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Weak field tests of metric gravity theories

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Zusammenfassung

Die Schwerkraft ist zweifellos die alltäglichste der vier grundlegenden Wechselwirkungen, welche wir heute kennen. Schon sehr früh in der Geschichte wurde sie auf ihre qualitativen Eigenschaften untersucht. Die populärsten Abhandlungen gravitativer Effekte, welche zur Zeit der Antike in der abendländischen Kultur verfasst wurden, gehen auf Archimedes zurück. Sie sind rund um das Jahr 250 v. Chr. entstanden und behalten bis heute ihre Gültigkeit. Isaac Newtons revolutionäre Gravitationstheorie aus dem Jahre 1687 stellte den Bezug zwischen der Schwerkraft als irdische Geläufigkeit und der Himmelsmechanik her. Die Entdeckung der Allgemeinen Relativität durch Albert Einstein rund 200 Jahre später unterzog das Verständnis der Schwerkraft erneut einem grundlegenden Wandel. Die Einsteinsche Theorie deutet einerseits die Gravitation als eine Erscheinung, welche durch die Geometrie der Raumzeit verursacht wird, und formuliert andererseits eine Vorschrift, wie Materie die Geometrie der Raumzeit beeinflusst. Sie liefert bis heute die präzisesten Voraussagen über gravitative Phänomene auf der Skala des Sonnensystems. Ihre Gültigkeit lässt sich gemäss dem heutigen Wissensstand jedoch nicht vorbehaltlos auf galaktische und universale Längenskalen extrapolieren. Rätselhafte Phänomene wie die Dunkle Materie und die Dunkle Energie motivierten in der jüngeren Vergangenheit die Entwicklung einer Vielfalt an heuristischen Erweiterungen der Allgemeinen Relativitätstheorie. Die vorliegende Arbeit befasst sich im wesentlichen mit den Konsequenzen solcher Theorien auf experimentell messbare Grössen. Wir konzentrieren uns dabei insbesondere auf metrische Gravitationstheorien.

Im einleitenden Kapitel 1 stellen wir zuerst die metrischen Gravitationstheorien in einen mathematischen Kontext und listen ihre physikalischen Postulate auf. Exakte Lösungen der grundlegenden Feldgleichungen sind bis heute nur in Spezialfällen bekannt, was eine näherungsweise Lösung der Gleichungen für allgemeinere Probleme motiviert. Wir erläutern daher zwei Methoden, welche insbesondere auf die Approximation schwacher Gravitationsfelder zugeschnitten sind. Die Entwicklung der Variablen nach inversen Potenzen der Lichtgeschwindigkeit c eignet sich einerseits für die Untersuchung isolierter Systeme auf der Skala des Sonnensystems. Die Linearisierung der Feldgleichungen andererseits führt zur Theorie der schwachen Gravitationswellen.

In Kapitel 2 gehen wir näher auf die Eigenschaften von Gravitationswellen ein. Wir zeigen, wie die möglichen Polarisationszustände von ebenen Wellen mittels einer Untersuchung gewisser algebraischer Eigenschaften des Riemanntensors hergeleitet werden können. Ferner stellen wir schwache Gravitationswellen im Rahmen der Allgemeinen Relativitätstheorie und der $f(R)$-Theorie vor.

Kapitel 3 befasst sich mit den Auswirkungen einer kosmologischen Konstanten Λ auf die Fortpflanzung und Messung von Gravitationswellen. Dazu untersuchen wir die linearisierten Einsteingleichungen mit Termen bis zur linearen Ordnung in Λ über einer de Sitter und einer anti-de Sitter Raumzeit. In dem vorgegebenen Rahmen führt der kosmologische Term nicht zu Änderungen der Polarisationszustände, während die Amplitude sich in Abhängigkeit von Λ ändert. Falls eine Quelle eine periodische Wellenform abstrahlt, so wird ausserdem die von einem entfernten Beobachter gemessene Periodizität modifiziert. Diese Effekte sind jedoch extrem klein und liegen rund zwanzig Grössenordnungen unter der Messgenauigkeit von existierenden Gravitationswellendetektoren wie *LIGO* oder dem in Zukunft geplanten Weltraumobservatorium *LISA*.

In Kapitel 4 leiten wir für Anwendungen auf isolierte Systeme auf der Skala des Sonnensystems die ersten relativistischen Korrekturen der $1/c$-Entwicklung der Raumzeitmetrik $g_{\mu\nu}$ für metrische $f(R)$-Gravitationstheorien her, wobei wir annehmen, dass f bei $R = 0$ analytisch ist. Für unsere Zwecke genügt es, $f(R) = R + aR^2$ zu betrachten, wobei a ein positiver dimensionsbehafteter Parameter ist. Im nichtrelativistischen Limes erhalten wir zum Newtonpotential eine zusätzliche Yukawakorrektur mit Kopplungsstärke $G/3$ und Comptonwellenlänge $\sqrt{6a}$, was bereits in der Literatur bekannt ist. Als Anwendung berechnen wir bis zur selben Ordnung die Korrektur zur geodätischen Präzession eines Kreisels in einem Gravitationsfeld und zur Präzession von Binärpulsaren. Das Resultat des Experiments *Gravity Probe B* liefert die Schranke $a \lesssim 5 \times 10^{11}\,\mathrm{m}^2$, während wir für den Pulsar B im PSR J0737-3039-System eine um 10^4 grössere Schranke erhalten. Andererseits ergibt sich aus dem Experiment *Eöt-Wash* die genaueste Laborschranke $a \lesssim 10^{-10}\,\mathrm{m}^2$. Obwohl die Schranken von der geodätischen Präzession viel grösser sind als die Laborschranke, so sind sie trotzdem sinnvoll, falls eine Art Chamäleoneffekt präsent ist, unter dem sich die effektiven Werte für verschiedene Längenskalen unterscheiden.

Die Emission von Gravitationsstrahlung durch ein isoliertes System in einer Gravitationstheorie mit Lagrangedichte $f(R) = R + aR^2$ wird in Kapitel 5 behandelt. Als formales Resultat erhalten wir eine Korrektur zur Quadrupolformel der Allgemeinen Relativitätstheorie. Wir verwenden dabei die Analogie zwischen $f(R)$–Theorien und Skalar–Tensor–Theorien, welche im Gegensatz zur Allgemeinen Relativität einen zusätzlichen skalaren Freiheitsgrad aufweisen. Während in der Allgemeinen Relativität die Gravitationsstrahlung in der führenden Ordnung von Quadrupolmomenten erzeugt wird, so prognostiziert der zusätzliche Freiheitsgrad die Strahlung von allen Multipolen, insbesondere von Monopol– und Dipolmomenten. Dies ist der Fall für fast alle alternativen Gravitationstheorien. In einem weiteren Punkt jedoch unterscheiden sich die $f(R)$–Theorien fundamental von anderen Theorien, welche bereits im Hinblick auf die Erzeugung von Gravitationsstrahlung untersucht worden sind. Da das skalare Feld massiv ist, führt es zu einer Yukawakorrektur im nichtrelativistischen Limes. Aus demselben Grund genügt der dynamische Anteil des Skalarfeldes einer Klein–Gordon–Gleichung anstelle einer Wellengleichung. Diese zwei Komplikationen überlagern sich bei der Behandlung der Erzeugung von Gravitationsstrahlung.

Abstract

Gravitation is certainly the most common of the four fundamental interactions which are known today. It has been tested for its qualitative behaviour very early in history. The most popular treatises of the ancient Occident about gravitational effects trace back to Archimedes. They were written around 250 B. C. and are still valid today. The revolutionary gravity theory published by Isaac Newton in 1687 established the relationship between gravity as a mundane phenomenon and celestial mechanics. About 200 years later, the discovery of General Relativity by Albert Einstein once again subjected the understanding of gravity to a fundamental change. On the one hand, Eintein's theory interprets gravitation as an appearance which is governed by the geometry of spacetime, and formulates on the other hand a directive on how matter influences the geometry of spacetime. It provides to date the most precise predictions for gravitational phenomena on the scale of the Solar System. However, according to the current state of knowledge, its validity cannot be extrapolated to galactic and universal scales without reservations. Enigmatic phenomena such as Dark Matter and Dark Energy motivated in the recent past the development of a variety of heuristic extensions of General Relativity theory. The present work essentially investigates the consequences of such theories on experimentally measurable quantities. In doing so we concentrate in particular on metric gravity theories.

In the introductory Chapter 1 we first put metric gravity theories into a mathematical context and list their physical postulates. Exact solutions of the fundamental field equations are to date known only for special cases. This motivates an approximation scheme for the solutions of the equations for generic problems. Therefore, we explain two methods that are particularly adapted to the approximation of weak fields. On the one hand, the expansion of the variables with respect to the inverse light speed c is appropriate to investigate isolated systems on the scale of the Solar System. The linearisation of the field equations on the other hand leads to the theory of weak gravitational waves.

In Chapter 2 we enlarge upon the properties of gravitational waves. We show how the possible polarisation modes of a plane wave can be determined by investigating certain algebraic properties of the Riemann tensor. Moreover, we present weak gravitational waves in the scope of General Relativity and $f(R)$ theory.

Chapter 3 addresses the effect of a cosmological constant Λ on the propagation and detection of gravitational waves. To this purpose we investigate the linearised Einstein's equations with terms up to linear order in Λ in a de Sitter and an anti–de Sitter background spacetime. In this framework the cosmological term does not induce changes in the polarisation states of the waves, whereas the amplitude gets modified by terms depending on Λ. Moreover, if a source emits a periodic waveform, its periodicity as measured by a distant observer gets modified. However, these effects are, extremely tiny and thus well below the detectability by some twenty orders of magnitude of present gravitational wave detectors such as *LIGO* or future planned ones such as *LISA*.

In Chapter 4 we derive the first relativistic terms in the $1/c$ expansion of the space time metric $g_{\mu\nu}$ in metric $f(R)$ gravity theories for applications to isolated systems on the scale of the Solar System. Thereby f is assumed to be analytic at $R = 0$. For our purposes it suffices to take into account terms up to quadratic order in the expansion

of $f(R)$, and we can thus approximate $f(R) = R + aR^2$ with a positive dimensional parameter a. In the non–relativistic limit, we get an additional Yukawa correction to the Newtonian potential with coupling strength $G/3$ and Compton wave length $\sqrt{6a}$, which is a known result in the literature. As an application, we derive to the same order the correction to the geodesic precession of a gyroscope in a gravitational field and the precession of binary pulsars. The result of the *Gravity Probe B* experiment yields the limit $a \lesssim 5 \times 10^{11}\,\mathrm{m}^2$, whereas for the pulsar B in the PSR J0737-3039 system we get a bound which is about 10^4 times larger. On the other hand the, *Eöt-Wash* experiment provides the best laboratory bound $a \lesssim 10^{-10}\,\mathrm{m}^2$. Although the former bounds from geodesic precession are much larger than the laboratory ones, they are still meaningful in case some type of chameleon effect is present, and thus the effective values could be different at different length scales.

Chapter 5 is devoted to the investigation of the gravitational radiation emitted by an isolated system for gravity theories with Lagrange density $f(R) = R + aR^2$. As a formal result we obtain leading order corrections to the quadrupole formula in General Relativity. We make use of the analogy of $f(R)$ theories with scalar–tensor theories, which in contrast to General Relativity feature an additional scalar degree of freedom. Unlike General Relativity, where the leading order gravitational radiation is produced by quadrupole moments, the additional degree of freedom predicts gravitational radiation of all multipoles, in particular monopoles and dipoles. Whereas this is the case for almost every alternative gravity theory, in one point the quadratic $f(R)$ theory considerably differs from other theories which already have been investigated in the context of the generation of gravitational radiation. Since the scalar field is massive, it gives rise to a Yukawa correction in the non–relativistic limit, and by the same reason the dynamical part of the scalar field suffices rather a Klein–Gordon equation than a wave equation. These two complications are superposed in the treatment of the generation of gravitational radiation.

Contents

Chapter 1

Metric Gravity Theories and Weak Fields

The discovery of General Relativity Theory (GR) by Albert Einstein at the beginning of the 20th century provided a so far unequalled insight in the nature of gravity. However, with regard to concrete problems in the vast area of astrophysics, it implied a big challenge in finding solutions of the underlying equations, the Einstein equations. Exact solutions of Einstein's equations are known only for highly symmetric problems, such as the Schwarzschild solution for a spherically symmetric vacuum spacetime. An overview of exact solutions can be found in [57]. The investigation of generic applications motivated the development of adequate approximation methods. These methods can even be generalised to applications of extensions of GR. Among them, we will consider so called metric gravity theories (MGT). As references for the foundations of GR and its applications one may consult the textbooks [58, 34, 62].

The mathematical setup for MGTs is concisely summarised in Section 1.1. In Section 1.2 the postulates of MGTs are formulated. Important examples of MGTs are introduced in Section 1.3. Section 1.4 is devoted to two common perturbative approximation methods.

1.1 Mathematical Preliminaries

In this subsection we premise the foundations of Riemannian and pseudo Riemannian geometry. The mathematical setting throughout all chapters of this thesis is as follows. Let M be a smooth 4-dimensional pseudo Riemannian manifold endowed with a metric $\boldsymbol{g}$ of signature $(-,+,+,+)$. $\mathcal{T}^q_{r,p}(M)$ denotes the set of tensors

$$\boldsymbol{T}_p : \otimes^q T^*_p M \otimes^r T_p M \to \mathbb{R} \tag{1.1}$$

at the point $p \in M$. The gravitational fields on M, i. e. the dynamical gravitational variables, are represented by smooth sections $\hat{\boldsymbol{T}}$ of the tensor bundle

$$\mathcal{T}^q_r(M) := \bigcup_{p \in M} \{p\} \times \mathcal{T}^q_{r,p}(M), \tag{1.2}$$

that is

$$\hat{\boldsymbol{T}} \in \hat{\mathcal{S}}^q_r(M) := \{\hat{\boldsymbol{S}} : M \to \mathcal{T}^q_r(M) \,|\, \hat{\boldsymbol{S}} \text{ smooth}, \pi_M \circ \hat{\boldsymbol{S}} = \mathrm{id}_M\}, \tag{1.3}$$

where $\pi_M : \mathcal{T}^q_r(M) \to M$ is the projection on M. More explicitly, we identify the dynamical variables with smooth maps

$$\tilde{\boldsymbol{T}} \in \mathcal{S}^q_r(M) := \{\boldsymbol{S} : M \to \mathcal{T}^q_{r,p}(M) \,|\, \hat{\boldsymbol{S}}(p) = (p, \boldsymbol{S}(p)), \hat{\boldsymbol{S}} \in \hat{\mathcal{S}}^q_r(M)\} \tag{1.4}$$

from M to the fibre at the corresponding point.

The metric $\tilde{\boldsymbol{g}}$ is an element of $\mathcal{S}^0_2(M)$ and plays the crucial role in MGTs. GR is an MGT with no additional dynamical variables other than $\tilde{\boldsymbol{g}}$. In the variety of MGTs which have been proposed since the discovery of GR, particularly scalar fields $\tilde{\phi} \in \mathcal{S}^0_0(M)$, vector fields $\tilde{\boldsymbol{K}} \in \mathcal{S}^0_1(M)$ and second rank tensor fields $\tilde{\boldsymbol{B}} \in \mathcal{S}^0_2(M)$ are paired with $\tilde{\boldsymbol{g}}$. We combine these nonmetric fields in the variable $\tilde{\boldsymbol{N}}_I \in \prod_{i\in I} \mathcal{S}^{q_i}_{r_i}(M)$, where I is an appropriate index set.

Let

$$\psi : U \subset M \to \mathbb{R}^4, \quad p \mapsto (x^0, x^1, x^2, x^3) \tag{1.5}$$

be a coordinate chart on an open subset U of M. For $\tilde{\boldsymbol{T}} \in \mathcal{S}^q_r(M)$ we then have a field on a subset of $\mathbb{R}^4$

$$\boldsymbol{T} = \tilde{\boldsymbol{T}} \circ \psi^{-1} : \psi(U) \subset \mathbb{R}^4 \to \mathcal{T}^q_{r,p}(M) \tag{1.6}$$

Moreover, we can expand the values of $\boldsymbol{T}$ with respect to a coordinate basis $\{\partial_\mu, \mu = 0, 1, 2, 3\}$ and its dual basis $\{dx^\mu, \mu = 0, 1, 2, 3\}$. We will use the summation convention, that is we sum over repeated indices in a single term. Greek indices range from 0 to 3, whereas latin indices range from 1 to 3. The expansion of $\boldsymbol{T}$ then reads

$$\boldsymbol{T} = T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \ldots \nu_r} \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_q} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_r}, \tag{1.7}$$

where we have introduced the smooth component functions

$$T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \ldots \nu_r} : \psi(U) \to \mathbb{R}. \tag{1.8}$$

We identify a gravitational tensor field $\tilde{\boldsymbol{T}} \in \mathcal{S}^q_r(M)$ by means of equations (1.6) and (1.7) always with the set of the corresponding coordinate components. According to the context we denote $\tilde{\boldsymbol{T}}$ itself as well as a single component by $T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \ldots \nu_r}$. Similarly, we denote the set of coordinates as well as a single component by x^μ. The component representation of the nonmetric field $\tilde{\boldsymbol{N}}_I$ is formally denoted by N_I. Similarly, we refer to matter and non–gravitational fields as A_J.

The possibility to freely choose the coordinates is referred to as the gauge freedom of MGTs. This freedom allows us to adapt the coordinates to the various applications.

For local problems it is useful to construct coordinates induced by a local freely falling frame. The free fall is defined as a motion that is subjected to the gravitational interaction exclusively. In the neighbourhood of the world line $l(t)$ of a freely falling observer, i. e. its trajectory in spacetime, we can construct local coordinates $(x^0, x^1, x^2, x^3) = (ct, \boldsymbol{x})$, where c is the vacuum speed of light, t the proper time along $l(t)$, and for a given time t_0, $\boldsymbol{x} = (x^1, x^2, x^3)$ are spatial Riemann normal coordinates. In such coordinates, spatial 3-vectors are usually denoted by bold face letters.

For the investigation of viable MGTs we will extensively make use of the Riemann tensor $R^\mu{}_{\nu\lambda\rho} \in \mathcal{S}^1_3(M)$ of $g_{\mu\nu}$,

$$R^\mu{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^\mu_{\rho\nu} - \partial_\rho \Gamma^\mu_{\lambda\nu} + \Gamma^\sigma_{\rho\nu}\Gamma^\mu_{\lambda\sigma} - \Gamma^\sigma_{\lambda\nu}\Gamma^\mu_{\rho\sigma}, \tag{1.9}$$

where the Levi–Civita connection coefficients are given by

$$\Gamma^\mu_{\nu\lambda} = \frac{g^{\mu\rho}}{2}(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\rho\nu} + \partial_\rho g_{\lambda\nu}), \tag{1.10}$$

and $g^{\mu\nu} \equiv (g^{-1})^{\mu\nu}$ denotes the inverse of the metric $g_{\mu\nu}$. The Ricci tensor and the Ricci scalar are given by $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \in \mathcal{S}^0_2(M)$ and $R = g^{\mu\nu}R_{\mu\nu} \in \mathcal{S}^0_0(M)$, respectively.

Notice that a single physical field might often be referred to as different tensor types. An index of a field can be raised or lowered by means of the metric to obtain a physically equivalent quantity. For instance, we will refer to the Riemann tensor also when we write $R_{\mu\nu\lambda\rho} = g_{\mu\sigma}R^{\sigma}{}_{\nu\lambda\rho}$.

1.2 Physical Definition of Metric Gravity Theories

MGTs are physically defined by three postulates [64]:

1. At least locally, spacetime can be modelled as a 4-dimensional smooth pseudo Riemannian manifold endowed with a metric $g_{\mu\nu}$ of signature $(-,+,+,+)$.

2. The world lines of test bodies are geodesics of $g_{\mu\nu}$.

3. In local freely falling frames, the non–gravitational laws of physics are those of Special Relativity.

An important consequence of the second postulate is that the equations of motion (EOM) of a test body only depend on $g_{\mu\nu}$, but not on the nonmetric fields. In other words, matter and non–gravitational fields couple only to the metric. In contrast, the nonmetric fields may be created by matter and non–gravitational fields, and they may account for the generation of the metric.

By postulate 2, gravity can be measured in terms of distances and angles in spacetime, or equivalently, in terms of time intervals on one hand, and distances and angles in space as well as their variation in time on the other hand.

We now focus on Lagrangian based MGTs, which means that they are derived from an action. To date, every known physically reasonable MGT is Lagrangian based.

In order to satisfy postulate 2, the Lagrangian density may be divided into two summands. Let $D(F_I)$ denote the set of partial derivatives of any order of the smooth component functions $\mathbb{R}^4 \to \mathbb{R}$ of a product tensor field $F_I \in \prod_{i\in I} \mathcal{S}^{q_i}_{r_i}(M)$, where – for the sake of simplicity – we have assumed that there exists a global coordinate chart, $\psi(U) = \mathbb{R}^4$. The EOM of non–gravitational fields should be derivable from a Lagrangian density which is a functional of $g_{\mu\nu}$ and A_J and their derivatives. Thus the corresponding action is

$$\mathcal{I}_{NG} = \int_{\mathbb{R}^4} d^4x\, \mathcal{L}_{NG}(D(g_{\mu\nu}), D(A_J)). \tag{1.11}$$

On the other hand, the Lagrangian density which leads to the EOM of the gravitational fields is a functional of N_I, and possibly $g_{\mu\nu}$, and their derivatives, such that the action is

$$\mathcal{I}_G = \int_{\mathbb{R}^4} d^4x\, \mathcal{L}_G(D(g_{\mu\nu}), D(N_I)), \tag{1.12}$$

Taking the sum $\mathcal{I} = \mathcal{I}_G + \mathcal{I}_{NG}$ for the total action ensures the absence of the coupling of N_I to A_J.

The variation of $\mathcal{I}$ with respect to the metric yields the equation

$$\mathcal{G}_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.13}$$

We have introduced the energy-momentum tensor

$$T_{\mu\nu} = -\frac{1}{\kappa}\frac{\delta \mathcal{I}_{NG}}{\delta g^{\mu\nu}} \tag{1.14}$$

and the tensor

$$\mathcal{G}_{\mu\nu} = \frac{\delta \mathcal{I}_G}{\delta g^{\mu\nu}}, \tag{1.15}$$

which carries the information of the measurable gravitational response to the field A_J. κ is the coupling constant, which measures basically the strength of the gravitational interaction. Depending on the theory, the EOM of the fields N_I take various forms [64]. In physically meaningful MGTs, the EOM of the gravitational fields are nonlinear and coupled with respect to the variables.

For viable MGTs, the tensor $\mathcal{G}_{\mu\nu}$ contains terms which depend on the Riemann tensor $R^{\mu}{}_{\nu\lambda\rho}$. In the context of fundamental physics, this fact reflects the deep connection

$$\text{Interaction} = \text{Curvature}.$$

Whereas gravity becomes manifest mathematically as the curvature of the tangent bundle, the electrodynamical interaction, e. g., is governed by the curvature of a $U(1)$ principal bundle.

1.3 Examples

Every reasonable MGT is designed as an extension of GR, which is still the most reliable gravity theory concerning high precision measurements on the scale of the Solar System. Basically, the extensions are heuristic approaches to problems on larger scales, such as the Dark Matter problem on the galactic scale and the Dark Energy problem on the universal scale. We will present here, along with GR, two representative theories which are derived from specific modifications of the GR Lagrangian.

1.3.1 General Relativity

The Lagrangian density of GR without cosmological constant is the scalar curvature of M. On $\mathbb{R}^4$ it is thus given as the Ricci scalar R multiplied by the Jacobian of $g_{\mu\nu}$. Hence the action is given by

$$\mathcal{I} = \int_{\mathbb{R}^4} d^4x\,\sqrt{-g}R + \mathcal{I}_{NG}, \tag{1.16}$$

where $g := \det g_{\mu\nu}$. The EOM are the so–called Einstein equations and can be written by means of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ as

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.17}$$

The coupling constant, composed of c and Newtons's constant G, is constrained by the Newtonian limit of GR.

1.3.2 f(R) Theory

One possibility to extend GR is to modify the Lagrangian density without introducing additional fields. Among the various possibilities to build meaningful Lagrangians out of the metric and its partial derivatives, there has been paid much attention to Lagrangians which are functionals of the scalar curvature R. An overview of such theories can be found for example in [10, 60]. Let f denote a differentiable real valued function defined on an appropriate subset of $\mathbb{R}$. The $f(R)$ action then reads

$$\mathcal{I} = \int_{\mathbb{R}^4} d^4x \sqrt{-g} f(R) + \mathcal{I}_{NG} \tag{1.18}$$

and leads to the EOM

$$\begin{aligned} f'(R)G_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box_g f'(R) & \\ + \frac{1}{2} g_{\mu\nu} \big(R f'(R) - f(R)\big) &= \frac{8\pi G}{c^4} T_{\mu\nu}, \end{aligned} \tag{1.19}$$

where we have introduced the Levi–Civita connection ∇_μ of $g_{\mu\nu}$ and the d'Alembert operator $\Box_g = \nabla^\mu \nabla_\mu$. If we assume the function f to be analytic in 0, we can write

$$f(R) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} R^k = -2\Lambda + a_1 R + a_2 R^2 + \cdots . \tag{1.20}$$

Λ is the cosmological constant. For the applications presented in this thesis we always assume f to be of the form (1.20).

A complicating fact of $f(R)$ theories is that the field equations (1.19) contain up to fourth derivatives of the metric, whereas the Einstein equations are second order partial differential equations. However, as we will see in the next paragraph, $f(R)$ theory is equivalent to a certain scalar tensor theory with EOM of second order.

1.3.3 Scalar Tensor Theory

The prototypic MGTs featuring additional fields besides the metric are scalar tensor theories. We will present the Bergmann–Wagoner theory [3, 61], which covers the most popular scalar tensor theories as special cases [64]. The dynamical fields are $g_{\mu\nu}$ and a scalar field $\phi \in \mathcal{S}_0^0(M)$. The Lagrangian also contains two arbitrary functionals of ϕ, the coupling functional $\omega(\phi)$ and the potential $U(\phi)$. The total action reads

$$\mathcal{I} = \int_{\mathbb{R}^4} d^4x \sqrt{-g} \left(\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + \mathcal{I}_{NG}. \tag{1.21}$$

The resulting field equations can be written as

$$\begin{aligned} \phi G_{\mu\nu} - \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \Box_g \phi + \frac{1}{2} g_{\mu\nu} U(\phi) & \\ + \frac{\omega(\phi)}{\phi} \left(\frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \partial^\lambda \phi - \partial_\mu \phi \partial_\nu \phi \right) &= \frac{8\pi G}{c^4} T_{\mu\nu}, \end{aligned} \tag{1.22}$$

$$\begin{aligned} \Box_g \phi + \frac{1}{3 + 2\omega(\phi)} \Big(2U(\phi) - \phi U'(\phi) & \\ + \omega'(\phi) \partial_\lambda \phi \partial^\lambda \phi \Big) &= \frac{1}{3 + 2\omega(\phi)} \frac{8\pi G}{c^4} T. \end{aligned} \tag{1.23}$$

By T we denote the trace of the energy momentum tensor $T_{\mu\nu}$. In the field equations (1.22) for $g_{\mu\nu}$, the potential U plays the role of a cosmological constant. Moreover, for isolated systems it gives rise to Yukawa like terms in the solutions ϕ of (1.23), and those also enter the solutions for $g_{\mu\nu}$ of (1.22). A discussion of some possible functionals ω and their consequences can be found in [64].

We assume now the coupling functional ω to vanish identically. A comparison of equations (1.19) and (1.22) then naturally suggests to introduce the scalar field $\phi := f'(R)$. We assume that f is at least locally convex. Then f' is invertible. We denote the inverse function of f' by $\mathcal{R}$. If we choose the potential U to be the Legendre transform of f,

$$U(\phi) = \mathcal{R}(\phi)\phi - f(\mathcal{R}(\phi)), \tag{1.24}$$

we see that equations (1.19) and (1.22) are equivalent. Furthermore, equation (1.23) is equivalent to the trace of equation (1.19).

For some applications, the formulation of $f(R)$ gravity as a metric scalar tensor theory can be very useful and convenient. In Chapters 2, 4 and 5, we will employ this equivalence for the case of

$$f(R) = R + aR^2. \tag{1.25}$$

The parameter a is positive and has dimension Length^{-2}. The Lagrangian is a small perturbation of the GR Lagrangian for $aR \ll 1$. This theory is discussed in more detail in the chapters mentioned above.

1.4 Weak Field Approximations

In this section we will introduce two approaches to approximate MGTs by the use of perturbative methods. The $1/c$ expansion discussed in the first subsection is appropriate for examining isolated weak field systems. In the second subsection we present the linearisation of the field theories which is a simple method to investigate gravitational radiation.

1.4.1 1/c Expansion

For systems which are characterised by small velocities compared to c, it makes sense to expand the gravitational field equations into powers of the inverse light speed $1/c$. If we assume more generally the energy-momentum densities to be correspondingly small, we can perform the so called Parametrised Post-Newtonian expansion (PPN) of MGTs [64]. The source is modelled as a perfect fluid. Let m be the total source mass, r the typical size of the system of sources, v the typical source velocity, p the pressure and ρ the mass density. The expansion parameter ε for the PPN formalism is usually chosen such that – after the choice of a set of physical units – the numerical values of the quantities

$$\left(\frac{Gm}{c^2 r}\right)^{1/2}, \ \frac{v}{c} \quad \text{and} \quad \frac{p}{\rho} \tag{1.26}$$

are at most of order ε. In particular, for the investigation of the gravitational mechanics of an isolated system, where the distances between the sources are large compared to their extension and the pressure is negligible, the $1/c$ expansion is equivalent to the ε expansion.

The gravitational fields are expanded around their asymptotic values,

$$\begin{aligned} g_{\mu\nu} &= {}^{(0)}g_{\mu\nu} + \sum_{n=1}^{\infty} {}^{(n)}h_{\mu\nu}, \\ N_I &= {}^{(0)}N_I + \sum_{n=1}^{\infty} {}^{(n)}N_I, \end{aligned} \tag{1.27}$$

where ${}^{(n)}F_I$ denotes a quantity of order $\mathcal{O}(1/c^n)$. The background fields ${}^{(0)}g_{\mu\nu}$ and ${}^{(0)}N_I$ are constrained by the cosmological boundary conditions.

The orders of the metric components are given by

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}h_{00} + {}^{(4)}h_{00} + \mathcal{O}\left(1/c^6\right), \\ g_{0i} &= {}^{(3)}h_{0i} + \mathcal{O}\left(1/c^5\right), \\ g_{ij} &= \delta_{ij} + {}^{(2)}h_{ij} + \mathcal{O}\left(1/c^4\right). \end{aligned} \tag{1.28}$$

Depending on the rank of the tensor fields N_I, the orders of their components have to be chosen accordingly. Generally, each time index corresponds to even orders, whereas each space index corresponds to an odd order, and scalar fields are of even order.

The field equations (1.13) are expanded in powers of $1/c$ by means of the ansatz (1.27). This typically leads to linear elliptic partial differential equations for the expansion coefficients. At least formally, these can be solved recursively.

The PPN expansion is an important tool to test MGTs with solar system measurements [64]. However, its classical form is not adapted to investigate theories with a non–relativistic limit which is not Newtonian, such as $f(R)$ gravity. The first relativistic order of the $1/c$ expansion of $f(R)$ gravity for the case (1.25) is discussed in detail in Chapter 4.

1.4.2 Linearised Metric Gravity

Linearised metric gravity is a useful tool to examine weak gravitational waves, which are discussed in detail in Chapter 2. As for the $1/c$ expansion, the linearisation is carried out around background fields ${}^{B}g_{\mu\nu}$ and ${}^{B}N_I$ that are constrained by the cosmological boundary conditions. We write the gravitational fields as the sum of the background fields and the perturbation fields,

$$\begin{aligned} g_{\mu\nu} &= {}^{B}g_{\mu\nu} + h_{\mu\nu}, \\ N_I &= {}^{B}N_I + K_I. \end{aligned} \tag{1.29}$$

The perturbations are assumed to be small compared to the background, that is

$$|h_{\mu\nu}|/|{}^{B}g_{\mu\nu}| \ll 1 \tag{1.30}$$

and

$$\begin{aligned} |K_I|/|{}^{B}N_I| &\ll 1, \quad \text{if} \quad |{}^{B}N_I| \neq 0, \\ |K_I| &\ll 1, \quad \text{if} \quad |{}^{B}N_I| = 0. \end{aligned} \tag{1.31}$$

Generally, the linearised equations are linear hyperbolic partial differential equations for the perturbation components $h_{\mu\nu}$ and K_I. In Chapter 2 we will see that linearised

GR with Minkowski background ${}^{B}g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ looks formally similar to electrodynamics. It states that dynamical variations of $h_{\mu\nu}$ propagate at the speed of light, and the corresponding waves feature two polarisation modes. In contrast, the linearised field equations of every known viable generalisation of GR are mostly complicated. Such theories usually predict additional polarisation modes. Some of them, such as $f(R)$ gravity, feature fields that propagate at a velocity slower than c.

Chapter 2

Gravitational Waves

It is natural to ask how the varying gravitational field generated by a nonstatic energy momentum density propagates through spacetime. In Newtonian gravity theory, the gravitational field acts instantaneously. Of course, such a behaviour is incompatible with the idea of Relativity theories. In Special Relativity the transmission velocity of any information, in particular the one carried by interaction fields, can not exceed the speed of light c. From this point of view, we arrive naturally at the phenomenon of gravitational waves as propagating dynamical gravitational fields. Indeed, as already mentioned at the end of the previous section, in GR weak perturbations of the metric propagate at velocity c on spacetime. In order to reasonably embed Relativity into its extension, we expect that the propagation velocity of gravitational waves in any MGT is at most the one of light.

While the detection of electromagnetic radiation such as the light of stars and galaxies is very common and even highly developed nowadays, it turned out to be a big challenge to measure weak gravitational waves predicted by linearised GR directly. This is due to the fact that, compared to the other three fundamental interactions, gravity is significantly weaker. Thus, even tiny perturbations governed by other interactions, for instance thermal noises, outrange the effects of a gravitational wave on a detector by far. Principally, the radiation of certain astrophysical events involving very massive objects, such as supernova collapses or the coalescence of two black holes, should be detectable by high precision measurements of the distance between two test masses. However, since the corresponding fields are expected to be very weak perturbations of the background spacetime, no attempt of direct measurement succeeded so far. Important experiments which are expected to obtain positive results are the Earth bound detector *Advanced LIGO* [26] or the future planned space observatory *LISA* [49].

In the first section, we will focus on the polarisations of gravitational waves propagating in a vacuum spacetime. In the second section we discuss the linearisations of GR and the scalar tensor theory inspired by $f(R)$ gravity.

2.1 Polarisations of Gravitational Waves

The gravitational fields can be measured by distances and angles and the time dependence thereof. Hence, an idealised gravity detector would be a set of test masses which are moving on geodesics and are uniformly distributed over a sphere initially. The various polarisations of gravitational waves lead to a set of characteristic dynamical deformations of the sphere which can be measured by the variation of the distances between all pairs of test masses.

Except for notations and conventions, the subsequent considerations follow the treatment of polarisations of gravitational waves in [21, 64]. Choose coordinates $(ct, \boldsymbol{x})$ induced by a local freely falling frame (see Section 1.1) and assume that the phase velocity of the gravitational wave is equal to c. We will work in geometrised units with $c = 1$ in this section exclusively. The evolution of the distance between two neighbouring geodesics is governed by the geodesic deviation formula for a separation vector $(0, \boldsymbol{n})$,

$$\frac{d^2 n^i}{dt^2} = -R^i{}_{0j0} n^j. \tag{2.1}$$

The polarisation is thus uniquely determined by the Riemann tensor. Consider a plane gravitational wave which is propagating in the direction x^3. Because of the planarity of the wave, the Riemann tensor components are functions of the retarded time $u = t - x^3$ solely,

$$R_{\mu\nu\lambda\rho} \equiv R_{\mu\nu\lambda\rho}(u), \tag{2.2}$$

whereas they are constant along the hyperplane orthogonal to the null direction normal to the wave vector $\partial^\mu u = (-1, 0, 0, -1)$. In order to investigate the properties of a Riemann tensor which satisfies (2.2), it is useful to introduce a nulltetrad which contains a vector proportional to the wave vector. With respect to the coordinate basis, this tetrad can be written as

$$\begin{aligned} k^\mu &= \frac{1}{\sqrt{2}}(1, 0, 0, 1), & \ell^\mu &= \frac{1}{\sqrt{2}}(1, 0, 0, -1), \\ m^\mu &= \frac{1}{\sqrt{2}}(0, 1, i, 0), & \bar{m}^\mu &= \frac{1}{\sqrt{2}}(0, 1, -i, 0). \end{aligned} \tag{2.3}$$

We index the tensor components with respect to this tetrad by the respective characters $k, \ell, m, \bar{m}$. In Appendix A we show by algebraic considerations that the Riemann tensor with the property (2.2) is determined by the four independent tetrad components

$$\begin{aligned} R_{\ell m \ell m},\, R_{\ell k \ell m} &\in \mathbb{C}\,, \\ R_{\ell m \ell \bar{m}},\, R_{\ell k \ell k} &\in \mathbb{R}\,, \end{aligned} \tag{2.4}$$

whereas the other components vanish. Up to constants and complex conjugation, the components (2.4) can be identified with four of the so–called Newman–Penrose quantities [42, 21, 64]. We can express the components (2.4) in terms of the six real coordinate components R_{i0j0},

$$R_{\ell m \ell m} = R_{1010} - R_{2020} + 2iR_{1020}, \tag{2.5}$$

$$R_{\ell k \ell m} = R_{1030} + iR_{2030}, \tag{2.6}$$

$$R_{\ell m \ell \bar{m}} = R_{1010} + R_{2020}, \tag{2.7}$$

$$R_{\ell k \ell k} = R_{3030}. \tag{2.8}$$

The polarisation modes are illustrated in Figure 2.1. The transversal modes given by (2.5) are the well–known $+$– and $\times$–modes of gravitational waves in GR. In a general MGT, there are up to four more polarisation modes, one transversal mode given by (2.7) and three longitudinal modes given by (2.6) and (2.8).

In general, the tetrad components (2.4) are not Lorentz invariant. Therefore, different observers possibly measure different polarisations. In Appendix B we determine

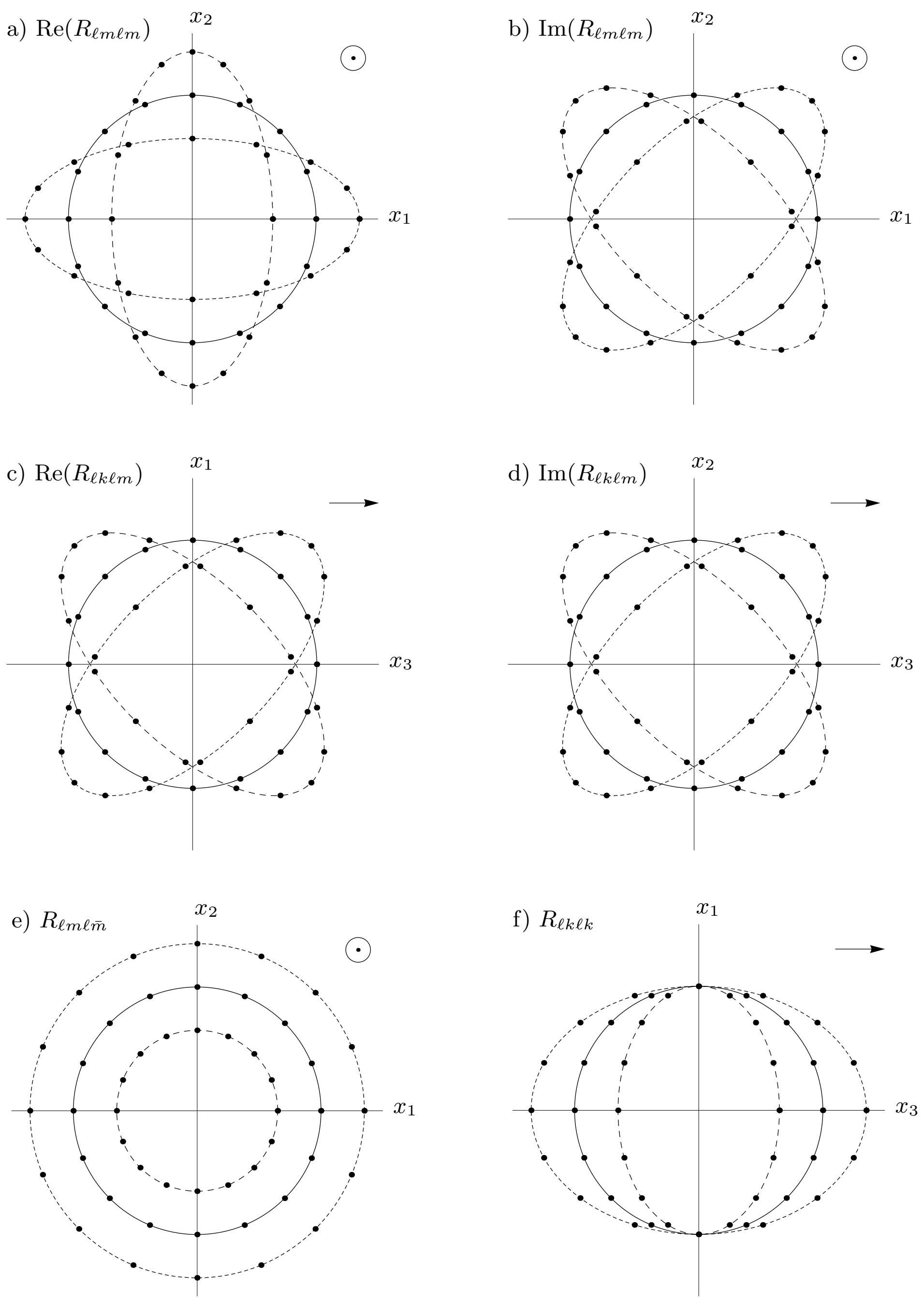

Figure 2.1: Polarisation modes of a plane gravitational wave in MGTs. Each mode induces a characteristic deformation of a sphere of uniformly distributed test masses. The wave propagates in x_3-direction and has time dependence $\sin\omega t$. Shown are cross sections through the sphere at $\omega t = 0$ (solid line), $\omega t = \pi/2$ (short–dashed line) and $\omega t = 3\pi/2$ (long–dashed line). There is no deformation perpendicular to the plane of the figure. In a), b) and e) the wave propagates out of the plane; in c), d) and f) the wave propagates in the plane to the right.

the most general Lorentz transformation that leaves k^μ unchanged. With respect to the tetrad (2.3), its matrix reads

$$L^\mu{}_\nu = \begin{pmatrix} 1 & \beta\bar{\alpha} & \beta e^{ir} & \bar{\beta}e^{-ir} \\ 0 & 1 & 0 & 0 \\ 0 & \bar{\beta} & e^{ir} & 0 \\ 0 & \beta & 0 & e^{-ir} \end{pmatrix}. \tag{2.9}$$

The parameter $\beta \in \mathbb{C}$ produces null rotations about k^μ, whereas the phase $r \in \mathbb{R}$ produces a rotation about the x_3-axis. The components (2.4) transform under L as

$$\begin{aligned} R_{\ell k\ell k} &\mapsto R_{\ell k\ell k}, \\ R_{\ell k\ell m} &\mapsto e^{ir}\left(R_{\ell k\ell m} + \beta R_{\ell k\ell k}\right), \\ R_{\ell m\ell m} &\mapsto e^{2ir}\left(R_{\ell m\ell m} + 2\beta R_{\ell k\ell m} + \beta^2 R_{\ell k\ell k}\right), \\ R_{\ell m\ell\bar{m}} &\mapsto R_{\ell m\ell\bar{m}} + \beta\bar{R}_{\ell k\ell m} + \bar{\beta}R_{\ell k\ell m} + \beta\bar{\beta}R_{\ell k\ell k}. \end{aligned} \tag{2.10}$$

Incidentally, the helicities s of the corresponding polarisation states can be read off from the prefactor e^{isr}. The transformation laws (2.10) motivate a classification of gravitational waves as shown in Table 2.1. The classes are labelled by the Petrov type corresponding to the non–vanishing Newman–Penrose quantity. The index counts the maximal number of polarisation modes detected by any observer.

Table 2.1: Classification of gravitational waves in metric gravity theories.

Class	Condition
II_6	$R_{\ell k\ell k} \not\equiv 0$
III_5	$R_{\ell k\ell k} \equiv 0$, $R_{\ell k\ell m} \not\equiv 0$
N_3	$R_{\ell k\ell k} \equiv R_{\ell k\ell m} \equiv 0$, $R_{\ell m\ell m} \not\equiv 0$, $R_{\ell m\ell\bar{m}} \not\equiv 0$
N_2	$R_{\ell k\ell k} \equiv R_{\ell k\ell m} \equiv R_{\ell m\ell\bar{m}} \equiv 0$, $R_{\ell m\ell m} \not\equiv 0$
O_1	$R_{\ell k\ell k} \equiv R_{\ell k\ell m} \equiv R_{\ell m\ell m} \equiv 0$, $R_{\ell m\ell\bar{m}} \not\equiv 0$
O_0	$R_{\ell k\ell k} \equiv R_{\ell k\ell m} \equiv R_{\ell m\ell m} \equiv R_{\ell m\ell\bar{m}} \equiv 0$

The predicted consequences of the classes on the polarisation detections are:

- II_6: the non–vanishing of $R_{\ell k\ell k}$ is independent of the observer, while the non–vanishing or vanishing of the other components is observer dependent.
- III_5: the vanishing of $R_{\ell k\ell k}$ and the non–vanishing of $R_{\ell k\ell k}$ are independent of the observer, while the non–vanishing or vanishing of the other components is observer dependent.
- N_3, N_2, O_1 and O_0: the non–vanishing or vanishing of all the components is independent of the observer.

Generally, one can determine for each MGT its wave class. We will demonstrate this for the cases of GR and $f(R)$ theory. The non–vanishing tetrad components of the Ricci tensor and the Ricci scalar are given by

$$\begin{aligned} R_{\ell\ell} &= 2R_{\ell m\ell\bar{m}}, \qquad & R_{\ell k} &= R_{\ell k\ell k}, \\ R_{\ell m} &= R_{\ell k\ell m}, \qquad & R_{\ell\bar{m}} &= \bar{R}_{\ell m}, \end{aligned} \tag{2.11}$$

$$R = -2R_{\ell k\ell k}. \tag{2.12}$$

In GR, the homogeneous Einstein equations can be reduced to the equations $R_{\mu\nu} = 0$. Therefore, by the conditions (2.11), the only non–vanishing tetrad component of the Riemann tensor is $R_{\ell m \ell m}$. Hence, the GR class is N_2.

Since $f(R)$ gravity features modes that are not null, there is the need for a generalisation of the scheme presented above to wave velocities smaller than c. This is achieved in [64] with the result that the classification in Table 2.1 is still correct asymptotically. The modes that are not null in $f(R)$ gravity correspond to fields for which the scalar curvature R does not vanish identically. The condition $R \not\equiv 0$ is by (2.12) equivalent to $R_{\ell k \ell k} \not\equiv 0$, and $f(R)$ theory is thus of class II_6.

2.2 Weak Gravitational Waves in General Relativity and $f(R)$ Inspired Scalar Tensor Theory

We consider the linearisations around the Minkowski background of the EOM of GR and of $f(R)$ inspired scalar tensor theory. We will therefore assume the cosmological constant to vanish. Gravitational waves in GR with a non–vanishing cosmological constant are examined in chapter 3. The asymptotic value for the scalar field is chosen such that a rescaling of Newton's constant for scalar tensor theory is unnecessary. Motivated by quadratic $f(R)$ gravity (1.25), we will also use the dimensional parameter a. The ansatz (1.29) for the gravitational fields then reads

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\ \phi &= 1 + 2a\varphi. \end{aligned} \tag{2.13}$$

The indices are raised and lowered with $\eta_{\mu\nu}$. Moreover, we write $h := h^{\mu}{}_{\mu}$ and $\Box := \Box_{\eta}$.

2.2.1 General Relativity

The linearisation of Einstein's equations (1.17) is given by

$$\begin{aligned} \Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\lambda \partial_\mu h^{\lambda}{}_{\nu} - \partial_\lambda \partial_\nu h^{\lambda}{}_{\mu} & \\ - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} &= -\frac{16\pi G}{c^4} T_{\mu\nu}. \end{aligned} \tag{2.14}$$

We define the trace reversed metric perturbation by $\gamma_{\mu\nu} := h_{\mu\nu} - \eta_{\mu\nu} h/2$ and choose the so-called harmonic gauge, $\partial^{\nu} \gamma_{\mu\nu} = 0$. Then the equations (2.14) simplify to the decoupled inhomogeneous wave equations

$$\Box \gamma_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{2.15}$$

The general solution of (2.15) is

$$\gamma_{\mu\nu}(t, \boldsymbol{x}) = {}^{\mathrm{hom}}\gamma_{\mu\nu}(t, \boldsymbol{x}) + \frac{4G}{c^4} \int_{\mathbb{R}^3} d^3x' \frac{T_{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{2.16}$$

where the retarded potentials are special solutions of the inhomogeneous wave equations and describe the waves radiated by the energy momentum field $T_{\mu\nu}$. The solutions of the homogeneous part of (2.15) can be written as a linear superposition of plane waves,

$${}^{\mathrm{hom}}\gamma_{\mu\nu}(t, \boldsymbol{x}) = \mathrm{Re} \int d^4k \, A_{\mu\nu}(k^{\mu}) e^{i k_\mu x^\mu}. \tag{2.17}$$

The integration in (2.17) is taken over all wave vectors that satisfy $k_\mu k^\mu = 0$. The quantities $A_{\mu\nu}(k^\mu)$ are the complex components of a tensor which carries the information about the amplitude and the polarisation of the corresponding plane wave.

Consider a plane wave propagating in the x^3 direction. The harmonic gauge can be extended to the so-called transverse traceless gauge. In the corresponding coordinates the perturbation field is given by

$$h_{\mu\nu}(t, x^3) = h_+(t - x^3/c)A^+_{\mu\nu} + h_\times(t - x^3/c)A^\times_{\mu\nu}, \tag{2.18}$$

where the polarisation tensors are

$$A^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.19}$$

The wave is thus a superposition of the polarisation states a) and b) shown in Figure 2.1.

2.2.2 $f(R)$ Inspired Scalar Tensor Theory

We assume f to be of the form (1.25). The potential (1.24) then evaluates to $U(\phi) = 1/(4a)(\phi - 1)^2$. Defining $\alpha := \sqrt{1/(6a)}$ and assuming $\omega(\phi) \equiv 0$, the linearisations of the field equations (1.22) and (1.23) read

$$\begin{aligned} \Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu - \eta_{\mu\nu}\Box h \\ + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} - \frac{1}{3\alpha^2}\left(\eta_{\mu\nu}\Box\varphi - \partial_\mu\partial_\nu\varphi\right) &= -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad (2.20) \\ \Box\varphi - \alpha^2\varphi &= \frac{8\pi G\alpha^2}{c^4}T. \quad (2.21) \end{aligned}$$

We define the quantity $\theta_{\mu\nu} := h_{\mu\nu} - (h/2)\eta_{\mu\nu} - 2a\varphi\eta_{\mu\nu}$ and choose a gauge such that $\partial^\nu\theta_{\mu\nu} = 0$. Then, as in GR, the equations (2.20) can be decoupled into the inhomogeneous wave equations

$$\Box\theta_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{2.22}$$

and the general solution is analogously given by

$$\theta_{\mu\nu}(t, \boldsymbol{x}) = {}^{\mathrm{hom}}\theta_{\mu\nu}(t, \boldsymbol{x}) + \frac{4G}{c^4}\int_{\mathbb{R}^3} d^3x' \frac{T_{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{2.23}$$

The Green's function of a Klein–Gordon equation of the form (2.21) is given by

$$\mathcal{G}(t, \boldsymbol{x}) = -\frac{1}{4\pi}\left[\frac{\delta\left(t - |\boldsymbol{x}|/c\right)}{|\boldsymbol{x}|} - \frac{\alpha J_1\left(\alpha c\sqrt{t^2 - (|\boldsymbol{x}|/c)^2}\right)H\left(t - |\boldsymbol{x}|/c\right)}{\sqrt{t^2 - (|\boldsymbol{x}|/c)^2}}\right], \tag{2.24}$$

where J_1 denotes the Bessel function of first order, δ the Dirac delta distribution and H the Heaviside distribution. Hence, the general solution of equation (2.21) can be written as

$$\begin{aligned}\varphi(t,\boldsymbol{x}) &= {}^{\mathrm{hom}}\varphi(t,\boldsymbol{x}) + \frac{2G\alpha^2}{c^4}\left[\int_{\mathbb{R}^3} d^3x' \frac{T\left(t-|\boldsymbol{x}-\boldsymbol{x}'|/c,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right. \qquad (2.25)\\ &\left.- \int_{-\infty}^{t-|\boldsymbol{x}-\boldsymbol{x}'|/c} dt' \int_{\mathbb{R}^3} d^3x' \frac{\alpha J_1\left(\alpha c\sqrt{(t-t')^2-(|\boldsymbol{x}-\boldsymbol{x}'|/c)^2}\right)}{\sqrt{(t-t')^2-(|\boldsymbol{x}-\boldsymbol{x}'|/c)^2}} T(t,\boldsymbol{x}')\right],\end{aligned}$$

The special solution in (2.25) may be interpreted by inspection of (2.24) as follows. On the one hand, a single excitation emitted by the source leads to a excitation that propagates with velocity c, whereas its amplitude is diminished by the inverse of the distance to the source. On the other hand, there exists a wake which follows the excitation at a slower velocity. The wake is represented by the last term on the right hand side of (2.24). For large $|z|$, the Bessel function $J_n(z)$ can be approximated as

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\left[\cos(z - n\pi/2 - \pi/4) + e^{|\mathrm{Im}(z)|}\mathcal{O}(|z|^{-1})\right], \qquad (2.26)$$

and the wake decays by the factor $(t^2 - (|\boldsymbol{x}|/c)^2)^{-3/4}$. The general solutions of the homogeneous part of (2.21) is a superposition

$${}^{\mathrm{hom}}\varphi(t,\boldsymbol{x}) = \mathrm{Re}\int d^4\bar{k}\, B(\bar{k}^\mu)e^{i\bar{k}_\mu x^\mu}, \qquad (2.27)$$

where the integration is taken over all vectors that satisfy $\bar{k}_\mu\bar{k}^\mu = \alpha^2$, and $B(\bar{k}^\mu)$ are complex amplitudes.

Since the wave vector $\bar{k}^\mu$ is not null, it is not possible to achieve the transversality for the full perturbation field. Moreover, for a nonvanishing φ the field can not be traceless. However, there exists an extension of the gauge chosen above such that the metric perturbation for a plane wave propagating in the x^3 direction can be put into the form [10, 17]

$$h_{\mu\nu}(t,x^3) = h_+(t - x^3/c)A^+_{\mu\nu} + h_\times(t - x^3/c)A^\times_{\mu\nu} + \frac{1}{3\alpha^2}\varphi(t,x^3)\eta_{\mu\nu}. \qquad (2.28)$$

The last term corresponds to a superposition of the polarisation modes e) and f) illustrated in Figure 2.1.

The theory of weak gravitational waves in scalar tensor theory motivated by $f(R)$ gravity is applied in Chapter 5 to investigate the gravitational radiation emitted by an isolated system.

Chapter 3

On Gravitational Waves in Spacetimes with a Nonvanishing Cosmological Constant

This chapter has been published in [41].

3.1 Introduction

The discovery that the expansion of the Universe is accelerating [47], which can be interpreted as due to a cosmological constant Λ, has triggered a lot of recent works with the aim to study how Λ affects e. g. celestial mechanics and the motion of massive bodies. In principle the cosmological constant should take part in phenomena on every physical scale. For instance, it has been studied which limits on Λ can be put from Solar System measurements, such as the effect on the perihelion precession of the Solar System's planets [28, 65, 31, 29, 54, 55, 27]. The cosmological constant could also influence gravitationallensing [50, 52] and play a role in the gravitational equilibrium of large astrophysical structures [1]. A natural question which arises is how the cosmological term affects gravitational waves. Clearly, we expect such an effect to be very tiny, nonetheless we believe that it is worthwhile to investigate it given the ongoing efforts in upgrading or building gravitational wave observatories either Earth bound or in space.

In this paper we study gravitational waves in spacetimes with a nonvanishing cosmological constant Λ in the framework of perturbation theory with respect to de Sitter (dS) and anti–de Sitter (AdS) metrics. There are few articles in the literature devoted to the question on how the cosmological constant affects gravitational waves. Some approaches consider exact solutions of the Einstein's equations with a cosmological term relying on the Kundt class of spacetimes, which admit a non-twisting and expansion–free null vectorfield [46, 4, 5, 2]. In [4, 5] these spacetimes are interpreted as plane gravitational waves with polarizations "+" and "×" which propagate on dS and AdS backgrounds.

A perturbative approach different from ours can be found in [36], where the Einstein equations with a cosmological term are linearised with respect to a Minkowski background metric. By choosing a particular non Hilbert gauge this leads then to a Klein–Gordon equation and thus to a nontrivial dispersion relation.

There exists a variety of works on the scalar wave equation in dS and Schwarzschild–dS spacetimes [48, 18, 66, 67, 6]. These treatments are, however, not directly connected to the present work, since the equations resulting from the linearisation of Einstein's equations are coupled partial differential equations for six independent variables.

The outline of the paper is as follows: in Section 3.2 we derive the linearised Einstein equations with respect to a dS or AdS background, which are represented by some generalized Klein–Gordon equations. Since a closed exact solution is not evident, we examine in Section 3.3 a perturbation expansion of these equations up to linear order with respect to Λ. In Section 3.4 we calculate the corresponding first order contributions to the amplitudes. The effects on directly measurable quantities are discussed in Section 3.5.

For the details of the linearisation of the Einstein equations with respect to an arbitrary differentiable background metric we refer e. g. to the textbooks [58, 34] or the review [23].

As far as notation is concerned: Greek letters denote spacetime indices and range from 0 to 3, whereas Latin letters denote space indices and range from 1 to 3. If not stated otherwise, we use geometrical units ($c = 1$ and $G = 1$).

3.2 Linearised Einstein's Equations with Cosmological Term

Let $(M, g_{\mu\nu})$ be a 4-dimensional pseudo Riemannian manifold with metric $g_{\mu\nu}$ of signature $(+, -, -, -)$. Let $R_{\mu\nu}$, and R, denote the Ricci tensor, and scalar, of $g_{\mu\nu}$, respectively. Then the vacuum Einstein equations with cosmological term read

$$R_{\mu\nu} - \left(\frac{R}{2} - \Lambda\right) g_{\mu\nu} = 0. \tag{3.1}$$

In what follows we consider a perturbed metric

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.2}$$

where $\tilde{g}_{\mu\nu}$ is a static background metric and $h_{\mu\nu}$ is a non–static perturbation with $|h_{\mu\nu}| \ll |\tilde{g}_{\mu\nu}|$. Up to first order in h the indices are uppered and lowered by $\tilde{g}_{\mu\nu}$. Indicating the unperturbed Riemann tensor by $\tilde{R}_{\mu\nu\lambda\rho}$ and consequently the Ricci tensor, and scalar, by $\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\nu\lambda}$, and $\tilde{R} = \tilde{R}^{\lambda}{}_{\lambda}$, respectively we can write the expansion of equation (3.1) up to linear order in $h_{\mu\nu}$ as

$$\tilde{R}_{\mu\nu} + R_{\mu\nu}(h) - \left(\frac{\tilde{R}}{2} + \frac{R(h)}{2} - \Lambda\right) \tilde{g}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) h_{\mu\nu} + \mathcal{O}(h^2) = 0, \tag{3.3}$$

where the linear contributions to the Ricci tensor [58] and Ricci scalar are

$$\begin{aligned} R_{\mu\nu}(h) &= \frac{1}{2}\Big(h^{\lambda}{}_{\mu;\nu;\lambda} + h^{\lambda}{}_{\nu;\mu;\lambda} - h_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - h^{\lambda}{}_{\lambda;\mu;\nu}\Big), \\ R(h) &= R^{\lambda}{}_{\lambda}(h) - h^{\lambda\rho}\tilde{R}_{\lambda\rho}. \end{aligned}$$

The semicolon denotes the covariant derivative with respect to $\tilde{g}_{\mu\nu}$. The terms in (3.3) which are independent of $h_{\mu\nu}$ satisfy equation (3.1) with $\tilde{g}_{\mu\nu}$,

$$\tilde{R}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) \tilde{g}_{\mu\nu} = 0. \tag{3.4}$$

The terms which are linear in $h_{\mu\nu}$ are determined by

$$R_{\mu\nu}(h) - \frac{R(h)}{2}\tilde{g}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) h_{\mu\nu} = 0. \tag{3.5}$$

In order to see explicitly the Klein–Gordon character of (3.5), we rewrite this equation using the expressions in equation (3.4) and the trace-reversed quantity $\tilde{\gamma}_{\mu\nu} := h_{\mu\nu} - \frac{h}{2}\tilde{g}_{\mu\nu}$, $h := h^{\lambda}{}_{\lambda}$. We are then left with [1]

$$\begin{aligned}\tilde{\gamma}_{\mu\nu}{}^{;\lambda}{}_{;\lambda} + \tilde{\gamma}_{\lambda\mu}{}^{;\lambda}{}_{;\nu} + \tilde{\gamma}_{\lambda\nu}{}^{;\lambda}{}_{;\mu} + 2\tilde{R}_{\lambda\mu\rho\nu}\tilde{\gamma}^{\lambda\rho} - \tilde{R}_{\lambda\mu}\tilde{\gamma}_{\nu}{}^{\lambda} - \tilde{R}_{\lambda\nu}\tilde{\gamma}_{\mu}{}^{\lambda} \\ - \tilde{R}_{\lambda\rho}\tilde{g}_{\mu\nu}\left(\tilde{\gamma}^{\lambda\rho} - \frac{\tilde{\gamma}^{\sigma}{}_{\sigma}}{2}\tilde{g}^{\lambda\rho}\right) + 2\Lambda\left(\tilde{\gamma}_{\mu\nu} - \frac{\tilde{\gamma}^{\lambda}{}_{\lambda}}{2}\tilde{g}_{\mu\nu}\right) &= 0,\end{aligned}$$

In contrast to the corresponding result for the Einstein's equations without cosmological term (where instead of eq.(3.5) we have $R_{\mu\nu}(h) = 0$ [58]), the equations (3.6) contain two additional terms on the left hand side.

In order to analyse further the equations (3.4) and (3.6) we fix the background as follows. It is well known that a dS and AdS metric, respcectively, solves the equations (3.1) exactly. For our purposes it is thus the natural choice for the background. We note that at this point a Schwarzschild–de Sitter solution might have been chosen as well. We avoid this since we are interested in a region of spacetime which is far from sources of gravitational radiation. We now choose an appropriate coordinate system for the background spacetime $(M, \tilde{g}_{\mu\nu})$. Let $p : I \subset \mathbb{R} \to M$ be the locus of an observer at rest [58] and let $\phi : M \to \mathbb{R}^4$, $m \mapsto (t, x, y, z)$ be a coordinate chart such that $\phi(p(t)) = (t, 0, 0, 0)$. An exact solution of (3.4) in the chart ϕ is given by

$$\tilde{g}_{00} = \left(\frac{1 - \frac{\Lambda}{12}r^2}{1 + \frac{\Lambda}{12}r^2}\right)^2, \qquad \tilde{g}_{ii} = \frac{-1}{\left(1 + \frac{\Lambda}{12}r^2\right)^2}, \qquad \tilde{g}_{ij} = 0 \quad (i \neq j), \tag{3.6}$$

where $r := \sqrt{x^2 + y^2 + z^2}$. The solution (3.6) is valid inside the null horizon $r^2 = 12/|\Lambda|$, which depends on the choice of the observer p. The apparent spacelike nature of the normal to this surface is due to the use of isotropic coordinates. For later use we denote the corresponding hypersurface in our coordinate chart by $\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid r^2 < 12/|\Lambda|\}$. However, for the following applications it suffices to consider a region which is much smaller than Ω. The metric (3.6) was first introduced in [20] and is therefore known as dS, and AdS metric according to $\Lambda > 0$ and $\Lambda < 0$, respectively. Its Riemann tensor is given by

$$\tilde{R}_{\mu\nu\lambda\rho} = \frac{\Lambda}{3}\left(\tilde{g}_{\mu\lambda}\tilde{g}_{\nu\rho} - \tilde{g}_{\mu\rho}\tilde{g}_{\nu\lambda}\right). \tag{3.7}$$

The equations (3.6) form a family of ten coupled generalised Klein–Gordon equations for which an algorithm providing closed solutions is not known. We point out that the high symmetry of dS and AdS, respectively, allows to derive exact solutions of equation

[1] At this point, it is possible to proceed with an approach that is slightly different to the one presented in [41]. In a short wave approximation [58] it is feasible to impose the Hilbert gauge condition $\tilde{\gamma}_{\mu\nu}{}^{;\nu} = 0$. Moreover, the terms of the form $\tilde{R}_{\lambda\mu}\tilde{\gamma}_{\nu}{}^{\lambda}$ can be neglected in this approximation. Using equation (3.7), we then can simplify (3.6) to $\tilde{\gamma}_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - (2\Lambda/3)\tilde{\gamma}_{\mu\nu} = 0$. From here, one might continue in the same way as in Section 3.3, whereby the gauge chosen there is automatically fulfilled to leading order. The result obtained from this approach differs from the solution (3.24) in the terms proportional to f and its primitives. The main term proportional to the derivative of f however remains unchanged.

(3.1) [4, 5]. Moreover, if we impose the Hilbert gauge condition $\tilde{\gamma}_{\mu\nu}{}^{;\nu} = 0$, then the trace of equation (3.6) turns into a simple Klein–Gordon equation for the trace of $\tilde{\gamma}_{\mu\nu}$ on dS and AdS space, respectively,

$$\tilde{\gamma}^{;\lambda}{}_{;\lambda} - \frac{2\Lambda}{3}\tilde{\gamma} = 0, \tag{3.8}$$

which may be solved exactly by using separation of variables [48]. However, since we are interested in the physical consequences of the cosmological constant for all the components $\tilde{\gamma}_{\mu\nu}$ (and not just for the trace) in the regime of a metric perturbation, equation (3.8) does not provide enough information. Moreover, the perturbed solutions derived below are traceless, thus only the trivial solution of (3.8) is relevant for our purposes.

Note that the contraction of the equations (3.6) with the stationary Killing field $(\partial_t)^\lambda$ might lead to simpler equations. However, the resulting equations are still non–scalar, as this is suggested by the equations (3.20) below. Thus the derivation of an analytic result, if possible, would be quite involved.

Although it would be useful to supplement the perturbative calculation below with analytic results in order to gain more confidence in the former, it seems that the effort for such a program would exceed the derivation of the perturbative results and definitely goes beyond the scope of the present work. We therefore content ourselves with an expansion of (3.6) with respect to Λ up to linear order.

We remark that such an expansion with respect to Λ is consistent with the expansion with respect to $h_{\mu\nu}$. Equation (3.4) may also be expanded with respect to Λ, and the coefficients of each order fulfill the equations subsequently. This point of view would correspond to a one–parameter perturbation of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \Lambda^n h^{(n)}_{\mu\nu}. \tag{3.9}$$

Each coefficient $h^{(n)}_{\mu\nu}$ could be written as

$$h^{(n)}_{\mu\nu} = \tilde{h}^{(n)}_{\mu\nu} + \bar{h}^{(n)}_{\mu\nu}, \tag{3.10}$$

where

$$\sum_{n=1}^{\infty} \Lambda^n \tilde{h}^{(n)}_{\mu\nu} = \tilde{g}_{\mu\nu} - \eta_{\mu\nu} \quad \text{and} \quad \sum_{n=1}^{\infty} \Lambda^n \bar{h}^{(n)}_{\mu\nu} = h_{\mu\nu}. \tag{3.11}$$

In other words, the $\tilde{h}^{(n)}_{\mu\nu}$ contain the contributions from the background, whereas the $\bar{h}^{(n)}_{\mu\nu}$ describe the waveform. Since Λ carries the physical unit of Length^{-2}, the n^{th} order coefficients of the expansions above carry the physical unit Length^{2n}.

3.3 Approximate Solution of the Linearised Equations

In particular the perturbation expansion with respect to Λ is based on the assumption $r \ll \sqrt{12/|\Lambda|}$. We collect terms proportional to Λ^n and denote them by $\mathcal{O}(\Lambda^n)$. For $r \ll \sqrt{12/|\Lambda|}$ equation (3.6) yields

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(\Lambda). \tag{3.12}$$

Then the connection coefficients are of order $\mathcal{O}(\Lambda)$ and so are $\tilde{R}_{\mu\nu\lambda\rho}$ and $\tilde{R}_{\mu\nu}$ by (3.7), such that equation (3.6) may be written as

$$\gamma_{\mu\nu}{}^{,\lambda}{}_{,\lambda} + \gamma_{\lambda\mu}{}^{,\lambda}{}_{,\nu} + \gamma_{\lambda\nu}{}^{,\lambda}{}_{,\mu} + \Lambda D_{\mu\nu}(\gamma) + \mathcal{O}(\Lambda^2) = 0, \tag{3.13}$$

where $\gamma_{\mu\nu} := h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}$, the comma denotes partial derivatives, the indices are uppered and lowered by $\eta_{\mu\nu}$, and $D_{\mu\nu}$ is a linear hyperbolic differential operator of second order. We are thus led to consider the variable $\gamma_{\mu\nu}$ instead of $\tilde{\gamma}_{\mu\nu}$. These variables differ if the trace h does notvanish. However, the solutions which are considered in the following sections are trace–free, and therefore they satisfy $\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} = h_{\mu\nu}$.

Hereafter we will neglect the terms of order $\mathcal{O}(\Lambda^2)$. Then Λ lends itself as expansion parameter for the following perturbation procedure.

Let $\Box := \partial^\lambda\partial_\lambda$ denote the d'Alembert operator. We assume that the exact solution of the equation

$$\Box\gamma_{\mu\nu} + \gamma_{\lambda\mu}{}^{,\lambda}{}_{,\nu} + \gamma_{\lambda\nu}{}^{,\lambda}{}_{,\mu} + \Lambda D_{\mu\nu}(\gamma) = 0 \tag{3.14}$$

can be written as a power series in Λ. Up to linear order we then have

$$\gamma_{\mu\nu} = \gamma^{(0)}_{\mu\nu} + \Lambda\gamma^{(1)}_{\mu\nu} + \mathcal{O}(\Lambda^2), \tag{3.15}$$

where the coefficients $\gamma^{(1)}_{\mu\nu}$ carry the physical unit (Length)^2. Thus a comparison to the case of a vanishing cosmological constant is achieved simply by considering only the zeroth order terms. Plugging (3.15) into (3.14) yields by comparing coefficients order by order

$$\begin{aligned} \Box\gamma^{(0)}_{\mu\nu} + \gamma^{(0)}_{\lambda\mu}{}^{,\lambda}{}_{,\nu} + \gamma^{(0)}_{\lambda\nu}{}^{,\lambda}{}_{,\mu} &= 0, \\ \Box\gamma^{(1)}_{\mu\nu} &= -D_{\mu\nu}(\gamma^{(0)}). \end{aligned} \tag{3.16}$$

On the first equation in (3.16) we impose the Hilbert gauge condition $\gamma_{\mu\nu}{}^{,\nu} = 0$ and afterwards choose the transverse traceless gauge. Thus the fundamental solutions are plane gravitational waves with the two linear polarization states "+" and "×".The solutions of the second equation in (3.16) are then determined by the expression

$$\gamma^{(1)}_{\mu\nu} = -\mathcal{G} * D_{\mu\nu}(\gamma^{(0)}), \tag{3.17}$$

where

$$\mathcal{G}(t,r) = \frac{\delta(t-r)\theta(t)}{4\pi r} \tag{3.18}$$

is the Green's function of the d'Alembert operator and the star denotes the convolution. The domain of integration in (3.17) is $\mathbb{R}^3$, which may be interpreted as lowest order approximation of Ω. In order to avoid divergences we need to choose an appropriate decrease for the amplitude of $\gamma^{(0)}_{\mu\nu}$ for $r \to \infty$. For our case a power counting argument requires the asymptotic behaviour $\left|\gamma^{(0)}_{\mu\nu}\right| \sim r^{-\alpha}$ for $\alpha > 2$. Accordingly one requires the boundary conditions

$$\lim_{r\to\infty} \gamma^{(1)}_{\mu\nu} = 0 \qquad \text{and} \qquad \lim_{r\to\infty} \gamma^{(1)}_{\mu\nu,\lambda} = 0. \tag{3.19}$$

However, in the following sections we will consider only a small region of spacetime, so that the question of the behaviour for $r \to \infty$ is not essential. We will, therefore, assume that the intersection of any spacelike hypersurface with the support of $\gamma_{\mu\nu}^{(0)}$ and thus the domain of integration in (3.17) is compact.

It is understood that in contrast to Minkowski space the notion of planarity of a wavefront has to be modified for waves in curved spacetime. In the framework of exact solutions of the Einstein's equations this is achieved by demanding that the spacetime admits a null vectorfield which is non-twisting and expansion-free.

However, in the perturbative approach we naturally assume that the wave front is a hyperplane up to lowest order. In a consistent perturbation expansion we are thus advised to assume that the fundamental solutions $\gamma_{\mu\nu}^{(0)}$ of the first equation in (3.16) is a Minkowski–plane wave. As mentioned above we restrict the support of $\gamma_{\mu\nu}^{(0)}$. In doing so we need to avoid further destruction of the symmetries of plane waves. Therefore we choose the domain of integration in (3.17) to be spherically symmetric and indicate it by $\Omega_{\mathcal{R}} := \{(x, y, z) \in \mathbb{R}^3 \mid r < \mathcal{R}\}$.

3.4 Plane Wave Propagation

We now choose the coordinate chart ϕ such that $\gamma_{\mu\nu}^{(0)}$ is a plane transverse traceless solution and the non–vanishing components are $\gamma_{11}^{(0)}$, $\gamma_{22}^{(0)} = -\gamma_{11}^{(0)}$ and $\gamma_{12}^{(0)}$. These components are functions of the retarded time $z - t$ and describe thus a plane wave propagating in z–direction. The non–vanishing components of $D_{\mu\nu}(\gamma^{(0)})$ are then given by

$$\begin{aligned}
D_{01}(\gamma^{(0)}) &= \frac{7}{6}\left(x\partial_t\gamma_{11}^{(0)} + y\partial_t\gamma_{12}^{(0)}\right), \qquad D_{02}(\gamma^{(0)}) = \frac{7}{6}\left(x\partial_t\gamma_{12}^{(0)} - y\partial_t\gamma_{11}^{(0)}\right), \\
D_{11}(\gamma^{(0)}) &= \frac{r^2}{6}\left(2\partial_{tt}\gamma_{11}^{(0)} - \partial_{zz}\gamma_{11}^{(0)}\right) - \frac{z}{6}\partial_z\gamma_{11}^{(0)} + \frac{2}{3}\gamma_{11}^{(0)}, \\
D_{22}(\gamma^{(0)}) &= -D_{11}(\gamma^{(0)}), \\
D_{12}(\gamma^{(0)}) &= \frac{r^2}{6}\left(2\partial_{tt}\gamma_{12}^{(0)} - \partial_{zz}\gamma_{12}^{(0)}\right) - \frac{z}{6}\partial_z\gamma_{12}^{(0)} + \frac{2}{3}\gamma_{12}^{(0)}, \\
D_{13}(\gamma^{(0)}) &= \frac{5}{6}\left(x\partial_z\gamma_{11}^{(0)} + y\partial_z\gamma_{12}^{(0)}\right), \qquad D_{23}(\gamma^{(0)}) = \frac{5}{6}\left(x\partial_z\gamma_{12}^{(0)} - y\partial_z\gamma_{11}^{(0)}\right),
\end{aligned} \tag{3.20}$$

We now restrict ourselves to the investigation of the contributions to the "+"–mode of $\gamma^{(0)}$. An analogous result may be derived for the "×"–mode. We have $\gamma_{11}^{(0)} = f(z - t)$ and $\gamma_{12}^{(0)} = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth function. Equations (3.20) yield

$$\begin{aligned}
D_{01}(\gamma^{(0)}) &= -\frac{7x}{6}f'(z - t), \qquad D_{02}(\gamma^{(0)}) = \frac{7y}{6}f'(z - t), \qquad (3.21) \\
D_{11}(\gamma^{(0)}) &= \frac{r^2}{6}f''(z - t) - \frac{z}{6}f'(z - t) + \frac{2}{3}f(z - t), \\
D_{22}(\gamma^{(0)}) &= -D_{11}(\gamma^{(0)}), \qquad D_{12}(\gamma^{(0)}) = 0, \\
D_{13}(\gamma^{(0)}) &= \frac{5x}{6}f'(z - t), \qquad D_{23}(\gamma^{(0)}) = -\frac{5y}{6}f'(z - t).
\end{aligned}$$

Thus we are able to calculate the first order corrections by using formula (3.17), i. e.

$$\gamma_{\mu\nu}^{(1)}(t, \vec{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathcal{R}}} \frac{D_{\mu\nu}(\gamma^{(0)})\left(t - |\vec{x} - \vec{\xi}|, \vec{\xi}\right)}{|\vec{x} - \vec{\xi}|}\, d^3\xi. \tag{3.22}$$

In particular, all components vanish except

$$\begin{aligned}\gamma_{11}^{(1)}(t,\vec{x}) &= \frac{1}{24\pi}\int_{\Omega_{\mathcal{R}}}\Bigg[(\vec{\xi})^2 f''\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right) \\ &\qquad - \xi_3 f'\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right) \\ &\qquad + 4f\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right)\Bigg]\frac{d^3\xi}{|\vec{x}-\vec{\xi}|} \\ \gamma_{22}^{(1)}(t,\vec{x}) &= -\gamma_{11}^{(1)}(t,\vec{x}).\end{aligned} \tag{3.23}$$

This result indicates that in contrast to the amplitude the polarization remains unchanged up to this order, thus preserving the quadrupole character of gravitational radiation. Though an evaluation of the equation (3.23) in general can hardly be carried out analytically for arbitrary events $(t,\vec{x})$, it still may be computed along the locus $p(t)$ of the observer using spherical coordinates. We now introduce physical units. Let $\gamma_{11}(t,\vec{0}) = f(\omega t)$, where ω denotes a frequency, and let c denote the speed of light. Then the non–vanishing components of the perturbation $h_{\mu\nu}$ in (3.2) are determined by

$$\begin{aligned}h_{11}(t,\vec{0}) &= \gamma_{11}(t,\vec{0}) \approx \left(\gamma_{11}^{(0)}+\Lambda\gamma_{11}^{(1)}\right)(t,\vec{0}) \\ &= f(\omega t)+\frac{\Lambda}{24\pi}\Bigg[\frac{\mathcal{R}^3\omega}{3c}f'(-\omega t)+\frac{\mathcal{R}^2}{2}\left(f(-\omega t)-f\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right) \\ &\qquad -\frac{\mathcal{R}c}{\omega}\left(5f^{\uparrow}(-\omega t)-f^{\uparrow}\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right) \\ &\qquad -\frac{2c^2}{\omega^2}\left(f^{\uparrow\uparrow}(-\omega t)-f^{\uparrow\uparrow}\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right)\Bigg],\end{aligned} \tag{3.24}$$

where we have denoted the primitive of any function $g : D \subset \mathbb{R} \to \mathbb{R}$ by

$$g^{\uparrow}(t) := \int^t g(t')dt'. \tag{3.25}$$

Due to the parameter $\mathcal{R}$, the formula (3.24) is not yet in a form which allows an immediate meaningful physical interpretation. A priori $\mathcal{R}$ is a positive real number which measures the dimension of the support of $\gamma_{\mu\nu}^{(0)}$ in Minkowski spacetime. A posteriori we gather from equation (3.24) that the perturbation expansion is reasonable if

$$\lim_{\Lambda\to 0}\frac{\Lambda\mathcal{R}^3\omega}{c} = \lim_{\Lambda\to 0}\Lambda\mathcal{R}^2 = \lim_{\Lambda\to 0}\frac{\Lambda\mathcal{R}c}{\omega} = \lim_{\Lambda\to 0}\frac{\Lambda c^2}{\omega^2} = 0. \tag{3.26}$$

Thus we formally obtain Λ–dependent constraints on $\mathcal{R}$ and ω. If we impose the the geometrical optics limit, $\mathcal{R} \gg c/\omega$, we have

$$\frac{\mathcal{R}^3\omega}{c} \gg \mathcal{R}^2 \gg \frac{\mathcal{R}c}{\omega} \gg \frac{c^2}{\omega^2}, \tag{3.27}$$

so that all the limites in (3.26) follow from the first one. In Section 3.5 we give more comments on the interpretation of $\mathcal{R}$. In particular we find that for our purposes we can assume $\mathcal{R} \ll 1/\sqrt{|\Lambda|}$. Since ω is a constant parameter, the limits in (3.26) are fullfilled.

The condition $\left|\Lambda\gamma_{11}^{(1)}\right| \ll \left|\gamma_{11}^{(0)}\right|$ yields then $\Lambda\mathcal{R}^3\omega/c \ll 1$, which gives an upper bound on ω.

As an illustration of the above results we consider the example $f(\omega t) := \sin(\omega t)$. Due to the conditions in equation (3.27) we can neglect the terms with coefficients proportional to c^2/ω^2, $\mathcal{R}c/\omega$ and $\mathcal{R}^2$, so that to leading order we get

$$h_{11}(t,\vec{0}) \quad\approx\quad \sin(\omega t) + \frac{\Lambda\mathcal{R}^3\omega}{72\pi c}\cos(\omega t). \tag{3.28}$$

Since $\Lambda\mathcal{R}^3\omega/c \ll 1$, this can also be written as

$$h_{11}(t,\vec{0}) \quad\approx\quad \sin\left[\omega\left(t + \frac{\Lambda\mathcal{R}^3}{72\pi c}\right)\right]. \tag{3.29}$$

For a periodic $\gamma_{\mu\nu}^{(0)}$, equation (3.28) shows that the correction $\gamma_{\mu\nu}^{(1)}$ features a modified amplitude, whereas (3.29) yields a modification of the frequency. In the following section we show that $\mathcal{R}$ depends on the proper time of the observer. In general the frequency therefore changes with varying time.

3.5 Effects on Measurable Quantities

The coordinate data in the in this section corresponds to the lowest order approximation of the chart ϕ, which represents a Minkowski background. Consider a source which starts to emit gravitational radiation at some event $(-t_0, \vec{x}_0)$ so that an observer at large distance $|\vec{x}_0| = t_0$ would start to perceive an approximate plane wave at the event $(0,\vec{0})$. Assume that the wave at this event had the shape of the function f up to lowest order. Let the observer at $p(t)$ carry out a measurement during a time interval $[0,\tau]$, such that $\tau \ll t_0$. In addition to the wave f, the observer would measure increasing retarded contributions $\gamma_{\mu\nu}^{(1)}$ with increasing τ. These contributions originate from a spherical region within $r \leq \tau$. For the present measurement we thus have $\mathcal{R} = \tau$. Reasonably we have $\tau \ll 1/\sqrt{|\Lambda|}$ and therefore $\mathcal{R} \ll 1/\sqrt{|\Lambda|}$. For $\Lambda \approx 10^{-52}\text{m}^{-2}$ this yields

$$\tau \ll 10^{18}\text{s} \approx 10^{11}\text{yr}. \tag{3.30}$$

Let τ_{yr} denote the length of the measurement in years, and let $c \approx 3\cdot 10^8\text{m/s}$. Then the condition

$$\Lambda c^2\tau^3\omega \ll 1 \tag{3.31}$$

and the geometrical optics limit $\tau \gg 1/\omega$ yield the following constraints on ω:

$$\frac{1}{\tau_{\text{yr}}}\cdot 10^{-7}\text{Hz} \ll \omega \ll \frac{1}{\tau_{\text{yr}}^3}\cdot 10^{15}\text{Hz}. \tag{3.32}$$

The condition (3.30) implies a non–vanishing range for the parameter ω in (4.55). For τ ranging from a couple of minutes up to several thousands of years, the radiation emitted by typical sources of gravitational waves features frequencies in this range.

The measurement via the equation for geodesic deviation is carried out analogously to the case $\Lambda = 0$ (cf. [58], e. g.). We have

$$\frac{d^2n^i}{dt^2} = -R^i{}_{00j}n^j, \tag{3.33}$$

where $\vec{n} = (n^1, n^2, n^3)$ is the separation vector between two neighbouring members of a congruence of timelike geodesics [58]. We expand the Riemann tensor with respect to the perturbation $h_{\mu\nu}$:

$$R_{\mu\nu\lambda\rho} = \tilde{R}_{\mu\nu\lambda\rho} + R_{\mu\nu\lambda\rho}(h) + \mathcal{O}(h^2), \tag{3.34}$$

where the linear contribution to the Riemann tensor is given by [34]

$$R^{\mu}{}_{\nu\lambda\rho}(h) = \frac{1}{2}\Big(h^{\mu}{}_{\nu;\rho;\lambda} + h^{\mu}{}_{\rho;\nu;\lambda} - h_{\nu\rho}{}^{;\mu}{}_{;\lambda} - h^{\mu}{}_{\nu;\lambda;\rho} - h^{\mu}{}_{\lambda;\nu;\rho} + h_{\nu\lambda}{}^{;\mu}{}_{;\rho}\Big). \tag{3.35}$$

For any measurement it is always possible to configure the detector such that it is sensitive only to the "+"–mode of the wave [23]. We assume that this is the case in the following paragraphs. Therefore, in the present case we consider only the following components of $R_{\mu\nu\lambda\rho}$:

$$\begin{aligned} \tilde{R}^{i}{}_{00j} &= -\frac{\Lambda}{3}\delta^{i}_{j} + \mathcal{O}(\Lambda^2) \qquad (3.36) \\ R^{1}{}_{001}(h) &= -R^{2}{}_{002}(h) \\ &= -\frac{1}{2}\left[\partial_{tt}\gamma^{(0)}_{11} + \Lambda\left(\partial_{tt}\gamma^{(1)}_{11} + \frac{1}{3}\vec{x}\cdot\nabla\gamma^{(0)}_{11}\right)\right] + \mathcal{O}(\Lambda^2). \end{aligned}$$

Along the locus of $p(t)$ the components of equation (3.33) thus read

$$\begin{aligned} \frac{d^2n^1}{dt^2} &= \left[\frac{1}{2}\frac{d^2\gamma^{(0)}_{11}}{dt^2} + \Lambda\left(\frac{1}{3} + \frac{1}{2}\frac{d^2\gamma^{(1)}_{11}}{dt^2}\right)\right]n^1, \qquad (3.37) \\ \frac{d^2n^2}{dt^2} &= \left[-\frac{1}{2}\frac{d^2\gamma^{(0)}_{11}}{dt^2} + \Lambda\left(\frac{1}{3} - \frac{1}{2}\frac{d^2\gamma^{(1)}_{11}}{dt^2}\right)\right]n^2, \\ \frac{d^2n^3}{dt^2} &= \frac{\Lambda}{3}n^3. \end{aligned}$$

Let $n^i(t) = n^i_{(0)} + \delta n^i(t)$ with $|\delta n^i(t)| \ll |n^i_{(0)}|$. We simplify the notation by setting $\gamma^{(i)}_{11}(t) \equiv \gamma^{(i)}_{11}(t, \vec{0})$. Since $\gamma^{(1)}_{11}(0) = \frac{d\gamma^{(1)}_{11}}{dt}(0) = 0$ we are then left with

$$\begin{aligned} \frac{n^1(\tau)}{n^1_{(0)}} &\approx 1 + \frac{\delta n^1(0)}{n^1_{(0)}} - \frac{1}{2}\gamma^{(0)}_{11}(0) + \tau\left(\frac{1}{n^1_{(0)}}\frac{d(\delta n^1)}{dt}(0) - \frac{1}{2}\frac{d\gamma^{(0)}_{11}}{dt}(0)\right) \qquad (3.38) \\ &\quad + \frac{1}{2}\gamma^{(0)}_{11}(\tau) + \Lambda\left(\frac{\tau^2}{6} + \frac{1}{2}\gamma^{(1)}_{11}(\tau)\right), \\ \frac{n^2(\tau)}{n^2_{(0)}} &\approx 1 + \frac{\delta n^2(0)}{n^2_{(0)}} + \frac{1}{2}\gamma^{(0)}_{11}(0) + \tau\left(\frac{1}{n^2_{(0)}}\frac{d(\delta n^2)}{dt}(0) + \frac{1}{2}\frac{d\gamma^{(0)}_{11}}{dt}(0)\right) \\ &\quad - \frac{1}{2}\gamma^{(0)}_{11}(\tau) + \Lambda\left(\frac{\tau^2}{6} - \frac{1}{2}\gamma^{(1)}_{11}(\tau)\right), \\ \frac{n^3(\tau)}{n^3_{(0)}} &\approx 1 + \frac{\delta n^3(0)}{n^3_{(0)}} + \frac{\tau}{n^3_{(0)}}\frac{d(\delta n^3)}{dt}(0) + \frac{\Lambda\tau^2}{6}. \end{aligned}$$

The contributions from the background thus induce an isotropic dilatation proportional to τ^2 which reflects the expansion of the universe. These terms may also be derived from the coefficient $\tilde{h}^{(1)}_{\mu\nu}$ in equation (3.10). From equation (3.24) we deduce that for $\mathcal{R} = \tau$ the dominant term in $\gamma^{(1)}_{\mu\nu}(\tau)$ is proportional to τ^3. In addition to a modification of the amplitude, for a periodic $\gamma^{(0)}_{\mu\nu}(\tau)$ this term leads to a loss of periodicity of the zeros of $\delta n^i(\tau)$. The term proportional to τ features the same consequences, whereas the term proportional to τ^2 only affects the amplitude.

In the following example we again introduce physical units and illustrate the qualitative behaviour of $\delta n^1(\tau)$. Consider a source which starts to emit a wave at an event $(-ct_0, 0, 0, z_0)$ with $ct_0 = |z_0|$ and $t_0 \gg \tau$. Let the source emit radiation during a time interval of length s. Moreover, assume that at the event $(0, \vec{0})$ the observer would perceive a sine wave up to lowest order. Then

$$\gamma^{(0)}_{11}(\omega t) = \varphi(\omega t) := \begin{cases} \sin(\omega t), & 0 \leq t \leq s \\ 0, & \text{otherwise.} \end{cases} \tag{3.39}$$

We choose the initial conditions

$$\begin{aligned} \delta n^1(0) &= \frac{n^1_{(0)}}{2}\gamma^{(0)}_{11}(0) \qquad \text{and} \\ \frac{d(\delta n^1)}{dt}(0) &= \frac{n^1_{(0)}}{2}\frac{d\gamma^{(0)}_{11}}{dt}(0). \end{aligned} \tag{3.40}$$

Equation (3.24) with $\tau = \mathcal{R}$ and $f = \varphi$ then leads to

$$\begin{aligned} \delta n^1(\tau) &\approx \frac{1}{2}\gamma^{(0)}_{11}(\omega\tau) + \Lambda\left(\frac{c^2\tau^2}{6} + \frac{1}{2}\gamma^{(1)}_{11}(\omega\tau)\right) \\ &= \begin{cases} \frac{1}{2}\sin(\omega\tau) + \frac{\Lambda}{24\pi}\left[c^2\tau^2\left(4\pi - \frac{1}{2}\sin(\omega\tau)\right)\right. \\ \qquad \left. + \frac{c^2\tau}{\omega}\cos(\omega\tau) + \frac{2c^2}{\omega^2}\sin(\omega\tau)\right], & 0 \leq \tau \leq s \\ \frac{\Lambda c^2\tau^2}{6}, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.41}$$

Figs. 3.1 and 3.2 show the contribution of Λ to the geodesic separation due to the wave. As shown in the plots, Λ affects both the amplitude and the frequency. In fig. 3.2 the contribution from the isotropic expansion is also included.

The shape of the amplitude as well as the approximate change of the frequency are explicitly apparent if we assume (3.31) and the geometrical optics limit and write the wave–dependent part for $0 \leq \tau \leq s$ in the form

$$\delta n^1_{\text{wave}}(\tau) = \frac{1}{2}\left(1 - \delta A_\Lambda\right)\sin\left(\omega\left(\tau + \delta\tau_\Lambda\right)\right) \tag{3.42}$$

with

$$\delta A_\Lambda = \frac{\Lambda c^2\tau^2}{24\pi} \qquad \text{and} \qquad \delta\tau_\Lambda = \frac{2\Lambda c^2\tau}{\omega^2}.$$

The functions δA_Λ and $\delta\tau_\Lambda$ are shown in fig. 3.3 and fig. 3.4, respectively, for a typical neutron star–neutron star inspiral in the LIGO band.

As seen in equation (3.42,) for a positive value of Λ the amplitude decreases, which might be due to the expansion induced by Λ. Indeed, we expect that an accelerated expansion stretches the wave.

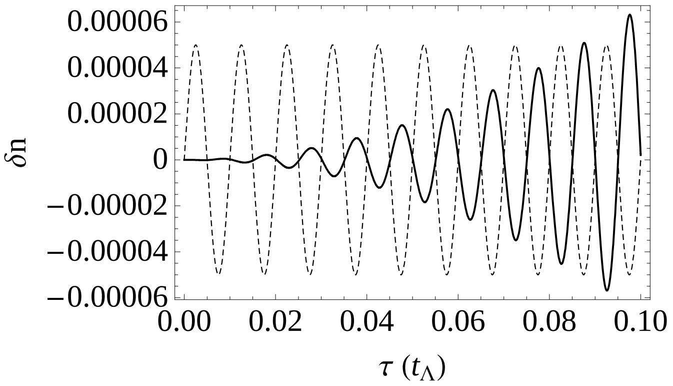

Figure 3.1: Magnified view of the contribution of Λ to the geodesic separation. The bold line is the contribution due to Λ coupled to the wave, whereas the dashed line is the unperturbed signal depleted by a factor 10^{-4}. Obviously, Λ affects in principle both amplitude and periodicity. While still preserving the ordering $1/\omega \ll \tau \ll t_\Lambda (\equiv 1/(c\sqrt{|\Lambda|}) \sim 10^{10}$ years), we are not considering realistic time-scales for the wave form, i.e. a duration event $\Delta\tau = 10^{-1} t_\Lambda$ and a frequency $f = 10/\Delta\tau$.

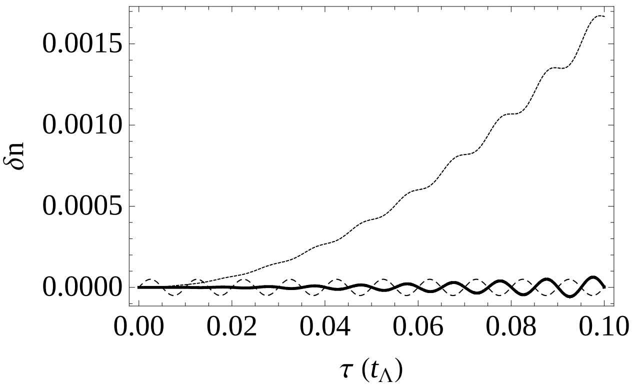

Figure 3.2: The same as fig. 3.1, but including the dotted line which accounts for the isotropic expansion too.

Let τ_{day} denote the length of the measurement in days. Then the relative weight of the leading Λ–dependent term for this example is of order

$$\left| \frac{\Lambda c^2 \tau^2}{48\pi} \right| \approx 5 \cdot 10^{-28} \cdot \tau_{\text{day}}^2. \tag{3.43}$$

If the amplitude of the wave does not vanish before the measurement starts, i. e. if the function $f(t)$ unlike $\varphi(t)$ does not vanish for $t < 0$, we gather from the general result (3.24) that then the leading term proportional to τ^3 is present. The relative weight of this term depends on ω and thus on the type of the source of radiation. We have

$$\left| \frac{\Lambda}{24\pi} \cdot \frac{\omega c^2 \tau^3}{3} \right| \approx 2.5 \cdot 10^{-23} \cdot \omega_{\text{Hz}} \tau_{\text{day}}^3, \tag{3.44}$$

where ω_{Hz} measures the frequency in Hertz. For compact sources ω is related to the size and the mass of the source. The size is bounded below by the Schwarzschild radius

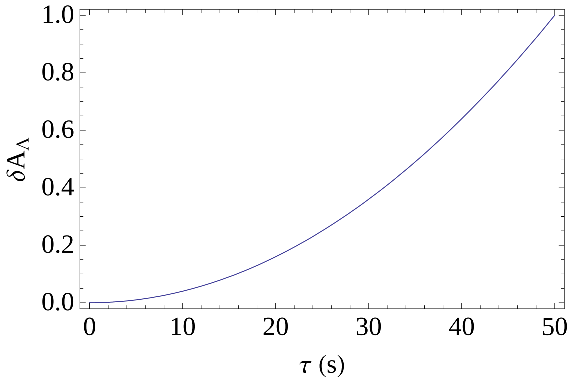

Figure 3.3: Contribution to the amplitude of the wave form due to Λ, δA_Λ, for a typical neutron star–neutron star inspiral in the LIGO band. We have considered $\Lambda = 10^{-52}\mathrm{m}^{-2}$, a frequency of $f(=\omega/2\pi) = 200$ Hz and a duration of 10^4 cycles. The amplitude is normalized as to be unitary at the end of the detection, when $\delta A_\Lambda \simeq 3 \times 10^{-34}$; time units are in seconds.

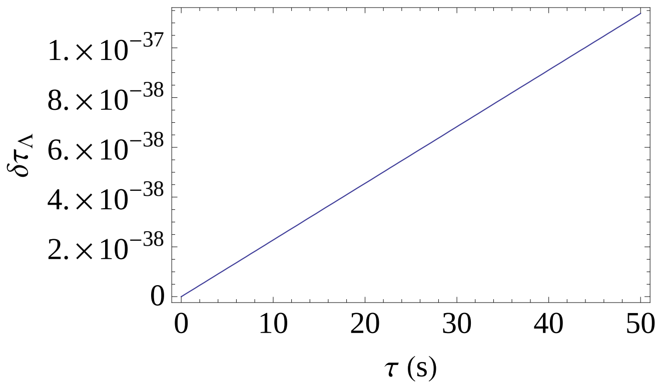

Figure 3.4: The same as fig. 3.3 for the phase shift $\delta\tau_\Lambda$. The time unit on the y-axis is given by the unperturbed period, $T = 2\pi/\omega 5 \times 10^{-3}$ s.

of the mass. This yields an upper bound on the frequency given by $\omega \approx 10^4$Hz [23]. Equation (3.44) then leads in the best case to

$$\left| \frac{\Lambda}{24\pi} \cdot \frac{\omega c^2 \tau^3}{3} \right| \lesssim 2.5 \cdot 10^{-19} \cdot \tau_{\mathrm{day}}^3. \tag{3.45}$$

In principle the effects of Λ are measurable if the signal to noise ratio (SNR) of the detector is sufficiently large. Present as well as planned observatories however do not feature the required accuracy. For example the Earthbounded detector advanced LIGO achieves a SNR ≈ 10 for the inspiral of compact objects of mass $m \approx 10^2 M_\odot$ at a frequency $\omega \approx 10^2$Hz [39]. Then the detectability of the effects of Λ may be measured by

$$\begin{aligned} \mathrm{SNR}_{\Lambda,\mathrm{LIGO}} &\approx 2.5 \cdot 10^{-23} \cdot \omega_{\mathrm{Hz}} \tau_{\mathrm{day}}^3 \mathrm{SNR}_{0,\mathrm{LIGO}} \\ &\approx 2.5 \cdot 10^{-20} \tau_{\mathrm{day}}^3. \end{aligned} \tag{3.46}$$

The planned spacebased observatory LISA on the other hand is expected to reach a SNR $\approx 10^4$ for the inspiral of supermassive black holes with $m \approx 10^6 M_\odot$ at a frequency $\omega \approx 10^{-2}$Hz [39]. This yields

$$\begin{aligned} \mathrm{SNR}_{\Lambda,\mathrm{LISA}} &\approx 2.5 \cdot 10^{-23} \cdot \omega_{\mathrm{Hz}} \tau_{\mathrm{day}}^3 \mathrm{SNR}_{0,\mathrm{LISA}} \\ &\approx 2.5 \cdot 10^{-21} \tau_{\mathrm{day}}^3 . \end{aligned} \tag{3.47}$$

The corresponding SNR for the example with $f = \varphi$ can be calculated by considering (3.43) instead of (3.44). Then

$$\begin{aligned} \mathrm{SNR}_{\Lambda,\mathrm{LIGO}} &\approx 5 \cdot 10^{-28} \cdot \tau_{\mathrm{day}}^2 \mathrm{SNR}_{0,\mathrm{LIGO}} \\ &\approx 5 \cdot 10^{-27} \tau_{\mathrm{day}}^2 , \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} \mathrm{SNR}_{\Lambda,\mathrm{LISA}} &\approx 5 \cdot 10^{-28} \cdot \tau_{\mathrm{day}}^2 \mathrm{SNR}_{0,\mathrm{LISA}} \\ &\approx 5 \cdot 10^{-24} \tau_{\mathrm{day}}^2 , \end{aligned} \tag{3.49}$$

respectively. For $\tau_{\mathrm{day}} = 1$ e. g., the aforesaid observatories would have to increase their accuracy by at least twenty orders of magnitude in order to detect the effects of Λ on the waveform radiated by the inspirals mentioned above. Thus even for a long but realistic period of measurement it is not possible to detect the effects of Λ within the existing technology.

3.6 Conclusions

We investigated the linearised Einstein's equations with a cosmological term and derived explicit expressions for the corrections to the plane gravitational waves up to linear order in Λ. The polarization states of a wave remain unchanged in the presence of the cosmological term. This conclusion is consistent with the result obtained in [4, 5]. The amplitude as well as the frequency (for periodic radiation) though are modified with increasing time. However, these effects are very tiny and thus not detectable by present or planned detectors.

We point out that one can not rule out the possibility that nonlinear effects originating from terms proportional to $h_{\mu\nu,\lambda} h_{\rho\sigma,\tau}$ in an expansion (3.3) could lead to effects on the waveform similar in size as the ones due to the cosmological term. However, as discussed for instance in [23], such a perturbation term can be split into a slowly varying piece, and a rapidly varying one. The latter one would induce modifications on a much shorter timescale than the contribution due to the cosmological constant as considered here, and should thus be easily discriminated. On the other hand the long timescale contribution would modify the background. However, its time dependence might be different from the one due to the cosmological constant and thus making it still possible to distinguish the various effects. A detailed analysis of effects due to quadratic terms in h is certainly quite involved and beyond the scope of the present work.

A mentionable phenomenon is eventually the connection between the cosmological constant and the mass of the graviton. Mass terms characterize Klein–Gordon equations and are connected to the dispersion relation. We do not go further into this question and refer to [24, 44, 35].

Chapter 4

On the $1/c$ Expansion of $f(R)$ Gravity

This chapter has been published in [40].

4.1 Introduction

Since the emergence of the concepts of dark matter (DM) and dark energy (DE), they still lack in a concrete and satisfying physical model. This open question motivated the development of new gravity theories. Most of them are direct modifications of general relativity (GR), which is still the simplest relativistic gravity theory fitting very accurately many precision measurements in astrophysics, such as Mercury perihelion shift or mass diagrams of double pulsars. Among such modified theories a lot of attention has been devoted to the so-called metric $f(R)$ theories with an action

$$S = \frac{c^3}{16\pi G} \int f(R)\sqrt{-g}\, d^4x + S_M, \tag{4.1}$$

where in contrast to GR the Einstein-Hilbert Lagrangian density is replaced by a nonlinear function $f(R)$. S_M is the standard matter action. For an overview one may consult e. g. [60] and references therein.

In the literature there are several approaches which address the question of the nonrelativistic limit as well as relativistic approximations of metric $f(R)$ theories, cf. for example [16]. A discussion of the first relativistic corrections after a transformation to the Einstein frame is given in [15, 19]. The nonrelativistic limit in the Jordan frame is investigated in [12, 13, 11], whereas a calculation in the Palatini formalism is given for example in [56]. In the present paper we work strictly in the Jordan frame. Our work is mainly motivated by the fact that the parametric post-Newtonian (PPN) formalism is not adapted to cover the $1/c$ expansion of $f(R)$ gravity [14]. As pointed out in [15, 12], the corresponding nonrelativistic limit indeed is not Newtonian, but contains a Yukawa type correction, too. We therefore derive the lowest order relativistic terms of the $1/c$-expansion of the space time metric governed by the Euler-Lagrange equations of (4.1). We thus achieve a "post-Yukawa" approximation of $f(R)$ gravity. This approximation is analogous to the complete first post-Newtonian approximation of GR, cf. for example [64, 62, 58, 34].

In Section 4.2 we present the field equations of the model. Section 4.3 is devoted to the calculation of the expansion coefficients, and in Section 4.4 we make some remarks on the nonrelativistic limit as well as the GR limit of the model. In Section 4.5 we derive the equations of motion for a test particle and determine the underlying potentials for a set of freely falling particles. In Section 4.6 we apply our results to the precession of orbiting gyroscopes, and by using the measurements of Gravity Probe B and of the pulsar B in the PSR J0737-3039 system, we get upper limits for the value of a.

As far as notation is concerned: Greek letters denote space time indices and range from 0 to 3, whereas Latin letters denote space indices and range from 1 to 3. We take the sum over repeated indices within a term. By an index "$,\mu$" we denote the partial differentiation with respect to x^μ, except for $\mu = 0$, where it denotes the differentiation with respect to the time coordinate t rather than the coordinate $x^0 = ct$.

4.2 The Field Equations

Consider a 4-dimensional pseudo Riemannian manifold with metric $g_{\mu\nu}$ of signature $(-,+,+,+)$. We write $g = \det g_{\mu\nu}$ and denote the Ricci tensor of $g_{\mu\nu}$ by $R_{\mu\nu}$. The variation of the action (4.1) with respect to the metric yields the Euler-Lagrange equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box_g f'(R) = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{4.2}$$

where $R = g^{\mu\nu}R_{\mu\nu}$, $T_{\mu\nu} = (-2c/\sqrt{-g})(\delta S_M/\delta g^{\mu\nu})$ is the energy-momentum tensor, c the vacuum speed of light, G Newton's constant, ∇_μ the covariant derivative for $g_{\mu\nu}$ and $\Box_g = \nabla^\mu\nabla_\mu$. Taking the trace of (4.2) we obtain

$$3\Box_g f'(R) + f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T, \tag{4.3}$$

where T is the trace of $T_{\mu\nu}$. Motivated by the post-Newtonian approximation of GR we calculate the coefficients of the expansion of $g_{\mu\nu}$ in powers of c^{-1}:

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}h_{00} + {}^{(4)}h_{00} + \mathcal{O}\left(c^{-6}\right), \\ g_{0i} &= {}^{(3)}h_{0i} + \mathcal{O}\left(c^{-5}\right), \\ g_{ij} &= \delta_{ij} + {}^{(2)}h_{ij} + \mathcal{O}\left(c^{-4}\right), \end{aligned} \tag{4.4}$$

where ${}^{(n)}h_{\mu\nu}$ denotes a quantity of order $\mathcal{O}(c^{-n})$. The Ricci scalar is at least of order $\mathcal{O}(c^{-2})$. Thus, if we assume the function f to be analytic at $R = 0$ with $f'(0) = 1$, it suffices to consider the expansion

$$f(R) = -2\Lambda + R + aR^2, \quad a \neq 0, \tag{4.5}$$

in order to calculate the coefficients of $g_{\mu\nu}$ up to the orders indicated in (4.4), since higher powers of R would only contribute to higher orders in the equations of the perturbation expansion. Moreover, since we adopt an expansion about a flat background space time in (4.4), we ignore a possible cosmological constant Λ in what follows. The influence of a nonvanishing Λ on the applications in Section 4.6 is discussed in [54]. As we will see later, the parameter a has to be positive for many reasons.

It is convenient to introduce the scalar field $\phi := f'(R)$. Since $f''(R) \neq 0$ holds for our choice of $f(R)$ in (4.5), we can invert $f'(R)$ in order to show that $f(R)$ gravity is equivalent to the Brans-Dicke theory with a non vanishing potential term and Brans-Dicke parameter $\omega_{BD} = 0$. We define the scalar field φ by $\phi = 1 + 2a\varphi$, where we have chosen the asymptotic value such that a renormalization of the Newton's constant will be redundant (cf. the end of Section 4.4.1). Then the equations (4.2) and (4.3) are equivalent to

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{1+2a\varphi}\left[\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{3}g_{\mu\nu}T\right)\right. && (4.6)\\
&\qquad\left.+\frac{1}{6}g_{\mu\nu}\varphi + a\left(\frac{1}{2}g_{\mu\nu}\varphi^2+2\nabla_\mu\nabla_\mu\varphi\right)\right]\\
\Box_g\varphi &= \frac{4\pi G}{3ac^4}T+\frac{1}{6a}\varphi. && (4.7)
\end{aligned}
$$

The field φ thus has the effective mass $\hbar/(c\sqrt{6a})$. From (4.5) we infer that the dimensionless quantity aR should be small compared to 1. This fact reflects the concept of the chameleon effect [32], which states the possibility that the Compton wave length $\lambda = \sqrt{6a}$ of the field φ is smaller or larger in regions with higher or lower matter density, respectively. We understand our $1/c$ expansion to be valid in a local region which has an approximately constant mean matter density, in the sense that we assume the parameter a to be constant on the length scale characteristic for later applications, in particular the geodetic precession. On the other hand, a may vary for applications which have different length scales.

4.3 The Expansion Coefficients

We introduce space time coordinates $(\boldsymbol{x}, t)$, where bold face letters denote three dimensional vectors. The expansion coefficients are functions of these coordinates. Denote the 3-dimensional Nabla operator by $\nabla = (\partial_1, \partial_2, \partial_3)$. We find the following expansions for $R_{\mu\nu}$ and φ,

$$
\begin{aligned}
R_{00} &= \frac{1}{2}\Bigg[-\nabla^2\,{}^{(2)}h_{00}-\nabla^2\,{}^{(4)}h_{00}+\frac{2}{c}{}^{(3)}h_{0i,0i}-\frac{1}{c^2}{}^{(2)}h_{ii,00} && (4.8)\\
&\qquad+{}^{(2)}h_{ij}{}^{(2)}h_{00,ij}+{}^{(2)}h_{00,i}\left({}^{(2)}h_{ij,j}-\frac{1}{2}{}^{(2)}h_{jj,i}-\frac{1}{2}{}^{(2)}h_{00,i}\right)\Bigg]+\mathcal{O}\left(c^{-6}\right),\\
R_{0i} &= \frac{1}{2}\left[-\nabla^2\,{}^{(3)}h_{0i}-\frac{1}{c}{}^{(2)}h_{jj,0i}+{}^{(3)}h_{j0,ij}+\frac{1}{c}{}^{(2)}h_{ij,0j}\right]+\mathcal{O}\left(c^{-5}\right),\\
R_{ij} &= \frac{1}{2}\left[-\nabla^2\,{}^{(2)}h_{ij}+{}^{(2)}h_{00,ij}-{}^{(2)}h_{kk,ij}+{}^{(2)}h_{ik,kj}+{}^{(2)}h_{kj,ki}\right]+\mathcal{O}\left(c^{-4}\right),\\
\varphi &= {}^{(2)}\varphi+{}^{(4)}\varphi+\mathcal{O}\left(c^{-6}\right),
\end{aligned}
$$

and for the energy-momentum tensor,

$$
\begin{aligned}
T^{00} &= {}^{(-2)}T^{00}+{}^{(0)}T^{00}+\mathcal{O}\left(c^{-2}\right), && (4.9)\\
T^{0i} &= {}^{(-1)}T^{0i}+\mathcal{O}\left(c^{-1}\right),\\
T^{ij} &= {}^{(0)}T^{ij}+\mathcal{O}\left(c^{-2}\right).
\end{aligned}
$$

Equation (4.7) then yields in leading order the inhomogeneous Helmholtz equation

$$\nabla^2\,{}^{(2)}\varphi-\alpha^2\,{}^{(2)}\varphi=-\frac{8\pi G\alpha^2}{c^4}{}^{(-2)}T^{00}, \tag{4.10}$$

where we have defined $\alpha^2 := 1/(6a)$ for a real α. Equation (4.10) has the solution

$$ {}^{(2)}\varphi(\boldsymbol{x},t) = \frac{1}{c^2} V(\boldsymbol{x},t) \tag{4.11} $$

with the potential

$$ V(\boldsymbol{x},t) := \frac{2G\alpha^2}{c^2} \int \frac{{}^{(-2)}T^{00}(\boldsymbol{x}',t) e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'. \tag{4.12} $$

The 00-component of equation (4.6) at order $\mathcal{O}(c^{-2})$ is given by

$$ \nabla^2 \, {}^{(2)}h_{00} = -\frac{32\pi G}{3c^4} {}^{(-2)}T^{00} + \frac{1}{3} {}^{(2)}\varphi \tag{4.13} $$

and has the solution

$$ {}^{(2)}h_{00}(\boldsymbol{x},t) = \frac{1}{c^2} \left(2U(\boldsymbol{x},t) - W(\boldsymbol{x},t) \right) \tag{4.14} $$

with the potentials

$$ \begin{aligned} U(\boldsymbol{x},t) &:= \frac{4G}{3c^2} \int \frac{{}^{(-2)}T^{00}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \\ W(\boldsymbol{x},t) &:= \frac{1}{12\pi} \int \frac{V(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'. \end{aligned} \tag{4.15} $$

In contrast to iterated Coulomb integrals, the potential W is well defined because of the exponential decay of the Yukawa term in V. Before we determine the coefficients of the $0i$- and ij-components, we impose the four gauge conditions

$$ g_{ij,j} - \frac{1}{2} \left(g_{jj} - g_{00} \right)_{,i} - \frac{1}{3\alpha^2} \varphi_{,i} = \mathcal{O}\left(c^{-4}\right), \tag{4.16} $$

$$ g_{0j,j} - \frac{1}{2c} g_{jj,0} - \frac{1}{3\alpha^2 c} \varphi_{,0} = \mathcal{O}\left(c^{-5}\right). \tag{4.17} $$

Using the condition (4.16), we find for the ij-component of equation (4.6), up to order $\mathcal{O}(c^{-2})$,

$$ \nabla^2 \, {}^{(2)}h_{ij} = -\left(\frac{16\pi G}{3c^4} {}^{(-2)}T^{00} + \frac{1}{3} {}^{(2)}\varphi \right) \delta_{ij} \tag{4.18} $$

with the solution

$$ {}^{(2)}h_{ij}(\boldsymbol{x},t) = \frac{\delta_{ij}}{c^2} \left(U(\boldsymbol{x},t) + W(\boldsymbol{x},t) \right). \tag{4.19} $$

Taking into account the gauge condition (4.17), the $0i$-component of equation (4.6), up to order $\mathcal{O}(c^{-3})$, simplifies to

$$ \nabla^2 \, {}^{(3)}h_{0i} = \frac{16\pi G}{3c^4} {}^{(-1)}T^{0i} + \frac{1}{c} {}^{(2)}h_{ij,0j} - \frac{1}{2c} {}^{(2)}h_{jj,0i}. \tag{4.20} $$

Defining the potentials

$$ \begin{aligned} \chi(\boldsymbol{x},t) &:= \frac{G}{3c^2} \int {}^{(-2)}T^{00}(\boldsymbol{x}',t) \left| \boldsymbol{x}-\boldsymbol{x}' \right| d^3x', \\ \psi(\boldsymbol{x},t) &:= \frac{1}{48\pi} \int V(\boldsymbol{x}',t) \left| \boldsymbol{x}-\boldsymbol{x}' \right| d^3x', \end{aligned} \tag{4.21} $$

we can write equation (4.20) as

$$\nabla^2 \left({}^{(3)}h_{0i} + \frac{1}{2c^3} \left(\chi + \psi\right)_{,0i} \right) = \frac{16\pi G}{3c^4} {}^{(-2)}T^{0i}. \tag{4.22}$$

The solution of equation (4.22) is given by

$${}^{(3)}h_{0i}(\boldsymbol{x}, t) = \frac{1}{c^3} \left(Y_i(\boldsymbol{x}, t) + Z_i(\boldsymbol{x}, t) \right) \ , \tag{4.23}$$

where

$$\begin{aligned} Y_i(\boldsymbol{x}, t) &:= -\frac{4G}{c} \int \frac{{}^{(-1)}T^{0i}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \\ Z_i(\boldsymbol{x}, t) &:= -\left(\chi(\boldsymbol{x}, t) + \psi(\boldsymbol{x}, t) \right)_{,0i}. \end{aligned} \tag{4.24}$$

Thus, the $f(R)$ correction to the shift is only due to the gradient field $\boldsymbol{Z}$.

For the derivation of the component ${}^{(4)}h_{00}$, we first address ${}^{(4)}\varphi$. In view of the gauge (4.16) the $\mathcal{O}(c^{-4})$ part of equation (4.7) is given by

$$\begin{aligned} \nabla^2 \, {}^{(4)}\varphi - \alpha^2 \, {}^{(4)}\varphi &= \frac{8\pi G \alpha^2}{c^4} \left(-{}^{(0)}T^{00} + {}^{(0)}T^{ii} \right) \\ &\quad + \frac{1}{c^2} {}^{(2)}\varphi_{,00} + {}^{(2)}h_{ij} \, {}^{(2)}\varphi_{,ij} + \frac{1}{6\alpha^2} {}^{(2)}\varphi_{,i} \, {}^{(2)}\varphi_{,i}. \end{aligned} \tag{4.25}$$

We use the identity

$$\tilde{U}_{,i}\tilde{V}_{,i} = \frac{1}{2} \left(\nabla^2 (\tilde{U}\tilde{V}) - \tilde{V}\nabla^2 \tilde{U} - \tilde{U}\nabla^2 \tilde{V} \right) \tag{4.26}$$

for two arbitrary potentials $\tilde{U}$ and $\tilde{V}$ as well as the equations (4.10), (4.11) and (4.19) to rewrite equation (4.25) as

$$\nabla^2 \left({}^{(4)}\varphi - \frac{1}{12\alpha^2 c^4} V^2 \right) - \alpha^2 \left({}^{(4)}\varphi - \frac{1}{12\alpha^2 c^4} V^2 \right) = \frac{1}{c^4} A, \tag{4.27}$$

where

$$\begin{aligned} A &:= 8\pi G \alpha^2 \left[-{}^{(0)}T^{00} + {}^{(0)}T^{ii} - \frac{1}{c^2} \left(U + W - \frac{1}{6\alpha^2} V \right) {}^{(-2)}T^{00} \right] \\ &\quad + V_{,00} + \alpha^2 UV + \alpha^2 VW - \frac{1}{12} V^2. \end{aligned} \tag{4.28}$$

From equation (4.27) we obtain

$${}^{(4)}\varphi(\boldsymbol{x}, t) = \frac{1}{c^4} B(\boldsymbol{x}, t), \tag{4.29}$$

with the potential

$$B(\boldsymbol{x}, t) = -\frac{1}{4\pi} \int \frac{A(\boldsymbol{x}', t) e^{-\alpha|\boldsymbol{x} - \boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' + \frac{1}{12\alpha^2} V^2(\boldsymbol{x}, t). \tag{4.30}$$

The c^{-4}-component of equation (4.6) can be simplified with equation (4.17):

$$\nabla^2\,{}^{(4)}h_{00} = {}^{(2)}h_{ij}{}^{(2)}h_{00,ij} + {}^{(2)}h_{00,i}\left({}^{(2)}h_{ij,j} - \frac{1}{2}{}^{(2)}h_{jj,i} - \frac{1}{2}{}^{(2)}h_{00,i}\right) - \frac{16\pi G\alpha^2}{3c^4}\left[2^{(0)}T^{00} + {}^{(0)}T^{ii} + \left({}^{(2)}h_{00} - \frac{2}{3\alpha^2}{}^{(2)}\varphi\right){}^{(-2)}T^{00}\right] + \frac{1}{3}{}^{(4)}\varphi + \frac{1}{3}{}^{(2)}h_{00}{}^{(2)}\varphi + \frac{1}{18\alpha^2}\left({}^{(2)}\varphi\right)^2 - \frac{1}{3\alpha^2}{}^{(2)}h_{00,i}{}^{(2)}\varphi_{,i}. \tag{4.31}$$

Using equations (4.10), (4.11), (4.13), (4.14), (4.15), (4.19), (4.26) and (4.29) we write equation (4.31) in the form

$$\nabla^2\left[{}^{(4)}h_{00} + \frac{1}{c^4}\left(\frac{3}{2}U^2 + \frac{1}{3\alpha^2}UV - \frac{3}{4}UW - \frac{1}{6\alpha^2}VW\right)\right] = \frac{1}{c^4}C \tag{4.32}$$

with

$$C := -\frac{16\pi G}{3}\left[2^{(0)}T^{00} + {}^{(0)}T^{ii} + \frac{1}{2c^2}\left(15U - \frac{2}{3\alpha^2}V\right){}^{(-2)}T^{00}\right] + \frac{19}{12}UV - \frac{1}{6}VW + \frac{1}{9\alpha^2}V^2 + \frac{1}{3}B. \tag{4.33}$$

Hence,

$${}^{(4)}h_{00}(\boldsymbol{x},t) = \frac{1}{c^4}D(\boldsymbol{x},t), \tag{4.34}$$

where

$$D(\boldsymbol{x},t) = -\frac{1}{4\pi}\int\frac{C(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' - \frac{3}{2}U^2(\boldsymbol{x},t) - \frac{1}{3\alpha^2}U(\boldsymbol{x},t)V(\boldsymbol{x},t) + \frac{3}{4}U(\boldsymbol{x},t)W(\boldsymbol{x},t) + \frac{1}{6\alpha^2}V(\boldsymbol{x},t)W(\boldsymbol{x},t). \tag{4.35}$$

The metric field, up to the first relativistic approximation, is thus determined by the fields $U, W, \boldsymbol{Y}, \boldsymbol{Z}, D$.

4.4 The GR Limit and the Nonrelativistic Limit

4.4.1 The GR Limit

By taking the limit $a \to 0$ resp. $\alpha \to \infty$, the theory converges to GR. Notice that with the assumption $a > 0$ we have the following representation of the Dirac delta function:

$$\lim_{\alpha\to\infty}\frac{\alpha^2}{4\pi}\int\xi(\boldsymbol{x}')\frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' = \int\xi(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')d^3x' = \xi(\boldsymbol{x}) \tag{4.36}$$

for an arbitrary test function $\xi(\boldsymbol{x})$. Hence

$$\lim_{\alpha\to\infty}W(\boldsymbol{x},t) = \frac{2G}{3c^2}\int\frac{{}^{(-2)}T^{00}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' = \frac{1}{2}U(\boldsymbol{x},t) \tag{4.37}$$

and thus

$$\lim_{\alpha\to\infty}{}^{(2)}h_{00}(\boldsymbol{x},t) = \lim_{\alpha\to\infty}{}^{(2)}h_{11}(\boldsymbol{x},t) = \frac{3}{2c^2}U(\boldsymbol{x},t) = \frac{2G}{c^4}\int\frac{{}^{(-2)}T^{00}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' = \frac{2}{c^2}U_N(\boldsymbol{x},t), \tag{4.38}$$

as expected. U_N is exactly the Newtonian potential. There is no need for a rescaling of Newton's constant G, as it is sometimes the case for other modified gravity theories.

4.4.2 The Nonrelativistic Limit

From the equations (4.12), (4.14) and (4.15) we gather that for the $f(R)$ model given by (4.5) the nonrelativistic limit is not Newtonian, since the component ${}^{(2)}h_{00}$ contains a Yukawa type term.

More explicitly, if we consider for instance a perfect, non viscous fluid with mass density ρ, pressure p and velocity field $\boldsymbol{v} = (v^1, v^2, v^3)$, we have

$$ {}^{(-2)}T^{00} = \rho c^2, \qquad {}^{(-1)}T^{0i} = \rho c v^i, \qquad {}^{(0)}T^{ij} = \rho v^i v^j + p\delta_{ij}. \tag{4.39}$$

In the nonrelativistic limit, the energy-momentum conservation

$$ T^{\mu\nu}{}_{;\nu} = 0 \tag{4.40}$$

then yields the equation of continuity,

$$ \partial_t \rho + \partial_i(\rho v^i) = 0, \tag{4.41}$$

and the analogous of the Euler equation,

$$ \rho\left(\partial_t v^i + v^j \partial_j v^i\right) = -\partial_i p + \rho \partial_i \left(U - \frac{1}{2}W\right). \tag{4.42}$$

Aside from a Newtonian term, the potential W contains also a Yukawa type term. The gravitational force, represented by the second term on the right hand side, therefore contains also the gradient of a Yukawa potential. Together with the Poisson equations

$$ \begin{aligned} \nabla^2 U &= -\frac{16\pi G}{3}\rho, \\ \nabla^2 W &= -\frac{2G\alpha^2}{3}\int \frac{\rho(\boldsymbol{x}', t)e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \end{aligned} \tag{4.43}$$

(4.41) and (4.42) are the basic equations of the nonrelativistic limit of hydrodynamics for metric $f(R)$ theory. The parameter $a = 1/(6\alpha^2)$ can be constrained by experiments which test a Yukawa type correction to the Newtonian potential. Experimental data and overviews can be found for example in [30, 43, 25, 53, 33]. Constraints on $f(R)$ theories are given for instance in [7]. From equation (4.42) (see also equation (4.49)) we find for our specific model the Yukawa field strength $G/3$. The Eöt-Wash experiment [30] thus yields the limit $a \lesssim 10^{-10}\,\mathrm{m}^2$.

4.5 Particle Dynamics

We can derive the equations of motion for a freely falling test particle in a field corresponding to the potentials $(U, W, \boldsymbol{Y}, \boldsymbol{Z}, D)$ by evaluating the geodesic equation,

$$ \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \tag{4.44}$$

where the $\Gamma^\lambda_{\mu\nu}$ are the connection coefficients of the metric $g_{\mu\nu}$, and $x^\mu(\tau) = (ct(\tau), \boldsymbol{x}(\tau))$ is the position of the test particle at proper time τ. Defining $\boldsymbol{v} := d\boldsymbol{x}/dt$, the equations of motion in vector notation read

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= \nabla\left(U - \frac{1}{2}W\right) + \frac{1}{c^2}\Bigg[(U+W)\nabla\left(U-\frac{1}{2}W\right) - \nabla D \\ &\quad + \partial_t(\boldsymbol{Y}+\boldsymbol{Z}) + \boldsymbol{v}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{Y}) - \boldsymbol{v}\partial_t\left(2U+\frac{1}{2}W\right) \\ &\quad - 3\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\nabla})U - \boldsymbol{v}^2\nabla(U+W)\Bigg] + \mathcal{O}\left(c^{-4}\right).\end{aligned} \tag{4.45}$$

The energy-momentum tensor of a set of point particles reads [62]

$$T^{\mu\nu}(\boldsymbol{x},t) = \frac{1}{\sqrt{-g}}\sum_n m_n \frac{dx_n^\mu}{dt}\frac{dx_n^\nu}{dt}\left(\frac{d\tau_n}{dt}\right)^{-1}\delta^3\left(\boldsymbol{x}-\boldsymbol{x}_n(t)\right), \tag{4.46}$$

where we have introduced the mass m_n of the n-th particle, its proper time τ_n and its postition $\boldsymbol{x}_n(t) = (x_n^1(t), x_n^2(t), x_n^3(t))$ at time t. This leads to the expansion

$$\begin{aligned}T^{00} &= \sum_n m_n\left[c^2 + \frac{1}{2}\left(U - 5W + \boldsymbol{v}_n^2\right)\right]\delta^3\left(\boldsymbol{x}-\boldsymbol{x}_n\right) + \mathcal{O}\left(c^{-2}\right), \\ T^{0i} &= \sum_n m_n c\, v_n^i \delta^3\left(\boldsymbol{x}-\boldsymbol{x}_n\right) + \mathcal{O}\left(c^{-1}\right), \\ T^{ij} &= \sum_n m_n v_n^i v_n^j \delta^3\left(\boldsymbol{x}-\boldsymbol{x}_n\right) + \mathcal{O}\left(c^{-2}\right),\end{aligned} \tag{4.47}$$

where $\boldsymbol{v}_n = d\boldsymbol{x}_n/dt$. Inserting (4.47) into the defining equations of the potentials U, W, χ, ψ and Y_i, we get

$$\begin{aligned}U(\boldsymbol{x},t) &= \frac{4G}{3}\sum_n \frac{m_n}{|\boldsymbol{x}-\boldsymbol{x}_n(t)|}, \\ W(\boldsymbol{x},t) &= \frac{2G}{3}\sum_n \frac{m_n}{|\boldsymbol{x}-\boldsymbol{x}_n(t)|}\left(1 - e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}_n(t)|}\right), \\ \chi(\boldsymbol{x},t) &= \frac{G}{3}\sum_n m_n\left|\boldsymbol{x}-\boldsymbol{x}_n(t)\right|, \\ \psi(\boldsymbol{x},t) &= \frac{G}{6}\sum_n m_n\left[|\boldsymbol{x}-\boldsymbol{x}_n(t)| + \frac{2}{\alpha^2\,|\boldsymbol{x}-\boldsymbol{x}_n(t)|}\left(1-e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}_n(t)|}\right)\right], \\ Y_i(\boldsymbol{x},t) &= -4G\sum_n \frac{m_n v_n^i(t)}{|\boldsymbol{x}-\boldsymbol{x}_n(t)|}.\end{aligned} \tag{4.48}$$

Using (4.47) and (4.48), one is able to calculate (at least formally) the potentials $\boldsymbol{Z}$ and D, which we however do not need for the applications we consider in this paper.

We investigate the non-relativistic limit by taking the $\mathcal{O}(1)$ part of equation (4.45). Considering each particle as a test particle in the field of the other ones, we replace the potentials U and W by their self-energy free parts. The equations of motion for the test particle then read

$$\frac{dv_n^i}{dt} = G\sum_{k\neq n}\frac{\partial}{\partial x_n^i}\left[\frac{m_k}{|\boldsymbol{x}_n-\boldsymbol{x}_k|}\left(1+\frac{1}{3}e^{-\alpha|\boldsymbol{x}_n-\boldsymbol{x}_k|}\right)\right]. \tag{4.49}$$

This is the analogue of the Newtonian equations of motion for a purely gravitating set of point particles.

4.6 Precession of Orbiting Gyroscopes

The following derivation of the precession and its applications is done in complete analogy with the corresponding computations in GR [58, 62]. A recent and detailed review of gravitomagnetism in physics and astrophysics is provided in [51].

Consider a gyroscope with spin $\boldsymbol{S} = (S_1, S_2, S_3)$. We define its 3-velocity $\boldsymbol{v}$ to be the velocity of the nonrelativistic center of mass of the gyroscope, the trajectory of which is assumed to be the one of a purely gravitating point particle. Notice that this last assumption includes the fact that the gyroscope moves along a geodesic and thus the Thomas precession due to a external force vanishes. The spin 4-vector $S_\mu := (S_0, \boldsymbol{S})$ precesses according to the equation of parallel transport,

$$\frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau}, \tag{4.50}$$

and satisfies the orthogonality condition

$$\frac{dx^\mu}{d\tau} S_\mu = 0. \tag{4.51}$$

Up to the lowest order, equation (4.50) reads

$$\frac{dS_i}{dt} = \left(c\,{}^{(3)}\Gamma^j_{i0} - {}^{(2)}\Gamma^0_{i0} v_j + {}^{(2)}\Gamma^j_{ik} v_k \right) S_j. \tag{4.52}$$

Similarly as in GR we define the intrinsic spin vector by

$$\boldsymbol{\mathcal{S}} := \left[1 - \frac{1}{2c^2} (U + W) \right] \boldsymbol{S} - \frac{1}{2c^2} \boldsymbol{v} (\boldsymbol{v} \cdot \boldsymbol{S}), \tag{4.53}$$

cf. Appendix C. Then $\boldsymbol{\mathcal{S}}^2$ is an integral of (4.52) up to the required order. Equations (4.52) and (4.45) then yield

$$\frac{d\boldsymbol{\mathcal{S}}}{dt} = \boldsymbol{\Omega} \wedge \boldsymbol{\mathcal{S}}, \tag{4.54}$$

and the precession angular velocity is given by

$$\boldsymbol{\Omega} = -\frac{1}{2c^2} (\boldsymbol{\nabla} \wedge \boldsymbol{Y}) + \frac{1}{c^2} \boldsymbol{v} \wedge \boldsymbol{\nabla} \left(U + \frac{1}{4} W \right). \tag{4.55}$$

Compared to GR, the Lense-Thirring precession represented by the first term remains unchanged. This was expected since, as mentioned in Section 4.3, a finite parameter α affects the shift only by a gradient field. On the other hand, the geodetic precession given by the second term is modified.

4.6.1 Gyroscope Orbiting Around the Earth

We now analyse the correction to the geodetic precession, since the Lense-Thirring precession is not modified. We model the Earth as a sphere with mass M which is at rest centered at the origin of our coordinate system. Consider the gyroscope to be in a circular orbit $\boldsymbol{x}(t)$ with radius $|\boldsymbol{x}(t)| \equiv r$ and unit normal $\boldsymbol{n}$, such that $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{n})$ is a positively oriented dreibein. Then (4.48) gives

$$U + \frac{1}{4} W = \frac{3GM}{2r} \left(1 - \frac{1}{9} e^{-\alpha r} \right), \tag{4.56}$$

and by equating the gravitational and centrifugal force on the gyroscope, we find for the velocity

$$\boldsymbol{v} = \left[\frac{GM}{r^3}\left(1 + \frac{1}{3}(1+\alpha r)\, e^{-\alpha r}\right)\right]^{1/2} \boldsymbol{n} \wedge \boldsymbol{x}. \tag{4.57}$$

Hence the geodetic precession angular velocity is

$$\begin{aligned}\boldsymbol{\Omega}_{\text{geodesic}} &:= \frac{1}{c^2}\boldsymbol{v} \wedge \boldsymbol{\nabla}\left(U + \frac{1}{4}W\right) \qquad (4.58)\\ &= \frac{3(GM)^{3/2}}{2c^2 r^{5/2}}\left[1 + \frac{1}{3}(1+\alpha r)\, e^{-\alpha r}\right]^{1/2}\left[1 - \frac{1}{9}(1+\alpha r)\, e^{-\alpha r}\right]\boldsymbol{n}.\end{aligned}$$

Obviously, $\boldsymbol{\Omega}_{\text{geodesic}}$ converges to its GR value for $\alpha \to \infty$. This result can be compared with the measurements of the Gravity Probe B experiment [22]. The measured value lies within a minimal residue of 30 mas/yr from the predicted GR value 6606 mas/yr. This allows to constrain the relative deviation from the GR value in (4.58) by approximately 0.45 %. Since this deviation decays faster than $e^{-\alpha r}$ with growing α, while $r \approx 7 \times 10^6$ m, we expect a much larger bound for a as the one given by the Eöt-Wash experiment. For the given accuracy of measurement, equation (4.58) yields $a \lesssim 5 \times 10^{11}\,\text{m}^2$. If we estimate an upper limit for the scalar curvature using the mean Earth mass density, we are left with $R \lesssim 10^{-22}\,\text{m}^{-2}$. Even in this very rough approximation we have $aR \ll 1$ for our constraint on a.

We remark that for a nonvanishing cosmological constant the residue in the Gravity Probe B measurements lead to the bound $\Lambda \lesssim 3 \times 10^{-27}\,\text{m}^{-2}$ [54]. When considering both $\Lambda \neq 0$ and the Yukawa term within perturbation theory, to leading order the corrections due to a and to Λ will add linearly.

4.6.2 Precession of Binary Pulsars

Consider a binary system with center of mass at the origin. We index the mass m_n and the position $\boldsymbol{x}_n$ by 1 for the pulsar and by 2 for the companion. Equation (4.48) gives for the fields of the companion

$$U(\boldsymbol{x},t) + \frac{1}{4}W(\boldsymbol{x},t) = \frac{3Gm_2}{2\,|\boldsymbol{x} - \boldsymbol{x}_2(t)|}\left(1 - \frac{1}{9}e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}_2(t)|}\right), \tag{4.59}$$

$$\boldsymbol{Y}(\boldsymbol{x},t) = -\frac{4Gm_2\boldsymbol{v}_2(t)}{|\boldsymbol{x} - \boldsymbol{x}_2(t)|}. \tag{4.60}$$

Define now $\boldsymbol{x} := \boldsymbol{x}_1 - \boldsymbol{x}_2$, $r := |\boldsymbol{x}|$, the reduced mass $\mu := m_1 m_2/(m_1 + m_2)$ and the angular momentum $\boldsymbol{L} := \mu\boldsymbol{x} \wedge (d\boldsymbol{x}/dt)$. Evaluating (4.55) at $\boldsymbol{x}_1$ then leads to

$$\boldsymbol{\Omega} = -\frac{G\boldsymbol{L}}{c^2 r^3}\left[2 + \frac{3m_2}{2m_1}\left(1 - \frac{1}{9}(1+\alpha r)\, e^{-\alpha r}\right)\right]. \tag{4.61}$$

In order to approximate the average of $\boldsymbol{\Omega}$ over a period, $\langle\boldsymbol{\Omega}\rangle = T^{-1}\int_0^T \boldsymbol{\Omega}(t)dt$, we need an expression of the trajectory in the nonrelativistic limit. Therefore we deduce the correction to the Kepler ellipse

$$r_0(\theta) := \frac{p}{1 + e\cos\theta}, \tag{4.62}$$

due to a small perturbation δU of the Newtonian potential U_N. In (4.62) we have introduced polar coordinates (r, θ) and the parameters $p := \boldsymbol{L}^2/(Gm_1m_2\mu)$ and $e := (1+2Ep/(Gm_1m_2))^{1/2}$, where E is the total energy of the two body system. We expand the equation of motion with respect to δU,

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{r^2}{|\boldsymbol{L}|}\left[2\mu\left(E+\frac{Gm_1m_2}{r}+\delta U\right)-\frac{\boldsymbol{L}^2}{r^2}\right]^{1/2} \qquad (4.63)\\ &= \frac{r^2}{|\boldsymbol{L}|}\Bigg[\left(2\mu\left(E+\frac{Gm_1m_2}{r}\right)-\frac{\boldsymbol{L}^2}{r^2}\right)^{1/2} \\ &\quad +\mu\left(2\mu\left(E+\frac{Gm_1m_2}{r}\right)-\frac{\boldsymbol{L}^2}{r^2}\right)^{-1/2}\delta U\Bigg]+\mathcal{O}(\delta U^2). \end{aligned}$$

Integration along the unperturbed trajectory $r_0(\theta)$ for $\delta U = Gm_1m_2e^{-\alpha r}/(3r)$ yields $r(\theta) \approx r_0(\theta) + \delta r(\theta)$ with

$$\delta r(\theta) = \frac{Gm_1m_2\mu}{3|\boldsymbol{L}|}\int_0^\theta r_0(\tilde{\theta})e^{-\alpha r_0(\tilde{\theta})}\left[2\mu\left(E+\frac{Gm_1m_2}{r_0(\tilde{\theta})}\right)-\frac{\boldsymbol{L}^2}{r_0^2(\tilde{\theta})}\right]^{-1/2}d\tilde{\theta}. \qquad (4.64)$$

We approximate the average of $\boldsymbol{\Omega}$ as

$$\begin{aligned} \langle\boldsymbol{\Omega}\rangle &\approx \frac{G\mu\hat{\boldsymbol{L}}}{c^2T}\int_0^{2\pi}\frac{1}{r}\left[2+\frac{3m_2}{2m_1}\left(1-\frac{1}{9}\left(1+\alpha r_0\right)e^{-\alpha r_0}\right)\right]d\theta \qquad (4.65)\\ &\approx \frac{G\mu\hat{\boldsymbol{L}}}{c^2T}\Bigg[\frac{2\pi}{p}\left(2+\frac{3m_2}{2m_1}\right) \\ &\quad -\int_0^{2\pi}\left(\frac{\delta r}{r_0^2}\left(2+\frac{3m_2}{2m_1}\right)+\frac{m_2}{6m_1r_0}\left(1+\alpha r_0\right)e^{-\alpha r_0}\right)d\theta\Bigg]. \end{aligned}$$

Because of the exponential decay the correction terms for r and $\langle\boldsymbol{\Omega}\rangle$ are very sensitive to variations of the parameter a. To give an idea of the orders of magnitude, we analyse (4.65) for the PSR 1913+16 data given in [63]. The predicted GR value given by the first term on the right hand side of (4.65) evaluates to $1.21°$/yr. The correction term reaches approximately 1% of the GR value for $a \approx 2.6 \times 10^{15}\,\mathrm{m}^2$. Simultaneously, the correction δr already after one period reaches 1% of the semi major axis, whose uncertainty can be measured with a much better accuracy of 4×10^{-5} %. From this last value we incidentally find the rough limit $a \lesssim 1.7 \times 10^{14}\,\mathrm{m}^2$, if we cumulate δr for one year.

The discovery of the double-pulsar binary PSR J0737-3039 paved the way to significantly improve the accuracy of binary pulsar measurements. An overview of the observed and derived parameters can be found in [9, 37]. With this data we evaluate the precession rate for the pulsar B predicted by GR to $5.07°$/yr. The measured value $\Omega_{\mathrm{B}} \approx 4.77^{+0.66}_{-0.65}\,°$/yr [8] then allows the correction to lie within a minimal residue of 7 % from $5.07°$/yr. This roughly yields the constraint $a \lesssim 2.3 \times 10^{15}\,\mathrm{m}^2$.

For other binary pulsars, even the required accuracy to precisely test GR is not yet reached by the corresponding experimental research, see e. g. [38] for the case of PSR J1141-6545. We finally conclude that a huge improvement of the accuracy of measurement would be necessary to put useful limits on a by the precession of binary pulsars.

4.7 Conclusions

We gave the general formula for the lowest relativistic order coefficients of the $1/c$ expansion for the metric $g_{\mu\nu}$ of $f(R)$ gravity, where we considered functions of the form $f(R) = R + aR^2$. Furthermore, we investigated the GR and nonrelativistic limits. The latter results in the Newtonian potential plus a Yukawa type correction with strength $G/3$ and Compton wave length $\sqrt{6a}$. As an application, we derived the $f(R)$-corrections to geodetic precession of orbiting gyroscopes. The Lense-Thirring precession is not affected.

While the laboratory bound from the Eöt-Wash experiment provides the small bound $a \lesssim 10^{-10}\,\mathrm{m}^2$, the results from Gravity Probe B imply the much larger limit $a \lesssim 5 \times 10^{11}\,\mathrm{m}^2$. The measurements of the precession of the pulsar B in the PSR J0737-3039 system provide instead the limit $a \lesssim 2.3 \times 10^{15}\,\mathrm{m}^2$. Even for these large values of a the quadratic term in (4.5) still induces a small correction of GR.

In principle, the coefficients for $g_{\mu\nu}$ can be used for the same applications as the PPN coefficients of metric gravity theories which have a Newtonian nonrelativistic limit. However, the computation of the applications which require the fourth order coefficient ${}^{(4)}h_{00}$ are challenging, because its formula is quite involved and contains up to three-fold iterated integrals. Therefore, a numerical analysis would be necessary for generic applications.

It would be interesting to take into account more general functions f. For instance one could extend the choice of f to functions which are not necessarily analytic at $R = 0$, but at a nonvanishing point $R = R_0$. Formally, this would imply to replace f given in (4.5) by the more general function

$$f(R) = -2\Lambda + a_1 R + a_2 R^2, \quad \Lambda, a_1, a_2 \neq 0. \tag{4.66}$$

While the possibility of $a_1 \neq 1$ would not cause much trouble in the derivation of the $1/c$ expansion, the nonvanishing cosmological constant requires an expansion about a de Sitter or anti-de Sitter background, thus leading to more complicated partial differential equations for the potentials. Nevertheless, the choice of f as in (4.66) would be needed to study many $f(R)$ models which are proposed in the literature.

Chapter 5

On Gravitational Radiation in Quadratic $f(R)$ Gravity

5.1 Introduction

One of the most impressive endorsements of General Relativity Theory (GR) is the agreement of the predictions of the famous quadrupole formula for gravitational radiation with indirect measurements of the energy loss of binary pulsars. It is thus natural to test modified gravity theories by deriving the corrections to the quadrupole formula and comparing them with experimental data. For many types of theories this has been done in the past [64]. Though this problem is still open for metric $f(R)$ theories with an action

$$S = \frac{c^3}{16\pi G} \int f(R)\sqrt{-g}\, d^4x + S_M, \tag{5.1}$$

where in contrast to GR the Einstein–Hilbert Lagrangian density is replaced by a non-linear function $f(R)$. S_M is the standard matter action. In the past years, this type of theories has become very popular to heuristically gain insight in the problem of dark energy. For an overview one may consult e. g. [10, 60] and references therein.

In this chapter we prepare the way to investigate the energy emission of binary systems by gravitational radiation. The basic equations of $f(R)$ gravity are given in Section 5.2. For our purposes it will be convenient to work in the scalar tensor formulation of quadratic $f(R)$ gravity. In Section 5.3 we employ the linearised field equations of quadratic $f(R)$ gravity to derive the weak gravitational fields emitted by a localised source. The linearised $f(R)$ gravity has been investigated for example in [10, 17]. In Section 5.4 we dwell on the energy–momentum complex in quadratic $f(R)$ gravity as an analogue to the Landau–Lifshitz complex in GR. In Section 5.5 we finally derive a correction to the quadrupole formula and express it in terms of momenta of the energy–momentum tensor.

Notational conventions: Greek letters denote space time indices and range from 0 to 3, whereas Latin letters denote space indices and range from 1 to 3. We take the sum over repeated indices within a term.

5.2 The Field Equations

Consider a 4-dimensional pseudo Riemannian manifold with metric $g_{\mu\nu}$ of signature $(-,+,+,+)$. We write $g = \det g_{\mu\nu}$ and denote the Ricci tensor of $g_{\mu\nu}$ by $R_{\mu\nu}$. The variation of the action (5.1) with respect to the metric yields the Euler–Lagrange equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box_g f'(R) = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{5.2}$$

where $R = g^{\mu\nu}R_{\mu\nu}$, $T_{\mu\nu} = (-2c/\sqrt{-g})(\delta S_M/\delta g^{\mu\nu})$ is the energy-momentum tensor, c the vacuum speed of light, G Newton's constant, ∇_μ the covariant derivative for $g_{\mu\nu}$ and $\Box_g = \nabla^\mu\nabla_\mu$. Taking the trace of (5.2) we obtain

$$3\Box_g f'(R) + f'(R)R - 2f(R) = \frac{8\pi G}{c^4}T, \tag{5.3}$$

where T is the trace of $T_{\mu\nu}$. We now assume

$$f(R) = R + aR^2 \tag{5.4}$$

and make use of the equivalence between $f(R)$ gravity and scalar tensor theory by defining the scalar field $\phi := f'(R)$. This identification is feasible since $f''(R) \neq 0$ holds for our choice of $f(R)$, and $f'(R)$ is thus invertible. We define the scalar field φ by $\phi = 1 + 2a\varphi$, where we have chosen the asymptotic value such that a renormalisation of the Newton's constant is redundant. Then the equations (5.2) and (5.3) are equivalent to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{1+2a\varphi}\left[\frac{8\pi G}{c^4}T_{\mu\nu} + a\left(2\nabla_\mu\nabla_\mu\varphi - 2g_{\mu\nu}\Box_g\varphi - \frac{1}{2}g_{\mu\nu}\varphi^2\right)\right] \tag{5.5}$$

$$\Box_g\varphi = \frac{4\pi G}{3ac^4}T + \frac{1}{6a}\varphi. \tag{5.6}$$

The field φ thus has the effective mass $\hbar/(c\sqrt{6a})$. From (5.4) we infer that the dimensionless quantity aR should be small compared to 1. This fact reflects the concept of the chameleon effect [32], which states the possibility that the Compton wave length $\lambda = \sqrt{6a}$ of the field φ is smaller or larger in regions with higher or lower matter density, respectively.

5.3 Gravitational Radiation in $f(R)$ gravity

Consider weak perturbations of the Minkowski spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{5.7}$$

where the coefficients of the perturbation satisfy $|h_{\mu\nu}| \ll 1$. In what follows the indices are raised and lowered by $\eta_{\mu\nu}$. For the field ϕ we have already chosen the asymptotic value 1, of which $2a\varphi$ is the perturbation. Moreover, the field equation (5.6) is inhomogeneous linear in φ, so that the linearisation in the perturbations is achieved simply through the replacement of $\Box_g$ by $\Box_\eta$. Let $h = h^\mu_\mu$, define

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} - 2a\varphi\eta_{\mu\nu} \tag{5.8}$$

and choose the gauge

$$\gamma_{\mu\nu}{}^{,\nu} = 0. \tag{5.9}$$

Up to linear order in $h_{\mu\nu}$ and φ, equation (5.5) can then be written as

$$\Box_\eta \gamma_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}. \tag{5.10}$$

In the slow motion approximation and at large distances from the localised sources, a special solution of (5.10) can be derived in analogy with the GR case. Since $T_{\mu\nu}$ is divergence free, we can to express the spatial components of $\gamma_{\mu\nu}$ in terms of the quadrupole momenta of T^{00}. Thus we obtain

$$\gamma^{ij}(t, \boldsymbol{x}) = \frac{2G}{c^6} \frac{1}{|\boldsymbol{x}|} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} d^3x' T^{00}(t - |\boldsymbol{x}|/c, \boldsymbol{x}') x'^i x'^j. \tag{5.11}$$

We write the linearisation of equation (5.6) as

$$\Box_\eta \varphi - \alpha^2 \varphi = \frac{8\pi G \alpha^2}{c^4} T, \tag{5.12}$$

where $\alpha := 1/\sqrt{6a}$. A special solution is the convolution of the source with the Green's function of the Klein–Gordon equation,

$$\mathcal{G}(t, \boldsymbol{x}) = -\frac{1}{4\pi} \left[\frac{\delta\left(t - |\boldsymbol{x}|/c\right)}{|\boldsymbol{x}|} - \frac{\alpha J_1\left(\alpha c \sqrt{t^2 - (|\boldsymbol{x}|/c)^2}\right) \theta\left(t - |\boldsymbol{x}|/c\right)}{\sqrt{t^2 - (|\boldsymbol{x}|/c)^2}} \right], \tag{5.13}$$

where δ is the Dirac delta distribution kernel, θ the Heaviside distribution kernel, and J_n the Bessel function of n^{th} order. Then

$$\begin{aligned} \varphi(t, \boldsymbol{x}) \;=\; & {}^{\text{hom}}\varphi(t, \boldsymbol{x}) + \frac{2G\alpha^2}{c^4} \left[\int_{\mathbb{R}^3} d^3x' \frac{T\left(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}'\right)}{|\boldsymbol{x} - \boldsymbol{x}'|} \right. \\ & \left. - \int_{-\infty}^{t - |\boldsymbol{x} - \boldsymbol{x}'|/c} dt' \int_{\mathbb{R}^3} d^3x' \frac{\alpha J_1\left(\alpha c \sqrt{(t - t')^2 - (|\boldsymbol{x} - \boldsymbol{x}'|/c)^2}\right)}{\sqrt{(t - t')^2 - (|\boldsymbol{x} - \boldsymbol{x}'|/c)^2}} T(t, \boldsymbol{x}') \right], \end{aligned} \tag{5.14}$$

If the source emits a single pulse, the field φ observed at a distance $|\boldsymbol{x}|$ consists of this pulse diminished by the factor $1/|\boldsymbol{x}|$ given by the first term on the right hand side of (5.14), and a wake represented by the second term.

For a fully consistent treatment of the emission of gravitational waves, we need to extend the source by the terms in the field equations that are quadratic in the perturbation fields. However, these contributions would lead to corrections in the energy emission formula which are quadratic in G. Therefore we will neglect these terms and proceed with $T^{\mu\nu}$ as the main contribution to the source. Formally, one can still take into account the quadratic terms in our main result (5.32) by replacing $T^{\mu\nu}$ with the full source.

Let $r := |\boldsymbol{x}|$ and $\boldsymbol{n} := \boldsymbol{x}/r$. At large distances to slowly moving sources, we can expand (5.14) asymptotically as

$$
\begin{aligned}
\varphi(t,\boldsymbol{x}) \quad = \quad & -\frac{2G\alpha^2}{c^4 r}\Bigg\{\int_{\mathbb{R}^3} d^3x' T\left(t-r/c,\boldsymbol{x}'\right)+\frac{n_i}{c}\frac{\partial}{\partial t}\int_{\mathbb{R}^3} d^3x' T\left(t-r/c,\boldsymbol{x}'\right)x'^i \\
& +\frac{n_i n_j}{c^2}\frac{\partial^2}{\partial t^2}\int_{\mathbb{R}^3} d^3x' T\left(t-r/c,\boldsymbol{x}'\right)x'^i x'^j \qquad (5.15)\\
& -\alpha r\int_{-\infty}^{0} dt'\Bigg[F_1(t',r)\int_{\mathbb{R}^3} d^3x' T\left(t'+t-r/c,\boldsymbol{x}'\right) \\
& +n_i\left(\frac{1}{c}F_1(t',r)\frac{\partial}{\partial t}+\alpha F_2(t',r)\right)\int_{\mathbb{R}^3} d^3x' T\left(t'+t-r/c,\boldsymbol{x}'\right)x'^i \\
& +n_i n_j\left(\frac{1}{c^2}F_1(t',r)\frac{\partial^2}{\partial t^2}+\frac{\alpha}{c}F_2(t',r)\frac{\partial}{\partial t}+\alpha^2 F_3(t',r)\right) \\
& \qquad \cdot\int_{\mathbb{R}^3} d^3x' T\left(t'+t-r/c,\boldsymbol{x}'\right)x'^i x'^j\Bigg]\Bigg\},
\end{aligned}
$$

where we have used the notations

$$
\begin{aligned}
F_1(t,r) &:= \frac{J_1(\alpha c f(t,r))}{f(t,r)}, \qquad & F_2(t,r) &:= \frac{t J_2(\alpha c f(t,r))}{f^2(t,r)}, \qquad (5.16)\\
F_3(t,r) &:= \frac{t^2 J_3(\alpha c f(t,r))}{2 f^3(t,r)}, \qquad & f(t,r) &:= \sqrt{t^2-\frac{2tr}{c}}.
\end{aligned}
$$

Note that through the functions $F_n(t,r)$, the asymptotic field (5.15) depends transcendentally on the parameters r, α and c. For fixed n and $|z|\to\infty$, we can approximate the Bessel functions as

$$
J_n(z)=\sqrt{\frac{2}{\pi z}}\left[\cos(z-n\pi/2-\pi/4)+e^{|\mathrm{Im}(z)|}\mathcal{O}(|z|^{-1})\right] \qquad (5.17)
$$

For $r\to\infty$ and fixed $t<0$, the functions (5.16) behave up to constants as

$$
\begin{aligned}
|\alpha r(F_1(t,r))| &\lesssim \alpha^{1/2}\left(\frac{cr}{|t|^3}\right)^{1/4}, \qquad & |\alpha r(F_2(t,r))| &\lesssim \alpha^{1/2}\left(\frac{c^3}{|t|r}\right)^{1/4}, \qquad (5.18)\\
|\alpha r(F_3(t,r))| &\lesssim \alpha^{1/2}\left(\frac{c^5|t|}{r^3}\right)^{1/4}.
\end{aligned}
$$

As we will see later, the dependence on r also enters the formula for the energy loss of an isolated system by the emission of gravitational radiation.

We introduce the following momenta of the energy–momentum tensor,

$$
\begin{aligned}
M(t) &:= \int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x}), \qquad & S^{ij}(t) &:= \int_{\mathbb{R}^3} d^3x T^{ij}(t,\boldsymbol{x}), \qquad (5.19)\\
M^i(t) &:= \int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x})\,x^i, \qquad & S^{ijk}(t) &:= \int_{\mathbb{R}^3} d^3x T^{ij}(t,\boldsymbol{x})\,x^k,\\
M^{ij}(t) &:= \int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x})\,x^i x^j, \qquad & S^{ijkl}(t) &:= \int_{\mathbb{R}^3} d^3x T^{ij}(t,\boldsymbol{x})\,x^k x^l.
\end{aligned}
$$

Moreover, we define the following quantities:

$$\begin{aligned}
\mathcal{M}(t,r) \;&:=\; M(t-r/c) - S^{ii}(t-r/c) \\
&\quad - \alpha r \int_{-\infty}^{0} dt' \Big[F_1(t',r) \left(M(t'+t-r/c) - S^{ii}(t'+t-r/c) \right) \Big], \\
\mathcal{M}^i(t,r) \;&:=\; \frac{1}{c}\frac{\partial}{\partial t} \left(M^i(t-r/c) - S^{jji}(t-r/c) \right) \\
&\quad - \alpha r \int_{-\infty}^{0} dt' \Bigg[\left(\frac{1}{c} F_1(t',r) \frac{\partial}{\partial t} + \alpha F_2(t',r) \right) \\
&\qquad \cdot \left(M^i(t'+t-r/c) - S^{jji}(t'+t-r/c) \right) \Bigg], \\
\mathcal{M}^{ij}(t,r) \;&:=\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \Big(M^{ij}(t-r/c) - S^{kkij}(t-r/c) \Big) \\
&\quad - \alpha r \int_{-\infty}^{0} dt' \Bigg[\left(\frac{1}{c^2} F_1(t',r) \frac{\partial^2}{\partial t^2} + \frac{\alpha}{c} F_2(t',r) \frac{\partial}{\partial t} + \alpha^2 F_3(t',r) \right) \\
&\qquad \cdot \Big(M^{ij}(t'+t-r/c) - S^{kkij}(t'+t-r/c) \Big) \Bigg].
\end{aligned} \tag{5.20}$$

We can write the asymptotic fields (5.11) and (5.15) in terms of the momenta (5.19) and the quantities (5.20) as

$$\gamma^{ij}(t,\boldsymbol{x}) \;=\; \frac{2G}{c^6 r}\frac{\partial^2}{\partial t^2} M^{ij}(t-r/c), \tag{5.21}$$

$$\varphi(t,\boldsymbol{x}) \;=\; \frac{2G\alpha^2}{c^4 r}\Big[\mathcal{M}(t,r) + n_i \mathcal{M}^i(t,r) + n_i n_j \mathcal{M}^{ij}(t,r) \Big]. \tag{5.22}$$

Unlike the quantities γ^{ij}, the asymptotic field ϕ has not a $1/r$ dependence by the estimates (5.18). This is due to the fact that the field ϕ has a range $1/\alpha$ as per equation (5.6), such that the solutions for ϕ for an isolated system contain Yukawa like terms $e^{-\alpha r}$ [40].

5.4 The Energy-Momentum Complex

In order to derive the energy flux of a gravitational wave in $f(R)$ inspired scalar tensor theory, we need an analogue to the Landau–Lifshitz complex $t^{\mu\nu}_{LL}$ in GR [34]. This can be obtained by using the method in [45], where an energy-momentum complex in the Brans–Dicke theory is presented. Defining

$$X^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \frac{1}{8a\phi}(\phi-1)^2 g^{\mu\nu} - \frac{1}{\phi}\left(\nabla^\mu \nabla^\mu \phi - g^{\mu\nu} \Box_g \phi \right) \tag{5.23}$$

and

$$U^{\mu\nu\lambda\sigma} := \phi^2(-g)\left(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\sigma} g^{\lambda\nu} \right), \tag{5.24}$$

we can write the generalisation of the Landau–Lifshitz complex as

$$t^{\mu\nu} = \frac{c^4 \phi}{8\pi G}\left(\frac{1}{2\phi^2(-g)} \partial_\lambda \partial_\sigma U^{\mu\lambda\nu\sigma} - X^{\mu\nu} \right). \tag{5.25}$$

The energy-momentum conservation laws then can be cast into the form

$$\partial_\mu \left[\phi(-g)\left(T^{\mu\nu} + t^{\mu\nu}\right)\right] = 0. \tag{5.26}$$

Using (5.23) and (5.24), the energy momentum complex (5.25) can be expressed in terms of the fields $g_{\mu\nu}$, ϕ, their first partial derivatives and the connection coefficients $\Gamma^\lambda_{\mu\nu}$ as

$$\begin{aligned} t^{\mu\nu} &= \phi t^{\mu\nu}_{LL} + \frac{c^4}{16\pi G\phi}\left[2g^{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi - 2\partial^\mu\phi\partial^\nu\phi - \frac{1}{4a}(\phi-1)^2 g^{\mu\nu}\right] \\ &+ \frac{c^4}{8\pi G}\left[g^{\mu\nu}\left(2\partial^\lambda\phi\Gamma^\sigma_{\lambda\sigma} - \partial_\lambda\phi\Gamma^\lambda_{\sigma\rho}g^{\sigma\rho}\right) + g^{\mu\lambda}g^{\nu\sigma}\partial_\rho\phi\Gamma^\rho_{\lambda\sigma}\right. \\ &+ \left(\partial^\mu\phi\Gamma^\nu_{\lambda\sigma} + \partial^\nu\phi\Gamma^\mu_{\lambda\sigma}\right)g^{\lambda\sigma} - \left(\partial^\mu\phi g^{\nu\lambda} + \partial^\nu\phi g^{\mu\lambda}\right)\Gamma^\sigma_{\lambda\sigma} \\ &\left. - \left(g^{\mu\lambda}\Gamma^\nu_{\lambda\sigma} + g^{\nu\lambda}\Gamma^\mu_{\lambda\sigma}\right)\partial^\sigma\phi\right]. \end{aligned} \tag{5.27}$$

The energy flux in an arbitrary direction x^i is given by the component t^{0i}.

5.5 Energy Emission of Isolated Systems

Consider a plane gravitational wave propagating in vacuum in the x^i direction. In addition to the gauge (5.9), it is possible to perform a further gauge transformation which makes the φ independent part of the perturbation transverse and traceless (TT) [10, 17], such that we can write

$$h^{\mu\nu}(t - x^i/c) = \gamma^{\mu\nu}_{TT_i}(t - x^i/c) - 2a\eta^{\mu\nu}\varphi(t, x^i) \tag{5.28}$$

In this gauge, we evaluate the energy flux to leading order in the perturbation fields $h_{\mu\nu}$ and φ. By angle brackets we denote the average over a four-dimensional spacetime region which is much larger than a typical wavelength. The formula (5.27) then yields

$$t^{0i} = \frac{c^4}{32\pi G}\left\langle \partial_0\gamma^{jk}_{TT_i}\partial_0\gamma^{jk}_{TT_i} + \frac{2}{3\alpha^4}(\partial_0\varphi)^2\right\rangle. \tag{5.29}$$

The first term on the right hand side of (5.29) can be evaluated in the same way as in GR by means of (5.21) and the trace–free quadrupole tensor

$$Q^{ij}(t) = \int_{\mathbb{R}^3} d^3x\rho(t, \boldsymbol{x})\left(x^i x^j - \frac{r^2}{3}\delta_{ij}\right), \tag{5.30}$$

where $\rho(t, \boldsymbol{x})$ is the energy density of the source. From (5.22) we obtain for the second term on the right hand side of (5.29) approximately

$$\begin{aligned} \frac{2}{3\alpha^4}(\partial_0\varphi)^2 &= \frac{8G^2}{3c^{10}r^2}\Big[\dot{\mathcal{M}}^2 + 2n_i\dot{\mathcal{M}}\dot{\mathcal{M}}^i + n_in_j(\dot{\mathcal{M}}^i\dot{\mathcal{M}}^j + 2\dot{\mathcal{M}}\dot{\mathcal{M}}^{ij}) \\ &+ 2n_in_jn_k\dot{\mathcal{M}}^i\dot{\mathcal{M}}^{jk} + n_in_jn_kn_l\dot{\mathcal{M}}^{ij}\dot{\mathcal{M}}^{kl}\Big], \end{aligned} \tag{5.31}$$

where the dot denotes the derivative with respect to t. The total power of the source is obtained by an integration over a sphere with radius r:

$$P = \frac{G}{c}\left\langle \frac{1}{5c^4}\dddot{Q}^{ij}\dddot{Q}^{ij} + \frac{1}{3}\Big[\dot{\mathfrak{M}}^2 + \frac{1}{3}(\dot{\mathfrak{M}}^i\dot{\mathfrak{M}}^j + 2\dot{\mathfrak{M}}\dot{\mathfrak{M}}^{ij}) + \frac{1}{15}(\dot{\mathfrak{M}}^{ij}\dot{\mathfrak{M}}^{ij} + 2\dot{\mathfrak{M}}^{ii}\dot{\mathfrak{M}}^{jj})\Big]\right\rangle, \tag{5.32}$$

where the momenta $\mathfrak{M}^I$ are obtained from the momenta $\mathcal{M}^I$ by a multiplication with c^{-2}. With this last replacement we emphasise the fact that – apart from the c dependence of the functions (5.16) – the correction terms are proportional to c^{-1} to leading order.

Note that the additional degree of freedom represented by the scalar field predicts radiation of all multipoles, in particular monopoles and dipoles. These are of lower order than the original quadrupole contribution in GR and could thus lead to a measurable contribution in concrete applications. The formula (5.32) also contains scalar field contributions that originate from the octupole and hexadecapole moments of the source. This is a result of the expansion (5.15). However, the corresponding contributions of the metric have been neglected. A consistent division of the formula (5.32) in multipoles is preferably obtained after the choice of a source.

5.6 Conclusion and Outlook

We have derived the $f(R)$ correction terms to the GR quadrupole formula for the emission of gravitational radiation to leading order. An important result is that, in contrast to GR, quadratic $f(R)$ theory predicts the radiation of monopoles and dipoles. This is the case for nearly every alternative metric gravity theory [64].

The most interesting application of the formal result (5.32) is the energy loss of binary systems by the emission of gravitational radiation, in particular binary pulsars such as the PSR J0737-3039 system [37, 57]. However, such an application requires some caution. The fact that the non–relativistic motion of compact objects is governed by the Newtonian potential with an additional Yukawa correction implies that the Keplerian orbits also need appropriate corrections, c. f. for example [40]. Moreover, the accumulation of the wake in the field φ could principally lead to the divergence of the power of the source. A viable choice of a cut off time after which the wake contributions should be taken into account is not evident. It is physically not trivial to fathom the time at which a system starts to radiate significantly.

In order to obtain rough estimates for the correction terms in (5.32), it might be useful to apply the formula to the radiation of rotating rigid bodies or binary systems in circular orbits. While for the rigid body it is reasonable to define a time at which the radiation starts, the radiation of circular binaries remains a thought experiment from this point of view.

Appendix A

Riemann Tensor of a Plane Wave

The standard references to this appendix are [21, 64]. We will work in geometrised units with $c = 1$. Consider a Riemann tensor whose components depend only on the retarded time $u = t - x^3$,

$$R_{\mu\nu\lambda\rho} \equiv R_{\mu\nu\lambda\rho}(u), \tag{A.1}$$

which is thus symmetric under translations orthogonal to the (real) null direction normal to the wave vector $\partial^\mu u$. In what follows we will inspect the Riemann components with respect to the nulltetrad

$$\begin{aligned} k^\mu &= \frac{1}{\sqrt{2}}(1,0,0,1), \qquad & \ell^\mu &= \frac{1}{\sqrt{2}}(1,0,0,-1), \\ m^\mu &= \frac{1}{\sqrt{2}}(0,1,i,0), \qquad & \bar{m}^\mu &= \frac{1}{\sqrt{2}}(0,1,-i,0). \end{aligned} \tag{A.2}$$

The vector k^μ is proportional to the wave vector, $\sqrt{2}k^\mu = -\partial^\mu u$. The tetrad is illustrated in Figure A.1. Because of the property (A.1), the derivatives in all tetrad directions vanish except the one in the direction of ℓ^μ,

$$\ell^\mu \partial_\mu R_{\nu\lambda\rho\sigma} \not\equiv 0, \quad n^\mu \partial_\mu R_{\nu\lambda\rho\sigma} \equiv 0 \;\; (n^\mu \in \{k^\mu, m^\mu, \bar{m}^\mu\})\,. \tag{A.3}$$

It is convenient to introduce two sets of indices,

$$a, b, c, d, e \in \{k, \ell, m, \bar{m}\}, \quad p, q \in \{k, m, \bar{m}\}.$$

In this notation, the property (A.3) implies

$$\partial_\ell R_{abcd} \not\equiv 0, \quad \partial_p R_{abcd} \equiv 0. \tag{A.4}$$

The second Bianchi identities for the Riemann tensor reads

$$\partial_e R_{abcd} + \partial_d R_{abec} + \partial_c R_{abde} \equiv 0. \tag{A.5}$$

In particular, we have

$$\partial_\ell R_{abpq} + \partial_q R_{ab\ell p} + \partial_p R_{ab\ell q} \equiv 0, \tag{A.6}$$

from what we immediately deduce with (A.4)

$$\partial_\ell R_{abpq} \equiv 0. \tag{A.7}$$

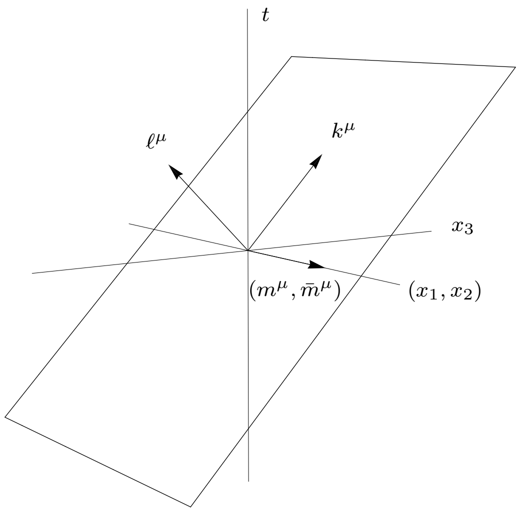

Figure A.1: Illustration of the nulltetrad (A.1). The (x_1, x_2)-plane is projected to one axis. The displayed plane represents the hypersurface $u = 0$. The derivatives of the Riemann tensor in directions parallel to this hypersurface vanish.

Thus the components R_{abpq} are constant. Since we analyse wave phenomena, we may assume $R_{abpq} \equiv 0$. Therefore, only the components $R_{ab\ell q}$ are non–vanishing. Using the symmetry $R_{abcd} \equiv R_{cdab}$, we finally conclude that all components vanish except the ones of the form $R_{\ell p \ell q}$. Up to complex conjugation, the Riemann tensor is thus determined by the four components

$$\begin{aligned} R_{\ell m \ell m},\, R_{\ell k \ell m} &\in \mathbb{C}\,, \\ R_{\ell m \ell \bar{m}},\, R_{\ell k \ell k} &\in \mathbb{R}\,. \end{aligned} \tag{A.8}$$

Appendix B

Null and Space Rotations About a Null Vector

In this appendix we will derive the most general proper orthochronous Lorentz Transformation $L^\mu{}_\nu \in SO^+(1,3)$ that preserves a given null vector n^μ. We use the conservation condition

$$L^\mu{}_\nu n^\nu = n^\mu \tag{B.1}$$

to determine a constraint on the elements of the Lie algebra of $SO^+(1,3)$. Assume $A^\mu{}_\nu \in so^+(1,3)$, $r \in \mathbb{R}$ and let

$$L^\mu{}_\nu = \exp(rA^\mu{}_\nu). \tag{B.2}$$

We will express the matrix of $A^\mu{}_\nu$ with respect to a coordinate basis. From $L_\mu{}^\lambda L^\rho{}_\nu \eta_{\lambda\rho} = \eta_{\mu\nu}$ it follows, with (B.2) after differentiation with respect to r and evaluation at $r = 0$, that

$$A^\mu{}_\nu = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix}, \tag{B.3}$$

where a, b, c, d, e and f are real numbers. Assume w.l.o.g. $n^\mu = (1, 0, 0, 1)$. Plugging (B.2) into (B.1) and differentiating with respect to r yields at the point $r = 0$

$$0 = \frac{d}{dr} n^\mu \Big|_{r=0} = \frac{d}{dr} \exp(rA^\mu{}_\nu) n^\nu \Big|_{r=0} = A^\mu{}_\nu n^\nu. \tag{B.4}$$

The matrix $A^\mu{}_\nu$ given in (B.3) fulfils (B.4) for $c = 0$, $e = -a$ and $f = -b$. Moreover, we can assume without loss of generality that $d = 1$, since an overall constant factor can be absorbed into the parameter r. Hence,

$$A^\mu{}_\nu = \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & 1 & -a \\ b & -1 & 0 & -b \\ 0 & a & b & 0 \end{pmatrix}. \tag{B.5}$$

The exponential of $rA^\mu{}_\nu$ can be evaluated using

$$(A^\mu{}_\nu)^2 = \begin{pmatrix} a^2+b^2 & -b & a & -a^2-b^2 \\ b & -1 & 0 & -b \\ -a & 0 & -1 & a \\ a^2+b^2 & -b & a & -a^2-b^2 \end{pmatrix}, \quad (A^\mu{}_\nu)^3 = -A^\mu{}_\nu. \tag{B.6}$$

Then we have

$$L^{\mu}{}_{\nu} = \exp(rA^{\mu}{}_{\nu}) = I + \sin(r)A^{\mu}{}_{\nu} + (1-\cos(r))(A^{\mu}{}_{\nu})^2 \tag{B.7}$$

with respect to the coordinate basis. By a change of basis, we find the matrix of $A^{\mu}{}_{\nu}$ with respect to the null tetrad

$$\begin{aligned} k^{\mu} &= \frac{1}{\sqrt{2}}(1,0,0,1), \qquad & \ell^{\mu} &= \frac{1}{\sqrt{2}}(1,0,0,-1), \\ m^{\mu} &= \frac{1}{\sqrt{2}}(0,1,i,0), \qquad & \bar{m}^{\mu} &= \frac{1}{\sqrt{2}}(0,1,-i,0), \end{aligned} \tag{B.8}$$

to be

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & \beta\bar{\beta} & \beta e^{ir} & \bar{\beta}e^{-ir} \\ 0 & 1 & 0 & 0 \\ 0 & \bar{\beta} & e^{ir} & 0 \\ 0 & \beta & 0 & e^{-ir} \end{pmatrix}, \tag{B.9}$$

where

$$\beta := \frac{1-e^{ir}}{\sqrt{2}}(b - ia). \tag{B.10}$$

Remark: The subgroup of $SO^{+}(1,3)$ that preserves a given null vector n^{μ} is called the *stabiliser* of n^{μ}. It is possible to show that this stabiliser is isomorphic to the Euclidean group $E(2)$ in two dimensions [59].

Appendix C

Intrinsic Spin Vector of an Orbiting Gyroscope

Let $S_\mu = (S_0, \boldsymbol{S})$ be the spin 4-vector of a gyroscope orbiting with a non–relativistic centre of mass 4-velocity $u^\mu = (u^0, u^0\boldsymbol{v})$. Up to third order in $1/c$, we want to express the intrinsic spin 3-vector $\boldsymbol{\mathcal{S}}$, as given in a local freely falling frame which is comoving with the gyroscope, in terms of S_μ in a space time endowed with a metric of the form

$$\begin{aligned} g_{\mu\nu} dx^\mu \otimes dx^\nu &= -\left(1 + \frac{2U_1}{c^2}\right) dx^0 \otimes dx^0 + \left(1 + \frac{2U_2}{c^2}\right) dx^i \otimes dx^i \\ &\quad + \frac{2W_i}{c^3} dx^0 \otimes dx^i, \end{aligned} \tag{C.1}$$

where U_1, U_2 and W_i are smooth functions of an appropriate open subset of $\mathbb{R}^2$. These functions represent potentials like the ones derived in Chapter 4. We express the metric (C.1) in terms of an orthonormal basis of 1-forms θ^μ,

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = -\theta^0 \otimes \theta^0 + \theta^i \otimes \theta^i. \tag{C.2}$$

This leads to

$$\begin{aligned} \theta^0 &= \left(1 + \frac{U_1}{c^2}\right) dx^0 - \frac{W_i}{c^3} dx^i, \\ \theta^i &= \left(1 + \frac{U_2}{c^2}\right) dx^i. \end{aligned} \tag{C.3}$$

We can approximate the connection between this basis and the comoving local freely falling frame by a Lorentz boost

$$L^\mu{}_\nu(\boldsymbol{w}) = \begin{pmatrix} \gamma & \gamma w^1/c & \gamma w^2/c & \gamma w^3/c \\ \gamma w^1/c & & & \\ \gamma w^2/c & & \delta_{ij} + \dfrac{w^i w^j(\gamma - 1)}{\boldsymbol{w}^2} & \\ \gamma w^3/c & & & \end{pmatrix}, \tag{C.4}$$

where $\boldsymbol{w}$ is the 3-velocity of the gyroscope with respect to the frame (C.3), and $\gamma := 1/\sqrt{1 - \boldsymbol{w}^2/c^2}$. The components of $\boldsymbol{w}$ can be expressed as

$$\begin{aligned} w^i &= \frac{\theta^i(u)}{\theta^0(u)} = \frac{(1 + U_2/c^2) u^i}{(1 + U_1/c^2) u^0 - W_i u^i / c^3} \\ &\approx \left(1 - \frac{U_1 + U_2}{c^2}\right) \frac{u^i}{u^0} = \left(1 - \frac{U_1 + U_2}{c^2}\right) v^i. \end{aligned} \tag{C.5}$$

Up to the required order, the spin 4-vector with respect to the frame (C.3), $\theta^\mu(S)$, is given in terms of the intrinsic spin 4-vector $\sigma^\mu = (0, \boldsymbol{\mathcal{S}})$ by

$$\theta^\mu(S) = L^\mu{}_\nu(\boldsymbol{w})\sigma^\nu = \left(\frac{w^j}{c}\mathcal{S}_j, \mathcal{S}_i + \frac{1}{2c^2}w^i(w^j\mathcal{S}_j)\right). \tag{C.6}$$

In terms of S^μ, on the other hand, its space components are given by

$$\theta^i(S) = \left(1 + \frac{U_2}{c^2}\right) S^i. \tag{C.7}$$

Combining (C.5), (C.6) and (C.7), we are left with

$$S^i = \left(1 - \frac{U_2}{c^2}\right)\mathcal{S}_i + \frac{1}{2c^2}v^i(v^j\mathcal{S}_j), \quad S_i = \left(1 + \frac{U_2}{c^2}\right)\mathcal{S}_i + \frac{1}{2c^2}v^i(v^j\mathcal{S}_j). \tag{C.8}$$

Hence, to a sufficient approximation,

$$\mathcal{S}_i = \left(1 - \frac{U_2}{c^2}\right) S_i - \frac{1}{2c^2}v^i(v^j S_j). \tag{C.9}$$

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