C}\Lambda^1$, where $\mathcal{C}$Diff is the algebra of scalar $\mathcal{C}$-differential operators. To show that from this "global" characteristic ideal one obtains the point-wise characteristic varieties as defined in the previous sections one would like to prove that the characteristic ideal restricted to points coincides with the annihilator of the symbolic module at that point. Nevertheless for non-regular PDEs this need not be the case at all points but only in generic ones, i.e. points in a dense open set. To prove this as well as a later result concerning the behaviour of characteristic under coverings we establish a property similar to Noetherianness for the $C^\infty(M)$-modules under consideration, which might be described as local Noetherianness in generic points.

Next we establish a fundamental result which geometrically states that passing to a covering of a diffiety, the characteristic variety may just grow (i.e. characteristics can become more), but if the covering has finite fibers then the two diffieties have the same characteristic varieties. This can be useful in deciding when two equations can cover each other. It implies in particular that (generic) equations of different order cannot cover each other with a finite covering (although this may also be deduced from the appendix).

We also work out in detail the example of the Heat equation covering the Burgers via the Cole-Hopf transformation and show how the fold-singularity equations behave under this covering.

As a last result in this section we exhibit a link between the characteristic variety of a diffiety and the existence of a vector field in the Cartan distribution of the diffiety which possesses a flow. More precisely we show that such a vector field may only exist if the characteristic varieties are contained in hyperplanes. This is as a partial generalization of the method of integrating a first order equation by the method of characteristics to arbitrary diffieties.

In the final section 7 we study in more detail fold-singularity equations of prolongations of second order hyperbolic scalar PDEs in two independent variables. We establish a result similar to the one obtained in [28, 23], which describes these singularity equations geometrically in each point as a pair of two transversal 2-dimensional planes which are orthogonal to each other with respect to the meta-symplectic structure on Cartan planes. Moreover we compute explicitly the fold-singularity equations of the first prolongation of hyperbolic Monge-Ampère equations.

In an appendix we give a local classification of pro-finite manifolds under the morphisms introduced in section 2. For those pro-finite manifolds which underlie diffieties this classification is very simple, namely the only invariants are the coefficient and leading term of the Hilbert polynomial of the PDE.

1. Jet Bundles and R-manifolds

In this section we review jet spaces and their geometry which are the starting point of the geometrical approach to nonlinear PDEs. Most of this material can be found in [2, 10], but we recall it to fix the background and notation. A slight expository difference is that we place more emphasis on what we call the R-distributions, instead of on the Cartan distributions. Even though both notions are equivalent this seems more natural from the point of view of infinite prolongations and allows one to prove some statements which characterize diffieties and their morphisms.

1.1. **Jet Bundles.** Throughout we let E denote a fixed smooth manifold of dimension $n+m$.

Definition 1.1. The **k-th jet** of an n-dimensional submanifold $L \subset E$ at $\theta_0 \in L$ is the equivalence class $[L]^k_{\theta_0}$ of all n-dimensional submanifolds tangent with order k to L at θ_0. The space of all k-th jets of n-dimensional submanifolds of E is denoted with $J^k(E,n)$ and is naturally a smooth finite dimensional manifold.

Example 1.2. $J^0(E,n) = E$ and $J^1(E,n)$ is the bundle of Grassmannians of n-dimensional subspaces of TE.

Frequently one considers the case that E is fibered over an n-dimensional manifold $\pi : E \to M$ and deals with jets of sections of π which are denoted with $J^k(\pi)$. Since sections of π are locally the same as n-dimensional submanifolds of E transversal to the fibers of π it follows that $J^k(\pi)$ is an open and dense subset of $J^k(E,n)$. If $\pi : E = M \times N \to M$ is a trivial bundle then $J^k(\pi)$ is the space of jets of maps from M to N.

As long as it is clear form the context we will suppress the reference to E or the bundle π and just write J^k. Most of the discussion will apply to both cases $J^k = J^k(E,n)$ and $J^k = J^k(\pi)$ but to distinguish them we might refer to them as the **projective** and the **fibered case** respectively.

There are **canonical projections**

$$\pi_{k,l} : J^k \to J^l$$

for $k > l$ which correspond to forgetting orders of tangency and these projections form smooth fiber bundles. In the fibered case there is moreover the canonical projection onto the base M denoted with $\pi_{k,-1} : J^k \to M$.

Notation 1.3. A point $\theta \in J^k$ will always be decorated with a lower index k to keep track of the jet space it lies in, i.e., $\theta_k \in J^k$. Moreover given θ_k it will be assumed without further mention that the points $\theta_{k-1}, \theta_{k-2}, \ldots \theta_0$ denote the projections of θ_k to lower jets, i.e., for $l < k$:

$$\theta_l = \pi_{k,l}(\theta_k).$$

In the fibered case we further put $\theta_{-1} = \pi_{k,-1}(\theta_k)$.

The algebra of smooth functions on J^k will be denoted by

$$\mathcal{F}^k = C^\infty(J^k).$$

The projections $\pi_{k,k-1}$ induce a chain of inclusions of algebras

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \ldots$$

whose direct limit is denoted with

$$\mathcal{F} = \bigcup_{k \in \mathbb{N}} \mathcal{F}^k$$

and is called the algebra of **smooth functions on** J^∞.

Given an n-dimensional submanifold $L \subset E$ its **k-th prolongation** $L^{(k)}$ is the n-dimensional smooth submanifold of J^k given by

$$L^{(k)} = \{[L]^k_{\theta_0} \mid \theta_0 \in L\}.$$

All the prolongations $L^{(k)}$, $k \in \mathbb{N}$ may be canonically identified as manifolds:

$$L \cong L^{(1)} \cong L^{(2)} \cong \ldots$$

since they map diffeomorphically onto each other via projections $\pi_{k,l}$ and in the fibered case these isomorphisms extend to an identification with the base M.

1.1.1. *Local coordinates.* A local chart $(x^1, \ldots, x^n, u^1, \ldots, u^m)$ on E which is **divided** into two subsets $(x^1, \ldots, x^n)$ and $(u^1, \ldots, u^m)$ called the **independent** and **dependent** coordinates respectively, induces local **canonical coordinates** in J^k denoted with

$$(u^j_\sigma)_{\substack{j=1,\ldots,n \\ |\sigma|\leq k}}$$

where the σ are a **multi-indices** $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n$ and

$$|\sigma| := \sum_{j=1}^{n} \sigma_j$$

denotes the **lenght** of the multi-inex. These coordinates are determined by the condition:

$$u^j_\sigma([L]^k_{\theta_0}) := \frac{\partial^{|\sigma|}}{\partial x^\sigma} f^j(x^1_0, \ldots, x^n_0)$$

where L is an n-dimensional submanifold of E transversal to the independent coordinates near $\theta_0 \in L$ and hence locally described as the graph of m functions $f^j(x^1, \ldots, x^n)$:

$$L = \{u^1 = f^1(x^1, \ldots, x^n), \ldots, u^m = f^m(x^1, \ldots, x^n)\}.$$

Accordingly the prolongations $L^{(k)}$ are described in local coordinates by the equations

$$L^{(k)} = \{u^j_\sigma = \frac{\partial^{|\sigma|}}{\partial x^\sigma} f^j \mid |\sigma| \leq k,\ j = 0, \ldots, m\}. \tag{1.1}$$

1.2. **R-Planes and the Cartan distribution.** An important notion in the geometry of jet spaces is the following

Definition 1.4. An **R-plane** at $\theta_k \in J^k$ is an n-dimensional subspace $R \subset T_{\theta_k} J^k$ tangent to a submanifold of the form $L^{(k)} \subset J^k$ for some n-fold $L \subset E$. I.e.

$$R = T_{\theta_k} L^{(k)}.$$

One easily verifies that if two submanifolds $L, \tilde{L} \subset E$ are tangent of order $k+1$ at $\theta_0 \in E$ then their k-th prolongations are tangent i.e. $T_{[L]^k_{\theta_0}} L^{(k)} = T_{[\tilde{L}]^k_{\theta_0}} \tilde{L}^{(k)}$ and hence to any point $\theta_{k+1} \in J^{k+1}$ one may associate in a unique way an R-plane at the point $\theta_k = \pi_{k+1,k}(\theta_{k+1}) \in J^k$ which we denote with $R_{\theta_{k+1}}$. The correspondence

$$\theta_{k+1} \mapsto R_{\theta_{k+1}}$$

establishes a bijection between points of J^{k+1} and R-planes in J^k, which allows one to think of J^{k+1} as constructed iteratively from J^k as its space of R-planes. One may also interpret this correspondence as a "relative" distribution of rank n along the projection $\pi_{k+1,k}$, which is the content of the following definition.

Definition 1.5. The **tautological relative distribution** R^k (or simply **R-distribution**) on J^k, $k \geq 1$ is the sub-bundle of the pulled-back bundle $\pi^*_{k,k-1}(TJ^{k-1})$ given by $R^k_{\theta_k} := R_{\theta_k}$ for $\theta_k \in J^k$. Its module of sections is the $\mathcal{F}^k$-module of relative vector fields along the projection $\pi_{k,k-1} : J^k \to J^{k-1}$ with values in the associated R-planes of J^{k-1}.

Remark 1.6. Recall that a **relative vector field** X along a map $\phi : M \to N$ of smooth manifolds is a section of the pulled-back bundle $\phi^*(TN)$ on M, i.e. geometrically a smooth map that associates to every point $p \in M$ a tangent vector $X_p \in T_{\phi(p)}N$. In algebraic terms it is equivalently described as a derivation $X : C^\infty(N) \to C^\infty(M)$ where $C^\infty(M)$ is understood as a $C^\infty(N)$-module via $\phi^* : C^\infty(N) \to C^\infty(M)$.

The $\mathcal{F}^k$-module R^k is projective finitely generated and has rank n. A basis in canonical coordinates is given by (truncated at order k) **total derivatives** $D_1^{[k]}, \ldots, D_n^{[k]}$:

$$D_i^{[k]} := \frac{\partial}{\partial x^i} + \sum_{\substack{j=1,\ldots,m \\ |\sigma| \leq k-1}} u^j_{\sigma+1_j} \frac{\partial}{\partial u^j_\sigma}$$

understood as relative fields along the projection. Here $\sigma + 1_j$ is the multi-index $(\sigma_1, \sigma_2, \ldots, \sigma_j + 1, \ldots \sigma_n)$.

Dually the annihilator of the R-distribution is described by so called **Cartan forms:**

$$\omega^j_\sigma := du^j_\sigma - \sum_{i=1}^n u^j_{\sigma+1_i} dx^i, \; j = 1, \ldots, m, \; |\sigma| < k$$

which may be understood as differential forms on J^{k-1} with coefficients in $\mathcal{F}^k$, or equivalently as differential forms on J^k horizontal to the projection $\pi_{k,k-1}$.

Remark 1.7. The R-distribution may also be interpreted as a smooth map $J^{k+1} \to J^1(J^k, n)$ and as such is a particular instance of the family of canonical inclusions

$$J^{k+l}(E, n) \to J^l(J^k(E, n), n). \tag{1.2}$$

These inclusions may be defined via $[L]^{k+l}_{\theta_0} \mapsto [L^{(k)}]^l_{[L]^k_{\theta_0}}$ and are easily verified to be independent of the choice of representative $L \subset E$.

Returning to R-distributions one has that for any $\theta_{k+1} \in J^{k+1}$ the R-planes $R_{\theta_{k+1}}$ and R_{θ_k} where $\theta_k = \pi_{k+1,k}(\theta_{k+1})$ project isomorphically onto each other via the map $d_{\theta_k}\pi_{k,k-1}$. Hence the vector bundle R^{k+1} is naturally identified with the pull-back of R^k to J^{k+1}. Algebraically this is expressed by saying that the homomorphism of $\mathcal{F}_{k+1}$ modules

$$R^{k+1} \to \mathcal{F}_{k+1} \otimes_{\mathcal{F}_k} R^k \tag{1.3}$$

which acts by taking derivation $X \in R^{k+1}$, $X : \mathcal{F}^k \to \mathcal{F}^{k+1}$ and restricting it to $\mathcal{F}^{k-1} \subset \mathcal{F}^k$, is an isomorphism.

It follows that the R^k's are pullbacks of R^1 along the projections $\pi_{k,1}$ and given an R-plane R_{θ_k} we will implicitly make the identifications $R_{\theta_k} = R_{\theta_{k-1}} = \ldots = R_{\theta_1}$.

Definition 1.8. The **Cartan distribution** C^k on J^k is the pre-image of the R-distribution R^k, i.e. at a point θ_k it is defined by

$$C^k_{\theta_k} = (d_{\theta_k}\pi_{k,k-1})^{-1}(R_{\theta_k}).$$

We emphasize that this is a "true" distribution on J^k and not a relative one. It is easy to check that $C^k_{\theta_k}$ may also be defined as the span of all R-planes at θ_k.

Example 1.9. In case that the number of dependent variables is 1, J^1 is of dimension $2n + 1$ and its Cartan distribution is the standard example of a contact structure.

Fixing canonical coordinates a basis of the Cartan distribution is given by vector fields

$$\frac{\partial}{\partial u^j_\sigma}, \;\; 1 \leq j \leq m, \; |\sigma| = k$$

together with (truncated) total derivatives $D_i^{[k]}$ $1 \leq i \leq n$ understood in local coordinates as true (i.e., not relative) vector fields.

Notation 1.10. We shall henceforth drop the truncation index $[k]$ from total derivatives when it is clear from the context on which order of jets we are working

Dually the annihilator of the Cartan distribution is just described by Cartan forms ω^j_σ, $|\sigma| < k$.

We remark that in general when fixing a divided chart, the total derivatives $D_1, \dots D_n$ together with all vertical fields $\frac{\partial}{\partial u^i_\sigma}$, $1 \leq i \leq m$, $|\sigma| \leq k$ form a frame on J^k which we call a **canonical non-holonomic frame** (associated to the chart). Its commutation relations are:

$$\begin{aligned} [D_i, D_j] &= 0 \\ \left[\frac{\partial}{\partial u^k_\rho}, \frac{\partial}{\partial u^j_\sigma}\right] &= 0 \\ \left[\frac{\partial}{\partial u^j_\sigma}, D_i\right] &= \begin{cases} \frac{\partial}{\partial u^j_{\sigma - 1_i}}, \text{ if } \sigma_i \geq 1 \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

From these relations one sees that Cartan distributions C^k are not involutive.

The dual basis of the non-holonomic canonical frame is given by: the one-forms $dx^1 \dots, dx^n$ the Cartan forms ω^j_σ, $j = 1, \dots, m$, $|\sigma| < k$ and the forms du^j_σ, $j = 1, \dots, m$, $|\sigma| = k$. Their differentials, expressed w.r.t. this co-frame are

$$d\omega^j_\sigma = \begin{cases} \sum_{i=1}^n dx^i \wedge \omega^j_{\sigma+1_i} \text{ if } |\sigma| < k-1 \\ \sum_{i=1}^n dx^i \wedge du^j_{\sigma+1_i} \text{ if } |\sigma| = k-1 \end{cases} .$$

1.3. **Symmetry group of the Cartan Distribution.** For completeness and since we will need it later we state here the Lie-Baecklund theorem which describes the structure of the automorphism group of the Cartan distribution on J^k.

Recall that a **point symmetry** of a distribution $P \subset TM$ is an automorphism of the underlying manifold M which preserves the distribution. An **infinitesimal symmetry** of P is vector field X on the manifold whose local flow consists of point symmetries, or equivalently satisfies the condition:

$$[X, Y] \in P, \; \forall Y \in P$$

Theorem 1.11. *If $m > 1$ then all finite symmetries of the Cartan distribution on J^k are prolongations of diffeomorphism of J^0. If $m = 1$ then they are prolongations of contact transformation of (J^1, C^1) .*

Here the prolongation of a symmetry is obtained by identifying J^{k+1} with its R-planes in J^k. A proof is given in [17] but may also be obtained from the description of maximal involutive subspaces given further down.

1.4. **R-manifolds and geometric singularities.** The importance of the Cartan distribution is that it allows one to recognize submanifolds $W \subset J^k$ which are of the form $W = L^{(k)}$ for a smooth n-submanifold $L \subset E$ by identifying them as certain integral submanifolds.

Recall that an **integral submanifold** of a distribution $P \subset TM$ on a manifold M is a submanifold $W \subset M$ such that $T_pW \subseteq P_p$ for all $p \in W$. We say that the integral submanifold is (locally) **maximal** if none of its open subsets are contained in an integral submanifold of strictly bigger dimension. Since we only deal with local aspects we will just call these manifolds maximal, though this should not be confused with the concept of globally maximal involutive manifolds analogous to the concept of maximal trajectories of vector fields.

Remark 1.12. A non-involutive distribution may possess maximal integral submanifolds of different dimensions even through a single point. In the case of the Cartan distribution C^k there is a nice description of all its "regular" maximal integral submanifolds [2]: they may be divided into types according to the dimension of their projection to J^{k-1} (this also follows from results in section 4).

For the current purposes only the following result is of importance.

Proposition 1.13. [2] *A smooth submanifold $W \subset J^k$ coincides everywhere locally with the prolongation $L^{(k)}$ of some n-submanifold $L \subset E$ if and only if the following two conditions are satisfied:*

1) W is an n-dimensional integral submanifold of C^k.

2) W is transversal to the projection $\pi_{k,k-1}$.

Moreover any such W is a maximal integral submanifold of C^k of smallest dimension among all maximal integral submanifolds.

Let us call a submanifold $W \subset J^k$ which satisfies the conditions of 1.13 a regular **R-manifold**, since it might be described as a smooth submanifold of J^k whose tangent planes are all R-planes.

Observe that even though locally a regular R-manifold $W \subset J^k$ is the prolongation of a smooth n-submanifold $L \subset E$ this need not be the case globally since $\pi_{k,0}(W)$ might have singularities like points of self-intersection. Hence R-manifolds may be considered a first generalization of smooth submanifolds of E. Indeed they can be thought of as "canonical parametrizations" of immersed n-dimensional manifolds $L \to E$.

As suggested by Vinogradov in [26] this generalization may be taken a step further by dropping condition 2) of proposition 1.13. The resulting concept is the geometric notion of generalized solutions for non-linear PDEs.

Definition 1.14. A **generalized R-manifold** $W \subset J^k$ is a smooth maximal integral submanifold of C^k of dimension n.

Remark 1.15. If the number of independent variables m is strictly bigger than 1, generalized R-manifolds are almost everywhere transversal to $\pi_{k,k-1}$ i.e. are almost everywhere regular. In the case $m = 1$ there appears another type of maximal integral submanifold of dimension n which project to $n-1$ dimensional submanifolds along $\pi_{k.k-1}$ (see section 4).

A point θ_k of a generalized R-manifold $W \subset J^k$ at which the projection $d_{\theta_k}\pi_{k,k-1}$: $T_{\theta_k}W \to T_{\theta_{k-1}}J^{k-1}$ has a non-trivial kernel of dimension $s > 0$ is called **singular point of W of type s**, and the tangent plane $T_{\theta_k}W$ at such a point is called a **singular R-plane in J^k of type s**. The set of all s-type singular points of W is denoted with

$$\Sigma_s W$$

and called the **s-singularity locus** of W. We stress that despite the terminology a generalized R-manifold is smooth everywhere and only the map $\pi_{k,k-1}|_W : W \to J^{k-1}$ has singularities.

We shall always assume that $\Sigma_s W$ is a smooth submanifold of W of dimension $n-s$ and that $\ker T\pi_{k.k-1}|_W$ is transversal to $\Sigma_s W$. Hence we consider only the case that the projection $\pi_{k,k-1}|_W$ has Thom-Boardman singularities of the simplest kind. Singularities of type 1 will also be referred to as **fold type singularities**.

Projecting a generalized R-manifold W down to E one generally obtains a submanifold with singularities, but unlike the singularities obtained from projections of regular R-manifolds these singularities do not disappear by restricting to sufficiently small regions of W. Geometrically one may think of generalized R-manifolds as n-dimensional submanifolds $\tilde{W} \subset E$ with singularities that can be resolved by prolonging $\tilde{W}$ to some $J^k(E, n)$.

Example 1.16. Consider the plane $E=\mathbb{R}^2$ with coordinates x,u. The 1-dimensional submanifold given by the equation $u^2=x^{2k+1}$ has a singularity at $x=u=0$ which is resolved when prolonging it to $J^k(E,1)$: the two regular branches may be written as

$$u=\pm x^{k+\frac{1}{2}},\ x>0$$

and hence their prolongations are given by $u_j=\pm(k+\frac{1}{2})\cdots(k-j+1+\frac{1}{2})x^{k-j+\frac{1}{2}}, x>0, j=1,\ldots,k$. The expression for x which results from the last of these equations $(j=k)$ may be substituted in the other equations to arrive at an equivalent system which describes the closure:

$$u_k^2-(k+\frac{1}{2})^2\cdots(1+\frac{1}{2})^2x=0$$

$$u_j-(k+\frac{1}{2})\cdots(k-j+1+\frac{1}{2})\left((k+\frac{1}{2})^{-1}\cdots(1+\frac{1}{2})^{-1}u_k\right)^{2(k-j)+1}=0,\ j=0,\ldots,k-1$$

This is easily seen to describe a smooth submanifold and hence a generalized R-manifold. At $x=u=u_1=\ldots=u_k=0$ the projection $\pi_{k,k-1}$ degenerates and so there is one singular point of type 1. Extending this example trivially by adding more independent variables $x_2,\ldots,x_n$ one obtains an example of fold-type singularities for any number of independent variables. One may of course also increase the number of dependent variables u by adding trivial equations $u^2=u^3=\ldots=u^j=0$.

While projecting a generalized R-manifold W to lower jet spaces makes it more singular, going the other direction i.e., taking prolongations of W makes the singularity locus "blow up". This ocurrs since if $\theta_k\in W\backslash\Sigma W$, then $T_{\theta_k}W$ is a regular R-plane and so determines a point of $W^{(1)}\subset J^{k+1}$, but letting θ_k approach the singularity locus ΣW the corresponding point in the prolongation $W^{(1)}$ must necessarily diverge in the fibers $J^{k+1}_{\theta_k}$ since the tangent planes T_pW approaches a singular R-plane which corresponds to no point in J^{k+1}. Hence when describing a non-singular component of the submanifold $\pi_{k,0}(W)\subset E$ in local coordinates as

$$\{u^i=f^i(x^1,\ldots,x^n)\mid i=1,\ldots,m\}$$

the $k+1$-th derivatives of the functions $f_1,\ldots,f_m$ must diverge when approaching the singularity. In particular a regular component of $\pi_{k,0}(W)$ may not be extended smoothly across its singularity locus and so these singularities are not avoidable (here by singularity locus we mean those singular points of $\pi_{k,0}(W)$ which are not of self intersecting type).

Remark 1.17. An obvious question that arises is whether singularities of affine algebraic subvarieties in $\mathbb{R}^{n+m}$ may be resolved by taking jet prolongations, i.e. if they belong to the class of generalized submanifolds from above. The author is not aware if the answer to this question is known.

2. PDEs their generalized solutions and singularity equations

After the review of jet spaces we turn to PDEs

2.1. PDEs and prolongations.

In local coordinates a system of nonlinear partial differential equations, or **PDE** for short, is usually expressed as

$$\begin{aligned} F^1(x^1,\ldots,x^n,u^1,\ldots,u^m,\ldots,u^j_\sigma,\ldots) &= 0\\ &\vdots\\ F^l(x^1,\ldots,x^n,u^1,\ldots,u^m,\ldots,u^j_\sigma,\ldots) &= 0 \end{aligned} \tag{2.1}$$

where $F^1 \dots F^l$ are smooth functions depending on jet coordinates x^i, u^j_σ up to some maximal order k i.e. $|\sigma| \leq k$. If we consider the F^i as functions on J^k they cut out a subset

$$\mathcal{E} = \{F^1 = 0, \dots, F^l = 0\} \subset J^k$$

which we may take as a preliminary coordinate free definition of a PDE. A **solution** of $\mathcal{E}$ is then just an n-dimensional submanifold $L \subset E$ such that $L^{(k)} \subset \mathcal{E}$. We will exclude singular PDEs from our considerations and assume that $\mathcal{E} \subset J^k$ is a smooth submanifold. Moreover whenever we express $\mathcal{E}$ locally as the zero set of some functions F^i we assume that their differentials dF^i are linearly independent along $\mathcal{E}$.

This preliminary coordinate free definition of a PDE has some conceptual as well as practical drawbacks and we roughly motivate what leads to a better definition: suppose we pick $\theta_k \in \mathcal{E}$ and ask if a solution of $\mathcal{E}$ can exist which passes through θ_k. An obviously necessary condition is that there exist at least one R-plane at θ_k which is tangent to the equation i.e. $R_{\theta_{k+1}} \subset T_{\theta_k}\mathcal{E}$. Another necessary condition is that for every $\theta_k \in \mathcal{E}$ the R-plane R_{θ_k} is tangent to the set $\pi_{k,k-1}(\mathcal{E})$. To take into consideration these conditions one introduces the following concept.

Definition 2.1. Let $\mathcal{E} \subseteq J^k$ be a submanifold. The set

$$\mathcal{E}^{(1)} := \{\theta_{k+1} \in J^{k+1} \mid R_{\theta_{k+1}} \subset T_{\theta_k}\mathcal{E}\}$$

is called the **first prolongation** of $\mathcal{E}$.

In local coordinates $\mathcal{E}^{(1)}$ will be given as the zero set of the functions F^j and $D_i(F^j)$ $j = 0, \dots, l$, $i = 0, \dots, n$, and hence in general, even if $\mathcal{E}$ is smooth $\mathcal{E}^{(1)}$ need not be smooth.

Remark 2.2. For n-dimensional submanifolds $L \subset J^0$ the notion of prolongation introduced here coincides with the one defined earlier.

Obviously $\mathcal{E}^{(1)}$ may also be considered as a PDE and one sees from the coordinate description that $\mathcal{E}$ and $\mathcal{E}^{(1)}$ have the same set of solutions.

Example 2.3. The system $u_x = f(x, y), u_y = g(x, y)$ where the functions g, f are fixed, give a typical example of an equation whose first prolongation can be empty if $g_x \neq f_y$.

According to the above remark one could drop the points $\mathcal{E} \smallsetminus \pi_{k+1,k}(\mathcal{E}^{(1)})$ since no solution can pass trough them and replace the original equation $\mathcal{E}$ with $\pi_{k+1,k}(\mathcal{E}^{(1)})$ (assuming it is smooth).

One should then repeat this procedure of prolongation and projection until one arrives at a stable situation where the projection $\mathcal{E}^{(1)} \to \mathcal{E}$ is surjective (and both $\mathcal{E}^{(1)}$ and $\mathcal{E}$ are non-empty). If this is satisfied and the projection $\mathcal{E} \to \pi_{k,k-1}(\mathcal{E})$ is submersive then the second necessary condition from above i.e.

$$(\pi_{k,k-1}(\mathcal{E}))^{(1)} \subseteq \mathcal{E}$$

will also be satisfied. Obviously one will also want that $(\mathcal{E}^{(1)})^{(1)} \to \mathcal{E}^{(1)}$ be surjective and submersive for similar reasons and so on for all further prolongations. This leads us to use the following definition of a PDE

Definition 2.4. A **system of nonlinear PDEs** $\mathcal{E}$ consist of a family of smooth submanifolds

$$\mathcal{E}^l \subseteq J^l,\ l \in \mathbb{N}, l \geq l_0$$

which project surjectively and submersively onto each other under the canonical projections $\pi_{l,l-1} : J^l \to J^{l-1}$ and satisfy the conditions

i) $\mathcal{E}^{l+1} \subseteq (\mathcal{E}^l)^{(1)}$ $\forall l \geq l_0$

ii) $\exists k_0 \in \mathbb{N}$ such that for all $l \geq k_0$ $\mathcal{E}^{l+1} = (\mathcal{E}^l)^{(1)}$

To be in agreement with the literature we should say that the above definition of PDE corresponds to formally integrable PDEs (and maybe "stably" non-singular, though this terminology does not exist), but only these will be considered here. The problem of how to decide when a given subset $\mathcal{E} \subset J^k$ gives rise to a family as above will not be discussed here, but see for example [10].

Notation 2.5. The regime of all $l \geq k$ where ii) holds will be referred to as the **stable regime** of the equation.

Hence a PDE $\mathcal{E}$ consist of a tower

$$\mathcal{E}^{l_0} \leftarrow \ldots \leftarrow \mathcal{E}^{l-1} \leftarrow \mathcal{E}^{l} \leftarrow \mathcal{E}^{l+1} \leftarrow \cdots \tag{2.2}$$

where the arrows are the restrictions of the projections $\pi_{l+1,l}$ and shall be denoted with the same symbol. Obviously for l sufficiently big we may recover this tower from the knowledge of any component $\mathcal{E}^l \subset J^l$ by either projecting or making iterated prolongations of $\mathcal{E}^l$. Observe that we could also extend the $\mathcal{E}^l$ to all $l < l_0$ by projecting them down, but these might be manifolds with singularities.

By our definition of PDEs the relative distributions R^l of J^l are restrictable to projections $\mathcal{E}^l \to \mathcal{E}^{l-1}$ in the sense that if $\theta_l \in \mathcal{E}^l$ then $R_{\theta_l} \subset T_{\theta_{l-1}}\mathcal{E}^{l-1}$. We shall call these restricted relative distributions the **R-distributions of $\mathcal{E}$** and denote them with the same symbol R^l or sometimes $R\mathcal{E}^l$. The restriction of the **Cartan distribution to $\mathcal{E}^l$** is denoted with

$$C\mathcal{E}^l := C^l \cap T\mathcal{E}^l$$

Observe that the Cartan distribution $C\mathcal{E}^l$ may equivalently be described as the inverse image of the R-distribution RE^l, i.e.

$$C_{\theta_l}\mathcal{E}^l = T_{\theta_l}\pi_{l,l-1}^{-1}(R_{\theta_{l-1}})$$

where $T_{\theta_l}\pi_{l,l-1} : T_{\theta_l}\mathcal{E}^l \to T_{\theta_{l-1}}\mathcal{E}^{l-1}$ and hence the R-distributions contain the same information as the Cartan distributions.

The tower $(\mathcal{E}^l)_{l \geq l_0}$ of smooth fiber bundles gives naturally rise to a (generally) infinite dimensional smooth manifold by considering the inverse limit, called the infinite prolongation of the PDE and denoted with $\mathcal{E}^\infty$. Seen from the infinite prolongation the family of relative distributions $R\mathcal{E}^l$ becomes a true distribution on $\mathcal{E}^\infty$ called the **Cartan distribution $C\mathcal{E}$ of $\mathcal{E}^\infty$**. The notation and name come from the fact that this distribution may also be seen as the inverse limit of the Cartan distributions $C\mathcal{E}^l$. It turns out that this Cartan distribution is n-dimensional and involutive (a basis is simply given by the (un-truncated) total derivatives D_i), and its n-dimensional integral submanifolds correspond precisely to smooth solutions of the equation $\mathcal{E}$ [2].

The precise way of treating $\mathcal{E}^\infty$ as an infinite dimensional manifold and doing calculus on it is not by introducing a Banach or Frechet manifold structure, but by extending the duality between differential geometry and commutative algebra to this case. Since the resulting category of manifolds and the associated differential calculus is of importance but not commonly known (some elements are given in [2, 14]) we review some basic facts in the following section.

2.2. The categories of co-filtered and pro-finite manifolds.

Definition 2.6. A **co-filtered manifold** $\mathcal{E}$ is a tower

$$\mathcal{E} = (\mathcal{E}^{l_0} \twoheadleftarrow \mathcal{E}^{l_0+1} \twoheadleftarrow \mathcal{E}^{l_0+2} \twoheadleftarrow \ldots)$$

of smooth fiber bundles i.e. each $\mathcal{E}^l$, $l \in \mathbb{N}$ is a smooth, finite dimensional manifold and the maps $\pi_{l,l-1}\mathcal{E}^l \twoheadrightarrow \mathcal{E}^{l-1}$ are smooth fiber bundles.

A morphism (or smooth map) $\phi : \mathcal{E} \to \tilde{\mathcal{E}}$ of co-filtered manifolds consist of a family of smooth maps

$$\phi_l : \mathcal{E}^l \to \tilde{\mathcal{E}}^{l-d},\ l \geq l_\phi \in \mathbb{N}$$

where $d \in \mathbb{N}$ is a fixed number called the **shift** of ϕ and all the ϕ_l are compatible with the projections:

$$\phi_l \circ \pi_{l+1,l} = \tilde{\pi}_{l-d+1,l-d} \circ \phi_{l+1},\ \forall l \geq l_\phi \in \mathbb{N}$$

For short we also write

$$\phi \circ \pi = \tilde{\pi} \circ \phi$$

Obviously each ϕ_l determines all the previous $\phi_{_i}$ $i \leq l$.

Example 2.7. Any finite dimensional smooth manifold M may be considered as a co-filtered manifold by putting $\mathcal{E}^i = M$ for all i and taking the projections to be the identity. Moreover any morphism of finite dimensional manifolds induces a morphism of shift 0 (or any shift) of the associated co-filtered manifolds.

yet another

Example 2.8. (The **standard** $\mathbb{R}^\chi$) Given any non decreasing function $\chi : \mathbb{N} \to \mathbb{N}$ define an associated co-filtered manifold by putting $\mathcal{E}^l = \mathbb{R}^{\chi(l)}$ with standard coordinates denoted by $y^1, \ldots, y^{\chi(l)}$ and projections $\pi_{l,l-1}$ given by the standard projections

$$(y^1, \ldots, y^{\chi(l)}) \mapsto (y^1, \ldots, y^{\chi(l-1)})$$

we shall denote this manifold with $\mathbb{R}^\chi$

Definition 2.9. The **growth function** of a co-filtered manifold $\mathcal{E}$ is the function $\chi_\mathcal{E} : \mathbb{N} \to \mathbb{N}$ given by

$$\chi_\mathcal{E}(l) = \dim(\mathcal{E}^l)$$

Obviously this is a non decreasing function and we say that $\mathcal{E}$ is **finite dimensional** if its growth function is bounded and **infinite dimensional** otherwise. Obviously the following is true

Proposition 2.10. *(Existence of local coordinates) Let $\theta_l \in \mathcal{E}^l, l \in \mathbb{N}$ be a sequence of points in a co-filtered manifold such that $\pi_{l+1,l}(\theta_{l+1}) = \theta_l,\ \forall l \in \mathbb{N}$. Then there exists a sequence of open neighborhoods $(U^l)_{l\in\mathbb{N}}$, $\theta_l \in U^l$ with $\pi_{l+1,l}(U^{l+1}) = U^l$ and a morphism ϕ of shift 0 from $(U^l)_{l\in\mathbb{N}}$ to $\mathbb{R}^{\chi_\mathcal{E}} = (\ldots \twoheadleftarrow \mathbb{R}^{\chi_\mathcal{E}(l-1)} \twoheadleftarrow \mathbb{R}^{\chi_\mathcal{E}(l)} \twoheadleftarrow \mathbb{R}^{\chi_\mathcal{E}(l+1)} \twoheadleftarrow \ldots)$ which is a diffeomorphism at each level, i.e. $\phi_l : U^l \xrightarrow{\sim} \mathbb{R}^{\chi_\mathcal{E}(l)}$*

Remark 2.11. In the appendix we give a local classification of co-filtered manifolds.

The inverse limit of a co-filtered manifold $\mathcal{E}$ is denoted with $\mathcal{E}^\infty$ and called the **infinite prolongation** of $\mathcal{E}$. As a set, the elements of $\mathcal{E}^\infty$ are by definition sequences of points

$$\theta = (\theta_l)_{l\geq l_0}$$

with $\theta_l \in \mathcal{E}^l$ and $\pi_{l,l-1}(\theta_l) = \theta_{l-1}, \forall l \geq l_0$. Geometrically one may think of $\mathcal{E}^\infty$ as lying above all the manifolds $\mathcal{E}^l$ and projecting down onto them by the rule $\pi_{\infty,l}(\theta) = \theta_l$.

A **topology** on $\mathcal{E}^\infty$ may be defined by declaring an open set to be a family $U^l \subset \mathcal{E}^l, l \geq l_0$ of open sets such that $U^j = \pi_{j+1,j}(U^{j+1})$ for all $j \in \mathbb{N}$.

One verifies immediately that a morphism of co-filtered manifolds induces a map in the infinite prolongations which is continuous. But two different morphism of co-filtered manifolds may induce the same map on the infinite prolongations. This is due to the fact that the projections $\pi = (\pi_{l+1,l})_{l\in\mathbb{N}}$ form a morphism of co-filtered manifolds of shift 1 but leave points of $\mathcal{E}^\infty$ fixed, i.e. they act like the identity on $\mathcal{E}^\infty$. In fact one easily proves the following

Lemma 2.12. *Let $\phi : \mathcal{E} \to \mathcal{E}$ be a morphism of shift d that acts like the identity on points of $\mathcal{E}^\infty$, then $\phi = \pi^d$.*

This leads one to introduce the following equivalence relation.

Definition 2.13. Two morphism ϕ_1, ϕ_2 between co-filtered manifolds $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are called **equivalent** $\phi_1 \sim \phi_2$, if there exist numbers $e_1, e_2 \in \mathbb{N}$ such that

$$\phi_1 \circ \pi^{e_1} = \phi_2 \circ \pi^{e_2}$$

One easily checks that this is indeed an equivalence relation and moreover if $\phi_1 \sim \phi_2$ and $\psi_1 \sim \psi_2$ then $\psi_1 \circ \phi_1 \sim \psi_2 \circ \phi_2$, hence we may consider the quotient category which by definition is the **category of pro-finite manifolds.** Hence objects in this category are co-filtered manifolds while morphism are equivalence classes of morphisms. Informally we will understand the infinite prolongation $\mathcal{E}^\infty$ as the pro-finite manifold associated to the co-filtered manifold $\mathcal{E}$.

Remark 2.14. A priori one might allow for a more general notion of morphism on the level of co-filtered manifolds than the one given here, namely morphism that have no fixed shift but for example a growing shift. However we will show that for diffieties, which are co-filtered manifolds supplied with a distribution of a certain kind, the maps preserving the distribution have always a fixed shift.

It is useful to introduce the dual point of view: to any co-filtered manifold we can associated a chain of inclusions of algebras

$$\ldots \subseteq \mathcal{F}^{l-1} \subseteq \mathcal{F}^l \subseteq \mathcal{F}^{l+1} \subseteq \ldots$$

where $\mathcal{F}^l = C^\infty(\mathcal{E}^l)$ and the inclusions are given by $\pi^*_{l,l-1} : \mathcal{F}^{l-1} \to \mathcal{F}^l$. This allows us to define an $\mathbb{R}-$algebra called the **algebra of smooth functions on $\mathcal{E}^\infty$** as

$$\mathcal{F} = \bigcup_l \mathcal{F}^l$$

This is an algebra supplied with a filtration by subalgebras. To distinguish this notion from what is commonly called a filtered algebra we shall sometimes call them **sub-filtered algebras** but if no confusion arises we just call them filtered algebras.

Lemma 2.15. *The real spectrum of $\mathcal{F}$ (i.e. the set of algebra morphisms from $\mathcal{F}$ to $\mathbb{R}$) is in one to one correspondence with points of $\mathcal{E}^\infty$ and the algebra $\mathcal{F}$ is geometric.*

Proof. Given a point $\theta \in \mathcal{E}^\infty$ one may evaluate it on an element $f \in \mathcal{F}$ as follows: chose a k such that $f \in \mathcal{F}^k$ and set $f(\theta) = f(\theta_k)$. This is well defined since if one considered $f \in \mathcal{F}^{k+c}$ and evaluated on θ_{k+c} the value remains the same. Hence every point of $\mathcal{E}$ gives rise to an element of the spectrum.

Assume conversely that an element $\tau : \mathcal{F} \to \mathbb{R}$ of the real spectrum is given. Then it gives rise to a family of algebra morphisms $\tau_l : \mathcal{F}^l \to \mathbb{R}$ by restricting to the subalgebras, and hence by the spectral theorem for finite dimensional manifolds [24] one obtains a sequence of points $\theta_k \in \mathcal{E}^k$. It is obvious that this sequence satisfies $\pi_{k,l}(\theta_k) = \theta_l$ and so constitutes a point of $\mathcal{E}^\infty$. It is an easy verification that this correspondences are inverse to each other.

We recall that an algebra is geometric if $\bigcap_{p \in \mathrm{Spec}_{\mathbb{R}}(\mathcal{F})} \mu_p = 0$ where μ_p is the maximal ideal corresponding to a point in the spectrum. So let $f \in \bigcap_{p \in \mathrm{Spec}_{\mathbb{R}}(\mathcal{F})} \mu_p$, then $f \in \mathcal{F}^k$ for some k but since f must vanish on all points of $\mathcal{E}^\infty$ is must also vanish on all points of $\mathcal{E}^k$ and hence $f = 0$. ☐

The duality between geometry and algebra holds even further as in the finite dimensional case since the association $\mathcal{E} \mapsto \mathcal{F}(\mathcal{E})$ becomes a fully faithful functor from the category of pro-finite manifolds into the category of sub-filtered algebras with a special kind of algebra morphism which we define next.

Definition 2.16. Given two sub-filtered algebras $\mathcal{F} = \bigcup_l \mathcal{F}^l$ and $\tilde{\mathcal{F}} = \bigcup_l \tilde{\mathcal{F}}^l$ we say that an algebra morphism $\phi : \tilde{\mathcal{F}} \to \mathcal{F}$ has a **finite shift** if there exists a $d \in \mathbb{Z}$ such that

$$\phi : \tilde{\mathcal{F}}^l \subset \mathcal{F}^{l+d},\ \forall l$$

In this case we say that ϕ has **shift** $\leq d$. It is immediately verified that filtered algebras with morphism of finite shift form a category which we called the **category of (sub)-filtered algebras**.

Let now $\phi : \mathcal{E} \to \tilde{\mathcal{E}}$ be a morphism of co-filtered manifolds of shift d, and let $\mathcal{F} = \mathcal{F}(\mathcal{E})$ and $\tilde{\mathcal{F}} = \mathcal{F}(\tilde{\mathcal{E}})$, then ϕ induces a morphism $\phi^* : \tilde{\mathcal{F}} \to \mathcal{F}$ as follows: for $f \in \tilde{\mathcal{F}}$ there is l such that $f \in \tilde{\mathcal{F}}^l$ and so $\phi_{l+d} : \mathcal{E}^{l+d} \to \tilde{\mathcal{E}}^l$ allows one to pull f back to $\mathcal{E}^{l+d}$ and put $\phi^*(f) = \phi^*_{l+d}(f) \in \mathcal{F}^{l+d} \subset \mathcal{F}$. Similarly as in the proof of the lemma 2.15 on checks that this is independent of the choice of l.

Obviously the operation $\phi \mapsto \phi^*$ is a co-variant functor from the category co-filtered manifolds into the category of sub-filtered algebras. Moreover if we replace the pre-morphism ϕ with $\phi \circ \pi^e$ the induced morphism of algebras remains the same. i.e. $\phi^* = (\phi \circ \pi^e)^*$ hence we have almost proved

Proposition 2.17. *Any morphism $[\phi] : \mathcal{E} \to \tilde{\mathcal{E}}$ of pro-finite manifolds induces a morphism of algebras of finite shift*

$$[\phi]^* : \mathcal{F}(\tilde{\mathcal{E}}) \to \mathcal{F}(\mathcal{E})$$

Conversely any morphism $\psi : \mathcal{F}(\tilde{\mathcal{E}}) \to \mathcal{F}(\mathcal{E})$ of finite shift determines a morphism of pro-finite manifolds.

Proof. It remains to show that a morphism ψ of filtered algebras of shift $\leq d$ determines an equivalence class of morphism between the cofiltered manifolds. By fixing d we construct a morphism of co-filtered manifolds using the spectral theorem in finite dimensions and taking $\psi^*_l : \mathcal{E}^l \to \widetilde{\mathcal{E}}^{l-d}$ to be the morphism on spectra induced by $\psi : \widetilde{\mathcal{F}}^{l-d} \to \mathcal{F}^l$. It is straightforward to check that the equivalence class of ψ^* is independent on the choice of d. □

In the category of sub-filtered algebras with morphism of finite shift, the filtration of an algebra is actually not an invariant but its equivalence class under the following equivalence relation is

Definition 2.18. Two filtrations by subalgebras $\{\mathcal{F}^l\}, \{\mathcal{F}'^l\}$ of an algebra $\mathcal{F}$ are called **equivalent** if there exist natural numbers $c, c' \in \mathbb{N}$ such that

$$\begin{aligned} \mathcal{F}^l &\subseteq \mathcal{F}'^{l+c'} \\ \mathcal{F}'^l &\subseteq \mathcal{F}^{l+c} \end{aligned}$$

for all $l \gg 0$.

Lemma 2.19. *Let $\phi : \mathcal{F} \to \mathcal{F}'$ be an isomorphism in the category of sub-filtered algebras, then the induced filtration $\phi(\mathcal{F}^k)$ on $\mathcal{F}'$ is equivalent to the filtration $\mathcal{F}'^k$*

Proof. Let ϕ be of shift $\leq c$. Obviously ϕ must be injective and surjective and its set theoretic inverse must coincide with the inverse ϕ^{-1} in the category of sub-filtered algebras, which means that the inverse is also of finite shift say c'. Then obviously

$$\begin{aligned} \phi(\mathcal{F}^k) &\subset \mathcal{F}'^{k+c} \\ \mathcal{F}'^k = \phi(\phi^{-1}(\mathcal{F}'^k)) &\subset \phi \quad (\mathcal{F}^{k+c'}) \end{aligned}$$

□

2.2.1. *Vector fields and differential forms on pro-finite manifolds.* Let $\mathcal{F}$ denote the filtered algebra of functions on a pro-finite manifold $\mathcal{E}^\infty$.

Definition 2.20. A derivation $X : \mathcal{F} \to \mathcal{F}$, (i.e. an $\mathbb{R}$-linear map such that $X(fg) = X(f)g + fX(g)\,\forall f, g \in \mathcal{F}$) is said to be of **finite shift** if there is a $r \in \mathbb{N}$ such that

$$X(\mathcal{F}^k) \subset \mathcal{F}^{k+r},\ \forall k \geq k_0 \tag{2.3}$$

In such a case we say that X has **shift** $\leq r$. The set of all derivations of shift $\leq r$ is denoted with $D_{\leq r}(\mathcal{F})$. Observe that if a derivation is of finite shift with respect to one sub-filtration of $\mathcal{F}$ it is also of finite shift with respect to any equivalent sub-filtration (but the shift may change) and and hence the set of all finite shift filtrations $D(\mathcal{F})$ is well behaved under isomorphism of sub-filtered algebras. By definition $D(\mathcal{F})$ will be called the module of **vector fields** on the pro-finite manifold $\mathcal{E}^\infty$.

Obviously if $X \in D_{\leq r}\mathcal{E}^\infty$ then $fX \in D_{\leq r}\mathcal{E}^\infty$ and hence the $D_{\leq r}\mathcal{E}^\infty$ are submodules of $D\mathcal{E}^\infty$. Moreover $D_{\leq r}\mathcal{E}^\infty \subset D_{\leq r+1}\mathcal{E}^\infty$ and so $D\mathcal{E}^\infty$ is filtered and $D\mathcal{E}^\infty = \bigcup_r D_{\leq r}\mathcal{E}^\infty$

Remark 2.21. Geometrically one may think of a derivation $X \in D_r\mathcal{E}^\infty$ as a family of relative vector fields along each projection $\pi_{k+r,k}$ such that each higher relative field is projectable onto the lower one. Nevertheless to make this correspondence bijective one should introduce an equivalence relation on such families of relative fields similar to the condition on morphism of co-filtered manifolds discussed earlier. We will not elaborate this point.

Example 2.22. The **total derivatives on** J^∞

$$D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma+1_i}\partial_{u^j_\sigma}$$

are derivations of shift ≤ 1. Here the sum is an infinite one, but this does not cause a problem of convergence since the derivations are applied to functions which only depend on a finite number of variables. Indeed restricting them to $\mathcal{F}^k$ one obtains the truncated total derivatives which are relative fields along $\pi_{k+1,k}$. These are also the typical examples of vector fields on a pro-finite manifold which don't possess a flow, as is seen in the simplest example $n = m = 1$: the infinite prolongation of any graph of a function $u = f(x)$ to J^∞ is tangent to D_x and should hence be a trajectory of D_x, but since there are smooth functions which are tangent of infinite order at some points but differ at others there cannot exist a flow of D_x.

Remark 2.23. The condition $X(\mathcal{F}^k) \subset \mathcal{F}^{k+d}$ means that when we think of a derivation X as a vector field on $\mathcal{E}^\infty$ and project it down via $\pi_{\infty,k} : \mathcal{E}^\infty \to \mathcal{E}^k$, then the obtained relative field is constant along fibers of the projection $\pi_{\infty,k+d} : \mathcal{E}^\infty \to \mathcal{E}^{k+d}$, i.e. depends only on the points in $\mathcal{E}^{k+d}$. One might say that one has "finite dimensional control" over the projected fields. Again we could allow more general derivations which have no bounded shift. But if one restricts to infinitesimal symmetries of the Cartan distribution the condition of bounded shift follows from some observations below.

Turning to differential forms, the projections $\pi_{k,k-1} : \mathcal{E}^k \to \mathcal{E}^{k-1}$ give rise to canonical inclusions

$$\Lambda^p\mathcal{E}^{k-1} \subset \Lambda^p\mathcal{E}^k$$

by pulling back, which allows us to give the following definition.

Definition 2.24. The **differential p-forms** on $\mathcal{E}^\infty$ are defined as

$$\Lambda^p\mathcal{E}^\infty := \bigcup_k \Lambda^p\mathcal{E}^k$$

We will not prove the following lemma, but it is easily verified by using the respective results on finite dimensional manifolds

Lemma 2.25. *$\Lambda^p\mathcal{E}^\infty$ is an $\mathcal{F}$-module in a natural way. Moreover the wedge product of differential forms and the De Rham differential can be defined by using the respective structures on the $\Lambda^p\mathcal{E}^k$'s.* If $\phi : \mathcal{E}^\infty \to \mathcal{G}^\infty$ is a morphism of pro-finite manifolds then the pullback of forms is well-defined.

Remark 2.26. There is a more conceptual definition of differential forms on $\mathcal{E}^\infty$ as the representative objects of certain functors [10] but this would require introducing a suitable category of modules over sub-filtered algebras which we avoid.

Since the subsets $\Lambda^p\mathcal{E}^k$ are not $\mathcal{F}$-submodules in $\Lambda^p\mathcal{E}$ we denote with

$$\Lambda^p_k\mathcal{E}^\infty := \mathcal{F}\cdot\Lambda^p\mathcal{E}^k$$

the $\mathcal{F}$-submodule generated by them, which gives $\Lambda\mathcal{E}^\infty$ a filtration by submodules. Note that these submodules are not closed with respect to the De Rham differential while the $\Lambda^p\mathcal{E}^k$'s are.

Lemma 2.27. *The natural maps*

$$\begin{array}{rcl} \mathcal{F}\otimes_{\mathcal{F}^k}\Lambda^p\mathcal{E}^k & \to & \Lambda^p_k\mathcal{E}^\infty \\ \sum f_i\otimes\omega_i & \mapsto & \sum f_i\omega_i \end{array}$$

are isomorphism of $\mathcal{F}$-modules for all $k\in\mathbb{N}$.

Proof. Surjectivity is clear while injectivity follows from injectivity of the inclusions $\mathcal{F}^{k+l}\otimes_{\mathcal{F}^k}\Lambda^p\mathcal{E}^k \to \Lambda^p\mathcal{E}^{k+l}$ for all $l,k\in\mathbb{N}$. □

Remark 2.28. Geometrically modules Λ^p_k might be thought of as "relative forms" along the projection $\pi_{\infty,k}$ in the sense that an element $\omega\in\Lambda^p_k$ associates smoothly to every point $\theta\in\mathcal{E}^\infty$ an element of $\Lambda^p_{\theta_k}\mathcal{E}^k$.

Proposition 2.29. On $\mathcal{E}^\infty$ the **insertion** of vector fields into differential forms is a well defined operation (by interpreting a vector field on $\mathcal{E}^\infty$ as sequences of relative fields on the co-filtered manifold) and has the property that for $X\in D_{\leq r}$ and $p>1$

$$i_X(\Lambda^p_k)\subseteq\Lambda^{p-1}_k \tag{2.4}$$

*The **Lie derivative** of a differential form along a vector field is defined by the Cartan formula*

$$L_X = [i_X, d]$$

and has the property that for $X\in D_{\leq r}$

$$L_X(\Lambda^p_k)\subseteq\Lambda^p_{k+r} \tag{2.5}$$

We will sometimes just write $X(\omega)$ for the lie derivative of $\omega\in\Lambda\mathcal{E}^\infty$ along $X\in D\mathcal{E}^\infty$

Proof. That insertion is well defined uses the fact that we may interpret finite shift derivations as collections of relative fields and will not be proved in detail. As mentioned above it may also be proven more generally that the functor of derivations on a certain category of modules is representable and the insertion is the natural isomorphism between the functor and its representation.

To show equations 2.4 and 2.5 one uses that

$$\begin{aligned} i_X(\Lambda^p\mathcal{E}^k) &\subseteq \mathcal{F}^{k+r}\otimes_{\mathcal{F}^k}\Lambda^{p-1}\mathcal{E}^k \\ L_X(\Lambda^p\mathcal{E}^k) &\subseteq \Lambda^p\mathcal{E}^{k+r} \end{aligned}$$

hold. Then since insertion and Lie derivative are derivations with respect to the wedge product and since any differential form $\omega \in \Lambda^p_k$ is locally a finite sum of the form $\omega = \sum f^i\omega_i$ with $\omega_i \in \Lambda^p\mathcal{E}^k$ and $f^i \in \mathcal{F}$ one obtains:

$$i_X\omega = \sum f^i i_X\omega_i \in \Lambda^{p-1}_k$$

and

$$L_X\omega = \sum X(f^i)\omega_i + \sum f^i X(\omega) \in \Lambda^p_{k+r}$$

□

Lemma 2.30. *The insertion of derivations induces an isomorphism of filtered modules*

$$D(\mathcal{E}^\infty) = Hom_{\mathcal{F}}(\Lambda^1\mathcal{E}^\infty, \mathcal{F}^\infty)$$

where on the right hand side the filtration is given by defining $\phi \in Hom_{\mathcal{F}}(\Lambda^1\mathcal{E}^\infty, \mathcal{F}^\infty)$ to have shift $\leq r$ if for all k sufficiently big

$$\phi(\Lambda^1\mathcal{E}^k) \subseteq \mathcal{F}^{k+r}$$

Proof. This is a special case of the above mentioned representation of the functor of derivations. Roughly the proof is as follows: the insertion of a derivation $X \in D_{\leq r}(\mathcal{E})$ is a homomorphism of shift $\leq r$, conversely any such homomorphism gives rise to a family of relative fields by the general theorem that the functor of derivations is represented by the module of one-forms on finite dimensional manifolds. These relative fields on the co-filtered manifold give rise to a derivation of shift $\leq r$. □

2.3. **Distributions on pro-finite manifolds and diffieties.** A **distribution** on a pro-finite manifold $\mathcal{E}^\infty$ in the most general sense is just a submodule $\mathcal{C}\Lambda^1 \subset \Lambda^1$ of the module of one-forms on $\mathcal{E}^\infty$. We say that a smooth map $\phi : \mathcal{E}^\infty \to \mathcal{E}'^\infty$ is compatible with the distributions if $\phi^*(\mathcal{C}\Lambda^1(\mathcal{E}')) \subset \mathcal{C}\Lambda^1(\mathcal{E})$ and hence we can speak of isomorphic manifolds with distributions.

The main example is the following

Example 2.31. Consider a PDE $\mathcal{E} = (\mathcal{E}^k)_{k\in\mathbb{N}}$ contained in jet spaces $\mathcal{E}^k \subseteq J^k$ and put $C\Lambda^1\mathcal{E}^\infty = \bigcup_k C\Lambda^1\mathcal{E}^k$, where $C\Lambda^1\mathcal{E}^k$ are the forms vanishing on the Cartan distributions $C\mathcal{E}^k$ and the inclusions are induced by the pullbacks along projections $\pi_{k,k-1}$. This is a well-defined $\mathcal{F}$-submodule in $\Lambda^1\mathcal{E}^\infty$ since successive Cartan distributions $C\mathcal{E}^{k+1}, C\mathcal{E}^k$ project into each other and so $\mathcal{F}^{k+l}\cdot C\Lambda^1\mathcal{E}^k \subset C\Lambda^1\mathcal{E}^{k+l}$.

Indeed these are the only distributions we shall be interested in, which leads us to the following definition

Definition 2.32. A **diffiety** is a pro-finite manifold $\mathcal{E}^\infty$ supplied with a distribution $\mathcal{C}\Lambda^1$ which is locally isomorphic to an infinite prolongation of a formally integrable PDE together with its restricted Cartan distribution.

We will continue calling the distribution of a diffiety its **Cartan distribution**.

The module of vector fields lying in the distribution of a diffiety is defined as

$$\mathcal{C}D(\mathcal{E}^\infty) := \mathrm{Ann}\mathcal{C}\Lambda^1(\mathcal{E}) = \{X \in D(\mathcal{E}^\infty) \mid i_X\omega = 0,\, \forall\omega \in \mathcal{C}\Lambda^1\}$$

We say that a submodule $P \subset D\mathcal{E}^\infty$ is involutive if $[P, P] \subset P$.

Proposition 2.33. *For a diffiety the module CD is locally free of rank n, generated by n derivations which are mutually in involution. In particular it is involutive.*

Proof. Realize $\mathcal{E}^\infty$ as the infinite prolongation of a PDE $\mathcal{E}^k \subset J^k$, and observe that the k-th module of Cartan forms $C\Lambda^1\mathcal{E}^k$ is a subset of $\mathcal{F}^k \otimes_{\mathcal{F}^{k-1}} \Lambda^1\mathcal{E}^{k-1}$ since the Cartan distribution is just the pre-image of the R-distribution R^k along $\pi_{k,k-1}$. Then let $X \in CD\mathcal{E}^\infty$ be of shift $\leq r$ and consider the restricted derivation $X_k : \mathcal{F}^k \to \mathcal{F}^{k+r}$ for a $k \gg 0$. This is a relative field along $\pi_{k+r,k}$ which must satisfy $i_{X_k}(C\Lambda^1\mathcal{E}^{k+1}) = 0$ and hence for $\theta_{k+r} \in \mathcal{E}^{k+r}$ one has $(X_k)_{\theta_{k+r}} \in R_{\theta_{k+1}}$. This condition holds also for all $k' \geq k$ and since R-planes of consecutive jet spaces map isomorphically onto each other it follows that X_k actually determines $X_{k'}$ for all $k' \geq k$. But since the R-distributions on $\mathcal{E}^{k+r}$ are spanned by the restrictions of the (truncated) total derivatives $D_i = D_i|_{\mathcal{E}^{k+r}}$ to $\mathcal{E}^{k+r}$, it follows that X_k is a linear combination of total derivatives $D_1, \ldots, D_n$ and so is all of X. Since the restricted total derivatives are in involution the statement is proven. □

Definition 2.34. Let $(\mathcal{E}^\infty, \mathcal{C}*^1)$ be a diffiety and $\theta \in \mathcal{E}^\infty$ a point, then the **Cartan plane at** θ is defined to be the fiber of the $\mathcal{F}$-module $\mathcal{C}D$ at θ:

$$\mathcal{C}D_\theta = \mathcal{C}D/\mu_\theta\mathcal{C}D$$

Corollary 2.35. *For any point $\theta \in \mathcal{E}^\infty$ the Cartan plane $\mathcal{C}D_\theta$ is canonically isomorphic to any R-plane $R_{\theta_k} \subseteq T_{\theta_{k-1}}\mathcal{E}^{k-1}$ via the projection $\pi_{\infty,k-1}$.*

Proof. Follows from the proof of the previous proposition. □

It is proven in [2] that for infinitely prolonged PDEs, the n-dimensional integral submanifolds of the Cartan distribution (if they exist) are locally in one-to-one correspondence with smooth solutions of the original equation.

Lemma 2.36. *The module of Cartan forms $C\Lambda^1\mathcal{E}^\infty$ may be recovered from $CD\mathcal{E}^\infty$ as*

$$C\Lambda^1\mathcal{E}^\infty = AnnCD\mathcal{E}^\infty = \{\omega \in \Lambda^1\mathcal{E}^\infty \mid i_X\omega = 0 \; \forall X \in CD\mathcal{E}^\infty\}$$

Proof. Obviously the inclusion "$\subset$" holds and the converse follows again from realizing the diffiety as an infinitely prolonged equation and considering the finite dimensional distributions $C\mathcal{E}^k$. □

Hence for a diffiety we may equivalently speak of $\mathcal{C}\Lambda^1$ or $\mathcal{C}D$ as its Cartan distribution.

The definition of the Cartan distribution on an infinitely prolongated PDE $\mathcal{E}^\infty$ is extrinsic in the sense that it is the restriction of the Cartan distribution of J^∞. This does not give much intuition on how to recognize intrinsically which distributions on co-filtered manifolds may be of this type. The following result gives an intrinsic characterization of the Cartan distribution of an infinitely prolonged PDE from a few simple properties.

Proposition 2.37. *Let $\mathcal{E} = (\mathcal{E}^l)_{l\in\mathbb{N}}$ be a co-filtered manifold with a distribution $\mathcal{C}D\mathcal{E}^\infty \subset D(\mathcal{E})$ which satisfies the following three properties:*

i) $\mathcal{C}D\mathcal{E}^\infty$ is locally free of rank n and involutive

ii) $\mathcal{C}D\mathcal{E}^\infty \subset D_{\leq 1}$, i.e. fields in the distribution are of shift ≤ 1

To formulate the third property observe first that for a distribution $\mathcal{C}D\mathcal{E}^\infty$ which satisfies properties i) and ii) there are, for all l sufficiently big, well defined relative distributions along $\pi_{l,l-1}$ locally free of rank n, spanned by the restrictions to $C^\infty(\mathcal{E}^{l-1})$ of a local basis of $\mathcal{C}D\mathcal{E}^\infty$. Lets denote these rlative distributions with $R\mathcal{E}^l$. The last property is

iii) The map

$$\begin{array}{rcl} \mathcal{E}^l_{\theta_{l-1}} & \to & Gr(T_{\theta_{l-1}}\mathcal{E}^{l-1}, n) \\ \theta_l & \mapsto & R_{\theta_l}\mathcal{E}^l \end{array}$$

is injective, where $Gr(V,n)$ is the Grassmannian of n-dimensional subspaces of a vector space V and $\mathcal{E}^l_{\theta_{l-1}}$ is the fiber over θ_{l-1} of the projection $\pi_{l,l-1} : \mathcal{E}^l \to \mathcal{E}^{l-1}$. Then there is locally an injective immersion $\mathcal{E}^\infty \to J^\infty$ into some jet space (with n-independent variables) such that the image is an infinitely prolonged PDE and the restricted Cartan distribution coincides with $\mathcal{CE}^\infty$. Conversely the Cartan distribution $CD\mathcal{E}^\infty$ of any infinitely prolonged PDE satisfies all of the above properties.

Proof. Given a PDE $\mathcal{E}^\infty \subset J^\infty$ we have already shown property i), while ii) follows from the fact that the module $CD\mathcal{E}^\infty$ is locally generated by total derivatives which are of shift ≤ 1. For property iii) it suffices to observe that the relative distributions $R\mathcal{E}^l$ are precisely the the R-distributions of the PDE introduced earlier. Since we know that for jet spaces the map $\theta_l \to R_{\theta_l}$ is injective the same holds true for the restricted R-distributions.

To show the converse let $(\mathcal{E})_{l\geq l_0}$ be a co-filtered manifold supplied with a distribution which satisfies the above properties, and let l' be sufficiently big so that the relative distributions $R\mathcal{E}^l$ are locally free of rank n for all $l \geq l'$. Then we put $E := \mathcal{E}^{l'}$ and shall define embedding $\mathcal{E}^{l'+s} \to J^s(E,n)$ inductively as follows: take the embedding $\mathcal{E}^{l'} \to J^0$ to be the identity. Then since $\mathcal{E}^{l'+1}$ is identified with a subset of $J^1(\mathcal{E}^{l'},n)$ due to the relative distribution $R\mathcal{E}^{l'+1}$ and condition iii), the embedding $\mathcal{E}^{l'+1} \to J^1(E,n)$ is also determined. Obviously the so constructed embeddings are compatible with the projections $\mathcal{E}^{l'+1} \to \mathcal{E}^{l'}$ and $J^1 \to J^0$, and satisfy$\left(\mathcal{E}^{l'}\right)^{(1)} \supset \mathcal{E}^{l'+1}$.

Suppose now that we have found embeddings $\mathcal{E}^{l'+k} \subset J^k(E,n)$ for $k = 0,\dots,s$ such that the relative distributions $R\mathcal{E}^{l'+k}$ are precisely the restrictions of the R-distributions of J^k to $\mathcal{E}^{l'+k}$, then the claim is that the planes $R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$ are R-planes in J^s and hence the embedding $\mathcal{E}^{l'+s+1} \subset J^{s+1}$ is also determined with property $\left(\mathcal{E}^{l'+s}\right)^{(1)} \supset \mathcal{E}^{l'+s+1}$. To show this we use proposition 4.3 from section 4 which states that R-planes in J^s are precisely n-dimensional involutive subspaces of the Cartan distribution transversal the projection $\pi_{s,s-1}$. Obviously the planes $R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$ (which belong to J^s by induction hypothesis) are transversal to the projection to J^{s-1} since they project non degenerately to $R_{\theta_{l'+s}}\mathcal{E}^{l'+s}$ which by induction hypothesis is an R-plane in J^{s-1}. By the same reason we see that the $R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$'s are contained in the Cartan distributions of J^s, hence it remains to show that they are involutive. For this we first show that the plane $R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$ is involutive in the distribution $C_{\theta_{l'+s}}\mathcal{E}^{l'+s}$, which follows from the involutivity of the distribution $CD\mathcal{E}^\infty$. Since for $\omega \in C\Lambda^1\mathcal{E}^{l'+s}$ and any $x,y \in R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$, we may extend the vectors x,y to derivations $X,Y \in CD\mathcal{E}^\infty$ such that $X_{\theta_{l'+s+1}} = x, Y_{\theta_{l'+s+1}} = y$, and obtain

$$d\omega_{\theta_{l'+s}}(x,y) = d\omega(X,Y)_{\theta_{l'+s+1}} = \omega([X,Y])_{\theta_{l'+s+1}} = 0$$

since $\omega \in C\Lambda^1\mathcal{E}^\infty$. Hence $R_{\theta_{l'+s+1}}\mathcal{E}^{l'+s+1}$ is an involutive plane of the distribution $C\mathcal{E}^{l'+s}$ which by induction coincides with the restriction of the Cartan distribution on J^s to $\mathcal{E}^{l'+s}$. Now the proposition follows from the next lemma. □

Lemma 2.38. *Let M be a finite dimensional manifold with a distribution $P \subset TM$, and $N \subset M$ a submanifold. Assume the restricted distribution $Q = P \cap TN$ on N is of constant rank. Then a plane $\Pi \subset Q$ is involutive with respect to Q on N if and only if it is involutive with respect to the distribution P on M.*

Proof. Let $P\Lambda^1$ denote the one forms vanishing on P. If $\omega \in P\Lambda^1$ then $\omega|_N \in Q\Lambda^1$ and conversely any $\tilde{\omega} \in Q\Lambda^1$ on N may be extended to a form $\omega \in P\Lambda^1$ on M by the regularity assumption. Hence by naturality of the de Rham differential we have for any $\omega \in P\Lambda$ $(d\omega)|_\Pi = d(\omega|_N)|_\Pi = d\tilde{\omega}|_\Pi$. And hence $d\omega|_\Pi = 0\ \forall \omega \in P\Lambda^1 \iff d\tilde{\omega}|_\Pi = 0\ \forall \tilde{\omega} \in Q\Lambda^1$. □

The proof of proposition 2.37 also shows why morphism of diffieties must be of finite shift, a result stated in an earlier remark: if the diffieties are realized as infinite prolongations of PDEs then a morphism $\mathcal{E}^\infty \to \mathcal{G}^\infty$ is already completely determined by any of the maps $\mathcal{E}^{l+d} \to \mathcal{G}^l$, since property iii) from the above proposition determines all the maps $\mathcal{E}^{l+d+1} \to \mathcal{G}^{l+1}$ etc..., and so the shift is bounded by d.

Expressing such a morphism of diffieties in local coordinates shows that they may be interpreted as infinite prolongations of maps induced by non-linear differential operators which send solutions of one equation into solutions of the other (see also [2] for a more detailed explanation of this construction). This is, in the authors opinion a strong point in favor of the naturality of the Diffiety approach to nonlinear PDE's.

Remark 2.39. The notion of diffiety $(\mathcal{E}, C)$ was introduced by Vinogradov and his school in the 70's as an adequate geometrical object representing a PDE. It allows in particular to put the theory of nonlinear PDEs on a functorial basis and opens the possibility to develop a differential calculus on the (virtual) space of solutions of $\mathcal{E}$ (called secondary calculus), as initiated and mainly developed by Vinogradov [30]. Moreover it seems that most well known constructions from the classical theory of PDEs fit naturally into this general setting, see for example [2, 10] (as well as this thesis for that matter).

2.4. **Geometric generalized solutions of a PDE and singularity equations.**

Definition 2.40. A **generalized solution** of a PDE $\mathcal{E} = (\mathcal{E}^l)_{l \geq l_0}$ $\mathcal{E}^\infty \subset J^\infty$ is a generalized R-manifold of J^k contained in $\mathcal{E}^k$ for some k.

Remark 2.41. It was shown in [28, 22] that in the case of linear PDEs, these geometric generalized solutions give rise to generalized solutions in the sense of Schwartz (i.e. distribution solutions). This is just one of the observations which supports the belief that geometric generalized solution are an adequate concept of generalized solutions for the non-linear theory.

Given a generalized solution $W \subset \mathcal{E}^k$ of a PDE the singularity locus $\Sigma_s W$ may not be arbitrary since if $\theta_k \in \Sigma_s W$ the singular R-plane $T_{\theta_k}\Sigma_s W$ has to be tangent to $\mathcal{E}^k$. Lets call a singular R-plane tangent to $\mathcal{E}^k$ a **singular R-plane of** $\mathcal{E}^k$. Projecting a singular R-plane of $\mathcal{E}^k$ down to $\mathcal{E}^{k-1}$ one obtains an $(n-s)$ dimensional plane in $\mathcal{E}^{k-1}$ and the collection of all the so obtained $(n-s)$ dimensional planes in $\mathcal{E}^{k-1}$ is a subset of $J^1(\mathcal{E}^{k-1}, n-s)$. As such it may be interpreted as a first order equation imposed on $(n-s)$-dimensional submanifolds of $\mathcal{E}^{k-1}$ and obviously the projection $\pi_{k,k-1}(\Sigma_s W)$ of the singularity locus of a generalized solution has to satisfy this equation. This leads one to make the following definition.

Definition 2.42. For a differential equation $\mathcal{E} = (\mathcal{E}^l)_{l\in\mathbb{N}}$ $\mathcal{E}^l \subset J^l$ the associated **k-th equation of singularities of type** s is the first order equation

$$\Sigma_{[s]}\mathcal{E}^k := \{T\pi_{k,k-1}(\Pi) \mid \Pi \text{ is a type } s \text{ singular R-plane of } \mathcal{E}^k\} \subset J^1(\mathcal{E}^{k-1}, n-s)$$

imposed on $n-s$-dimensional submanifolds of $\mathcal{E}^{k-1}$.

This thesis we will be mainly dedicated to describing these equations in the case of type 1-singularities. We shall also relate the k-th singularity equations with the $(k+1)$-st and explain how they are seen from the point of view of the infinite prolongation. Finally we show that they are in some sense intrinsic to the diffiety and show how they behave under coverings of nonlinear PDEs. To carry out the first part we will first introduce the symbolic system of a non-linear PDE. After that we will describe the structure of involutive subspaces of the Cartan distribution on J^k and relate them to the symbols of a PDE.

3. The symbolic system of a PDE and characteristics

3.1. **The affine structure on fibers $J^k_{\theta_{k-1}}$.** In this section we recall a well know result (at least in the fibered case) which states that for $k \geq 2$ the fibers of the projections $J^k \to J^{k-1}$ are naturally affine spaces over certain vector spaces of symmetric tensors. We give a new proof of this by first constructing the canonical trivializations $TJ^k_{\theta_{k-1}} \cong W \times J^k_{\theta_{k-1}}$ where W will be the vector space modeling the affine structure and then showing that the obtained "connection" is flat.

We denote with $V^{k-r}J^k$ the module of $\pi_{k,k-r}$**-vertical fields** on J^k i.e.,

$$V^{k-r}J^k = \{X \in TJ^k \mid X(\mathcal{F}^{k-r}) = 0\}.$$

We have the inclusions $V^{k-1}J^k \subset V^{k-2}J^k \subset \ldots \subset V^0J^k$ and all these distributions are integrable. In coordinates they are given by

$$V^{k-l}J^k = \left\langle \frac{\partial}{\partial u^j_\sigma} \mid 1 \leq j \leq m,\ k-l < |\sigma| \leq k \right\rangle.$$

We also need the following

Definition 3.1. The **normal bundle** N^k on J^k is the vector bundle

$$N^k = (\mathcal{F}_k \otimes_{\mathcal{F}_0} TJ^0)/R^k$$

where the natural inclusion $R^k \to \mathcal{F}_k \otimes_{\mathcal{F}_0} TJ^0$ is given by restricting a derivation $X \in R^k$ to $\mathcal{F}^0 \subset \mathcal{F}^{k-1}$.

Point-wise we have have

$$N^k_{\theta_k} = T_{\theta_0}E/R_{\theta_k}$$

where R_{θ_k} is identified with $R_{\theta_1} \subset T_{\theta_0}E$ and so N^k is the pullback of N^1 and hence of rank m.

The main statement of this subsection is the following

Theorem 3.2. *For $k \geq 2$ there is a canonical isomorphism*

$$V^{k-1}J^k \cong S^k\left(R^{k*}\right) \otimes N^k$$

were S^k denotes the k-th symmetric tensor power.

In the case when E is fibered over M and we consider jets of sections of $\pi : E \to M$, this isomorphism becomes

$$V^{k-1}J^k \cong \mathcal{F}^k \otimes_{C^\infty(M)} S^k\left(T^*M\right) \otimes_{C^\infty(M)} VE$$

where VE is the vertical tangent bundle to the fibration $\pi : E \to M$. Moreover in the fibered case it also holds for $k = 1$.

The proof of 3.2 is based on the following simple facts.

Lemma 3.3. *The following relations hold between the Cartan–, vertical–, and R–distributions:*

$$C^k \cap V^{k-r}J^k = V^{k-1}J^k \tag{3.1}$$

$$(C^k)^{(l)} = C^k + V^{k-(l+1)}J^k \tag{3.2}$$

$$\pi^{-1}_{k,l}(C^l) = (C^k)^{(k-l-1)} \tag{3.3}$$

$$C^k/V^{k-1}J^k \cong R^k \tag{3.4}$$

$$V^{k-r-l}J^k/V^{k-r}J^k \cong \mathcal{F}_k \otimes_{\mathcal{F}_{k-r}} V^{k-r-l}J^{k-r} \tag{3.5}$$

Here $C^{(r)}$ denotes the r-th ***derived distribution*** *defined iteratively by $C^{(r)} = (C^{(r-1)})^{(1)}$ and $C^{(1)} = \langle [X,Y] \mid X, Y \in C \rangle$. Moreover the last two isomorphisms are canonical: the first one by taking a derivation in C^k and restricting it to $\mathcal{F}_{k-1} \subset \mathcal{F}_k$, the second one by taking a derivation in $V^{k-r-l}J^k$ and restricting it to $\mathcal{F}_{k-r} \subset \mathcal{F}_k$.*

Proof. The first two are checked immediately in local coordinates using the canonical non-holonomic frame on J^k all the others are trivial consequences of the definitions. □

Corollary 3.4. *For all $r = 0, \ldots, k-1$*

$$T_{\theta_k} J^k/(C^k)^{(r)}_{\theta_k} \cong T_{\theta_{k-r-1}} J^{k-r-1}/R_{\theta_k}$$

In particular for $r = k-1$

$$T_{\theta_k} J^k/C^{(k-1)}_{\theta_k} \cong N_{\theta_k}$$

Proof. Use 3.2, 3.1 and 3.4. □

From this corollary and the identification $C^k/V^{k-1}J^k \cong R^k$, theorem 3.2 follows from the following

Proposition 3.5. *For $k \geq 2$ there is a canonical isomorphism of $\mathcal{F}^k$ modules*

$$V^{k-1}J^k \cong S^k \left(C^k/V^{k-1}J^k\right)^* \otimes_{\mathcal{F}_k} \left(TJ^k/\left(C^k\right)^{(k-1)}\right)$$

given by

$$V^{k-1}J^k \ni Y \mapsto \left((\overline{X}_1, \ldots, \overline{X}_k) \mapsto \overline{[[\cdots[[Y, X_1], X_2]\ldots], X_{k-1}], X_k]}\right)$$

where $X_i \in \mathcal{C}_k$ and the bars denote the equivalence classes in the adequate spaces. In local coordinates this map works as follows:

$$\frac{\partial}{\partial u^i_\sigma} \mapsto \frac{1}{\sigma!}(dx^1)^{\sigma_1} \cdot \ldots \cdot (dx^n)^{\sigma_n} \otimes \frac{\partial}{\partial u^i} \tag{3.6}$$

Before passing to the proof of 3.5 we recall how the natural identification

$$S^k(R^*) \cong (S^k R)^* \tag{3.7}$$

used in the statement works: for a module P over a commutative algebra A there is a natural pairing between $S^k(P^*)$ and S^kP given by:

$$\begin{aligned} \langle \cdot, \cdot \rangle : S^k R \otimes S^k(R^*) \quad &\to \quad A \\ w_1 \cdot \ldots \cdot w_k \otimes \alpha_1 \cdot \ldots \cdot \alpha_k \quad &\mapsto \quad \langle w_1 \cdot \ldots \cdot w_k, \alpha_1 \cdot \ldots \cdot \alpha_k \rangle = \sum_{\varsigma} \prod_{i=1}^{k} \langle w_{\varsigma(i)}, \alpha_i \rangle \end{aligned}$$

where ς runs through all permutations of the set $\{1, \ldots, k\}$. In our case (since R^k is the module of sections of a finite rank vector bundle over J^k) this pairing is non-degenerate and induces the stated isomorphism. Locally if $r_1, \ldots, r_n$ is a basis of R and the associated dual basis of R^* is denoted with $r^*_1, \ldots, r^*_n$, then the $\binom{n+k-1}{k}$ monomials

$$r^\sigma = r_1^{\sigma_1} \cdot \ldots \cdot r_n^{\sigma_n}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a multi-index with $|\sigma| = k$ form a basis of S^kR and analogously

$$(r^*)^\sigma = r_1^{*\sigma_1} \cdot \ldots \cdot r_n^{*\sigma_n}, \ |\sigma| = k$$

form a basis of $S^k(R^*)$. A computation shows that $\langle r^\sigma, (r^*)^\sigma \rangle = \sigma!$ where

$$\sigma! := \prod_{i=1}^{n} \sigma_i!$$

which implies that under identification 3.7 the dual basis of r^σ is mapped to $\frac{1}{\sigma!}(r^*)^\sigma$.

We return to the proof of proposition 3.5

Proof. To check the independence on the choice of representatives suppose $v \in V^{k-1}J^k$ and use the Jacobi identity repeatedly:

$$\begin{aligned}
&[[\cdots[[Y,X_1]\cdots],X_{i-1}],v],X_{i+1}]\cdots],X_k]\\
=&\overbrace{[\underbrace{[\cdots[[Y,X_1]\cdots]}_{\in C_k^{(i-2)}},[X_{i-1},v]]}^{\in C_k^{(i-1)}},X_{i+1}]\cdots],X_k]\in C_k^{(k-1)}\\
&+[[\cdots[[Y,X_1]\cdots],v],X_{i-1}],X_{i+1}]\cdots],X_k]\\
=&\dots\\
=&\text{terms in } C_k^{(k-1)}+[[\cdots\underbrace{[[Y,v]}_{\in C_k},X_1]\cdots],X_{i-1}],X_{i+1}]\cdots],X_k]\in C_k^{(k-1)}
\end{aligned}$$

which shows that the application is well defined. That it is indeed symmetric follows similarly from the Jacobi identity:

$$\begin{aligned}
&[[\cdots[[Y,X_1]\cdots],X_i],X_{i+1}]\cdots],X_k]-[[\cdots[[Y,X_1]\cdots],X_{i+1}],X_i]\cdots],X_k]=\\
&\qquad=[[\cdots[[Y,X_1]\cdots],[X_{i+1},X]_i]\cdots],X_k]\in C_k^{(k-1)}.
\end{aligned}$$

Lastly one checks $\mathcal{F}_k$-linearity in each argument. For the X's it suffices to verify this in the last argument because of symmetry:

$$\begin{aligned}
[[\cdots[[Y,X_1],X_2]\ldots],X_{k-1}],fX_k] &= \underbrace{[[\cdots[[Y,X_1],X_2]\ldots],X_{k-1}](f)\cdot X_k}_{\in C_k}\\
&\quad+f[[\cdots[[Y,X_1],X_2]\ldots],X_{k-1}],X_k].
\end{aligned}$$

For the argument Y it holds similarly since

$$\begin{aligned}
&[[\cdots[[fY,X_1],X_2]\ldots],X_{k-1}],X_k]\\
=&f[[\cdots[[Y,X_1],X_2]\ldots],X_{k-1}],X_k]+\text{terms with at most } k-1 \text{ Lie brackets.}
\end{aligned}$$

□

Corollary 3.6. *The fibers $J^k_{\theta_{k-1}}$ of the projection $\pi_{k,k-1}$ carry an affine structure over the vector space $S^k\left(R^*_{\theta_0}\right)\otimes N_{\theta_0}$.*

Proof. That the obtained connection on $TJ^k_{\theta_{k-1}}$ through the above trivialization is flat follows from the coordinate description above. It remains to see that $J^k_{\theta_{k-1}}$ is contractible which is easily verified by deforming any n-submanifold $L\subset E$ to any other (locally and keeping one point fixed). □

3.2. δ-Spencer operator.

Proposition 3.7. *There is a canonical $\mathcal{F}_k$-linear injection*

$$\begin{aligned}
\delta: V^{k-1}J^k &\to Hom_{\mathcal{F}_k}(R^k,\mathcal{F}_k\otimes V^{k-2}J^{k-1})\\
v&\mapsto \delta v
\end{aligned}$$

*called the **first δ-Spencer operator** and given by*

$$\delta_{\overline{X}}v := \delta v(\overline{X}) := \overline{[v,X]} \tag{3.8}$$

where $X\in C^k$ and $v\in V^{k-1}J^k$ and we use the canonical identifications $R^k = C^k/V^{k-1}J^k$ and $\mathcal{F}_k\otimes V^{k-1}J^{k-1}\cong\frac{\left(C^k\right)^{(1)}}{C^k}$. The last isomorphism is obtained more *explicitly via*

$$(3.9)\qquad \frac{\left(C^k\right)^{(1)}}{C^k} = \frac{V^{k-2}J^k + C^k}{C^k} \cong \frac{V^{k-2}J^k}{V^{k-2}J^k \cap C^k} \cong \frac{V^{k-2}J^k}{V^{k-1}J^k} = \mathcal{F}_k \otimes V^{k-1}J^{k-1}$$

In coordinates it acts as

$$\delta_{D_i}\partial_{u^j_\sigma} = \partial_{u^j_{\sigma-1_i}}$$

Proof. Well-definedness: suppose $X \in V^{k-1}J^k$ then $[v, X] \in V^{k-1}J^k$ and hence $\overline{[v, X]} = 0$. Moreover for $f \in \mathcal{F}^k$and $X \in C^k$ we have $\delta v(\overline{fX}) = \overline{[v, fX]} = \overline{\underbrace{v(f)X}_{\in C^k} + f[v, X]}$ and analogously for $\delta f v = f\delta v$ □

Using the identification $V^{k-1}_{\theta_k}J^k = S^k(R_{\theta_k}) \otimes N_{\theta_k}$ one sees that this δ-Spencer operator coincides with the following δ-Spencer operator defined in terms of commutative algebra:

Let R be an n-dimensional vector space, then the **δ-Spencer complex** of $S(R^*) = \bigoplus_{k=0}^{\infty} S^k(R^*)$ (also called the polynomial de Rham complex) is the complex

$$0 \to S(R^*) \otimes R^{*\wedge 0} \to S(R^*) \otimes R^{*\wedge 1} \to \ldots \to S(R^*) \otimes R^{*\wedge n}$$

where the differential δ acts by:

$$(3.10)\qquad \delta : S^k(R^*) \otimes R^{*\wedge l} \to S^{k-1}(R^*) \otimes R^{*\wedge l+1}$$

$$\alpha_1 \cdot \ldots \cdot \alpha_k \otimes \beta_1 \wedge \ldots \wedge \beta_l \mapsto \sum_{i=1}^{k} \alpha_1 \cdot \ldots \cdot \hat{\alpha}_i \cdot \ldots \cdot \alpha_k \otimes \alpha_i \wedge \beta_1 \wedge \ldots \wedge \beta_l$$

or in terms of a basis $r_1, \ldots, r_n$ of R:

$$\delta(r^{*\sigma} \otimes r^*_{i_1} \wedge \ldots \wedge r^*_{i_l}) = \sum_{m=1}^{n} \sigma_m r^{*\sigma-1_m} \otimes r^*_m \wedge r^*_{i_1} \wedge \ldots \wedge r^*_{i_l}$$

Given another vector space N of dimension m, the **δ-Spencer complex of SR^* with values in** N is just the $\mathbb{R}$-tensor product of N with the δ-Spencer complex of SR^*. One verifies easily that the δ-Spencer operator 3.8 defined in the beginning of this section coincides with the first algebraic $\delta-$Spencer operator 3.10.

It is well-known that the cohomology of the δ-Spencer complex of $S(R^*)$ comes only from the piece $0 \to \mathbb{R} \to 0$ in the 0-th row which is therefore also the case for the δ-Spencer complex with values in N. Hence for $k \geq 1$ the first δ-Spencer operator $\delta : S^k(R^*) \otimes N \to R^* \otimes S^{k-1}(R^*) \otimes N$ is injective and allows one to identify a tensor $v \in S^k(R^*) \otimes N$ with a linear map $\delta v \in \mathrm{Hom}_{\mathbb{R}}(R, S^{k-1}(R^*) \otimes N)$.

3.3. **Symbolic system associated to a PDE.** Let $\mathcal{E} \subset J^\infty$ be a PDE, i.e. $\mathcal{E} = (\mathcal{E}^l)_{l\in\mathbb{N}}$ with $\mathcal{E}^l \subset J^l$.

Definition 3.8. The **l-th symbol** of $\mathcal{E}$ at $\theta_l \in \mathcal{E}^l$ is the vertical tangent space

$$g^l_{\theta_l} := T_{\theta_l}\mathcal{E}_{\theta_{l-1}}$$

which we may understand as a subspace of $S^l(R^*_{\theta_l}) \otimes N_{\theta_l}$.

Lemma 3.9. *The first δ-Spencer operator restricts to symbols of a PDE, i.e. for any $X \in R_{\theta_l}$ and $v \in g^l_{\theta_l}$ one has $\delta_X v \in g^{l-1}_{\theta_{l-1}}$.*

Proof. We need to show that for any $f \in C^\infty(J^{l-1})$ which vanishes on $\mathcal{E}^{l-1}$ we have $\delta_X v(f) = 0$. For this we locally around θ_l extend v to a vector field $\tilde{v} \in V^{j-1}$

J^l and X to a section $\tilde{X} \in R^l$. Obviously $\tilde{X}$ will be tangent to $\mathcal{E}^l$ and hence we find by equation 3.8 that on $\mathcal{E}^{l-1}$

$$\delta_{\tilde{X}}\tilde{v}(f) = \tilde{X} \circ \underbrace{\tilde{v}(f)}_{=0} - \tilde{v} \circ \underbrace{\tilde{X}(f)}_{=0} = 0.$$

□

The above property of the symbols of a PDE makes them into what is known as a symbolic system.

Definition 3.10. Let R and N be finite dimensional $\mathbb{R}$-vector spaces. A graded vector subspace $g \subseteq S(R^*) \otimes N$, i.e.

$$g = \bigoplus_{k \in \mathbb{N}} g^k$$

with $g^k \subseteq S^k(R^*) \otimes N$ is called a **symbolic system** if for all $v \in g^k$ and all $X \in R$

$$\delta_X v \in g^{k-1}$$

Hence by lemma 3.9 for any point $\theta = (\theta_l)_{l \in \mathbb{N}}$ in the infinite prolongation of a PDE $\mathcal{E}$ the associated graded vector space

$$g_\theta = \bigoplus_{l=0}^{\infty} g^l_{\theta_l} \subset S(R^*_\theta) \otimes N_\theta$$

is a symbolic system called the **symbolic system of $\mathcal{E}$ at θ**.

We introduce some further terminology regarding symbolic systems which will be of importance later on.

Definition 3.11. Let $g_k \subset S^k(R^*) \otimes N$ be a subspace, then its **first prolongation** is defined to be

$$g_k^{(1)} := \left\{ v \in S^{k+1}R^* \otimes N \mid \delta_X v \subseteq g_k \ \forall X \in R \right\}$$

and the **l-th prolongation** is defined recursively as

$$g_k^{(l)} = \left(g_k^{(l-1)} \right)^{(1)}$$

The **derivation** $\partial g_k \subset S^{k-1}R^* \otimes N$ is defined to be

$$\partial g_k = \langle \delta_X v \mid v \in g, \, X \in R \rangle$$

One immediately verifies the lemma

Lemma 3.12. *g is a symbolic system if and only if $\partial g \subseteq g$ or equivalently $g^{(1)} \supseteq g$*

3.4. **The symbolic module of a PDE.** In this section we recall that the $\mathbb{R}$-dual of any symbolic system is always an $S(R)$-module and vice versa [15], and hence the study of symbolic system is essentially a part of commutative algebra (in the sense of algebraic geometry).

Let us suppose again that R and N are $n-$ and m-dimensional $\mathbb{R}$-vector spaces, respectively. Recall that $SR = \bigoplus S^k R$ is a commutative graded algebra which is naturally identified with the algebra of polynomial functions on R^*. The evaluation of a polynomial $f \in S^k R$ at a point $\alpha \in R^*$ is expressed using the pairing $(S^k R) \cong S^k(R^*)$ as

$$f(\alpha) = \frac{1}{k!}\langle f, \alpha^k \rangle$$

where α^k is the $k-$th symmetric power of α. Moreover $SR \otimes N^*$ is naturally a graded SR module, and may be interpreted as polynomial maps from R^* to N^*.

Lemma 3.13. *For any $X \in R$ the map*

$$\delta_X^* : S^{k-1}R \otimes N^* \to S^k R \otimes N^*$$

($\mathbb{R}$-dual to $\delta_X : S^k(R^*) \otimes N \to S^{k-1}(R^*) \otimes N$) is the multiplication by $X \in R$.

Proof. This follows from the computation

$$\begin{aligned} \langle X \cdot w_1 \cdot \ldots \cdot w_{k-1}, \alpha_1 \cdot \ldots \cdot \alpha_k \rangle &= \sum_{j=1}^{k} \langle w_1 \cdot \ldots \cdot w_{k-1}, \alpha_1 \cdot \ldots \cdot \hat{\alpha_j} \ldots \cdot \alpha_k \rangle \langle X, \alpha_j \rangle \\ &= \langle w_1 \cdot \ldots \cdot w_{k-1}, \delta_X(\alpha_1 \cdot \ldots \cdot \alpha_k) \rangle \end{aligned}$$

□

Corollary 3.14. *Interpreting an element $v \in S^k R^* \otimes N$ as a k-multi-linear symmetric map from R to N we have*

$$\delta_X v(\cdot, \ldots, \cdot) = v(X, \cdot, \ldots, \cdot)$$

Proof. By the previous lemma we obtain

$$\begin{aligned} \delta_X v(w_1, \ldots, w_{k-1}) &= \langle w_1 \cdot \ldots \cdot w_{k-1}, \delta_X v \rangle \\ &= \langle X \cdot w_1 \cdot \ldots \cdot w_{k-1}, v \rangle \\ &= v(X, w_1 \ldots, w_{k-1}) \end{aligned}$$

□

The next two results state the above mentioned characterizations of symbolic systems as the dual notion to finitely generated SR-modules (see also [15]). A more conceptual way to formulate these results would be by making the observation that $SR^* \otimes N$ carries a natural co-module structure over the co-algebra SR^* and that $g \subset ST^* \otimes N$ is a symbolic system iff it is a sub-co-module. By the duality of the categories of SR-modules and SR^*-co-modules it then follows that g^* is a quotient module of $ST \otimes N^*$. But we will not elaborate on this point of view to avoid reviewing the notions of co-modules and co-algebras.

Proposition 3.15. *Let $g = \bigoplus_{k \in \mathbb{N}} g_k$ be a graded subspace of $SR^* \otimes N$. Then g is a symbolic system if and only if the annihilator $g^\circ \subset SR \otimes N^*$ is an SR submodule.*

Proof. By definition g is symbolic if and only if the δ_X restrict to g for all $X \in R$, which is by lemma 3.13 equivalent to saying that g° is closed under multiplication with all $X \in R$ and therefore an SR-submodule. □

Corollary 3.16. *$g \subset SR^* \otimes N$ is a symbolic system if and only if g^* is an SR-module (with the module structure induced from the projection $SR \otimes N^* \to g^*$).*

Proof. This follows immediatley from the identity $g^* = SR \otimes N^* / g^\circ$ □

Definition 3.17. Given a PDE $\mathcal{E} = (\mathcal{E}^l)_{l \in \mathbb{N}}$ and a point $\theta = (\theta_l)_{l \in \mathbb{N}}$ in it, then the graded $S(R_\theta)$-module g_θ^* is called the **symbolic module** of $\mathcal{E}$ at θ.

Proposition 3.18. *If $g \subset SR \otimes N$ is a symbolic system, then there exists a k_0 such that for all $k \geq k_0$*

$$g_{k+1} = g_k^{(1)}$$

The smallest k_0 which satisfies the above property is called the ***order of the symbolic system.*** *Moreover there is a unique polynomial $P_g \in \mathbb{Q}[x]$ called the* ***Hilbert polynomial*** *of g such that*

$$\dim(g_k) = P_g(k)$$

for all k sufficiently big.

Proof. $SR \otimes N^*$ is a Noetherian module since it is finitely generated over SR which is a Noetherian algebra. Hence g° is a finitely generated SR-module. Fix a set of homogeneous generators and let k_0 be bigger than the degree of any of the generators. Then obviously $g^\circ_{k+1} = R \cdot g^\circ_k$ for all $k \geq k_0$ which means that any $w \in \mathrm{Ann}(g_{k+1})$ is of the form $w = \sum X_i \cdot u$ with $X_i \in R$ and $u_i \in g^\circ_k$. Hence for $v \in g^{(1)}_k$ and $w \in \mathrm{Ann}(g_{k+1})$ we have

$$\begin{aligned}\langle w, v\rangle &= \langle \sum X_i \cdot u_i, v\rangle \\ &= \sum \langle u_i, \delta_{X_i} v\rangle \\ &= 0\end{aligned}$$

which implies $v \in g_{k+1}$ and so $g^{(1)}_k = g_{k+1}$. The statement about the Hilbert polynomial follows from the existence of the Hilbert polynomial for the finitely generated SR-module g^*. □

Even though we will show later that the symbolic module g^* is "intrinsic" to the PDE $\mathcal{E}$ if considered as a co-filtered manifold with a distribution (without the knowledge of the embedding $\mathcal{E}^\infty \subset J^\infty$), it should be stressed that it is nevertheless *not* an invariant of the PDE considered as a diffiety (i.e. as a pro- finite manifold) since there is no natural way of making g^* behave functorial under maps of diffieties.

3.5. **Kernel and image of tensors in $S^k R^* \otimes N$.** It is convenient at this point to introduce some notions related with tensors in $S^k(R^*) \otimes N$ which will be of importance in what follows. So let again R and N denote n and m-dimensional vector spaces, respectively.

Definition 3.19. The **kernel** of a tensor $v \in S^k(R^*) \otimes N$ is

$$\ker(v) := \ker \delta v$$

and its **image** is

$$\mathrm{im}(v) := \mathrm{im}\delta v$$

where $\delta v : R \to S^{k-1}(R^*) \otimes N$.

The **rank** of v is defined to be

$$\mathrm{rank}(v) := \dim(\mathrm{im}(v))$$

Obviously $0 \leq \mathrm{rank}(v) \leq n$ and v is called **non-degenerate** if it has maximal rank, and degenerate otherwise. The cone of tensors of rank r in $S^k(R^*) \otimes N$ will be denoted with

$$Q^k_r := \{v \in S^k(R^*) \otimes N \mid \mathrm{rank}(v) = r\}$$

The following simple lemma allows one to reduce the study of arbitrary tensors to the study of non-degenerate ones. To state it recall that if $K \subset R$ is a subspace then $K^\circ = \{\alpha \in R^* \mid \alpha|_K = 0\} \cong (R/K)^*$ and so there is a natural inclusion

$$S^k((R/K)^*) \otimes N = S^k(K^\circ) \otimes N \subset S^k(R^*) \otimes N$$

A tensor $v \in S^k(R^*) \otimes N$ which is contained in this subspace $S^k((R/K)^*) \otimes N$ is called **reducible to** R/K. Its **reduction** $v_{R/K}$ is then just the same tensor v understood as an element in $S^k((R/K)^*) \otimes N$.

Lemma 3.20. *Given a tensor $v \in S^k(R^*) \otimes N$ and a subspace $K \subset R$, then the following two conditions are equivalent:*

i) $K \subset \ker v$

ii) v is reducible to R/K

Moreover for the kernel of the reduction $v_{R/K}$ one has

$$\ker v_{R/K} = \frac{\ker v}{K} \subset R/K$$

In particular if one takes $K = \ker v$ then the reduced tensor $v_{R/K}$ is non degenerate.

Proof. Pick a basis $r_1, \dots, r_n$ of R such that the first s of these vectors span K. Expand v in the corresponding basis $(r^*)^\sigma \otimes e^j$ where e^j is any basis of N. Condition i) is then equivalent to $\delta_{r_i} v = 0$, $i = 1, \dots, s$ which is equivalent to v being a linear combination of tensors of the form $(r^*)^\sigma \otimes e^j$ with $\sigma_1 = \sigma_2 = \ldots = \sigma_s = 0$, which is obviously equivalent to $v \in S^k(K^\circ) \otimes N \subset S^k(R^*) \otimes N$. The rest of the statement is also straightforwardly proven. □

3.6. **Characteristics of a PDE.**

3.6.1. *Tensors of rank 1 and characteristics of a symbolic system.* Obviously the kernel, image and rank of a tensor $v \in S^k(R^*) \otimes N$ remain invariant when multiplying it with a non zero scalar. Hence we may speak of the kernel, image and rank of a line $l \subset S^k(R^*) \otimes N$.

Proposition 3.21. *Let $l \subset S^k(R^*) \otimes N$ be a one-dimensional subspace, then it is of rank 1 if and only if it is generated by a vector of the form*

$$v = \alpha^k \otimes e$$

for some $\alpha \in R^* \backslash \{0\}$ and $e \in N \backslash \{0\}$. Moreover α and e are unique up to multiplication with an non-zero scalar and the prolongation $l^{(1)}$ and derived space ∂l are also lines of rank 1 generated by $\alpha^{k+1} \otimes e$ and $\alpha^{k-1} \otimes e$, respectively.

Proof. Obviously tensors of the form $v = \alpha^k \otimes e$ are of rank one since $\ker v = \langle \alpha \rangle^\circ$ is of co-dimension 1. Conversely suppose that v is of rank one, then $\ker v$ is of co-dimension one and hence there is a one form α unique up to scalar multiple such that $\ker v = \langle \alpha \rangle^\circ$. By lemma 3.20 we must have $v \in S^k(\langle \alpha \rangle) \otimes N$ and hence $v = \alpha^k \otimes e$ for some $e \in N$. It is a direct computation that the spaces ∂l and $l^{(1)}$ are of the form stated in the proposition. □

This implies the following

Corollary 3.22. *Let $L = \bigoplus l_j \subseteq S(R^*) \otimes N$ be a symbolic system with Hilbert polynomial $P_L = 1$ then there exist a unique one-dimensional subspace $\langle \alpha \rangle \in R^*$ and $\langle e \rangle \in N$ such that for all j big enough*

$$l_j = \langle \alpha^j \otimes e \rangle$$

Let us denote with

$$L_{\alpha,e} = \bigoplus_{j \in \mathbb{N}} l_j \subset SR^* \otimes N$$

the symbolic system defined by $l_j := \langle \alpha^j \otimes e \rangle \; \forall l \in \mathbb{N}$.

Observe that lines of rank one in $S^k R^* \otimes N$ are therefore parametrized by $\mathrm{Gr}(R^*, 1) \oplus \mathrm{Gr}(N, 1)$ via the map $\langle \alpha \rangle, \langle e \rangle \mapsto \langle \alpha^k \otimes e \rangle$, which in the case that $\dim N = 1$ is the well known Veronese embedding [8].

Definition 3.23. Let $g \subset SR^* \otimes N$ be a symbolic system, then $\alpha \in R^*$ is called a **(real) characteristic** of g if there exists an $e \in N \backslash \{0\}$ s.t. $L_{\alpha,e} \subseteq g$. We denote the set of real characteristics of g with

$$\mathrm{char}^{\mathbb{R}}(g) := \{\alpha \in R^* \mid L_{\alpha,e} \subseteq g\}.$$

Remark 3.24. If $g_k \subset S^k(R^*) \otimes N$ is a subspace and $l \subset S^k(R^*) \otimes N$ a line of rank one then it follows immediately from proposition 3.21 that $l \subset g_k \Leftrightarrow l^{(1)} \subset g_k^{(1)}$ and so by proposition 3.18 to test whether $\alpha \in \mathrm{char}^{\mathbb{R}}(G)$ is a characteristic of g it suffices to find an $e \in N$ and a $k \in \mathbb{N}$ in the stable regime s.t. $\alpha^k \otimes e \in g_k$.

Definition 3.25. Given a PDE $\mathcal{E}$ the **(real) characteristics** of $\mathcal{E}$ at a point $\theta = (\theta_l)_{l\in\mathbb{N}} \in \mathcal{E}$ are the characteristics of the symbolic system g_θ of the PDE and are denoted with

$$\mathrm{char}^{\mathbb{R}}_\theta(\mathcal{E}) := \mathrm{char}^{\mathbb{R}}(g_\theta)$$

Recall that by corollary 2.35 for any point $\theta = (\theta_l)_{l\in\mathbb{N}}$ in the infinite prolongation of a PDE there is a canonical identification of the Cartan plane at infinity with any R_{θ_k} and hence the set of characteristics should be understood as a subset of the dual of the Cartan plane at θ. Moreover the kernel of a characteristic may be understood as a plane of co-dimension one contained in the Cartan plane. Such a plane will by abuse of terminology also be called **characteristic.**

Obviously the characteristics of a symbolic system are a conic subset in R^* and indeed they are a conic algebraic subvariety of R^* as we explain next.

Notation 3.26. If we neglect the first few components of a graded SR- module the remaining graded SR module is denoted with

$$M_{\geq k_0} := \bigoplus_{k\geq k_0} M_k$$

Definition 3.27. The **annihilator** of a graded SR-module M is the homogeneous ideal

$$\mathrm{Ann}(M) = \{f \in SR \mid fM = 0\} \subset SR$$

Let us also denote the radical ideal of $\mathrm{Ann}(M)$ with

$$I_{\mathrm{char}}(M) = \sqrt{\mathrm{Ann}(M)}$$

and call it the **characteristic ideal** of the SR-module M (the more standard terminology would be the support of M).

Obviously truncating the module M does not change the characteristic ideal except possibly in its 0-th term, i.e. $(I_{\mathrm{char}}M_{\geq k_0})_j = (I_{\mathrm{char}}M)_j$ for any $k_0 \in \mathbb{N}$ and any $j \geq 1$.

Proposition 3.28. *The co-vector $\alpha \in R^*$ is a characteristic of the symbolic system g if and only if $\alpha \in V(I_{char}(g^*))$, where $V(I_{char}(g^*)) \subseteq R^*$ denotes the algebraic variety determined by the ideal $I_{char}(g^*)$.*

This result is due to Quillen [7] but we prove it here along some different lines. We state some preliminaries. The first is an analog of corollary 3.22 in terms of graded modules over SR

Proposition 3.29. *Let M be a finitely generated graded SR-module with Hilbert polynomial $P_M = 1$, then $Ann(M) = I_{char}(M) = I(\langle\alpha\rangle) = (\langle\alpha\rangle^\circ)$ for some $\alpha \in R^*$. Here the round brackets denote the ideal generated by $\langle\alpha\rangle^\circ$ while $I(V)$ denotes the vanishing ideal of a subset $V \subseteq R^*$.*

Proof. Since M is finitely generated there is a k_0 such that M_{k_0} generates all M_k with $k \geq k_0$ and $\dim M_{k_0} = 1$. The multiplication by elements of $R \subset SR$ defines a non-degenerate map $\phi \in \mathrm{Hom}_{\mathbb{R}}(R, \mathrm{Hom}_{\mathbb{R}}(M_{k_0}, M_{k_0+1}))$, $\phi(X)(m) = X\cdot m$ and since $\dim(\mathrm{Hom}_{\mathbb{R}}(M_{k_0}, M_{k_0+1})) = 1$ the kernel of ϕ is of co-dimension 1 and determines a unique line $\langle\alpha\rangle \subset R^*$. The elements $X \in \langle\alpha\rangle^\circ = \ker(\phi)$ are obviously in $\mathrm{Ann}(M)$ and hence $(\langle\alpha\rangle) \subseteq \mathrm{Ann}(M)$. Note that the ideal $(\langle\alpha\rangle^\circ)$ has co-dimension 1 in each S^kR and hence if $\mathrm{Ann}(M)$ was strictly bigger it would coincide with SR after a certain degree and so $SR/\mathrm{Ann}(M)$ would be a finite dimensional algebra. But since M is a finitely generated $SR/\mathrm{Ann}(M)$-module this would imply that it is finite dimensional over $\mathbb{R}$ contradicting that it has Hilbert polynomial 1. We conclude that $\mathrm{Ann}(M) = (\langle\alpha\rangle^\circ)$. ◻

a partial inverse result we need is

Lemma 3.30. *Let M be a finitely generated SR-module with $Ann(M) = (\langle\alpha\rangle^\circ)$ then its Hilbert polynomial is a constant $P_M = c \in \mathbb{N}$ in the stable regime it is the direct sum of c finitely generated modules with Hilbert polynomial equal to* 1.

Proof. Neglecting the first k_0 terms $M_{\geq k_0}$ is canonically a finitely generated $SR/(\langle\alpha\rangle^\circ)$ module and this algebra is isomorphic to a polynomial algebra in one variable say $S[x]$. Such modules must always have constant Hilbert polynomial since the multiplication by x must induce an isomorphism from M_k to M_{k+1} for k big enough. Moreover choosing a splitting of M_k into a direct sum of one dimensional subspaces induces a splitting of all further M_{k+l}, which are submodules of M with Hilbert polynomial equal to 1 □

Finally we need the following

Lemma 3.31. *Let*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of graded SR-modules (the maps are homogeneous but might have a shift) then $I_{char}(M) = I_{char}(L) \cap I_{char}(N)$

Proof. If $f \in I_{\text{char}}(M)$ and $j \in \mathbb{N}$ is such that $f^j M_k = 0$ for all k, then for $l \in L_k$ obviously $f^j l = 0$ and for $n \in N_k$ take $m \in M_k$ with $\pi(m) = n$ so that $f^j n = f^j \pi(m) = \pi(f^j m) = 0$ for $k \geq k_0$. This shows $I_{\text{char}}(M) \subseteq I_{\text{char}}(L) \cap I_{\text{char}}(N)$. Conversely let $f \in I_{\text{char}}(L) \cap I_{\text{char}}(N)$ then for $m \in M$ we have $0 = f^j \pi(m) = \pi(f^j m)$, which means that $f^j m \in L$, and so $f^{k+j} m = 0$ for some $j, k \in N$. □

Now we prove proposition 3.28

Proof. [of prop. 3.28] Obviously the proposition is true if $g_k = 0$ for all $k \geq k_0$, hence we only consider the cases when g has infinitely many nonzero components.

So let $g \subset S(R^*) \otimes N$ be such a symbolic system and $\alpha \in R^*$ a characteristic of it. By definition there is $e \in N$ such that the symbolic system $L_{\alpha,e}$ is contained in g. From corollary 3.16 we deduce that the projection dual to the inclusion gives a surjective map of SR-modules $g^* \to L^*_{\alpha,e}$. It is easy to check that $I_{\text{char}} L^*_{\alpha,e} = (\langle\alpha\rangle^\perp)$ and by proposition 3.31 we conclude that $I_{\text{char}}(g^*) \subset (\langle\alpha\rangle^\perp)$ which means $\alpha \in V(I_{\text{char}}(g^*))$.

Suppose now conversely that $\alpha \in V(I_{\text{char}}(g^*))$. Since $I_{\text{char}}(g^*)$ is homogeneous we have $\langle\alpha\rangle \subset V(I_{\text{char}}(g^*))$ which is equivalent to $I_{\text{char}}(g^*) \subset (\langle\alpha\rangle^\perp)$. Consider now the natural projection of SR-modules $\pi : g^* \to \frac{g^*}{(\langle\alpha\rangle^\perp) g^*}$. By the graded Nakayama lemma and our assumption made in the beginning of the proof the module $\frac{g^*}{(\langle\alpha\rangle^\perp) g^*}$ has infinitely many non-zero homogeneous components and characteristic ideal $(\langle\alpha\rangle^\perp)$. So by lemma 3.30 it is asymptotically a direct sum of modules with Hilbert polynomial equal to 1. Projecting onto one of its summands h we obtain a surjective SR-module morphism $\pi : g^* \to h$ where h has Hilbert polynomial $P_h = 1$ and $I_{\text{char}}(h) = (\langle\alpha\rangle^\perp)$. Applying 3.16 we find that $h^* \subset g$ is a symbolic subsystem with Hilbert polynomial equal to 1 and hence by corollary 3.22 it must be asymptotically of the form $L_{\alpha,e}$. □

Since we are working over the real numbers one may recover the set $\text{char}_{\mathbb{R}} g$ from the ideal $I_{\text{char}}(g)$ but not conversely, so $I_{\text{char}}(g)$ contains more information. Obviously this is the same information contained in the algebraic variety $\text{char}_{\mathbb{C}}(g)$ of complex characteristics, i.e. the zero locus of $I_{\text{char}}(g)$ in the dual of the complexified Cartan planes. We will further down see that $I_{\text{char}}(g)$ is indeed invariantly associated to the diffiety and not only to the concrete realization of it as a PDE.

3.6.2. *Computation of characteristics in local coordinates:* Suppose we are given a PDE $\mathcal{E} = (\mathcal{E}^l)_{l\geq l_0}$ and let k be in the stable regime of $\mathcal{E}$. Then all $\mathcal{E}^{k+l}$ are prolongations of $\mathcal{E}^k$ and all lower $\mathcal{E}^{k-l}$ are projections thereof. Hence to compute the characteristics of a point $\theta \in \mathcal{E}^\infty$ by remark 3.24 it suffices to compute the tensors of rank 1 which are tangent to $\mathcal{E}^k$. Suppose we have introduced local canonical coordinates (x^i, u^j_σ) in J^k and the equation is given as

$$\mathcal{E}^k = \{F^1 = 0, \dots, F^l = 0\}$$

If $\alpha = \sum_{i=1}^n \alpha_i dx^i \in R^{k*}$ and $e = \sum_{j=1}^m e_j \partial_{u^j} \in N^k$ then interpreting $\alpha^k \otimes e$ as a vertical tangent vector in J^k with formula 3.6 it becomes

$$\alpha^k \otimes e = k! \sum_{|\sigma|=k, j=1,\dots,m} \alpha^\sigma e_j \partial_{u^j_\sigma}$$

where $\alpha^\sigma = \alpha_1^{\sigma_1} \cdots \alpha_n^{\sigma_n}$. Hence for α to be characteristic at a point $\theta_k \in \mathcal{E}^k$ there must exist nontrivial e_j's such that

$$\begin{aligned} \sum_{|\sigma|=k, j=1,\dots,m} \alpha^\sigma e_j \partial_{u^j_\sigma} F^1 &= 0 \\ &\vdots \\ \sum_{|\sigma|=k, j=1,\dots,m} \alpha^\sigma e_j \partial_{u^j_\sigma} F^l &= 0 \end{aligned}$$

in θ_k. Considering the matrix $(A_{ij})_{\substack{1\leq i\leq l \\ 1\leq j\leq m}}$ given by:

$$A_{ij} := \sum_{|\sigma|=k} \alpha^\sigma \partial_{u^j_\sigma} F^i$$

this occurs iff $\operatorname{rank}(A_{ij}) < m$. Obviously, only the case $m \leq l$ (i.e. the determined and overdetermined case) needs to be considered since in the underdetermined case all directions are characteristic. In the case $m \leq l$ we may compute all all $m \times m$ minors of A_{ij} and set them equal to zero which gives a set homogeneous polynomial equations in the variables $\alpha_1, \dots, \alpha_n$ of degree mk whose solutions are the characteristic variety.

Remark 3.32. From the above one observes that the dimension of the characteristic variety is a measure of overdetermined-ness of the equation (the smaller the more overdetermined) while its degree is a measure of the degree of the equation. Since we will see that characteristics are an invariant of the diffiety and not only of its realization as a PDE this opens the possibility of introducing the concepts of degree and degree of overdetermined-ness for diffieties.

4. Involutive subspaces of the Cartan distribution

Having introduced the symbolic module we now relate it to singular R-planes in J^k, which are at the heart of the definition of the singularity equations. This requires first a study of involutive subspaces of the Cartan distribution.

4.1. Structure of involutive subspaces of CJ^k.

4.1.1. *Involutive subspaces of a distribution.* Consider a manifold M with distribution $P \subset TM$ and take two vector fields $X, Y \in P$. One easily verifies that the equivalence class $\overline{[X,Y]} \in TM/P$ of the Lie bracket $[X,Y]$ is $C^\infty(M)$-linear in the arguments X, Y and hence the tensor

$$\begin{aligned} \Omega : P \wedge_{C^\infty(M)} P &\rightarrow TM/P \\ X \wedge Y &\rightarrow \overline{[X,Y]} \end{aligned}$$

called the **curvature** of the distribution is well-defined. A plane (i.e. a subspace) in the distribution $\Pi \subset P_p$ $p \in M$ is said to be **involutive** if the curvature tensor vanishes on it, i.e.

$$\Omega_p(v,w) = 0,\ \forall v,w \in \Pi$$

Equivalently one might say that $\Pi \subset P_p$ is involutive if for all $\omega \in P\Lambda^1$ (forms vanishing on the distribution) one has

$$d\omega_p|_\Pi = 0$$

Obviously tangent planes to integral submanifolds are involutive, and so involutiveness is a necessary condition for the existence of an integral submanifold to a prescribed tangent plane. An involutive subspace is said to be **maximal** if it is not contained in any strictly bigger involutive subspace.

4.1.2. *The metasymplectic structure.* Observe that one may also think of the curvature as taking values in the smaller space $P^{(1)}/P \subset TM/P$ which in the case of the Cartan distribution is canonically isomorphic to $S^{k-1}(R^{k*}) \otimes N^k$ (see 3.9), which leads to the following

Definition 4.1. The **metasymplectic structure** on J^k is the curvature tensor of the Cartan distribution with values in $S^{k-1}(R^{k*}) \otimes N^k$:

$$\begin{array}{rcl} \Omega^k : C^k \wedge C^k & \to & S^{k-1}(R^{k*}) \otimes N^k \\ X \wedge Y & \mapsto & \overline{[X,Y]} \end{array}$$

where the bar is the equivalence class in $C^{(1)}/C \cong S^{k-1}(R^{k*}) \otimes N^k$

In local coordinates the metasymplectic structure acts as

$$\begin{array}{rcl} \Omega(D_i, D_j) & = & 0 \\ \Omega(\partial_{u^j_\sigma}, \partial_{u^l_\mu}) & = & 0 \\ \Omega(\partial_{u^j_\sigma}, D_i) & = & \partial_{u^j_{\sigma-1_i}} \end{array}$$

from which one sees for example that the vertical spaces $V^{k-1}J^k$ are involutive.

4.1.3. *Structure of involutive subspaces of CJ^k.* We shall say that two involutive subspaces $P_i \subset T_{\theta_i}J^k$, $i = 1,2$ are **equivalent** if there is a local symmetry of the Cartan distribution sending θ_1 to θ_2 and inducing an isomorphism between P_1 and P_2.

Obviously R-planes are involutive and one has the simple lemma.

Lemma 4.2. *Any two R-planes in J^k are equivalent.*

Proof. Given an R-plane $R_{\theta_{k+1}}$ pick a submanifold $L \subset E$ with $R_{\theta_{k+1}} = T_{\theta_k}L^{(k)}$ and choose divided coordinates x^i, u^j in E such that $L = \{u^1 = \ldots = u^m = 0\}$. Then in the corresponding canonical coordinates on J^k the R-plane has the standard form $R_{\theta_{k+1}} = \langle D_1, \ldots, D_n \rangle$. □

We say that an involutive subspace $H \subset C^k_{\theta_k}$ is **horizontal** if $H \cap V^{k-1}_{\theta_k}J^k = 0$. Since a Cartan plane C_{θ_k} in J^k is the inverse image of the R-plane R_{θ_k} in J^k any horizontal involutive subspace must be of dimension $\leq n$. Obviously R-planes are horizontal and indeed we have the next proposition.

Proposition 4.3. *Any horizontal involutive subspace is contained in an R-plane.*

Proof. Let $U \subset C^k_{\theta_k}$ be such a horizontal involutive plane of dimension $s \leq n$. Choose a divided chart on J^0 such that $T\pi_{k,0}(U) = \langle \partial_{x_1}, \dots, \partial_{x_s} \rangle$. Hence U will be spanned by s vectors of the form

$$X_i = D_i + \sum_{\substack{j=1,\dots,m \\ |\sigma|=k}} \alpha^j_{i,\sigma} \partial_{u^j_\sigma}, \ i = 1, \dots, s$$

with $\alpha^j_{i,\sigma} \in \mathbb{R}$, where we used $D_i = \partial_{x^i} + \sum_{j=1\dots m, |\sigma|<k} u^j_{\sigma+1_i} \partial_{u^j_\sigma}$ and $\partial_{u^j_\sigma}$, $i = 1, \dots, n$, $j = 1, \dots, m$, $|\sigma| = k$ as basis of the Cartan plane C_{θ_k}. Then involutivity of U implies that

$$\alpha^j_{i,\sigma} = \alpha^j_{l,\rho}$$

for any two multi-indices ρ,σ of length k with $\rho - 1_i = \sigma - 1_l$. Since $\rho - 1_i = \sigma - 1_l$ is equivalent to $\rho + 1_l = \sigma + 1_i$, the following definition is well-posed:

$$u^j_{\sigma+1_i} := \alpha^j_{i,\sigma}$$

Setting $u^j_\mu = 0$ for those multi-indices μ of length $k+1$ which are not obtained as $\mu = \sigma + 1_i$, $i = 1, \dots, s$ we obtain

$$U = \langle D_i + \sum_{|\sigma|=k} u^j_{\sigma+1_i} \partial_{u^j_\sigma} \mid i = 1, \dots, s \rangle$$

which is obviously contained in the R-plane $\langle D_i + \sum_{|\sigma|=k} u^j_{\sigma+1_i} \partial_{u^j_\sigma} \mid i = 1, \dots, n \rangle$. □

Corollary 4.4. *Any two horizontal involutive subspaces in C^k of dimension s are equivalent.*

Proof. Similarly as in the proof of 4.2 they can be brought into standard form $\langle D_1, \dots, D_s \rangle$ by choosing the independent coordinates x^i such a way that $T\pi_{k,0}(U) = \langle dx^{s+1}, \dots, dx^n \rangle^\circ$. □

Returning to the case of a general involutive subspace $H \subset C^k_{\theta_k}$ we let

$$H_{\text{vert}} := H \cap V^{k-1}_{\theta_k} J^k \subset S^k(R^*_{\theta_k}) \otimes N^k_{\theta_k}$$

denote its vertical component and call it the **label** of H. Obviously the projection $T_{\theta_k}\pi_{k,k-1}(H)$ is contained in R_{θ_k} and will therefore be a horizontal involutive subspace which we denote

$$H_{\text{hor}} := T_{\theta_k}\pi_{k,k-1}(H)$$

(We emphazise that H_{hor} is not a subspace of H). One may always split (non uniquely) $H = U \oplus H_{\text{vert}}$ choosing a horizontal subspace U which will obviously be involutive. But an arbitrary direct sum of a vertical space and a horizontal involutive space don't need to be involutive since both components have to be orthogonal to each other with respect to the metasymplectic structure. In this direction the following observation which links the metasymplectic structure to the δ-Spencer operator is useful.

Lemma 4.5. *Given $X \in C_{\theta_k}$ and avertical vector $v \in S^k(R^*_{\theta_{k+1}}) \otimes N_{\theta_k}$, then X and v are in involution iff*

$$\delta_{\overline{X}} v = 0$$

where $\overline{X} = T\pi_{k,k-1}X$.

Proof. This follows immediately by comparing the definitions of the first δ-Spencer operator 3.8 and the metasymplectic structure Ω 4.1. □

From this one obtains

Proposition 4.6. *For any involutive subspace $H \subset C^k_{\theta_k}$ the relation*

$$H_{vert} \subset S^k(H^\circ_{hor}) \otimes N \tag{4.1}$$

holds and H is maximally involutive if and only if equality is achieved:

$$H_{vert} = S^k(H^\circ_{hor}) \otimes N$$

Proof. The inclusion 4.1 follows from lemma 4.5 and lemma 3.20.

As for the second statement consider an involutive subspace H for which $H_{\text{vert}} = S^k(\text{Ann}(H_{\text{hor}})) \otimes N$ and suppose $H \subseteq G$ with G involutive, then $H_{\text{hor}} \subseteq G_{\text{hor}}$ implies $S^k(H^\circ_{\text{hor}}) \otimes N \supseteq S^k(G^\circ_{\text{hor}}) \otimes N$. But the inclusion $H_{\text{vert}} \subseteq G_{\text{vert}}$ together with equation (4.1) and dimensional reasons implies $H = G$ and hence H is maximal.

It remains to show that any involutive space H is contained in an involutive G with $G_{\text{vert}} = S^k(G^\circ_{\text{hor}}) \otimes N$. To this end choose a splitting $H = U \oplus H_{\text{vert}}$. Then obviously U is an horizontal involutive subspace and as such contained in an R-plane R_{θ_k} by proposition 4.3. We may choose a local divided chart in which R_{θ_k} is given by $D_1, \dots, D_n$ while U is given by the first s of these vectors, then it is easily checked that the space $G = \langle D_1, \dots, D_s, \partial_{u^j_\sigma} \mid j = 1, \dots, m, |\sigma| = k, \sigma_1 = \dots = \sigma_s = 0 \rangle$ is an involutive subspace with $H \subseteq G$ and $G_{\text{vert}} = S^k(G^\circ_{\text{hor}}) \otimes N$. □

We can conclude that maximal involutive subspaces of C^k are of the dimensions $s + m\binom{n-s+k-1}{k}$ where $s = 0, \dots, n$ is the dimension of the horizontal part (for $\alpha < \beta$ we set $\binom{\alpha}{\beta} = 0$). Observing that $s_1 < s_2$ implies $s_1 + m\binom{n-s_1+k-1}{k} \geq s_2 + m\binom{n-s_2+k-1}{k}$ (equality beeing achieved only in the cases $mn = 1$ or $mk = 1$ or $m = 1 \wedge s_1 = n-1, s_2 = n$) one obtains

Corollary 4.7. *If $m > 1$ then regular R-planes are precisely the maximal involutive subspaces of the smallest dimension among maximal involutive spaces.*

and similarly

Corollary 4.8. *Except for the cases when* $mk = 0$ or $mn = 1$*, vertical subspaces $V^{k-1}_{\theta_k} J^k$ are precisely the maximal involutive spaces of C^k of highest dimension among maximal involutive subspaces.*

Hence, excluding the particular cases $mk = 1$ and $mn = 1$, the vertical distributions $V^{k-1}J^k$ are intrinsic to the Cartan distribution and hence fibers $J^k_{\theta_{k-1}}$ are preserved by symmetries. Moreover a refined argument shows that also in the case when $mn = 1, k > 1$ vertical spaces are recovered from the Cartan distribution. This leads to a proof of the Lie-Baecklund theorem from above. Only in the case $mk = 1$ which corresponds to contact geometry one may not recover R-planes from the Cartan distribution since there are contact transformations mixing vertical and horizontal spaces.

For those cases where vertical spaces are preserved by a symmetry $\phi : J^k \to J^k$ one observes from the definition of the isomorphism $V^{k-1}_{\theta_k} J^k \cong S^k(R^*_{\theta_k}) \otimes N_{\theta_k}$, that the action of $T_{\theta_k}\phi$ on $S^k(R^*_{\theta_k}) \otimes N_{\theta_k}$ is just the tensorial extension of the linear actions on $R_{\theta_k} = C_{\theta_k}/V^{k-1}_\theta J^k$ and $N_{\theta_k} = T_{\theta_k} J^k / C^{(k-1)}_{\theta_k}$.

The following result (mentioned in [28] but there seems to be no proof available in the literature) shows that the classification of involutive subspaces reduces to the classification of their labels.

Proposition 4.9. *Let $mk > 1$, then two involutive subspaces of C^k of the same dimension are equivalent if and only if their labels, understood as subspaces of $S^k(R^*) \otimes N$ are equivalent under the action of $Aut(R) \otimes Aut(N)$.*

Proof. By the above remark it is obvious that if two involutive subspaces are equivalent then their labels are equivalent under the action of $\mathrm{Aut}(R) \otimes \mathrm{Aut}(N)$.

Conversely suppose that two involutive subspaces $H_i \subset C^k_{(\theta_k)_i}, i = 1,2$ of the same dimension have isomorphic labels, and let $A : R_{(\theta_0)_1} \to R_{(\theta_0)_2}$, $B : N_{(\theta_0)_1} \to N_{(\theta_0)_2}$ be linear maps inducing an isomorphism between the labels.

Let us assume first the case that A sends the horizontal parts $H_{\mathrm{hor},i}$ into each other, then we can choose a splitting $H_i = H_{i,\mathrm{vert}} \oplus U_i$ where U_i are horizontal involutive subspaces contained in some R-planes $R_{(\theta_{k+1})_i}$, and pick any extension of A and B to $T_{\theta_0} J^0$. By the process of prolongation (see the Lie-Baecklund Theorem) we may then construct a local diffeomorphism preserving the Cartan distributions sending $R_{(\theta_{k+1})_1}$ to $R_{(\theta_{k+1})_2}$ and acting like A on $R_{(\theta_{k+1})_i}$ and like $A^k \otimes B$ on the vertical component, which proves the statement in this case.

In the case that A does not send the horizontal parts $H_{\mathrm{hor},i}$ into each other we aim to show that we can choose a modified $\tilde{A} : R_{(\theta_0)_1} \to R_{(\theta_0)_2}$ which maps the horizontal parts of the involutive planes into each other without changing the isomorphism on the vertical parts (i.e. such that $A^k \otimes B = \tilde{A}^k \otimes B$). For this consider the subspaces of $R_{(\theta_k)_i}$, $i = 1, 2$ given by

$$K_i := \{X \in R_{(\theta_k)_i} \mid X \in \ker v \ \forall v \in H_{i,\mathrm{vert}}\} = \bigcap_{v \in H_{i,\mathrm{vert}}} \ker v$$

Obviously A must induce an isomorphism between V_1 and V_2, and by proposition 4.6 the horizontal parts satisfy $H_{i,\mathrm{hor}} \subset K_i$. Now observe that by lemma 3.20 the labels satisfy

$$H_{i,\mathrm{vert}} \subset S^k K_i^\circ \otimes N = S^k (R/K_i)^* \otimes N$$

hence modifying A only on the subspaces V_i so that it maps $H_{1,\mathrm{hor}} \subset K_1$ to $H_{2,\mathrm{hor}} \subset K_2$ without changing its action on the quotient $R_{(\theta_k)_i}/K_i$ we achieve the desired property. Obviously this is always possible. □

4.2. **Blocks and singular R-planes.** In this short section we recall that for a given PDE $(\mathcal{E}^l)_{l\in\mathbb{N}}$, $\mathcal{E}^l \subset J^l$ in the stable regime the fibers of the projection $\mathcal{E}^{l+1} \to \mathcal{E}^l$ are affine spaces modeled on the vector spaces g_{l+1}.

Consider an affine vertical line $l \subset J^k_{\theta_{k-1}}$, i.e. a subspace of the form $l = \{\theta_k + \lambda v \mid \lambda \in \mathbb{R}\}$ where $\theta_k \in l$ and $v \in S^k(R^*) \otimes V$ is a tangent vector generating the line. To it corresponds the one parameter family of R-planes in J^{k-1}

$$(R_{\theta_k})_{\theta_k \in l} \subset T_{\theta_{k-1}} J^{k-1}$$

which we call the **block** associated to l and denoted with $B(l)$. Let $B^\cup(l) \subset T_{\theta_{k-1}} J^{k-1}$ be the subspace spanned by the totality of R-planes belonging to the block and $B^\cap(l) \subset T_{\theta_{k-1}} J^{k-1}$ the intersection of all R-planes of the block. Then the following result holds

Proposition 4.10. *(See* [10]*) For a line $l = \{\theta_k + \lambda v \mid \lambda \in \mathbb{R}\}$ one has that*

- $B^\cup(l) = R_{\theta_k} \oplus \mathrm{im}(v) = R_{\theta_k} + R_{\theta'_k}$ for any $\theta_k, \theta'_k \in l$, $\theta_k \neq \theta'_k$
- $B^\cap(l) = \ker(v) = R_{\theta_k} \cap R_{\theta'_k}$ for any $\theta_k, \theta'_k \in l$, $\theta_k \neq \theta'_k$
- $R^\infty(l) := \lim_{\substack{\theta_k \in l \\ \theta_k \to \infty}} R_{\theta_k} = \ker(v) \oplus \mathrm{im}(v)$

Proposition 4.11. *Let $\mathcal{E} \subset J^k$ a submanifold then for any $\theta_k \in \mathcal{E}$ the spaces $\mathcal{E}^{(1)}_{\theta_k}$ are affine subspaces of $J^{k+1}_{\theta_k}$.*

Proof. It suffices to show that if $\theta_{k+1}, \theta'_{k+1} \in \mathcal{E}^{(1)}_{\theta_k}$ then the affine line l connecting them is also contained in $\mathcal{E}^{(1)}_{\theta_k}$, but this follows from proposition 4.10 since if $L_{\theta_{k+1}}, L_{\theta'_{k+1}} \subset T_{\theta_k}\mathcal{E}$ then also $L_{\theta_{k+1}} + L_{\theta'_{k+1}} \subset T_{\theta_k}\mathcal{E}$ and hence the whole block $B(l)$ is tangent to $\mathcal{E}$ □

Note that by the previous proposition the symbol of an equation in the stable regime at θ_{k+1} depends only on the point θ_k and not on $\theta_{k+1} \in \mathcal{E}^{(1)}$ Moreover the symbol of the prolongation $g^{k+1}_{\theta_k}$ is completely determined by the symbol $g^k_{\theta_k}$ as shows the following result.

Proposition 4.12. *Let $\theta_k \in \mathcal{E}$ and suppose that $\mathcal{E}^{(1)}_{\theta_k}$ is non empty, then $g^{k+1}_{\theta_k}$ is the prolongation of the symbol $g^k_{\theta_k}$:*

$$g^{k+1}_{\theta_k} = (g^k_{\theta_k})^{(1)}$$

Proof. "$\subseteq$": Suppose $\theta_{k+1} \in \mathcal{E}^{(1)}_{\theta_k}$ and $v \in V^k\mathcal{E}^{(1)}_{\theta_k}$ then the line $l = \theta_{k+1} + \lambda v$, $\lambda \in \mathbb{R}$ is contained in $\mathcal{E}^{(1)}_{\theta_k}$ by proposition 4.11 and hence the block $B(l)$ is contained in $T_{\theta_k}\mathcal{E}$, but then by proposition 4.10 the space $\mathrm{im}(v) \subseteq T_{\theta_k}\mathcal{E}$ and hence $v \in (g^k_{\theta_k})^{(1)}$.

"$\supseteq$": Conversely suppose $v \in (g^k_{\theta_k})^{(1)}$ and $\theta_{k+1} \in \mathcal{E}^{(1)}_{\theta_k}$ then $R_{\theta_{k+1}} \oplus \mathrm{im}(v) \subseteq T_{\theta_k}\mathcal{E}$ by definition of the prolongation and the symbol of the equation, but then again by proposition 4.10 the block $B(l)$ associated to the line $l = \theta_{k+1} + \lambda v$, $\lambda \in \mathbb{R}$ is contained in $T_{\theta_k}\mathcal{E}$ and therefore the line l is contained in $\mathcal{E}^{(1)}$ implying that v is contained in $g^{(k+1)}_{\theta_k}$. □

5. The relation between singularity equations and characteristics

In this section we establish how 1-singularity equations are related to characteristics of the PDE.

Proposition 5.1. *An n-dimensional subspace $\Pi \subset C_{\theta_k}$ is a singular R-plane in J^k of type 1 if and only if it is involutive and its vertical part Π_{vert} is a one dimensional subspace in $S^kR^*_{\theta_k} \otimes N$ of rank 1.*

Proof. Let Π be an n-dimensional singular R-plane then it is involutive since it is tangent to an integral submanifold. Moreover if it is of type 1 then its vertical part Π_{vert} is a one dimensional subspace in $S^kR^*_{\theta_k} \otimes N_{\theta_k}$ and by proposition 4.6 we have $\Pi_{\mathrm{vert}} \subset S^k(\Pi^\circ_{\mathrm{hor}}) \otimes N_{\theta_k}$. Since $\dim(\Pi^\circ_{\mathrm{hor}}) = 1$ it is generated by a co-vector $\alpha \in R^*_{\theta_k}$ and hence Π_{vert} is generated by $\alpha^k \otimes e$ for some $e \in N$ and of rank 1 by proposition 3.21.

Conversely suppose Π is involutive and has a vertical component of dimension one and rank 1. Since all tensors of rank one are equivalent by proposition 3.21, using proposition 4.9we conclude that any two involutive subspaces of this type are equivalent. That they are realized as tangent planes to a generalized R-manifold follows from example 1.16. □

Before stating the next result which describes the relation between fold type singularity equations and characteristics we make the following observation. Given a point $\theta_k \in \mathcal{E}^k$ then a real characteristic $\alpha \in \mathrm{char}^{\mathbb{R}}_{\theta_k}(\mathcal{E})$ determines an $(n-1)$-dimensional plane at θ_{k-1} namely

$$\ker\alpha \subset R_{\theta_k}$$

and conversely this plane determines the characteristic (up to scaling). In other words the set of all characteristics at θ_k may be thought of as a subvariety of $\mathrm{Gr}(R_{\theta_k}, n-1)$. Varying the point θ_k in the fiber over $\mathcal{E}_{\theta_{k-1}}$ all the so obtained $n-1$ planes give a subset in $\mathrm{Gr}(C_{\theta_{k-1}}, n-1)$. The union of all these subsets over $\theta_{k-1} \in \mathcal{E}^{k-1}$ may be considerd as a first order PDE imposed on $n-1$ dimensional submanifolds of $\mathcal{E}^{k-1}$. The next result states that this equation coincides with the k-th equation of singularities of type 1 of $\mathcal{E}$.

Theorem 5.2. In the stable regime the k-th singularity equations of type 1 of a PDE $\mathcal{E}$ are given by

$$\Sigma_{[1]}\mathcal{E}^k = \{\ker(\alpha) \subset R_{\theta_k} \mid \alpha \in \mathrm{char}^{\mathbb{R}}_{\theta_k}(\mathcal{E}^k) \text{ and } \theta_k \in \mathcal{E}^k\}$$

Proof. "$\subseteq$" By proposition 5.1 we have that if Π is a singular R-plane of type 1 of $\mathcal{E}^k$ at $\theta_k \in \mathcal{E}^k$ then its vertical component is generated by a tensor of the form $\alpha^k \otimes e$ and by proposition 4.6 we know $\pi_{k,k-1}(\mathcal{E}^k) = \ker \alpha$. Obviously in the stable regime α is a characteristic of $\mathcal{E}$ as stated in remark 3.24.

"$\supseteq$" Suppose $\alpha \in \mathrm{char}^{\mathbb{R}}_{\theta_k}(\mathcal{E}^k)$ then there is an $e \in N$ such that $\alpha^k \otimes e$ is tangent to the equation at θ_k. Choose a point $\theta_{k+1} \in \mathcal{E}^{k+1}_{\theta_k}$, then $R_{\theta_{k+1}}$ will be tangent to $\mathcal{E}^k$ and

$$\Pi := \ker(\alpha) \oplus \langle \alpha^k \otimes e \rangle \subset C_{\theta_k}$$

(here $\ker(\alpha) \subset R_{\theta_{k+1}}$) will be a singular R-plane of type 1 of the equation by proposition 5.1 which satisfies $\pi_{k,k-1}(\Pi) = \ker(\alpha)$. □

This theorem may also be interpreted as an analog of a theorem in the linear theory of differential equations which states that the singular support of distribution solutions occurs along characteristics.

Another way to reformulate the theorem is by saying that singularity equations are projections of characteristics of the infinite prolongation.

Corollary 5.3. *Let $\mathcal{E}$ be a PDE, then in the stable regime the k-th 1-singularity equations of type one are the projections of characteristics from the infinite prolongation*

$$\Sigma_{[1]}\mathcal{E}^k = \{\ker(\alpha) \subset R_{\theta_k} \mid \alpha \in \mathit{char}^{\mathbb{R}}_{\theta}(\mathcal{E}^\infty) \text{ and } \theta \in \mathcal{E}^\infty\}$$

Proof. This follows from the previous theorem and remark 3.24 since characteristics do not change when passing to higher prolongations in the stable regime. □

Corollary 5.4. *In the stable regime of an equation, subsequent 1-singularity equations $\Sigma_{[1]}\mathcal{E}^{k+1}$ and $\Sigma_{[1]}\mathcal{E}^k$ project onto each other in the sense that*

$$\Sigma_{[1]}\mathcal{E}^k(\theta_{k-1}) = \{T\pi_{k,k-1}(\Sigma_{[1]}\mathcal{E}^{k+1}(\theta_k)) \mid \theta_k \in \mathcal{E}^k_{\theta_{k-1}}\}$$

Proof. Since characteristics of $\mathcal{E}^k$ and $\mathcal{E}^{k+1}$ remain the same by remark 3.24 this is a direct consequence the previous results. □

The converse of proposition 5.2 is also true in the sense that real characteristics can be recovered from the 1-singularity equations at any level. To prove this let us introduce a notation. Suppose $\theta_k \in \mathcal{E}^k$ then we set

$$\Xi_{\theta_k} = \{Q \in \mathrm{Gr}(R_{\theta_k}, n-1) \mid Q \in \Sigma_{[1]}\mathcal{E}^k(\theta_{k-1})\}$$

In other words Ξ_{θ_k} consist of all $n-1$ planes in R_{θ_k} which are projections of some 1-singular R-plane of $\mathcal{E}^k$

Lemma 5.5. *Let k be in the stable regime of a PDE $\mathcal{E}$, then for any $\theta_k \in \mathcal{E}^k$ we have that*

$$\Xi_{\theta_k} = \mathit{char}^{\mathbb{R}}_{\theta_k}(\mathcal{E}^k)$$

where $\mathit{char}^{\mathbb{R}}_{\theta_k}(\mathcal{E}^k)$ is understood as a subvariety of $Gr(R_{\theta_k}, n-1)$.

Proof. Let $Q \in \Xi_{\theta_k}$ be the $n-1$ dimensional plane obtained from projecting a 1-singular R-plane $\Pi \subset T_{\tilde{\theta}_k}\mathcal{E}^k$ of the equation at some point $\tilde{\theta}_k \in \mathcal{E}^k_{\theta_{k-1}}$. Then $\Pi_{\mathrm{vert}} = \langle \alpha^k \otimes e \rangle$ and α is a characteristic at $\tilde{\theta}_k$. But since in the stable regime the symbols g_{θ_k} and $g_{\tilde{\theta}_k}$ coincide we see that $\alpha^k \otimes e$ is also tangent to the equation at point θ_k and hence is also a characteristic at θ_k. The converse, i.e. to show that if α is a characteristic at θ then $\ker(\alpha)$ is part of the singularity equation was already done. □

Hence 1-singularity equations and real characteristics contain the same information. Nevertheless singularity equations are not an invariant of the diffiety but a contact invariant of the PDE, i.e. an invariant under the smaller group of diffeomorphism of the PDE which preserve the co-filtration. Since we will show that characteristics, or better the characteristic ideal of a PDE are invariants of the diffiety, they might be be considered the more fundamental concept behind 1-singularity equations.

Remark 5.6. Vinogradov observed in [28] that under some favorable conditions (namely that the equation be determined, of pure degree k and strongly hyperbolic), the knowledge of the 1-singularity equations $\Sigma_{[1]}\mathcal{E}^k$ allows one to recover the whole PDE $\mathcal{E}^k$. Stated differently this suggests that observing singularities of a physical system allows one to deduce the laws governing the system.

Another important application of 1-singularity equations is that they allow one to construct scalar differential invariants of classes of PDEs under contact transformation, which in principle allows a local contact classification of such PDEs. This is for example carried out in [23] for the case of hyperbolic Monge-Ampère equations and in [3] for the case of parabolic Monge-Ampère equations.

Example 5.7. Let's consider an example of the computation of fold type singularity equations for a third order scalar equation in two independent variables. Let x, y denote independent coordinates and u the dependent coordinate equation $\mathcal{E} \subset J^3$ is given by

$$F = u_{0,3} - u_{2,1}^2 + u_{3,0}u_{1,2} = 0$$

Writing:

$$u_{0,3} = u_{2,1}^2 - u_{3,0}u_{1,2} \tag{5.1}$$

one may parametrized the equation in any fiber $J^3_{\theta_2}$ by coordinates $u_{3,0}, u_{2,1}, u_{1,2}$

To compute the characteristics we associate to any non-zero co-vector on the base M:

$$\omega = adx + bdy,\ a, b \in \mathbb{R}$$

the vertical vector

$$X = a^3\partial_{3,0} + a^2b\partial_{2,1} + ab^2\partial_{1,2,} + b^3\partial_{0,3}$$

in J^3 corresponding to $\omega^3 \in S^3(T^*M)$. The values of a, b where X is tangent to the equation at $\theta_3 \in \mathcal{E}$ gives the characteristics at θ_3. To find them we need to solve the equation

$$X(F) = b^3 - 2a^2bu_{2,1} + a^3u_{1,2} + ab^2u_{3,0} = 0 \tag{5.2}$$

for any $(u_{3,0}, u_{2,1}, u_{1,2}) \in \mathcal{E}$. Note that coordinate $u_{0,3}$ does not appear in this equation hence by equation 5.1 values of $(u_{3,0}, u_{2,1}, u_{1,2})$ are arbitrary. Moreover a may not be zero since otherwise $b = 0$, so we may scale $a = 1$. This reduces equation 5.2 to an (arbitrary) cubic polynomial

$$b^3 + u_{3,0}b^2 - 2u_{2,1}b + u_{1,2} = 0$$

One can avoid writing down explicitly the roots of this polynomial since we shall need them for arbitrary parameters $(u_{3,0}, u_{2,1}, u_{1,2})$ one may take $b \in \mathbb{R}$ as a new parameter and eliminate $u_{1,2}$:

$$u_{1,2} = -b^3 - u_{3,0}b^2 + 2u_{2,1}b \tag{5.3}$$

Now the 1-Singularity equations in J^2 are the lines generated by

$$Y = b(\theta_3)D_x(\theta_3) - D_y(\theta_3)$$

where $\theta_3 \in \mathcal{E}$, and D_x, D_y are total derivatives understood as relative fields and $b(\theta_3)$ are solutions of the charactericity equation above.

To write this explicitly consider the following basis of the Cartan distribution in J^2

$$\begin{aligned}
\Delta_x &= \partial_x + u_{1,0}\partial_{0,0} + u_{2,0}\partial_{1,0} + u_{1,1}\partial_{0,1} \\
\Delta_y &= \partial_y + u_{0,1}\partial_{0,0} + u_{1,1}\partial_{1,0} + u_{0,2}\partial_{0,1} \\
Z_0 &= \partial_{0,2} \\
Z_1 &= \partial_{1,1} \\
Z_2 &= \partial_{2,0}
\end{aligned}$$

and denote with g_x, g_y, f_0, f_1, f_2 associated coordinates on each Cartan plane. Then

$$\begin{aligned}
Y \;=\; & b(\underbrace{\partial_x + u_{1,0}\partial_{0,0} + u_{2,0}\partial_{1,0} + u_{1,1}\partial_{0,1}}_{\Delta_x} + u_{3,0}\partial_{2,0} + u_{2,1}\partial_{1,1} + u_{1,2}\partial_{0,2}) \\
& -(\underbrace{\partial_y + u_{0,1}\partial_{0,0} + u_{1,1}\partial_{1,0} + u_{0,2}\partial_{0,1}}_{\Delta_y} + u_{2,1}\partial_{2,0} + u_{1,2}\partial_{1,1} + u_{0,3}\partial_{0,2})
\end{aligned}$$

and using equations 5.1 and 5.3 we find

$$Y = bD_x - D_y + (bu_{3,0} - u_{2,1})Z_2 + b(b^2 + bu_{3,0} - u_{2,1})Z_1 - (b^2 + bu_{3,0} - u_{2,1})^2 Z_0$$

Introducing the scaling parameter l the cone describing the singularity equations is given by all vectors lY. In coordinates

$$\begin{aligned}
g_x &= lb \\
g_y &= -l \\
f_0 &= -l(b^2 + bu_{3,0} - u_{2,1})^2 \\
f_1 &= lb(b^2 + bu_{3,0} - u_{2,1}) \\
f_2 &= l(bu_{3,0} - u_{2,1})
\end{aligned}$$

the parameters may easily be eliminated (observing that the expression $bu_{3,0} - u_{2,1}$ appears repeatedly) leading to two homogeneous equations describing the singularity equation

$$\begin{aligned}
-g_x^3 + g_x g_y f_2 + g_y^2 f_1 &= 0 \\
g_x^4 - 2g_x^2 g_y f_2 + g_y^2 f_2^2 - g_y^3 f_0 &= 0
\end{aligned}$$

5.1. **Grassmannian of horizontal involutive subspaces.** Singularity equations were defined as lying in $J^1(J^k, n-s) = \mathrm{Gr}(TJ^k, n-s)$ but may actually be consider to be part of a smaller Grassmanian namely the **Grassmannian of s-dimensional horizontal involutive subspaces** of $C^k_{\theta_k}$ which will be denoted with

$$I^{k,s}_{\theta_k} := \{U \subseteq C^k_{\theta_k} \mid U \text{ involutive horizontal and } \dim U = s\}$$

Any horizontal involutive subspace $U \in I^{k,s}_{\theta_k}$ may be projected to J^{k-1} giving again an involutive horizontal subspace of the same dimension s and contained in R_{θ_k}. This projected plane will simply be denoted with $U_{k-1} := d_{\theta_k}\pi_{k,k-1}(U)$.

Definition 5.8. Given a horizontal involutive subspace $U \in I^{k,s}_{\theta_k}$ we define its **ray** to be

$$\ell(U) := \{\theta_{k+1} \in J^{k+1}_{\theta_k} \mid U \subset R_{\theta_{k+1}}\}$$

Proposition 5.9. [10] *The ray $\ell(U)$ is an affine subspace in $J^{k+1}_{\theta_k}$ whose tangent space is given by*

$$S^{k+1}(U^\circ) \otimes N$$

Definition 5.10. Let $V \subset C^k_{\theta_k}$ be a subspace then we define its **orthogonal** $V^\perp$ to be

$$V^\perp = \{w \in C^k_{\theta_k} \mid \Omega(w, v) = 0 \,\forall v \in V\}$$

Now observe that given a horizontal involutive subspace $U \in I^{k,s}_{\theta_k}$ the space $U^\perp$ determines a subspace of the tangent space $T_U I^{k,s}_{\theta_k}$ consisting of all deformations of U which are to first order contained in $U^\perp$, which gives rise to a distribution on $I^{k,s}_{\theta_k}$. Due to lack of time we don't enter into more details of this general construction but illustrate it in the next subsection for the case of two independent variables and one dependent variable. There we show that the so obtained distribution is isomorphic to the Cartan distribution on jet spaces with one independent and one dependent variable. This gives the possibility of associating to the fold-singularity equations at a point θ_k an ODE which by construction is a contact invariant of the equation.

5.2. **The distribution on $I^{k,1}_{\theta_k}$ for two independent and one dependent variable.** Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the trivial bundle and $J^k = J^k(\pi)$. In these paragraphs it is shown that the distribution on $I^{k,1}_{\theta_k}$ for π as above is locally isomorphic to the Cartan distribution on $J^k(\xi)$ for the bundle $\xi : \mathbb{R}^2 \to \mathbb{R}$.

Recall that by lemma 4.2 for any two horizontal involutive subspaces of the same dimension there exist a local diffeomorphism preserving the Cartan distribution and sending one subspace to the other (i.e. any two such spaces are equivalent). So $I^{k,s}_{\theta_k}$ is homogeneous and hence it suffices to give a description of the distribution on $I^{k,1}$ in local coordinates around any point.

Projective coordinates are chosen as follows: recall that any vector $v \in C^k_{\theta_k}$ is given in standard coordinates by

$$v = aD_x + bD_y + \sum_{l=0}^{k} f_l \partial_{l,k-l}$$

where $a, b, f_l \in \mathbb{R}$ and $\partial_{l,k-l} = \partial_{u_{l,k-l}}$. Fixing $a = 1$ the remaining coefficients $b, f_0, \dots f_k$ define a local chart on $I^{k,1}$ and a vector field

$$w = B\partial_b + \sum_{l=0}^{k} g_l \partial_{f_l}$$

in these charts with B, g_l functions of the $b, f_0, \dots f_k$ is in the distribution if and only if

$$\Omega(D_x + bD_y + \sum_{l=0}^{k} f_l \partial_{l,k-l}, BD_y + \sum_{l=0}^{k} g_l \partial_{l,k-l}) = 0$$

where Ω is the metasymplectic structure. Recalling that the only nontrivial pairings are given by

$$\Omega(\partial_{u^j_\sigma}, D_i) = \partial_{u^j_{\sigma-1_i}}$$

when $\sigma_i \geq 1$, we arrive at the following conditions on the coefficients B, g_l

$$g_{l+1} + bg_l = Bf_l,\ l = 0, \dots, k-1.$$

One sees that by choosing for example B and g_0 arbitrarily the other g_l, $l \geq 1$ are determined uniquely by the formula

$$g_l = (-b)^l g_0 + B \sum_{j=0}^{l} (-b)^{l-j-1} f_j$$

Making a change of coordinates ($\alpha \leftrightarrow -b$) and choosing $(B, g_0) = (0, 1)$ and $(B, g_0) = (1, 0)$ respectively, the two basis vector fields of our distribution in these local coordinates are

$$\begin{aligned} X &= \sum_{l=0}^{k} \alpha^l \partial_{f_l} \\ Y &= -\partial_\alpha + \sum_{j<l} \alpha^{l-j-1} f_j \partial_{f_l} \end{aligned}$$

where in the last sum the indices j, l run through $0, \ldots, k$.

Define for $j = 1, \ldots k$

$$X_j = [\cdots [X, \underbrace{Y], Y], \ldots], Y}_{j\text{-times}}]$$

and put $X_0 = X$. Obviously for $j < k$

$$X_{j+1} = [X_j, Y] \tag{5.4}$$

.

Proposition 5.11. *For $j = 0, \ldots, k$*

$$X_j = (j+1) \sum_{l=j}^{k} (\alpha^l)^{(j)} \partial_{f_l} \tag{5.5}$$

where $(\alpha^l)^{(j)} = \frac{d^j}{d\alpha^j} \alpha^l = l(l-1) \cdots (l-j+1) \alpha^{l-j}$

Corollary 5.12. *The fields $Y, X_0, \ldots, X_k$ are linearly independent and hence form a local non-holonomic frame field on $I^{k,1}$, moreover the only nontrivial commutation relations are the ones given by equations 5.4. This shows that (rearranging indices by $j \leftrightarrow k - j$) the frame satisfies the same commutation relations as the non-holonomic frame $D_t, \partial_p, \partial_{p^{(1)}}, \ldots, \partial_{p^{(k)}}$ on $J^k(\xi)$, showing in particular that the distributions are isomorphic.*

The proof of the corollary is by checking the explicit expressions 5.5 given above. Proposition 5.11 is proven by induction and uses the following formula

Lemma 5.13. *For any $k \in \mathbb{N} \backslash \{0\}$*

$$\sum_{i=1}^{m} i(i+1)(i+2) \cdots (i+k-1) = \frac{m(m+1)(m+2) \cdots (m+k)}{k+1}$$

Proof. [of the lemma] By induction on m: the case $m = 1$ is clear and the induction $m \to m+1$ is

$$\begin{aligned} \sum_{i=1}^{m+1} i(i+1)(i+2) \cdots (i+k-1) &= \sum_{i=1}^{m} i(i+1)(i+2) \cdots (i+k-1) \\ &\quad + (m+1)(m+2) \cdots (m+k) \\ &= \frac{m(m+1)(m+2) \cdots (m+k)}{k+1} \\ &\quad + (m+1)(m+2) \cdots (m+k) \\ &= \frac{(m+1)(m+2) \cdots (m+k+1)}{k+1} \end{aligned}$$

□

now we prove the proposition

Proof. For $j = 0$ the formula is true, and inductively we obtain

$$
\begin{aligned}
X_{j+1} &= [X_j, Y] \\
&= [(j+1)\sum_{l=j}^{k}(\alpha^l)^{(j)}\partial_{f_l}, -\partial_\alpha + \sum_{m<i}\alpha^{i-m-1}f_m\partial_{f_i}] \\
&= (j+1)\left(\sum_{l=j+1}^{k}(\alpha^l)^{(j+1)}\partial_{f_l} + \sum_{l=j,}^{k}\sum_{m<i}(\alpha^l)^{(j)}\alpha^{i-m-1}\partial_{f_l}(f_m)\partial_{f_i}\right) \\
&= (j+1)\left(\sum_{l=j+1}^{k}(\alpha^l)^{(j+1)}\partial_{f_l} + \sum_{l=j}^{k}\sum_{l<i} l(l-1)\cdots(l-j+1)\alpha^{i-j-1}\partial_{f_i}\right) \\
&= (j+1)\left(\sum_{l=j+1}^{k}(\alpha^l)^{(j+1)}\partial_{f_l} + \sum_{i=j+1}^{k}\underbrace{\sum_{l=j}^{i-1} l(l-1)\cdots(l-j+1)}_{\frac{(i-j)(i-j+1)\cdots(i-1)i}{j+1}}\alpha^{i-j-1}\partial_{f_i}\right) \\
&= (j+1)\left(\sum_{l=j+1}^{k}(\alpha^l)^{(j+1)}\partial_{f_l} + \sum_{i=j+1}^{k}\frac{1}{j+1}(\alpha^i)^{(j+1)}\partial_{f_i}\right) \\
&= (j+2)\sum_{l=j+1}^{k}(\alpha^l)^{(j+1)}\partial_{f_l}
\end{aligned}
$$

□

Lemma 5.14. *The dual one forms to the fields X_j are given by the formulas*

$$
\omega^r = \frac{1}{r+1}\left(\frac{1}{r}\sum_{j=0}^{r-1}\frac{(-\alpha)^{r-j-1}}{j!(r-j-1)!}f_j d\alpha + \sum_{t=0}^{r}\frac{1}{t!(r-t)!}(-\alpha)^{r-t}df_t\right)
$$

for $r > 0$ and $\omega^0 = df_0$ while the one form dual to Y is $-d\alpha$

To prove this we need the following two formulas involving factorials

Proposition 5.15. *For all $r \in \mathbb{N}$*

$$
\sum_{s=0}^{r}\frac{(-1)^{r-s}}{s!(r-s)!} = \begin{cases} 1 & , r = 0 \\ 0 & , r \geq 1 \end{cases} \tag{5.6}
$$

and for the partial sums with $q < r$

$$
\sum_{s=0}^{q}\frac{(-1)^{r-s}}{s!(r-s)!} = \frac{1}{r}\frac{(-1)^{r-q}}{q!(r-q-1)!} \tag{5.7}
$$

Proof. [of proposition 5.15] The first formula is proven using the well known formula $(a+b)^r = \sum_{s=0}^{r}\binom{r}{s}a^s b^{r-s}$ with $a = 1$ and $b = -1$. The second formula is proven by induction on q, we just check the induction step $q \to q+1$:

$$
\begin{aligned}
\sum_{s=0}^{q+1}\frac{(-1)^{r-s}}{s!(r-s)!} &= \frac{1}{r}\frac{(-1)^{r-q}}{q!(r-q-1)!} + \frac{(-1)^{r-q-1}}{(q+1)!(r-q-1)!} \\
&= (-1)^{r-q-1}\left(-\frac{q+1}{r(q+1)!(r-q-1)!} + \frac{r}{r(q+1)!(r-q-1)!}\right) \\
&= \frac{1}{r}\frac{(-1)^{r-q}}{(q+1)!(r-q-2)!}
\end{aligned}
$$

□

now we prove lemma 5.14

Proof. One immediately checks the claim for the one forms ω^0 and $-d\alpha$ while for $r > 0$ we obtain:

$$\begin{aligned}\omega^r(X_j) &= \frac{j+1}{r+1}\sum_{l=j}^{k}\sum_{t=0}^{r}\frac{1}{t!(r-t)!}(\alpha^l)^{(j)}(-\alpha)^{r-t}df_t(\partial_{f_l})\\ &= \frac{j+1}{r+1}\sum_{l=j}^{k}\sum_{t=0}^{r}\frac{(-1)^{r-t}l(l-1)\cdots(l-j+1)}{t!(r-t)!}\alpha^{r+l-j-t}\delta_{t,l}\\ &= \begin{cases}0 & ,r<j\\ \frac{j+1}{r+1}\alpha^{r-j}\sum_{l=j}^{r}\frac{(-1)^{r-l}}{(l-j)!(r-l)!} & ,r\geq j\end{cases}\end{aligned}$$

using formula 5.6 the last sum becomes

$$\sum_{l=j}^{r}\frac{(-1)^{r-l}}{(l-j)!(r-l)!} = (-1)^{r-j}\sum_{l=0}^{r-j}\frac{(-1)^l}{l!(r-j-l)!} = \delta_{r,j}$$

which implies

$$\omega^r(X_j) = \delta_{r,j}$$

moreover

$$\begin{aligned}\omega^r(Y) &= \frac{1}{r+1}\left(-\frac{1}{r}\sum_{j=0}^{r-1}\frac{(-\alpha)^{r-j-1}}{j!(r-j-1)!}f_j + \sum_{t=0}^{r}\sum_{j<l}\frac{1}{t!(r-t)!}(-\alpha)^{r-t}\alpha^{l-j-1}f_j df_t(\partial_{f_l})\right)\\ &= \frac{1}{r+1}\left(-\frac{1}{r}\sum_{j=0}^{r-1}\frac{(-\alpha)^{r-j-1}}{j!(r-j-1)!}f_j + \sum_{j<l}^{r}\frac{(-1)^{r-l}}{l!(r-l)!}\alpha^{r-j-1}f_j\right)\\ &= \frac{1}{r+1}\left(-\frac{1}{r}\sum_{j=0}^{r-1}\frac{(-\alpha)^{r-j-1}}{j!(r-j-1)!}f_j + \sum_{j=0}^{r-1}\alpha^{r-j-1}f_j\underbrace{\sum_{l=j+1}^{r}\frac{(-1)^{r-l}}{l!(r-l)!}}_{=\star}\right)\end{aligned}$$

where we can use formulas 5.6 and 5.7 to write

$$\star = -\sum_{l=0}^{j}\frac{(-1)^{r-l}}{l!(r-l)!} = \frac{1}{r}\frac{(-1)^{r-j-1}}{j!(r-j-1)!}$$

and so $\omega^r(Y) = 0$ □

Using this we may explicitly give an isomorphism between the distribution on $I^{k,1}$ and $J^k(\xi)$

Proposition 5.16. *The change of coordinates*

$$\begin{aligned}t &= -\alpha\\ p_r &= \frac{1}{r+1}\sum_{l=0}^{r}\frac{(-\alpha)^{r-l}}{l!(r-l)!}f_l,\ r=0,\dots,k\end{aligned}\tag{5.8}$$

(after a re-indexing $u_{k-r} = p_r$) puts the distribution on $I^{k,1}$ into the standard form of the Cartan distribution on $J^k(\xi)$.

Proof. The Cartan forms in the coordinates $(t, p_0, \dots, p_k)$(without re-indexing) are given by

$$\begin{aligned} dp_r - p_{r-1}dt &= \frac{1}{r+1}\sum_{l=0}^{r}\frac{(-\alpha)^{r-l}}{l!(r-l)!}df_l - \frac{1}{r+1}\sum_{l=0}^{r-1}\frac{(-\alpha)^{r-l-1}}{l!(r-l-1)!}f_l d\alpha + \frac{1}{r}\sum_{l=0}^{r-1}\frac{(-\alpha)^{r-l-1}}{r!(r-l-1)!}f_l d\alpha \\ &= \frac{1}{r+1}\sum_{l=0}^{j}\frac{(-\alpha)^{r-l}}{l!(r-l)!}df_l + \frac{1}{(r+1)r}\sum_{l=0}^{r-1}\frac{(-\alpha)^{r-l-1}}{l!(r-l-1)!}f_l d\alpha \\ &= \omega^r \end{aligned}$$

□

Lemma 5.17. *The inverse transformation to the change of coordinates above is*

$$\begin{aligned} \alpha &= -t \\ f_l &= l!\sum_{j=0}^{l}\frac{(j+1)(-t)^{l-j}}{(l-j)!}p_j \end{aligned} \tag{5.9}$$

Proof. Plugging the formula 5.8 for p_j into $f_l(p_0 \dots, p_k)$ we find

$$\begin{aligned} l!\sum_{j=0}^{l}\frac{(j+1)(-t)^{l-j}}{(l-j)!}\frac{1}{j+1}\sum_{i=0}^{j}\frac{t^{j-i}}{i!(j-i)!}f_i &= l!\sum_{j=0}^{l}\frac{(-t)^{l-j}}{(l-j)!}\sum_{i=0}^{j}\frac{t^{j-i}}{i!(j-i)!}f_i \\ &= l!\sum_{i=0}^{l}\frac{t^{l-i}}{i!}f_i\underbrace{\sum_{j=i}^{l}\frac{(-1)^{l-j}}{(l-j)!(j-i)!}}_{\star} \end{aligned}$$

using formula 5.6 we find $\star = \sum_{j=0}^{l-i}\frac{(-1)^j}{j!(l-i-j)!} = \delta_{l,i}$ and hence

$$f_l(p_0(f_0, \dots, f_k), \dots p_k(f_0, \dots, f_k)) = f_l$$

□

Example 5.18. Let us apply this to fold-singularity equations of the third order equation from the example 5.7. Recall that the singularity equation at each point may be seen as a subvariety of $I^{2,1}$ and restricting to it the distribution on $I^{2,1}$ we obtain, using the coordinates from above the system of second order ODEs

$$\begin{aligned} -1 - 2t^2\dot{u} + 6tu &= 0 \\ (1 - t^3\ddot{u} + 4t^2\dot{u} + 6tu)^2 - u &= 0 \end{aligned}$$

6. The characteristic ideal of a Diffiety

6.1. **An intrinsic approach to the characteristic variety.** In the previous sections we used the structure of the surrounding jet spaces to define the characteristic variety of a nonlinear PDE. In this section we show how the characteristic variety of an abstract diffiety can be described intrinsically.

The approach is based on the following purely algebraic preliminaries which are well known to people familiar with D-modules (see for instance [5, 6])

6.1.1. *Almost commutative filtered algebras, filtered modules and characteristic ideals.* Let A be a **filtered algebra**, i.e. A is a unital, associative, not necessarily commutative $\mathbb{R}$-algebra supplied with an increasing filtration of vector subspaces $(A_i)_{i\in\mathbb{N}}$:

$$A_0 \subset A_1 \subset \dots \subset A_i \subset \dots \subset A$$

such that $1 \in A_0$, $A = \bigcup_{i\in\mathbb{N}} A_i$ and $A_l \cdot A_k \subset A_{k+l}$. Here the notation $A_l \cdot A_k$ denotes the set of all finite sums of the form $\sum_{j=1}^{r} f_j \cdot g_j$ with $f_j \in A_l$ and $g_j \in A_k$.

We use the same notation $B \cdot Q$ for the general case that B is a subset of the algebra A and Q is a subset of a left A-module P:

$$B \cdot Q := \{\sum_{j=1}^{r} b_j \cdot q_j \mid r \in \mathbb{N},\ b_j \in B,\ q_j \in Q\}$$

Example 6.1. An example of a filtered algebra is the algebra of scalar differential operators $A = \mathrm{Diff}(C^\infty(M))$ on a smooth manifold which is filtered by the order of operators.

The **associated graded** vector space

$$\mathrm{Gr}A = \bigoplus_{i=0}^{\infty} \mathrm{Gr}_i A$$

where

$$\mathrm{Gr}_i A := A_i / A_{i-1}$$

of a filtered algebra is naturally a graded algebra with the product defined by

$$(a)_k (b)_l := (a \cdot b)_{k+l}$$

where we use round brackets with a lower index to denote the equivalence class $(a)_k \in A_k/A_{k-1} = \mathrm{Gr}_k(A)$ of an element $a \in A_k$. If $\mathrm{Gr}(A)$ is commutative then A is called **almost commutative**, which is equivalent to saying that the commutator $[a, b] = ab - ba$ of two elements $a \in A_k, b \in A_l$ lies is in A_{k+l-1}. In such a case $\mathrm{Gr}(A)$ is naturally supplied with a Poisson bracket given by

$$\{(a)_k, (b)_l\} := ([a, b])_{k+l-1}$$

The example $A = \mathrm{Diff}(C^\infty(M))$ from above is such an almost commutative algebra and it is well known that the associated graded algebra is naturally isomorphic to $S(D(M))$, the symmetric algebra over the module of vector fields on M, which may also be interpreted as the (sub)-algebra of functions on the co-tangent space T^*M which are polynomial along the fibers of $\pi : T^*M \to M$. The algebraic Poisson bracket introduced above coincides in this case with the well known Poisson bracket on $C^\infty(T^*M)$.

Returning to the general situation let P be a left module over A (in what follows when we speak of modules we will always mean left modules) and suppose that it is supplied with a **compatible increasing filtration**, i.e. a filtration of vector spaces $P_0 \subset P_1 \subset \cdots P_i \subset \cdots \subset P$, $i \in \mathbb{N}$ such that

$$A_k P_l \subset P_{l+k}$$

and $P = \bigcup_i P_i$. Then it is immediately verified that the associated graded

$$\mathrm{Gr}(P) := \bigoplus_{i \in \mathbb{N}} P_i / P_{i-1}$$

is a graded $\mathrm{Gr}(A)$-module with the obvious scalar multiplication

$$(a)_k (p)_l := (ap)_{k+l}$$

From now on we assume that A is almost commutative.

Recall that the annihilator of a graded module over a commutative algebra is

$$\mathrm{Ann}(\mathrm{Gr}P) = \{f \in \mathrm{Gr}A \mid fq = 0\, \forall q \in \mathrm{Gr}P\}$$

Lemma 6.2. *$Ann(GrP)$ is a graded ideal and co-isotropic with respect to the Poisson bracket, i.e.*

$$\{Ann(GrP), Ann(GrP)\} \subset Ann(GrP)$$

Proof. That it is graded is seen by taking $f \in \mathrm{Ann}(\mathrm{Gr}P)$ and splitting it into homogeneous components

$$f = f_{i_1} + f_{i_2} + \ldots + f_{i_s}$$

then for any homogeneous $q \in \mathrm{Gr}_k P$ the equation $fq = 0$ implies $f_{i_j} q = 0$ and hence all the homogeneous components of f are also in $\mathrm{Ann}(\mathrm{Gr}P)$.

For the closedness under the Poisson bracket consider two homogeneous $(a)_k, (b)_l \in \mathrm{Ann}(\mathrm{Gr}P)$, then for any homogeneous $(p)_s \in \mathrm{Gr}P$ one has $ap \in P_{s+k-1}$, $bp \in P_{s+l-1}$, $abp \in P_{s+k+l-2}$ and similarly $bap \in P_{s+k+l-2}$. With this we compute:

$$\begin{aligned} \{(a)_k, (b)_l\}(p)_s &= ([a,b]p)_{k+l+s-1} \\ &= (abp - bap)_{k+l+s-1} \\ &= 0 \end{aligned}$$

which implies that $\{\mathrm{Ann}(\mathrm{Gr}P), \mathrm{Ann}(\mathrm{Gr}P)\} \subset \mathrm{Ann}(\mathrm{Gr}P)$ □

The question whether the radical of $\mathrm{Ann}(\mathrm{Gr}(P))$ i.e. the characteristic ideal of $\mathrm{Gr}(P)$ is still co-isotropic has been addressed in several places. In [5] it has been shown that in the algebraic setting i.e. when the algebra $\mathrm{Gr}(A)$ is Noetherian this is generally true. In the setting where $A = \mathrm{Diff}(M)$ for a smooth manifold M there are some partial results in. Maybe one might use the Noetherian case together with our result of local generical Noetherianness (section 6.3) to deduce at least some "generical" co-isotropy in the smooth case.

Remark 6.3. The Lie algebra $\mathrm{Ann}(\mathrm{Gr}P)$ acts naturally on the the module $\mathrm{Gr}P$ by defining the action as

$$[(a)_k, (p)_s] = (ap)_{k+s-1}$$

for $(a)_k \in \mathrm{Ann}(\mathrm{Gr}P), (p)_s \in \mathrm{Gr}P$. This action satisfies

$$\begin{aligned} [f, gp] &= \{f, g\}p + g[f, p] \\ [f, [h, p]] &= [\{f, h\}, p] + [h, [f, p]] \end{aligned}$$

for $f, h \in \mathrm{Ann}(\mathrm{Gr}P)$, $g \in \mathrm{Gr}A$ and $p \in \mathrm{Gr}P$. Using this one checks that $N = \mathrm{Ann}(\mathrm{Gr}P)/\mathrm{Ann}(\mathrm{Gr}P)^2$ is naturally a Lie Reinhart algebra (or a Lie algebroid) over the algebra $\mathrm{Gr}A/\mathrm{Ann}(\mathrm{Gr}P)$ and this Lie algebroid acts on the $\mathrm{Gr}A/\mathrm{Ann}(\mathrm{Gr}P)$-module $\mathrm{Gr}P/\mathrm{Ann}(\mathrm{Gr}P) \cdot \mathrm{Gr}P$.

In the situation of interest to us the filtration on P is not invariantly associated to P, but it's equivalence class under the following equivalence relation is.

Definition 6.4. Given an A-module P, then two compatible filtrations P_k and P'_k are called **equivalent** if there exist numbers $c, c' \in \mathbb{N}$ such that

$$\begin{aligned} P_k &\subset P'_{k+c'} \\ P'_k &\subset P_{k+c} \end{aligned}$$

for all $k \in \mathbb{N}$

It is easily verified that this is indeed an equivalence relation. It is important for our purposes that the characteristic ideal of the associated graded modules is not affected by the change from one filtration to an equivalent one.

Lemma 6.5. *Let P be an A-module and P_k and P'_k two A-compatible filtrations of P. Let GrP' and GrP denote the graded modules with respect to the primed filtration and un-primed filtration respectively. If the two filtrations are equivalent then*

$$I_{char}(GrP) = I_{char}(GrP')$$

i.e. the radical ideal of the annihilator is independent of the choice of equivalent filtrations.

Proof. It suffices to show just one inclusion $\sqrt{\mathrm{Ann}(\mathrm{Gr}P)} \subset \sqrt{\mathrm{Ann}(\mathrm{Gr}P')}$, the other follows by switching the role of the filtrations. Moreover it is enough to show that $\mathrm{Ann}(\mathrm{Gr}P) \subset \sqrt{\mathrm{Ann}(\mathrm{Gr}P')}$. So let $f \in \mathrm{Ann}(\mathrm{Gr}P)$ be homogeneous of degree l. i.e. $f = (a)_l$ with $a \in A_l$, this means that we have the relations

$$\begin{aligned} aP_k &\subset P_{k+l-1} \\ a^n P_k &\subset P_{k+ln-n} \end{aligned}$$

for all $n \in \mathbb{N}$. We need to show the existence of an $n \in \mathbb{N}$ such that $a^n P'_k \subset P'_{k+ln-1}$ for all k sufficiently big. Using the equivalence of the filtrations we compute

$$a^n P'_k \subset a^n P_{k+c} \subset P_{k+c+ln-n} \subset P'_{k+c+c'+ln-n}$$

hence setting $n = c + c' + 1$ we obtain the result. ☐

Definition 6.6. A compatible filtration P_k of an A-module P is called **good** if for all $k \in \mathbb{N}$ the A_0- modules P_k are a finitely generated over A_0 and there is a $k_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ and all $k \geq k_0$

$$A_l \cdot P_k = P_{k+l} \tag{6.1}$$

The filtration of an algebra A is called good if it is good as a module over itself and $k_0 = 0$.

Lemma 6.7. *If the filtration A_i of the algebra A is good then A is generated as an A_0-algebra by finitely many elements $a_1, \dots, a_n \in A_1$.*

Proof. Let $a_1, \dots, a_n$ be generators of A_1 as A_0-module. We claim that these generate the whole of A as A_0-algebra. For $a \in A_i$ we need to show that it is a polynomial expression of $a_1, \dots, a_n$ with coefficients in A_0. If $i = 1$ this is obviously true. Suppose now $i > 1$ then by assumption $A_i = A_1 \cdot A_{i-1}$ and the result follows by induction. ☐

We will henceforth assume that the filtration of the algebra A is good.

Proposition 6.8. *For a module P over an almost commutative, good filtered algebra A the following properties hold:*

i) P possesses a good filtration if and only if it is finitely generated as A-module.

ii) Any two good filtrations of P are equivalent.

iii) A filtration P_k is good if and only if GrP is finitely generated over GrA.

Proof. i) Suppose P_i is a good filtration and let $p_1, \dots, p_m$ be generators of P_{k_0} as A_0-module then from $P_{k_0+i} = A_i \cdot P_{k_0}$ it follows that the $p_1, \dots, p_m$ generate all of P. Conversely suppose that P is finitely generated as A-module by $p_1, \dots, p_m$. Then we define a filtration by $P_i := A_i \cdot \{p_1, \dots, p_m\}$ this is obviously a compatible filtration, and since $A_j A_i = A_{i+j}$ it follows that the filtration is good.

ii) Suppose that P_i and P'_i are two good filtrations and $p_1, \dots, p_m$ generate P_{k_0} as A_0-module. Obviously there is a $c \in \mathbb{N}$ such that $\{p_1, \dots, p_m\} \subset P'_c$. But then $P_{k_0+j} \subset P'_{k_0+j+c}$ for all $j \in \mathbb{N}$. Exchanging the roles of the filtrations gives the result.

iii) Suppose the filtration is good then $\oplus_{i=0}^{k_0} \mathrm{Gr}P_i$ has finitely many generators as A_0-module and since $A_l \cdot P_k = P_{k+l}$ implies $\mathrm{Gr}_l A \cdot \mathrm{Gr}_k P = \mathrm{Gr}_{k+l} P$ these obviously generate all of $\mathrm{Gr}P$ as $\mathrm{Gr}A$-module . Conversely if $p_1, \dots, p_m \in \oplus_{i=0}^{k_0} \mathrm{Gr}P_i$ generate $\mathrm{Gr}P$ as $\mathrm{Gr}A$-module for some k_0 then from the exact sequence $0 \to P_i \to P_{i+1} \to \mathrm{Gr}_{i+1}P \to 0$ one deduces inductively that all the P_i are finitely generated over A_0. To show condition (6.1) observe that since the filtration of A is good it suffices to show that $P_{k+1} = A_1 \cdot P_k$ for all $k \geq k_0$. Obviously for $k \geq k_0$ we have that $\mathrm{Gr}_1 A \cdot \mathrm{Gr}_k P = \mathrm{Gr}_{k+1} P$ so if $p \in P_{k+1}$ then $(p)_{k+1} = \sum (a_i)_1 (p_i)_k$ for some

$a_i \in A_1$ and $p_i \in P_k$. Hence $p - \sum a_i p_i = q \in P_k$ and so $p = q + \sum a_i p_i$ showing $A_1 \cdot P_k = P_{k+1}$. □

These results allows us to define in a unique way the **characteristic ideal** $I_{\text{char}}(P) \subset \text{Gr}(A)$ of a *finitely generated* A-module P as

$$I_{\text{char}}(P) = \sqrt{\text{Ann}(\text{Gr}P)}$$

where $\text{Gr}P$ is the graded associated to *any* good filtration of P.

In the previous proof we made use of the following lemma.

Lemma 6.9. *Let* $0 \to P \xrightarrow{\iota} Q \xrightarrow{\pi} R \to 0$ *be a short exact sequence of* A*-modules. Then if* P *and* R *are finitely generated so is* Q

Proof. Let $\{p_1, \dots, p_l\}$ and $\{r_1, \dots, r_m\}$ be generators of P and R respectively. Choose pre-images $\{q_1, \dots, q_m\}$ of $\{r_1, \dots, r_m\}$ in Q, then we claim that $\{\iota(p_1), \dots, \iota(p_l), q_1, \dots, q_m\}$ generate Q. For this let $q \in Q$, then $\pi(q) = \sum a^i r_i$, $a^i \in A$ and so $\pi(q - \sum a^i q_i) = 0$ which means that $q - \sum a^i q_i = \sum b^j \iota(p_j)$ for some $b^j \in A$ showing the claim.

Another lemma we will need later on is the following □

Lemma 6.10. *Let* P *be a* A*-module and* P_k *a good filtration on it, then the following are equivalent:*

1. *P is finitely generated over A_0*
2. *The filtration P_k becomes stationary, i.e. $P_k = P_{k+1}$ for $k \gg 0$*
3. *for all $k \gg 0$ $Gr(P)_k = 0$*

Proof. 1.⇒2. Choose elements $\{p_1 \dots, p_l\}$ which generate P over A_0 then there is a k_0 such that $\{p_1 \dots, p_l\} \subset P_{k_0}$ and hence $P_{k_0} = P$.

2.⇒1. by assumption $P = P_k$ for k sufficiently big and P_k is finitely generated over A_0.

The equivalence 3.⇔2. is obvious. □

6.2. **Characteristic ideal of a Diffiety.** Having these preliminaries at hand lets sketch the plan of the remaining section: to any diffiety $\mathcal{E}$ are associated the module $\mathcal{C}\Lambda^1(\mathcal{E})$ of one forms vanishing on the Cartan distribution and the algebra of scalar $\mathcal{C}$-differential operators $\mathcal{C}\text{Diff}(\mathcal{E})$ consisting of differential operators generated by vector fields lying in the Cartan distribution. The algebra $\mathcal{C}\text{Diff}(\mathcal{E})$ is filtered (by the order of the operators) and almost commutative. Moreover $\mathcal{C}\Lambda^1$ is naturally a finitely generated left $\mathcal{C}\text{Diff}(\mathcal{E})$ module, with vector fields acting by the Lie derivative. So by the previous generalities we may define the characteristic ideal of $\mathcal{E}$ to be the characteristic ideal of $\mathcal{C}\Lambda^1\mathcal{E}$. It then remains to explain how this characteristic ideal is related to the characteristics discussed earlier. We now carry these steps out in detail.

Definition 6.11. A **scalar differential operator** of order $\leq k$ on a pro-finite manifold $\mathcal{E}$ which is represented by a co-filtered manifold $\dots \mathcal{E}^{i+1} \twoheadrightarrow \mathcal{E}^i \twoheadrightarrow \dots$ is an $\mathbb{R}$-linear map $\Delta : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ which satisfies the usual conditions of a scalar differential operator [24] plus the additional condition that there is a $d \in \mathbb{N}$ such that $\forall i \in \mathbb{N}$:

$$\Delta(C^\infty(\mathcal{E}^i)) \subseteq C^\infty(\mathcal{E}^{i+d})$$

The set of scalar differential operators on $\mathcal{E}$ is denoted with $\text{Diff}(\mathcal{E})$.

It is easy to convinces oneself that this definition is independent of the choice of co-filtered manifold representing $\mathcal{E}$.

Moreover one verifies that $\text{Diff}(\mathcal{E})$ is a filtered (by the order of the operators) almost commutative algebra. The associated graded commutative algebra is denoted with

$$\text{Symb}(\mathcal{E}) := \text{Gr}(\text{Diff}(\mathcal{E}))$$

and called the **algebra of symbols**.

Definition 6.12. Let $(\mathcal{E}, \mathcal{C}\Lambda^1)$ be a diffiety, then the algebra of **$\mathcal{C}$-differential** operators $\mathcal{C}$Diff is defined to be the $\mathcal{F}$-subalgebra of Diff$(\mathcal{E})$ generated by $\mathcal{C}D(\mathcal{E}) \oplus \mathcal{F}$. A filtration on $\mathcal{C}$Diff is given by setting $\mathcal{C}\mathrm{Diff}_r = \mathrm{Diff}_r \cap \mathcal{C}\mathrm{Diff}$.

Lemma 6.13. *As an associative (non-commutative) unital left $\mathcal{F}$-algebra, $\mathcal{C}$Diff is isomorphic to the free associative (non-commutative) unital $\mathcal{F}$-algebra generated over the $\mathcal{F}$-module $\mathcal{C}D$ modulo the relations:*

$$\begin{aligned} X \circ Y &= Y \circ X + [X, Y] \\ X \circ f &= f \circ X + X(f) \end{aligned}$$

with $X, Y \in \mathcal{C}D$ and $f \in \mathcal{F}$. Moreover the filtration $\mathcal{C}\mathit{Diff}_r$ coincides with the filtration obtained by considering elements which are sums of compositions of at most r elements in $\mathcal{C}D \oplus \mathcal{F}$.

Proof. The statement is true for the case of the algebra of linear differential operators on a finite dimensional smooth manifold M, and since in local coordinates $\mathcal{C}$Diff$(\mathcal{E})$ is the pullback [27] of Diff(M) to the infinite prolongation it also holds for $\mathcal{C}$Diff. □

Definition 6.14. The associated graded Gr($\mathcal{C}$Diff) is denoted with $\mathcal{C}$Symb and called the algebra of **$\mathcal{C}$-symbols.**

Proposition 6.15. *The algebra $\mathcal{C}$Symb is canonically isomorphic to $S(\mathcal{C}D)$, i.e. the symmetric algebra generated by the module of vector fields lying in the Cartan distribution.*

Proof. The isomorphism $\psi : S(\mathcal{C}D) \to \mathcal{C}$Symb is given by

$$\psi(X_1 \cdot X_2 \cdots X_k) = (X_1 \circ X_2 \circ \cdots \circ X_k)_k$$

where the round brackets with lower index k denote the equivalence class in $\mathcal{C}\mathrm{Diff}_k/\mathcal{C}\mathrm{Diff}_{k-1}$. This is well defined by lemma 6.13 since exchanging two adjacent elements X_i, X_{i+1} on the right hand side introduces an additional term of lower degree. It is an isomorphism because the defining relations of $S(\mathcal{C}D)$ are the same as those obtained from 6.13 by passing to the graded. □

To give the reader some geometric intuition of the algebra of $\mathcal{C}$-symbols we provide the following results.

Proposition 6.16. *The $\mathbb{R}-$spectrum of $\mathcal{C}$Symb consist of co-vectors on Cartan planes, i.e. elements $\alpha \in \mathcal{C}D^*_\theta$ where $\theta \in \mathcal{E}$ is a point on the diffiety, $\mathcal{C}D_\theta$ is the Cartan plane at θ and $\mathcal{C}D^*_\theta = Hom(\mathcal{C}D_\theta, \mathbb{R})$*

Proof. Let $\alpha : \mathcal{C}\mathrm{Symb} \to \mathbb{R}$ be an algebra morphism. i.e. a point in the $\mathbb{R}-$spectrum. Then obviously since there is a natural inclusion of algebras $\mathcal{F} \to \mathcal{C}$Symb the composition determines a point $\theta \in \mathcal{E}$ by lemma 2.15. Let $\mu_\theta \subset \mathcal{F}$ be its vanishing ideal, then α has $\mu_\theta\mathcal{C}$Symb in its kernel and hence induces a map $\alpha \in \mathrm{Hom}_{\mathcal{F}/\mathfrak{m}_\theta}(\mathcal{C}\mathrm{Symb}/\mu_\theta\mathcal{C}\mathrm{Symb}, \mathbb{R})$. But by the next lemma 6.17 $S(\mathcal{C}D)/\mu_\theta S(\mathcal{C}D) \cong S(\mathcal{C}D_\theta)$ and hence α determines an element of the $\mathbb{R}$ spectrum of the symmetric algebra on $\mathcal{C}D_\theta$ which is well known to be $\mathcal{C}D^*_\theta$. Reversing the argument shows that any co-vector $\alpha \in \mathcal{C}D^*_\theta$ determines a unique algebra morphism $\alpha : \mathcal{C}\mathrm{Symb} \to \mathbb{R}$. □

Lemma 6.17. *Let $\theta \in \mathcal{E}$ then the vector spaces $S^k(\mathcal{C}D)_\theta$ and $S^k(\mathcal{C}D_\theta)$ are canonically isomorphic*

Proof. We have morphisms

$$\phi : S^k(\mathcal{C}D)/\mu_\theta S^k(\mathcal{C}D) \to S^k(\mathcal{C}D/\mu_\theta \mathcal{C}D)$$

and

$$\psi : S^k(\mathcal{C}D/\mu_\theta \mathcal{C}D) \to S^k(\mathcal{C}D)/\mu_\theta S^k(\mathcal{C}D)$$

given by $\phi(\overline{X_1 \cdot \ldots \cdot X_k}) = \overline{X_1} \cdot \ldots \cdot \overline{X_k}$ and $\psi(\overline{X_1} \cdot \ldots \cdot \overline{X_k}) = \overline{X_1 \cdot \ldots \cdot X_k}$ where $X_i \in \mathcal{C}D$ and the over-line denotes the equivalence class in adequate quotients The first one is well defined since for $f \in \mu_\theta$ $\phi(\overline{f(X_1 \cdot \ldots \cdot X_k)}) = \overline{fX_1} \cdot \ldots \cdot \overline{X_k} = 0 \cdot X_2 \ldots \cdot \overline{X_k} = 0$ while similarly the second is well defined since $\psi(\overline{X_1} \cdot \ldots \cdot \overline{fX_j} \cdot \ldots \cdot \overline{X_k}) = \overline{f(X_1 \cdot \ldots \cdot X_k)} = 0$. Obviously they are inverse to one another. □

Lemma 6.18. *$\mathcal{C}\Lambda^1$ is naturally a left $\mathcal{C}$Diff module with functions acting by multiplication and vector fields acting by Lie derivative.*

Proof. Obviously if such an action exists it is unique by lemma 6.13 hence one only needs to see that it is well defined and satisfies the properties of an algebra representation. For this let $\omega \in \mathcal{C}\Lambda^1$, $X \in \mathcal{C}D$ and $f \in \mathcal{F}$ and let $X\omega$ denote the Lie derivative. Observe that

$$X\omega = i_X d\omega + d\underbrace{i_X\omega}_{=0} = i_X d\omega$$

form which follows that

$$(f \circ X)\omega = i_{fX} d\omega = f i_X d\omega = f(X\omega)$$

while the relation $(X \circ f)\omega = X(f\omega)$ is obvious. To extend the action to all of $\mathcal{C}$Diff we use lemma 6.13 which allow us to write any $\mathcal{C}-$differential operator ∇ as a sum of compositions of derivations and functions $\nabla = \sum X^{i_1} \circ \ldots \circ X^{i_k}$. One then sets

$$\nabla\omega = \sum X^{i_1}(X^{i_2}(\ldots(X^{i_k}\omega)\ldots)$$

That this is well defined follows from the well known fact that the action of vector fields on differential forms satisfies both relations appearing in lemma 6.13 , i.e.

$$\begin{aligned} (X \circ f)\omega &= (f \circ X)\omega + X(f)\omega \\ (X \circ Y)\omega &= (Y \circ X)\omega + [X,Y]\omega \end{aligned}$$

Obviously the so obtained action of $\mathcal{C}$Diff on $\mathcal{C}\Lambda^1$ satisfies the property $(\nabla \circ \Box)\omega = \nabla(\Box\omega)$ by construction. □

Proposition 6.19. *For a diffiety $\mathcal{E}$ the CDiff($\mathcal{E}$)-module $C\Lambda^1(\mathcal{E})$ is finitely generated.*

Proof. This is obviously true for J^∞ since there it follows from the formula

$$D_i(\omega^j_\sigma) = \omega^j_{\sigma+1_i}$$

and hence $C\Lambda^1(J^\infty)$ is generated by $\omega^j_\sigma, |\sigma| = 0, j = 1, \ldots, m$. Now if $\mathcal{E}$ is represented as the infinite prolongation of a PDE then any $\omega \in C\Lambda^1(\mathcal{E})$ is the restriction of a form $\tilde{\omega} \in C\Lambda^1(J^\infty)$ which may be written as a linear combination $\sum_{j=1,\ldots,m} \Delta_j(\omega^j_0)$ where $\Delta_j \in C$Diff. But the Δ_j are restrictable to the equation (see [27]) and since the Lie derivatives commute with the pullback we have $\omega = \sum_{j=1,\ldots,m} \Delta_j|_{\mathcal{E}}(i^*(\omega^j_0))$ where $i : \mathcal{E} \to J^\infty$ denotes the inclusion. Hence $\mathcal{C}\Lambda^1(\mathcal{E})$ is generated by $i^*(\omega^j_0)$. □

Definition 6.20. The **characteristic ideal** $I_{\text{char}}(\mathcal{E}) \subset \mathcal{C}\text{Symb}(\mathcal{E})$ of a diffiety is the characteristic ideal of the finitely generated $\mathcal{C}$Diff($\mathcal{E}$)-module $\mathcal{C}\Lambda^1(\mathcal{E})$.

6.3. **Local Noetherianness in generic points.** Before we can prove the relation of I_{char} with the point-wise characteristics introduced in previous sections we need to establish a useful property of the characteristic ideal $I_{\text{char}}(\mathcal{E})$.

Recall that a module is called Noetherian if every submodule of it is finitely generated. This condition is of central importance in algebraic geometry and commutative algebra but for the modules of interest to us which are modules over algebras of smooth functions, Noetherianness does not hold as the following simple example shows.

Example 6.21. Take $C^\infty(\mathbb{R})$ as a module over itself and consider the submodule of $C^\infty(\mathbb{R})$ generated by countably many functions $f_i \in C^\infty(\mathbb{R})$ $i \in \mathbb{N}$ which satisfy $\operatorname{supp}(f_i) \subset [\frac{1}{i+1}, \frac{1}{i}]$ and $f_i(\frac{1}{2}(\frac{1}{i+1} + \frac{1}{i})) = 1$. It is not finitely generated since any section of it has support in a finite union of the intervals $[\frac{1}{i+1}, \frac{1}{i}]$, and so a finite amount of sections will also have support in only finitely many of these intervals, hence never generating all the f_i.

The example also shows that not even locally every submodule of $C^\infty(\mathbb{R})$ is finitely generated (a condition we will call local Noetherianness) since the above submodule is not finitely generated in any neighbourhood of 0. The aim of this section is to prove that nevertheless for the modules of interest to us, local Noetherianness holds almost everywhere, i.e. around points of dense open subsets of M.

We start by fixing the terminology.

Let A be a commutative ring and P an A-module. Recall that for a multiplicative subset $S \in A$ one denotes with $S^{-1}P$ the module of fractions $\frac{p}{h}$ with $p \in P$, $h \in S$. These are equivalence classes under the relation $\frac{p}{h} \sim \frac{q}{f} :\Leftrightarrow \exists s \in S :\ s(fp - hq) = 0$. It is well known that $S^{-1}A$ is a commutative ring and $S^{-1}P$ is a module over $S^{-1}A$.

Lemma 6.22. *The functor $P \mapsto S^{-1}P$ is exact. Moreover if P is finitely generated over A then $S^{-1}P$ is finitely generated over $S^{-1}A$.*

Proof. First observe that the map

$$\begin{array}{ccc} S^{-1}A \otimes_A P & \to & S^{-1}P \\ \dfrac{f}{g} \otimes p & \mapsto & \dfrac{fp}{g} \end{array}$$

is an isomorphism and hence the functor S^{-1} is just the tensor product with $S^{-1}A$ and is therefore right exact. It only remains to show that an inclusion $Q \hookrightarrow P$ of A-modules induces an inclusion $S^{-1}Q \to S^{-1}P$, but this is obvious since if $\frac{q}{h} \in S^{-1}Q$ such that $\frac{q}{h} = 0$ as an element of $S^{-1}P$ then there exists $s \in S$ such that $sq = 0$ which implies that $\frac{q}{g} = 0$ in $S^{-1}Q$ as well.

For the finite generated-ness one may choose generators of P and check that the corresponding elements in $S^{-1}P$ generate the module. □

A consequence of this is that localization commutes with taking quotients, i.e. $S^{-1}(P/Q) = S^{-1}P/S^{-1}Q$. Another well know fact we'll need is that localization commutes with taking radicals of ideals, i.e. if $I \subset A$ is an ideal then $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ see [1].

From now on we consider the case where the ring A is in addition an $C^\infty(M)$-algebra (the cases of interest to us will be either $A = C^\infty(M)$, or A is the symmetric algebra over the module of sections of a vector bundle on M). In particular A and P are then also $C^\infty(M)$-modules. Let $U \subseteq M$ be an open subset and denote with S_U the multiplicative subset of A consisting of functions $f \in C^\infty(M)$ which are nowhere zero on U (more precisely we understand such an $f \in C^\infty(M)$ as an element of A via the natural map $C^\infty(M) \to A$). It is well know [24] that for the case that $A = C^\infty(M)$ one has

$$S_U^{-1}C^\infty(M) = C^\infty(U)$$

and if $\Gamma(\pi)$ is the $C^\infty(M)$-module of sections of a vector bundle $\pi : E \to M$ then the $S_U^{-1}C^\infty(M)$-module $S_U^{-1}\Gamma(\pi)$ is the same as the module of sections of the vector bundle π restricted to U. We will therefore use the somewhat more suggestive notation

$$P|_U := S_U^{-1}P$$

for the localization of an A-module P at S_U and call it the **restriction** of P to U. We will also denote the action of the natural map $P \to P|_U$ with $p \mapsto p|_U$ and call it the restriction to U.

Definition 6.23. An A-module P is called **finitely generated near** $\theta \in M$ if there is an open neighbourhood $U_\theta \subset M$ of θ such that the restriction $P|_{U_\theta}$ is finitely generated over $S|_{U_\theta}$. The module P is called **locally finitely generated** if it is finitely generated near each point of M. It is called **generically finitely generated** if it is finitely generated near all points of a dense open subset $U \subset M$.

Finally we say that P is **generically Noetherian** if every submodule $Q \subset P$ is generically finitely generated. The algebra A is called generically Noetherian if it is generically Noetherian as a module over itself.

Theorem 6.24. *The $C^\infty(M)$-module $\Gamma(\pi)$ of sections of a smooth vector bundle $\pi : E \to M$ of rank r is generically Noetherian.*

Proof. Given a $C^\infty(M)$-submodule $Q \subseteq \Gamma(\pi)$ define subsets $B_l \subset M$ $l = 1, \dots, r$ by

$$B_l = \{\theta \in M \mid \exists q_1, \dots q_l \in Q \text{ with } q_{1\theta}, \dots q_{l\theta}\ \mathbb{R}\text{-lin. indep.}\}$$

where $q_\theta \in E_\theta$ is the value of $q \in Q$ in θ. Set $B_0 := M$. Observe that $B_r \subset B_{r-1} \subset \dots \subset B_0$. It is also clear that the sets B_l are open since if there are l sections of Q linearly independent in θ, then these sections will also be linearly independent in all points of a small neighbourhood of θ. Next consider the disjoint open sets

$$A_l := B_l \backslash \overline{B}_{l+1}, l = 0, \dots, r$$

Obviously their union $U := \bigcup_{l=0}^r A_l$ is an open dense subset of M. We claim that Q is finitely generated near all points of U. So let $\theta \in A_l$ and choose sections $q_1, \dots, q_l \in Q$ which are linearly independent in θ. Then pick an open neighbourhood $U_\theta \subset A_l$ of θ such that $q_{1\theta'}, \dots q_{l\theta'}$ are linearly independent at all $\theta' \in U_p$. We claim that the restrictions of $q_1, \dots, q_l$ to U_θ generate $Q|_{U_\theta}$. Observe first that the module generated by $q_1|_{U_\theta}, \dots, q_l|_{U_\theta}$ is the module of sections of a smooth subvector bundle $V \subset E|_{U_\theta}$ of $\pi|_{U_\theta}$ of rank l. Suppose now there is a $\frac{q}{h} \in Q|_{U_\theta}$ which is not a $C^\infty(U_\theta)$-linear combination of $q_1|_{U_\theta}, \dots, q_l|_{U_\theta}$, then this section cannot lie in the sub-bundle V and hence there must be at least one point $\theta' \in U_\theta$ where $q_{\theta'} \notin V_{\theta'}$ but then the vectors $q_{\theta'}, q_{1\theta'}, \dots q_{l\theta'}$ are linearly independent, contradicting the fact that $\theta' \in B_l \backslash \overline{B}_{l+1}$. ☐

Remark 6.25. We have actually shown more, namely that any submodule of a module of sections is locally generically a subbundle of the vector bundle.

The aim of the next section is to prove the following theorem.

Theorem 6.26. *Let M be a smooth manifold and let R be finitely generated projective $C^\infty(M)$-module (i.e the module of sections of smooth vector bundles). Let $S := SR = \bigoplus_{i\in\mathbb{N}} S^iR$ be the symmetric algebra on R (here the symmetric tensor product is understood over $C^\infty(M)$), and suppose that $g = \bigoplus g_i$ is a finitely generated graded S-module. Then g is generically Noetherian over S.*

6.3.1. *General properties of generical Noetherianness.*

Lemma 6.27. *Suppose*

$$0 \to Q \to P \to P/Q \to 0$$

is a short exact sequence of S modules. Then:
i) If P is generically finitely generated so is P/Q
ii) If Q and P/Q are generically finitely generated so is P.

Proof. i) If P is locally finitely generated near all points in a dense open set $U \subset M$ then we claim that also P/Q is finitely generated near all points of U. So let $\theta \in U$ and U_θ be a open neighbourhood where $P|_{U_\theta}$ is finitely generated then the result follows from the fact that $\pi|_{U_\theta} : P|_{U_\theta} \to (P/Q)|_{U_\theta}$ is surjective by exactness of the localization.

ii) Suppose Q and P/Q are locally finitely generated in open dense sets $U, V \subset M$ respectively. Obviously $W := U \cap V$ is also open and dense and we claim that P is locally finitely generated in W. Choose for $\theta \in W$ neighbourhoods $U_\theta$and V_θ where $Q|_{U_\theta}$ and $(P/Q)|_{V_\theta}$ are finitely generated. Then restricted to $W_\theta := U_\theta \cap V_\theta$ both Q and P/Q will still be finitely generated and the result follows from the exactness of the localization. □

Corollary 6.28. *If*

$$0 \to Q \to P \to P/Q \to 0$$

is a short exact sequence of S-modules, then
i) If P is generically Noetherian so are Q and P/Q.
ii) If Q and P/Q are generically Noetherian then so is P.

Proof. i) Obviously every submodule Q of a generically Noetherian module inherits the property. Suppose now $R \subset P/Q$ is a submodule then its inverse image $\tilde{R}$ under the projection $P \to P/Q$ is locally finitely generated for all points in a dense open $U \subset M$ and by i) of lemma 6.27 so is R.

ii) Suppose Q and P/Q are generically Noetherian and let $\tilde{R} \subset P$ be a submodule. The image of $\tilde{R}$ under $P \to P/Q$, denoted with R, as well as the module $Q \cap \tilde{R} \subset Q$ are generically finitely generated and obviously we have the exact sequence $0 \to Q \cap \tilde{R} \to \tilde{R} \to R \to 0$. Hence by ii) of lemma 6.27 also $\tilde{R}$ is generically finitely generated. □

Corollary 6.29. *Finite sums of generically Noetherian modules are generically Noetherian.*

Proof. Direct application of lemma 6.28 part ii) □

Corollary 6.30. *If the algebra S is generically Noetherian and P is finitely generated module over S then P is generically Noetherian.*

Proof. Under the assumption P will be the quotient of a finite sum of modules isomorphic to S and hence by the previous corollary and corollary 6.28 part i) it will be generically Noetherian. □

Proposition 6.31. *Suppose that P is a generically Noetherian module then any ascending chain of submodules $Q_0 \subset Q_1 \subset \ldots \subset P$ becomes locally generically stationary, i.e. there is an open dense subset $U \subset M$ such that for each point $\theta \in U$ there is a closed neighbourhood U_θ such that the restricted sequence sequence* $Q_0|_{U_\theta} \subset Q_1|_{U_\theta} \subset \ldots$ becomes stationary.

Proof. Suppose P is generically Noetherian and Q_i is an ascending chain of submodules. Let $Q := \bigcup Q_i$ and choose $U \subseteq M$ open and dense in which Q is locally finitely generated. Then for any $\theta \in U$ choose a closed neighbourhood U_p in which

$Q|_{U_p}$ is finitely generated by generators $q_1 \dots q_l$. Then there must be a $k \in \mathbb{N}$ such that $q_1 \dots q_l \in Q_k|_{U_k}$ and hence the restricted sequence is stationary. □

The next theorem is the analog of Hilbert's theorem in our setting

Theorem 6.32. *Suppose that S is a generically Noetherian $C^\infty(M)$ algebra, then the polynomial algebra $S[X]$ is also generically Noetherian.*

Proof. The proof is an adaptation of the proof of Hilbert's theorem as found for example in [16]. We will provide the parts that vary from that proof and refer to [16] for the remaining ones.

Let $\mathfrak{A} \subset S[X]$ be an ideal and define the sets $\mathfrak{a}_i = \mathfrak{a}_i(\mathfrak{A}) \subset S$ consisting of all elements $a \in S$ which appear as leading coefficient of a degree i polynomial of $\mathfrak{A}$. Then the $\mathfrak{a}_i$ are all ideals in S and form an ascending chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$. By the previous proposition the sequence becomes locally stationary near all points of a dense open subset $U \subset M$.

Choose an open covering $(U_\iota)_{\iota \in I}$ of U such that when restricted to each U_ι the sequence $\mathfrak{a}_0|_{U_\iota} \subset \mathfrak{a}_1|_{U_\iota} \subset \dots$ becomes stationary say after $\mathfrak{a}_{r_\iota}$. The $\mathfrak{a}_i|_{U_\iota}$ $0 \le i \le r_\iota$ need not be finitely generated but we may choose in each U_ι an open dense subset $V_\iota \subset U_\iota$ such that all $\mathfrak{a}_i|_{U_\iota}$ $0 \le i \le r_\iota$ are locally finitely generated near all points $\theta \in V_\iota$ (one may first do this for each single $\mathfrak{a}_i|_{U_\iota}$ $0 \le i \le r_\iota$ since S is generically Noetherian and then take the intersection of these (finitely many) dense open subsets). Then obviously the set $V := \bigcup_{\iota \in I} V_\iota$ is open and dense in M and we will show that $\mathfrak{A}$ is finitely generated near all points of V.

By construction for each $\theta \in V$ there is a neighbourhood V_θ such that the restricted sequence $\mathfrak{a}_0|_{V_\theta} \subset \mathfrak{a}_1|_{V_\theta} \subset \dots$ becomes stationary say after $\mathfrak{a}_r$ and all $\mathfrak{a}_i|_{V_\theta}$ $0 \le i \le r$ are finitely generated.

Now observe that for any open $V \subset M$ we have $S|_V[X] \cong S[X]|_V$ and moreover the construction of the ideals $\mathfrak{a}_i(\mathfrak{A})$ commutes with taking restrictions, i.e. $(\mathfrak{a}_i(\mathfrak{A}))|_V = \mathfrak{a}_i(\mathfrak{A}|_V)$.

Hence we have the situation that $\mathfrak{A}|_{V_\theta} \subset S|_{V_\theta}[X]$ is an ideal for which the chain of ideals $\mathfrak{a}_i(\mathfrak{A}|_{V_\theta})$ becomes stationary and each of these ideals is finitely generated. Now we may carry on the standard proof as in [16] to conclude that $\mathfrak{A}|_{V_\theta}$ is finitely generated. □

Corollary 6.33. *If S is generically Noetherian then the polynomial algebra in several variables $S[X_1, \dots, X_n]$ is generically Noetherian.*

Proof. Apply theorem 6.32 repeatedly. □

This also implies

Theorem 6.34. *Let R be a finitely generated projective module over $C^\infty(M)$ then the symmetric algebra $S = S(R)$ is generically Noetherian.*

Proof. Locally S is of the form $C^\infty(M)[X_1, \dots, X_n]$ and since we have shown in theorem 6.24 that $C^\infty(M)$ is generically Noetherian the result follows from the previous corollary. □

Finally we may give a proof of theorem 6.26

Proof. (of theorem 6.26) Since S is generically Noetherian by the previous theorem and by assumption g is finitely generated over S the result follows from corollary 6.30. □

6.4. **Relation of the characteristic ideal $\mathrm{I}_{\mathrm{char}}(\mathcal{E})$ with point-wise characteristics.** Suppose now that our diffiety $\mathcal{E}$ is realized as an infinitely prolonged PDE $(\mathcal{E}^j) \subset (J^j)$. Then let us introduce a filtration on $\mathcal{C}\Lambda^1(\mathcal{E})$ by setting

$$\mathcal{C}\Lambda^1_k(\mathcal{E}) := \Lambda^1_k(\mathcal{E}) \cap \mathcal{C}\Lambda^1(\mathcal{E})$$

which are obviously $\mathcal{F}$-submodules (for the definition of Λ^1_k see section 2.2.1). We will call this the **canonical filtration** associated to representation of the diffiety $\mathcal{E}$ as a PDE.

Lemma 6.35. *For $X \in \mathcal{C}D$ we have*

$$X(\mathcal{C}\Lambda^1_k) \subseteq \mathcal{C}\Lambda^1_{k+1}$$

Proof. A direct consequence of the fact that fields in $\mathcal{C}D(\mathcal{E})$ are of shift at most one, (proposition 2.37) and equation 2.5. □

Corollary 6.36. *The filtration $\{\mathcal{C}\Lambda^1_k\}_{k\in\mathbb{N}}$ of $\mathcal{C}\Lambda^1$ is compatible with the action of $\mathcal{C}Diff$, i.e.*

$$\mathcal{C}Diff^r \cdot \mathcal{C}\Lambda^1_k \subseteq \mathcal{C}\Lambda^1_{r+k}$$

Proof. By lemma 6.13 any element $\nabla \in \mathcal{C}\mathrm{Diff}^r$ is a sum of compositions of at most r elements in $\mathcal{C}D \oplus \mathcal{F}$ which all act by shift at most 1, hence ∇ acts by shift at most r. □

Lemma 6.37. *The filtration $\{\mathcal{C}\Lambda^1_k(\mathcal{E})\}_{k\in\mathbb{N}}$ of $\mathcal{C}\Lambda^1(\mathcal{E})$ is good.*

Proof. Same as in proposition 6.19. □

Hence the characteristic ideal associated to the canonical filtration coincides with the characteristic ideal of the diffiety.

Recall now that we introduced the symbols g^k of a PDE as the kernels of the projection $T\mathcal{E}^k \to T\mathcal{E}^{k-1}$ and showed that (when pulled back to $\mathcal{E}^\infty$) the sum of duals $g^* = \bigoplus_{k\in\mathbb{N}} \hom_{C^\infty(\mathcal{E}^\infty)}(g^k, C^\infty(\mathcal{E}^\infty))$ is naturally a module over $S(\mathcal{C}D(\mathcal{E}))$.

The next result shows that the graded module associated to the canonical filtration is the same as the symbolic module g^*.

Proposition 6.38. *Let $\mathcal{E}^\infty \subset J^\infty$ be an infinitely prolonged equation and $Gr(\mathcal{C}\Lambda^1(\mathcal{E}))$ the graded module of the canonical filtration. There is a canonical isomorphism of $S(\mathcal{C}D)$ modules*

$$Gr(\mathcal{C}\Lambda^1(\mathcal{E})) \cong g^*$$

Proof. Observe that the $\mathcal{F}$-dual of the exact sequence

$$0 \to \mathcal{C}\Lambda^1_k \to \Lambda^1_k \to \Lambda^1_k/\mathcal{C}\Lambda^1_k \to 0$$

is

$$0 \leftarrow \mathcal{F} \otimes_{\mathcal{F}^{k+1}} \left(\frac{\mathcal{F}^{k+1} \otimes T\mathcal{E}^k}{R^{k+1}}\right) \leftarrow \mathcal{F} \otimes_{\mathcal{F}^k} T\mathcal{E}^k \leftarrow \mathcal{F} \otimes_{\mathcal{F}^{k+1}} R^{k+1} \leftarrow 0$$

where R^{k+1} is the R-distribution on $\mathcal{E}^{k+1}$.

Now recall that R-distributions R^{k+1} are always transversal to the projections $T\mathcal{E}^k \to T\mathcal{E}^{k-1}$ and project isomorphically onto each other. Hence we can write the following commutative diagram of $\mathcal{F}$-modules with exact columns and rows (everything should be read as pulled back to $\mathcal{E}^\infty$)

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & g_k(\mathcal{E}) & \to & g_k(\mathcal{E}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R^{k+1}(\mathcal{E}) & \to & T\mathcal{E}^k & \to & T\mathcal{E}^k/R^{k+1}(\mathcal{E}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R^k(\mathcal{E}) & \to & T\mathcal{E}^{k-1} & \to & T\mathcal{E}^{k-1}/R^k(\mathcal{E}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

dualizing the last column we obtain

$$\frac{\mathcal{C}\Lambda^1_k}{\mathcal{C}\Lambda^1_{k-1}} \cong g_k^*$$

This establishes the canonical isomorphism of g^* and $\mathrm{Gr}(\mathcal{C}\Lambda^1)$ as $\mathcal{F}$ modules and it remains to show that this isomorphism is compatible with the $S(\mathcal{C}D)$-module structures. But it suffices to show this for the case of the "empty" equation J^∞ since the module structures on $\mathrm{Gr}(C\Lambda^1(\mathcal{E}))$ and $g^*(\mathcal{E})$ are just the ones induced from the natural projections $\mathrm{Gr}(C\Lambda^1(J^\infty)) \to \mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{E}))$ and $g^*(J^\infty) \to g^*(\mathcal{E})$ respectively.

For J^∞ it follows from a direct check in local coordinates: since the relation $D_i(\omega^j_\sigma) = \omega^j_{\sigma+1_i}$ holds, and moreover the equivalence classes $(\omega^j_\sigma)_k \in \mathcal{C}\Lambda^1_k/\mathcal{C}\Lambda^1_{k-1}$ correspond to $(\frac{\partial}{\partial x})^\sigma \otimes e^{j*} \in S^k(TM) \otimes E^*$ under the isomorphism 3.6 we obtain the result. □

Recall now that we actually defined the characteristic variety $\mathrm{char}_\theta(\mathcal{E})$ in a point $\theta \in \mathcal{E}$ as the zero set of the characteristic ideal of the *restricted* symbolic module g^*_θ. The following sections explains how this is related with the global characteristic ideal.

6.4.1. *Point-wise and global characteristics.* For this subsection let $g = \bigoplus g_i$ be a graded module over $SR = \bigoplus S^k R$ generated in degree 0, where R, and $g_i, i \in \mathbb{N}$ are all finitely generated projective $C^\infty(M)$-modules and the symmetric product is taken over $C^\infty(M)$. Then we might consider the "global" characteristic ideal $I_{\mathrm{char}}(g) \subset SR$ (which in our setting is an invariant of the diffiety) or we might consider the family of SR_θ-modules g_θ where θ ranges over the points of M and their characteristic ideals. The aim of this section is to prove that from the global characteristic ideal one may recover the point-wise characteristic ideal in generic points.

We start by observing that for any $\theta \in M$ there is a map

$$\begin{array}{rcl} \mathrm{ev}_\theta : I_{\mathrm{char}}(g) & \to & SR_\theta \\ f & \mapsto & f_\theta \end{array}$$

where f_θ is the evaluation of f at θ. Algebraically this means that ev_θ is the composition of the inclusion $I_{\mathrm{char}}(g) \subset SR$ with the restriction $SR \to SR/\mu_\theta SR \cong SR_\theta$ to θ. Obviously the image of ev_θ is an ideal in SR_θ and it is also clear that $\mathrm{im}(\mathrm{ev}_\theta) \subset I_{\mathrm{char}}(g_\theta)$.

Theorem 6.39. *There is a dense open subset $U \subset M$ such that for all $\theta \in U$ we have*

$$\sqrt{im(ev_\theta)} = I_{char}(g_\theta)$$

This states that for generic points of M we recover the point-wise characteristics from the global characteristic variety.

To prove it we need two preliminary results, the first of which is

Lemma 6.40. *Let $\phi : P \to Q$ be a morphism of finitely generated projective $C^\infty(M)$-modules then there is a dense open subset $U \subset M$ such that for all $\theta \in M$ we have*

$$(\ker \phi)_\theta = \ker(\phi_\theta)$$

i.e. almost everywhere the restriction to a point commutes with taking kernels. (More precisely we should have written $im(ev_\theta) = \ker(\phi_\theta)$ where $ev_\theta : \ker(\phi) \to P_\theta$ is the evaluation).

Proof. We may use the remark at the end of theorem 6.24 which implies that $(\ker \phi)|_{U_\theta}$ is the module of sections of a sub-vector bundle of P for generic θ in small a open neighbourhood U_θ of θ. Obviously in that neighbourhood ϕ is of constant rank and for morphism of constant rank one has $(\ker \phi)_\theta = \ker(\phi_\theta)$. □

We also need a result from commutative algebra. To state it let us introduce a notation: let P be a graded module over a symmetric algebra SR with R an n-dimensional vector space. We say that P is **generated in degree** $\leq c$ and write

$$\mathrm{gen}(P) \leq c$$

if $P_0 \oplus P_1 \oplus \ldots \oplus P_c$ generates P as an SR module.

Lemma 6.41. *Let R be an n-dimensional $\mathbb{R}$-vector space and $H : \mathbb{Z} \to \mathbb{Z}$ be a polynomial-like function and consider all graded SR-modules $g = \bigoplus_{i=0}^{\infty} g_i$ with Hilbert function H and generated in degree 0. Then there exists a constant $c \in \mathbb{N}$ only depending on H and n such that for any such module g one has*

$$gen(Ann(g)) \leq c$$

i.e., c bounds from above the maximal degree of generators of the annihilators of such g's.

Proof. The proof has been obtained in collaboration with Prof. Markus Brodmann and will appear elsewhere. □

Now we are able to prove the theorem 6.39

Proof. We will show that for generic points $\mathrm{Ann}(g)_\theta = \mathrm{Ann}(g_\theta)$ which implies the result.

Observe that if a graded SR module g is generated in degree 0 then the i-th homogeneous component of the annihilator $\mathrm{Ann}(g)_i$ may be described as $\ker_{S^0R} m_i$ where $m_i : S^iR \to \hom_{S^0R}(g_0, g_i)$ is the map which associates to $f \in S^iR$ the multiplication with f.

Let us choose by theorem 6.26 an open dense $U \subset M$ such that for every $\theta \in U$ there is a neighbourhood U_θ in which $\mathrm{Ann}(g)$ is finitely generated by generators of degree less than $r(\theta) \in \mathbb{N}$. For any $\theta' \in U_\theta$ we have $\mathrm{ev}_\theta(p|_{U_\theta}) = \mathrm{ev}_\theta(p)$ with $p \in \mathrm{Ann}(g)$ and so we conclude that $\mathrm{im}(\mathrm{ev}_{\theta'})$ is also generated by elements of degree $\leq r(\theta)$ for all $\theta' \in U_\theta$. Since all the $g_{\theta'}$ have the same Hilbert function we also conclude by lemma 6.41 that $\mathrm{Ann}(g_{\theta'})$ is generated in degrees lower than $c(H_{g_\theta}, n)$.

Let now $j = \max\{c(H_{g_\theta}, n), r(\theta)\}$ then by lemma 6.40 and by the fact that a finite number of intersections of open dense subset is still open and dense, we find an open dense $V_\theta \subset U_\theta$ such that for all $\theta' \in V_\theta$ and all $i = 0, \ldots, j$ we have $(\ker_{C^\infty(M)} m_i)_{\theta'} = \ker_{\mathbb{R}}(m_{i\theta'})$. Since both $\mathrm{Ann}(g)_\theta$ and $\mathrm{Ann}(g_\theta)$ are generated in degrees less then j for all $\theta' \in V_\theta$ we conclude $\mathrm{Ann}(g)_\theta = \mathrm{Ann}(g_\theta)$. □

6.5. **Behaviour of the characteristic ideal under morphisms.** The behaviour of the characteristic ideal under morphisms of diffieties is most naturally tested for morphisms which induce isomorphism on the Cartan planes.

Before introducing the notion we first recall a well known fact from algebra about change of rings [10].

Proposition 6.42. *(Change of rings) Let $\chi : A \to B$ be a morphism of commutative algebras, P an A-module and Q a B-module, then there is a canonical isomorphism*

$$Hom_A(P,Q) \cong Hom_B(B \otimes_A P, Q)$$

natural in P, Q. Moreover if P is either finitely generated or projective over A then so is $B \otimes_A P$ over B.

A fact about geometric modules we recall is

Lemma 6.43. *Submodules of geometric modules are geometric and if the algebra A is geometric then free modules are geometric, in particular any projective module over an geometric A algebra is geometric.*

Now we make the following definition.

Definition 6.44. A smooth map $\varphi : \mathcal{E} \to \mathcal{G}$ between two diffieties is called a **morphism of diffieties** if

$$\varphi * (\mathcal{C}\Lambda^1(\mathcal{G})) \subset \mathcal{C}\Lambda^1(\mathcal{E})$$

and if the induced homomorphism of $C^\infty\mathcal{E}$-modules

$$\begin{array}{rcl} \overline{\varphi}^* : C^\infty(\mathcal{E}) \otimes_{C^\infty(\mathcal{G})} \overline{\Lambda}^1(\mathcal{G}) & \to & \overline{\Lambda}^1(\mathcal{E}) \\ f \otimes \overline{\omega} & \mapsto & f\overline{\varphi^*(\omega)} \end{array}$$

is an isomorphism. Here $\overline{\Lambda}^1 = \Lambda^1/\mathcal{C}\Lambda^1$.

Obviously for a morphism to exist between to diffieties they must be of the same Cartan-dimension n. In what follows we denote the function algebras of our two model diffieties with $\mathcal{F} = C^\infty(\mathcal{E})$ and $\mathcal{H} = C^\infty(\mathcal{G})$.

Lemma 6.45. *A morphism of diffieties $\varphi : \mathcal{E} \to \mathcal{G}$ induces a well defined pullback of Cartan fields*

$$\begin{array}{rcl} \varphi^* : \mathcal{C}D(\mathcal{G}) & \to & \mathcal{C}D(\mathcal{E}) \\ X & \mapsto & \phi^*(X) \end{array}$$

which is a morphism of $\mathcal{H}$-modules determined uniquely by the property

$$T_\theta\varphi(\varphi^*(X)_\theta) = X_{\varphi(\theta)}, \forall \theta \in \mathcal{E} \tag{6.2}$$

This pullback satisfies:

$$\varphi^*(X)(\varphi^*(f)) = \varphi^*(X(f)) \tag{6.3}$$

$$\varphi^*([X,Y]) = [\varphi^*(X), \varphi^*(Y)] \tag{6.4}$$

and the induced map of $\mathcal{F}$-modules $\mathcal{F} \otimes_{\mathcal{H}} \mathcal{C}D(\mathcal{G}) \to \mathcal{C}D(\mathcal{E})$ is an isomorphism

Proof. If the pullback exist it is obviously uniquely determined by condition 6.2 since the module $\mathcal{C}D(\mathcal{E})$ is geometric. To show existence we define it as the composition of the following chain of natural morphism $\mathcal{C}D(\mathcal{G}) = \mathrm{Hom}_{\mathcal{H}}(\overline{\Lambda}^1(\mathcal{G}), \mathcal{H}) \overset{\varphi^*\circ}{\to} \mathrm{Hom}_{\mathcal{H}}(\overline{\Lambda}^1(\mathcal{G}), \mathcal{F}) = \mathrm{Hom}_{\mathcal{F}}(\mathcal{F} \otimes_{\mathcal{H}} \overline{\Lambda}^1(\mathcal{G}), \mathcal{F}) = \mathrm{Hom}_{\mathcal{F}}(\overline{\Lambda}^1(\mathcal{E}), \mathcal{F}) = \mathcal{C}D(\mathcal{E})$It follows immediately that it is a morphism of $\mathcal{H}$-modules and the remaining properties are easily verified. □

Proposition 6.46. *A morphism of diffieties $\varphi : \mathcal{E} \to \mathcal{G}$ induces a well defined morphism of filtered algebras*

$$\varphi^* : \mathcal{C}\mathit{Diff}(\mathcal{G}) \to \mathcal{C}\mathit{Diff}(\mathcal{E})$$

by

$$\varphi^*(\sum_{|\sigma|\leq k} f_\sigma D^\sigma) := \sum \varphi^*(f_\sigma)\varphi^*(D_1)^{\sigma_1} \circ \cdots \circ \varphi^*(D_n)^{\sigma_n}$$

where $f_\sigma \in C^\infty(\mathcal{G})$ and $D^\sigma = D_1^{\sigma_1} \circ \cdots \circ D_n^{\sigma_n}$ with $\sigma = (\sigma_1, \ldots, \sigma_n)$ a multi-index and $D_1, \ldots, D_n \in \mathcal{C}D$ a basis of $\mathcal{C}D$. Moreover the map $\varphi^* : \mathcal{C}\Lambda^1(\mathcal{G}) \to \mathcal{C}\Lambda^1(\mathcal{H})$ is compatible with the $\mathcal{C}$Diff-module structures

Proof. By lemma 6.13 $\mathcal{C}$Diff$(\mathcal{G})$ is freely generated over $\mathcal{C}D(\mathcal{G})$ modulo certain relations which are obviously preserved by the pullback of vector fields according to 6.45, hence the map is well defined. It is obviously an algebra morphism and preserves the filtration. The compatibility with the map on the Cartan forms follows also from 6.45. □

from these facts we obtain

Corollary 6.47. *The image of the map $\varphi^* : \mathcal{F} \otimes_{\mathcal{H}} \mathcal{C}\Lambda^1(\mathcal{G}) \to \mathcal{C}\Lambda^1(\mathcal{E})$ is a finitely generated $\mathcal{C}\mathit{Diff}(\mathcal{E})$-submodule.*

and

Corollary 6.48. *A morphism of diffieties $\varphi : \mathcal{E} \to \mathcal{G}$ induces a morphism of graded algebras $\mathcal{C}\mathit{Symb}(\varphi^*) : \mathcal{C}\mathit{Symb}(\mathcal{G}) \to \mathcal{C}\mathit{Symb}(\mathcal{E})$.*

Hence we know that $\mathcal{C}$Diff(), $\mathcal{C}$Symb() and $\mathcal{C}\Lambda^1()$ are functorial on the category of diffieties and we can test what happens with the characteristic ideals under $\mathcal{C}$Symb(φ^*). Lets denote with

$$I_{\text{char}}(\mathcal{G})^{\varphi^*} \subset \mathcal{C}\text{Symb}(\mathcal{E})$$

the radical ideal generated by $\mathcal{C}$Symb$(\varphi^*)(I_{\text{char}}(\mathcal{G}))$.

Expecting functoriality of characteristics one might hope for one of the following inclusions to hold:

$$I_{\text{char}}(\mathcal{G})^{\varphi^*} \subseteq I_{\text{char}}(\mathcal{E}) \tag{6.5}$$

or

$$I_{\text{char}}(\mathcal{G})^{\varphi^*} \supseteq I_{\text{char}}(\mathcal{E}) \tag{6.6}$$

But neither of these is true in general as the following examples show.

Example 6.49. Consider a trivial bundle $\pi : M \times \mathbb{R}^m \to M$ and let $\pi_{\infty,-1} : J^\infty(\pi) \to M$ be the canonical projection. We claim that $\pi_{\infty,-1}$ may be understood as a morphism of diffieties if we supply M with the Cartan distribution $\mathcal{C}D(M) = D(M)$ and consider J^∞ as the trivial (or empty) equation. To convince the reader that M is indeed the infinite prolongation of a PDE let $\iota : M \to M \times \mathbb{R}^m$ be the zero section, then the image of ι can be considered as a 0-th order PDE which we just denote by $M = \{u^i = 0\}$. Obviously all prolongations of this PDE are isomorphic to M hence $M = M^{(\infty)}$. Moreover the map $\iota \circ \pi$ can be infinitely prolonged giving back $\pi_{\infty,-1} : J^\infty(\pi) \to M$. Now $I_{\text{char}}(J^\infty) = 0$ while $I_{\text{char}}(M) = S(D(M))$. It is moreover easy to see that $\mathcal{C}$Symb$(\pi^*_{\infty,-1})(I_{\text{char}}(M)) \neq 0$ and hence inclusion 6.5 is false.

To disprove the second inequality one may consider the inclusion $M \to J^\infty$ given by the restriction of the identity id $: J^\infty \to J^\infty$ to $M^{(\infty)} \subseteq J^\infty$.

Nevertheless we will show that locally in generic points inclusion 6.6 holds for a certain type of morphism which include coverings (which play an important role in the theory of PDEs [11, 2, 9])

Definition 6.50. We say that a morphism of diffieties $\varphi : \mathcal{E} \to \mathcal{G}$ is a **regular submersion** if $\varphi^* : \mathcal{F} \otimes_{\mathcal{H}} \Lambda^1(\mathcal{G}) \to \Lambda^1(\mathcal{E})$ is injective and $\mathcal{F}$ is a flat $\mathcal{H}$-module. We say that the regular submersion is **finite** if the quotient $\Lambda^1(\mathcal{E})/\varphi^*(\mathcal{F} \otimes \Lambda^1(\mathcal{G}))$ is finitely generated as $\mathcal{F}$-module.

One may verify that a finite covering $\varphi : \mathcal{E} \to \mathcal{G}$ in the sense of [9], is a finite regular submersion.

The main result of this section is

Theorem 6.51. *Let $\varphi : \mathcal{E} \to \mathcal{G}$ be a regular submersion of diffieties, then there is an open and dense $U \subset \mathcal{E}$ such that for all $\theta \in U$ there is an open neighbourhood U_θ of θ on which*

$$I_{char}(\mathcal{G})^{\varphi^*}|_{U_\theta} \supseteq I_{char}(\mathcal{E})|_{U_\theta}$$

moreover if the regular submersion is finite we have

$$I_{char}(\mathcal{G})^{\varphi^*}|_{U_\theta} = I_{char}(\mathcal{E})|_{U_\theta}$$

Geometrically this implies that for a covering $\varphi : \mathcal{E} \to \mathcal{G}$ in generic points $\theta \in \mathcal{E}$ the map $T_\theta\varphi^* : \mathcal{C}^*_{\varphi(\theta)}\mathcal{G} \to \mathcal{C}^*_\theta\mathcal{E}$ induces an inclusion of characteristics $\mathrm{char}_{\varphi(\theta)}\mathcal{G} \hookrightarrow \mathrm{char}_\theta\mathcal{E}$ and if the covering is finite this is an isomorphism. Hence characteristics may grow under coverings but remain the same for finite coverings.

Corollary 6.52. *Consider two second order PDEs in two independent variables, then a finite covering may exist between them only if they are of the same type (elliptic, parabolic, hyperbolic).*

a more interesting consequence is

Corollary 6.53. *From remark 3.32 we know that for generic determined equations the order of the equations coincides with the degree of $char_\theta(\mathcal{E})$, hence if two such equations have different degrees then there cannot be a finite covering between them.*

To prove the above theorem we start by an observation.

Lemma 6.54. *The characteristic ideal of the finitely generated $\mathcal{C}Diff(\mathcal{E})$-module $\mathcal{F} \otimes_{\mathcal{H}} \mathcal{C}\Lambda^1(\mathcal{G})$ (as a submodule of $\mathcal{C}\Lambda^1(\mathcal{E})$) coincides with $I_{char}(\mathcal{G})^{\varphi^*}$*

Proof. Observe that a good filtration on $\mathcal{C}\Lambda^1(\mathcal{G})$ induces a good filtration on $\mathcal{F} \otimes_{\mathcal{H}} \mathcal{C}\Lambda^1(\mathcal{G})$, and because of flatness of $\mathcal{F}$ we obtain $\mathrm{Gr}(\mathcal{F} \otimes \mathcal{C}\Lambda^1(\mathcal{G})) = \mathcal{F} \otimes \mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{G}))$. Tensorising the exact sequence

$$0 \to \mathrm{Ann}(\mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{G})) \to \mathcal{C}\mathrm{Symb}(\mathcal{G}) \to \mathrm{End}_{\mathcal{C}\mathrm{Symb}(\mathcal{G})}(\mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{G}))$$

with $\mathcal{F}$ we obtain

$$0 \to \mathcal{F} \otimes \mathrm{Ann}(\mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{G})) \to \mathcal{C}\mathrm{Symb}(\mathcal{E}) \to \mathrm{End}_{\mathcal{C}\mathrm{Symb}(\mathcal{E})}(\mathrm{Gr}(\mathcal{F} \otimes \mathcal{C}\Lambda^1(\mathcal{G})))$$

by flatness of $\mathcal{F}$ we obtain $\mathcal{F} \otimes \mathrm{Ann}(\mathrm{Gr}(\mathcal{C}\Lambda^1(\mathcal{G})) = \mathrm{Ann}(\mathrm{Gr}(\mathcal{F} \otimes \mathcal{C}\Lambda^1(\mathcal{G})))$ which implies the result. □

Now theorem 6.51 is a direct application of the next result

Theorem 6.55. *Let $\mathcal{E}$ be the infinite prolongation of a PDE and $Q \subset P$ both finitely generated $CDiff(\mathcal{E})$ modules. Then there is an open dense subset $U \subset \mathcal{E}$ such that for any $\theta \in U$ there exist a neighbourhood U_θ of θ such that*

$$I_{char}(P)|_{U_\theta} \subseteq I_{char}(Q)|_{U_\theta}$$

If moreover P/Q is finitely generated over $\mathcal{F}$ then the above inclusion becomes an equality.

To prove this we also need some remarks.

If P is a $\mathcal{C}$Diff-module and $(P_k)_{k\in\mathbb{N}}$ a compatible (not necessarily good) filtration and if $Q \subset P$ is a $\mathcal{C}$Diff-submodule, then we define a filtration on Q by setting

$$Q_k := Q \cap P_k$$

Similarly we may define a filtration on P/Q by observing that the canonical maps $P_k/Q_k \to P_{k+1}/Q_{k+1} \to P/Q$ are injective for all k and so define a filtration on the $\mathcal{C}$Diff-module P/Q by setting

$$(P/Q)_k := P_k/Q_k$$

One easily verifies that the filtrations Q_k and $(P/Q)_k$ are compatible with the $\mathcal{C}$Diff -module structures on Q and P/Q.

Lemma 6.56. *The induced sequence* $0 \to Gr(Q) \to Gr(P) \to Gr(P/Q) \to 0$ *of* $\mathcal{C}$*Symb-modules is exact.*

Proof. Using $Q_{i-1} = Q_i \cap P_{i-1}$ one easily checks by diagram chasing that all rows and colums in the following diagram are exact

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \frac{Q_i}{Q_{i-1}} & \to & \frac{P_i}{P_{i-1}} & \to & \frac{P_{i+1}}{Q_{i+1}} / \frac{P_i}{Q_i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q_i & \to & P_i & \to & \frac{P_i}{Q_i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q_{i-1} & \to & P_{i-1} & \to & \frac{P_{i-1}}{Q_{i-1}} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the highest row of the diagram is just the i-th component of the sequence $0 \to \mathrm{Gr}(Q) \to \mathrm{Gr}(P) \to \mathrm{Gr}(P/Q) \to 0$ this proves the statement. □

Lemma 6.57. *Let Q be a finitely generated left $\mathcal{C}$Diff($\mathcal{E}$)-module and $U \subset \mathcal{E}$ open, then*

$$I_{char}(Q|_U) = I_{char}(Q)|_U$$

Proof. Let Q_i be a good filtration of Q then because of the exactness of localization we have that $\mathrm{Gr}(Q)|_U = \mathrm{Gr}(Q|_U)$ where on the right hand side we took the filtration $Q_i|_U$ on $Q|_U$. Since the filtration Q_i was good we have that $\mathrm{Gr}(Q)|_U$ is finitely generated and hence the filtration $Q_i|_U$ is also good. Now $\mathrm{Ann}(\mathrm{Gr}(Q))$ may be represented as the kernel of the natural map $C\mathrm{Symb} \to \hom(\mathrm{Gr}(Q), \mathrm{Gr}(Q))$ which commutes with localization and so again by exactness of the localization we have $\mathrm{Ann}(\mathrm{Gr}(Q))|_U = \mathrm{Ann}(\mathrm{Gr}(Q|_U))$. The result now follows since localization commutes with taking radicals. □

we come to the proof of theorem 6.55

Proof. Let P_i be good filtration on P. Introduce the filtration $Q_i = Q \cap P_i$ on Q. Then the induced map in the graded modules $\mathrm{Gr}(Q) \to \mathrm{Gr}(P)$ is an injection. Now $\mathrm{Gr}(Q)$ need not be finitely generated (because the filtration Q_i need not be good) but by theorem 6.26 there is an open dense $U \subset M$ in which $\mathrm{Gr}(Q)$ is locally finitely generated. Restricting to a small neighbourhood U_θ for $\theta \in U$ we obtain that the filtration $Q_i|_{U_\theta}$ is good. By lemmas 3.31 and 6.57 we have that $I_{\mathrm{char}}(Q)|_{U_\theta} = I_{\mathrm{char}}(Q|_{U_\theta}) \supseteq I_{\mathrm{char}}(P|_{U_\theta}) = I_{\mathrm{char}}(P)|_{U_\theta}$. If moreover $\mathrm{Gr}(P/Q)$ is finitely generated over $\mathcal{F}$ then $I_{\mathrm{char}}(P/Q)|_{U_\theta} = \mathcal{C}\mathrm{Symb}(\mathcal{E})|_{U_\theta}$ and hence $I_{\mathrm{char}}(Q)|_{U_\theta} = I_{\mathrm{char}}(P)|_{U_\theta}$ lemma 3.31. □

6.5.1. *The example of the Cole-Hopf transformation.* Let's consider an example of a covering which is the heat equation covering the Burgers equation via the Cole-Hopf transformation [2] and see how it affects the fold type singularity equations.

We fix a two dimensional base manifold with independent coordinates x, t.
Heat equation: The dependent coordinate is a and jet coordinates are denoted with

$$a_{l,k} = a_{\underbrace{x\cdots x}_{l}\underbrace{t\cdots t}_{k}}$$

The Heat equation $\mathcal{H}$ is

$$a_t = a_{xx}$$

and differential consequences are

$$\begin{aligned} a_{2l,k} &= a_{0,k+l} \\ a_{2l+1,k} &= a_{1,k+l} \end{aligned}$$

On the l-th prolongation $\mathcal{H}^{(l)}$ we use as internal coordinates $x, t, a, a_{0,k}, a_{1,k-1}$ $k = 1\ldots l+2$

Cartan fields on $\mathcal{H}^{(\infty)}$ in internal coordinates are expressed as:

$$\begin{aligned} D_t &= \partial_t + a_{0,1}\partial_a + \sum_{k=1}^{\infty}(a_{0,k+1}\partial_{a_{0,k}} + a_{1,k}\partial_{a_{1,k-1}}) \\ D_x &= \partial_x + a_{1,0}\partial_a + \sum_{k=1}^{\infty}(a_{1,k}\partial_{a_{0,k}} + a_{0,k}\partial_{a_{1,k-1}}) \end{aligned}$$

while Cartan forms are

$$\begin{aligned} \alpha_{0,k} &= da_{0,k} - a_{1,k}dx - a_{0,k+1}dt \\ \alpha_{1,k-1} &= da_{1,k-1} - a_{0,k}dx - a_{1,k}dt \end{aligned}$$

The $\mathcal{C}$Diff module $\mathcal{C}\Lambda^1$ is generated by $\alpha_{0,0}$ since

$$\begin{aligned} \alpha_{0,k} &= D_t\alpha_{0,k-1} \\ \alpha_{1,k-1} &= D_x\alpha_{0,k-1} = D_t\alpha_{1,k-2} \end{aligned}$$

hence the symbolic module is generated over $X_1 = \widehat{D_x}$ and $X_2 = \widehat{D_t}$ by $\widehat{\alpha_{0,0}}$, and has Hilbert polynomial equal to 2. Obviously the element X_1^2 is in the annihilator and from dimensional reasons it follows that $\mathrm{Ann}(\mathcal{C}\mathrm{Symb}(\mathcal{H}) = (X_1^2)$ which is already a radical ideal. Hence at every point the characteristics are given by dt or, as subspaces of TM spanned by ∂_x.
Burgers equation: Here the dependent coordinate is denoted with u, and jet coordinates are introduced as foe the heat equation. The burgers equation $\mathcal{B}$ is

$$u_t = u_{xx} + uu_x$$

and as internal coordinates we use $u_{0,k}, u_{1,k-1}$ $k = 1\ldots\infty$. The relations on the infinite prolongation are

$$u_{2,k} = u_{0,k+1} - \sum_{j=0}^{k}\binom{k}{j}u_{0,,j}u_{1,k-j} = A_{k+1}$$

Cartan fields are given in internal coordinates by

$$\begin{aligned} D_t &= \partial_t + u_{0,1}\partial_u + \sum_{k=1}^{\infty}(u_{0,k+1}\partial_{u_{0,k}} + u_{1,k}\partial_{u_{1,k-1}}) \\ D_x &= \partial_x + u_{1,0}\partial_u + \sum_{k=1}^{\infty}(u_{1,k}\partial_{u_{0,k}} + A_k\partial_{u_{1,k-1}}) \end{aligned}$$

while Cartan forms are given by

$$\begin{array}{rcl} \omega_{0,k} & = & du_{0,k} - u_{1,k}dx - u_{0,k+1}dt \\ \omega_{1,k-1} & = & du_{1,k-1} - A_{k-1}dx - u_{1,k}dt \end{array}$$

again the $\mathcal{C}$Diff module $\mathcal{C}\Lambda^1$ is generated by $\omega_{0,0}$ since

$$\begin{array}{rcl} \omega_{0,k} & = & D_t\omega_{0,k-1} \\ \omega_{1,k-1} & = & D_x\omega_{0,k-1} = D_t\omega_{1,k-2} \end{array}$$

and the symbolic module is again isomorphic to

$$C^\infty(\mathcal{B}) \otimes S(TM)/(\partial_x^2)$$

so characteristics are again in direction ∂_x
Cole Hopf transformation: The Cole Hopf transformation is a finite covering obtained by prolonging the nonlinear differential operator

$$u = 2\frac{a_x}{a}$$

The induced map $\phi^\infty : \mathcal{H}^\infty \to \mathcal{B}^\infty$ is given by

$$\phi^*(u_{l,k}) = D_x^l D_t^k (2\frac{a_x}{a})$$

Now the first prolongation goes from $\phi^{(1)} : \mathcal{H} \to J^1(u, x, t)$ and is given by

$$\begin{array}{rcl} u_x & = & 2(\frac{a_t}{a} - \frac{a_x^2}{a^2}) \\ u_t & = & 2(\frac{a_{xt}}{a} - \frac{a_x a_t}{a^2}) \end{array}$$

the Jacobian of this map is

	∂_x	∂_t	∂_a	∂_{a_x}	∂_{a_t}	$\partial_{a_{xt}}$	$\partial_{a_{tt}}$
x	1						
t		1					
u			$-2\frac{a_x}{a^2}$	$\frac{2}{a}$			
u_x			$\left(-2\frac{a_t}{a^2} + 4\frac{a_x^2}{a^3}\right)$	$-4\frac{a_x}{a^2}$	$\frac{2}{a}$		
u_t			$\left(-2\frac{a_{xt}}{a^2} + 4\frac{a_x a_t}{a^3}\right)$	$-2\frac{a_t}{a^2}$	$-2\frac{a_x}{a^2}$	$\frac{2}{a}$	

with two dimensional kernel given by

$$\langle \partial_{a_{tt}}, a\partial_a + a_x\partial_{a_x} + a_t\partial_{a_t} + a_{xt}\partial a_{xt} \rangle$$

hence the fibers of $\phi^{(1)}$ may be parametrized as $(x, t, \lambda a, \lambda a_x, \lambda a_t, \lambda a_{xt}, a_{tt})$ where the parameters are a_{tt} and λ
Fold-Singularity equations: The singularity equation of the first prolongation $\mathcal{H}^{(1)}$ is a two dimensional plane in each point of the equation $\mathcal{H} \subset J^2$ spanned by vectors (using internal coordinates):

$$\partial_{a_{tt}}, \partial_x + a_x\partial_a + a_t\partial_{a_x} + a_{xt}\partial_{a_t} + a_{tt}\partial_{a_{xt}}$$

while the singularity equations of the Burgers equation are spanned at each point of $J^1(u, x, t)$ by the two directions

$$\partial_{u_t}, \partial_x + u_x\partial_u + (u_t - uu_x)\partial_{u_x}$$

To study the behaviour of the singularity equation of the heat equation under the Cole Hopf transformation observe that one direction of this singularity equation is in the kernel of $\phi^{(1)}$, namely $\partial_{a_{tt}}$. Hence the singularity plane is projected to a line which is the spanned by the image of the second direction $\partial_x + a_x\partial_a + a_t\partial_{a_x} + a_{xt}\partial_{a_t} + a_{tt}\partial_{a_{xt}}$. This image vector is computed to be

$$\partial_x + u_x\partial_u + (u_t - uu_x)\partial_{u_x} + 2(-2\frac{a_{xt}a_x}{a^2} + 2\frac{a_x^2 a_t}{a^3} - \frac{a_t^2}{a^2} + \frac{a_{tt}}{a})\partial_{u_t}$$

where the values of u, u_t, u_x are determined by values of $a, a_x, a_t, a_{tx}, a_{tt}$ under the Cole-Hopf map. Obviously this vector is in the singularity equation of the Burgers equation. Observe that while varying in the fiber of the Cole Hopf map the last vector will vary and may assume any value $\mu\partial_{u_t}$. Moreover the coefficient in front of ∂_{u_t} is nothing but $\phi^{(2)*}(u_{x,t})$

Corollary 6.58. *Under the Cole Hopf transformation the fold-type singularity equations are mapped into each other, but with degeneration. Nevertheless the whole singularity equation of the Burgers equation is obtained by varying the image of the singularity of the Heat equation along the fiber of the map.*

6.6. **A relation to the method of integration by characteristics.** Consider an infinitely prolonged PDE $\mathcal{E}$. If one could find a nowhere zero vector field $X \in \mathcal{C}D(\mathcal{E})$ which possesses a flow, then solving the Cauchy problem for $\mathcal{E}$ would be reduced to transporting an initial data along the flow of the field, as in the case of the method of characteristics known for first order scalar equations. Unfortunately it is well known that not many vector fields on the infinite dimensional manifolds $\mathcal{E}$ posses a flow. Indeed there is a nice result due to Chetverikov which states the following

Theorem 6.59. *(See [4]) Let $(\mathcal{E}^k)$ be a PDE and let $X \in \mathcal{C}D(\mathcal{E})$ then X possesses a flow if and only if there is a $l \in \mathbb{N}$ such that for all $r, k \in \mathbb{N}$ we have*

$$\underbrace{X \circ X \circ \ldots \circ X}_{r-times}(C^\infty(\mathcal{E}^k)) \subset C^\infty(\mathcal{E}^{k+l})$$

In other words the shift of all the differential operators X^r is bounded by l.

Interestingly this is related with the characteristic ideal of $\mathcal{E}$.

Corollary 6.60. *Suppose $X \in \mathcal{C}D(\mathcal{E})$ possesses a flow then $X \in I_{char}(\mathcal{E})$.*

Remark 6.61. Let us call a vector field $X \in I_{\text{char}}(\mathcal{E})$ a **characteristic field**. Geometrically it is clear that a nowhere zero characteristic field can only exist if the characteristic varieties $\text{char}_\theta^{\mathbb{C}}(\mathcal{E})$ are all contained in hyper-planes, which is a rather strong condition and excludes the existence of such integrating fileds for many examples of equations.

A well know example where the characteristic varieties are contained in hyper-planes and where such an integrating field is know to exist is precisely the case of first order scalar PDEs which are integrated with the method of characteristics.

We come to the proof of the above corollary

Proof. Suppose X possesses a flow and let l bound the shift of all the operators X^r. Then the differential operator X^{l+1} is of shift $\leq l$. Now any $\omega \in \mathcal{C}\Lambda_k^1$ can be expressed as a finite sum $\omega = \sum f^i dg^i$ with $f^i \in C^\infty(\mathcal{E}^\infty)$ and $g^i \in C^\infty(\mathcal{E}^k)$ and by assumption $X^r(g^i) \subset C^\infty(\mathcal{E}^{k+l})$ and so by definition of the action of X^{l+1}on ω we find

$$X^{l+1}(\omega) = \sum_{j=0}^{l+1} \sum X^j(f^i) d(X^{l+1-j} g^i) \in \mathcal{C}\Lambda_{k+l}^1$$

which shows that $(X^{l+1}) \in \text{Ann}(\text{Gr}(\mathcal{C}\Lambda^1))$ and hence $X \in I_{\text{char}}(\mathcal{E})$. ☐

7. Second order hyperbolic and parabolic PDEs and their 1-singularities

In this section we explore in more detail the structure of the fold singularity equations for hyperbolic second order equations in two independent and one dependent variable, and give a detailed descrition of the fold-singularity equations of hyperbolic Monge Ampère equations.

7.1. **Second order hyperbolic nonlinear PDEs.** We shall consider PDEs in two independent and one dependent variable hence we consider the trivial bundle $\pi : M \times \mathbb{R} \to M$ where $M = \mathbb{R}^2$. Coordinates in M are denoted with (x, y) and the single fiber coordinate is denoted with u. Coordinates in $J^k(\pi)$ are denoted as usual with $u_{r,s}$where the first index corresponds to the number of derivatives with respect to x and the second with respect to y.

By a **hyperbolic second order nonlinear PDE** we will understand a smooth submanifold $\mathcal{E} \subset J^2(\pi)$ of co-dimension 1 such that the projection $\mathcal{E} \to J^1(\pi)$ is a surjective submersion and its symbol g_2 (see definition 3.8) is generated at each point by two linearly independent characteristics (see definition 3.23).

Proposition 7.1. *Given any hyperbolic second order nonlinear PDE $\mathcal{E}$ in two independent and one dependent variable, then all the prolongations of the symbol $g_2^{(l)}$ (see definition 3.11) are 2 dimensional and generated by the prolongations of the two characteristics of g_2.*

Proof. The condition may be checked point-wise for fixed $\theta_2 \in \mathcal{E}$ and for such a point we may introduce coordinates (x, y) in the base M such that the characteristics in $g_2(\theta_2)$ are

$$(dx)^2, (dy)^2$$

(see proposition 3.21). We know that characteristics dx^l and dy^l are contained in $g_2^{(l-2)}$ for $l \geq 2$, hence it remains to show that any $v \in g_2^{(l-2)}$ is of the form $t_{l,0}dx^l + t_{0,l}dy^l$ with $t_{l,0}, t_{0,l} \in \mathbb{R}$ which is proven by induction on l: suppose that $g_2^{(l-3)} = \langle dx^{l-1}, dy^{l-1}\rangle$ and suppose that $v \in g_2^{(l-2)}$

$$v = \sum_{j=0}^{l} t_{j,l-j} dx^j dy^{l-j} \in g_{l-1}^{(1)}$$

with $t_{\alpha,\beta} \in \mathbb{R}$ then

$$\delta_{\partial_x} v = \sum_{j=0}^{l} j t_{j,l-j} dx^j dy^{l-j} \in g_2^{(l-3)}$$

implies $v = t_{0,l}dy^l + t_{1,l-1}dxdy^{l-1} + t_{l,0}dx^l$ and

$$\delta_{\partial_y} v \in g_2^{(l-3)}$$

implies $v = t_{l,0}dx^l + t_{0,l}dy^l$. □

Proposition 7.2. *For any hyperbolic second order nonlinear PDE $\mathcal{E}$ in two independent and one dependent variable all second δ-Spencer cohomologies vanish*

$$H^{k,2}(g) = 0, \ \forall k \geq 0$$

Proof. As before we only need to check this at any fixed point $\theta_2 \in \mathcal{E}$, hence we introduce coordinates x, y in M such that $g_k(\theta_2) = \langle dx^k, dy^k\rangle$, $\forall k \geq 1$. To show is that the sequence

$$g_{k+1} \otimes T^*_\theta M \xrightarrow{\delta} g_k \otimes \bigwedge^2 T^* M \to 0$$

is exact, i.e that the second δ-Spencer operator is surjective. To avoid confusion with the skew symmetric part of the tensors we will change the notation for the components of the symmetric part $dx \leftrightarrow \overline{x}$ and $dy \leftrightarrow \overline{y}$. Let now $\omega \in g_{k+1} \otimes T^*_\theta M$

$$\omega = (c_x\overline{x}^{k+1} + c_y\overline{y}^{k+1}) \otimes dx + (b_x\overline{x}^{k+1} + b_y\overline{y}^{k+1}) \otimes dy$$

where $c_x, c_y, b_x, b_y \in \mathbb{R}$ then

$$\delta\omega = (k+1)(-c_y\overline{y}^k + b_x\overline{x}^k) \otimes dx \wedge dy$$

which immediately shows that δ_2 is surjective. □

By a theorem of Goldschmidt [25] one obtains

Corollary 7.3. *Any hyperbolic second order nonlinear PDE $\mathcal{E}$ in two independent and one dependent variable whose first prolongation $\mathcal{E}^{(1)} \to \mathcal{E}$ is a smooth fiber bundle is formally integrable.*

Until the end of this paragraph we will assume that the equation under consideration is formally integrable and hence by proposition 4.12 we know that the fibers of the prolongations $\mathcal{E}^l_{\theta_l}$, $l > 2$ are all 2 dimensional affine spaces over $g_l(\theta_l) = g_2^{(l-2)}(\theta_2)$. This also implies that the restriction of the Cartan distribution $C\mathcal{E}^l$ is 4 dimensional at each point. A particular, a (non canonical) base of $C_{\theta_l}\mathcal{E}^l$ may be introduces by fixing a point $\theta_{l+1} \in \mathcal{E}^{l+1}_{\theta_l}$ and splitting

$$C_{\theta_l}\mathcal{E}^l = R_{\theta_{l+1}} \oplus g_{l,\theta_l}$$

Then there are two linearly independent characteristics

$$\alpha^l_+, \alpha^l_- \in g_l$$

with $\alpha_\pm \in T^*_\theta M$ giving a basis of g_l, and if $X_\pm \in T_\theta M$ denotes the dual basis of $\alpha_\pm$, then the lifting $D_{X_\pm}$ to $R_{\theta_{l+1}}$ gives a basis there.

Proposition 7.4. *For all prolongations $\mathcal{E}^l$ $l \geq 3$ of a hyperbolic PDE, the* 1*-singularity equations $\Sigma_{[1]}\mathcal{E}^l(\theta_{l-1}) \subset I^{l-1,1}_\theta$ at a point $\theta_{l-1} \in \mathcal{E}^{l-1}$ are given by two* 2*-dimensional transversal planes $\Sigma_\pm \subset C_{\theta_{l-1}}\mathcal{E}^{l-1}$spanned in the above basis by*

$$\Sigma_\pm = \langle \alpha^{l-1}_\pm, D_{X_\pm} \rangle$$

moreover these two planes are orthogonal to each other with respect to the metasymplectic structure.

Proof. Consider one of the characteristic directions say α_+ (the other is handled exactly the same). Together with each point $\theta_l \in \mathcal{E}^l_{\theta_{l-1}}$ it generates a horizontal line $k_{\theta_l} \subset R_{\theta_l}$ by

$$k_{\theta_l} = \ker \alpha_+ \cap R_{\theta_l} = \langle D_{X_-}(\theta_l) \rangle$$

Varying the point θ_l in the fiber $\mathcal{E}^l_{\theta_{l-1}}$ the lines k_{θ_l} constitute one of the two sought after cones of the singularity equation. By proposition 5.9 moving a point θ_l in direction of the characteristic α^l_+ will not change the line k_{θ_l} hence effectively one just has to move one point $\tilde{\theta}_l$ in direction of the other characteristic α^l_-. For this let us introduce coordinates x, y, u such that $\tilde{\theta}_l = 0$ and $\alpha_+ = dx$ and $\alpha_- = dy$. Then $D_{X_-}(\tilde{\theta}_l) = \partial_y$and $\alpha^l_- = l!\partial_{u_{0,l}}$, then

$$k_{\tilde{\theta}_l + \tau\alpha^l_-} = \langle \partial_y + \tau l!\partial_{u_{0,l-1}} \rangle,\ \tau \in \mathbb{R}$$

which implies that $\Sigma_- = \langle D_{x_-}(\tilde{\theta}_l), \alpha^{l-1}_- \rangle$. That both planes are orthogonal with respect to the metasymplectic structure follows immediately from its description 4.1. □

Example 7.5. The **hyperbolic Monge-Ampère equation** $\mathcal{E}$ is a second order equation in two independent and one dependent variables given in coordinates by the zero set of a function of the form

$$\Phi := N(u_{2,0}u_{0,2} - u_{1,1}^2) + Au_{2,0} + Bu_{1,1} + Cu_{0,2} + D$$

where $N, A, B, C, D \in \mathcal{F}^1$ and hyperbolicity is expressed by the condition:

$$\Delta := B^2 - 4AC + 4ND > 0$$

Proposition 7.6. *The Monge Ampère equation $\mathcal{E} = \{\Phi = 0\}$ is a smooth submanifold of $J^2(\pi)$ of co-dimension 1 and the projection*

$$\pi_{2,1} \mid_{\mathcal{E}} : \mathcal{E} \to J^1(\pi)$$

is a surjective submersion where each fiber is a smooth 2 dimensional submanifold.

Proof. For surjectivity of the projection it suffices to show that the polynomial equation

$$N(u_{2,0}u_{0,2} - u_{1,1}^2) + Au_{2,0} + Bu_{1,1} + Cu_{0,2} + D = 0$$

has always a solution in $\mathbb{R}^3$ for fixed scalars N, A, B, C, D, with $(N, A, B, C) \neq (0, 0, 0, 0)$. Without loss of generality we may assume $N > 0$ (if $N = 0$ one obtains a linear equation and if $N < 0$ one may multiply with -1), then setting $u_{2,0} = u_{0,2} = 0$ and $u_{1,1} \gg 0$ large enough the left hand side becomes negative, while setting $u_{1,1} = 0$ and $u_{2,0} = u_{0,2} \gg 0$ large enough the left hand side becomes positive implying the existence of a solution. Moreover 0 is a regular value of Φ since the functions

$$\begin{aligned} a := \partial_{u_{2,0}}\Phi &= Nu_{0,2} + A \\ b := \partial_{u_{1,1}}\Phi &= -2Nu_{1,1} + B \\ c := \partial_{u_{0,2}}\Phi &= Nu_{2,0} + C \end{aligned}$$

don't vanish all at the same time on $\Phi = 0$, implying $d\Phi_{\theta_2} \neq 0$ for $\theta_2 \in \mathcal{E}$. This follows from the hyperbolicity condition as is seen by considering the function $b^2 - 4ac$:

$$\begin{aligned} b^2 - 4ac =& B^2 - 4CA - 4N \underbrace{\left(N(u_{2,0}u_{0,2} - u_{1,1}^2) + Au_{2,0} + Bu_{1,1} + Cu_{0,2}\right)}_{=-D} \\ =& B^2 - 4CA + 4ND > 0 \end{aligned}$$

These facts imply that the equation $\mathcal{E}$ is a 7 dimensional smooth submanifold of $J^2(\pi)$ projecting surjectively and submersively onto $J^1(\pi)$ and the fibers $F_{\theta_1} \cap \mathcal{E}$ of the projection are smooth 2 dimensional submanifolds. $\square$

Proposition 7.7. *The symbols $g_2(\theta_2) \subset S^2\left(\left(\mathbb{R}^2\right)^*\right)$ of the hyperbolic Monge Ampère equation are everywhere 2 dimensional and generated by two characteristic vectors.*

Proof. The 2 dimensionality follows from the previous proposition and a computation shows that for $\theta_2 \in \mathcal{E}$ the tensors

$$\alpha_+^2, \alpha_-^2 \in g_2(\theta_2)$$

where

$$\alpha_\pm = \delta_\pm dx + \gamma_\pm dy$$

and

$$\begin{aligned} \delta_\pm(\theta_2) &= Nu_{2,0} + C \\ \gamma_\pm(\theta_2) &= Nu_{1,1} - \frac{B \pm \sqrt{\Delta}}{2} \end{aligned}$$

or equivalently (unless both of these functions vanish):

$$\begin{aligned}\delta_\pm &= Nu_{1,1} - \frac{B \mp \sqrt{\Delta}}{2}\\ \gamma_\pm &= Nu_{0,2} + A\end{aligned}$$

□

The singularity equations of the Monge-Ampère equation are well known . So let us compute explicitly the singularity equation for the first prolongation of the Monge Ampère equation.

The first prolongation of the Monge-Ampère equation is given by the two additional equations

$$\begin{aligned}0 = D_x(\Phi) =& D_x(N)(u_{2,0}u_{0,2} - u_{1,1}^2) + D_x(A)u_{2,0} + D_x(B)u_{1,1} + D_x(C)u_{0,2} + D_x(D)\\ &+ N(u_{3,0}u_{0,2} + u_{2,0}u_{1,2} - 2u_{2,1}u_{1,1}) + Au_{3,0} + Bu_{2,1} + Cu_{1,2}\end{aligned}$$

and

$$\begin{aligned}0 = D_y(\Phi) =& D_y(N)(u_{2,0}u_{0,2} - u_{1,1}^2) + D_y(A)u_{2,0} + D_y(B)u_{1,1} + D_y(C)u_{0,2} + D_y(D)\\ &+ N(u_{2,1}u_{0,2} + u_{2,0}u_{0,3} - 2u_{1,2}u_{1,1}) + Au_{2,1} + Bu_{1,2} + Cu_{0,3}\end{aligned}$$

The intersection of $\mathcal{E}^{(1)}$ with a fiber $J^3_{\theta_2}$ where $\theta_2 \in \mathcal{E}$ with coordinates $\theta_2 = (\overline{x}, \overline{y}, \overline{u}, , \ldots, \overline{u}_{1,1}, \overline{u}_{0,2})$, is a system of 2 linear non-homogeneous equations in the coordinates $(u_{3,0}, u_{2,1}, u_{1,2}, u_{0.3})$ hence an affine subspace in $J^3_{\theta_2}$. Its tangent space at a fixed point $\theta_3 = (\overline{x}, \overline{y}, \overline{u}, , \ldots, \overline{u}_{1,2}, \overline{u}_{0,3}) \in \mathcal{E}^{(1)} \cap J^3_{\theta_2}$ is given by the equations

$$\begin{aligned}N(\bar{u}_{0,2}t_{3,0} + \bar{u}_{2,0}t_{1,2} - 2\bar{u}_{1,1}t_{2,1}) + At_{3,0} + Bt_{2,1} + Ct_{1,2} &= 0\\ N(\bar{u}_{0,2}t_{2,1} + \bar{u}_{2,0}t_{0,3} - 2\bar{u}_{1,1}t_{1,2}) + At_{2,1} + Bt_{1,2} + Ct_{0,3} &= 0\end{aligned} \tag{7.1}$$

which may also be written as

$$\begin{pmatrix} \partial_{u_{2,0}}\Phi & \partial_{u_{1,1}}\Phi & \partial_{u_{0,2}}\Phi & 0\\ 0 & \partial_{u_{2,0}}\Phi & \partial_{u_{1,1}}\Phi & \partial_{u_{0,2}}\Phi \end{pmatrix} \cdot \begin{pmatrix} t_{3,0}\\ t_{2,1}\\ t_{1,2}\\ t_{0,3}\end{pmatrix} = 0$$

From 7.6 it follows that the rank of the matrix on the left is maximal and hence $\mathcal{E}^{(1)}_{\theta_2}$ is a 2 dimensional affine space in the 4 dimensional fibers $J^3_{\theta_2}$.

7.2. **Rays in $J^2(\pi)$.** To compute the **1-ray cones** (the envelope of all rank one tensors) in $V^{3,2}$, fix a point $\theta_3 = (\overline{x}, \overline{y}, \overline{u}, , \ldots, \overline{u}_{1,2}, \overline{u}_{0,3}) \in J^2(\pi)$ and let $\theta_2, \theta_1, \theta_0, \theta_{-1}$ denote its corresponding projections to $J^2(\pi), \ldots, M$. Fix also a one-dimensional subspace in $P \subset T^*_{\theta_{-1}}M$ generated by a one form $\alpha dx + \beta dy$. The corresponding one dimensional subspace in $T_{\theta_{-1}}M$ annihilated by P is given by $\beta\partial_x - \alpha\partial_y$. The R-plane $R_{\theta_3} \subset C_{\theta_2}$ is generated by the two total derivatives $D_x(\theta_3)$, while the lifting of P to R_{θ^3} is given by

$$\beta D_x(\theta_3) - \alpha D_y(\theta_3)$$

To obtain the ray associated to it we need to find all the points $\tilde{\theta}_3 \in F_{\theta_2}$ such that

$$\beta D_x(\theta_3) - \alpha D_y(\theta_3) = \beta D_x(\tilde{\theta}_3) - \alpha D_y(\tilde{\theta}_3) \tag{7.2}$$

denoting the only free coordinates of the point $\tilde{\theta}_3$ with $(u_{3,0}, u_{2,1}, u_{1,2}, u_{0,3})$ and equating coefficients in 7.2 we obtain the non-homogeneous equations

$$\begin{aligned}\beta u_{3,0} - \alpha u_{2,1} &= \beta\bar{u}_{3,0} - \alpha\bar{u}_{2,1}\\ \beta u_{2,1} - \alpha u_{1,2} &= \beta\bar{u}_{2,1} - \alpha\bar{u}_{1,2}\\ \beta u_{1,2} - \alpha u_{0,3} &= \beta\bar{u}_{1,2} - \alpha\bar{u}_{0,3}\end{aligned} \tag{7.3}$$

Conversely any affine line in F_{θ_2} given by a non homogeneous system of equations of the form

$$\begin{aligned}\beta u_{3,0} - \alpha u_{2,1} &= \kappa_1\\ \beta u_{2,1} - \alpha u_{1,2} &= \kappa_2\\ \beta u_{1,2} - \alpha u_{0,3} &= \kappa_3\end{aligned} \tag{7.4}$$

where $\alpha, \beta, \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ and $(\alpha, \beta) \neq (0,0)$ is a ray, and the line $P \subset C_{\theta_2}$ that generates it is spanned by the vector

(7.5)
$$\beta(\partial_x + u_{1,0}\partial_{u_{0,0}} + u_{2,0}\partial_{u_{1,0}} + u_{1,1}\partial_{u_{0,1}}) - \alpha(\partial_y + u_{0,1}\partial_{u_{0,0}} + u_{1,1}\partial_{u_{1,0}} + u_{0,2}\partial_{u_{0,1}}) + \kappa_1\partial_{u_{2,0}} + \kappa_2\partial_{u_{1,1}} + \kappa_3\partial_{u_{0,2}}$$

The associated homogeneous equations to 7.3 describe the tangent space to the ray

$$\begin{aligned}\beta t_{3,0} - \alpha t_{2,1} &= 0\\ \beta t_{2,1} - \alpha t_{1,2} &= 0\\ \beta t_{1,2} - \alpha t_{0,3} &= 0\end{aligned}$$

where $t_{i,j}$ denote coordinates in $V^{3,2}$ with respect to the base $\partial_{u_{3,0}}, \partial_{u_{2,1}}, \partial_{u_{1,2}}, \partial_{u_{0,3}}$ its solutions are given by

$$(t_{3,0}, t_{2,1}, t_{1,2}, t_{03}) = (\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3)$$

hence the "ray plane" associated to P is a line generated by

$$\alpha^3\partial_{u_{3,0}} + \alpha^2\beta\partial_{u_{2,1}} + \alpha\beta^2\partial_{u_{1,2}} + \beta^3\partial_{u_{0,3}}$$

By rotating the line P in $T_{\theta_{-1}}M$ we obtain a one parameter family of ray planes $V^3_{\theta_3}$ generating a 2-dimensional cone. This cone is determined by the equations

$$\begin{aligned}t_{2,1}^2 &= t_{3,0}t_{1,2}\\ t_{1,2}^2 &= t_{2,1}t_{0,3}\\ t_{3,0}t_{0,3} &= t_{2,1}t_{1,2}\end{aligned}$$

which are the minors of the matrix

$$\begin{pmatrix} t_{3,0} & t_{2,1}\\ t_{2,1} & t_{1,2}\\ t_{1,2} & t_{0,3}\end{pmatrix}$$

7.3. **Rays tangent to the equation.** Observe that since $\mathcal{E}^{(1)} \cap F_{\theta_2}$is affine it suffices to calculate the intersection of its tangent space with the ray-cone in any point, then the others are obtained by affine transport.

A ray-line at $\theta_3 \in J^2(\pi)$ is given by a tangent vector

$$\alpha^3\partial_{u_{3,0}} + \alpha^2\beta\partial_{u_{2,1}} + \alpha\beta^2\partial_{u_{1,2}} + \beta^3\partial_{u_{0,3}}$$

to see which of these are tangent to $\mathcal{E}^{(1)}$ we must insert them into the equations which determine the vertical tangent spaces of $\mathcal{E}^{(1)}$ 7.1 and solve for $\alpha$and β

Case 1 ($\alpha \neq 0$): One may fix $\alpha = 1$ obtaining the two equations

$$\begin{aligned}N(u_{0,2} + u_{2,0}\beta^2 - 2u_{1,1}\beta) + A + B\beta + C\beta^2 &= 0\\ N(u_{0,2}\beta + u_{2,0}\beta^3 - 2u_{1,1}\beta^2) + A\beta + B\beta^2 + C\beta^3 &= 0\end{aligned}$$

where the second equals the first multiplied by β and may be discarded. Hence only the polynomial

$$\underbrace{(C + Nu_{2,0})}_{=c=\partial_{u_{0,2}}\Phi}\beta^2 + \underbrace{(B - 2Nu_{1,1})}_{=b=\partial_{u_{1,1}}\Phi}\beta + \underbrace{(A + Nu_{0,2})}_{=a=\partial_{u_{2,0}}\Phi} = 0 \tag{7.6}$$

needs to be solved.

Case 1 ($c \neq 0$): The number of solutions is determined by the sing of $b^2 - 4ac$ which is strictly positive as follows from the proof of 7.6, hence there are exactly two solutions, i.e. the ray-cone intersects the equation in two lines. These lines are determined by

$$\beta = \frac{-b \pm \sqrt{\Delta}}{2c}$$

Case 2 ($c = 0$): In this case $b \neq 0$ and the equation 7.6 is linear and has exactly one solution.

7.4. **The 1-singularity equation in $J^2(\pi)$:** The singularity equations $\mathcal{E}^{(1)}_{\Sigma_1} \subset J^1(\mathcal{E}, 1)$ are those lines $P \subset T\mathcal{E}$ such that $l(P)$ is tangent to $\mathcal{E}^{(1)}$but since $\mathcal{E}^{(1)} \cap F_{\theta_2}$ is affine this implies $l(P) \subset \mathcal{E}^{(1)} \cap F_{\theta_2}$

Recall that the equations describing $\mathcal{E}^{(1)} \cap F_{\theta_2}$ in F_{θ_2} are given by

$$\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \end{pmatrix} \cdot \begin{pmatrix} u_{3,0} \\ u_{2,1} \\ u_{1,2} \\ u_{0,3} \end{pmatrix} = \begin{pmatrix} -W_x \\ -W_y \end{pmatrix}$$

where $a = \partial_{u_{2,0}} \Phi, \; b = \partial_{u_{1,1}} \Phi, \; c = \partial_{u_{0,2}} \Phi$ and

$$\begin{aligned} W_x &:= D_x(N)(u_{2,0}u_{0,2} - u_{1,1}^2) + D_x(A)u_{2,0} + D_x(B)u_{1,1} + D_x(C)u_{0,2} + D_x(D) \\ W_y &:= D_y(N)(u_{2,0}u_{0,2} - u_{1,1}^2) + D_y(A)u_{2,0} + D_y(B)u_{1,1} + D_y(C)u_{0,2} + D_y(D) \end{aligned}$$

while equations describing a ray in F_{θ_2} are:

$$\begin{aligned} \beta u_{3,0} - \alpha u_{2,1} &= \kappa_1 \\ \beta u_{2,1} - \alpha u_{1,2} &= \kappa_2 \\ \beta u_{1,2} - \alpha u_{0,3} &= \kappa_3 \end{aligned}$$

To find those rays which are contained in $\mathcal{E}^{(1)} \cap F_{\theta_2}$ the conditions on the coefficients α, β have already been deduced in the previous section. Assuming $\alpha \neq 0$ and $c \neq 0$ they were

$$\alpha = 1$$

$$\beta_\pm = \frac{-b \pm \sqrt{\Delta}}{2c}$$

hence it remains to find $(\kappa_1, \kappa_2, \kappa_3)$ s.t. the system

$$\underbrace{\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ \beta_\pm & -1 & 0 & 0 \\ 0 & \beta_\pm & -1 & 0 \\ 0 & 0 & \beta_\pm & -1 \end{pmatrix}}_{=:\Theta} \begin{pmatrix} u_{3,0} \\ u_{2,1} \\ u_{1,2} \\ u_{0,3} \end{pmatrix} = \begin{pmatrix} -W_x \\ -W_y \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix}$$

has a solution, i.e. $(-W_x, -W_y, \kappa_1, \kappa_2, \kappa_3) \in \mathrm{im}(\Theta)$. The kernel of Θ is by construction 1 dimensional hence the image is 3 dimensional. A operator $\Xi \in M(\mathbb{R}^5, \mathbb{R}^2)$ s.t. $\ker \Xi = \mathrm{im}\Theta$ is given by

$$\Xi \;=\; \begin{pmatrix} 1 & 0 & b + \beta_\pm c & c & 0 \\ 0 & 1 & 0 & b + \beta_\pm c & c \end{pmatrix}$$

This leads to the the condition

$$\begin{pmatrix} b + \beta_\pm c & c & 0 \\ 0 & b + \beta_\pm c & c \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix} = \begin{pmatrix} W_x \\ W_y \end{pmatrix}$$

which has all solutions given by

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{W_x}{c} \\ \frac{1}{c}(W_y - \frac{1}{c}(b + \beta_\pm c)W_x) \end{pmatrix} + \tau \begin{pmatrix} 1 \\ -\left(\frac{b}{c} + \beta_\pm\right) \\ \left(\frac{b}{c} + \beta_\pm\right)^2 \end{pmatrix}, \ \tau \in \mathbb{R}$$

Hence the singularity equation is given by two one-parameter family of lines $P_\tau \subset C_{\theta_2}(\mathcal{E})$. Using coordinates associated to the basis $(D_x, D_y, \partial_{u_{2,0}}\partial_{u_{1,1}}, \partial_{u_{0,2}})$ in C_{θ_2} and formula 7.5 to recover the line P associated to a ray $l(P)$ these cones are two planes spanned by

$$\Pi^{(1)}_\pm = \left\langle \begin{pmatrix} \beta_\pm \\ -1 \\ 0 \\ \frac{W_x}{c} \\ \frac{1}{c}(W_y + \beta_\mp W_x) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \beta_\mp \\ {\beta_\mp}^2 \end{pmatrix} \right\rangle \tag{7.7}$$

The metasymplectic structure applied to the two basis vectors of each plane is proportional to

$$\beta_\mp \partial_{0,1} + \partial_{1,0}$$

and in particular is non degenerate

Proposition 7.8. *The two planes Π_+ and Π_- are transversal and orthogonal complements of each other.*

Proof. A direct computation shows that they are orthogonal to each other, which using the fact that the metasymplectic structure restricted to each one is non-degenerate implies that they are transversal. By dimensional reasons it follows that they are othogonal complements. □

Proposition 7.9. *Every R-plane in $\mathcal{E}$ intersects $\Pi^{(1)}_\pm$ in exactly two lines.*

Now we want to reveal the relation between $\Pi^{(1)}_\pm$ and the singularity equation $\Pi_\pm$ in $J^{(1)}$.

Remark 7.10. According to Vinogradov the two planes on $J^1(\pi)$ are given by

$$\Pi_\pm = \langle X_\pm, Y_\pm \rangle \tag{7.8}$$

where

$$\begin{aligned} X_\pm &= \partial_x + u_{1,0}\partial_u - \frac{C}{N}\partial_{u_{1,0}} + \frac{B \mp \sqrt{\Delta}}{2N}\partial_{u_{0,1}} \\ Y_\pm &= \partial_y + u_{0,1}\partial_u + \frac{B \pm \sqrt{\Delta}}{2N}\partial_{u_{1,0}} - \frac{A}{N}\partial_{u_{0,1}} \end{aligned}$$

Proposition 7.11. *Projecting the planes $\Pi^{(1)}_{\pm,\theta_2}$ to $J^1(\pi)$ one obtains two lines contained in $\Pi_{\pm,\theta_1}$, to be precise:*

$$\pi_{2,1*}\left(\Pi^{(1)}_{\pm,\theta_2}\right) = R_{\theta_2} \cap \Pi_{\pm,\theta_1} \tag{7.9}$$

Moreover $\Pi_\pm$ can be reconstructed knowing $\Pi^{(1)}_\pm$ by taking the cone of all lines $\pi_{2,1*}\left(\Pi^{(1)}_{\pm,\theta_2}\right)$ while varying $\theta_2 \in \mathcal{E} \cap F_{\theta_1}$, i.e.

$$\Pi_{\pm,\theta_1} = Cone\left(\bigcup_{\theta_2 \in \mathcal{E}\cap F_{\theta_1}} \pi_{2,1*}\left(\Pi^{(1)}_{\pm,\theta_2}\right)\right)$$

Proof. From the description 7.7 of $\Pi_{\pm}^{(1)}$ it immediately follows that the second basis vector is tangential to the fiber of the projection, hence these planes project to lines. Using coordinates in the Cartan space C_{θ_1} with respect to the basis $(\partial_x + u_{1,0}\partial_u, \partial_y + u_{0,1}\partial_u, \partial_{u_{1,0}}, \partial_{u_{0,1}})$these lines are spanned by the basis vectors

$$\begin{pmatrix} \beta_{\pm} \\ -1 \\ \beta_{\pm}u_{2,0} - u_{1,1} \\ \beta_{\pm}u_{1,1} - u_{0,2} \end{pmatrix}$$

where $\beta_{\pm} = \frac{2Nu_{1,1} - B \pm \sqrt{\Delta}}{2(Nu_{2,0}+C)}$ and the values of $(u_{2,0}, u_{1,1}, u_{0,2})$ are determined by the choice of the point $\theta_2 \in \mathcal{E}$ projecting to θ_1. Considering now description 7.8 we need to check if this vector coincides with

$$\begin{pmatrix} \beta_{\pm} \\ -1 \\ \frac{-2\beta_{\pm}C - B \mp \sqrt{\Delta}}{2N} \\ \frac{2A + \beta_{\pm}(B \mp \sqrt{\Delta})}{2N} \end{pmatrix}$$

which is verified in a direct computation. Obviously from $\pi_{2,1*}(C_{\theta_2}) = R_{\theta_1}$ and the fact that $\Pi_{\pm}^{(1)} \subset C_{\theta_2}$ follows 7.9. □

8. Appendix: Local classification of Pro-finite manifolds

In the category of pro-finite manifolds the co-filtration is not actually an invariant of the manifold, i.e there are isomorphic pro-finite manifolds whose filtrations are not isomorphic. An example is given by the two isomorphic pro-finite manifolds $\mathbb{R}^1 \twoheadleftarrow \mathbb{R}^3 \twoheadleftarrow \mathbb{R}^5 \twoheadleftarrow \ldots$ and $\mathbb{R}^2 \twoheadleftarrow \mathbb{R}^4 \twoheadleftarrow \mathbb{R}^6 \twoheadleftarrow \ldots$. But there are indeed non isomorphic pr-finite manifolds as for example the co-filtration $\mathbb{R}^2 \twoheadleftarrow \mathbb{R}^4 \twoheadleftarrow \mathbb{R}^8 \twoheadleftarrow \mathbb{R}^{16} \twoheadleftarrow \ldots$is not isomorphic as a pro-finite manifold to the two previous ones. Nevertheless a local classification of pro-finite manifolds is very simple and for this the following notion is useful

Definition 8.1. Two non decreasing functions $\chi_i : \mathbb{Z} \to \mathbb{N}, \;\; i = 1, 2$ are said to have **equivalent growth** if there exist two numbers $l_i \in \mathbb{N}, \;\; i = 1, 2$such that for all l big enough i.e. $\forall l \geq l_0$

$$\begin{aligned} \chi_1(l) &\leq \chi_2(l + l_2) \\ \chi_2(l) &\leq \chi_1(l + l_1) \end{aligned}$$

This is obviously an equivalence relation and the equivalence class is called the **growth type** of a function χ

This allows to state the

Proposition 8.2. *Two co-filtered manifolds are locally isomorphic if and only if they have the same growth type*

Proof. Obviously for these the statement is true in the category of linear pro-finite vector spaces. One may then reduce the problem to the linear case by regarding the co-tangent space at a point, which is a filtered pro-finite vector spaces. □

A more explicit classification is possible in the case we restrict the type of growth to that of interest to us

Definition 8.3. A co-filtered manifold $\mathcal{E}$ will be called of **polynomial growth** if there exists a polynomial $Q \in \mathbb{R}[x]$ such that for all $l \in \mathbb{Z}$ big enough the growth function of $\mathcal{E}$ coincides with Q i.e.

$$\dim \mathcal{E}^l = Q(l), \; \forall l \geq l_0$$

From the existence of the Hilbert polynomial for the symbolic system of a PDE it follows that the infinite prolongations of formally integrable partial differential equations are of polynomial growth.

Next we make the simple observation that a polynomial is a growth function (i.e. is positive and non decreasing for sufficiently big values) if and only if it has a positive leading coefficient.

Lemma 8.4. *Let $p(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_o$ be a polynomial, then $p(x) \geq 0$ for all x sufficiently big if and only if $a_n > 0$.*

Proof. Obviously $\frac{p(x)}{x^n} \to a_n$ with $x \to \infty$, and since the denominator is greater than zero for $x > 0$ we see that $a_n > 0$ if and only if $p(x)$ is greater than zero for sufficiently big x □

This also implies the that a polynomial $p(x)$ with positive leading coefficient is non decreasing for sufficiently big x since the derivative $p'(x)$ will again have a positive leading coefficient and hence will be positive for big x by the previous lemma.

Proposition 8.5. *Two polynomials with positive leading term have the same growth type if and only if they have the same leading term i.e. are of the same order and have the same leading coefficient.*

Proof. Observe that changing a polynomial $p(x)$ to $p(x + c)$ does not change its leading term. Next suppose to have two polynomials $p(x), q(x)$ with positive leading coefficients and different leading terms. Consider first the case $\deg(p) > \deg(q)$, then there is no $c \in \mathbb{Z}$ such that $p(x) \leq q(x + c)$ for sufficiently big x, since $p(x) - q(x + c)$ will always be a polynomial with the same leading term as $p(x)$ and hence by lemma 8.4 will always be positive for big x. Suppose now that $\deg(p) = \deg(q)$ but the leading coefficient of p is strictly bigger than that of q then for any $c \in \mathbb{N}$ the polynomial $p(x) - q(x + c)$ is of $\deg(p)$ and has leading positive term hence again $p(x) \geq q(x + c)$ for all x sufficiently big. It remains to show that two polynomials with the same leading term are equivalent. For this it suffices to show that a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_0$ is equivalent to its leading term monomial $a_n x^n$. First consider the case that the second highest non zero coefficient satisfies $a_k > 0$, then obviously $a_n x^n \leq p(x)$ for sufficiently big x by lemma 8.4, and to show the second inequality choose $c \in \mathbb{N}$ big enough so that $a_n c^{n-k} > a_k$ then the second highest nonzero coefficient of $a_n(x + c)^n$will certainly be bigger than a_{n-1} and hence $p(x) \leq a_n(x + c)^n$. If the second nonzero coefficient is negative we may apply a similar reasoning. □

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