Cohomotopy Invariants and Gauge Theoretical Gromov-Witten Theory

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Markus Bader

aus

Winterthur ZH

Promotionskomitee

Prof. Dr. Christian Okonek (Vorsitz)
Prof. Dr. Andrei Teleman (Marseille)
Prof. Dr. Andrew Kresch

Zürich, 2009

Contents

Chapter 1

Introduction

The goal of this work is to introduce and classify certain classes of non-linear Fredholm maps in terms of stable homotopy groups and to define cohomotopy invariants in gauge theoretical Gromov-Witten theory.

1.1 A homotopy classification of certain classes of non-linear Fredholm maps

In a now classical result in non-linear analysis, Svarc gives an interpretation of a certain type of non-linear perturbations of linear Fredholm maps between Banach spaces in terms of stable homotopy groups of spheres ([42], see also [8]). Svarc associates with every coercive compact perturbation μ of a fixed linear Fredholm map d between two infinite dimensional Banach spaces a map between finite dimensional spheres, well defined up to suspension. Here we call a map coercive if preimages of bounded sets under this map are bounded. This idea to approximate maps between infinite dimensional spaces by maps between finite dimensional spaces in a certain way has been introduced by Furuta to gauge theory: Gauge theoretical moduli problems often lead to maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ between Hilbert bundles $\mathcal{E}$ and $\mathcal{F}$ over a compact base manifold B, equivariant with respect to a symmetry group G which acts fiberwise on the bundles $\mathcal{E}$ and $\mathcal{F}$. The classical approach to define invariants of such moduli problems is to consider the space of solutions $\mu^{-1}(0)$ modulo the G-action and to associate with it a (virtual) fundamental class, which is the invariant. In Seiberg-Witten theory, the map μ is called the monopole map; it is defined between Hilbert bundles over a compact torus and the symmetry group is S^1. In his work on the $\frac{11}{8}$-conjecture [17], Furuta studies a finite-dimensional approximation of the monopole map instead of the moduli space of monopoles itself. Subsequently, Bauer and Furuta [7] more formally introduced cohomotopy refined Seiberg-Witten invariants. Their joint publication is based on their independent preprints [4] and [16]. The underlying idea is the formalization of this new approach: to replace the moduli space of monopoles with a finite dimensional approximation of the monopole map, defined as an element in a stable cohomotopy group. In [34], Okonek and Teleman present a different framework for cohomotopy invariants. Let us point out some differences between the approach of Bauer and Furuta and of Okonek and Teleman which will be of importance:

Let G be a compact Lie group and let $\mathcal{E}$ and $\mathcal{F}$ be two G-Hilbert bundles over a compact base B with trivial G-action. Let $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be a G-equivariant fiberwise map, which is a compact perturbation of a linear Fredholm map d. Bauer and Furuta require the map μ to be coercive and they fix the linear Fredholm map d. Furthermore, they fix an infinite dimensional G-Hilbert space H (a universe) and assume that the fibers of $\mathcal{E}$ and $\mathcal{F}$ respectively are isomorphic to H. Then they associate with the map μ an element $\{\mu\} \in \pi^0_{G,H}(B; \operatorname{ind} d)$ in a certain stable homotopy group. The definition of this group uses a presentation of the index of the G-Fredholm morphism d.

Okonek and Teleman's invariants are defined in the case $G = S^1$ and when the action of S^1 is semifree. By that we mean that the bundles $\mathcal{E}$ and $\mathcal{F}$ split as fiber products $\mathcal{E} = \mathcal{E}_{\mathbb{R}} \times_B \mathcal{E}_{\mathbb{C}}$ and

$\mathcal{F} = \mathcal{F}_\mathbb{R} \times_B \mathcal{F}_\mathbb{C}$ consisting of a real bundle with trivial S^1-action and of a complex bundle with standard S^1-action. The real parts $\mathcal{E}_\mathbb{R}$ and $\mathcal{F}_\mathbb{R}$ are assumed to be trivialized.

The maps considered by Okonek and Teleman are smooth S^1-equivariant fiberwise maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ which satisfy a properness condition and which are compact perturbations of a linear Fredholm map d. This Fredholm morphism is the fiberwise differential of μ at the zero-section. It is allowed to vary on the complex part, whereas on the real part of the bundles, it is supposed to be a fixed fiberwise constant isometric embedding. Furthermore, a non-vanishing property is imposed: the image of the S^1-fixed locus under μ is assumed to avoid a neighbourhood of the zero section. This non-vanishing property already appeared in Furuta's preprint [16].

Okonek and Teleman associate with such a map μ an element $\{\mu\}$ in a group $\alpha^{p-1}(y)$, which is defined by an involved stabilization process. Here $(-p, y) \in \mathbb{Z} \times K(B)$ is the index of the linear Fredholm map d. In particular, Okonek and Teleman also stabilize with respect to the presentation $(E_\mathbb{C}, F_\mathbb{C})$ of the element $y \in K(B)$ in order to avoid a dependence of the groups on this presentation of y.

In [5, Theorem 2.1], Bauer proves that the map that associates with a homotopy class $[\mu]$ its Bauer-Furuta invariant $\{\mu\}$ is a bijection between the set of homotopy classes of maps μ they are considering and the group $\pi^0_{G,H}(B; \operatorname{ind} d)$ in which the invariants are defined. This result gives a geometric description of the stable homotopy groups as homotopy classes of maps between Hilbert bundles. It generalizes the result of Svarc obtained in the non-equivariant case when the base space B is a point.

We ask the following questions:

1. Is there a similar description for Okonek and Teleman's invariants?

2. Can the Okonek-Teleman invariants be generalized for general compact Lie groups G and arbitrary reprensentations?

3. Do the two approaches fit into a common framework and under what conditions can they be compared?

We will answer these questions completely. To do so, we present a general framework for constructing cohomotopy invariants that leads to a generalization of both Bauer and Furuta's and Okonek and Teleman's construction. This generalization allows us to explain the conceptual differences between the respective approaches both on the level of homotopy classes of maps between infinite dimensional bundles and on the level of stable homotopy groups.

We use a new approach to define cohomotopy invariants. It starts with the following two questions:

1. What is finite dimensional approximation?

2. What are the finest possible cohomotopy invariants?

The leading idea is that the finite dimensional approximation of an admissible map μ should capture its homotopy type. This rather vague statement is made precise by a reformulation of the notion of finite dimensional approximation. We define a suitable set-valued functor on a category whose objects are composed of finite dimensional G-bundles. Then we realize the set of homotopy classes of maps μ between two fixed infinite dimensional G-Hilbert bundles as a colimit of this functor. This approach leads to a conceptually different understanding of what finite dimensional approximation is. It allows the reduction to a finite dimensional setting in a formal way: by using the universal property of the colimit. Thus, the usual arguments to show that a construction is independent of a particular choice of approximation become superfluous. Second, this approach allows the construction of group valued invariants in a natural way: The idea is to replace the original set valued functor by a group valued functor and to take the colimit of this new functor. Using the language of functors we can formulate precisely in what sense the group valued invariants are the finest possible.

We now explain our main results in more details. We consider G-equivariant maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ between a special class of G-equivariant Hilbert bundles. Our bundles may contain a real summand, not necessarily trivialized, equipped with the trivial G-action, and a complex summand consisting of a unitary G-bundle. In fact we fix the irreducible representations of G that may occur by choosing an arbitrary subset $\rho \subset \mathrm{Irr}(G, \mathbb{C}) \cup \{\mathbb{R}\}$. We introduce a non-vanishing property in terms of a fixed isotropy family Ω of subgroups of G. It generalizes the non-vanishing property imposed by Okonek and Teleman on their class of maps. The maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ we study have the following three properties:

1. The map μ is coercive.
2. There is a linear G-Fredholm morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$, such that $\mu - d$ is compact.
3. The image of $\mathcal{E}(\Omega)$ under μ avoids a neighbourhood of the 0-section in $\mathcal{F}$. Here $\mathcal{E}(\Omega)$ denotes the subset of $\mathcal{E}$ consisting of the points whose stabilizer is an element of Ω.

Such a map μ we call Ω-(Fredholm) map. There is a natural notion of Ω-(Fredholm) homotopy and we denote the set of homotopy classes of Ω-maps by ${}_G[\mathcal{E}, \mathcal{F}]_B^\Omega$. Our first result concerns finite dimensional approximation. We introduce a category ${}_\rho\mathcal{H}_B$, whose objects are pairs of finite dimensional G-Hilbert bundles. The notion of Ω-maps leads to a natural functor $\mathfrak{p}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Set}$ which associates with a pair (E, F) of G-bundles the set of homotopy classes of Ω-maps $m : E \longrightarrow F$. We can now formulate the principle of finite dimensional approximation in the following way:

Theorem 1.1.1. *There is a natural transformation $p : \mathfrak{p}_\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega$ between the functor $\mathfrak{p}_\Omega$ and the constant functor ${}_G[\mathcal{E}, \mathcal{F}]_B^\Omega$. This natural transformation is a colimit.*

It is important to observe that we concretely construct the natural transformation p and that it does not depend on any further choices.

There is now a natural way to produce group valued invariants: Let (π_Ω, τ) be a pair consisting of a functor $\pi_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Group}$ and a natural transformation $\tau : \mathfrak{p}_\Omega \longrightarrow U\pi_\Omega$, where $U : \mathrm{Group} \longrightarrow \mathrm{Set}$ denotes the forgetful functor. We obtain a natural map

$$ {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega = \mathrm{colim}\, \mathfrak{p}_\Omega \longrightarrow \mathrm{colim}\, U\pi_\Omega \longrightarrow U \,\mathrm{colim}\, \pi_\Omega. \tag{1.1} $$

Here the first map is induced by the natural transformation τ and the second map is the natural map given by the universal property of $\mathrm{colim}\, U\pi_\Omega$. The image of the homotopy class $[\mu]$ is then the group valued invariant. In order to loose as little information as possible, it is natural to look for a pair (π_Ω, τ), such that the map $\mathrm{colim}\, \mathfrak{p}_\Omega \longrightarrow \mathrm{colim}\, U\pi_\Omega$ induced by τ is a bijection. In our situation, there is a natural model for such a pair (π_Ω, τ). The idea is to replace proper maps between finite dimensional bundles with maps between their one point compactifications: Recall that Ω-maps between bundles E and F are by definition proper. They extend to maps between the fiberwise one point compactified bundles E_B^+ and F_B^+.

We define a functor

$$ \mathfrak{p}_\Omega^+ : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Set},\ (E, F) \mapsto {}_G[E_B^+, F_B^+]_B^\Omega. \tag{1.2} $$

Clearly, there is a natural transformation $+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}_\Omega^+$ that associates with an Ω-map $m : E \longrightarrow F$ its extension $m^+ : E_B^+ \longrightarrow F_B^+$. We obtain the following result:

Theorem 1.1.2. *The natural transformation $+$ is stably bijective. In particular, the induced map*

$$ \mathrm{colim}\, \mathfrak{p}_\Omega \longrightarrow \mathrm{colim}\, \mathfrak{p}_\Omega^+ \tag{1.3} $$

is a bijection.

We prove this theorem using a very concrete geometric construction that shows essentially that after stabilization with a trivial real summand, any Ω-map $n : E_B^+ \longrightarrow F_B^+$ becomes homotopic to a map of the form m^+. There are two important points to make: First, it is essential that we are allowed to stabilize with a trivial real summand. Second, we work with Ω-homotopies fibered over a base. Because our construction is very concrete, it is straightforward to verify that it yields indeed a fiberwise Ω-homotopy in the sense we need. Notice that Bauer uses a similar statement to prove surjectivity in [5, Theorem 2.1].

In the next step, we define a group valued functor π_Ω in a straightforward way, so that $U\pi_\Omega$ is naturally isomorphic to $\mathfrak{p}_\Omega^+$. Then we put ${}_\rho\mathbb{P}_\Omega(B) := \mathrm{colim}\, \pi_\Omega$. The induced map

$$ \{\,\} : {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega = \mathrm{colim}\, \mathfrak{p}_\Omega \xrightarrow{\hat{=}} \mathrm{colim}\, \mathfrak{p}_\Omega^+ \xrightarrow{\hat{=}} \mathrm{colim}\, U\pi_\Omega \longrightarrow U \,\mathrm{colim}\, \pi_\Omega = U\,{}_\rho\mathbb{P}_\Omega(B). \tag{1.4} $$

is our cohomotopy invariant.

Asking for a version of Bauer's theorem [5, Theorem 2.1] in this context means asking in what sense forming the colimit of π_Ω commutes with the forgetful functor. The observation one has to make is that the category ${}_\rho\mathcal{H}_B$ is not connected. It has a decomposition ${}_\rho\mathcal{H}_B = \coprod_{x \in K_\rho(B)} \mathcal{H}_B(x)$ into connected components indexed by a group $K_\rho(B)$. This group is by definition the subgroup of $KO(B) \times K_G(B)$, generated by $KO(B)$ and by elements of the form $[W_\chi] \otimes K(B)$ where W_χ is an irreducible representation with character $\chi \in \rho$. The decomposition into connected components induces decompositions

- $\pi_\Omega = \oplus_{x \in K_\rho(B)} \operatorname{colim} \pi_\Omega(x)$
- $\operatorname{colim} U\pi_\Omega = \coprod_{x \in K_\rho(B)} \operatorname{colim} U\pi_\Omega(x)$

where $\pi_\Omega(x)$ is the functor obtained as the restriction of π_Ω to the subcategory ${}_\rho\mathcal{H}_B(x)$. We find a similar decomposition ${}_G[\mathcal{E}, \mathcal{F}]^\Omega_B = \coprod_{x \in K_\rho(B)} {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B(x)$. Here ${}_G[\mathcal{E}, \mathcal{F}]^\Omega_B(x)$ is the set of homotopy classes of Ω-maps μ which are of the form $\mu = d + c$ with a linear Fredholm map of index x. Similarly to the definition of $K_\rho(B)$ we can also define $K^{-1}_\rho(B)$. We put ${}_\rho\mathbb{P}_\Omega(B, x) := \operatorname{colim} \pi_\Omega(x)$. Then we obtain the following result:

Theorem 1.1.3. *Assume that $K^{-1}_\rho(B) = 0$. Then the map*

$$ {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B(x) \longrightarrow {}_\rho\mathbb{P}_\Omega(B, x) \tag{1.5} $$

is bijective.

The maps considered by Bauer and Furuta are compact perturbations of a fixed linear Fredholm map. Okonek and Teleman consider compact perturbations of a linear Fredholm map whose real part is fixed. In order to generalize their constructions, we introduce the notion of framed Ω-Fredholm maps. Let $\rho_0 \subset \rho$ be a subset and put $\rho_* := \rho - \rho_0$. Then every ρ-bundle $\mathcal{E}$ has a canonical decomposition $\mathcal{E} = \mathcal{E}_0 \times_B \mathcal{E}_*$, where $\mathcal{E}_0$ is a ρ_0-bundle and $\mathcal{E}_*$ is a ρ_*-bundle. Fix now a linear Fredholm map $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$. Then an Ω-Fredholm map μ is l-framed if it has a decomposition $\mu = d + c$ with a compact map c and a linear Fredholm map d of the form $d = l \times d_*$. Let ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B$ denote the set of homotopy classes of l-framed Ω-Fredholm maps.

Similarly to the unframed case, we can define a category whose objects are composed of finite dimensional G-bundles. Let now $V \subset \mathcal{F}_0$ be a finite dimensional subbundle with the properties that $l^{-1}(V)$ is a finite dimensional subbundle of $\mathcal{E}_0$ and that the pair $(l^{-1}(V), V)$ represents the index of l (we call such a bundle l-adapted). Then we can define a set valued functor $\mathfrak{p}^{l,V}_\Omega$ by considering homotopy classes of Ω-maps between finite dimensional G-bundles. We obtain the following result:

Theorem 1.1.4. *There is a natural transformation $p_{l,V} : \mathfrak{p}^{l,V}_\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B$ between the functor $\mathfrak{p}^{l,V}_\Omega$ and the constant functor ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B$. This natural transformation is a colimit.*

The following two observations are important: In order to define the functor $\mathfrak{p}^{l,V}_\Omega$ only the presentation $(l^{-1}(V), V)$ of the index of l is needed. However we need the embeddings $l^{-1}(V) \subset \mathcal{E}_0$ and $V \subset \mathcal{F}_0$ to define the natural transformation $p_{l,V}$. This means that our finite dimensional approximation and therefore our invariants depend on this data.

Very similarly to the unframed case, we define a group valued functor $\pi^{l,V}_\Omega$ and a stably bijective natural transformation $\mathfrak{p}^{l,V}_\Omega \longrightarrow U\pi^{l,V}_\Omega$. Again, there is a natural map induced by this data:

$$ \{\ \}_l : {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B = \operatorname{colim} \mathfrak{p}^{l,V}_\Omega \longrightarrow \operatorname{colim} U\pi^{l,V}_\Omega \longrightarrow U \operatorname{colim} \pi^{l,V}_\Omega =: {}_\rho\mathbb{P}^{l,V}_\Omega(B). \tag{1.6} $$

It realizes our invariants in a group.

As before, there are natural decompositions of the set ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B$ and of the group ${}_\rho\mathbb{P}^{l,V}_\Omega(B)$:

- ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B = \coprod_{y \in K_{\rho_*}(B)} {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y)$
- ${}_\rho\mathbb{P}^{l,V}_\Omega(B) = \oplus_{y \in K_{\rho_*}(B)} \mathbb{P}^{l,V}_\Omega(B)$

It turns out that the restriction imposed on the linear part has a consequence to the bijectivity of the invariant map:

Theorem 1.1.5. *Assume that $K^{-1}_{\rho_*}(B) = 0$. Then the invariant map*

$$\{\ \}_l : {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \longrightarrow {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \tag{1.7}$$

is bijective for any $y \in K_{\rho_*}(B)$.

The assumption of the previous theorem is of course satisified when $\rho = \rho_0$. The statement then becomes an analogon to Bauer's theorem [5, Theorem 2.1]. Last, we compare the framed with the unframed invariants with the following outcome:

Theorem 1.1.6. *Let $y \in K_{\rho_*}(B)$ and set $x := (\mathrm{ind}_{\rho_0} l, y) \in K_\rho(B)$. Then there are natural surjective maps*

- ${}_G[\mathcal{E}, \mathcal{F}]^l_B(y) \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B(x)$
- ${}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}_\Omega(B, x)$

They are bijective when $K^{-1}_{\rho_0}(B) = 0$.

Assume now that we are in the situation where the Okonek-Teleman invariants are defined: Let $G = S^1$, let $\mathcal{E} = \mathcal{E}_\mathbb{R} \times_B \mathcal{E}_\mathbb{C}$ and $\mathcal{F} = \mathcal{F}_\mathbb{R} \times_B \mathcal{F}_B$ be two Hilbert bundles, where $\mathcal{E}_\mathbb{R}$ and $\mathcal{F}_\mathbb{R}$ are trivial real bundles with trivial S^1-action and where $\mathcal{E}_\mathbb{C}$ and $\mathcal{F}_\mathbb{C}$ are complex Hilbert bundles with standard S^1-action. Let $l : \mathcal{E}_\mathbb{R} \longrightarrow \mathcal{F}_\mathbb{R}$ be a fiberwise constant linear embedding and let $V \subset \mathcal{F}_\mathbb{R}$ be a complement of the image of l of dimension p. Furthermore, fix an element $y \in K(B)$. In [34], Okonek and Teleman introduce the stable cohomotopy group $\alpha^{p-1}(y)$. We set $\rho = \{\mathbb{R}, \mathrm{id}_{S^1}\}$, and $\Omega = \{S^1\}$. Then we prove:

Theorem 1.1.7. *There is a natural isomorphism*

$$\alpha^{p-1}(y) \xrightarrow{\cong} {}_\rho\mathbb{P}^{l,V}_\Omega(B, y). \tag{1.8}$$

The Bauer-Furuta cohomotopy groups $\pi^0_{G,H}(B; \mathrm{ind}\, d)$ are obtained by putting $\Omega = \emptyset$ and by fixing the linear map $d : \mathcal{E} \longrightarrow \mathcal{F}$ on the whole space (i.e by putting $\rho_0 := \rho$).

It is clear from this description that there are two conceptual differences between the Okonek-Teleman and the Bauer-Furuta invariants: First, the choice of ρ_0, i.e. the locus where the linear Fredholm map fixed. Second, the choice of Ω. We show that when $K^{-1}(B) = 0$, there is a natural comparison map

$$\alpha^{p-1}(y) \longrightarrow \pi^0_{G,H}(B; \mathrm{ind}\, d), \tag{1.9}$$

between the two stable cohomotopy groups which maps the Okonek-Teleman invariant to the Bauer-Furuta invariant.

1.2 Cohomotopy invariants in gauge theoretical Gromov Witten theory

Gauge theoretical Gromov-Witten theory is concerned with the moduli space of symplectic vortices on Riemann surfaces. They have been studied by Mundet motivated by his work on the Kobayashi-Hitchin correspondence ([30], [29], [28]), by Cielebak, Gaio, Mundet, and Salamon ([12] and [11]) in an attempt to describe Gromov-Witten invariants of symplectic quotients, by Okonek and Teleman in [32], and by Mundet and Tian in [31].

Bauer asks whether there are homotopy interpretations of Donaldson or Gromov-Witten invariants ([5, page 43]). We define cohomotopy invariants in Abelian gauge theoretical Gromov-Witten theory. Let G be a compact Abelian Lie group acting unitarily on a Hermitian vector space V and let μ :

$V \longrightarrow \mathfrak{g}^*$ be a moment map for this action. Let Σ be a compact Riemann surface. Then the vortices over Σ are the solutions of the following vortex equations:

$$\begin{aligned} \bar{\partial}_a \varphi &= 0 \\ *F_a + \mu(\varphi) &= 0 \end{aligned} \tag{1.10}$$

Here a is a connection in a principal G-bundle P and $\varphi : P \longrightarrow V$ is a G-equivariant map. This is the moduli problem explained for instance in [12]. In fact, we work with a more general moduli problem introduced by Okonek and Teleman in [32]. It takes into account an additional symmetry coming from the fact that G is assumed to be a subgroup of a (not necessarily Abelian) compact Lie group $\hat{G}$, and that the action of G is induced by a unitary action of $\hat{G}$ on V. In this introduction we restrict to the simpler case $G = \hat{G}$ in order to simplify the notation.

The vortex equations give rise to the vortex map

$$v : \mathcal{A}(P) \times A^0(P \times_G V) \longrightarrow A^0(\Sigma, \mathfrak{g}) \times A^{0,1}(P \times_G V),\ (a, \varphi) \mapsto \left(*F_a + \mu(\varphi), \bar{\partial}_a \varphi\right). \tag{1.11}$$

This map is invariant with respect to the natural action of the gauge group $C^\infty(\Sigma, G)$. In order to apply the cohomotopy theory developed in the first part, we are faced with the following questions:

1. Can the vortex map V be replaced by a map $\nu : \mathcal{E} \longrightarrow \mathcal{F}$ between G-Hilbert bundles, defined over a compact base manifold?

2. Is the map ν a Fredholm map?

The solution to the first problem is to reduce the setup in a natural way. We want to avoid complications arising from considering the infinite dimensional gauge group $C^\infty(\Sigma, G)$. Because we are in the Abelian case, this can be achieved in a standard way (see [43] and [15]). To obtain Hilbert bundles, we need to work with Sobolev completions with respect to L^2_k-norms throughout. This is a technical, but important point. We produce G-Hilbert bundles $\mathcal{E}_k$ and $\mathcal{F}_k$ over a compact torus B of dimension $b_1(\Sigma) \dim \mathfrak{g}$ in a natural way. The answer to the first question is then given by the following proposition:

Proposition 1.2.1. *The vortex map induces a map*

$$\nu_k : \mathcal{E}_{k+1} \longrightarrow \mathcal{F}_k \tag{1.12}$$

fibered over the compact torus B.

A careful analysis of the map ν_k shows that it is indeed a compact perturbation of a linear Fredholm map. Our main result in this section says that when $k \geq 3$, then the map ν_k so obtained is coercive, i.e. preimages of bounded sets under ν are bounded. This result generalizes the compactness results obtained by Okonek and Teleman in [32, Theorem 2.12] and by Cielebak, Gaio, and Salamon in [12, Proposition 3.5]. In fact, we prove the following stronger statement:

Theorem 1.2.2. *There exists a polynomial $P \in \mathbb{R}[X]$, such that*

$$||e||_{k+1} \leq P(||\nu_k(e)||_k) \text{ for all } e \in \mathcal{E}_{k+1}. \tag{1.13}$$

Therefore, the map ν_k is a Fredholm map. Let $\rho \subset \{\mathbb{R}\} \cup \mathrm{Irr}(G, \mathbb{C})$ be a subset containing $\mathbb{R}$ and every irreducible character that appears in the representation of G on the Hermitian vector space V. Then we obtain a cohomotopy invariant $\{\nu_k\} \in {}_\rho\mathbb{P}_\emptyset(B)$.

Our analysis yields more: The real part of the bundles $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ is trivial and the fibers of the bundles $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ over a point b are given as follows:

- $\mathcal{E}_{k+1}(b) \cong (\mathrm{im}\, d^*)_{k+1} \times A^0 \left(P \times_G V\right)_{k+1}$;
- $\mathcal{F}_k(b) \cong A^0(\Sigma, \mathfrak{g})_k \times A^{0,1} \left(P \times_G V\right)_k$.

Furthermore, the map ν_k is a compact perturbation of a linear Fredholm map, whose real part is given by the map $l : (\operatorname{im} d^*)_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k,\ s \mapsto *ds$. The image of l is $(\operatorname{im} d^*)_k$, which has the complement $V := \ker d \cong \mathfrak{g}$. Additionally, let Ω be the isotropy family of all positive dimensional subgroups of G. We prove that in many interesting situations ν_k is a Ω-Fredholm map. Therefore the map ν_k defines an element $[\nu_k]_l \in {}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}$ and thus the following cohomotopy invariant:

$$\{\nu_k\}_l \in {}_\rho\mathbb{P}_\Omega^{l,V}(B). \tag{1.14}$$

Chapter 2

Preliminaries

In this chapter, we collect known results from topology, fiberwise topology, representation theory, index theory, and category theory.

2.1 Proper maps

We collect several well known facts about proper maps. The main reference is [9, §I.10]. Map always means continuous map and X, Y denote topological spaces if not specified otherwise.

Definition 2.1.1. A map $f : X \longrightarrow Y$ is proper if for all topological spaces Z, the map $f \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$ is closed.

Theorem 2.1.2. *A map $f : X \longrightarrow Y$ is proper if and only if f is closed and $f^{-1}(y) \subset X$ is quasi-compact for every $y \in Y$.*

Proof. [9, §I.10.2, Theorem 1]. □

Proposition 2.1.3. *Let $f : X \longrightarrow Y$ be an injective map. Then the following three statements are equivalent:*

1. *f is proper.*
2. *f is closed.*
3. *f is a homeomorphism of X onto a closed subset of Y.*

Proof. [9, §I.10.2, Proposition 2]. □

Proposition 2.1.4. *Let $f : X \longrightarrow Y$ be a proper map, and let K be a quasi-compact subset of Y. Then $f^{-1}(K)$ is quasi-compact.*

Proof. [9, §I.10.2, Proposition 6]. □

There are two classes of spaces for which the converse is true as well:

Proposition 2.1.5. *Let X be a Hausdorff space and let Y be a locally compact space. Then a map $f : X \longrightarrow Y$ is proper if and only if $f^{-1}(K)$ is compact for every compact subset K of Y. Furthermore, if f is proper, then X is locally compact.*

Proof. [9, §I.10.2, Proposition 7]. □

Corollary 2.1.6. *Let X and Y be two locally compact spaces with Alexandroff compactifications X^+ and Y^+. Then a map $f : X \longrightarrow Y$ is proper if and only if $f^+ : X^+ \longrightarrow Y^+$ is continuous.*

Proposition 2.1.7. *Let X and Y be Banach spaces. Then a map $f : X \longrightarrow Y$ is proper if and only if $f^{-1}(K)$ is compact for every compact subset K of Y.*

Proof. [8, Theorem 2.7.1]. □

Definition 2.1.8. A map $f : X \longrightarrow Y$ between Banach spaces is coercive if $f^{-1}(B)$ is bounded for every bounded subset $B \subset Y$.

Lemma 2.1.9. *A map $f : X \longrightarrow Y$ is coercive if and only if $f(x_n) \to \infty$ for every sequence $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \to \infty$.*

Proof. Let f be coercive and let (x_n) be a sequence in X with $x_n \to \infty$. Let $R > 0$. Since f is coercive, $f^{-1}(D_R(Y)) \subset X$ is bounded. Since $x_n \to \infty$, there exists $n_0 \in \mathbb{N}$, such that $x_n \notin f^{-1}(D_R(Y))$ for all $n \geq n_0$. Hence $|f(x_n)| > R$ for all $n \geq n_0$. Therefore $f(x_n) \to \infty$. If f is not coercive, there exists $R > 0$, such that $f^{-1}(D_R(Y)) \subset X$ is unbounded. Thus, for every $n \in \mathbb{N}$ there exists $x_n \in f^{-1}(D_R(Y))$ with $|x_n| > n$. Then $x_n \to \infty$, but $|f(x_n)| < R$ for all $n \in \mathbb{N}$. □

Proposition 2.1.10. *Let $f : X \longrightarrow Y$ be a coercive map between Banach spaces. Then f is proper if either*

1. *f is a compact perturbation of a proper mapping; or*
2. *X is reflexive, and $x_n \rightharpoonup x$ weakly in X with $(f(x_n))$ strongly convergent, implies that $x_n \to x$ strongly.*

Proof. [8, §2.7.2]. □

Following Bauer & Furuta [7], we define

Definition 2.1.11. Let X and Y be Banach spaces. A map $f : X \longrightarrow Y$ is Fredholm, if there exists a linear Fredholm map $d : X \longrightarrow Y$, such that $f - d : X \longrightarrow Y$ is compact.

Proposition 2.1.12. *Let X, Y be Banach spaces and let $f : X \longrightarrow Y$ be a Fredholm map.*

1. *Let $A \subset X$ be a bounded and closed subspace. Then $f|_A : A \longrightarrow Y$ is proper.*
2. *If f is coercive, then it is proper.*

Proof. [7, Lemma 2.2]: They formulate it for Hilbert spaces, but the proof holds also for Banach spaces. □

In a remark following [7, Lemma 2.2], Bauer and Furuta give an example of a Fredholm map $f : H \longrightarrow H$ with H a Hilbert space, which is proper, but not coercive.

2.2 Fiberwise topology

We collect the basic definitions and results from fiberwise topology that we need in this work. The references are [13] and [19]. For simplicity we assume all our spaces to be Hausdorff. We fix a base space B. A fiberwise space (with base B) is a topological space X, together with a map $p : X \longrightarrow B$. When W is a topological space then we write $\underline{W} := W \times B$ for the trivial fiberwise space with base B and fiber W.

Definition 2.2.1. A fiberwise space $p : X \longrightarrow B$ is fiberwise open (closed, compact), if the projection p is open (closed, proper).

Proposition 2.2.2. *A fiberwise space X is fiberwise closed if and only if for each fiber X_b of X and each neighbourhood U of X_b in X, there exists a neighbourhood W of b, such that $X_W \subset U$.*

Proof. [19, Proposition 1.8]. □

Proposition 2.2.3. *Let X be a fiberwise space, let $(B_j)_{j \in J}$ be an open cover of B. If $X_{B_j} \longrightarrow B_j$ is fiberwise open (closed, compact) for all $j \in J$, then X is fiberwise open (closed, compact).*

Proof. [19, Proposition 1.15, Proposition 3.9]. □

Example 2.2.4. Fiber bundles are fiberwise open because every trivial fiberwise space is fiberwise open. Fiber bundles with compact typical fiber are fiberwise compact.

Proposition 2.2.5. *A fiberwise space X is fiberwise compact if and only if it is fiberwise closed and the fiber X_b is compact for all $b \in B$.*

Proof. Immediate consequence of Theorem 2.1.2. □

Proposition 2.2.6. *A fiberwise space X is fiberwise compact if and only if for each fiber X_b of X and each overing Γ of X_b by open sets of X there exists a neighbourhood W of b, such that a finite subfamily of Γ covers X_W.*

Proof. [19, Proposition 3.3]. □

Proposition 2.2.7. *Let $A \subset X$ be a fiberwise compact subspace of a fiberwise space X. Then A is closed.*

Proof. Let $x \in X - A$ and set $b := p(x) \in B$. For each $a \in A_b$ choose open and disjoint neighbourhoods $U_a \subset X$ of a and $V_a \subset X$ of x. It follows that $A_b \subset \cup_{a \in A_b} U_a$. By the previous proposition, we may choose a finite number of points $a_1, \dots, a_n \in A_b$ and an open neighbourhood $W \subset B$ of b, such that $A_W \subset \cup_{i=1}^n U_{a_i} =: U$. Put $V := \cap_{i=1}^n V_{a_i} \subset X$. It is an open neighbourhood of $x \in X$ and $U \cap V = \emptyset$. Then $V_W := V \cap X_W$ is still an open neighbourhood of x and $V_W \cap A = V_W \cap A_W = \emptyset$. Hence $X - A \subset X$ is open. □

Proposition 2.2.8. *Let $f : X \longrightarrow Y$ be a fiberwise map between fiberwise spaces. If X is fiberwise compact, then so is $f(X)$.*

Proof. [19, Proposition 3.7]. □

The following observation shows that for compact base spaces B there is no difference between the notions "compact" and "fiberwise compact".

Lemma 2.2.9. *Let $p : X \longrightarrow B$ be a fiberwise space.*

1. *If X is compact, then X is fiberwise compact.*
2. *If B is compact and if X is fiberwise compact, then X is compact.*

Proof. If X is compact, then the projection $p : X \longrightarrow B$ is closed and has compact fibers. If $p : X \longrightarrow B$ is proper, then $X = p^{-1}(B)$ is compact if B is compact. □

Lemma 2.2.10. *Let $f : X \longrightarrow Y$ be a proper fiberwise map between fiberwise spaces. Let $C \subset Y$ be fiberwise compact. Then $f^{-1}(C)$ is fiberwise compact.*

Proof. Since f is proper, so is $f|_{f^{-1}(C)} : f^{-1}(C) \longrightarrow C$. The subspace C is fiberwise compact, i.e. the projection $C \longrightarrow B$ is proper. The statement follows from the fact that f is a fiberwise map and that the composition of two proper maps is proper. □

Definition 2.2.11. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A fiberwise topological $\mathbb{K}$-vector space is a fiberwise space X, together with fiberwise maps

$$X \times_B X \longrightarrow X \text{ and } \mathbb{K} \times X \longrightarrow X, \tag{2.1}$$

which induce in each fiber X_b the structure of a topological vector space, compatible with the relative topology of X_b.

A fiberwise normed ($\mathbb{K}$-vector) space is a fiberwise topological $\mathbb{K}$-vector space X, together with a map

$$|\,| : X \longrightarrow \mathbb{R}, \tag{2.2}$$

which induces in each fiber X_b a norm compatible with the topology on X_b. It is a fiberwise ($\mathbb{K}$-)Banach space if the fibers equipped with that norm are Banach spaces.

A fiberwise ($\mathbb{K}$-)Hilbert space is a fiberwise topological $\mathbb{K}$-vector space, together with a map

$$\langle\,,\,\rangle : X \times_B X \longrightarrow \mathbb{K}, \tag{2.3}$$

which induces in each fiber X_b a scalar product making the fiber into a Hilbert space.

Notation 2.2.12. Let X be a fiberwise normed space and let $c \in \mathbb{R}$. Then we write $X_{\geq c} := \{x \in X \,\big|\, |x| \geq c\}$.

Definition 2.2.13. Let X be a fiberwise normed space. A subset $A \subset X$ is fiberwise bounded, if there exists a map $b : B \longrightarrow \mathbb{R}$, such that $|a| \leq b(p(a))$ for all $a \in A$.

Definition 2.2.14. Let $f : X \longrightarrow Y$ be a fiberwise map between fiberwise normed spaces. It is fiberwise compact, if for any fiberwise bounded subset $S \subset X$, there exists a fiberwise compact subspace $C \subset Y$, such that $f(S) \subset C$.

Definition 2.2.15. A fiberwise map $f : X \longrightarrow Y$ between fiberwise normed spaces is fiberwise coercive, if $f^{-1}(T) \subset X$ is fiberwise bounded for every fiberwise bounded subset $T \subset Y$.

Definition 2.2.16. A fiberwise map $f : X \longrightarrow Y$ between fiberwise normed spaces is fiberwise bounded, if $f(S)$ is fiberwise bounded for every fiberwise bounded subset $S \subset X$.

Lemma 2.2.17. *Let X be a fiberwise normed space with finite dimensional fibers. Let $A \subset X$ be a fiberwise bounded, fiberwise closed, and closed subspace. Then A is fiberwise compact.*

Proof. It follows from the assumptions that each fiber $A_b \subset X_b$ is a bounded and closed subspace of a finite dimensional normed vector space. Hence it is compact. Since A is fiberwise closed, the statement follows from Proposition 2.2.5. □

Proposition 2.2.18. *A closed subspace $A \subset B \times \mathbb{R}^n$ is fiberwise compact if it is fiberwise bounded.*

Proof. [19, Proposition 3.10]. □

Proposition 2.2.19. *Let B be a paracompact space and let X be a fiberwise normed vector space. Let $A \subset X$ be a fiberwise compact subspace. Then A it is fiberwise bounded.*

Proof. For $n \in \mathbb{N}$ set $U_n := (n-1/2, n+1) \subset \mathbb{R}$ and let $X_n := \{x \in X \,\big|\, |x| \in U_n\}$. Then $X = \cup_{n\in\mathbb{N}} X_n$. For each $b \in B$, the family $X_n \cap A$ covers the fiber A_b. Since A is fiberwise compact, there is an open neighbourhood $W_b \subset B$ of b and a number $n_b \in \mathbb{N}$, such that $A_{W_b} \subset \cup_{n\leq n_b} X_n \cap A$. The space B is paracompact, therefore there exists a subset $B' \subset B$, such that $B = \cup_{b'\in B'} W_{b'}$ is a locally finite cover that admits a partition of unity $(\tau_{b'})_{b'\in B'}$. Define $\beta : B \longrightarrow \mathbb{R}$ by $\beta(b) := \sum_{b'\in B'} \tau_{b'}(b)(n_{b'}+1)$. We claim that the function β bounds A: let $x \in A_b$ for some $b \in B$. We know that there are only finitely many $b' \in B'$ with $b \in W_{b'}$. And for those $b' \in B'$ we have $x \in A_{W_{b'}}$ and therefore $|x| < n_{b'}$. Then:

$$\beta(b) = \sum_{b'\in B'} \tau_{b'}(b) n_{b'} \geq \sum_{b'\in B'} \tau_{b'}(b)|x| = |x|. \tag{2.4}$$

□

Corollary 2.2.20. *Let B be compact and let X, Y be fiberwise normed vector spaces over B. A fiberwise map $f : X \longrightarrow Y$ is*

1. *fiberwise coercive if and only if $f^{-1}(T)$ is bounded for all bounded subsets $T \subset Y$;*
2. *fiberwise bounded if and only if $f(S)$ is bounded for all bounded subset $S \subset X$.*

Proof. When B is compact, a closed subset $T \subset Y$ is fiberwise bounded if and only if it is bounded. And the closure of a (fiberwise) bounded set is (fiberwise) bounded. The statement follows. □

2.3 Representation theory

In this section, we collect the results on representation theory that we will need. The main reference is [10, Chapters 2 & 3]. See also [22, §1.5], [47], and [14].

Let G be a compact Lie group. By a (continuous, complex) G-module V we mean a complex Banach space V, together with a continuous action $G \times V \longrightarrow V$, such that left translation $l_g : V \longrightarrow V$ is a bounded linear operator for all $g \in G$. The G-module V is called unitary when V is a Hilbert space and when all left translations are unitary operators. A closed G-invariant subspace is called a G-submodule. A G-module is called irreducible if it contains no proper G-submodules.

Proposition 2.3.1. *All irreducible G-modules are finite dimensional. When G is abelian they are one-dimensional.*

Proof. By [10, Chapter 2, Corollary 5.8] they are finite dimensional, by [10, Chapter 2, Proposition 1.13] one dimensional when G is abelian. □

Proposition 2.3.2 (Schur's Lemma)**.**

1. *Let V and V' be irreducible G-modules. Let $f : V \longrightarrow V'$ be an equivariant, bounded linear map. Then $f = 0$ or f is an isomorphism.*
2. *Let V be an irreducible G-module. Let $f : V \longrightarrow V$ be an equivariant, bounded linear map. Then $f = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*

Proof. [10, Chapter 2, Theorem 1.10]. □

Let V be a finite dimensional G-module. Its character is by definition the map $\chi_V : G \longrightarrow \mathbb{C},\ g \mapsto \mathrm{tr}(l_g)$. The character of an irreducible G-module is called an irreducible character. A finite dimensional G-module is determined up to isomorphism by its character ([10, Chapter 2, Theorem 4.12]). Let $\mathrm{Irr}(G, \mathbb{C})$ be a complete list of pairwise non-isomorphic irreducible unitary G-modules. We will often identify it with the set of irreducible characters of G.

Proposition 2.3.3 (Isotypical decomposition)**.** *Let V be a finite dimensional G-module. There is a natural isomorphism of G-modules*

$$d : \oplus_{W\in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \longrightarrow V \tag{2.5}$$

Proof. [10, Chapter 2, Proposition 1.14]. Observe that the direct sum appearing in the proposition is finite, since V is finite dimensional. □

In the infinite dimensional situation, similar results can be deduced from the Theorem of Peter and Weyl. Let V be a G-module. Let V_s be the subspace generated by the finite dimensional G-subspaces of V.

Proposition 2.3.4.

1. *There is a natural morphism of G-modules*

$$c : \oplus_{W\in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \longrightarrow V. \tag{2.6}$$

   It is injective and has image V_s.

2. *Let $W \in \mathrm{Irr}(G, \mathbb{C})$ and let $U \subset V_s$ be a G-invariant linear subspace. Then $U \subset c(W)$ if and only if the irreducible submodules of U are isomorphic to W.*

Proof. [10, Chapter 3, Proposition 1.7]. □

Theorem 2.3.5. *Let V be a G-module and let V_s be the subspace generated by the finite dimensional G-subspaces of V. Then V_s is dense in V.*

Proof. [10, Chapter 3, Theorem 5.7]. □

For unitary G-modules more can be said: Before doing so we recall the notion of an infinite direct sum of Hilbert spaces from [14, page 314] (see also [36, page 338]). Let H be a Hilbert space and let $(H_\alpha)_{\alpha\in A}$ be a family of pairwise orthogonal closed subspaces. We say that H is the (internal) direct (Hilbert) sum of the H_α, if $\overline{\sum_{\alpha\in A} H_\alpha} = H$. We define the (external) direct (Hilbert) sum of a family of Hilbert spaces $(H_\alpha)_{\alpha\in A}$ by

$$\hat{\oplus}_{\alpha\in A} H_\alpha := \left\{ (h_\alpha) \in \prod_{\alpha\in A} H_\alpha \mid \sum_{\alpha\in A} ||h_\alpha||^2 < \infty \right\} \subset \prod_{\alpha\in A} H_\alpha. \tag{2.7}$$

The metric on $\hat{\oplus}_{\alpha\in A} H_\alpha$ is defined by $\langle (h_\alpha), (h'_\alpha)\rangle := \sum_{\alpha\in A}\langle h_\alpha, h'_\alpha\rangle$. When a Hilbert space H is the internal direct sum of a family of subspaces $(H_\alpha)_{\alpha\in A}$, there is a natural isometry $H \cong \hat{\oplus}_{\alpha\in A} H_\alpha$ between H and the external direct sum of the family (H_α). Therefore we will drop the distinction between internal and external direct sum.

Let H be a G-module, and let $h \in H$ and $f \in C^0(G) := C^0(G, \mathbb{C})$. We define

$$f * h := \int_G f(g)ghdg \in H, \tag{2.8}$$

where $\int_G dg$ denotes the invariant integral (see [10, Chapter 3 §5.1]).

Let $\chi : G \longrightarrow \mathbb{C}$ be an irreducible character. With $\dim\chi$ we denote the dimension of an irreducible representation with character χ. Put $e_\chi := \dim\chi \cdot \bar{\chi} \in C^0(G)$ and define

$$P_\chi : H \longrightarrow H \text{ by } h \mapsto e_\chi * h. \tag{2.9}$$

Then we have the following theorem:

Theorem 2.3.6. *Let H be a unitary G-module and let $\chi \neq \psi \in \mathrm{Irr}(G, \mathbb{C})$ be irreducible characters.*

1. *The operator P_χ is an orthogonal projection operator onto a closed subspace H_χ.*
2. *H_χ and H_ψ are orthogonal.*
3. *H is the Hilbert sum of the H_χ for $\chi \in \mathrm{Irr}(G, \mathbb{C})$.*
4. *H_χ is the smallest closed subspace of H containing all irreducible submodules with character χ.*

Proof. [10, Chapter, 3 Theorem 5.10]. □

Remark 2.3.7. Let H be a unitary G-module and let $W \in \mathrm{Irr}(G, \mathbb{C})$ be an irreducible unitary G-module with character χ_W. Let $c_W : \mathrm{Hom}_G(W, H) \otimes_{\mathbb{C}} W \longrightarrow H$ be the natural map. Then $\overline{\mathrm{im}\, c_W} = H_{\chi_W}$.

Proof. It follows directly from Proposition 2.3.4 that the subspace $\mathrm{im}\, c_W$ is the smallest G-submodule that contains all irreducible submodules with character χ_W. Therefore its closure is the smallest closed G-submodule that contains all irreducible submodules with character χ_W. Therefore $\overline{\mathrm{im}\, c_W} = H_{\chi_W}$. □

Let H be a unitary G-module and let W be an irreducible unitary G-module. We define a scalar product on $\mathrm{Hom}_G(W, H)$ as follows: Let $\varphi, \psi : W \longrightarrow H$ be G-equivariant linear maps. Then $\psi^* \circ \varphi \in \mathrm{Hom}_G(W, W)$. Schur's Lemma implies that there is a unique $\lambda \in \mathbb{C}$, such that $\psi^* \circ \varphi = \lambda\, \mathrm{id}_W$. We put $\langle \varphi, \psi\rangle := \lambda$.

Lemma 2.3.8. *The space $\mathrm{Hom}_G(W, H)$ together with the scalar product $\langle , \rangle$ is a Hilbert space.*

Proof. [14, Lemma 7.3.1]. □

The tensor product $\mathrm{Hom}_G(W, H) \otimes W$ is naturally a Hilbert space (see [14, page 314] and [47, page 66]) and we have the following statement:

Lemma 2.3.9. *The canonical map* $c_W : \mathrm{Hom}_G(W,H) \otimes W \longrightarrow H$ *is an isometric embedding.*

Proof. [14, Lemma 7.3.1]. □

Corollary 2.3.10. *Let* H *be a unitary* G*-module. Then there is a natural isomorphism of* G*-modules*

$$\hat{c} : \hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W \longrightarrow H \tag{2.10}$$

Proof. The Hilbert space direct sum $\hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W$ is the completion of the (algebraic) direct sum $\bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W$. Since H is complete, the map

$$c : \bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W \longrightarrow H \tag{2.11}$$

extends uniquely to a bounded linear map

$$\hat{c} : \hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W \longrightarrow H. \tag{2.12}$$

This map is injective: Let $\varphi \in \ker \hat{c}$. Then there is a sequence $\varphi_n \in \bigoplus_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W$ with $\varphi_n \to \varphi$. The element $\hat{c}(\varphi)$ is by definition the limit of the sequence $c(\varphi_n) \in H$. Therefore $c(\varphi_n) \longrightarrow 0$. By the previous lemma, for every $W \in \mathrm{Irr}(G,\mathbb{C})$, the map c_W is an isometry. Therefore c is also an isometry and $|\varphi_n| = |c(\varphi_n)| \to 0$. Therefore $\varphi = 0$.

Since the image of c is H_s and H_s is dense in H, and because c is an isometry, the map $\hat{c}$ must be surjective and therefore an isomorphism. □

The abelian case

Let G be a compact abelian Lie group and let H be a unitary G-module. Fix an irreducible character $\chi : G \longrightarrow S^1$ of G.

Corollary 2.3.11. *Let* $H_\chi \subset H$ *be the subspace introduced in Theorem 2.3.6. Then*

$$H_\chi = \left\{ h \in H \,\middle|\, gh = \chi(g)h \ \forall g \in G \right\}. \tag{2.13}$$

Furthermore, let W *be an irreducible representation with character* χ *and let*

$$c_W : \mathrm{Hom}_G(W,H) \otimes_{\mathbb{C}} W \longrightarrow H \tag{2.14}$$

be the natural map. Then $\mathrm{im}\, c_W = H_\chi$.

Proof. Let us denote the space on the right with H'_χ. It is closed, G-invariant and obviously contains all irreducible submodules with character χ. Therefore $H_\chi \subset H'_\chi$. Now let $h \in H'_\chi$. Then any irreducible representation W with character χ is isomorphic to $\mathbb{C}h$. This isomorphism defines an equivariant map $\varphi : W \cong \mathbb{C}h \subset H$ with $\varphi(W) = \mathbb{C}h$. Thus $h \in \mathrm{im}\, c_W$. Therefore $H'_\chi \subset \mathrm{im}\, c_W$. Together we obtain $H_\chi \subset H'_\chi \subset \mathrm{im}\, c_W \subset \overline{\mathrm{im}\, c_W} = H_\chi$. □

2.4 Kuiper's theorem

Theorem 2.4.1 (Kuiper). *Let* H *be an infinite dimensional real or complex Hilbert space. The space* $\mathrm{GL}(H)$ *of continuous linear automorphisms of* H *equipped with the norm-topology is contractible.*

Proof. [24, Theorem 2]. □

Corollary 2.4.2 (Segal). *Let* H *be an infinite dimensional unitary* G*-module, such that* $\dim H_\chi \in \{0, \infty\}$ *for every irreducible character* χ*. Then the group of continuous equivariant isomorphisms* $\mathrm{GL}_G(H)$ *equipped with the norm-topology is contractible.*

Proof. [38, Proof of Proposition 1]. □

2.5 Banach and Hilbert bundles

Fix a finite dimensional topological manifold B. When E, F are Banach spaces, we denote with $L(E, F)$ the space of continuous linear maps between E and F.

Definition 2.5.1. Let E be a Banach (Hilbert) space, and let $p : \mathcal{E} \longrightarrow B$ be a fiberwise normed (Hilbert) space. A trivializing Banach (Hilbert) cover with fiber E for p is a family $(U_i, \tau_i)_{i\in I}$ of pairs, consisting of open subsets $U_i \subset B$ and fiberwise homeomorphisms

$$\begin{array}{ccc} \mathcal{E}_{U_i} := p^{-1}(U_i) & \xrightarrow{\quad\tau_i\quad} & U_i \times E \;, \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U_i & \end{array} \tag{2.15}$$

such that

1. for each $b \in U_i$, the induced map $\tau_i(b) : \mathcal{E}_b \longrightarrow E$ is an (isometric) isomorphism.
2. for each i, j, the map
$$g_{ij} : U_{ij} := U_i \cap U_j \longrightarrow L(E, E),\ b \mapsto \tau_i(b)\tau_j(b)^{-1} \tag{2.16}$$
is continuous.

Two trivializing covers are called equivalent, if their union is still a trivializing cover. We call a fiberwise normed (Hilbert) space together with an equivalence class of trivalizing Banach (Hilbert) covers with fiber E a Banach (Hilbert) bundle of type E. We call it finite dimensional if the fiber E is of finite dimension.

Notice that in our definition of Banach bundles, the fibers $\mathcal{E}_b$ are Banach spaces. This differs from the definition in [25], where the fibers are Banachable spaces. If needed, we will call such a bundle a Banachable bundle. In the case of Hilbert bundles, there is no ambiguity.

Definition 2.5.2. Let $\mathcal{E}$ be a Banach bundle of type E. Let $\mathcal{F} \subset \mathcal{E}$ be a subspace, such that $\mathcal{F}_b \subset \mathcal{E}_b$ is closed for all $b \in B$. It is a subbundle if there exists a closed subspace $F \subset E$ and a trivializing cover (U_i, τ_i) of $\mathcal{E}$, such that $\tau_i\left(\mathcal{F} \cap \mathcal{E}_{U_i}\right) = U_i \times F$ for all $i \in I$.

Definition 2.5.3. Let $\mathcal{E}$ and $\mathcal{E}'$ be Banach (Hilbert) bundles over B of type E and E', respectively. A fiberwise map $f : \mathcal{E} \longrightarrow \mathcal{E}'$ is called a morphism of Banach (Hilbert) bundles if the following statements are true:

1. for any $b \in B$, the induced map $f_b : \mathcal{E}_b \longrightarrow \mathcal{E}'_b$ is a bounded linear operator;
2. for each $b_0 \in B$, there exist a neighbourhood $U \subset B$ of b_0, and there exist trivializations
$$\tau : \mathcal{E}_U \longrightarrow U \times E \text{ and } \tau' : \mathcal{E}'_U \longrightarrow U \times E', \tag{2.17}$$
such that the induced map
$$U \longrightarrow L(E, E'), b \mapsto \tau'(b) \circ f_b \circ \tau(b)^{-1} \tag{2.18}$$
is continuous.

Lemma 2.5.4. *Let $p : \mathcal{E} \longrightarrow B$ be a finite dimensional Banach bundle. Then a subspace $A \subset \mathcal{E}$ is fiberwise compact if and only if it is fiberwise bounded and closed.*

Proof. Let A be fiberwise bounded. By definition this means that there is a map $\beta : B \longrightarrow \mathbb{R}$, such that $A \cap \mathcal{E}_b \subset D_{\beta(b)}(\mathcal{E}_b)$ for all $b \in B$. Let $(U_i, \tau_i)_{i\in I}$ be a trivializing Banach cover for $\mathcal{E}$. Put $\mathcal{E}_{U_i} := p^{-1}(U_i)$ and $A_i := A \cap \mathcal{E}_{U_i}$. Then $\tau_i(A_i) \subset U_i \times \mathbb{R}^n$ is a closed and bounded subset. By Proposition 2.2.18, $\tau_i(A_i) \subset U_i \times E$ is fiberwise compact. Therefore, A_i is fiberwise compact. The condition of being fiberwise compact is local on the base (Proposition 2.2.3), and therefore A is fiberwise compact. The converse follows from Proposition 2.2.19 and Proposition 2.2.7. □

Lemma 2.5.5. *Every morphism of Banach bundles $f : \mathcal{E} \longrightarrow \mathcal{E}'$ is fiberwise bounded.*

Proof. The statement is local on the base, so we may assume that $\mathcal{E} = B \times E$ and $\mathcal{E}' = B \times E'$ for Banach spaces E, E'. Write $f : \mathcal{E} \longrightarrow \mathcal{E}',\ (b, e) \mapsto (x, f_b(e))$. The map $B \longrightarrow L(E, E'),\ b \mapsto f_b$ is continuous by assumption and so is $B \longrightarrow \mathbb{R},\ b \mapsto ||f_b||$.

Let $S \subset \mathcal{E}$ be a fiberwise bounded subset and let $\beta : B \longrightarrow \mathbb{R}$ be a map, such that $\beta(b)$ is a bound for $S \cap \mathcal{E}_b$ for every $b \in B$. Define

$$\tilde{\beta} : B \longrightarrow \mathbb{R},\ b \mapsto ||f_b||\, \beta(b). \tag{2.19}$$

It is again continuous, and for any $b \in B$ and $s \in S \cap \mathcal{E}_b$, the inequality $|f(s)| \leq ||f_b||\, |s| \leq ||f_b||\, \beta(b) = \tilde{\beta}(b)$ holds. Thus $f(S)$ is fiberwise bounded. □

Definition 2.5.6. Let $\mathcal{E}$ be a Banach bundle. A morphism of Banach bundles $P : \mathcal{E} \longrightarrow \mathcal{E}$ is a projection operator if $P \circ P = P$.

A projection operator P on a Banach bundle $\mathcal{E}$ induces a decomposition $\mathcal{E} = P(\mathcal{E}) \oplus (\mathrm{id}_{\mathcal{E}} - P)(\mathcal{E})$. For finite dimensional bundles, a proof is given in [2, Lemma 1.4]. For Banach bundles, the statement also seems to be well known ([48, Page 120]), because of a lack of reference we include a proof.

Proposition 2.5.7. *Let $\mathcal{E}$ be a Banach bundle with fiber E. Let $P : \mathcal{E} \longrightarrow \mathcal{E}$ be a projection operator. Then $\operatorname{im} P = \ker(\mathrm{id}_{\mathcal{E}} - P)$ and $\operatorname{im}(\mathrm{id}_{\mathcal{E}} - P) = \ker P$ are closed subbundles of $\mathcal{E}$ and $\mathcal{E} = \operatorname{im} P \oplus \operatorname{im}(\mathrm{id}_{\mathcal{E}} - P)$.*

Proof. The equations $\operatorname{im} P = \ker(\mathrm{id}_{\mathcal{E}} - P)$ and $\operatorname{im}(\mathrm{id}_{\mathcal{E}} - P) = \ker P$ are immediately verified. Therefore these subsets are closed. We now prove that $\operatorname{im} P$ is a subbundle of $\mathcal{E}$.

Let $b_0 \in B$. Choose an open neighbourhood $U \subset B$ of b_0 and a local trivialization $\tau : \mathcal{E}_U \xrightarrow{\cong} U \times E$. Then $P_\tau := \tau \circ P \circ \tau^{-1}$ is a projection operator of the bundle $U \times E$. It suffices to prove the statement for this operator.

Therefore we can assume without loss of generality that $\mathcal{E} = B \times E$ and that $\tau = \mathrm{id}$. The projection operator is then of the form $(b, e) \mapsto (b, P_b(x))$ and the map $B \longrightarrow L(E, E), b \mapsto P_b$ is continuous. Therefore, also the maps

1. $\varphi : B \longrightarrow L(E, E), b \mapsto P_b + (\mathrm{id}_E - P_{b_0})$;
2. $\psi : B \longrightarrow L(E, E), b \mapsto P_{b_0} + (\mathrm{id}_E - P_b)$

are continuous. Furthermore, $\varphi(b_0) = \psi(b_0) = \mathrm{id}_E$. Therefore there is an open subset $V \subset B$, such that $b_0 \in V$ and that $\varphi(b)$ and $\psi(b)$ are automorphisms for every $b \in V$. We claim that

$$P_{b_0} \circ P_b(E) = P_{b_0}(E) \text{ for every } b \in V. \tag{2.20}$$

Let $e \in E$. Then because $\varphi(b)$ is surjective, there is $e' \in E$, such that $e = \varphi(b)e' = P_b e' + (\mathrm{id}_E - P_{b_0})e'$. It follows that

$$P_{b_0} e = P_{b_0} P_b e' + P_{b_0} e' - P_{b_0} e' = P_{b_0} P_b e'. \tag{2.21}$$

This shows that $P_{b_0}(E) \subset P_{b_0} \circ P_b(E)$. The other inclusion is tautological. Thus we have proven the claim.

By definition, $\psi(b) \circ P_b = [P_{b_0} + (\mathrm{id}_E - P_b)] \circ P_b = P_{b_0} \circ P_b$. As a consequence

$$\psi(b) \circ P_b(E) = P_{b_0} \circ P_b(E) = P_{b_0}(E) \text{ for every } b \in V. \tag{2.22}$$

The morphism ψ defines an automorphism of the bundle $\mathcal{E}_V = V \times E$:

$$\Psi : V \times E \longrightarrow V \times E,\ (b, e) \mapsto (b, \psi(b)e). \tag{2.23}$$

It has the property that $\Psi(P(\mathcal{E}_V)) = V \times P_{b_0}(E) \subset V \times E$. Therefore $P(\mathcal{E})$ is a subbundle of $\mathcal{E}$. □

Let $(\mathcal{E}_\lambda)_{\lambda\in\Lambda}$ be a family of Hilbert bundles over B modelled on a family (E_λ) of Hilbert spaces. We define the Hilbert direct sum $\hat{\oplus}_{\lambda\in\Lambda}\mathcal{E}_\lambda$ first set-theoretically:

$$\hat{\oplus}_{\lambda\in\Lambda}\mathcal{E}_\lambda := \cup_{b\in B}\Big(\hat{\oplus}(\mathcal{E}_\lambda)_b\Big). \tag{2.24}$$

Let $U \subset B$ be a contractible open subspace. Then for every $\lambda \in \Lambda$ there is a trivialization

$$\varphi_{U,\lambda} : (\mathcal{E}_\lambda)_U \xrightarrow{\cong} U \times E_\lambda \tag{2.25}$$

of $(\mathcal{E}_\lambda)_U$. These trivializations define a bijection

$$\varphi_U := \hat{\oplus}_{\lambda\in\Lambda}\,\varphi_{U,\lambda} : \big(\hat{\oplus}_{\lambda\in\Lambda}\mathcal{E}_\lambda\big)_U = \hat{\oplus}_{\lambda\in\Lambda}\big(\mathcal{E}_\lambda\big)_U \xrightarrow{\hat{=}} U \times \hat{\oplus}_{\lambda\in\Lambda}E_\lambda. \tag{2.26}$$

Therefore $\hat{\oplus}_{\lambda\in\Lambda}\mathcal{E}_\lambda$ is a set theoretical Hilbert bundle over B. Given another contractible open subset $V \subset B$ and choices of trivializations $\varphi_{V,\lambda} : (E_\lambda)_U \xrightarrow{\cong} U \times E_\lambda$ we obtain transition maps

$$\varphi_{UV} : U \cap V \longrightarrow U(\hat{\oplus}_{\lambda\in\Lambda}E_\lambda), \tag{2.27}$$

defined by the equation

$$\varphi_U \circ \varphi_V^{-1}(b,e) = (b, \varphi_{UV}(b)e) \text{ for } (b,e) \in U \cap V \times \hat{\oplus}_{\lambda\in\Lambda}E_\lambda. \tag{2.28}$$

These transition maps are of the form $\varphi_{UV} = \hat{\oplus}_{\lambda\in\Lambda}\varphi_{UV,\lambda}$ with $\varphi_{UV,\lambda} \in U(E_\lambda)$. The latter maps are the transition maps of the bundles $\mathcal{E}_\lambda$, hence continuous by assumption. Therefore also φ_{UV} is continuous.

Now by [25, Proposition 1.2] there exists a unique topological structure on $\hat{\oplus}_{\lambda\in\Lambda}\mathcal{E}_\lambda$ making it into a topological Hilbert bundle with the maps φ_U a trivializing cover. We call the induced Hilbert bundle the Hilbert direct sum of the bundles $\mathcal{E}_\lambda$.

2.6 Equivariant Hilbert bundles

Fix a compact Lie group G and let B be a topological space on which G acts trivially.

Definition 2.6.1. A G-equivariant Banach (Hilbert) bundle is a Banach (Hilbert) bundle $\mathcal{E} \longrightarrow B$, together with a fiberwise G-action $G \times \mathcal{E} \longrightarrow \mathcal{E}$, which induces in each fiber $\mathcal{E}_b$ the structure of a G-module. When $\mathcal{E}$ is a Hilbert bundle and when all fibers are unitary G-modules it is called a unitary G-(Hilbert) bundle.

Observe that this definition of a G-equivariant Banach or Hilbert bundle does not require the bundles to be locally G-isomorphic to a trivial G-bundle. We will now prove that every equivariant Hilbert bundle is locally G-isomorphic to a trivial G-bundle.

We start by recalling the notion of invariant integration:

Theorem 2.6.2 (Invariant Integration)**.** *Let E be a Banach space and let $C^0(G,E)$ be the vector space of continuous functions $G \longrightarrow E$ endowed with the compact-open topology.*

There exists a continuous linear map

$$\int_G : C^0(G,E) \longrightarrow E,\ f \mapsto \int_G f = \int_G f(g)dg \tag{2.29}$$

which is both left and right invariant and which satisfies

1. $\int_G e dg = e$ *for all* $e \in E$,
2. $\int_G f(g)dg = \int_G f(g^{-1})dg$,
3. $|\int_G f(g)dg| \leq \int_G |f(g)|dg$,

4. *if* $L : E \longrightarrow E'$ *is a continuous linear map between Banach spaces, then* $L \int_G f = \int_G Lf$ *for all* $f \in C^0(G, E)$.

Proof. [10, III. §5.1]. □

Let now $\mathcal{E}$ be a G-equivariant Banach bundle over B. We define an averaging operator $P : \mathcal{E} \longrightarrow \mathcal{E}$ in the following way: Let $e \in \mathcal{E}_b$. Then by definition $P(e) := \int_G gedg \in \mathcal{E}_b$.

Lemma 2.6.3. *Averaging defines a morphism of Banach bundles*

$$P : \mathcal{E} \longrightarrow \mathcal{E}. \tag{2.30}$$

It is a projection operator with image $\mathcal{E}^G$.

Proof. Let $\varphi : \mathcal{E}_U \longrightarrow U \times E$ be a local trivialization of the bundle $\mathcal{E}$. We calculate the map $P_\varphi := \varphi \circ P \circ \varphi^{-1}$: Let $(x, e) \in U \times E$. Then:

$$\varphi \circ P \circ \varphi^{-1}(x, e) = \varphi(\int\limits_G g\varphi^{-1}(x, e)dg). \tag{2.31}$$

We claim that this map is continuous. Let $\pi_E : U \times E \longrightarrow E$ and $\pi_U : U \times E \longrightarrow U$ be the projections. The map

$$F : G \times U \times E \longrightarrow E,\ (g, x, e) \mapsto \pi_E \circ \varphi(g\varphi^{-1}(x, e)) \tag{2.32}$$

is continuous because it can be written as the composition

$$F = \pi_E \circ \varphi \circ \mu \circ (\mathrm{id}_G \times \varphi^{-1}) : G \times U \times E \longrightarrow U \times E, \tag{2.33}$$

of continuous maps. Here $\mu : G \times \mathcal{E} \longrightarrow \mathcal{E}$ denotes the multiplication map.

The map $F : G \times U \times E \longrightarrow E$ induces a map $f : U \times E \longrightarrow C^0(G, E)$ by the formula

$$f(x, e)(g) = F(g, x, e) = \pi_E \circ \varphi(g\varphi^{-1}(x, e)) \text{ for } (g, x, e) \in G \times U \times E. \tag{2.34}$$

The map f is continuous because F is continuous ([10, III. §5.11, 1.i)]). Now observe that

$$\begin{aligned} \varphi \circ P \circ \varphi^{-1}(x, e) &= \left(x, \pi_E \circ \varphi\left(\int\limits_G g\varphi^{-1}(x, e)dg\right)\right) \\ &= \left(x, \int\limits_G \pi_E \circ \varphi(g\varphi^{-1}(x, e))dg\right) = \left(x, \int\limits_G f(x, e)(g)dg\right). \end{aligned} \tag{2.35}$$

Here we used the property that the invariant integral commutes with continuous linear maps. Therefore

$$P_\varphi = \varphi \circ P \circ \varphi^{-1} = \pi_U \times \left(\int\limits_G \circ f\right) : U \times E \longrightarrow U \times E. \tag{2.36}$$

is continuous. We still have to prove that also the induced map $\bar{P}_\varphi : U \longrightarrow L(E, E)$ that is defined by $\bar{P}_\varphi(x)(e) = \pi_E \circ P_\varphi(x, e)$ is continuous, where $L(E, E)$ is equipped with the operator norm. Let $x_n \in U$ be a sequence converging to x and let $e \in E$. Then $\varphi(g\varphi^{-1}(x_n, e)) \to \varphi(g(\varphi^{-1}(x, e)) \in E$. But from the inequality

$$\left|\int\limits_G \varphi(g\varphi^{-1}(x_n, e)dg\right| \leq \int\limits_G \left|\varphi(g\varphi^{-1}(x_n, e)\right| dg \tag{2.37}$$

it follows that $\bar{P}_\varphi(x_n) \to \bar{P}_\varphi(x)$ in the norm topology. This proves that P is a morphism of Banach bundles.

Next we prove that P is a projection operator. Let $e \in \mathcal{E}_g$. Then using the properties of the integral (Theorem 2.6.2) we conclude that

$$PP(e) = \int\limits_G g \left(\int\limits_G hedh \right) dg = \int\limits_G \left(\int\limits_G ghedh \right) dg = \int\limits_G \left(\int\limits_G hedh \right) dg = \int\limits_G hedh = P(e). \tag{2.38}$$

Therefore P is a projection operator. For $e \in \mathcal{E}^G$, we have $P(e) = \int_G gedg = \int_G edg = e$. Therefore $\mathcal{E}^G \subset P(\mathcal{E})$. On the other hand, let $e \in \mathcal{E}$. Then $gP(e) = g\int_G hedh = \int_G ghedh = \int_G hedh = P(e)$. Therefore $P(\mathcal{E}) \subset \mathcal{E}^G$. □

Corollary 2.6.4. *Let $\mathcal{E}$ be a G-equivariant Banach bundle. Then $\mathcal{E}^G \subset \mathcal{E}$ is a subbundle.*

Proof. By Lemma 2.6.3, the averaging operator $P : \mathcal{E} \longrightarrow \mathcal{E}$ is a projection operator with image $\mathcal{E}^G$. By Proposition 2.5.7, its image is a subbundle. □

Corollary 2.6.5 (Isotypical decomposition)**.** *Let $\mathcal{E}$ be a G-equivariant Hilbert bundle. There exists a canonical isomorphism of G-Hilbert bundles*

$$\hat{c} : \hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(\underline{W}, \mathcal{E}) \otimes_{\mathbb{C}} \underline{W} \xrightarrow{\cong} \mathcal{E}. \tag{2.39}$$

Proof. The bundle $\mathrm{Hom}(\underline{\mathrm{W}}, \mathcal{E})$ is naturally a Banach bundle and $\mathrm{Hom}_G(\underline{\mathrm{W}}, \mathcal{E}) = \mathrm{Hom}(\underline{\mathrm{W}}, \mathcal{E})^G$. By Corollary 2.6.4 it is a subbundle. We define fiberwise a Hilbert metric as in Lemma 2.3.8. Then $\mathrm{Hom}_G(\underline{\mathrm{W}}, \mathcal{E})$ is a Hilbert bundle.

There is a canonical map $\hat{c} : \hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(\underline{W}, \mathcal{E}) \otimes_{\mathbb{C}} \underline{W} \longrightarrow \mathcal{E}$. For each $b \in B$, the induced map between the fibers $\hat{c}_b : \hat{\bigoplus}_{W \in \mathrm{Irr}(G,\mathbb{C})} \mathrm{Hom}_G(\{b\} \times W, \mathcal{E}_b) \otimes_{\mathbb{C}} (\{b\} \times W) \longrightarrow \mathcal{E}_b$ is an isometry by Corollary 2.3.10. Therefore the map $\hat{c}$ is an isomorphism of Hilbert bundles. □

Corollary 2.6.6. *Every G-equivariant Hilbert bundle $\mathcal{E}$ is locally G-isomorphic to a trivial G-Hilbert bundle.*

2.7 The index of Fredholm morphisms between equivariant Banach bundles

Let G be a compact Lie group and let B be a compact space on which G acts trivially. From now on we will use latin letters $E, F, \ldots$ for finite dimensional bundles and caligraphic letters $\mathcal{E}, \mathcal{F}, \ldots$ for (possibly) infinite dimensional bundles.

Definition 2.7.1. Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a Fredholm G-morphism between G-equivariant Banach bundles. Let $F \subset \mathcal{F}$ be a finite dimensional G-subbundle and write $\iota_F : F \hookrightarrow \mathcal{F}$ for the inclusion. The subbundle F is d-adapted if the induced map

$$d \oplus \iota_F : \mathcal{E} \oplus F \longrightarrow \mathcal{F} \tag{2.40}$$

is surjective.

Proposition 2.7.2. *Every Fredholm G-morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$ between G-equivariant Banach bundles admits a d-adapted subbundle.*

Proof. [39, Proposition 3.7]. See also [3, Proposition A5] for the case of a compact family of Fredholm operators between Hilbert spaces. □

Proposition 2.7.3. *Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a surjective Fredholm G-morphism between G-equivariant Banach bundles. Then* $\ker d \subset \mathcal{E}$ *is a finite dimensional G-subbundle.*

Proof. [39, Proposition 3.5]. □

Corollary 2.7.4. *Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a Fredholm G-morphism between G-Banach bundles and let $F \subset \mathcal{F}$ be d-adapted. Then $d^{-1}(F) \subset \mathcal{E}$ is a finite dimensional G-subbundle.*

Proof. By definition, the map $d \oplus \iota_F : \mathcal{E} \oplus F \longrightarrow \mathcal{F}$ is surjective. By the previous proposition, $\ker(d \oplus \iota_F) \subset \mathcal{E} \oplus F$ is a subbundle.

But the projection $\mathcal{E} \oplus F \longrightarrow \mathcal{E}$ induced an fiberwise linear isomorphism $\ker(d \oplus \iota_F) \cong d^{-1}(F)$: Its inverse is given by $e \mapsto e - d(e)$. □

The index of a Fredholm G-morphism between G-equivariant Banach bundles is defined to be

$$\mathrm{ind}_G\, d := [d^{-1}(F)] - [F] \in K_G(B) \tag{2.41}$$

for a d-adapted subbundle $F \subset \mathcal{F}$. This definition is independent of the choice of F ([39, Theorem 3.1 and proof of Theorem 3.3]). See [20] and [3] for the case of families of Fredholm operators between Hilbert spaces. See also [48] for the (non-equivariant) Banach case.

Proposition 2.7.5. *Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a Fredholm G-morphism between G-Banach bundles. Let $F_0 \subset \mathcal{F}$ be a finite dimensional subbundle. Then there exists a d-adapted subbundle $F \subset \mathcal{F}$, such that $F_0 \subset F$.*

Proof. Let $\pi_0 : \mathcal{F} \longrightarrow \mathcal{F}/F_0$ be the canonical projection. It is a Fredholm morphism of index $[F_0]$. The composition $\pi_0 \circ d : \mathcal{E} \longrightarrow \mathcal{F}/F_0$ is again a Fredholm morphism. Choose a $(\pi_0 \circ d)$-adapted subbundle $\bar{F} \subset \mathcal{F}/F_0$. It is obviously π_0-adapted, and therefore $\pi_0^{-1}(\bar{F}) \subset \mathcal{F}$ is a finite dimensional subbundle. Moreover it is d-adapted by construction: let $b \in B$ and let $f \in \mathcal{F}_b$. Since $\bar{F}$ is $(\pi_0 \circ d)$-adapted, there exist $e \in \mathcal{E}_b$ and $f' \in \pi_0^{-1}(\bar{F})$, such that $\bar{f} = \overline{d(e)} + \overline{f'} \in \mathcal{F}_b/F_{0b}$. Hence there is $f'' \in F_{0b}$, such that $de + f' + f'' = f$. □

Lemma 2.7.6. *Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a Fredholm G-morphism between G-Banach bundles. Let $F \subset \mathcal{F}$ be a d-adapted subbundle and let $\tilde{\mathcal{E}}$ be a closed complement of $E := d^{-1}(F)$. Then $d(\tilde{\mathcal{E}})$ is a closed complement of F and $d|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \longrightarrow d(\tilde{\mathcal{E}})$ is an isomorphism.*

Proof. By definition, the map $d \oplus \iota_F : \mathcal{E} \oplus F \longrightarrow \mathcal{F}$ is surjective. Let $\tilde{d}$ be the restriction of d to $\tilde{\mathcal{E}}$. We claim that $\tilde{d} \oplus \iota_F : \tilde{\mathcal{E}} \oplus F \longrightarrow \mathcal{F}$ is an isomorphism. The statement of the lemma would then follow. We prove this statement fiberwise. Let $\tilde{e} + f \in \tilde{\mathcal{E}}_b \oplus F_b$ with $d(\tilde{e}) + f = 0 \in \mathcal{F}_b$. Then $\tilde{e} \in d^{-1}(F_b) \cap \tilde{\mathcal{E}}_b = \{0\}$. Therefore the morphism is injective. Surjectivity follows immediately from the surjectivity of $d \oplus \iota_F$. □

Proposition 2.7.7 (Segal)**.** *Let B be a compact space equipped with the trivial G-action. Then there is a natural isomorphism of rings*

$$\mu : R(G) \otimes K(B) \longrightarrow K_G(B). \tag{2.42}$$

Proof. [37, Proposition 2.2]. □

2.8 A special class of equivariant Hilbert bundles

Let G be a compact Lie group and let B be a connected compact CW-complex with trivial G-action. We are interested in a special class of unitary G-equivariant Hilbert bundles: Let $\rho \subset \mathrm{Irr}(G, \mathbb{C}) \cup \{\mathbb{R}\}$ be a non empty subset.

Definition 2.8.1. A ρ-bundle is a pair $\mathcal{E}. = (\mathcal{E}_{\mathbb{R}}, \mathcal{E}_{\mathbb{C}})$ consisting of a real Hilbert bundle $\mathcal{E}_{\mathbb{R}} \longrightarrow B$ equipped with the trival G-action and a unitary G-bundle $\mathcal{E}_{\mathbb{C}} \longrightarrow B$. We write $\mathcal{E}_{\mathbb{C}} = \hat{\oplus}_{\chi \in \mathrm{Irr}(G,\mathbb{C})} \mathcal{E}_\chi$ for its isotypical decomposition. We assume the following conditions:

1. Let $\chi \in \mathrm{Irr}(G, \mathbb{C}) \cup \{\mathbb{R}\}$. If $\chi \notin \rho$, then $\mathcal{E}_\chi = 0_B$.
2. The fibers of $\mathcal{E}_\chi$ are either finite dimensional or infinite dimensional for all $\chi \in \rho$; we call the bundle finite dimensional or infinite dimensional, respectively.

We write $\mathcal{E} := \mathcal{E}_{\mathbb{R}} \times_B \mathcal{E}_{\mathbb{C}}$. When $\mathbb{R} \notin \rho$ there is no difference between $\mathcal{E}.$ and $\mathcal{E}$; abusing the notation we will often write $\mathcal{E}$ for a ρ-bundle. No confusion is possible.

Definition 2.8.2. Let $\mathcal{E}.$ and $\mathcal{F}.$ be two ρ-bundles. A G-equivariant fiberwise map $f : \mathcal{E} \longrightarrow \mathcal{F}$ is a ρ-morphism if

1. $f(\mathcal{E}_{\mathbb{R}}) \subset \mathcal{F}_{\mathbb{R}}$ and $f(\mathcal{E}_{\mathbb{C}}) \subset \mathcal{F}_{\mathbb{C}}$;
2. the induced map $f_{\mathbb{R}} : \mathcal{E}_{\mathbb{R}} \longrightarrow \mathcal{F}_{\mathbb{R}}$ is a morphism of real Hilbert bundles and the map $f_{\mathbb{C}} : \mathcal{E}_{\mathbb{C}} \longrightarrow \mathcal{F}_{\mathbb{C}}$ is a G-equivariant morphism of Hilbert bundles.

Corollary 2.8.3. *Let B be a connected compact CW-complex with trivial G-action. Every infinite dimensional ρ-bundle over B is trivializable.*

Proof. Let $\mathcal{E}$ be an infinite dimensional ρ-bundle. Let $\mathcal{E}(b)$ be a fiber and let $\mathcal{I}som_G(\underline{\mathcal{E}(b)}, \mathcal{E})$ be the fiber bundle of ρ-isomorphisms between the two ρ-bundles $\underline{\mathcal{E}(b)}$ and $\mathcal{E}$. Its typical fiber is isomorphic to $\mathrm{GL}(\mathcal{E}_{\mathbb{R}}(b)) \times \mathrm{GL}_G(\mathcal{E}_{\mathbb{C}}(b))$ which is contractible by Theorem 2.4.1 and Corollary 2.4.2. Therefore this fiber bundle has a section by [41, Corollary 29.3]. □

Corollary 2.8.4. *Let B be a connected compact CW-complex with trivial G-action. Any two ρ-isomorphisms between infinite-dimensional ρ-bundles over B are isotopic.*

Proof. This follows from [41, Theorem 34.8]. □

Proposition 2.8.5. *Let B be a connected compact CW-complex with trivial G-action. Let E be a finite dimensional ρ-bundle over B and let $\mathcal{E}$ be an infinite dimensional ρ-bundle over B.*

1. *There exists an embedding $i : E \hookrightarrow \mathcal{E}$.*
2. *Any two such embeddings are homotopic as embeddings.*

Proof. Every finite dimensional ρ-bundle E is stably trivializable, i.e. there exists a finite dimensional ρ-bundle E', such that $E \oplus E'$ is trivializable. Then the first statement follows from the fact that every infinite dimensional ρ-bundle is trivializable.

Let $i, j : E \hookrightarrow \mathcal{E}$ be two embeddings. Set $\iota := j \circ i^{-1} : i(E) \xrightarrow{\cong} j(E)$ and choose a ρ-isomorphism $\pi^{\perp} : i(E)^{\perp} \xrightarrow{\cong} j(E)^{\perp}$. Then $\varphi := \iota \oplus \pi^{\perp} : \mathcal{E} \longrightarrow \mathcal{E}$ is an automorphism of the bundle $\mathcal{E}$. Choose an isotopy $\varphi_t : \mathcal{E} \longrightarrow \mathcal{E}$ with $\varphi_0 = \mathrm{id}_{\mathcal{E}}$ and $\varphi_1 = \varphi$. Then $\iota_t := \varphi_t \circ i$ defines a homotopy of embeddings and $\iota_0 = i$ and $\iota_1 = \varphi \circ i = j$. □

Proposition 2.8.6. *Let B be a connected compact CW-complex with trivial G-action. Let $\mathcal{E}$ be an infinite dimensional ρ-bundle over B, let $E \subset \mathcal{E}$ be a finite dimensional subbundle, and let $\varphi : \mathcal{E} \longrightarrow \mathcal{E}$ be a ρ-isomorphism with $\varphi|_E = \mathrm{id}_E$. Then there exists a ρ-isotopy $\varphi_t : \mathcal{E} \longrightarrow \mathcal{E}$, such that*

1. *$\varphi_0 = \mathrm{id}_{\mathcal{E}}$ and $\varphi_1 = \varphi$;*
2. *$\varphi_t|_E = \mathrm{id}_E$.*

Proof. Let $\pi_{E^{\perp}} : \mathcal{E} \longrightarrow E^{\perp}$ and $\pi_E : \mathcal{E} \longrightarrow E$ be the orthogonal projections. Define

$$\varphi_E := \pi_E \circ \varphi|_{E^{\perp}} : E^{\perp} \longrightarrow E \text{ and } \varphi^{\perp} := \pi_{E^{\perp}} \circ \varphi|_{E^{\perp}} : E^{\perp} \longrightarrow E^{\perp}. \tag{2.43}$$

Then $\varphi^{\perp}$ is an isomorphism and $\varphi^{\perp} + \varphi_E = \varphi|_{E^{\perp}}$. The bundle $E^{\perp}$ is an infinite dimensional ρ-bundle. By Corollary 2.8.4 any two ρ-automorphisms of $E^{\perp}$ are therefore isotopic. Choose an isotopy $\varphi_t^{\perp} : E^{\perp} \xrightarrow{\cong} E^{\perp}$, such that $\varphi_0^{\perp} = \mathrm{id}_{E^{\perp}}$ and $\varphi_1^{\perp} = \varphi^{\perp}$.

Now we define $\varphi_t := \mathrm{id}_E + \left(t\varphi_E + \varphi_t^{\perp}\right) \circ \pi_{E^{\perp}} : \mathcal{E} \longrightarrow \mathcal{E}$ for $t \in [0,1]$. Clearly, $\varphi_t|_E = \mathrm{id}_E$. Furthermore $\varphi_0 = \mathrm{id}_{\mathcal{E}}$ and $\varphi_1 = \varphi$. We claim that φ_t defines a isotopy. Fix $(b,t) \in B \times [0,1]$. We prove that $\varphi_t|_{\mathcal{E}_b} : \mathcal{E}_b \longrightarrow \mathcal{E}_b$ is an isomorphism. Let $e + e_0 \in E_b \oplus E_b^{\perp} = \mathcal{E}_b$.

If $\varphi_t(e + e_0) = 0$, then both $e + t\varphi_E(e_0) = 0 \in E$ and $\varphi_t^{\perp}(e_0) = 0 \in E^{\perp}$ hold. Since $\varphi_t^{\perp}$ is an isomorphism, $e_0 = 0$, and thus the first equation implies that $e = 0$. Hence φ_t is injective on the fibers.

Choose $\hat{e}_0 \in E^{\perp}$, such that $\varphi_t^{\perp}(\hat{e}_0) = e_0$ and put $\hat{e} := e - t\varphi_E(\hat{e}_0) \in E$. Then

$$\varphi_t(\hat{e} + \hat{e}_0) = \hat{e} + t\varphi_E(\hat{e}_0) + e_0 = e + e_0. \tag{2.44}$$

Therefore φ_t is surjective on the fibers. □

Let $\mathcal{E}$ be a ρ-bundle. We call a subbundle $\tilde{\mathcal{E}} \subset \mathcal{E} \times [0,1]$ over the base $B \times [0,1]$ a homotopy of subbundles. We will usually write $\tilde{\mathcal{E}} = \cup_{t\in[0,1]}\mathcal{E}_t$ - or just $\mathcal{E}_t$ -, where $\mathcal{E}_t := i_t^*\tilde{\mathcal{E}}|_{B\times\{t\}}$ and where $i_t : B \xrightarrow{\cong} B \times \{t\}$ denotes the natural isomorphism.

Corollary 2.8.7. *Let B be a connected compact CW-complex with trivial G-action. Let $E \subset \mathcal{E}$ be a finite-dimensional subbundle of an infinite dimensional ρ-bundle over B. Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be two closed complements of E. Then they fit in a homotopy of closed complements $\mathcal{E}_t \subset \mathcal{E}$.*

Proof. Choose a ρ-isomorphism $\varphi_{01} : \mathcal{E}_0 \longrightarrow \mathcal{E}_1$. Put $\varphi := \mathrm{id}|_E \oplus \varphi_{01} : \mathcal{E} \longrightarrow \mathcal{E}$. It is an isomorphism with the property that $\varphi|_E = \mathrm{id}_E$. By the previous proposition, there exists an isotopy $\varphi_t : \mathcal{E} \longrightarrow \mathcal{E}$ with $\varphi_t|_E = \mathrm{id}_E$, and such that $\varphi_0 = \mathrm{id}_{\mathcal{E}}$ and $\varphi_1 = \varphi$. Put $\mathcal{E}_t := \varphi_t(\mathcal{E}_0)$. By construction $E \oplus \mathcal{E}_t = \mathcal{E}$ for all $t \in [0,1]$. □

2.9 Colimits

In this section we collect results from category theory. See [26] as a main reference, and also [1] and the appendix of [34].

Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor between two categories. Let $A \in \mathrm{ob}\,\mathcal{A}$ be an object. With $\underline{A} : \mathcal{D} \longrightarrow \mathcal{A}$ we denote the associated functor that maps every object in $\mathcal{D}$ to A and every morphism to the identity. A natural transformation $a : F \longrightarrow \underline{A}$ is also called a F-cocone. It is a family $a = (a_D : F(D) \longrightarrow A)_{D\in\mathrm{ob}\,\mathcal{D}}$ of morphisms with the property that for every morphism $d : D \longrightarrow D'$ in $\mathcal{D}$ the diagram

$$\begin{array}{ccc} F(D) & & \\ {\scriptstyle F(d)}\big\downarrow & \searrow^{a_D} & \\ F(D') & \xrightarrow[a_{D'}]{} & A \end{array} \tag{2.45}$$

commutes. A morphism of two F-cocones a, a' is by definition a morphism $\alpha : A \longrightarrow A'$, such that for every $D \in \mathrm{ob}\,\mathcal{D}$ the diagram

$$\begin{array}{ccc} F(D) & \xrightarrow{a_D} & A \\ & {\scriptstyle a'_D}\searrow & \big\downarrow{\scriptstyle \alpha} \\ & & A' \end{array} \tag{2.46}$$

commutes. We denote the category of F-cocones with $\mathrm{CC}(F)$. A colimit for the functor F is by definition an initial object in the category $\mathrm{CC}(F)$. If it exists, it is unique up to unique isomorphism.

We are mainly interested in colimits for functors $F : \mathcal{D} \longrightarrow \mathcal{A}$ where $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Group}, \mathrm{Ab}\}$. But when no additional work is required we formulate our results generally.

Definition 2.9.1. Let $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Ab}, \mathrm{Group}\}$ and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. We call two elements $x \in F(D)$ and $x' \in F(D')$ F-connected if there is a sequence of morphisms

$$\begin{array}{ccccccccccccccccc} D = D_0 & & & & D_1 & & & & \cdots & & & & D_{n-1} & & & & D_n = D' \\ & \searrow^{r_0} & & \swarrow_{l_1} & & \searrow^{r_1} & & \swarrow & & \searrow & & \swarrow_{l_{n-1}} & & \searrow^{r_{n-1}} & & \swarrow_{l_n} & \\ & & D'_1 & & & & \cdots & & & & \cdots & & & & D'_n & & \end{array} \tag{2.47}$$

and a sequence of elements $x_i \in F(D_i)$ for $i = 0, \ldots, n$ such that

1. $x_0 = x \in F(D)$ and $x_n = x' \in F(D')$;
2. $F(l_i)x_i = F(r_{i-1})x_{i-1}$ for all $i = 1, \ldots, n$.

Remark 2.9.2. Let $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Ab}, \mathrm{Group}\}$ and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. Let $a : F \longrightarrow \underline{A}$ be a F-cocone. Let $x \in F(D)$ and $x' \in F(D')$ be two F-connected elements. Then $a_D(x) = a_{D'}(x') \in A$.

Proposition 2.9.3. *Let $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Ab}, \mathrm{Group}\}$, let $\mathcal{D}$ be a small category, and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. Then F admits a colimit.*

Proof. The category $\mathcal{D}$ is small and therefore the coproduct $\coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D)$ exists in $\mathcal{A}$. We denote the correponding inclusions with $i_D : F(D) \longrightarrow \coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D)$. We define an equivalence relation $\sim$ on the coproduct $\coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D)$. When $\mathcal{A} = \mathrm{Set}$, then we define two elements to be equivalent if they are F-connected. When $\mathcal{A} = \mathrm{Ab}$ or $\mathcal{A} = \mathrm{Group}$, then we define $\sim$ to be the smallest equivalence relation that is compatible with the group structure and identifies any two F-connected elements. In other words, two elements are equivalent if their difference lies in the (normal) subgroup generated by all elements of the form xy^{-1} where x and y are F-connected.

We put $A := \coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D)/\sim$ and let $\varepsilon_\sim : \coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D) \longrightarrow A$ be the canonical map. We define $a_D := \varepsilon_\sim \circ i_D : F(D) \longrightarrow \coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D) \longrightarrow A$ for $D \in \mathrm{ob}\,\mathcal{D}$. We claim that $a = (a_D)_D$ is a colimit for the functor F. So let $a' = (a'_D : F(D) \longrightarrow A')$ be another F-cocone. There is a unique morphism $\tilde{\alpha} : \coprod_{D \in \mathrm{ob}\,\mathcal{D}} F(D) \longrightarrow A'$, such that $\tilde{\alpha} \circ i_D = a'_D$ for all objects $D \in \mathrm{ob}\,\mathcal{D}$. Since a' is a F-cocone, F-connected elements have the same image in A'. Hence $\tilde{\alpha}$ identifies F-connected elements. Furthermore, $\tilde{\alpha}$ is a morphism, so it identifies any two equivalent elements. Thus it descends to a unique morphism $\alpha : A \longrightarrow A'$ with $\alpha \circ a_D = a'_D$ for all $D \in \mathrm{ob}\,\mathcal{D}$. □

Remark 2.9.4. The preceeding proposition can be easily generalized to the following well known statement: Let $\mathcal{A}$ be a category with all small coproducts and with all coequalizers. Then all small colimits exist in $\mathcal{A}$.

Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{A}$ be categories and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ and $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. The functor θ induces a functor $\theta^* : \mathrm{CC}(F) \longrightarrow \mathrm{CC}(F \circ \theta)$. We say that θ identifies F-colimits if θ^* is an isomorphism of categories. (Observe that θ^* is an isomorphism of categories if and only if it is an equivalence of categories.) We say that θ identifies (all) colimits if θ identifies all F-colimits for all functors F.

When θ identifies F-colimits then a cocone $a : F \longrightarrow \underline{A}$ is a F-colimit if and only if $\theta^* a : F \circ \theta \longrightarrow \underline{A}$ is a $F \circ \theta$-colimit.

Now let $F, G : \mathcal{D} \longrightarrow \mathcal{A}$ be two functors between two categories $\mathcal{D}$ and $\mathcal{A}$. A natural transformation $p : F \longrightarrow G$ induces a functor $p^* : \mathrm{CC}(G) \longrightarrow \mathrm{CC}(F)$. We say that p identifies colimits if p^* is an isomorphism of categories.

Lemma 2.9.5.

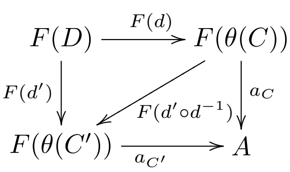
1. *An equivalence of categories $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ identifies colimits.*

2. *An isomorphism of functors $p : F \longrightarrow G$ identifies colimits.*

Proof. Let $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence of categories. Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. We define a push-forward functor

$$\theta_* : \mathrm{CC}(F \circ \theta) \longrightarrow \mathrm{CC}(F). \tag{2.48}$$

Let $a = (a_C : F(\theta(C) \longrightarrow A)_C$ be a $F \circ \theta$-cocone. The functor θ is an equivalence of categories, therefore it is essentially surjective. To $D \in \mathrm{ob}\,\mathcal{D}$, we choose $C \in \mathrm{ob}\,\mathcal{C}$ and an isomorphism $d : D \xrightarrow{\cong} \theta(C)$. Then we put $(\theta_* a)_D := a_C \circ F(d) : F(D) \longrightarrow A$. This map does not depend on the choice of d, since any other choice leads to a diagram:



(2.49)

It commutes because a_C is natural in C and because $d' \circ d^{-1} = \theta(c)$ for an isomorphism $c : C \longrightarrow C'$. One verifies that $\theta_* a$ is a F-cocone and that the functor θ_* inverts the functor θ.

To prove the second statement, let $p^{-1} : G \longrightarrow F$ be the inverse of p. It follows immediately that $p_* := (p^{-1})^*$ inverts the functor p^*. □

Definition 2.9.6. Let $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Group}, \mathrm{Ab}\}$ and let $\mathcal{C}$ be a category. Let $F, G : \mathcal{D} \longrightarrow \mathcal{A}$ be two functors. A natural transformation $p : F \longrightarrow G$ is called

1. stably surjective, if for every $D \in \mathrm{ob}\,\mathcal{D}$ and for every $g \in G(D)$, there is an element $f' \in F(D')$, such that $p_{D'}(f')$ is G-connected to g.
2. stably injective, if any two elements $f \in F(D)$ and $f' \in F(D')$ are F-connected if and only if $p_D(f)$ and $p_{D'}(f')$ are G-connected.

Proposition 2.9.7. *Let $\mathcal{A} \in \{\mathrm{Set}, \mathrm{Group}, \mathrm{Ab}\}$ and let $\mathcal{C}$ be a category. Let $F, G : \mathcal{D} \longrightarrow \mathcal{A}$ be two functors. A stably bijective natural transformation $p : F \longrightarrow G$ identifies colimits.*

Proof. The natural transformation p induces a morphism of categories

$$p^* : \mathrm{CC}(G) \longrightarrow \mathrm{CC}(F). \tag{2.50}$$

We define a push-forward functor $p_* : \mathrm{CC}(F) \longrightarrow \mathrm{CC}(G)$. Let $a = (a_D : F(D) \longrightarrow A)_D$ be a F-cocone. Let $D \in \mathrm{ob}\,\mathcal{D}$ be an object. We define the map $(p_*a)_D : G(D) \longrightarrow A$ as follows: let $g \in G(D)$. The tranformation p is stably surjective, hence there is an element $f' \in F(D')$, such that $p_{D'}(f')$ is G-connected to g. We put $(p_*a)_D(g) := a_{D'}(f')$. This is independent of the choice made: For let $f'' \in F(D'')$ be another choice. Then $p_{D''}(f'')$ is G-connected to $p_{D'}(f')$. By stable injectivity, f' is F-connected to f''. But by naturality $a_{D'}(f') = a_{D''}(f'')$.

We still have to prove that p_*a is indeed a F-cocone. Therefore, let $d : D \longrightarrow \hat{D}$ be a morphism. Let $g \in G(D)$ be an element and choose $f' \in F(D')$ such that $p_{D'}(f')$ is G-equivalent to g. Then $G(d)(g)$ is G-equivalent to g and also to $p_{D'}(f')$. Hence $(p_*a)_{\hat{D}}(G(d)g) = (p_*a)_D(g)$.

Now we prove that the two functors invert each other. Let $a = (a_D)_D$ be a F-cocone. We consider the map $(p^*p_*a)_D : F(D) \longrightarrow A$. Let $f \in F(D)$. By definition $(p^*p_*a)_D(f) = (p_*a)_D(p_D(f))$. But it follows directly from the definition of p_* that $(p_*a)_D(p_D(f)) = a_D(f)$.

On the other hand, let $(b_D : G(D) \longrightarrow B)_D$ be a G-cocone. We consider the map $(p_*p^*b)_D : G(D) \longrightarrow B$. Let $g \in G(D)$. Choose $f' \in F(D')$, such that $p_{D'}(f')$ is G-equivalent to g. By definition $(p_*p^*b)_D(g) = (p^*b)_{D'}(f') = b_{D'}(p_{D'}(f'))$. The G-equivalence of $p_{D'}f'$ and g implies $b_D(g) = b_{D'}(p_{D'}(f'))$. □

Corollary 2.9.8. *Let $F : \mathcal{D} \longrightarrow \mathrm{Set}$ be a functor, let A be a set, and let $p : F \longrightarrow \underline{A}$ be a F-cocone. Then p is a F-colimit if and only if it is stably bijective.*

Proof. Assume that the transformation p is stably bijective. Let A' be another set and let $p' : F \longrightarrow \underline{A}'$ be a natural transformation. Then we define a map $\alpha : A \longrightarrow A'$ in the following way: let $a \in A$. Since p is stably surjective, there is an object $D \in \mathrm{ob}\,\mathcal{D}$ and an element $f \in F(D)$, such that $p_D(f) = a$. We put $\alpha(a) := p'_D(f) \in A'$. Let $f' \in F(D')$ with $p_{D'}(f') = a$ be a different choice. Then $f \in F(D)$ and $f' \in F(D')$ are F-connected. Therefore $p'_D(f) = p'_{D'}(f')$. This shows that the map α is well defined. By construction α is characterized by the property that

$$\alpha \circ p_D = p'_D \text{ for all } D \in \mathrm{ob}\,\mathcal{D}. \tag{2.51}$$

This proves that p is a colimit for the functor F.

Now we assume that p is a F-colimit. If p is not stably surjective, then there is an element $a \in A$, so that there is no object $D \in \mathrm{ob}\,\mathcal{D}$ with $a \in \mathrm{im}\,p_D$. If $A = \{a\}$, then necessarily $F = \underline{\emptyset}$. But this functor has $\mathrm{id} : F \longrightarrow \underline{\emptyset}$ as a colimit. Therefore there is $a' \neq a \in A$. We define $\alpha : A \longrightarrow A$ to be the identity everywhere except that $\alpha(a) := a'$. Then $\alpha \circ p_D = p_D$ and $\mathrm{id}_A \circ p_D = p_D$ for all $D \in \mathrm{ob}\,\mathcal{D}$. This contradicts the fact that p is a colimit. Hence p is stably surjective. Now let $f \in F(D)$ and $f' \in F(D')$ with $p_D(f) = p_{D'}(f')$. We define a natural transformation $q : F \longrightarrow \underline{S_2}$ where $S_2 := \{0, 1\}$ as follows: Let $\hat{D} \in \mathrm{ob}\,\mathcal{D}$ and $\hat{f} \in F(\hat{D})$ and put

$$q_{\hat{D}}(\hat{f}) := \begin{cases} 0, & \text{if } \hat{f} \text{ is } F\text{-connected to } f \\ 1, & \text{else.} \end{cases} \tag{2.52}$$

This defines a natural transformation, so by definition there is a map $\alpha : A \longrightarrow S_2$, such that $\alpha \circ p_D = q_D$ for all $D \in \mathrm{ob}\,\mathcal{D}$. But this implies that $0 = q_D(f) = \alpha(p_D(f)) = \alpha(p_{D'}(f')) = q_{D'}(f')$. Therefore f' is F-connected to f. □

Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. We say that a diagram in the category $\mathcal{D}$ is F-commutative if the induced diagram in $\mathcal{A}$ commutes. For instance the diagram

$$\begin{array}{ccc} D & \xrightarrow{d} & \hat{D} \\ & \searrow^{d'} & \downarrow{\scriptstyle \hat{d}} \\ & & D' \end{array} \tag{2.53}$$

F-commutes, if the diagram

$$\begin{array}{ccc} F(D) & \xrightarrow{F(d)} & F(\hat{D}) \\ & \searrow^{F(d')} & \downarrow{\scriptstyle F(\hat{d})} \\ & & F(D') \end{array} \tag{2.54}$$

commutes.

We will need several notions of a (co)final functor between two categories. The notion of a cofinal functor appeared in [1] and was used in [34]. Our notion of a strongly cofinal functor is a special case of what is usually called a final functor (see [26]).

Definition 2.9.9. A functor $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ is weakly cofinal if for every object $D \in \operatorname{ob}\mathcal{D}$ there is an object $C \in \operatorname{ob}\mathcal{C}$ and a morphism $d : D \longrightarrow \theta(C)$.

Definition 2.9.10. Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. A functor $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ is

1. F-cofinal, if it is weakly cofinal and if for every $C \in \operatorname{ob}\mathcal{C}$, $D \in \operatorname{ob}\mathcal{D}$, and for every morphism $d : \theta(C) \longrightarrow D$, there exists an object $C' \in \operatorname{ob}\mathcal{C}$ and morphisms $c : C \longrightarrow C'$ and $d' : D \longrightarrow \theta(C')$, such that the diagram

$$\begin{array}{ccc} \theta(C) & \xrightarrow{d} & D \\ & \searrow^{\theta(c)} & \downarrow{\scriptstyle d'} \\ & & \theta(C') \end{array} \tag{2.55}$$

   F-commutes.

2. strongly F-cofinal, if it is weakly cofinal and if for any two morphisms $d_i : D \longrightarrow \theta(C_i)$ $i = 0, 1$, there is an object $C \in \operatorname{ob}\mathcal{C}$ and there are morphisms $c_i : C_i \longrightarrow C$, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{d_0} & \theta(C_0) \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle \theta(c_0)} \\ \theta(C_1) & \xrightarrow[\theta(c_1)]{} & \theta(C) \end{array} \tag{2.56}$$

   F-commutes.

We call θ (strongly) cofinal if it is (strongly) $\mathrm{id}_{\mathcal{D}}$-cofinal.

Lemma 2.9.11. *Every strongly F-cofinal functor $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ is F-cofinal.*

Proof. Let $d : \theta(C) \longrightarrow D$ be a morphism. Since θ is weakly cofinal, we may choose $d' : D \longrightarrow \theta(C')$. Now we apply the fact that θ is strongly cofinal to the two morphisms $d' \circ d : \theta(C) \longrightarrow \theta(C')$ and $\mathrm{id} : \theta(C) \longrightarrow \theta(C)$: It yields two morphisms $c : C \longrightarrow \hat{C}$ and $c' : C' \longrightarrow \hat{C}$, such that the diagram

$$\begin{array}{ccccc} \theta(C) & \xrightarrow{d} & D & \xrightarrow{d'} & \theta(C') \\ {\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \theta(c')} \\ \theta(C) & & \xrightarrow[\theta(c)]{} & & \theta(\hat{C}) \end{array} \tag{2.57}$$

F-commutes. This shows that θ is F-cofinal. $\square$

Proposition 2.9.12. *Every strongly F-cofinal functor $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ identifies F-colimits.*

Proof. The functor θ induces a functor $\theta^* : \mathrm{CC}(F) \longrightarrow \mathrm{CC}(F \circ \theta)$. We define a push-forward functor

$$\theta_* : \mathrm{CC}(F \circ \theta) \longrightarrow \mathrm{CC}(F). \tag{2.58}$$

Let $a = (a_C : F(\theta(C)) \longrightarrow A)_C$ be a $F \circ \theta$-cocone. We define $\theta_* a$ as follows: Let $D \in \mathrm{ob}\,\mathcal{D}$. Choose a morphism $d : D \longrightarrow \theta(C)$ for some object $C \in \mathrm{ob}\,\mathcal{C}$ and put

$$(\theta_* a)_D := a_{F(\theta(C))} \circ F(d) : F(D) \longrightarrow F(\theta(C)) \longrightarrow A. \tag{2.59}$$

This definition is independent of the choice, since for any other choice $d' : D \longrightarrow \theta(C')$ we can find morphisms $c : C \longrightarrow \hat{C}$ and $c' : C' \longrightarrow \hat{C}$, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\;d\;} & \theta(C) \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle \theta(c)} \\ \theta(C') & \xrightarrow[\theta(c')]{} & \theta(\hat{C}) \end{array} \tag{2.60}$$

F-commutes. But then

$$a_{F(\theta(C))} \circ F(d) = a_{F(\theta(\hat{C}))} \circ F\left(\theta(c) \circ d\right) = a_{F(\theta(\hat{C}))} \circ F(\theta(c') \circ d') = a_{F(\theta(C'))} \circ F(d'). \tag{2.61}$$

The system of morphisms $(\theta_* a)_D$ defines a F-cocone: let $d : D \longrightarrow D'$ be a morphism. Then we choose a morphism $d' : F(D') \longrightarrow F(\theta(C'))$. By definition

$$(\theta_* a)_D = a_{\theta(C')} \circ F(d' \circ d) = (\theta_* a)_{D'} \circ F(d). \tag{2.62}$$

Similarly, one shows that the functor θ_* inverts the functor θ^*. ☐

Definition 2.9.13. A category $\mathcal{D}$ is strongly connected if for any two objects D_0 and D_1 there is a diagram $D_0 \longrightarrow D \longleftarrow D_1$.

Definition 2.9.14. A category $\mathcal{D}$ is connected if for any two objects D, D' there is a sequence

$$\begin{array}{ccccccccccccc} D = D_0 & & & & D_1 & & & & \cdots & & D_{n-1} & & & & D_n = D' \\ & \searrow^{r_0} & & \swarrow_{l_1} & & \searrow^{r_1} & & \swarrow & & \searrow \quad \swarrow_{l_{n-1}} & & \searrow^{r_{n-1}} & & \swarrow_{l_n} & \\ & & D_1' & & & & \cdots & & & \cdots & & & D_n' & & \end{array} \tag{2.63}$$

Definition 2.9.15. Let $\mathcal{D}$ be a category and $F : \mathcal{D} \longrightarrow \mathcal{A}$ a functor.

1. The category $\mathcal{D}$ is F-filtered, if it is strongly connected and if for any two morphisms $d, d' : D \longrightarrow D'$, there is a morphism $\hat{d} : D' \longrightarrow \hat{D}$, such that $F(\hat{d} \circ d) = F(\hat{d} \circ d')$.

2. The category $\mathcal{D}$ is weakly F-filtered, if it is strongly connected and if for any two morphisms $d_i : D \longrightarrow D_i$ $(i = 0, 1)$, there are morphisms $d_i' : D_i \longrightarrow D'$ $(i = 0, 1)$, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\;d_0\;} & D_0 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_0'} \\ D_1 & \xrightarrow[d_1']{} & D' \end{array} \tag{2.64}$$

   F-commutes.

The category $\mathcal{D}$ is called (weakly) filtered if it is (weakly) $\mathrm{id}_{\mathcal{D}}$-filtered.

Proposition 2.9.16. *Let $\theta : \mathcal{C} \longrightarrow \mathcal{D}$ and $F : \mathcal{D} \longrightarrow \mathcal{A}$ be functors. Assume that $\mathcal{C}$ is strongly connected and that $\mathcal{D}$ is F-filtered. If θ is F-cofinal, then it is strongly F-cofinal.*

Proof. For $i = 0, 1$ let $C_i \in \mathrm{ob}\,\mathcal{C}$ be objects and let $d_i : D \longrightarrow \theta(C_i)$ be morphisms. Because the category $\mathcal{C}$ is strongly connected there are morphisms $c_i : C_i \longrightarrow \hat{C}$. Because the category $\mathcal{D}$ is F-filtered, we may choose a morphism $\hat{d} : \theta(\hat{C}) \longrightarrow \hat{D}$, such that the diagram

$$\begin{array}{ccccc} D & \xrightarrow{d_0} & \theta(C_0) & \xrightarrow{\theta(c_0)} & \theta(\hat{C}) \\ \downarrow{\scriptstyle d_1} & & & & \downarrow{\scriptstyle \hat{d}} \\ \theta(C_1) & \xrightarrow[\theta(c_1)]{} & \theta(\hat{C}) & \xrightarrow[\hat{d}]{} & \hat{D} \end{array} \tag{2.65}$$

F-commutes. Now we use the fact that θ is F-cofinal: There is an object $\bar{C} \in \mathrm{ob}\,\mathcal{C}$ and there are morphisms $\bar{d} : \hat{D} \longrightarrow \theta(\bar{C})$ and $\bar{c} : \hat{C} \longrightarrow \bar{C}$, such that the diagram

$$\begin{array}{ccc} \theta(\hat{C}) & \xrightarrow{\hat{d}} & \hat{D} \\ & {\scriptstyle \theta(\bar{c})}\searrow & \downarrow{\scriptstyle \bar{d}} \\ & & \theta(\bar{C}) \end{array} \tag{2.66}$$

F-commutes. This implies that the diagram

$$\begin{array}{ccc} D & \xrightarrow{d_0} & \theta(C_0) \\ \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle \theta(\bar{c} \circ c_0)} \\ \theta(C_1) & \xrightarrow[\theta(\bar{c} \circ c_1)]{} & \theta(\bar{C}) \end{array} \tag{2.67}$$

F-commutes. Therefore θ is strongly F-cofinal. $\square$

The following property appeared in [34] as axiom $S3$ for a so called category with automorphism push-forward.

Definition 2.9.17. A category $\mathcal{D}$ is filtered up to automorphisms if it is strongly connected and if for any two morphisms $d, d' : D \longrightarrow D'$, there exists a morphism $\hat{d} : D' \longrightarrow \hat{D}$ and an automorphism $\hat{a} : \hat{D} \xrightarrow{\cong} \hat{D}$, such that

$$\hat{d} \circ d = \hat{a} \circ \hat{d} \circ d. \tag{2.68}$$

The following property is similar to axiom TSA introduced in [34]. This axiom is slightly weaker then our condition.

Definition 2.9.18. Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. The automorphisms of $\mathcal{D}$ act stably trivially on the functor F if for every $D \in \mathrm{ob}\,\mathcal{D}$ and for every $a \in \mathrm{Aut}_{\mathcal{D}}(D)$ there is a morphisms $d : D \longrightarrow D'$, such that $F(d \circ a) = F(d)$.

Lemma 2.9.19. *Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor on which the automorphisms of $\mathcal{D}$ act stably trivially. If $\mathcal{D}$ is filtered up to automorphisms then it is F-filtered.*

Proof. Let $d, d' : D \longrightarrow D'$ be two morphisms. Since $\mathcal{D}$ is filtered up to automorphisms, there is a morphism $\hat{d} : D' \longrightarrow \hat{D}$ and an automorphism $\hat{a} : \hat{D} \xrightarrow{\cong} \hat{D}$, such that

$$\hat{a} \circ \hat{d} \circ d = \hat{d} \circ d'. \tag{2.69}$$

The automorphisms of $\mathcal{D}$ act stably trivially on the functor F, hence there is a morphism $\bar{d} : \hat{D} \longrightarrow \bar{D}$, such that $F(\bar{d} \circ \hat{a}) = F(\bar{d})$. Therefore

$$F\big(\bar{d} \circ \hat{d} \circ d\big) = F(\bar{d} \circ \hat{a}) \circ F(\hat{d} \circ d) = F\big(\bar{d}\big) \circ F\big(\hat{a} \circ \hat{d} \circ d\big) = F\big(\bar{d} \circ \hat{d} \circ d'\big) \tag{2.70}$$

This shows that $\mathcal{D}$ is F-filtered. $\square$

Proposition 2.9.20. *Let $F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{A}$ be a bifunctor. Assume that for every $D \in \operatorname{ob} \mathcal{D}$, the induced functor $F_D : \mathcal{C} \longrightarrow \mathcal{A}$ admits a colimit. Then there is a natural functor $\mathbb{F} : \mathcal{D} \longrightarrow \mathcal{A}$, $D \mapsto \operatorname{colim} F_D$. The functor $\mathbb{F}$ admits a colimit if and only if the functor F admits a colimit, in which case they are naturally identified.*

Proof. The first statement is a straightforward consequence from the universal property of the colimit. The second statement is proven as follows: The objects in the category of $\mathbb{F}$-cocones are families of morphisms $(a_D : \operatorname{colim} F_D \longrightarrow A)_D$, natural in D. Every morphism $a_D : \operatorname{colim} F_D \longrightarrow A$ corresponds uniquely to a family of morphisms $a_D^C : F(C, D) \longrightarrow A$, natural in C. The family of morphisms a_D is natural in D if and only if for every morphism $d : D \longrightarrow D'$, the diagram

$$\begin{array}{ccc} \operatorname{colim} F_D & & \\ {\scriptstyle \mathbb{F}(d)}\downarrow & \searrow^{a_D} & \\ \operatorname{colim} F_{D'} & \xrightarrow[a_{D'}]{} & A \end{array} \tag{2.71}$$

commutes. Now consider the diagram:

$$\begin{array}{ccccc} F(C, D) & & \xrightarrow{F(\mathrm{id}, d)} & & F(C, D') \\ & \searrow & & \swarrow & \\ {\scriptstyle a_D^C}\downarrow & \operatorname{colim} F_D & \xrightarrow{\mathbb{F}(d)} & \operatorname{colim} F_{D'} & \downarrow{\scriptstyle a_{D'}^C} \\ & \swarrow_{a_D} & & \searrow_{a_{D'}} & \\ A & & \xrightarrow[=]{} & & A \end{array} \tag{2.72}$$

All outer triangles in that diagram commute by definition. Therefore the diagram

$$\begin{array}{ccc} \operatorname{colim} F_D & \xrightarrow{\mathbb{F}(d)} & \operatorname{colim} F_{D'} \\ & \searrow_{a_D} & \downarrow{\scriptstyle a_{D'}} \\ & & A \end{array} \tag{2.73}$$

commutes if and only if for every $C \in \operatorname{ob} \mathcal{C}$, the diagram

$$\begin{array}{ccc} F(C, D) & \xrightarrow{F(\mathrm{id}, d)} & F(C, D') \\ & \searrow_{a_D^C} & \downarrow{\scriptstyle a_{D'}^C} \\ & & A \end{array} \tag{2.74}$$

commutes. This shows that the family of morphism $(a_D : \operatorname{colim} F_D \longrightarrow A)_D$ is natural in D if and only if the corresponding family $(a_D^C : F(C, D) \longrightarrow A)_{C,D}$ is natural in both C and D. It follows that there is a natural isomorphism between the category of $\mathbb{F}$-cocones and the category of F-cocones. The statement follows. □

Lemma 2.9.21. *Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. Assume that $\mathcal{A}$ admits all small coproducts and that $\mathcal{D} = \coprod_{i \in I} \mathcal{D}_i$ is the decomposition into connected components, where I is a set. Let $F_i : \mathcal{D}_i \subset \mathcal{D} \longrightarrow \mathcal{A}$ be the induced functors and assume that F_i admits a colimit for every $i \in I$. Then F admits a colimit and*

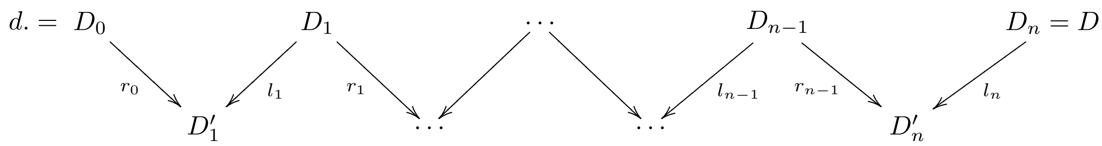
$$\operatorname{colim} F = \coprod_{i \in I} \operatorname{colim} F_i. \tag{2.75}$$

Proof. A F-cocone is a natural transformation $p : F \longrightarrow \underline{A}$. These correspond bijectively to families $p_i : F_i \longrightarrow \underline{A}$ of natural transformations. Now by the universal property of the coproduct and the colimit, a morphism $\alpha : \coprod_{i \in I} \operatorname{colim} F_i \longrightarrow A$ is the same as a family of morphisms $\alpha_i : \operatorname{colim} F_i \longrightarrow A$, and each of these corresponds bijectively to a natural transformation $p_i : F_i \longrightarrow \underline{A}$, and thus the morphism α corresponds bijectively to a natural tranformation $p : F \longrightarrow \underline{A}$. □

The following proposition is similar to [34, Lemma 5.4].

Proposition 2.9.22. *Let $\mathcal{D}$ be a category with decomposition $\mathcal{D} = \coprod_{i\in I} \mathcal{D}_i$ into connected components $\mathcal{D}_i$, where I is a set. Let $\mathcal{A} = \{\mathrm{Set}, \mathrm{Group}, \mathrm{Ab}\}$ and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor. Assume that for every morphism $d : D \longrightarrow D'$, the morphism $F(d)$ is an isomorphism. Then the functor F admits a colimit.*

Proof. By Lemma 2.9.21 we can assume that $\mathcal{D}$ is connected. First we assume that $\mathcal{A} = \mathrm{Set}$. Choose an object $D_0 \in \mathrm{ob}\,\mathcal{D}$ and put $A_0 := F(D_0)/\sim_F$, where $x \sim_F x' \in F(D_0)$ if x and x' are F-connected. Let $D \in \mathrm{ob}\,\mathcal{D}$ be any other object. We define a map $a_D : F(D) \longrightarrow F(D_0)$ as follows: choose a sequence



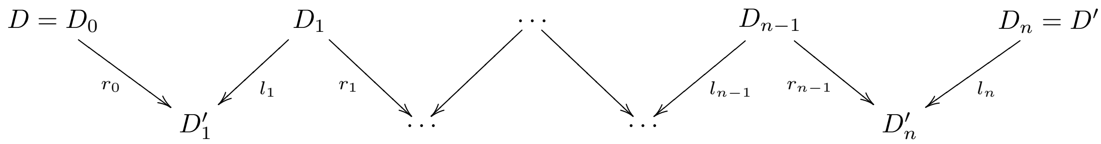
(2.76)

and let $F(d.) := F(r_0)^{-1} \circ F(l_1) \circ \ldots \circ F(l_n) : F(D) \longrightarrow F(D_0)$. Any other choice leads to a map $F(d'.) : F(D) \longrightarrow F(D_0)$ with the property that $F(d'.)x$ is F-connected to $F(d.)x$ for every $x \in F(D)$. Therefore the induced map $a_D := \varepsilon \circ F(d.) : F(D) \longrightarrow A_0$ is independent of the choice of $d.$. The maps $(a_D)_D$ clearly define an F-cocone. We need to verify its universal property: Let $a' = (a'_D : F(D) \longrightarrow A')$ be another F-cocone. Necessarily $a'_{D_0}(x) = a'_{D_0}(x') \in A'$ for any two F-connected elements $x, x' \in F(D_0)$. Therefore the map a'_{D_0} descends to a map $\alpha : A_0 \longrightarrow A$ which has the property that $\alpha \circ a_D = a'_D$ for all objects $D \in \mathrm{ob}\,\mathcal{D}$. Furthermore, α is uniquely determined by this property.

Now assume $\mathcal{A} \in \{\mathrm{Group}, \mathrm{Ab}\}$. The equivalence relation $\sim_F$ generates an equivalence relation compatible with the group structure. We take A_0 to be the group $F(D_0)$ modulo this equivalence relation. The other arguments hold verbatim. ☐

Lemma 2.9.23. *Let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor and let $\mathcal{D}$ be weakly F-filtered. Let $D, D' \in \mathcal{D}$ be objects. Two elements $x \in F(D)$ and $x' \in F(D')$ are F-equivalent, if and only if there are morphisms $d : D \longrightarrow \hat{D}$ and $d' : D' \longrightarrow \hat{D}$, such that $F(d)x = F(d')x'$.*

Proof. By assumption, there is a sequence



(2.77)

and a sequence of elements $x_i \in F(D_i)$ for $i = 0, \ldots, n$ such that

1. $x_0 = x \in F(D)$ and $x_n = x' \in F(D')$;

2. $F(l_i)x_i = F(r_{i-1})x_{i-1}$ for all $i = 1, \ldots, n$.

When $n = 1$, then we have nothing to prove. So assume that $n \geq 2$. The category $\mathcal{D}$ is weakly F-filtered, therefore there are morphisms $\hat{d}_1 : D'_1 \longrightarrow \hat{D}_1$ and $\hat{d}_2 : D'_2 \longrightarrow \hat{D}_1$, so that the diagram

$$\begin{array}{ccc} D_1 & \xrightarrow{r_1} & D'_2 \\ {\scriptstyle l_1}\downarrow & & \downarrow{\scriptstyle \hat{d}_2} \\ D'_1 & \xrightarrow[\hat{d}_1]{} & \hat{D}_1 \end{array} \tag{2.78}$$

F-commutes. Set $\hat{x}_1 := F(\hat{d}_2 \circ r_1)x_1 \in F(\hat{D}_1)$. Now consider the sequence

$$\begin{array}{ccccccccccccc} D = D_0 & & & & D_2 & & & & \cdots & & D_{n-1} & & & & D_n = D' \\ & \searrow^{\hat{d}_1 r_0} & & \swarrow^{\hat{d}_2 \circ l_2} & & \searrow^{r_2} & & \swarrow & & \searrow & & \swarrow^{l_{n-1}} & & \searrow^{r_{n-1}} & & \swarrow^{l_n} \\ & & \hat{D}_1 & & & & \cdots & & & & \cdots & & & & D'_n & \end{array} \tag{2.79}$$

and the sequence $(x_0, \hat{x}_1, x_2, \dots x_n)$. It has length $n-1$. The statement follows by induction on n. ☐

Corollary 2.9.24. *Let $\mathcal{A} \in \{\mathrm{Group}, \mathrm{Ab}\}$ and let $F : \mathcal{D} \longrightarrow \mathcal{A}$ be a functor which maps all morphisms to isomorphisms. Assume that $\mathcal{D}$ is F-filtered. Let $U : \mathcal{A} \longrightarrow \mathrm{Set}$ be the forgetful functor. Then the natural morphism $\operatorname{colim} UF \longrightarrow U \operatorname{colim} F$ is bijective.*

Proof. Recall the construction of the colimit from Proposition 2.9.22. We choose $D_0 \in \operatorname{ob} \mathcal{D}$. Then the $\operatorname{colim} F$ is the quotient of $F(D_0)$ modulo the equivalence relation compatible with the group structure generated by F-connectedness. Let $x, x' \in F(D_0)$ be F-connected. Then there are morphisms $d, d' : D_0 \longrightarrow D$, such that $F(d)x = F(d')x'$. But since $\mathcal{D}$ is F-filtered, there is a morphism $\hat{d} : D \longrightarrow \hat{D}$, such that $F(\hat{d} \circ d) = F(\hat{d} \circ d')$. Then $F(d) = F(d')$ and therefore $x = x'$. It follows that $\operatorname{colim} F = F(D_0)$ and $\operatorname{colim} UF = UF(D_0)$. ☐

Lemma 2.9.25. *Let $F : \mathcal{D} \longrightarrow \mathrm{Ab}$ be a functor and let $V : \mathrm{Ab} \longrightarrow \mathrm{Group}$ be the forgetful functor. Assume that $\operatorname{colim} VF$ exist. Let C be its commutator. Then $(\operatorname{colim} VF)/C$ is naturally a colimit for the functor F.*

Proof. Let $L := \operatorname{colim} VF$ and $C := [L, L] \subset L$. Morphisms $l : L/C \longrightarrow A$ into an abelian group A correspond bijectively to morphisms $l : L \longrightarrow A$. Since $L = \operatorname{colim} VF$, they correspond bijectively to VF-cocones $(a_D : VF(D) \longrightarrow A)$. But every VF-cocone $(a_D : VF(D) \longrightarrow A)$ with A abelian is also an F-cocone. ☐

Proposition 2.9.26. *Let $\mathcal{D}$ be a small category and let $F : \mathcal{D} \longrightarrow \mathrm{Ab}$ be a functor. Let $U : \mathrm{Ab} \longrightarrow \mathrm{Set}$ and $V : \mathrm{Ab} \longrightarrow \mathrm{Group}$ be the forgetful functors.*

1. *Assume that $\mathcal{D}$ is F-filtered. Then the natural map $\operatorname{colim} UF \longrightarrow U \operatorname{colim} F$ is bijective.*
2. *Assume that $\mathcal{D}$ is strongly connected. Then the natural map $\operatorname{colim} VF \longrightarrow V \operatorname{colim} F$ is an isomorphism.*

Proof. Recall that the coproduct in the category Ab is the direct sum and that in Group it is the free product. There is a natural map

$$c : \coprod_{D \in \operatorname{ob} \mathcal{D}} UF(D) \longrightarrow \oplus_{D \in \operatorname{ob} \mathcal{D}} F(D). \tag{2.80}$$

Let $\sim_{\coprod}$ be the equivalence relation on $\coprod_{D \in \operatorname{ob} \mathcal{D}} UF(D)$ that identifies any two F-connected elements and let $\sim_{\oplus}$ be the smallest equivalence relation on $\oplus_{D \in \operatorname{ob} \mathcal{D}} F(D)$ that is compatible with the group structure and that identifies any two F-connected elements.

Let $x \in \oplus_{D \in \operatorname{ob} \mathcal{D}} F(D)$. Then it has a unique description as a sum $x = x_1 + \ldots + x_n$ with $x_i \in F(D_i)$ for objects $D_1, \ldots, D_n \in \operatorname{ob} \mathcal{D}$. By assumption, the category $\mathcal{D}$ is strongly connected. Therefore, there is an object $D \in \operatorname{ob} \mathcal{D}$ and there are morphisms $d_i : D_i \longrightarrow D$. Every element x_i is F-connected to $F(d_i)x_i \in F(D)$. Therefore $x = x_1 + \ldots + x_n \sim_{\oplus} F(d_1)x_1 + \ldots F(d_n)x_n \in \oplus_{D \in \operatorname{ob} \mathcal{D}} F(D)$. It follows that the induced map

$$\bar{c} : \operatorname{colim} UF = \coprod_{D \in \operatorname{ob} \mathcal{D}} UF(D)/ \simeq_{\coprod} \longrightarrow \oplus_{D \in \operatorname{ob} \mathcal{D}} F(D)/ \sim_{\oplus} = \operatorname{colim} F \tag{2.81}$$

is surjective.

We claim that it is injective. Let $x, x' \in \oplus_{D \in \mathrm{ob}\,\mathcal{D}} F(D)$ be two elements. Write $x = x_1 + \ldots + x_n$ with $x_i \in F(D_i)$ and $x' = x'_1 + \ldots x'_{n'}$ with $x'_i \in F(D'_i)$. We claim that $x \sim_\oplus x'$ if and only if there are morphisms $d_i : D_i \longrightarrow D$ and $d'_i : D'_i \longrightarrow D'$, such that

$$F(d_1)x_1 + \ldots + F(d_n)x_n \in F(D) \text{ is } F\text{-connected to } F(d'_1)x'_1 + \ldots + F(d'_n)x'_{n'} \in F(D'). \tag{2.82}$$

If this property is satisfied, then clearly $x \sim_\oplus x'$. To prove the other implication, we need to show that this property indeed defines an equivalence relation, compatible with the group structure. First we show that it defines an equivalence relation. Transitivity is the only problematic point. Therefore, let $x, x', x'' \in \oplus_{D \in D} F(D)$ be three elements with respective decompositions $x = x_1 \ldots + x_n$, $x' = x'_1 + \ldots + x'_{n'}$, and $x'' = x''_1 + \ldots + x''_{n''}$ with $x_i \in F(D_i)$, $x'_i \in F(D'_i)$ and $x''_i \in F(D''_i)$. By assumption there are

1. morphisms $d_i : D_i \longrightarrow D$, $d'_i : D'_i \longrightarrow D'$, such that $\sum F(d_i)x_i \in F(D)$ is F-connected with $\sum F(d'_i)x'_i \in F(D')$
2. morphisms $e'_i : D'_i \longrightarrow E'$ and $e''_i : D''_i \longrightarrow E''$, such that $\sum F(e'_i)x'_i \in F(E')$ is F-connected with $\sum F(e''_i)x''_i \in F(E'')$.

Because the category $\mathcal{D}$ is strongly connected, we can assume that $D = D' = E = E'$. Furthermore, we can assume that $n = n' = n''$. The category $\mathcal{D}$ is F-filtered, therefore there is a morphism $d : D \longrightarrow \hat{D}$, such that $F(d \circ d'_i) = F(d \circ e'_i)$. Then $\sum F(d'_i)x'_i \in F(D)$ is F-connected to $\sum F(d \circ d'_i)x'_i = \sum F(d \circ e'_i)x'_i$. The latter is F-connected to $\sum F(e'_i)x'_i$ which again is F-connected to $\sum F(e''_i)x''_i$, and thus to $\sum F(d \circ e''_i)x''_i$. This proves transitivity.

Now we prove that the equivalence relation defined by this property is compatible with the group structure. We only need to prove the following statement: Let $x, y \in F(D)$ and $x', y' \in F(D')$, such that x is F-connected to x' and y is F-connected to y'. Then $x+y$ is F-connected to $x'+y'$. By Lemma 2.9.23 x is F-connected to x' if and only if there are morphisms $d : D \longrightarrow \hat{D}$ and $d' : D' \longrightarrow \hat{D}$, such that $F(d)x = F(d')x'$. Since y is F-connected to y' we find morphisms $e : D \longrightarrow \hat{E}$ and $e' : D' \longrightarrow \hat{E}$, such that $F(e)y = F(e')y'$. As before we can assume without loss of generality that $\hat{E} = \hat{D}$, that $d = e$ and $d' = e'$. But then $F(d)(x+y) = F(d)x + F(d)y = F(e)x' + F(e)y' = F(e)(x'+y')$, which shows that $x + y$ is F-connected to $x' + y'$.

Now that we have a concrete description of $\sim_\oplus$, injectivity of $\bar{c}$ is an immediate consequence: let $x \in F(D)$ and $x' \in F(D')$, such that $c(x) \sim_\oplus c(x')$. This means that there are morphisms $d : D \longrightarrow \hat{D}$ and $d' : D' \longrightarrow \hat{D}'$, such that $F(d)x$ is F-connected to $F(d')x'$. But x is F-connectd to $F(d)x$ and x' is F-connected to $F(d')x'$. Transitivity of the notion F-connectedness implies that $x \sim_\coprod x'$.

Now we prove the second statement. Let $\sim_*$ be the smallest equivalence relation on the free product $*_{D \in VF(D)} VF(D)$, that identifies any to F-connected elements and that is compatible with the group structure. We need to prove that $*_{D \in \mathrm{ob}\,\mathcal{D}} F(D)/\sim_*$ is abelian. Every element in $*_{D \in \mathrm{ob}\,\mathcal{D}} F(D)$ can be written as a finite product $x_1 * \ldots * x_n$ with $x_i \in F(D_i)$. Let $x'_1 * \ldots x'_{n'}$ be another element with $x'_i \in F(D'_i)$. Since the category $\mathcal{D}$ is strongly connected, there are morphisms $d_i : D \longrightarrow D$ and $d'_i : D'_i \longrightarrow D$. For every i the element $x_i \in F(D_i)$ is F-connected to $F(d_i)x_i \in F(D)$. The same holds for $x'_i \in F(D'_i)$ and $F(d'_i)x'_i \in F(D)$. Therefore $x_1 * \ldots * x_n * x'_1 * \ldots * x'_{n'}$ is equivalent to

$$\sum_i F(d_i)x_i + \sum_i F(d'_i)x'_i = \sum_i F(d'_i)x'_i + \sum_i F(d_i)x_i \in F(\hat{D}). \tag{2.83}$$

But the latter element is equivalent to $x'_1 * \ldots * x'_{n'} * x_1 * \ldots * x_n$. This shows that $*_{D \in \mathrm{ob}\,\mathcal{D}} F(D)/\sim_*$ is an abelian group. The statement follows from the preceeding lemma. □

Chapter 3

A homotopy classification of certain classes of non-linear Fredholm maps

In this chapter we study certain classes of non-linear equivariant Fredholm maps between equivariant Hilbert bundles. The background for this work is threefold: It is provided first by a classical result in non-linear analysis due to Svarc, where a certain type of non-linear perturbations of linear Fredholm maps between Banach spaces is interpreted in terms of stable homotopy groups of spheres ([42], see also [8]). Second, by Bauer and Furuta's work on cohomotopy refined Seiberg-Witten invariants ([17], [7], [6], [5]) and third by Okonek and Teleman's recent article [34], where they present a general framework for cohomotopy invariants, which is then applied to Seiberg-Witten theory.

All three sources study compact perturbations of linear Fredholm maps and associate with them a homotopy invariant. In the case of Svarc, this invariant is an element of a stable homotopy group of spheres. In the other cases the groups also arise as stabilized cohomotopy groups.

Svarc studies compact perturbations of a fixed linear Fredholm map between two Banach spaces. The maps are required to satisfy a properness condition. Bauer and Furuta work with G-equivariant maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$, where G is a compact Lie group and where $\mathcal{E}$ and $\mathcal{F}$ are unitary Hilbert bundles over a compact base B. Okonek and Teleman work with S^1-equivariant maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ between Hilbert bundles over a compact base B. The Hilbert bundles are assumed to have a real summand equipped with the trivial S^1-action and a complex summand equipped with the standard S^1-action. The maps they consider are roughly speaking compact perturbations of linear Fredholm morphisms with a properness condition. Okonek and Teleman assume the maps to be smooth and the Fredholm morphism to be given by the fiberwise differential at the zero-section. On the real part it is supposed to be a fixed isometric embedding. Furthermore, they impose a non-vanishing property: the image of the S^1-fixed point locus under μ is assumed to avoid a neighbourhood of the zero section. This contrasts Bauer and Furuta's construction which does not impose this non-vanishing property, but fixes the Fredholm map also on the complex part.

Let now G be a compact Lie group. We study G-equivariant maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ between G-equivariant Hilbert bundles. Our bundles may contain a real summand, not necessarily trivialized, equipped with the trivial G-action, and a complex summand consisting of a unitary G-bundle. In fact we fix the irreducible complex representations of G that may occur by choosing an arbitrary subset $\rho \subset \mathrm{Irr}(G, \mathbb{C})$. The maps μ we consider are coercive, compact perturbations of linear Fredholm maps. We also include a treatment for the case where the linear Fredholm morphism is partially fixed: Fixing a subset $\rho_0 \subset \rho$ induces decompositions $\mathcal{E} = \mathcal{E}_0 \times_B \mathcal{E}_*$ and $\mathcal{F} = \mathcal{F}_0 \times_B \mathcal{F}_*$. We then fix a Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and modify our definition of Fredholm map by allowing only the part defined on $\mathcal{E}_*$ to vary. Furthermore, we introduce a non-vanishing property in terms of a fixed isotropy family Ω of subgroups of G. It generalizes the non-vanishing property imposed by Okonek and Teleman on their class of maps. These maps we call Ω-Fredholm maps and l-framed Ω-Fredholm maps, respectively.

The leading idea behind our approach is the formalization of the notion of finite dimensional approximation. We define a suitable functor $\mathfrak{p}_\Omega$ involving only finite-dimensional data and we realize its colimit as the set ${}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$ of homotopy classes of Ω-Fredholm maps. In order to produce group valued invariants we relate the functor $\mathfrak{p}_\Omega$ with a group valued functor π_Ω by means of a natural transformation in such a way that the colimits ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega = \operatorname{colim} \mathfrak{p}_\Omega$ and $\operatorname{colim} U\pi_\Omega$ are identified. Here U : Group $\longrightarrow$ Set denotes the forgetful functor. We put ${}_\rho\mathbb{P}_\Omega(B) := \operatorname{colim} \pi_\Omega$ and obtain a natural map

$$ {}_G[\mathcal{E},\mathcal{F}]_B^\Omega = \operatorname{colim} U\pi_\Omega \longrightarrow \operatorname{colim} \pi_\Omega = {}_\rho\mathbb{P}_\Omega(B). \tag{3.1}$$

Associating with a homotopy class $[\mu]$ the corresponding element $\{\mu\} \in {}_\rho\mathbb{P}_\Omega(B)$ defines the invariant.

Let us point out several advantages of this approach. The natural homotopy invariant of an Ω-map μ is its homotopy class itself. This leads to the study of the set of homotopy classes of Ω-maps. Our description of this set as a colimit of a functor defined on finite dimensional data allows the reduction to a finite dimensional setting in a very formal way: Working with the more common formulations of finite dimensional approximations always involves showing that the constructions made are independent of the choice of approximations. This can be tedious. But once the homotopy set realizes a colimit, one can work with the universal property of that colimit and thus avoids all such arguments. Second, one often is interested in group valued invariant. In our case, we exhibit the set ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega$ as the colimit of the functor $U\pi_\Omega$. Since π_Ω is group valued, its colimit is the natural candidate. The corresponding map ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega \longrightarrow {}_\rho\mathbb{P}_\Omega(B)$ enjoys a natural universal property which can be interpreted in the following way: The map defines the finest group valued invariants for the class of Ω-Fredholm maps, compatible with the natural group structure on the set of Ω-maps between finite dimensional fiberwise one-point compactified bundles. This property is explained in Corollary 3.4.14.

The definition of a suitable functor in the framed case is more involved. We need to fix the Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and an l-adapted subbundle $V \subset \mathcal{F}_0$ in order to define a functor $\mathfrak{p}_\Omega^{l,V}$ and to realize its colimit as the set ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}$ of homotopy classes of l-framed Fredholm maps. A careful analysis of this situation shows that the functor $\mathfrak{p}_\Omega^{l,V}$ in fact only depends on the presentation $(l^{-1}(V), V)$ of the index $\operatorname{ind}_{\rho_0} l \in K_{\rho_0}(B)$. But the natural transformation that realizes the colimit additionally depends on the embeddings $l^{-1}(V) \subset \mathcal{E}_0$ and $V \subset \mathcal{F}_0$. We then proceed in analogy to the unframed case and obtain a map

$$ {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_\rho\mathbb{P}_\Omega^{l,V}(B) = \operatorname{colim} \pi_\Omega^{l,V}. \tag{3.2}$$

We show that this map is always surjective and that it is injective when $K_{\rho_*}^{-1}(B) = 0$. Here the set $\rho_* = \rho - \rho_0$ specifies the part where the Fredholm morphism is allowed to vary. Furthermore, we let $K_{\rho_*}^{-1}(B) \subset KO^{-1}(B) \times K_G^{-1}(B)$ be the subgroup generated by $[W] \otimes K^{-1}(B)$ for all $W \in \rho_* \cap \operatorname{Irr}(G,\mathbb{C})$ and additionally by $KO^{-1}(B)$ when $\mathbb{R} \in \rho_*$.

So, in the case $K_{\rho_*}^{-1}(B) = 0$, the groups ${}_\rho\mathbb{P}_\Omega^{l,V}(B)$ classify the homotopy classes of Ω-Fredholm maps. When $\rho_* = \emptyset$, i.e. when the Fredholm morphism is fully fixed, then the hypothesis $K_{\rho_*}^{-1}(B) = 0$ is automatically satisfied. This result corresponds to a theorem of Bauer [5, Theorem 2.1] and in the case where $B = \{*\}$ to Svarc's result (see [8, Theorem 5.3.20]).

We compare the framed with the unframed approach by means of the natural maps

$$ {}_G[\mathcal{E},\mathcal{F}]_B^l \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B \text{ and } {}_\rho\mathbb{P}_\Omega^{l,V}(B) \longrightarrow {}_\rho\mathbb{P}_\Omega(B) \tag{3.3}$$

between framed and unframed classes of maps. With the index $\operatorname{ind}_{\rho_0} l \in K_{\rho_0}(B)$ of l we can associate a subset of ${}_G[\mathcal{E},\mathcal{F}]_B$ and a subgroup of ${}_\rho\mathbb{P}_\Omega(B)$. The maps above then naturally take values in this subset (subgroup). We show that the corestrictions so obtained are surjective and that they are injective when $K_{\rho_0}^{-1}(B) = 0$.

We then explain a second, more explicit construction of these colimits of the respective group valued functors. This construction is based on ideas in [34]. We give a third description of the groups ${}_\rho\mathbb{P}_\Omega(B)$ by following closely Okonek and Teleman's construction of the cohomotopy groups in [34]. This description will then allow us to specialize to the invariants defined by Okonek and Teleman. On the other hand, this gives a new interpretation of Okonek and Teleman's groups. In the case $K^{-1}(B) = 0$, we obtain a description of their groups as a set of homotopy classes of Ω-maps.

Last, we explain how also Bauer and Furuta's construction ([7]) fits into our framework. We use this description to produce a comparison map between the Okonek-Teleman and the Bauer-Furuta groups when $K^{-1}(B) = 0$.

This chapter is organized in the following way: We start by explaining the notion of Ω-Fredholm maps in a detailed way. Then we realize the set ${}_G[\mathcal{E},\mathcal{F}]_B$ as a colimit of a functor $\mathfrak{p}_\Omega$. After that we explain the setup for framed homotopy maps and formulate and prove the analogous results.

In the next section we prove a result concerning Ω-maps between sphere bundles. We show that up to stabilization they arise as one-point compactifications of (proper) Ω-maps between finite dimensional bundles. This result will be used to relate the functor $\mathfrak{p}_\Omega$ with the group valued functor π_Ω. The latter functor is subject of the last section of this chapter, where the cohomotopy groups are introduced and studied.

Fix a connected and compact CW-complex B, a compact Lie group G, and a subset $\rho \subset \mathrm{Irr}(G,\mathbb{C}) \cup \{\mathbb{R}\}$. We work with fiberwise G-bundles over the base B. By G-map we mean a G-equivariant fiberwise map. Recall that a ρ-bundle $\mathcal{E}$ is a bundle of the form $\mathcal{E} = \mathcal{E}_\mathbb{R} \times_B \mathcal{E}_\mathbb{C}$, where $\mathcal{E}_\mathbb{R}$ is a real Hilbert bundle equipped with the trivial G-action, and where $\mathcal{E}_\mathbb{C}$ is a unitary G-bundle. A ρ-morphism between two ρ-bundles $\mathcal{E}$ and $\mathcal{F}$ is a G-equivariant map $f : \mathcal{E} \longrightarrow \mathcal{F}$, such that $f(\mathcal{E}_\mathbb{R}) \subset \mathcal{F}_\mathbb{R}$ and $f(\mathcal{E}_\mathbb{C}) \subset \mathcal{F}_\mathbb{C}$. We denote the induced maps by $f_\mathbb{R} : \mathcal{E}_\mathbb{R} \longrightarrow \mathcal{F}_\mathbb{R}$ and $f_\mathbb{C} : \mathcal{E}_\mathbb{C} \longrightarrow \mathcal{F}_\mathbb{C}$, respectively. They are required to be morphisms of Hilbert bundles. See [Preliminaries 2.8] for more details. Let $K_\rho(B) \subset KO(B) \times K_G(B)$ be the subgroup generated by $[W] \otimes K(B)$ for all $W \in \rho \cap \mathrm{Irr}(G,\mathbb{C})$ and additionally by $KO(B)$ when $\mathbb{R} \in \rho$. A ρ-morphism $f : \mathcal{E} \longrightarrow \mathcal{F}$ is called Fredholm if $f_\mathbb{R}$ and $f_\mathbb{C}$ are fiberwise linear Fredholm maps. When f is a ρ-Fredholm morphism, then $\mathrm{ind}_\rho f := (\mathrm{ind}\, f_\mathbb{R}, \mathrm{ind}_G f_\mathbb{C}) \in K_\rho(B)$.

A (closed) isotropy family ([44, page 46]) for G is a set Ω of closed subgroups of G, such that for all closed subgroups $H, K \subset G$, the following hold:

1. if $H \in \Omega$ and K is conjugate to H, then $K \in \Omega$;
2. if $H \in \Omega$ and $H \subset K$, then $K \in \Omega$.

With an isotropy family Ω for G and a ρ-bundle $\mathcal{E}$ we associate the G-subspace

$$\mathcal{E}(\Omega) := \{e \in \mathcal{E} \,\big|\, G_e \in \Omega\}.$$

Remark 3.0.27. Let Ω be an isotropy family for G, and let $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be a G-map between two ρ-bundles. Then $\mu(\mathcal{E}(\Omega)) \subset \mathcal{F}(\Omega)$.

Proof. Equivariance of μ implies that $G_e \subset G_{\mu(e)}$ for every $e \in \mathcal{E}$. The statement follows from the second axiom of an isotropy family. □

We are mainly interested in the following families:

$$\emptyset \text{ and } \Omega_n := \{H \subset G \,\big|\, \dim H > n\} \text{ for } n \in \mathbb{N}. \tag{3.4}$$

We now fix two infinite dimensional and separable ρ-bundles $\mathcal{E}$ and $\mathcal{F}$ and an isotropy family Ω for G.

Definition 3.0.28. A G-map $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ is called an Ω-(Fredholm) map if it satisfies the following properties:

1. It is coercive.
2. There exists a neighbourhood $U \subset \mathcal{F}$ of the 0-section, such that $\mu(\mathcal{E}(\Omega)) \subset \mathcal{F} - U$.
3. There exists a ρ-Fredholm morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$, such that $\mu - d$ is compact.

Definition 3.0.29. An Ω-(Fredholm) homotopy is an Ω-Fredholm map $\tilde{\mu} : \mathcal{E} \times [0,1] \longrightarrow \mathcal{F} \times [0,1]$ between the ρ-bundles $\mathcal{E} \times [0,1]$ and $\mathcal{F} \times [0,1]$ over the base $B \times [0,1]$.

Abusing notation we will usually write $(\mu_t)_{t\in[0,1]}$ or even μ_t, instead of $\tilde{\mu}$.

Lemma 3.0.30. *Let $\mu_t : \mathcal{E} \longrightarrow \mathcal{F}$ be an Ω-homotopy with $\mu_0 = d + c$ and $\mu_1 = d' + c'$. Then there exists a homotopy $d_t : \mathcal{E} \longrightarrow \mathcal{F}$ of ρ-Fredholm morphisms, such that $d_0 = d$, $d_1 = d'$, and that $\mu_t - d_t$ is compact.*

Proof. Since μ_t is an Ω-homotopy, there exists a decomposition $\mu_t = \hat{d}_t + \hat{c}_t$. We consider the equations $d + c = \hat{d}_0 + \hat{c}_0$ and $d' + c' = \hat{d}_1 + \hat{c}_1$. They imply that $\hat{d}_0 - d = c - \hat{c}_0$ and $\hat{d}_1 - d' = c' - \hat{c}_1$ are compact ρ-morphisms. Therefore $(1-t)(\hat{d}_0 - d) + t(\hat{d}_1 - d') =: k_t$ defines a homotopy of compact ρ-morphisms. It follows that

$$\mu_t = \big(\hat{d}_t - k_t\big) + \big(\hat{c}_t + k_t\big). \tag{3.5}$$

We put $d_t := \hat{d}_t - k_t$. Then $d_0 = \hat{d}_0 - k_0 = d$, $d_1 = \hat{d}_1 - k_1 = d'$, and $\mu_t - d_t = \hat{c}_t + k_t$ is compact. □

Lemma 3.0.31. *The notion of Ω-homotopy defines an equivalence relation on the set of Ω-Fredholm maps.*

Proof. The only problematic point is transitivity. Thus, let $\mu_t, \mu'_t : \mathcal{E} \longrightarrow \mathcal{F}$ be two Ω-homotopies with $\mu_1 = \mu'_0$. We define μ''_t by

$$\mu''_t := \begin{cases} \mu_{2t} & \text{if } 0 \leq t \leq 1/2; \\ \mu'_{2t-1} & \text{else.} \end{cases} \tag{3.6}$$

We need to check whether it defines an Ω-homotopy. By the preceeding lemma we may choose ρ-Fredholm homotopies d_t and d'_t with $d_1 = d'_0$, such that $\mu_t - d_t$ and $\mu'_t - d'_t$ are compact. We can glue d_t and d'_t to a ρ-Fredholm morphism d''_t using the obvious formula. All other points are immediately verified and we have thus shown that μ''_t is an Ω-map. □

We write ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B$ for the set of Ω-homotopy classes of Ω-Fredholm maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$, and we put ${}_G[\mathcal{E}, \mathcal{F}]_B := {}_G[\mathcal{E}, \mathcal{F}]^{\emptyset}_B$.

Remark 3.0.32.

1. There is a natural map ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B$.

2. There exists a unique map $\operatorname{ind}_\rho : {}_G[\mathcal{E}, \mathcal{F}]_B \longrightarrow K_\rho(B)$ with the following property: Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a ρ-Fredholm morphism and let $c : \mathcal{E} \longrightarrow \mathcal{F}$ be a compact map, such that $d + c$ is an Ω-Fredholm map. Then $\operatorname{ind}_\rho[d + c] = \operatorname{ind}_\rho d$.

Proof. Let $[\mu] \in {}_G[\mathcal{E}, \mathcal{F}]_B$ be a homotopy class with representative $\mu : \mathcal{E} \longrightarrow \mathcal{F}$. We choose a ρ-Fredholm morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$, such that $\mu - d$ is compact and we put $\operatorname{ind}_\rho([\mu]) := \operatorname{ind}_\rho(d)$.

Let $\mu = d + c = d' + c'$ be two decompositions of an Ω-map μ, where $d, d' : \mathcal{E} \longrightarrow \mathcal{F}$ are ρ-Fredholm morphisms and where $c, c' : \mathcal{E} \longrightarrow \mathcal{F}$ are compact. Then $d' - d = c - c' : \mathcal{E} \longrightarrow \mathcal{F}$ is a compact ρ-morphism. Therefore $\operatorname{ind}_\rho d = \operatorname{ind}_\rho(d + d' - d) = \operatorname{ind}_\rho d'$. Let $\mu_t : \mathcal{E} \longrightarrow \mathcal{F}$ be an Ω-homotopy between μ_0 and μ_1. There exists $d_t : \mathcal{E} \longrightarrow \mathcal{F}$, a family of ρ-Fredholm morphisms, such that $\mu_t - d_t$ is compact. Then $\operatorname{ind}_\rho d_0 = \operatorname{ind}_\rho d_1$ by the homotopy invariance of the ρ-Fredholm index. □

Lemma 3.0.33. *A subset $U \subset \mathcal{F}$ is a neighbourhood of the 0-section if and only if there exists $\varepsilon > 0$, such that $\mathring{D}_\varepsilon(\mathcal{F}) \subset U$.*

Proof. First, we prove the following statement: For $x \in B$, there exist an open neighbourhood $V_x \subset B$ of x and $\varepsilon_x > 0$, such that

$$\mathring{D}_{\varepsilon_x}(\mathcal{F}_y) \subset U \cap \mathcal{F}_y \text{ for all } y \in V_x. \tag{3.7}$$

Choose an open neighbourhood $V'_x \subset B$ of x and a fiberwise isometry $\varphi_x : \mathcal{F}_{V'_x} \xrightarrow{\cong} V'_x \times \mathcal{F}_x$. The subset $\varphi_x(U \cap \mathcal{F}_{V'_x}) \subset V'_x \times \mathcal{F}_x$ is an open neighbourhood of $V'_x \times \{0\}$. Hence there exist $\varepsilon_x > 0$ and $V_x \subset V'_x$ open, such that $V_x \times \mathring{D}_{\varepsilon_x}(\mathcal{F}_x) \subset \varphi_x(U \cap \mathcal{F}_{V'_x})$. This implies $\mathring{D}_{\varepsilon_x}(\mathcal{F}_{V_x}) \subset U \cap \mathcal{F}_{V_x}$.

For every point $x \in B$, we choose $\varepsilon_x > 0$ and $V_x \subset B$ as above. Compactness of B allows us to choose a finite covering $B = V_{x_1} \cup \ldots \cup V_{x_r}$. Put $\varepsilon := \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_r}\}$. Then $\mathring{D}_\varepsilon(\mathcal{F}) \subset U$. □

3.1 Finite dimensional approximation of Ω-Fredholm maps

In this section we realize the set ${}_G[\mathcal{E},\mathcal{F}]_B$ as the colimit of a functor $\mathfrak{p}_\Omega$. We start by introducing a category ${}_\rho\mathcal{H}_B$ on which the functor $\mathfrak{p}_\Omega$ is defined. It is defined in analogy to the category $\mathcal{T}(x)$ in [34]. The realization of the set ${}_G[\mathcal{E},\mathcal{F}]_B$ as a colimit of the functor $\mathfrak{p}_\Omega$ is done in two steps: First a natural transformation between the functor $\mathfrak{p}_\Omega$ and the set ${}_G[\mathcal{E},\mathcal{F}]_B$ (understood as constant functor) is defined (Theorem 3.1.4). Then this transformation is proven to be a colimit (Theorem 3.1.11). The proof of the second theorem relies on methods used by Bauer ([5, Theorem 2.1]) and by Okonek and Teleman [34, Lemma 3.12].

We define the category ${}_\rho\mathcal{H}_B$ as follows: its objects are pairs (E,F) of finite-dimensional ρ-bundles E,F. Let (E,F) and (E',F') be two such objects. A morphism $f:(E,F)\longrightarrow(E',F')$ is a family $f=(i_E,i_F,\tilde{E},\tilde{F},\tau)$, consisting of the following data:

1. two closed ρ-embeddings $i_E:E\hookrightarrow E'$ and $i_F:F\hookrightarrow F'$;
2. closed ρ-complements $E'=i_E(E)\oplus\tilde{E}$ and $F'=i_F(F)\oplus\tilde{F}$;
3. a ρ-isomorphism $\tau:\tilde{E}\xrightarrow{\cong}\tilde{F}$.

We compose two morphisms $f=(i_E,i_F,\tilde{E},\tilde{F},\tau):(E,F)\longrightarrow(E',F')$ and $f'=(i_{E'},i_{F'},\tilde{E}',\tilde{F}',\tau'):(E',F')\longrightarrow(E'',F'')$ to a morphism $f'':(E,F)\longrightarrow(E'',F'')$ in the following natural way: By definition f'' consists of

1. the ρ-embeddings $i_{E'}\circ i_E:E\hookrightarrow E''$ and $i_{F'}\circ i_F:F\hookrightarrow F''$;
2. the ρ-complements $\tilde{E}'\oplus i_{E'}(\tilde{E})$ of $i_{E'}\,(i_E(E))$ and $\tilde{F}'\oplus i_{F'}(\tilde{F})$ of $i_{F'}\,(i_F(F))$, respectively;
3. the ρ-isomorphism $\tau'\oplus\left[i_{F'}\circ\tau\circ(i_{E'}|_{\tilde{E}})^{-1}\right]$.

Given an object (E,F) in the category ${}_\rho\mathcal{H}_B$, we define the notions of Ω-maps and Ω-homotopies as in the infinite-dimensional setting. Thus a G-map $m:E\longrightarrow F$ is an Ω-map if it is proper and if $m(E(\Omega))\subset F-U$ for a neighbourhood $U\subset F$ of the 0-section. An Ω-homotopy is then an Ω-map $\tilde{m}:E\times[0,1]\longrightarrow F\times[0,1]$ over the base $B\times[0,1]$. We denote the set of Ω-homotopy classes of Ω-maps by ${}_G[E,F]^\Omega_B$ and we introduce the notation ${}_G[E,F]_B:={}_G[E,F]^\emptyset_B$.

With each object (E,F) we associate its ρ-index $[E]-[F]:=([E_\mathbb{R}]-[F_\mathbb{R}],[E_\mathbb{C}]-[F_\mathbb{C}])\in K_\rho(B)\subset KO(B)\times K_G(B)$.

Lemma 3.1.1. *Let (E,F) and (E',F') be objects in ${}_\rho\mathcal{H}_B$. There exists a diagram in ${}_\rho\mathcal{H}_B$*

$$\begin{array}{ccc} (E,F) & & (E',F') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & (\hat{E},\hat{F}) & \end{array} \tag{3.8}$$

if and only if $[E]-[F]=[E']-[F']\in K_\rho(B)$.

Proof. If there is a morphism $f:(E,F)\longrightarrow(\hat{E},\hat{F})$, then $[E]-[F]=[\hat{E}]-[\hat{F}]\in K_\rho(B)$. On the other hand, if $[E]-[F]=[E']-[F']$, then there exists a finite dimensional ρ-bundle H and a ρ-isomorphism $\varphi:E\oplus F'\oplus H\xrightarrow{\cong}E'\oplus F\oplus H$. We put $\hat{E}:=E\oplus F'\oplus H$ and $\hat{F}:=F\oplus F'\oplus H$. Then there are natural embeddings

- $E\hookrightarrow E\oplus F'\oplus H=\hat{E}$ and $F\hookrightarrow F\oplus F'\oplus H=\hat{F}$;
- $E'\hookrightarrow E'\oplus F\oplus H\cong_\varphi E\oplus F'\oplus H=\hat{E}$ and $F'\hookrightarrow F\oplus F'\oplus H=\hat{F}$.

We obtain morphisms $f:(E,F)\longrightarrow(\hat{E},\hat{F})$ and $f':(E',F')\longrightarrow(\hat{E},\hat{F})$ in the obvious way. □

Remark 3.1.2. A morphism $f = (i_E, i_F, \tilde{E}, \tilde{F}, \tau) : (E, F) \longrightarrow (E', F')$ in ${}_\rho\mathcal{H}_B$ induces an isomorphism

$$\hat{f} : (E', F') \longrightarrow (E \oplus \tilde{E}, F \oplus \tilde{E}), \tag{3.9}$$

such that $\hat{f} \circ f : (E, F) \longrightarrow (E \oplus \tilde{E}, F \oplus \tilde{E})$ is the natural morphism.

Proof. The isomorphism $\hat{f} : (E', F') \longrightarrow (E \oplus \tilde{E}, F \oplus \tilde{E})$ is given by the isomorphisms

$$i_E^{-1} \oplus \mathrm{id}_{\tilde{E}} : E' = i_E(E) \oplus \tilde{E} \longrightarrow E \oplus \tilde{E} \text{ and } i_F^{-1} \oplus \tau^{-1} : F' = i_F(F) \oplus \tilde{F} \longrightarrow F \oplus \tilde{E}. \tag{3.10}$$

It is an immediate consequence of the definition of $\hat{f}$ that the composition $\hat{f} \circ f$ is the natural morphism $(E, F) \longrightarrow (E \oplus \tilde{E}, F \oplus \tilde{E})$. □

For $x \in K_\rho(B)$ let ${}_\rho\mathcal{H}_B(x) \subset {}_\rho\mathcal{H}_B$ be the full subcategory consisting of the objects (E, F) with $[E] - [F] = x \in K_\rho(B)$.

Corollary 3.1.3. *The category ${}_\rho\mathcal{H}_B$ has a decomposition into connected components*

$${}_\rho\mathcal{H}_B = \coprod_{x \in K_\rho(B)} {}_\rho\mathcal{H}_B(x). \tag{3.11}$$

For every $x \in K_\rho(B)$, the category ${}_\rho\mathcal{H}_B(x)$ is weakly filtered.

Proof. The first statement follows immediately from the preceeding lemma, as does the fact that ${}_\rho\mathcal{H}_B(x)$ is strongly connected. For $l = 0, 1$ let now $f_l : (E, F) \longrightarrow (E_l, F_l)$ be a morphism. We write $f_l = (i_{E,l}, i_{F,l}, \tilde{E}_l, \tilde{F}_l, \tau_l)$. Let $\hat{f}_l : (E_l, F_l) \longrightarrow (E \oplus \tilde{E}_l, F \oplus \tilde{E}_l)$ be the isomorphism induced by f_l (Remark 3.1.2). Furthermore, let $g_l : (E \oplus \tilde{E}_l, F \oplus \tilde{E}_l) \longrightarrow (E \oplus \tilde{E}_0 \oplus \tilde{E}_1, F \oplus \tilde{E}_0 \oplus \tilde{E}_1)$ be the natural inclusion. Then

$$g_0 \circ \hat{f}_0 \circ f_0 = g_1 \circ \hat{f}_1 \circ f_1. \tag{3.12}$$

Thus the category ${}_\rho\mathcal{H}_B(x)$ is weakly filtered. □

We define a functor $\mathfrak{p}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Set}$: Given $(E, F) \in \mathrm{ob}\, {}_\rho\mathcal{H}_B$, we put $\mathfrak{p}_\Omega(E, F) := {}_G[E, F]_B^\Omega$. Given a morphism $f = (i_E, i_F, \tilde{E}, \tilde{F}, \tau) : (E, F) \longrightarrow (E', F')$ and given an Ω-map $m : E \longrightarrow F$, we put

$$f_* m := \left[i_F \circ m \circ (i_E)^{-1} \right] \oplus \tau : E' = i_E(E) \oplus \tilde{E} \longrightarrow i_F(F) \oplus \tilde{F}. \tag{3.13}$$

This defines a map $\mathfrak{p}_\Omega(f) : {}_G[E, F]_B^\Omega \longrightarrow {}_G[E', F']_B^\Omega$, $[m] \mapsto [f_* m]$.

For $x \in K_\rho(B)$, let $\mathfrak{p}_\Omega(x) : {}_\rho\mathcal{H}_B(x) \longrightarrow \mathrm{Set}$ be the induced functor and let ${}_G[\mathcal{E}, \mathcal{F}]_B^\Omega(x) := \mathrm{ind}_\rho^{-1}(x)$.

Theorem 3.1.4. *There exists a natural transformation*

$$p : \mathfrak{p}_\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega, \tag{3.14}$$

which for each $x \in K_\rho(B)$ induces a natural transformation

$$p_x : \mathfrak{p}_\Omega(x) \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega(x). \tag{3.15}$$

Proof. Let (E, F) be a pair of finite dimensional ρ-bundles. Define

$$p_{(E,F)} : {}_G[E, F]_B^\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^\Omega \tag{3.16}$$

as follows: Fix a family $C = (\iota_E, \iota_F, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{d})$ consisting of

- closed ρ-embeddings $\iota_E : E \hookrightarrow \mathcal{E}$ and $\iota_F : F \hookrightarrow \mathcal{F}$ (Proposition 2.8.5);
- closed ρ-complements $\iota_E(E) \oplus \tilde{\mathcal{E}} = \mathcal{E}$ and $\iota_F(F) \oplus \tilde{\mathcal{F}} = \mathcal{F}$;
- a ρ-isomorphism $\tilde{d} : \tilde{\mathcal{E}} \xrightarrow{\cong} \tilde{\mathcal{F}}$ (Corollary 2.8.3).

We define $\pi_E : \mathcal{E} \longrightarrow \iota_E(E) \cong E$ by first projecting to $\iota_E(E)$ using the decomposition $\mathcal{E} = \iota_E(E) \oplus \tilde{\mathcal{E}}$ and then identifying $\iota_E(E)$ with E using the embedding ι_E. Similarly we define $\pi_F : \mathcal{F} \longrightarrow F$. The projections on the respective complements we denote with $\pi_{\tilde{\mathcal{E}}}$ and $\pi_{\tilde{\mathcal{F}}}$.

Put $d := 0_E \oplus \tilde{d} : \mathcal{E} \longrightarrow \mathcal{F}$. By definition d is a ρ-Fredholm morphism of ρ-index $\operatorname{ind}_\rho d = [E] - [F] \in K_\rho(B)$. With an Ω-map $m : E \longrightarrow F$ we associate the map

$$\mu := \mu(C, m) := \iota_F \circ m \circ \pi_E + d. \tag{3.17}$$

We now prove that $\mu(C, m)$ is an Ω-Fredholm map. By construction $\mu - d = \iota_F \circ m \circ \pi_E$ is bounded with image contained in the finite dimensional ρ-subbundle $\iota_F(F) \subset \mathcal{F}$, and hence compact. Next we prove that μ is coercive. Let $S \subset \mathcal{F}$ be a bounded subset. We need to show that $\mu^{-1}(S)$ is bounded. A subset $T \subset \mathcal{E}$ is bounded if and only if both $\pi_{\tilde{\mathcal{E}}}(T) \subset \tilde{\mathcal{E}}$ and $\pi_E(T) \subset E$ are bounded. Applying this property to $S \subset \mathcal{F}$ we see that both $\pi_{\tilde{\mathcal{F}}}(S) \subset \tilde{\mathcal{F}}$ and $\pi_F(S) \subset F$ are bounded. Now notice that

$$\pi_{\tilde{\mathcal{F}}} \circ \mu = \mu \circ \pi_{\tilde{\mathcal{E}}} = \tilde{d} \circ \pi_{\tilde{\mathcal{E}}} \text{ and } \pi_F \circ \mu = m \circ \pi_E. \tag{3.18}$$

Thus, if $e \in \mu^{-1}(S)$, then $\tilde{d}(\pi_{\tilde{\mathcal{E}}}(e)) = \pi_{\tilde{\mathcal{F}}}(\mu(e)) \in \pi_{\tilde{\mathcal{F}}}(S)$. Hence $\pi_{\tilde{\mathcal{E}}}(\mu^{-1}(S)) \subset \tilde{d}^{-1}(\pi_{\tilde{\mathcal{F}}}(S))$. Since $\tilde{d}$ is an isomorphism, this subset is bounded. On the other hand, if $e \in \mu^{-1}(S)$, then $m(\pi_E(e)) = \pi_F(\mu(e)) \in \pi_F(S)$. Hence $\pi_E(\mu^{-1}(S)) \subset m^{-1}(\pi_F(S))$. Since $m : E \longrightarrow F$ is proper, this subset is bounded. It follows that $\mu^{-1}(S)$ is bounded.

Last, we need to show that $\mu(\mathcal{E}(\Omega)) \subset \mathcal{F}$ avoids a neighbourhood of the 0-section. The projection map $\pi_E : \mathcal{E} \longrightarrow E$ is a ρ-morphism, and therefore $\pi_E(\mathcal{E}(\Omega)) \subset E(\Omega)$. Since $m : E \longrightarrow F$ is an Ω-map, there is an open neighbourhood of the 0-section $V \subset F$, such that $m(\pi_E(\mathcal{E}(\Omega))) \subset m(E(\Omega)) \subset F - V$. From $\pi_F \circ \mu(\mathcal{E}(\Omega)) = m \circ \pi_E(\mathcal{E}(\Omega)) \subset F - V$ it follows that $\mu(\mathcal{E}(\Omega)) \subset \pi_F^{-1}(F - V)$. The latter is a closed subset of $\mathcal{F}$ which does not meet the 0-section. Therefore $\mu(\mathcal{E}(\Omega))$ avoids a neighbourhood of the 0-section.

The association $m \mapsto \mu(C, m)$ carries over to Ω-homotopies, therefore it descends to a map $[m] \mapsto [\mu(C, m)]$. Next we show that the Ω-homotopy class of $\mu(C, m)$ is independent of the choice of C. We prove this claim in three steps. Step 1: Let $C' = (\iota_E, \iota_F, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{d}')$, where $\tilde{d}' : \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{F}}$ is a different choice of isomorphism. Consider the ρ-isomorphism $\varphi := \tilde{d}' \circ \tilde{d}^{-1} : \tilde{\mathcal{F}} \xrightarrow{\cong} \tilde{\mathcal{F}}$. By Theorem 2.8.5, there exists a ρ-isotopy $\varphi_t : \tilde{\mathcal{F}} \xrightarrow{\cong} \tilde{\mathcal{F}}$, such that $\varphi_0 = \mathrm{id}_{\tilde{\mathcal{F}}}$ and $\varphi_1 = \varphi$. Put $\mu_t := (\varphi_t \oplus \mathrm{id}_{\iota_F(F)}) \circ \mu$. It defines an Ω-homotopy between $\mu = \mu_0$ and $\mu_1 = \mu(C', m)$.

Step 2: Let $C' = (\iota_E, \iota_F, \tilde{\mathcal{E}}', \tilde{\mathcal{F}}', \tilde{d}')$, where $\tilde{\mathcal{E}}'$ and $\tilde{\mathcal{F}}'$ are closed ρ-complements of $\iota_E(E)$ and $\iota_F(F)$, respectively, and where $\tilde{d}' : \tilde{\mathcal{E}}' \xrightarrow{\cong} \tilde{\mathcal{F}}'$ is a ρ-isomorphism. By Corollary 2.8.7, there exists a homotopy of complements $\hat{\mathcal{E}}_t \subset \mathcal{E}$ and $\hat{\mathcal{F}}_t \subset \mathcal{F}$ inducing homotopies of projections $\pi_{E,t} : \mathcal{E} \longrightarrow E$, $\pi_{\hat{\mathcal{E}}_t,t} : \mathcal{E} \longrightarrow \hat{\mathcal{E}}_t$, $\pi_{F,t} : \mathcal{F} \longrightarrow F$, and $\pi_{\hat{\mathcal{F}}_t,t} : \mathcal{F} \longrightarrow \hat{\mathcal{F}}_t$ with $\hat{\mathcal{E}}_0 = \tilde{\mathcal{E}}$, $\hat{\mathcal{E}}_1 = \tilde{\mathcal{E}}'$, $\hat{\mathcal{F}}_0 = \tilde{\mathcal{F}}$, and $\hat{\mathcal{F}}_1 = \tilde{\mathcal{F}}'$. Choose a ρ-isotopy $\gamma_t : \hat{\mathcal{E}}_t \xrightarrow{\cong} \hat{\mathcal{F}}_t$ (by that we mean a ρ-isomorphism between the respective bundles over the base $B \times [0, 1]$). Define $C_t := (\iota_E, \iota_F, \hat{\mathcal{E}}_t, \hat{\mathcal{F}}_t, \gamma_t)$. Then $\mu(C_t, m)$ defines an Ω-homotopy between $\mu(C_0, m)$ and $\mu(C_1, m)$. Observe that $C_0 = (\iota_E, \iota_F, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \gamma_0)$ and $C_1 = (\iota_E, \iota_F, \tilde{\mathcal{E}}', \tilde{\mathcal{F}}', \gamma_1)$. By step 1, there exists a Ω-homotopy between $\mu(C_0, m)$ and $\mu(C, m)$, as well as between $\mu(C_1, m)$ and $\mu(C', m)$. This shows that $\mu(C, m)$ and $\mu(C', m)$ are Ω-homotopic.

Step 3: Let $C' = (\iota'_E, \iota'_F, \tilde{\mathcal{E}}', \tilde{\mathcal{F}}', \tilde{d}')$. By Lemma 2.8.5, there exist homotopies $\iota_{E,t} : E \hookrightarrow \mathcal{E}$ and $\iota_{F,t} : F \hookrightarrow \mathcal{F}$ of ρ-embeddings with $\iota_{E,0} = \iota_E$, $\iota_{E,1} = \iota'_E$ and $\iota_{F,0} = \iota_F$, $\iota_{F,1} = \iota'_F$. Furthermore, we choose homotopies of closed ρ-complements $\iota_{E,t}(E) \oplus \hat{\mathcal{E}}_t = \mathcal{E}$ and $\iota_{F,t}(F) \oplus \hat{\mathcal{F}}_t = \mathcal{F}$: We could for example take the orthogonal complements. Choose furthermore an isotopy $\hat{d}_t : \hat{\mathcal{E}}_t \xrightarrow{\cong} \hat{\mathcal{F}}_t$. We put $\hat{C}_t := (\iota_{E,t}, \iota_{F,t}, \hat{\mathcal{E}}_t, \hat{\mathcal{F}}_t, \hat{d}_t)$. This data then defines an Ω-homotopy $\mu(\hat{C}_t, m)$ between $\mu(\hat{C}_0, m)$ and $\mu(\hat{C}_1, m)$. By the argument made in the previous step, $\mu(\hat{C}_0, m)$ is Ω-homotopic to $\mu(C, m)$, and $\mu(\hat{C}_1, m)$ is Ω-homotopic to $\mu(C', m)$. Therefore the Ω-homotopy class of $\mu(C, m)$ is independent of the choice of C.

Let $f = (i_E, i_F, \tilde{E}, \tilde{F}, \tau) : (E,F) \longrightarrow (E',F')$ be a morphism in ${}_\rho\mathcal{H}_B$. We have to show that the diagram

$$\begin{array}{ccc} {}_G[E,F]_B^\Omega & & \\ {\scriptstyle \mathfrak{p}_\Omega(f)}\downarrow & \searrow^{p_{(E,F)}} & \\ {}_G[E',F']_B^\Omega & \xrightarrow[p_{(E',F')}]{} & {}_G[\mathcal{E},\mathcal{F}]_B^\Omega \end{array} \tag{3.19}$$

commutes. Let $m : E \longrightarrow F$ be an Ω-map. Choose a family $C' = (\iota_{E'}, \iota_{F'}, \tilde{\mathcal{E}}', \tilde{\mathcal{F}}', \tilde{d}')$, consisting of

1. closed ρ-embeddings $\iota_{E'} : E' \hookrightarrow \mathcal{E}$, $\iota_{F'} : F' \hookrightarrow \mathcal{F}$;
2. closed ρ-complements $\mathcal{E} = \iota_{E'}(E') \oplus \tilde{\mathcal{E}}'$ and $\mathcal{F} = \iota_{F'}(F') \oplus \tilde{\mathcal{F}}'$, and
3. a ρ-isomorphism $\tilde{d}' : \tilde{\mathcal{E}}' \xrightarrow{\cong} \tilde{\mathcal{F}}'$.

Then put $C := (\iota_E, \iota_F, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{d})$, with

1. $\iota_E := \iota_{E'} \circ i_E : E \hookrightarrow \mathcal{E}$, $\iota_F := \iota_{F'} \circ i_F : F \hookrightarrow \mathcal{F}$;
2. $\tilde{\mathcal{E}} := \tilde{\mathcal{E}}' \oplus \iota_{E'}(\tilde{E})$ and $\tilde{\mathcal{F}} := \tilde{\mathcal{F}}' \oplus \iota_{F'}(\tilde{F})$;
3. $\tilde{d} := \tilde{d}' \oplus \iota_{F'} \circ \tau \circ (\iota_{E'}|_{\tilde{E}})^{-1}$.

With these choices, we claim that $\mu(C,m) = \mu(C', f_* m)$. To see this, let $e = \iota_E(e_0) + \iota_{E'}(e_1) + e_2 \in \mathcal{E} = \iota_E(E) \oplus \iota_{E'}(\tilde{E}) \oplus \tilde{\mathcal{E}}'$. Then

$$\begin{aligned} \mu(C,m)(e) &= \iota_F(m(e_0)) + \tilde{d}(\iota_{E'}(e_1) + e_2) = \iota_F(m(e_0)) + \iota_{F'}(\tau(e_1)) + \tilde{d}'(e_2) \\ &= \iota_{F'}(i_F(m(e_0)) + \tau(e_1)) + d'_0(e_2) \\ &= \iota_{F'}(f_* m(i_E(e_0) + e_1)) + \tilde{d}'(e_2) = \mu(C', f_* m). \end{aligned} \tag{3.20}$$

□

Remark 3.1.5. Let $V \subset \mathcal{V}$ be a finite dimensional vector subspace of a Banach space $\mathcal{V}$. Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be closed complements of V and let $\pi_i : \mathcal{V} = V \oplus \mathcal{V}_i \longrightarrow \mathcal{V}_i$ be the respective projections. Then $\pi_{10} := \pi_1|_{\mathcal{V}_0} : \mathcal{V}_0 \longrightarrow \mathcal{V}_1$ is an isomorphism with inverse $\pi_{01} := \pi_0|_{\mathcal{V}_1} : \mathcal{V}_1 \longrightarrow \mathcal{V}_0$.

Proof. We prove that $\pi_{01} \circ \pi_{10} = \mathrm{id}_{\mathcal{V}_0}$. Let $v_0 \in \mathcal{V}_0$. It has a unique decomposition $v_0 = v + v_1$ with $v \in V$ and $v_1 \in \mathcal{V}_1$. Furthermore, $\pi_{10}(v_0) = v_1$. Then $v_1 = -v + v_0 \in V \oplus \mathcal{V}_0$ and therefore $\pi_{01}(v_1) = v_0$. □

Remark 3.1.6. Let $V \subset V' \subset \mathcal{V}$ be finite dimensional vector subspaces of a Banach space $\mathcal{V}$, and let $\mathcal{V}_0$ be a closed complement of V in $\mathcal{V}$. Let $\pi_0 : \mathcal{V} = V \oplus \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ be the projection. Then:

$$\mathcal{V}_0 \cap V' = \pi_0(V') \text{ and } V' = V \oplus \pi_0(V'). \tag{3.21}$$

Proof. Let $v_0 \in \mathcal{V}_0 \cap V'$. Then $v_0 = \pi_0(v_0) \in \pi_0(V')$. On the other hand, let $v' \in V' \subset V \oplus \mathcal{V}_0$. Write $v' = v + v_0$ with $v \in V$ and $v_0 \in \mathcal{V}_0$. Then $\pi_0(v') = v_0 = v' - v \in V' \cap \mathcal{V}_0$. This proves the first equality. Let $v' \in V'$. Then $v' - \pi_0(v') \in V$ and hence $v' = (v' - \pi_0(v')) + \pi_0(v') \in V + \pi_0(V')$. On the other hand we have clearly that $V \cap (\mathcal{V}_0 \cap V') = \{0\}$. □

We will need a bundle version of the previous remark.

Lemma 3.1.7. *Let $E \subset E' \subset \mathcal{E}$ be finite dimensional subbundles of a Hilbert bundle $\mathcal{E}$. Let $\tilde{\mathcal{E}}$ be a closed complement of E in $\mathcal{E}$ and let $\pi_{\tilde{\mathcal{E}}} : \mathcal{E} = E \oplus \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}$ be the projection. Then:*

$$\tilde{\mathcal{E}} \cap E' = \pi_{\tilde{\mathcal{E}}}(\mathcal{E}) \subset \tilde{\mathcal{E}} \text{ is a closed subbundle.} \tag{3.22}$$

Proof. Let $p_{E'} : \mathcal{E} \longrightarrow \mathcal{E}$ be the orthogonal projection to E' and define $P := \pi_{\tilde{\mathcal{E}}} \circ p_{E'}|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}$. We claim that P is a projection operator with image $\tilde{\mathcal{E}} \cap E'$.

We start by proving that $P \circ P = P$. Let $e \in \tilde{\mathcal{E}}$. Then $P(e) \in \tilde{\mathcal{E}} \cap E'$. Therefore $p_{E'}(P(e)) = P(e) \in \tilde{\mathcal{E}} \cap E'$ and hence $\pi_{\tilde{\mathcal{E}}}(p_{E'}(P(e))) = \pi_{\tilde{\mathcal{E}}}(P(e)) = P(e)$. This shows that $P \circ P = P$.

Now we prove the second claim. By the previous remark, we know that $\tilde{\mathcal{E}} \cap E' = \pi_{\tilde{\mathcal{E}}}(E')$. Therefore $P(\tilde{\mathcal{E}}) \subset \tilde{\mathcal{E}} \cap E'$. Let $e \in \tilde{\mathcal{E}} \cap E'$. Then

$$P(e) = \pi_{\tilde{\mathcal{E}}}(p_{E'}(e)) = \pi_{\tilde{\mathcal{E}}}(e) = e. \tag{3.23}$$

Therefore $\tilde{\mathcal{E}} \cap E' \subset P(\tilde{\mathcal{E}})$.

It now follows from Proposition 2.5.7 that $\tilde{\mathcal{E}} \cap E' \subset \tilde{\mathcal{E}}$ is a subbundle. □

Definition 3.1.8. Let $\varepsilon > 0$ and let $F \subset \mathcal{F}$ be a finite dimensional ρ-subbundle of an infinite dimensional ρ-bundle $\mathcal{F}$. A ρ-invariant subspace $C \subset \mathcal{F}$ is ε-close to F if

$$|p_F(c) - c| \leq \varepsilon \text{ for all } c \in C \text{ where } p_F \text{ is the orthogonal projection to } F. \tag{3.24}$$

Lemma 3.1.9. *Let $C \subset \mathcal{F}$ be a relatively compact G-invariant subspace of an infinite dimensional ρ-bundle $\mathcal{F}$, let $F \subset \mathcal{F}$ be a finite dimensional ρ-subbundle with closed ρ-complement $\tilde{\mathcal{F}} \subset \mathcal{F}$, and let $\varepsilon > 0$. Then there exists a finite dimensional ρ-subbundle $\tilde{F} \subset \tilde{\mathcal{F}}$, such that C is ε-close to $F \oplus \tilde{F}$.*

Proof. Let $C \subset \mathcal{F}$ be a relatively compact G-invariant subspace. First, we prove the statement for $F = 0_B$. By Corollary 2.8.3, we may assume $\mathcal{F} = B \times \mathcal{H}$, where $\mathcal{H}$ is a ρ-Hilbert space. Recall that $\mathcal{H}_s \subset \mathcal{H}$ is the subspace generated by all finite dimensional G-submodules. It is dense in $\mathcal{H}$ by Theorem 2.3.5. Let $\varepsilon > 0$. We can cover C as follows:

$$C \subset B \times \mathring{D}_\varepsilon(h_1) \cup \ldots \cup B \times \mathring{D}_\varepsilon(h_r), \tag{3.25}$$

with $h_1, \ldots, h_r \in \mathcal{H}_s$. Let $H \subset \mathcal{H}$ be a finite dimensional G-submodule containing $h_1, \ldots, h_r$. Then $F' := B \times H \subset B \times \mathcal{H}$ is a ρ-subbundle. Let $(b, h) \in C$ be an arbitrary element. Then $(b, h) \in B \times \mathring{D}_\varepsilon(h_i)$ for some $1 \leq i \leq r$. Hence $|(b, h) - (b, h_i)| < \varepsilon$ and by definition $(b, h_i) \in F'_b$. Hence C is ε-close to F'.

Now let $F \subset \mathcal{F}$ be a finite dimensional ρ-subbundle. First we consider the case where its complement is the orthogonal complement. Let $p_{F^\perp} : \mathcal{F} \longrightarrow F^\perp$ and $p_F : \mathcal{F} \longrightarrow F$ be the orthogonal projections. Then $p_{F^\perp}(C) \subset F^\perp$ is a relatively compact G-invariant subspace. Therefore there exists a finite dimensional ρ-subbundle $F' \subset F^\perp$, ε-close to $p_{F^\perp}(C)$. Let $c \in C$. Then

$$\begin{aligned} |p_{F \oplus F'}(c) - c| &= |p_{F \oplus F'}(p_{F^\perp}(c)) + p_{F \oplus F'}(p_F(c)) - p_F(c) - p_{F^\perp}(c)| \\ &= |p_{F'}(p_{F^\perp}(c)) - p_{F^\perp}(c)| \leq \varepsilon. \end{aligned} \tag{3.26}$$

Hence C is ε-close to $F \oplus F'$.

Now let $\tilde{\mathcal{F}} \subset \mathcal{F}$ be an arbitary closed complement. By the last step we find a finite dimensional subbundle $F' \subset F^\perp$, such that C is ε-close to $F \oplus F'$. Now let $\pi_{\tilde{\mathcal{F}}} : \mathcal{F} = F \oplus \tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}}$ be the projection. Notice that the induced morphism $\pi_{\tilde{\mathcal{F}}}|_{F^\perp} : F^\perp \longrightarrow \tilde{\mathcal{F}}$ is an isomorphism with inverse $p_{F^\perp}|_{\tilde{\mathcal{F}}}$ (Remark 3.1.5). In particular, $\pi_{\tilde{\mathcal{F}}}(F') =: \tilde{F} \subset \tilde{\mathcal{F}}$ is a finite dimensional subbundle. We claim that $F \oplus \tilde{F} = F \oplus F'$. Let $f + f' \in F \oplus F'$. Then $f' = \bar{f} + \tilde{f}$ with $\bar{f} \in F$ and $\tilde{f} = \pi_{\tilde{F}}(f')$. Hence $f + f' = (f + \bar{f}) + \tilde{f} \in F \oplus \tilde{F}$. This shows that $F \oplus F' \subset F \oplus \tilde{F}$. Both bundles have the same rank, therefore equality holds. □

We will frequently use the following criteria for coerciveness.

Remark 3.1.10. A map $f : X \longrightarrow Y$ between fiberwise normed vector spaces over a compact base is coercive if and only if one of the following holds:

1. For every sequence $(x_n)_n \in X$ with $|x_n| \to \infty$, the sequence $|f(x_n)|$ converges to ∞.

2. For every sequence $(x_n)_n \in X$ with $|x_n| \to \infty$, the sequence $|f(x_n)|$ is unbounded.

3. For every sequence $(x_n)_n \in X$ with $|x_n|$ unbounded, the sequence $|f(x_n)|$ is unbounded.

Proof. Let f be coercive and let (x_n) be a sequence with $|x_n| \to \infty$. Assume that $|f(x_n)|$ does not converge to ∞. Then it admits a subsequence $|f(x_{n_k})|$ bounded by $C > 0$. But $|x_{n_k}| \to \infty$ and $x_{n_k} \in f^{-1}(Y_{\leq C})$. This is a contradiction since the latter set is bounded because f is coercive. The first property obviously implies the second. So assume that f satisfies the second property. Let $(x_n)_n$ be a sequence with $|x_n|$ unbounded. Then it admits a subsequence (x_{n_k}) with $|x_{n_k}|$ converging to ∞. By assumption $f(x_{n_k})$ is unbounded. This implies that $f(x_n)$ is unbounded. Now assume that f satisfies the third property and let $A \subset Y$ be a bounded subset. If $f^{-1}(A)$ is unbounded, then there is a sequence (x_n) with $x_n \in f^{-1}(A)$ and $(|x_n|)$ unbounded. But then $f(x_n)$ is unbounded, which is impossible because $f(x_n) \in A$. □

Next we are going to prove that the natural transformation $p : \mathfrak{p}_\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$ is stably bijective. Recall that this means

1. that for every $\mu \in {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$, there is $(E, F) \in {}_\rho\mathcal{H}_B$ and there is $m \in {}_G[E, F]^\Omega_B$, such that $p_{(E,F)}(m) = \mu$;
2. that for any $(E_i, F_i) \in \text{ob}\,{}_\rho\mathcal{H}_B$ and for any $m_i \in {}_G[E_i, F_i]^\Omega_B$ $(i = 0, 1)$, the following holds: If $p_{(E_0,F_0)}(m_0) = p_{(E_1,F_1)}(m_1) \in {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$, then there are morphisms $f_i : (E_i, F_i) \longrightarrow (E, F)$ $(i = 0, 1)$, such that
$$\mathfrak{p}_\Omega(f_0)(m_0) = \mathfrak{p}_\Omega(f_1)(m_1) \in {}_G[E, F]^\Omega_B. \tag{3.27}$$

Theorem 3.1.11. *The natural transformation $p : \mathfrak{p}_\Omega \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$ is stably bijective. In particular, it is a colimit.*

Proof. First we prove that it is stably surjective. Let $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be an Ω-map. We will construct an Ω-homotopy
$$\tilde{\mu} : [0, 3] \times \mathcal{E} \longrightarrow \mathcal{F}, \ (t, e) \mapsto \mu_t(e) \tag{3.28}$$
in three steps. Before doing so, we choose the following:

- a decomposition $\mu = d + c$, where $d : \mathcal{E} \longrightarrow \mathcal{F}$ is ρ-Fredholm and $c : \mathcal{E} \longrightarrow \mathcal{F}$ is compact;
- constants $\varepsilon > 0$ and $\delta > 1$, such that $\mu\left(\mathcal{E}_{\geq\delta} \cup \mathcal{E}(\Omega)\right) \subset \mathcal{F}_{>\varepsilon}$;
- a finite dimensional ρ-subbundle $F \subset \mathcal{F}$ which is d-adapted and to which $c\left(D_\delta(\mathcal{E})\right)$ is $\varepsilon/2$-close;
- We put $E := d^{-1}(F)$ and we choose a complement $E \oplus \tilde{\mathcal{E}} = \mathcal{E}$.

In the first step we use an idea of Bauer [5, Theorem 2.1] and define μ_t for $t \in [0, 1]$: For $t \in [0, 1]$ and $e \in \mathcal{E}$, we put
$$\mu_t(e) := \begin{cases} \mu(e) & \text{when } |e| \leq \delta; \\ \frac{|e|^t}{\delta^t}\mu\left(\frac{\delta^t}{|e|^t}e\right) & \text{else.} \end{cases} \tag{3.29}$$

This map is an Ω-homotopy: Continuity is clear. Also $\mu_t - d =: c_t$ is compact by construction: Let $A \subset \mathcal{E}$ be a subset bounded by $C > 1$. Then we claim that for each $t \in [0, 1]$ the set $c_t(A)$ is contained in the relatively compact subset
$$\tilde{A} := \{\lambda f \mid f \in c(D_{\delta C}(\mathcal{E})), \lambda \in [0, C]\} \subset \mathcal{F}. \tag{3.30}$$

Let $a \in A$. If $|a| \leq \delta$, then $c_t(a) = c(a) \in \tilde{A}$, because $A \subset D_{\delta C}(\mathcal{E})$. If $|a| > \delta$, then $|a|^t\delta^{-1} \leq C$ and $\delta^t|a|^{-t}|a| \leq \delta|a| \leq \delta C$. The latter inequality implies $\delta^t|a|^{-t}a \in D_{\delta C}(\mathcal{E})$. Therefore $c_t(a) = |a|^t\delta^{-t}c(\delta^t|a|^{-t}a) \in \tilde{A}$. This proves compactness of c_t.

We now prove coerciveness. Let $(t_n, e_n) \in [0, 1] \times \mathcal{E}$ be a sequence converging to (t, ∞_b). We can assume that $|e_n| > \delta$ for all $n \in \mathbb{N}$. We treat first the case $t \neq 0$: From $\frac{\delta^{t_n}}{|e_n|^{t_n}}|e_n| = \delta^{t_n}|e_n|^{1-t_n} \geq \delta$, we know that $\left|\mu\left(\frac{\delta^{t_n}}{|e_n|^{t_n}}e_n\right)\right| > \varepsilon$. Furthermore, $\frac{|e_n|^{t_n}}{\delta^{t_n}} \to \infty$. Therefore $|\mu_{t_n}(e_n)| \to \infty$. Assume now that $t = 0$. Then $\delta^{t_n}|e_n|^{1-t_n} \to \infty$. Since μ is coercive, it follows that $|\mu\left(\delta^{t_n}|e_n|^{-t_n}e_n\right)| \to \infty$. Furthermore $\frac{|e_n|^{t_n}}{\delta^{t_n}} \geq 1$. Therefore $|\mu_{t_n}(e_n)| \to \infty$. By Remark 3.1.10, the homotopy μ_t is coercive.

Now let $t \in [0,1]$ and $e \in \mathcal{E}(\Omega)$. When $|e| > \delta$, then $e' := \frac{\delta^t}{|e|^t} e \in \mathcal{E}(\Omega)$, and hence $|\mu_t(e)| = \left| \frac{|e|^t}{\delta^t} \mu(e') \right| > \frac{|e|^t}{\delta^t} \varepsilon > \varepsilon$. When $|e| \leq \delta$, then $|\mu_t(e)| = |\mu(e)| > \varepsilon$.

We come to the second step of the homotopy: We denote the orthogonal projection with $p_F : \mathcal{F} \longrightarrow F$ and define

$$\mu_t := \mu_1 + (t-1)\left(p_F c_1 - c_1\right) \text{ for } t \in [1,2]. \tag{3.31}$$

Remember that c_1 is defined by scaling c as in (3.29). As before we need to prove that μ_t defines an Ω-homotopy. Again continuity is immediate and by construction $\mu_t - d =: c_t$ is compact. We prove that μ_t is coercive: Let $C > 0$ and let $(t,e) \in [1,2] \times \mathcal{E}$, such that $|\mu_t(e)| \leq C$. Then

$$C \geq |\mu_t(e)| = |\mu_1(e) + (t-1)\left(p_F c_1(e) - c_1(e)\right)| \geq |\mu_1(e)| - (t-1)\left|p_F c_1(e) - c_1(e)\right|. \tag{3.32}$$

Assume that $|e| \geq \delta$. Then $c_1(e) = |e|\delta^{-1} c(\delta|e|^{-1}e)$ and $|p_F c(\delta|e|^{-1}e) - c(\delta|e|^{-1}e)| \leq \varepsilon/2$ by the choice of F. Therefore

$$C \geq |\mu_1(e)| - |p_F c_1(e) - c_1(e)| = \frac{|e|}{\delta}\left[\left|\mu\left(\frac{\delta}{|e|}e\right)\right| - \left|p_F c\left(\frac{\delta}{|e|}e\right) - c\left(\frac{\delta}{|e|}e\right)\right|\right] \geq \frac{|e|}{\delta}[\varepsilon - \varepsilon/2]. \tag{3.33}$$

Therefore $|e| \leq 2\delta C/\varepsilon$ when $|e| \geq \delta$. This shows that μ_t is coercive.

Now let $e \in \mathcal{E}(\Omega)$. First assume that $|e| \leq \delta$. Then

$$\begin{aligned} |\mu_t(e)| = \left|\mu_1(e) + (t-1)\left(p_F c_1(e) - c_1(e)\right)\right| &\geq |\mu_1(e)| - |p_F c_1(e) - c_1(e)| \\ &= |\mu(e)| - |p_F c(e) - c(e)| > \frac{\varepsilon}{2}. \end{aligned} \tag{3.34}$$

When $|e| > \delta$, we set $\hat{e} := \frac{\delta}{|e|} e$. Note that $\hat{e} \in \mathcal{E}(\Omega)$. We now argue as follows:

$$\begin{aligned} |\mu_t(e)| = |\mu_1(e) + (t-1)(p_F c_1(e) - c_1(e))| &= \frac{|e|}{\delta}\left|\mu(\hat{e}) + (t-1)(p_F c(\hat{e}) - c(\hat{e}))\right| \\ &\geq \frac{|e|}{\delta}\left[\left|\mu\left(\hat{e}\right)\right| - \frac{\varepsilon}{2}\right] > \frac{\varepsilon}{2}. \end{aligned} \tag{3.35}$$

Therefore $\mu_t\big(\mathcal{E}(\Omega)\big) \subset \mathcal{F}_{>\varepsilon/2}$ for all $t \in [1,2]$.

Remember that we have a decomposition $\mathcal{E} = E \oplus \tilde{\mathcal{E}}$. Let π_E and $\pi_{\tilde{\mathcal{E}}}$ be the respective projections. For $t \in [2,3]$, we define $g_t : \mathcal{E} \longrightarrow \mathcal{E}$ by $g_t := \pi_E + (3-t)\pi_{\tilde{\mathcal{E}}}$. Now we are able to define μ_t for $t \in [2,3]$:

$$\mu_t := \mu_2 \circ g_t + (t-2) d \circ \pi_{\tilde{\mathcal{E}}} = d + c_2 \circ g_t = d + p_F c_1 \circ g_t \text{ for } t \in [2,3]. \tag{3.36}$$

Again, we need to check that this defines an Ω-homotopy. As before, continuity is immediate and $\mu_t - d =: c_t$ is compact by construction. Let $(t_n, e_n) \in [2,3] \times \mathcal{E}$ be a sequence with $|e_n|$ converging to ∞. We prove that $\mu_{t_n}(e_n)$ is unbounded. Assume first that $\pi_{\tilde{\mathcal{E}}}(e_n)$ is unbounded. Then $d \circ \pi_{\tilde{\mathcal{E}}}(e_n)$ is unbounded because $d|_{\tilde{\mathcal{E}}}$ is an isomorphism, and since

$$\mu_{t_n}(e_n) = \left[d \circ \pi_E(e_n) + p_F \circ c_1(g_{t_n}(e_n))\right] + d \circ \pi_{\tilde{\mathcal{E}}}(e_n) \in F \oplus d(\tilde{\mathcal{E}}), \tag{3.37}$$

we conclude that $\mu_{t_n}(e_n)$ is unbounded. So assume that $\pi_{\tilde{\mathcal{E}}}(e_n)$ is bounded. Then $|\pi_E(e_n)| \to \infty$ and also $|g_{t_n}(e_n)| = |\pi_E(e_n) + (3-t_n)\pi_{\tilde{\mathcal{E}}}(e_n)| \to \infty$. Therefore $|\mu_2(g_{t_n}(e_n))| \to \infty$. Together with the boundedness of $(t_n - 2) d \circ \pi_{\tilde{\mathcal{E}}}(e_n)$ this allows us to deduce $|\mu_{t_n}(e_n)| \to \infty$ from the equality $\mu_t = \mu_2 \circ g_t + (t-2) d \circ \pi_{\tilde{\mathcal{E}}}$. Therefore μ_t is coercive (Remark 3.1.10).

Now let $e \in \mathcal{E}(\Omega)$. Remember that $\mathcal{F} = F \oplus d(\tilde{\mathcal{E}})$ (Lemma 2.7.6). Let $\pi_{d(\tilde{\mathcal{E}})}$ be the projection to the second factor. In the fiber over a point $b \in B$ it induces a projection $\pi_{d_b(\tilde{\mathcal{E}}_b)}$ of norm $||\pi_{d_b(\tilde{\mathcal{E}}_b)}||$. Let $C := \max_{b \in B}\{||\pi_{d_b(\tilde{\mathcal{E}}_b)}||\}$. Assume first that $|d \circ \pi_{\tilde{\mathcal{E}}} e| \geq \varepsilon/4$. In that case

$$\varepsilon/4 \leq |d \circ \pi_{\tilde{\mathcal{E}}} e| = |\pi_{d(\tilde{\mathcal{E}})} \mu_t(e)| \leq C\, |\mu_t(e)|. \tag{3.38}$$

This shows that $|\mu_t(e)| \geq C^{-1}\varepsilon/4$. Therefore we may assume that $|d \circ \pi_{\tilde{\mathcal{E}}} e| < \varepsilon/4$. Now we use that g_t is a ρ-map and therefore $g_t(e) \in \mathcal{E}(\Omega)$. This implies that $|\mu_2(g_t(e))| > \varepsilon/2$. Then:

$$|\mu_t(e)| = |\mu_2(g_t(e)) + (t-2) d \circ \pi_{\tilde{\mathcal{E}}} e| \geq |\mu_2(g_t(e))| - (t-2)\varepsilon/4 > \varepsilon/2 - \varepsilon/4 = \varepsilon/4. \tag{3.39}$$

Therefore μ_t is an Ω-homotopy.

It follows that μ is Ω-homotopic to $\mu_3 = d + c_3 = d + \pi_F \circ c_1 \circ \pi_E$. Now let $\iota_E : E \subset \mathcal{E}$ and $\iota_F : F \subset \mathcal{F}$ be the inclusions, and let $\tilde{d} := d|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \longrightarrow d(\tilde{\mathcal{E}}) =: \tilde{F}$ be the isomorphism induced by d. Set $C := (\iota_E, \iota_F, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{d})$. Then $m := \mu_3|_E : E \longrightarrow F$ is an Ω-map and $\mu(C, m) = \mu_3$: Let $e + \tilde{e} \in E \oplus \tilde{\mathcal{E}}$. Then

$$\mu(C,m)(e+\tilde{e}) = \mu_3(e) + d(\tilde{e}) = de + c_3(e) + d(\tilde{e}) = d(e+\tilde{e}) + c_3(e+\tilde{e}) = \mu_3(e+\tilde{e}). \tag{3.40}$$

This proves that the natural transformation p is stably surjective.

Now we prove that it is stably injective: Let (E_0, F_0) and (E_1, F_1) be two objects in ${}_\rho\mathcal{H}_B$ and let $m_i \in {}_G[E_i, F_i]^\Omega_B$ for $i = 0, 1$, such that $p_{(E_0,F_0)}(m_0) = p_{(E_1,F_1)}(m_1) \in {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B$. It follows from Theorem 3.1.4 that

$$[E_0] - [F_0] = \operatorname{ind}_\rho\big(p_{(E_0,F_0)}(m_0)\big) = \operatorname{ind}_\rho\big(p_{(E_1,F_1)}(m_1)\big) = [E_1] - [F_1]. \tag{3.41}$$

Thus, by Lemma 3.1.1 there exists a diagram

$$\begin{array}{ccc} (E_0, F_0) & & (E_1, F_1) \\ & {\searrow}^{f_0} \qquad {}^{f_1}{\swarrow} & \\ & (E, F) & \end{array} \tag{3.42}$$

in the category ${}_\rho\mathcal{H}_B$. Since p is a natural transformation, the equality

$$p_{(E,F)}\left(\mathfrak{p}_\Omega(f_0)m_0\right) = p_{(E_0,F_0)}(m_0) = p_{(E_1,F_1)}(m_1) = p_{(E,F)}\left(\mathfrak{p}_\Omega(f_1)m_1\right) \tag{3.43}$$

results. This proves that we can assume without loss of generality that $(E_0, F_0) = (E_1, F_1) = (E, F)$. Choose ρ-embeddings $\iota_E : E \hookrightarrow \mathcal{E}$ and $\iota_F : F \hookrightarrow \mathcal{F}$, closed ρ-complements $\iota_E(E) \oplus \tilde{\mathcal{E}} = \mathcal{E}$ and $\iota_F(F) \oplus \tilde{\mathcal{F}} = \mathcal{F}$, as well as a ρ-isomorphism $\tilde{\delta} : \tilde{\mathcal{E}} \xrightarrow{\cong} \tilde{\mathcal{F}}$. Put $d := \tilde{d} \oplus 0_E : \mathcal{E} \longrightarrow \mathcal{F}$ and $\mu_i := \mu(C, m_i) = d + \iota_F \circ m_i \circ \pi_E$ for $i = 0, 1$.

By assumption the Ω-maps μ_i $(i = 0, 1)$ fit in an Ω-homotopy $\mu_t : \mathcal{E} \longrightarrow \mathcal{F}$ $(t \in [0, 1])$. We want to apply stable surjectivity to this Ω-homotopy. By Lemma 3.0.30, there is a homotopy of ρ-Fredholm maps $d_t : \mathcal{E} \longrightarrow \mathcal{F}$, such that $d_0 = d_1 = d$ and that $c_t := \mu_t - d_t$ is compact.

By Lemma 2.7.5, there exists a finite dimensional subbundle $\cup_{t\in[0,1]} F'_t \subset \mathcal{F} \times [0, 1]$ that is d_t-adapted and that contains $\iota_F(F) \times [0, 1] \subset \mathcal{F} \times [0, 1]$. As above we choose

- constants $\varepsilon > 0$, $\delta > 1$, such that $\mu_t\left(\mathcal{E}_{\geq\delta} \cup \mathcal{E}(\Omega)\right) \subset \mathcal{F}_{>\varepsilon}$ for all $t \in [0, 1]$;
- a finite dimensional subbundle $\cup_{t\in[0,1]} F_t \subset \mathcal{F} \times [0, 1]$ that contains $\cup_{t\in[0,1]} F'_t$ and such that $c_t(D_\delta(\mathcal{E}))$ is $\varepsilon/2$-close to F'_t for all $t \in [0, 1]$;
- Put $E_t := d_t^{-1}(F_t)$. Notice that $\cup_{t\in[0,1]} E_t \subset \mathcal{E} \times [0, 1]$ is a subbundle which contains $\iota_E(E) \times [0, 1]$. By Remark 3.1.6 we have the equality $E_t = \iota_E(E) \oplus (E_t \cap \tilde{\mathcal{E}})$ and by Lemma 3.1.7 $\cup_{t\in[0,1]}\big(E_t \cap \tilde{\mathcal{E}}\big) \subset \tilde{\mathcal{E}} \times [0, 1]$ is a subbundle. Choose a closed complement $\cup_{t\in[0,1]} \tilde{\mathcal{E}}_t$ of the subbundle $\cup_{t\in[0,1]}\big(E_t \cap \tilde{\mathcal{E}}\big)$ in $\tilde{\mathcal{E}} \times [0, 1]$. Then:

$$\mathcal{E} = \iota_E(E) \oplus \tilde{\mathcal{E}} = \iota_E(E) \oplus \big(\tilde{\mathcal{E}} \cap E_t\big) \oplus \tilde{\mathcal{E}}_t = E_t \oplus \tilde{\mathcal{E}}_t \text{ for all } t \in [0, 1]. \tag{3.44}$$

Now we are able to repeat the construction from above: it yields an Ω-homotopy which corresponds to a family of maps $\mu_{t,s} : \mathcal{E} \longrightarrow \mathcal{F}$ with $t \in [0, 1]$ and $s \in [0, 3]$, where:

1. When $s \in [0, 1]$, then $\mu_{t,s}(e) = \frac{|e|^s}{\delta^s} \mu_t\left(\frac{\delta^s}{|e|^s} e\right)$ for $|e| \geq \delta$ and $\mu_{t,s}(e) = \mu_t(e)$, else.
2. When $s \in [1, 2]$, then $\mu_{t,s} = \mu_{t,1} + (s - 1)\left(p_{F_t} c_{t,1} - c_{t,1}\right)$. (Here $p_{F_t} : \mathcal{F} \longrightarrow F_t$ denotes the orthogonal projection.)
3. When $s \in [2, 3]$, then $\mu_{t,s} = d_t + p_{F_t} \circ c_{t,1} \circ \big(\pi_{E_t} + (3 - s)\pi_{\tilde{\mathcal{E}}_t}\big)$.

For $i \in \{0,1\}$ the homotopies $\mu_{i,s}$ are of the following special form:

1. When $s \in [0,1]$, then: $\mu_{i,s}(e) = d(e) + \frac{|e|^s}{\delta^s} c_i\left(\frac{\delta^s}{|e|^s} e\right)$ for $|e| \geq \delta$ and $\mu_{i,s}(e) = \mu_i(e) = d(e) + c_i(e)$, else.

2. When $s \in [1,2]$, then $\mu_{i,s} = \mu_{i,1}$ because $c_i(\mathcal{E}) \subset \iota_F(F) \subset F_i$.

3. When $s \in [2,3]$, then $\mu_{i,s} = d + c_{i,1} \circ (\pi_{E_i} + (3-s)\pi_{\tilde{\mathcal{E}}_t})$.

In particular, we see that $\mu_{i,s}(\iota_E(E)) \subset \iota_F(F)$ for all $s \in [0,3]$. Then for $i = 0,1$ the family

$$m'_s := \iota_F^{-1} \circ \mu_{i,s} \circ \iota_E : E \cong \iota_E(E) \longrightarrow \iota_F(F) \cong F \tag{3.45}$$

defines an Ω-homotopy between m_i and m'_i.

We are now going to show that m'_0 and m'_1 are stably Ω-homotopic. We know that $\mu_{t,3} : \mathcal{E} \longrightarrow \mathcal{F}$ is an Ω-homotopy between $\mu_{0,3}$ and $\mu_{1,3}$. It is of the form

$$\mu_{t,3} = d_t + p_{F_t} \circ c_{t,1} \circ \pi_{E_t}. \tag{3.46}$$

We choose isomorphisms

$$\varphi : E_0 \times [0,1] \xrightarrow{\cong} \cup_{t\in[0,1]} E_t \text{ and } \psi : F_0 \times [0,1] \xrightarrow{\cong} \cup_{t\in[0,1]} F_t \tag{3.47}$$

of ρ-bundles over $B \times [0,1]$. Let $\varphi_t : E_0 \xrightarrow{\cong} E_t$ and $\psi_t : F_0 \xrightarrow{\cong} F_t$ be the ρ-isomorphisms obtained by restricting φ and ψ, respectively to $B \cong B \times \{t\}$ for $t \in [0,1]$. We put

$$\hat{m}_t := \psi_t^{-1} \circ \mu_{t,3} \circ \varphi_t : E_0 \longrightarrow F_0. \tag{3.48}$$

The family $\hat{m}_t : E_0 \longrightarrow F_0$ defines an Ω-homotopy between $\hat{m}_0$ and $\hat{m}_1$. Let $l \in \{0,1\}$. Then by assumption $\iota_F(F) \subset F_l$, indeed $F_l = \iota_F(F) \oplus \Big(F_l \cap \tilde{\mathcal{F}}\Big)$. Furthermore $E_l = d^{-1}(F_l) = \iota_E(E) \oplus \Big(E_l \cap \tilde{\mathcal{E}}\Big)$.

Define for $l = 0,1$ morphisms

$$f_l = (i_{E,l}, i_{F,l}, \tilde{E}_l, \tilde{F}_l, \tau_l) : (E,F) \longrightarrow (E_0, F_0) \tag{3.49}$$

as follows:

- $i_{E,l} := \varphi_l^{-1} \circ \iota_E : E \hookrightarrow E_0$ and $i_{F,l} := \psi_l^{-1} \circ \iota_F : F \hookrightarrow F_0$;
- $\tilde{E}_l := \varphi_l^{-1}(E_l \cap \tilde{\mathcal{E}})$ and $\tilde{F}_l := \psi_l^{-1}(F_l \cap \tilde{\mathcal{F}})$;
- $\tau_l := \psi_l^{-1} \circ \tilde{d} \circ \varphi_l$.

We display the situation in the following commutative diagram:

$$\begin{array}{ccccccc}
E_0 & \xrightarrow{\varphi_l} & E_l & \xrightarrow{\mu_{l,3}} & F_l & \xrightarrow{\psi_l^{-1}} & F_0 \\
\downarrow = & & \downarrow = & & \downarrow = & & \downarrow = \\
i_{E,l}(E) \oplus \tilde{E}_l & & \iota_E(E) \oplus \Big(E_l \cap \tilde{\mathcal{E}}\Big) & & \iota_F(F) \oplus \Big(F_l \cap \tilde{\mathcal{F}}\Big) & & i_{F,l}(F) \oplus \tilde{F}_l \\
\uparrow i_{E,l} & \nearrow \iota_E & & & & \nwarrow \iota_F & \uparrow i_{F,l} \\
E & & & \xrightarrow{m'_l} & & & F
\end{array}
\qquad (\hat{m}_l : E_0 \longrightarrow F_0) \tag{3.50}$$

We claim that $(f_l)_* m'_l = \hat{m}_l$. Let $e_0 \in E_0$. We write $e_0 = i_{E,l}(e) + \tilde{e} \in i_{E,l}(E) \oplus \tilde{E}_l = E_0$. We know that $\mu_{l,3} = d + c_{l,1} \circ \pi_{\tilde{E}_l}$. It implies that

$$\mu_{l,3}(E + \varphi_l(\tilde{e})) = d(\varphi_l(\tilde{e})) + c_{l,1}(e + \varphi_l(\tilde{e})) = d(\varphi_l(\tilde{e})) + \iota_F(m'_l(e)). \tag{3.51}$$

Then

$$\begin{aligned} ((f_l)_* m'_l)(e_0) &= i_{F,l}(m'_l(e)) + \tau_l(\tilde{e}) = i_{F,l}(m'_l(e)) + \psi_l^{-1}(d(\varphi_l(\tilde{e}))) \\ &= \psi_l^{-1}(\iota_F(m'_l(e))) + \psi_l^{-1}(d(\varphi_l(\tilde{e}))) = \psi_l^{-1}(\iota_F(m'_l(e)) + d(\varphi_l(\tilde{e}))) \\ &= \psi_l^{-1}(\mu_{l,3}(\iota_E(e) + \varphi_l(\tilde{e}))) = \psi_l^{-1} \circ \mu_{l,s} \circ \varphi_l\big(i_{E,l}(e) + \tilde{e}\big) == \hat{m}_l(e_0). \end{aligned} \tag{3.52}$$

We conclude that

$$(f_0)_* [m_0] = [\hat{m}_0] = [\hat{m}_1] = (f_1)_* [m_1] \in {}_G[E_0, F_0]^{\Omega}_B \tag{3.53}$$

This proves that the transformation p is stably injective. □

3.2 Framed homotopy classes

In this section we introduce the notion of framed Ω-Fredholm maps and study how the homotopy classes of these maps relate to the (unframed) homotopy classes of (unframed) Ω-maps.

Fix a subset $\rho_0 \subset \rho$ and write $\rho_* := \rho - \rho_0$. Then $\mathcal{E} = \mathcal{E}_0 \times_B \mathcal{E}_*$ for a ρ_0-bundle $\mathcal{E}_0$ and a ρ_*-bundle $\mathcal{E}_*$, and similarly $\mathcal{F} = \mathcal{F}_0 \times_B \mathcal{F}_*$. A ρ-morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$ induces morphisms $d_0 : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and $d_* : \mathcal{E}_* \longrightarrow \mathcal{F}_*$. Furthermore, there is a canonical splitting $K_\rho(B) = K_{\rho_0}(B) \times K_{\rho_*}(B)$.

Let $(U, V) \in {}_{\rho_0}\mathcal{H}_B$. We define the functor

$$\mathfrak{p}^{U,V}_\Omega : {}_{\rho_0}\mathcal{C}_B \times {}_{\rho_*}\mathcal{H}_B \longrightarrow \text{Set} \tag{3.54}$$

as the composition of the functor $\mathfrak{p}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \text{Set}$ with the natural functor

$$I^{U,V} : {}_{\rho_0}\mathcal{C}_B \times {}_{\rho_*}\mathcal{H}_B \longrightarrow {}_\rho\mathcal{H}_B,\ (W, (E, F)) \mapsto (U \oplus W \oplus E, V \oplus W \oplus F). \tag{3.55}$$

The functor $\mathfrak{p}^{U,V}_\Omega$ admits a colimit which we now describe.

Definition 3.2.1. Let $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ be a ρ_0-Fredholm morphism. A Ω-map $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ is l-framed if there exists a ρ-Fredholm morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$, such that $\mu - d$ is compact and that $d_0 = l$.

We write ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B$ for the set of l-framed Ω-homotopy classes of l-framed Ω-maps μ. Clearly there is a map ${}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B$.

Lemma 3.2.2. *Let $l, d : \mathcal{E} \longrightarrow \mathcal{F}$ be two ρ-Fredholm morphisms with index $\mathrm{ind}_\rho\, d = \mathrm{ind}_\rho\, l \in K_\rho(B)$. Then there exist ρ-automorphisms $\varphi : \mathcal{F} \longrightarrow \mathcal{F}$, $\psi : \mathcal{E} \longrightarrow \mathcal{E}$, and a compact ρ-morphism $k : \mathcal{E} \longrightarrow \mathcal{F}$, such that*

$$\varphi \circ d - l \circ \psi = k. \tag{3.56}$$

Proof. We choose a finite dimensional ρ-subbundle $F \subset \mathcal{F}$ which is both d- and l-adapted. Put $E_d := d^{-1}(F)$, $E_l := l^{-1}(F)$, and choose closed complements $\mathcal{E} = E_d \oplus \mathcal{E}_d = E_l \oplus \mathcal{E}_l$. Then both $\mathcal{F}_d := d(\mathcal{E}_d)$ and $\mathcal{F}_l := l(\mathcal{E}_l)$ are closed complements of $F \subset \mathcal{F}$ (Lemma 2.7.6). Let $\beta_{l,d} : \mathcal{F}_d \longrightarrow \mathcal{F}_l$ be defined by composing the inclusion $\mathcal{F}_d \subset \mathcal{F}$ with the projection $\pi_{\mathcal{F}_l} : \mathcal{F} = F \oplus \mathcal{F}_l \longrightarrow \mathcal{F}_l$. Similarly, we define $\beta_{d,l} : \mathcal{F}_l \longrightarrow \mathcal{F}_d$. Then $\beta_{l,d} : \mathcal{F}_d \longrightarrow \mathcal{F}_l$ is an isomorphism with inverse $\beta_{d,l}$ (Remark 3.1.5). Furthermore, we obtain an induced isomorphism $\alpha_{l,d} := l^{-1} \circ \beta_{l,d} \circ d : \mathcal{E}_d \longrightarrow \mathcal{E}_l$. By definition $l \circ \alpha_{l,d} = \beta_{l,d} \circ d : \mathcal{E}_d \longrightarrow \mathcal{F}_l$.

By assumption $[E_d] - [F] = \mathrm{ind}_\rho\, d = \mathrm{ind}_\rho\, l = [E_l] - [F] \in K_\rho(B)$. Therefore there is a finite dimensional ρ-bundle G, and an isomorphism $\alpha_G : E_d \oplus G \xrightarrow{\cong} E_l \oplus G$. Choose an embedding $\iota_G : G \hookrightarrow \mathcal{E}_d$ and a closed complement $\iota_G(G) \oplus \tilde{\mathcal{E}}_d = \mathcal{E}_d$. We write $\iota^{l,d}_G := \alpha_{l,d} \circ \iota_G$ and $\tilde{\mathcal{E}}_l := \alpha_{l,d}(\tilde{\mathcal{E}}_d)$ Then $\mathcal{E}_l = \iota^{l,d}_G(G) \oplus \tilde{\mathcal{E}}_l$ and

$$\mathcal{E} = E_d \oplus \big(\iota_G(G) \oplus \tilde{\mathcal{E}}_d\big) = E_l \oplus \alpha_{l,d}\big(\iota_G(G) \oplus \tilde{\mathcal{E}}_d\big) = E_l \oplus \iota^{l,d}_G(G) \oplus \tilde{\mathcal{E}}_l. \tag{3.57}$$

Using the isomorphisms $\alpha_G : E_d \oplus G \xrightarrow{\cong} E_l \oplus G$ and $\alpha_{l,d}|_{\tilde{\mathcal{E}}_d} : \tilde{\mathcal{E}}_d \longrightarrow \tilde{\mathcal{E}}_l$ we obtain an automorphism

$$\psi : \mathcal{E} = E_d \oplus \iota_G(G) \oplus \tilde{\mathcal{E}}_d \cong E_d \oplus G \oplus \tilde{\mathcal{E}}_d \cong E_l \oplus G \oplus \tilde{\mathcal{E}}_l \cong E_l \oplus \iota_G^{l,d}(G) \oplus \tilde{\mathcal{E}}_l = \mathcal{E}, \tag{3.58}$$

defined by $\psi := \big(\mathrm{id}_{E_l} \oplus \iota_G^{l,d} \oplus \mathrm{id}_{\tilde{\mathcal{E}}_l}\big) \circ \big(\alpha_G \oplus \alpha_{l,d}|_{\tilde{\mathcal{E}}_d}\big) \circ \big(\mathrm{id}_{E_d} \oplus \iota_G^{-1} \oplus \mathrm{id}_{\tilde{\mathcal{E}}_d}\big)$. We put $\varphi := \mathrm{id}_F \oplus \beta_{l,d} : \mathcal{F} \xrightarrow{\cong} \mathcal{F}$. We now claim that the image of $k := \varphi \circ d - l \circ \psi$ is contained in the finite dimensional subbundle $F \oplus l\big(\iota_G^{l,d}(G)\big) \subset \mathcal{F}$. This would imply the compactness of k.

To prove this statement, let $e = e_1 + \iota_G(g) + e_2 \in E_d \oplus \iota_G(G) \oplus \tilde{\mathcal{E}}_d = \mathcal{E}$. Then

- $\varphi \circ d(e_1 + \iota_G(g)) = d(e_1) + \beta_{l,d}(d\iota_G(g)) = d(e_1) + l\alpha_{l,d}(\iota_G(g)) = d(e_1) + l\iota_G^{l,d}(g) \in F \oplus l(\iota_G^{l,d}(G)$.
- $l \circ \psi(e_1 + \iota_G(g)) \in l\big(E_l \oplus \iota_G^{l,d}(G)\big) = F \oplus l(\iota_G^{l,d}(G))$ by definition of ψ.
- $\varphi \circ d(e_2) = \beta_{l,d}(d(e_2)) = l(\alpha_{l,d}(e_2)) = l(\psi(e_2))$.

Therefore $[\varphi \circ d - l \circ \psi](e_1 + \iota_G(g) + e_2) = [\varphi \circ d - l \circ \psi](e_1 + \iota_G(g)) \in F \oplus l(\iota_G^{l,d}(G))$. □

Proposition 3.2.3. *The natural map* ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega}$ *has image* $\mathrm{ind}_\rho^{-1}\big(\{\mathrm{ind}_\rho l\} \times K_{\rho_*}(B)\big)$.

Proof. Let $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be an Ω-Fredholm map with $\mathrm{ind}_\rho \mu \in \{\mathrm{ind}_\rho l\} \times K_{\rho_*}(B)$. Let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a ρ-Fredholm morphism, such that $c := \mu - d$ is compact. We assumed that $\mathrm{ind}_{\rho_0}(d_0) = \mathrm{ind}_{\rho_0} l$. It follows from Lemma 3.2.2 that there are ρ_0-isomorphisms $\psi_0 : \mathcal{E}_0 \longrightarrow \mathcal{E}_0$ and $\varphi_0 : \mathcal{F}_0 \longrightarrow \mathcal{F}_0$, such that $k_0 : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ is compact, where $k_0 := \varphi_0 \circ d_0 - l \circ \psi_0$. Therefore:

$$\varphi_0 \circ d_0 \circ \psi_0^{-1} - l = k_0 \circ \psi_0^{-1} \text{ is compact.} \tag{3.59}$$

We define ρ-automorphisms of $\mathcal{F}$ and $\mathcal{E}$, respectively by $\varphi := \varphi_0 \oplus \mathrm{id}_{\mathcal{F}_*}$ and $\psi := \psi_0 \oplus \mathrm{id}_{\mathcal{E}_*}$. Since all ρ-automorphisms are isotopic to the identity (Corollary 2.8.4), the map $\varphi \circ \mu \circ \psi^{-1}$ is Ω-homotopic to μ. We claim that $\varphi \circ \mu \circ \psi^{-1}$ is l-framed. Let $e_0 + e_* \in \mathcal{E}_0 \oplus \mathcal{E}_* = \mathcal{E}$. Then:

$$\begin{aligned}
\varphi \circ \mu \circ \psi^{-1}(e_0 + e_*) &= \varphi \circ d \circ \psi^{-1}(e_0 + e_*) + \varphi \circ c \circ \psi^{-1}(e_0 + e_*) \\
&= \varphi_0 \circ d_0 \circ \psi_0^{-1}(e_0) + d_*(e_*) + \varphi \circ c \circ \psi^{-1}(e_0 + e_*) \\
&= [l + k_0 \circ \psi_0^{-1}](e_0) + d_*(e_*) + \varphi \circ c \circ \psi^{-1}(e_0 + e_*) \\
&= [l \oplus d_*](e_0 + e_*) + k_0 \circ \psi_0^{-1}(e_0) + \varphi \circ c \circ \psi^{-1}(e_0 + e_*).
\end{aligned} \tag{3.60}$$

This shows that $\varphi \circ \mu \circ \psi^{-1}$ is l-framed. Therefore $[\mu] = [\varphi \circ \mu \circ \psi^{-1}]$ lies in the image of the map ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega}$. □

Let us now return to the problem of constructing a colimit for the functor $\mathfrak{p}_\Omega^{U,V}$: Recall that we are given a pair $(U,V) \in {}_{\rho_*}\mathcal{H}_B$. We will use the letters U, V, W for ρ_0-bundles and E, F, G for ρ_*-bundles in order to avoid the use of too many indices.

Theorem 3.2.4. *Let $\gamma = (\iota_U, \iota_V, l)$ be the choice of embeddings $\iota_U : U \hookrightarrow \mathcal{E}_0$, $\iota_V : V \hookrightarrow \mathcal{F}_0$, and of a ρ_0-Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$, such that $i_V(V)$ is l-adapted and such that $\iota_U(U) = l^{-1}(\iota_V(V))$. Then there is a natural tranformation*

$$p_\gamma : \mathfrak{p}_\Omega^{U,V} \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \tag{3.61}$$

This transformation is stably bijective. In particular, it is a colimit for the functor $\mathfrak{p}_\Omega^{U,V}$.

Proof. We assume that $\rho_0 = \rho$. The general case is achieved by combining this proof with the proof of Theorem 3.1.11 in a straightforward way. Let $W \in {}_\rho\mathcal{C}_B$. We define the map

$$(p_\gamma)_W : \mathfrak{p}_\Omega^{U,V}(W) \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \tag{3.62}$$

as follows: Fix a family $C = (\tilde{U}, \iota_W, \tilde{\mathcal{W}})$ consisting of

- a closed complement $\iota_U(U) \oplus \tilde{\mathcal{U}} = \mathcal{E}_0 = \mathcal{E}$;
- a closed embedding $\iota_W : W \hookrightarrow \tilde{\mathcal{U}}$;
- a closed complement $\tilde{\mathcal{U}} = \iota_W(W) \oplus \tilde{\mathcal{W}}$.

Then $l(\tilde{\mathcal{U}})$ is a closed complement of $\iota_V(V)$ (Lemma 2.7.6), the composition $l \circ \iota_W : W \hookrightarrow l(\tilde{\mathcal{U}}) \subset \mathcal{F}$ is a closed embedding, and $l(\tilde{\mathcal{W}})$ is a closed complement of $l \circ \iota_W(W)$ in $l(\tilde{U})$. We obtain closed embeddings

- $\iota_E := \iota_U \oplus \iota_W : E := U \oplus W \hookrightarrow \iota_U(U) \oplus \iota_W(W) \subset \mathcal{E}$;
- $\iota_F := \iota_V \oplus \iota_W : F := V \oplus W \hookrightarrow \iota_V(V) \oplus l \circ \iota_W(W) \subset \mathcal{F}$.

We are given closed complements $\iota_E(E) \oplus \tilde{\mathcal{W}} = \mathcal{E}$ and $\iota_F(F) \oplus l(\tilde{\mathcal{W}}) = \mathcal{F}$ and an isomorphism $l|_{\tilde{\mathcal{W}}} : \tilde{\mathcal{W}} \longrightarrow l(\tilde{\mathcal{W}})$. Let $m : U \oplus W \longrightarrow V \oplus W$ be an Ω-map. As in Theorem 3.1.4 we define

$$\mu(C, m) := l|_{\tilde{\mathcal{W}}} + \iota_F \circ m \circ \pi_E : \mathcal{E} = \iota_E(E) \oplus \tilde{\mathcal{W}} \longrightarrow \iota_F(F) \oplus l(\tilde{\mathcal{W}}) = \mathcal{F}. \tag{3.63}$$

In Theorem 3.1.4 we proved that $\mu(C, m)$ is an Ω-map. Furthermore, the image of $\mu(C, m) - l$ is contained in $\iota_F(F)$: let $e = \iota_U(u) + \iota_W(w) + \tilde{w} \in \iota_U(U) \oplus \iota_W(W) \oplus \tilde{\mathcal{W}} = \mathcal{E}$. Recall that $\iota_F(F) = \iota_V(V) \oplus l(\iota_W(W)$. By assumption $l^{-1}(\iota_V(V)) = \iota_U(U)$ and therefore $l(\iota_U(u)) \in \iota_V(V) \subset \iota_F(F)$. Also $l(\iota_W(w)) \in l(\iota_W(W)) \subset \iota_F(F)$. Therefore:

$$\begin{aligned}[\mu(C, m) - l](e) &= \iota_F \circ m(u + w) + l(\tilde{w}) - l(\iota_U(u)) - l(\iota_W(w)) - l(\tilde{w}) \\ &= \iota_F \circ m(u + w) - l(\iota_U(u)) - l(\iota_W(w)) \in \iota_F(F).\end{aligned} \tag{3.64}$$

Therefore $\mu(C, m)$ is l-framed. As in the proof of Theorem 3.1.4 one shows that the map $m \mapsto \mu(C, m)$ descends to a map

$${}_G[U \oplus W, V \oplus W]^\Omega_B \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^\Omega_B,\ [m] \mapsto [\mu(C, m],\tag{3.65}$$

independent of the choice C, and that the family of maps $(p_\gamma)_W$ defines a natural transformation.

Now we prove stable surjectivity: Let $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be a l-framed Ω-map. Then $c := \mu - l$ is compact by assumption. We choose:

- constants $\varepsilon > 0$ and $\delta > 1$, such that $\mu(D_\delta(\mathcal{E}) \cup \mathcal{E}(\Omega)) \subset \mathcal{F}_{>\varepsilon}$;
- a closed complement $\iota_U(U) \oplus \tilde{\mathcal{U}} = \mathcal{E}$;
- a finite dimensional subbundle $W \subset \tilde{\mathcal{U}}$, such that $c(D_\delta(\mathcal{E}))$ is $\varepsilon/2$-close to $\iota_V(V) \oplus l(W) \subset \mathcal{F}$.
- a closed complement $W \oplus \tilde{\mathcal{W}} = \tilde{\mathcal{U}}$.

The penultimate choice is possible because we have the decomposition $\mathcal{F} = \iota_V(V) \oplus l(\tilde{U})$ and because $l|_{\tilde{U}} : \tilde{U} \longrightarrow l(\tilde{U})$ is an isomorphism (Lemma 3.1.9 and Lemma 2.7.6). We put $F := \iota_V(V) \oplus l(W)$ and $E := l^{-1}(F) = \iota_U(U) \oplus W$. We have decompositions

$$\mathcal{E} = E \oplus \tilde{\mathcal{W}} \text{ and } \mathcal{F} = F \oplus l(\tilde{\mathcal{W}}). \tag{3.66}$$

In Theorem 3.1.11, we associated with this data a Ω-homotopy $(\mu_t : \mathcal{E} \longrightarrow \mathcal{F})_{t \in [0,3]}$ between μ and $\mu_3 = l + p_F \circ c_1 \circ \pi_E$, where $c_1 : \mathcal{E} \longrightarrow \mathcal{F}$ is the compact map obtained by scaling c. It is clear from the construction that the Ω-homotopy μ_t is l-framed. We now define

$$m := \iota_F^{-1} \circ \mu_3 \circ \iota_E : U \oplus W = E \longrightarrow F = V \oplus W. \tag{3.67}$$

Let $\iota_W : W \hookrightarrow \tilde{\mathcal{U}} \subset \mathcal{E}$ be the inclusion and let $C = (\tilde{U}, \iota_W, \tilde{W})$. Now let $\iota_U(u) + \iota_W(w) + \tilde{w} \in \iota_U(U) \oplus \iota_W(W) \oplus \tilde{W}$. Then

$$\begin{aligned}\mu(C, m)(\iota_U(u) + \iota_W(w) + \tilde{w}) &= l(\tilde{w}) + \iota_F \circ \left(\iota_F^{-1} \circ \mu_3 \circ \iota_E\right) \circ \iota_E^{-1}(\iota_U(u) + \iota_W(w)) \\ &= l(\tilde{w}) + \mu_3(\iota_U(u) + \iota_W(w)) \\ &= l(\tilde{w}) + l(\iota_U(u) + \iota_W(w)) + p_F \circ c_1(\iota_U(u) + \iota_W(w)) \\ &= l(\iota_U(u) + \iota_W(w) + \tilde{w}) + p_F \circ c_1 \circ \pi_E(\iota_U(u) + \iota_W(w) + \tilde{w}) \\ &= \mu_3(\iota_U(u) + \iota_W(w) + \tilde{w}).\end{aligned} \tag{3.68}$$

This proves stable surjectivity. Stable injectivity is deduced from stable surjectivity in the same way as in the unframed case. □

Corollary 3.2.5. *Let $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ be a ρ_0-Fredholm morphism and let $V \subset \mathcal{F}_0$ be an l-adapted subbundle. This data determines a stably bijective natural transformation*

$$p_{l,V} : \mathfrak{p}_\Omega^{l,V} := \mathfrak{p}_\Omega^{l^{-1}(V),V} \longrightarrow {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}. \tag{3.69}$$

Therefore, if we fix both l and $V \subset \mathcal{F}_0$, then we have a natural way of understanding the set ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}$ as the colimit of the functor $\mathfrak{p}_\Omega^{l,V}$ defined on the category ${}_{\rho_0}\mathcal{C}_B \times {}_{\rho_*}\mathcal{H}_B$.

3.3 Fiberwise one point compactification

In the previous section we have described the homotopy set ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega$ as a colimit for a functor $\mathfrak{p}_\Omega$ which was defined on a category of pairs of finite dimensional bundles. Our ultimate goal is to produce group valued invariants. The first step to that end is to replace the functor $\mathfrak{p}_\Omega$ by a functor $\mathfrak{p}_\Omega^+$ which eventually will take values in Group. The idea is simple: Instead of considering a certain class of proper maps $m : E \longrightarrow F$ between finite dimensional ρ-bundles we consider a certain class of maps $n : E_B^+ \longrightarrow F_B^+$ between the fiberwise one point compactified bundles E_B^+ and F_B^+. From now on, we assume that $\mathbb{R} \in \rho$.

Every proper map $m : E \longrightarrow F$ induces a map $m^+ : E_B^+ \longrightarrow F_B^+$. In this section we are going to prove that the converse is true up to stabilization. We now decribe the situation in a more formal way.

We have already defined the functor $\mathfrak{p}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Set}$. It associates with a pair $(E,F) \in \mathrm{ob}\,{}_\rho\mathcal{H}_B$ the set ${}_G[E,F]_B^\Omega$ of Ω-homotopy classes of Ω-maps $m : E \longrightarrow F$.

Definition 3.3.1. Let $(E,F) \in \mathrm{ob}\,{}_\rho\mathcal{H}_B$ be a pair of finite dimensional ρ-bundles. A fiberwise pointed G-map $n : E_B^+ \longrightarrow F_B^+$ is an Ω-map, if $n(E(\Omega))$ avoids a neighbourhood of the 0-section in F_B^+. An Ω-homotopy is an Ω-map $\tilde{n} = (n_t)_{t\in[0,1]} : [0,1] \times E_B^+ \longrightarrow [0,1] \times F_B^+$ over the base $B \times [0,1]$.

Now we can define the functor $\mathfrak{p}_\Omega^+ : {}_\rho\mathcal{H}_B \longrightarrow \mathrm{Set}$. By definition it associates with a pair $(E,F) \in \mathrm{ob}\,{}_\rho\mathcal{H}_B$ the pointed set ${}_G[E_B^+,F_B^+]_B^\Omega$ of Ω-homotopy classes of Ω-maps $n : E_B^+ \longrightarrow F_B^+$. Let $f = (i_E, i_F, \tilde{E}, \tilde{F}, \tau) : (E,F) \longrightarrow (E',F')$ be a morphism in the category ${}_\rho\mathcal{H}_B$. We define

$$\mathfrak{p}_\Omega^+(f) : {}_G[E_B^+,F_B^+]_B^\Omega \longrightarrow {}_G[E_B'^+,F_B'^+]_B^\Omega \tag{3.70}$$

as follows: Let $n : E_B^+ \longrightarrow F_B^+$ be an Ω-map. We set

$$f_* n := \left(i_F^+ \circ n \circ (i_E^+)^{-1}\right) \wedge_B \tau^+ : E_B'^+ = i_E^+\left(E_B^+\right) \wedge_B \tilde{E}_B^+ \longrightarrow F_B'^+ = i_F^+\left(F_B^+\right) \wedge_B \tilde{F}_B^+. \tag{3.71}$$

An Ω-map $m : E \longrightarrow F$ is by definition proper and hence extends to an Ω-map $m^+ : E_B^+ \longrightarrow F_B^+$. This induces a natural transformation

$$+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}_\Omega^+. \tag{3.72}$$

The goal of this section is to prove the following theorem:

Theorem 3.3.2. *Assume that $\mathbb{R} \in \rho$. Then the natural transformation $+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}_\Omega^+$ is stably bijective.*

The main point of our proof is the following: Given an Ω-map $n : E_B^+ \longrightarrow F_B^+$, we will show that the induced map $n \wedge_B \mathrm{id}_{\underline{\mathbb{R}}} : [E \oplus \underline{\mathbb{R}}]_B^+ \longrightarrow [F \oplus \underline{\mathbb{R}}]_B^+$ is Ω-homotopic to a map n' with the property $n'^{-1}(\infty_B) = \infty_B$. We will construct this homotopy in a concrete way. For simplicity we do so in the case $B = \{*\}$. The generalization is straight forward.

Let E be a ρ-Hilbert space. Let $r \in \mathbb{R}_{\geq 0}$ be a non negative real number. Set $m_r := (0,r) \in E \times \mathbb{R}$. We start by defining a family of maps

$$\Phi_r = \Phi_{r,E} : (E \times \mathbb{R}) - m_r \longrightarrow E^+ \text{ for } r \in [0,\infty). \tag{3.73}$$

Before defining it formally, we give a simple geometric description. Let $r \in [0,\infty)$ and let $(e,x) \in E\times\mathbb{R}$. Consider the segment joining (e,x) with $(0,r)$. It intersects the unit sphere $S_r(E \times \mathbb{R})$ with center $(0,r)$ in a unique point $(\hat{e},\hat{x})$. Now use the stereographic projection from the north pole $(0,r+1)$ to E^+ to map $(\hat{e},\hat{x})$ to the element $\Phi_r(e,x)/2 \in E^+$. (See figure (3.3)).

We now give the formal definition of the map Φ_r: Consider the stereographic projection

$$\varphi_r = \varphi_{r,E} : S_r(E \times \mathbb{R}) \xrightarrow{\approx} E^+,\ (e,x) \mapsto \frac{r+1}{r+1-x} e. \tag{3.74}$$

We define the map $\Phi_{r,E} : \left(E \times \mathbb{R}\right) - m_r \longrightarrow E_B^+$ for $r \in [0,\infty)$ by

$$\Phi_{r,E}(e,x) := \begin{cases} 2\varphi_{r,E}\left(\frac{(e,x-r)}{|(e,x-r)|} + m_r\right) & \text{when } e \neq 0 \text{ or } x < r \\ \infty & \text{when } e = 0 \text{ and } x > r. \end{cases} \tag{3.75}$$

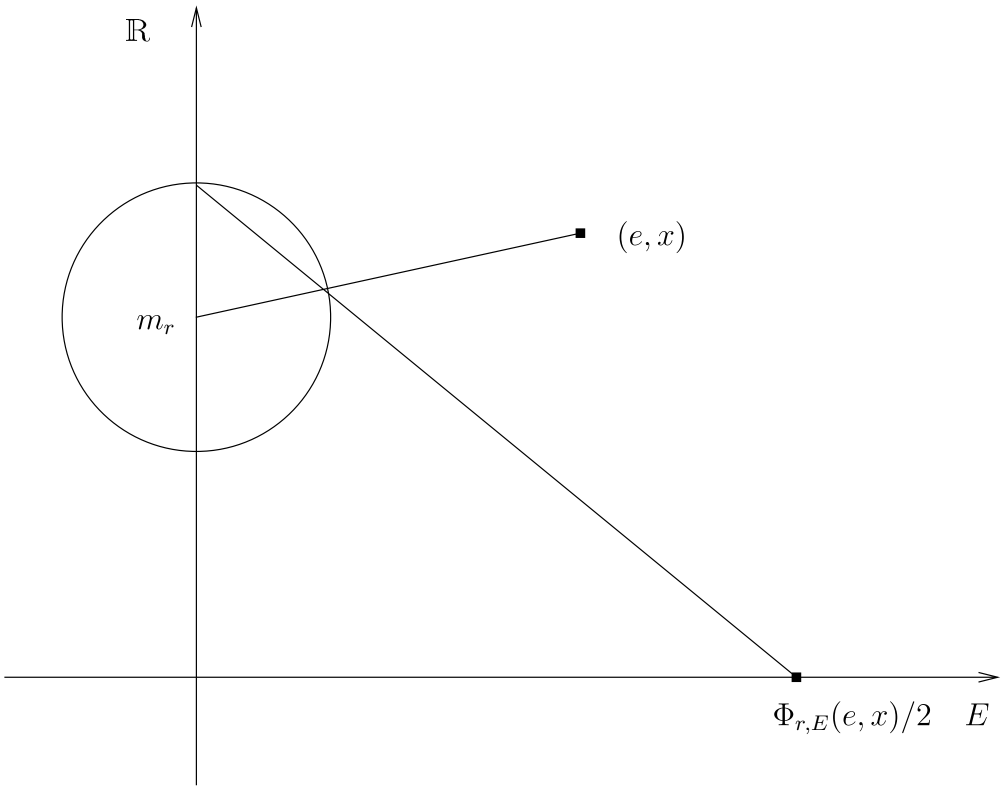


Figure 3.1: The map $\Phi_{r,E}$

Remark 3.3.3. Let $e \neq 0$ or $x < r$. Then

$$\Phi_r(e,x) = \frac{2(r+1)e}{|(e,x-r)| + r - x}. \tag{3.76}$$

Proof.

$$\begin{aligned}\Phi_r(e,x) &= 2\varphi_r\left(\frac{(e,x-r)}{|(e,x-r)|} + m_r\right) = 2\varphi_r\left(\frac{(e,x-r+r|(e,x-r)|)}{|(e,x-r)|}\right)\\ &= 2\frac{r+1}{r+1+\frac{-x+r-r|(e,x-r)|}{|(e,x-r)|}}\ \frac{e}{|(e,x-r)|} = 2\frac{r+1}{\frac{(r+1)|(e,x-r)|-x+r-r|(e,x-r)|}{|(e,x-r)|}}\ \frac{e}{|(e,x-r)|}\\ &= \frac{2(r+1)e}{|(e,x-r)|+r-x}.\end{aligned} \tag{3.77}$$

□

Last, we define $\Phi_{\infty,E} : E \times \mathbb{R} \longrightarrow E \subset E^+$ to be the orthogonal projection $(e,x) \mapsto e$.

Lemma 3.3.4. *The family of maps $(\Phi_{r,E})_{r\in[0,\infty]}$ defines a continuous G-map*

$$\Phi = \Phi_E : ([0,\infty] \times E \times \mathbb{R}) - \{(r,0,r) \mid r \in \mathbb{R}_{\geq 0}\} \longrightarrow [E \times \mathbb{R}]^+,\ (r,e,x) \mapsto \Phi_r(e,x). \tag{3.78}$$

Proof. The G-equivariance is clear from the definition of Φ since the action of G on E is unitary. Put $m_E := \{(r,0,r) \mid r \in [0,\infty)\}$. Note that Φ is obviously continuous on the open subset

$$U := \{(r,e,x) \in [0,\infty) \times E \times \mathbb{R} \mid e \neq 0 \text{ or } r > x\}. \tag{3.79}$$

Let $(r_n, e_n, x_n)_n \in ([0,\infty] \times E \times \mathbb{R}) - m_E$ be a sequence converging to $(r, e, x) \in ([0,\infty] \times E \times \mathbb{R}) - m_E$. We will show that $\Phi(r_n, e_n, x_n) \to \Phi(r, e, x)$.

We distinguish three cases:

Case 1: Assume that $r < \infty$, $e = 0$, and that $r < x$. Then $r_n < x_n$ for almost all $n \in \mathbb{N}$ so that we may assume $r_n < x_n$ for all $n \in \mathbb{N}$. When $e_n = 0$, then by definition $\Phi(r_n, e_n, x_n) = \infty = \Phi(r, e, x)$. So we can assume that $e_n \neq 0$ for all $n \in \mathbb{N}$. Then:

$$\begin{aligned} \Phi(r_n, e_n, x_n) &= \left| \frac{2(r_n+1)e_n}{|(e_n, x_n - r_n)| + r_n - x_n} \right| = \frac{2(r_n+1)}{\left| \frac{\sqrt{|e_n|^2 + |x_n - r_n|^2}}{|e_n|} - \frac{|x_n - r_n|}{|e_n|} \right|} \\ &= \frac{2(r_n+1)}{\sqrt{1 + \frac{|x_n - r_n|^2}{|e_n|^2}} - \frac{|x_n - r_n|}{|e_n|}}. \end{aligned} \tag{3.80}$$

The numerator of this fraction converges to $2(r+1) < \infty$, whereas the denominator is of the form $\sqrt{1 + \hat{x}_n^2} - \hat{x}_n$ where $\hat{x}_n = \frac{|x_n - r_n|}{|e_n|} \in \mathbb{R}_{\geq 0}$. From $e_n \to e = 0$ and $|x_n - r_n| \to |x - r| > 0$ we deduce that $\hat{x}_n \to \infty$. But then:

$$\begin{aligned} \sqrt{1 + \hat{x}_n^2} - \hat{x}_n &= \left(\sqrt{1 + \hat{x}_n^2} - \hat{x}_n \right) \frac{\sqrt{1 + \hat{x}_n^2} + \hat{x}_n}{\sqrt{1 + \hat{x}_n^2} + \hat{x}_n} \\ &= \frac{1 + \hat{x}_n^2 - \hat{x}_n^2}{\sqrt{1 + \hat{x}_n^2} + \hat{x}_n} \to 0. \end{aligned} \tag{3.81}$$

Therefore $\Phi(r_n, e_n, x_n) \to \infty = \Phi(r, e, x)$.

Case 2: Assume that $r < \infty$, $e \neq 0$ or $r > x$. In that case $(r, e, x) \in U$ and also $(r_n, e_n, x_n) \in U$ for almost all $n \in \mathbb{N}$. But Φ is continuous on the open set U. Therefore $\Phi(r_n, e_n, x_n) \to \Phi(r, x, e)$.

Case 3: Assume that $r = \infty$. Then $r > x$ and therefore $r_n > x_n$ for almost all $n \in \mathbb{N}$. We can assume that $r_n > x_n$ for all $n \in \mathbb{N}$. By definition $\Phi(r, e, x) = e$. When $r_n = \infty$, then $\Phi(r_n, e_n, x_n) = e_n \to e$, so we can assume $r_n < \infty$ for all $n \in \mathbb{N}$. Then:

$$\Phi(r_n, e_n, x_n) = \frac{2(r_n+1)e_n}{|(e_n, x_n - r_n)| + r_n - x_n} = \frac{2e_n + \frac{2e_n}{r_n}}{\left| \left(\frac{e_n}{r_n}, \frac{x_n}{r_n} - 1 \right) \right| + 1 - \frac{x_n}{r_n}}. \tag{3.82}$$

Now $\frac{e_n}{r_n} \to 0$ and $\frac{x_n}{r_n} \to 0$. Therefore $\Phi(r_n, e_n, x_n) \to e = \Phi(r, e, x)$. □

In the following lemma we study the behaviour of Φ when both r and $|(e, x)|$ go to infinity. This will be used later on.

Lemma 3.3.5. *Let $(r_n, e_n, x_n) \in (\mathbb{R} \times E \times \mathbb{R}) - \{(r, 0, r) \mid r \in [0, \infty)\}$. Assume that:*

1. *The sequences r_n and $|(e_n, x_n)|$ converge to ∞.*
2. *The sequence $\big| |(e_n, x_n - r_n)| - r_n \big|$ is bounded.*

Then:

$$\Phi_{r_n}(e_n, x_n) \to \infty. \tag{3.83}$$

Proof. We first observe that it is enough to prove that $\Phi_{r_n}(e_n, x_n)$ is unbounded: If $\Phi_{r_n}(e_n, x_n)$ does not converge to ∞, there is a subsequence $(r_{n_k}, e_{n_k}, x_{n_k})$, such that $\Phi_{r_{n_k}}(e_{n_k}, x_{n_k})$ is bounded. But the subsequence $(r_{n_k}, e_{n_k}, x_{n_k})$ still satisfies the assumption of the Lemma and we obtain a contradiction.

Notice that when $e_n = 0$ and $x_n > r_n$, then by definition $\Phi_{r_n}(e_n, x_n) = \infty$. So we can assume that $e_n \neq 0$ or that $x_n < r_n$ for all $n \in \mathbb{N}$.

By assumption $\big| |(e_n, x_n - r_n)| - r_n \big|$ is bounded and $r_n \to \infty$. This implies that

$$\frac{|(e_n, x_n - r_n)|}{r_n} \longrightarrow 1 \text{ and that } \frac{|x_n - r_n|}{r_n} \text{ is bounded.} \tag{3.84}$$

First assume that e_n is unbounded. Then:

$$
\begin{aligned}
|\Phi_{r_n}(e_n,x_n)| &= \frac{2(r_n+1)|e_n|}{|\,|(e_n,x_n-r_n)|+r_n-x_n|} \\
&= \frac{\frac{2(r_n+1)}{r_n}|e_n|}{\left|\frac{|(e_n,x_n-r_n)|}{r_n}+\frac{r_n-x_n}{r_n}\right|}
\end{aligned}
\tag{3.85}
$$

The numerator is unbounded and the denominator is bounded. Therefore the fraction is unbounded.

So we can assume that e_n is bounded. Then $|x_n| \to \infty$ because $|(e_n,x_n)| \to \infty$. We prove that $x_n > r_n$ for all but finitely many $n \in \mathbb{N}$: Assume the contrary. Then $x_n \leq r_n$ for infinitely many $n \in \mathbb{N}$. By choosing a subsequence we assume that this inequality holds for all $n \in \mathbb{N}$. Now:

$$
\left|\left(\frac{e_n}{r_n},\frac{x_n}{r_n}-1\right)\right| = \frac{|(e_n,x_n-r_n)|}{r_n} \to 1 \text{ and } \frac{e_n}{r_n} \to 0. \tag{3.86}
$$

As a consequence $1-\frac{x_n}{r_n} = \left|\frac{x_n}{r_n}-1\right| \to 1$ and therefore $\frac{x_n}{r_n} \to 0$. Now we use that $x_n \to \infty$ and that $|\,|(e_n,x_n-r_n)|-r_n|$ is bounded. These facts imply:

$$
\left|\left|\left(\frac{e_n}{x_n},1-\frac{r_n}{x_n}\right)\right|-\frac{r_n}{x_n}\right| = \frac{|\,|(e_n,x_n-r_n)|-r_n|}{x_n} \to 0. \tag{3.87}
$$

Set $a_n := \frac{e_n}{x_n}$ and $b_n := \frac{r_n}{x_n}$. We have seen that $b_n \to \infty$. The sequence e_n is bounded and $x_n \to \infty$, therefore $a_n \to 0$. But then:

$$
\begin{aligned}
|\,|(a_n,1-b_n)\,|-b_n| &= \left|\sqrt{a_n^2+(1-b_n)^2}-b_n\right| \\
&= \left|\left(\sqrt{a_n^2+(1-b_n)^2}-b_n\right)\frac{\sqrt{a_n^2+(1-b_n)^2}+b_n}{\sqrt{a_n^2+(1-b_n)^2}+b_n}\right| \\
&= \left|\frac{a_n^2+1-2b_n+b_n^2-b_n^2}{\sqrt{a_n^2+(1-b_n)^2}+b_n}\right| = \frac{\left|\frac{a_n^2}{b_n}+\frac{1}{b_n}-2\right|}{\sqrt{\frac{a_n^2}{b_n^2}+\frac{(1-b_n)^2}{b_n^2}}+1}.
\end{aligned}
\tag{3.88}
$$

Both the numerator and the denominator of this fraction converge to 2. Therefore the fraction converges to 1. But we have already concluded that it converges to 0. This is the contradiction we were looking for.

So $x_n > r_n$ for all but finitely many $n \in \mathbb{N}$. At the beginning of the proof we assumed that either $e_n \neq 0$ or $x_n < r_n$. Therefore $e_n \neq 0$ for all $n \in \mathbb{N}$. Then:

$$
\begin{aligned}
|\Phi_{r_n}(e_n,x_n)| &= \frac{2(r_n+1)|e_n|}{|\,|(e_n,x_n-r_n)|+r_n-x_n|} \\
&= \frac{2(r_n+1)}{\left|\sqrt{1+\frac{(x_n-r_n)^2}{|e_n|^2}}-\frac{(x_n-r_n)}{|e_n|}\right|}.
\end{aligned}
\tag{3.89}
$$

We know that e_n is bounded, $r_n \to \infty$, and that $|\,|(e_n,x_n-r_n)|-r_n|$ is bounded. Therefore $|x_n-r_n| \to \infty$ and also $\frac{x_n-r_n}{|e_n|} \to \infty$. Then (see (3.81)):

$$
\sqrt{1+\frac{(x_n-r_n)^2}{|e_n|^2}}-\frac{x_n-r_n}{|e_n|} = \frac{1}{\sqrt{1+\frac{(x_n-r_n)^2}{|e_n|^2}}+\frac{x_n-r_n}{|e_n|}} \to 0. \tag{3.90}
$$

Going back to $|\Phi_{r_n}(e_n,x_n)|$, we conclude that this sequence is unbounded. □

In the next step, we define a a family of maps $\Psi_r = \Psi_{r,F} : [0,\infty)\times F_B^+ \longrightarrow F\times\mathbb{R}$ for $r\in[0,\infty)$. Again, we first describe it geometrically. Let $r\in[0,\infty)$ and let $(y,f)\in[0,\infty)\times F_B^+$. Use the inverse stereographic projection to map the element $f/2 \in F_B^+$ to an element $\hat{f} \in S_r(F \times \mathbb{R}) \subset F \times \mathbb{R}$ in the unit sphere with center $(0, r)$. Then consider the half line through $(0, r)$ and $\hat{f} \in F \times \mathbb{R}$. We define $\Psi_r(y, f)$ to be the element of norm y on that half line.

Formally, we define the map $\Psi_r = \Psi_{r,F} : [0, \infty) \times F_B^+ \longrightarrow F \times \mathbb{R}$ as follows: Let $(y, f) \in [0, \infty) \times F_b^+$. Then

$$\Psi_r(y, f) := \left(\varphi_{r,F}^{-1}(f/2) - (0, r)\right) y + (0, r) \in F \times \mathbb{R}. \tag{3.91}$$

Observe that for $f = \infty$, we have $\Psi_{r,F}(y, f) = (0, r + y) \in F \times \mathbb{R}$.

Remark 3.3.6. Let $(y, f) \in [0, \infty) \times F$ and let $r \in [0, \infty)$. Then:

$$\begin{aligned}\Psi_r(y, f) &= \left(\frac{(r+1)yf}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\left[\frac{|f|^2}{4} - (r+1)^2\right] y}{\frac{|f|^2}{4} + (r+1)^2} + r\right) \\ &= \left(\frac{(r+1)fy}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\frac{|f|^2}{4}[r+y] + (r+1)^2[r-y]}{\frac{|f|^2}{4} + (r+1)^2}\right).\end{aligned} \tag{3.92}$$

Proof. First we recall the formula for the inverse stereographic projection $\varphi_{r,F}^{-1}$. Let $f \in F$. Then:

$$\varphi_{r,F}^{-1}(f) = (0, r+1) + \frac{2(r+1)}{|f|^2 + (r+1)^2}(f, -(r+1)). \tag{3.93}$$

To verify this formula, we need to check that this element lies on the unit sphere with center $(0, r)$. We have:

$$\begin{aligned}&\left[|f|^2 + (r+1)^2\right]^2 \left|(0,1) + \frac{2(r+1)}{|f|^2 + (r+1)^2}(f, -(r+1))\right|^2 \\ &= \left|\left(2(r+1)f,\ |f|^2 + (r+1)^2 - 2(r+1)^2\right)\right|^2 \\ &= 4(r+1)^2|f|^2 + \left[|f|^2 - (r+1)^2\right]^2 = 2(r+1)^2|f|^2 + |f|^4 + (r+1)^4 \\ &= \left[|f|^2 + (r+1)^2\right]^2.\end{aligned} \tag{3.94}$$

Therefore $\left|(0,1) + \frac{2(r+1)}{|f|^2+(r+1)^2}(f, -(r+1))\right| = 1$. This proves the formula for $\varphi_{r,F}^{-1}$.

Then:

$$\begin{aligned}\varphi_{r,F}^{-1}(f/2) - (0, r) &= \left((0,1) + \frac{2(r+1)}{\frac{|f|^2}{4} + (r+1)^2}(f/2, -(r+1))\right) \\ &= \left(\frac{(r+1)f}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\frac{|f|^2}{4} - (r+1)^2}{\frac{|f|^2}{4} + (r+1)^2}\right).\end{aligned} \tag{3.95}$$

Therefore

$$\begin{aligned}\Psi_r(y, f) &= \left(\frac{(r+1)f}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\frac{|f|^2}{4} - (r+1)^2}{\frac{|f|^2}{4} + (r+1)^2}\right) y + (0, r) \\ &= \left(\frac{(r+1)fy}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\left[\frac{|f|^2}{4} - (r+1)^2\right] y + r\left[\frac{|f|^2}{4} + (r+1)^2\right]}{\frac{|f|^2}{4} + (r+1)^2}\right) \\ &= \left(\frac{(r+1)fy}{\frac{|f|^2}{4} + (r+1)^2}, \frac{\frac{|f|^2}{4}[r+y] + (r+1)^2[r-y]}{\frac{|f|^2}{4} + (r+1)^2}\right).\end{aligned} \tag{3.96}$$

□

Lemma 3.3.7. *The family of maps $(\Psi_{r,F})_{r\in[0,\infty)}$ defines a continuous G-map*

$$\Psi_F : [0,\infty)\times[0,\infty)\times F^+ \longrightarrow F\times\mathbb{R},\ (r,y,f)\mapsto \Psi_{r,F}(y,f). \tag{3.97}$$

Proof. The group G acts unitarily on F and trivially on $\mathbb{R}$. Equivariance is therefore clear from the definition of Ψ_F. It is also clear from the previous remark that the map Ψ_F is continuous on the open subset $U := [0,\infty)\times[0,\infty)\times F$. So let $(r_n,y_n,f_n)\in U$ be a sequence converging to $(r,y,\infty)\in[0,\infty)^2\times F_B^+$. Clearly

$$\left|\frac{(r_n+1)y_nf_n}{\frac{|f_n|^2}{4}+(r_n+1)^2}\right| = \frac{(r_n+1)y_n}{\frac{|f_n|}{4}+\frac{(r_n+1)^2}{|f_n|}} \to 0 \tag{3.98}$$

when $n\to\infty$. On the other hand

$$\frac{\left[\frac{|f_n|^2}{4}-(r_n+1)^2\right]y_n}{\frac{|f_n|^2}{4}+(r_n+1)^2}+r_n = \frac{\left[\frac{1}{4}-\frac{(r_n+1)^2}{|f_n|^2}\right]y_n}{\frac{1}{4}+\frac{(r_n+1)^2}{|f_n|^2}}+r_n \to y+r. \tag{3.99}$$

Therefore $\Psi_F(r_n,y_n,f_n)\to(0,y+r)=\Psi_F(r,y,\infty)$. This proves continuity. □

Remark 3.3.8. Let V be a normed vector space and let $a,b\in V$. Then:

1. $|a|\geq|\,|a-b|-|b|\,|$;
2. $|a+b|\geq|\,|a|-|b|\,|$.

Proof. The triangle inequality implies $|a-b|\leq|a|+|b|$ and $|b|=|b-a+a|\leq|a-b|+|a|$. The statements follow. □

Lemma 3.3.9. *Let $(r_n,y_n,f_n)\in[0,\infty)\times[0,\infty)\times F_B^+$ converge to $(\infty,y,\infty)\in[0,\infty]^2\times F_B^+$. Then $\Psi_{r_n}(y_n,f_n)\to\infty$.*

Proof. Again, it is enough to prove that $\Psi_{r_n}(y_n,f_n)$ is unbounded. First, we consider the case when $|y_n-r_n|$ is unbounded. Then:

$$|\Psi_{r_n}(y_n,f_n)| = |\left(\varphi^{-1}(f_n/2)-(0,r_n)\right)y_n+(0,r_n)| \geq |y_n-r_n| \tag{3.100}$$

is unbounded.

Second, we assume that $|y_n-r_n|$ is bounded and that $r_n/|f_n|$ is bounded. Then

$$\frac{\frac{|f_n|^2}{4}[r_n+y_n]+(r_n+1)^2[r_n-y_n]}{\frac{|f|^2}{4}+(r+1)^2} = \frac{\frac{1}{4}[r_n+y_n]+\frac{(r_n+1)^2}{|f_n|^2}[r_n-y_n]}{\frac{1}{4}+\frac{(r_n+1)^2}{|f_n|^2}}. \tag{3.101}$$

The denominator of this fraction is bounded. The numerator consists of two summands: the second one is bounded, the first one goes to ∞. Therefore the fraction converges to ∞. This shows that $|\Psi_{r_n}(y_n,f_n)|\to\infty$.

Third, we assume that $|y_n-r_n|$ is bounded by $C>0$ and that $r_n/|f_n|$ is unbounded. By choosing a subsequence we can assume without loss of generality that $r_n/|f_n|\longrightarrow\infty$. Then:

$$\begin{aligned}\frac{(r_n+1)|f_n|y_n}{\frac{|f_n|^2}{4}+(r_n+1)^2} &\geq \frac{(r_n+1)|f_n|(r_n-C)}{\frac{|f_n|^2}{4}+(r_n+1)^2}\\ &= \frac{\frac{(r_n+1)r_n}{r_n^2}|f_n|}{\frac{|f_n|^2}{4r_n^2}+\frac{(r_n+1)^2}{r_n^2}} - \frac{\frac{r_n+1}{r_n}\frac{C|f_n|}{r_n}}{\frac{|f_n|^2}{4r_n^2}+\frac{(r_n+1)^2}{r_n^2}}.\end{aligned} \tag{3.102}$$

We first look at the first fraction: Its denumerator converges to 1, whereas its numerator converges to ∞. Next we look at the second fraction: Its denumerator converges to 1 and its numerator converges to 0. This shows that $\Psi_{r_n}(y_n,f_n)$ is unbounded. □

Now let $n : E^+_B \longrightarrow F^+_B$ be an Ω-map. We define for $r \in [0, \infty)$ a map $N_r : [E \times \mathbb{R}]^+_B \longrightarrow [F \times \mathbb{R}]^+_B$ as follows:

- $N_r(e, x) := \Psi_{F,r}\left(|(e, x - r)|,\ n(\Phi_r(e, x))\right)$ when $(e, x) \in (E \times \mathbb{R}) - m_r$;
- $N_r(m_r) := (0, r)$;
- $N_r(\infty) := \infty$.

Furthermore, we set $N_\infty := n \wedge \mathrm{id}_{\mathbb{R}} : [E \times \mathbb{R}]^+_B \longrightarrow [F \times \mathbb{R}]^+_B$.

Remark 3.3.10. Let $(r, e, x) \in [0, \infty) \times E \times \mathbb{R}$. Then

1. $|N_r(e, x) - (0, r)| = |(e, x - r)|$;
2. $|N_r(e, x)| \geq ||(e, x - r)| - r|$.

Proof. The first statement follows immediately from the definition of the map Ψ_r. The second statement follows follows from the first:

$$|N_r(e, x)| = |N_r(e, x) - (0, r) + (0, r)| \geq ||N_r(e, x) - (0, r)| - r|\,. \tag{3.103}$$

□

Lemma 3.3.11. *The map*

$$N : [0, \infty] \times [E \times \mathbb{R}]^+_B \longrightarrow [F \times \mathbb{R}]^+_B,\ (r, (e, x)) \mapsto N_r(e, x) \tag{3.104}$$

is continuous.

Proof. Let $(r_n, e_n, x_n) \in [0, \infty] \times [E \times \mathbb{R}]^+_B$ be a sequence converging to (r, e, x).

Case 1: Assume that $r < \infty$ and $(e, x) = (0, r)$. Then $N_r(e, x) = (0, r)$. We can assume that $(e_n, x_n) \neq (0, r_n)$ for all $n \in \mathbb{N}$, because otherwise $N_{r_n}(e_n, x_n) = (0, r_n)$. Then by definition:

$$|N_{r_n}(e_n, x_n) - (0, r_n)| = |(e_n, x_n) - (0, r_n)| \longrightarrow 0. \tag{3.105}$$

Therefore $N_{r_n}(e_n, x_n) \to (0, r)$.

Case 2: Assume that $r < \infty$ and $(e, x) = \infty \in [E \times \mathbb{R}]^+$. Then $N_r(e, x) = \infty$ by definition. It is enough to show that $N_{r_n}(e_n, x_n)$ is unbounded. We can assume that $(e_n, x_n) \neq \infty$ for all $n \in \mathbb{N}$. The case $e_n = 0$ and $r_n = x_n$ can only appear for finitely many $n \in \mathbb{N}$: Otherwise $x_n = r_n$ would be unbounded in contradiction to the assumption that $r_n \to r < \infty$. So we can assume without loss of generality that either $e_n \neq 0$ or $r_n \neq x_n$ for all $n \in \mathbb{N}$. Then (Remark 3.3.10):

$$|N_{r_n}(e_n, x_n)| \geq |\ |(e_n, x_n - r_n)| - r_n| \to \infty. \tag{3.106}$$

Case 3: Assume that $r = \infty$ and $(e, x) \neq \infty$. Then $y_n := |(e_n, x_n - r_n)| \to \infty$. Furthermore we can assume that $r_n > x_n$ for all $n \in \mathbb{N}$. Set $f_n := n(\Phi_{r_n}(e_n, x_n))$. We know from Lemma 3.3.4 that $\Phi_{r_n}(e_n, x_n) \to e$. Therefore $f_n \to n(e) =: f \in F^+_B$. When $f = \infty$, then it follows from Lemma 3.3.9 that $N_{r_n}(e_n, x_n) = \Psi_{r_n}(y_n, f_n) \to \infty = N_\infty(e, x)$. So we assume that $f \neq \infty$. Then:

$$N_{r_n}(e_n, x_n) = \left(\frac{(r_n + 1) f_n y_n}{\frac{|f_n|^2}{4} + (r_n + 1)^2}, \frac{\frac{|f_n|^2}{4}[r_n + y_n] + (r_n + 1)^2 [r_n - y_n]}{\frac{|f_n|^2}{4} + (r_n + 1)^2} \right). \tag{3.107}$$

We consider first the F-coordinate:

$$\frac{(r_n + 1) f_n y_n}{\frac{|f_n|^2}{4} + (r_n + 1)^2} = \frac{\frac{r_n + 1}{r_n} f_n \frac{y_n}{r_n}}{\frac{|f_n|^2}{4 r_n^2} + \frac{(r_n + 1)^2}{r_n^2}}. \tag{3.108}$$

Notice that $y_n = |(e_n, x_n - r_n)|$ and therefore $y_n/r_n \to 1$. Therefore the numerator of the fraction converges to f, whereas the denominator converges to 1. This shows that the F-coordinate of $N_{r_n}(e_n, x_n)$ converges to $f = n(e)$. Next we consider the $\mathbb{R}$-coordinate:

$$\frac{\frac{|f_n|^2}{4}[r_n + y_n] + (r_n+1)^2[r_n - y_n]}{\frac{|f_n|^2}{4} + (r_n+1)^2} = \frac{\frac{|f_n|^2}{4r_n}\frac{r_n+y_n}{r_n} + \frac{(r_n+1)^2}{r_n^2}[r_n - y_n]}{\frac{|f_n|^2}{r_n^2} + \frac{(r_n+1)^2}{r_n^2}}. \tag{3.109}$$

The denominator of that fraction converges to 1 and $y_n/r_n \to 1$ and therefore $\frac{|f_n|^2}{4r_n}\frac{r_n+y_n}{r_n} \to 0$. We claim that $r_n - y_n \to x$. This would complete case 3, for then it would follow that $\frac{(r_n+1)^2}{r_n^2}(r_n - y_n) \to x$ and therefore that the numerator of the fraction above converges to x.

Our claim is proven as follows:

$$\begin{aligned} r_n - y_n &= r_n - \sqrt{e_n^2 + (r_n - x_n)^2} = \left(r_n - \sqrt{e_n^2 + (r_n - x_n)^2}\right)\frac{r_n + \sqrt{e_n^2 + (r_n - x_n)^2}}{r_n + \sqrt{e_n^2 + (r_n - x_n)^2}} \\ &= \frac{r_n^2 - e_n^2 - r_n^2 + 2r_n x_n - x_n^2}{r_n + \sqrt{e_n^2 + (r_n - x_n)^2}} \\ &= \frac{-\frac{e_n^2}{r_n} + 2x_n - \frac{x_n^2}{r_n}}{1 + \sqrt{\frac{e_n^2}{r_n^2} + \frac{(r_n - x_n)^2}{r_n^2}}}. \end{aligned} \tag{3.110}$$

The numerator of this fraction converges to $2x$ and the denominator to 2. This proves the claim.

Case 4: Assume that $r = \infty$ and $(e, x) = \infty$. We want to prove that $N_{r_n}(e_n, x_n) \to \infty$. It is enough to prove that this sequence is unbounded. First we observe that whenever $e_n = 0$ and $r_n = x_n$, then $N_{r_n}(e_n, x_n) = (0, r_n)$ by definition. So we can assume that $e_n \neq 0$ or $r_n \neq x_n$ for all $n \in \mathbb{N}$. As usual we set $y_n := |(e_n, x_n - r_n)|$ and $f_n := n(\Phi_{r_n}(e_n, x_n))$.

We distinguish two cases: First the case that $|y_n - r_n|$ is unbounded: Then by Remark 3.3.10

$$|N_{r_n}(e_n, x_n)| \geq |y_n - r_n|. \tag{3.111}$$

It follows that $N_{r_n}(e_n, x_n)$ is unbounded. So we have to treat the case that $|y_n - r_n|$ is bounded. In that case Lemma 3.3.5 implies that $f_n := \Phi_{r_n}(e_n, x_n) \to \infty$. By Lemma 3.3.9, $N_{r_n}(e_n, x_n) = \Psi_{r_n}(y_n, f_n) \to \infty$. □

Proposition 3.3.12. *The map*

$$N : [2, \infty] \times [E \times \mathbb{R}]^+ \longrightarrow [F \times \mathbb{R}]^+, \; (r, (e, x)) \mapsto N_r((e, x)) \tag{3.112}$$

is an Ω-homotopy. It has the property that $N_r(E \times \mathbb{R}) \subset F \times \mathbb{R}$ for all $r \neq \infty$.

Proof. We have already proven that N is continuous. Equivariance follows from the equivariance of Φ, Ψ, and n, and because $(0, r)$ and ∞ are fixed by the G-action. We claim that there is $\varepsilon > 0$, such that for all $(r, e, x) \in [2, \infty] \times E \times \mathbb{R}$ with $(e, x) \in [E \times \mathbb{R}](\Omega)$, we have $|N_r(e, x)| > \varepsilon$. This statement can be easily deduced from the geometric description of the homotopy. We give a formal proof. The map $n : E^+ \longrightarrow F^+$ is an Ω-map, therefore there is $\varepsilon > 0$, such that $|n(e)| \geq \varepsilon$ for all $e \in E(\Omega)$.

First notice that when $r = \infty$, then $N_\infty(e, x) = (n(e), x)$. Since $(e, x) \in [E \times \mathbb{R}](\Omega)$ implies that $e \in E(\Omega)$, it follows that $|n(e)| > \varepsilon$. So we can assume that $r < \infty$. Next consider the case $e = 0$, $r = x$. Then $|N_r(e, x)| = |(0, r)| = r \geq 2$. So we can also assume that $r \neq x$ or $e \neq 0$.

Assume that the claim is wrong. Then we find a sequence $(r_n, e_n, x_n) \in [0, \infty) \times E \times \mathbb{R}$ with $e_n \neq 0$ or $r_n \neq x_n$ for all $n \in \mathbb{N}$, such that $|N_{r_n}(e_n, x_n)| < \frac{1}{n} \to 0$. The map Φ is G-equivariant and therefore $(e_n, x_n) \in [E \times \mathbb{R}](\Omega)$ implies that $\Phi_{r_n}(e_n, x_n) \in E(\Omega)$ or that $\Phi_{r_n}(e_n, x_n) = \infty$. We set $f_n := n(\Phi_{r_n}(e_n, x_n) \in F^+$ and $y_n := |(e_n, x_n - r_n)|$. It follows that $|f_n| \geq \varepsilon$ for all $n \in \mathbb{N}$. We assumed that

$$N_{r_n}(e_n, x_n) = \Psi_{r_n}(y_n, f_n) \to 0. \tag{3.113}$$

Recall that $|\Psi_{r_n}(y_n, f_n)| \geq |y_n - r_n| \geq 0$. Therefore $|y_n - r_n| \to 0$. Since $r_n \geq 2$ for all $n \in \mathbb{N}$, we can assume that $y_n \geq 1$ for all $n \in \mathbb{N}$. We recall:

$$\Psi_{r_n}(y_n, f_n) = \left(\frac{(r_n+1)f_n y_n}{\frac{|f_n|^2}{4} + (r_n+1)^2}, \frac{\frac{|f_n|^2}{4}[r_n + y_n] + (r_n+1)^2[r_n - y_n]}{\frac{|f_n|^2}{4} + (r_n+1)^2} \right) \to 0. \tag{3.114}$$

We first consider the F-coordinate:

$$\left| \frac{(r_n+1)f_n y_n}{\frac{|f_n|^2}{4} + (r_n+1)^2} \right| = \frac{\frac{(r_n+1)y_n}{r_n^2}|f_n|}{\frac{|f_n|^2}{4r_n^2} + \frac{(r_n+1)^2}{r_n^2}} = \frac{\frac{(r_n+1)y_n}{r_n^2}}{\frac{|f_n|}{4r_n^2} + \frac{(r_n+1)^2}{|f_n|r_n^2}}. \tag{3.115}$$

The numerator of the last fraction converges to 1 (remember that $|r_n - y_n| \to 0$.) Since the whole fraction converges to 0, its denominator converges to ∞. But $\frac{(r_n+1)^2}{|f_n|r_n^2} = \frac{1}{|f_n|}\frac{(r_n+1)^2}{r_n^2}$ is bounded. Therefore $\frac{|f_n|}{4r_n^2} \to \infty$.

Now we look at the $\mathbb{R}$-coordinate of $\Psi_{r_n}(y_n, f_n)$:

$$\frac{\frac{|f_n|^2}{4}[r_n + y_n] + (r_n+1)^2[r_n - y_n]}{\frac{|f_n|^2}{4} + (r_n+1)^2} = \frac{\frac{1}{4}(r_n + y_n) + \frac{(r_n+1)^2}{|f_n|^2}(r_n - y_n)}{\frac{1}{4} + \frac{(r_n+1)^2}{|f_n|^2}}. \tag{3.116}$$

From $|f_n|/r_n^2 \to \infty$ it follows that $(r_n+1)^2/|f_n|^2 \to 0$. Since $(r_n - y_n) \to 0$, also $(r_n - y_n)(r_n + 1)^2/|f_n|^2 \to 0$. Furthermore $(r_n + y_n) \geq 3$ for all $n \in \mathbb{N}$. Therefore the numerator of the right hand side of (3.116)

$$\frac{1}{4}(r_n + y_n) + \frac{(r_n+1)^2}{|f_n|^2}(r_n - y_n) \tag{3.117}$$

is bounded below. Since $\Psi_{r_n}(y_n, f_n) \to 0$, the denominator converges to ∞. But

$$\frac{1}{4} + \frac{(r_n+1)^2}{|f_n|^2} \to \frac{1}{4}. \tag{3.118}$$

This is the contradiction we were looking for. Therefore N is a Ω-homotopy. The property $N_r(E \times \mathbb{R}) \subset F \times \mathbb{R}$ for $r < \infty$ follows immediately from the geometric description of N_r. □

Corollary 3.3.13. *Let $n : E_B^+ \longrightarrow F_B^+$ be an Ω-map. Then there is an Ω-homotopy $(N_t : [E \times \mathbb{R}]_B^+ \longrightarrow [F \times \mathbb{R}]_B^+)_{t \in [0,1]}$ with*

$$N_0 = n \wedge \mathrm{id}_{\underline{\mathbb{R}}} \text{ and } N_t(E \times \mathbb{R}) \subset F \times \mathbb{R} \text{ for all } t \neq 0. \tag{3.119}$$

In particular, $N_1 = (N_1|_{E \times \mathbb{R}})^+$.

Furthermore, if $n(E) \subset F$, then $N_t(E \times \mathbb{R}) \subset F \times \mathbb{R}$ for all $t \in [0,1]$.

Proof. The first statement is a reformulation of the previous proposition. If $n(E) \subset F$, then

$$\left(n \wedge \mathrm{id}_{\underline{\mathbb{R}}}\right)(E \times \mathbb{R}) \subset F \times \mathbb{R}. \tag{3.120}$$

This proves the second statement because $N_0 = n \wedge \mathrm{id}_{\underline{\mathbb{R}}}$. □

Now we are able to prove that the natural transformation $+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}_\Omega^+$ is stably bijective.

Proof of Theorem 3.3.2. Let $n : E_B^+ \longrightarrow F_B^+$ be an Ω-map. By Corollary 3.3.13, there exists an Ω-homotopy $N_t : [E \times \mathbb{R}]_B^+ \longrightarrow [F \times \mathbb{R}]_B^+$, such that $N_0 = n \wedge \mathrm{id}_{\mathbb{R}}$ and $N_1 = (N_1|_{E \times \mathbb{R}})^+$. Hence the transformation is stably surjective.

Let $m_0, m_1 : E \longrightarrow F$ be Ω-maps and let $n_s : E_B^+ \longrightarrow F_B^+$ $s \in [0,1]$ be an Ω-homotopy between $m_0^+ = n_0$ and $m_1^+ = n_1$. We apply Corollary 3.3.13 to the homotopy n_s, viewed as an Ω-map between $E_B^+ \times [0,1]$ and $F_B^+ \times [0,1]$ over the base $B \times [0,1]$. We obtain an Ω-homotopy $N_{s,t} : [E \times \mathbb{R}]_B^+ \longrightarrow$

$[F \times \mathbb{R}]_B^+$, such that $N_{s,0} = n_s \wedge \mathrm{id}_{\underline{\mathbb{R}}}$ and $N_{s,1}(E \times \mathbb{R}) \subset F \times \mathbb{R}$ for all $s \in [0,1]$. As a consequence we obtain a string of Ω-homotopies

$$m_0^+ \wedge \mathrm{id}_{\underline{\mathbb{R}}} = n_0 \wedge \mathrm{id}_{\underline{\mathbb{R}}} = N_{0,0} \simeq N_{0,1} \simeq N_{1,1} \simeq N_{1,0} = m_1^+ \wedge \mathrm{id}_{\underline{\mathbb{R}}}\,. \tag{3.121}$$

The first and the last homotopy is $N_{0,t}$ and $N_{1,t}$, respectively. Because $n_i(E) \subset F$, we have $N_{i,t}(E \times \mathbb{R}) \subset F \times \mathbb{R}$ for $i = 0, 1$. The middle homotopy is $N_{s,1}$ and it has this property by construction. Therefore, the three homotopies restrict to $E \times \mathbb{R}$ and induce a Ω-homotopy between $m_0 \oplus \mathrm{id}_{\underline{\mathbb{R}}}$ and $m_1 \oplus \mathrm{id}_{\underline{\mathbb{R}}}$. Therefore, the natural transformation is stably injective. ☐

Corollary 3.3.14. *Let $\mathcal{E}$ and $\mathcal{F}$ be two infinite dimensional ρ-bundles. Then:*

$$\operatorname{colim} \mathfrak{p}_\Omega^+ = \operatorname{colim} \mathfrak{p}_\Omega = {}_\rho[\mathcal{E}, \mathcal{F}]_B^\Omega. \tag{3.122}$$

Proof. This follows from Proposition 2.9.7 and Theorem 3.1.11. ☐

3.4 Group valued cohomotopy invariants

We have given an interpretation of the set ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega$ of homotopy classes of Ω-Fredholm maps as colimit of the functor $\mathfrak{p}_\Omega$. In the previous section, we constructed a functor $\mathfrak{p}_\Omega^+$ and a stably bijective transformation between $\mathfrak{p}_\Omega$ and $\mathfrak{p}_\Omega^+$. This describes the set ${}_G[\mathcal{E},\mathcal{F}]_B^\Omega$ also as colimit of the functor $\mathfrak{p}_\Omega^+$. In this section we define a group valued functor π_Ω, very closely related to $\mathfrak{p}_\Omega^+$ and with the property that the colimits $\operatorname{colim} U\pi_\Omega = \operatorname{colim}\mathfrak{p}_\Omega^+$ are naturally identified. Here U : Group $\longrightarrow$ Set denotes the forgetful functor. In this way we obtain a natural map

$$ {}_G[\mathcal{E},\mathcal{F}]_B^\Omega \longrightarrow \operatorname{colim}\mathfrak{p}_\Omega = \operatorname{colim}\mathfrak{p}_\Omega^+ = \operatorname{colim} U\pi_\Omega \longrightarrow U\operatorname{colim}\pi_\Omega. \tag{3.123}$$

This map assigns to every homotopy class of Ω-maps μ an invariant $\{\mu\}$ in the group $\operatorname{colim}\pi_\Omega =: {}_\rho\mathbb{P}_\Omega(B)$. We will show that (after fixing the index of the Ω-maps) this map is surjective and that it is injective when $K_\rho^{-1}(B) = 0$. Here we let $K_\rho^{-1}(B) \subset KO^{-1}(B)\times K_G^{-1}(B)$ be the subgroup generated by $[W]\otimes K^{-1}(B)$ for all $W \in \rho\cap \operatorname{Irr}(G,\mathbb{C})$ and additionally by $KO^{-1}(B)$ when $\mathbb{R}\in\rho$.

After fixing a subset $\rho_0 \subset \rho$, a ρ_0-Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$, and a l-adapted subbundle $V \subset \mathcal{F}_0$, we obtain similarly a map

$$ {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_\rho\mathbb{P}_\Omega^{l,V}(B) \tag{3.124}$$

that associates with every l-framed Ω-map μ an invariant $\{\mu\}_l$ in the framed cohomotopy group ${}_\rho\mathbb{P}_\Omega^{l,V}(B)$. We go on to compare the framed with the unframed invariants and see that when $K_{\rho_0}^{-1}(B) = 0$, they are canonically identified.

We then give another description of the cohomotopy groups that is closer to the construction used by Okonek and Teleman in [34]. In the last part, we explain how Okonek and Teleman's, Bauer and Furuta's ([7],[5]) , and Svarc's ([42]) construction fit into our framework. We use these descriptions to produce a comparison map in the case $K^{-1}(B) = 0$.

In this section, we assume that $\mathbb{R}\in\rho$. All spaces will be fiberwise pointed G-spaces over the fixed compact base space B.

We start by discussing three functors:

$$ \mathfrak{p}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \text{Set},\ \ \mathfrak{p}_\Omega^+ : {}_\rho\mathcal{H}_B \longrightarrow \text{Set},\ \text{and}\ \pi_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \text{Set}\,. \tag{3.125}$$

The first two functors have already been introduced and discussed before. Let us recall how they act on objects: Let (E,F) be a pair of ρ-bundles. Then

$$ \mathfrak{p}_\Omega(E,F) = {}_G[E,F]_B^\Omega \text{ and } \mathfrak{p}_\Omega^+(E,F) = {}_G[E_B^+,F_B^+]_B^\Omega. \tag{3.126}$$

Before defining the third functor we introduce an additional notation: Given a G-space X over B, we set $X(\Omega) := \{x\in X \mid G_x \in \Omega\}$ and we put furthermore $X/\Omega := X/_B X(\Omega)$. Let (E,F) be an object in ${}_\rho\mathcal{H}_B$. We write

$$ \pi_\Omega(E,F) := {}_G\pi_B^0\left(E_B^+/\Omega, F_B^+\right) \tag{3.127}$$

for the set of homotopy classes of pointed G-equivariant maps from E_B^+/Ω to F_B^+. Given a morphism $f = (i_E, i_F, \tilde E, \tilde F, \tau) : (E,F) \longrightarrow (E',F')$ there is a natural map

$$ \pi_\Omega(f) : {}_G\pi_B^0\left(E_B^+/\Omega, F_B^+\right) \longrightarrow {}_G\pi_B^0\left(E_B'^+/\Omega, F_B'^+\right), \tag{3.128}$$

defined as follows: Let $m : E_B^+/\Omega \longrightarrow F_B^+$ be a given (fiberwise pointed) G-map. Let $c_\Omega : E_B^+ \longrightarrow E_B^+/\Omega$ denote the contraction map. The map

$$ \left(i_F^+\circ m\circ c_\Omega\circ\left(i_E^{-1}\right)^+\right)\wedge\tau^+ : E_B'^+ = i_E(E)_B^+\wedge_B \tilde E_B^+ \longrightarrow i_F(F)_B^+\wedge_B\tilde F_B^+ = F_B'^+ \tag{3.129}$$

contracts the subspace $E_B'^+(\Omega)$ by construction: Recall that the projection map $E' = i_E(E)\oplus\tilde E \longrightarrow i_E(E)\cong E$ maps $E'(\Omega)$ to $E(\Omega)$. Therefore we obtain an induced map

$$ f_* m : E_B'^+/\Omega \longrightarrow F_B'^+. \tag{3.130}$$

Mapping $[m] \in {}_G\pi^0_B(E^+_B/\Omega, F^+_B)$ to $[f_* m] =: \pi_\Omega(f)([m]) \in {}_G\pi^0_B(E'^+_B/\Omega, F'^+_B)$ defines the map $\pi_\Omega(f)$. This makes π_Ω into a functor.

We have already observed that there is a natural transformation $+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}^+_\Omega$ which in fact is stably bijective: this was Theorem 3.3.2. There is a natural transformation $c_\Omega : \pi_\Omega \longrightarrow \mathfrak{p}^+_\Omega$ that for a given object (E, F) is defined by composing on the right with c_Ω (we refrain from introducing a new symbol).

Lemma 3.4.1. *Let $\varepsilon > 0$. There is a natural G-homotopy equivalence of pairs*

$$\big(F^+_B, F^+_B - \mathring{D}_\varepsilon(F)\big) \simeq_B \big(F^+_B, \infty_B\big). \tag{3.131}$$

Proof. There is a natural inclusion $I_\varepsilon : \big(F^+_B, \infty_B\big) \subset \big(F^+_B, F^+_B - \mathring{D}_\varepsilon(F)\big)$. To define its homotopy inverse, consider the fiberwise homotopy $\pi^\varepsilon : [0,1] \times F^+_B \longrightarrow F^+_B$, $(t, f) \mapsto \pi^\varepsilon_t(f)$, defined as follows: Let $t \in [0,1]$ and let $f \in F^+_b$ for some $b \in B$. We put:

$$\pi^\varepsilon_t(f) := \begin{cases} \frac{\varepsilon f}{\varepsilon - t|f|} & \text{when } t|f| < \varepsilon \\ \infty_b & \text{else.} \end{cases} \tag{3.132}$$

This homotopy has the following properties:

1. $\pi^\varepsilon_0 = \mathrm{id}$;
2. $\pi^\varepsilon_1\big(F^+_B - \mathring{D}_\varepsilon(F)\big) = \infty_B$;
3. $\pi^\varepsilon_t(\infty_B) = \infty_B$ and $\pi^\varepsilon_t\big(F^+_B - \mathring{D}_\varepsilon(F)\big) \subset F^+_B - \mathring{D}_\varepsilon(F)$ for all $t \in [0,1]$.

It follows immediately from these properties that π^ε_1 is homotopy inverse to the inclusion I_ε. The only statement we need to prove is that π^ε is continuous. It is obviously continuous on the subsets

$$\{(t, e) \in [0,1] \times F \,|\, tf < \varepsilon\} \text{ and } \{(t, e) \in [0,1] \times F \,|\, tf \geq \varepsilon\}. \tag{3.133}$$

Let $(t_n, f_n) \in [0,1] \times F^+_B$ be a sequence converging to $(t, f) \in [0,1] \times F^+_B$. Consider the expression:

$$\left|\frac{\varepsilon f_n}{\varepsilon - t_n|f_n|}\right| = \left|\frac{\varepsilon}{\frac{\varepsilon}{|f_n|} - t_n}\right|. \tag{3.134}$$

First assume that $|f| < \infty$. We need only to consider the case $t|f| = \varepsilon$. Furthermore, we can assume that $t_n|f_n| < \varepsilon$ and $f_n \neq 0$ for all $n \in \mathbb{N}$. Then $\frac{\varepsilon}{|f_n|} - t_n \to 0$ and from equation (3.134) we deduce that $\frac{\varepsilon f_n}{\varepsilon - t_n|f_n|} \to \infty_b$. Now assume that $|f| = \infty$. If $t \neq 0$, there is nothing to prove. When $t = 0$, then in equation (3.134) we use that $t_n \to 0$ and $\frac{\varepsilon}{|f_n|} \to 0$, therefore $\frac{\varepsilon f_n}{\varepsilon - t_n|f_n|} \to \infty_b$. □

Lemma 3.4.2. *The natural transformation c_Ω is an isomorphism of functors.*

Proof. Fix an object $(E, F) \in {}_\rho\mathcal{H}_B$. We have to prove that the map

$${}_G\pi^0_B\big(E^+_B/\Omega, F^+_B\big) \longrightarrow {}_G[E^+_B, F^+_B]^\Omega_B,\ [m] \mapsto [m \circ c_\Omega] \tag{3.135}$$

is a bijection. Let $m : E^+_B \longrightarrow F^+_B$ be an Ω-map. There is $\varepsilon > 0$, such that $m\big(E^+_B(\Omega)\big) \subset F^+_B - \mathring{D}_\varepsilon(F)$. Therefore $\pi^\varepsilon_1 \circ m : E^+_B \longrightarrow F^+_B$ maps $E^+_B(\Omega)$ to ∞_B. Hence it descends to a map $\overline{\pi^\varepsilon_1 \circ m} : E^+_B/\Omega \longrightarrow F^+_B$. The family of maps $\pi^\varepsilon_t \circ m : E^+_B \longrightarrow F^+_B$ is an Ω-homotopy between

$$m = \pi^\varepsilon_0 \circ m \text{ and } \pi^\varepsilon_1 \circ m = \overline{\pi^\varepsilon_1 \circ m} \circ c_\Omega. \tag{3.136}$$

This proves surjectivity.

Now let $m_0, m_1 : E^+_B/\Omega \longrightarrow F^+_B$ be two maps, such that $m_0 \circ c_\Omega$ and $m_1 \circ c_\Omega$ are Ω-homotopic. Then there is an Ω-homotopy $m'_t : E^+_B \longrightarrow F^+_B$ with $m'_0 = m_0 \circ c_\Omega$ and $m'_1 = m_1 \circ c_\Omega$. It is an Ω-homotopy, therefore there is $\varepsilon > 0$, such that $m'_t(E^+_B(\Omega)) \subset F^+_B - \mathring{D}_\varepsilon(F)$ for all $t \in [0,1]$. As before, the composition $\pi^\varepsilon_1 \circ m'_t$ descends to a homotopy $\overline{\pi^\varepsilon_1 \circ m'_t} : E^+_B/\Omega \longrightarrow F^+_B$. Furthermore $\overline{\pi^\varepsilon_1 \circ m'_i} = m_i$ for $i = 0, 1$. This proves injectivity. □

Corollary 3.4.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be two infinite dimensional ρ-bundles. Then*

$$_G[\mathcal{E},\mathcal{F}]_B^\Omega = \operatorname{colim} \mathfrak{p}_\Omega = \operatorname{colim} \mathfrak{p}_\Omega^+ = \operatorname{colim} \pi_\Omega \tag{3.137}$$

Proof. This follows from Corollary 3.3.14 and Lemma 3.4.2 together with Lemma 2.9.5. □

Recall that the homotopy set $_G\pi_B^0(X,Y)$ is a group whenever X is a cogroup-like space ([44, p.129] and [13, p. 64-65 and p.178]). Furthermore, when $X = \underline{S}^1 \wedge_B X'$, then it is cogroup like ([13, p. 64]). Let now $_\rho\mathcal{H}_B^* \subset {_\rho\mathcal{H}_B}$ be the full subcategory of objects $(E,F) \in \operatorname{ob} {_\rho\mathcal{H}_b}$ with the property that E has a trivial real summand, i.e. is of the form $E = E' \oplus \underline{\mathbb{R}}$ for some ρ-bundle E'.

Remark 3.4.4. Let E be a finite dimensional ρ-bundle with trivial real summand. Then $E_B^+/_B\Omega$ is cogroup like.

Proof. When $\Omega = \emptyset$, then $E_B^+/\Omega = E_B^+ = \underline{S}^1 \wedge_B [E']_B^+$. When $\Omega \neq \emptyset$, then $\Omega \ni G$, and thus $E_\mathbb{R} \subset E(\Omega)$. If follows from the next Lemma (take $E_1 = E'$ and $E_2 = \underline{\mathbb{R}}$) that there is a natural isomorphism $E_B^+/\Omega \cong \underline{S}^1 \wedge_B [E']_B^+/\Omega$. □

Lemma 3.4.5. *Let $E = E_1 \oplus E_2$ be a direct sum of two ρ-bundles. Then*

1. *There is a natural isomorphism $\big[E_1 \oplus E_2\big]_B^+/_B\big([E_1]_B^+(\Omega) \wedge_B [E_2]_B^+\big) \xrightarrow{\approx} \big([E_1]_B^+/\Omega\big) \wedge_B [E_2]_B^+$.*
2. *There is a natural contraction map $c_{E,E_1} : E_B^+/\Omega \longrightarrow [E_1]_B^+/\Omega \wedge_B [E_2]_B^+$.*

Proof. Let $c_1 : [E_1]_B^+ \longrightarrow [E_1]_B^+/\Omega$ denote the contraction map. We obtain an induced map

$$f := c_1 \wedge \mathrm{id} : [E_1 \oplus E_2]_B^+ = [E_1]_B^+ \wedge_B [E_2]_B^+ \longrightarrow [E_1]_B^+/\Omega \wedge_B [E_2]_B^+. \tag{3.138}$$

The map f is surjective and contracts the subspace $[E_1]_B^+(\Omega) \wedge_B [E_2]_B^+$. Hence it descends to a bijective map

$$\varphi : \big[E_1 \oplus E_2\big]_B^+/_B\big([E_1]_B^+(\Omega) \wedge_B [E_2]_B^+\big) \longrightarrow \big([E_1]_B^+/_B[E_1]_B^+(\Omega)\big) \wedge_B [E_2]_B^+. \tag{3.139}$$

Since its source is fiberwise compact, the map φ is a fiberwise homeomorphism.

There is an inclusion $\big([E_1]_B^+ \wedge_B [E_2]_B^+\big)(\Omega) \subset [E_1]_B^+(\Omega) \wedge_B [E_2]_B^+$. Using the identification from the first part we obtain the contraction map $c_{E,E_1} : E_B^+/\Omega \longrightarrow [E_1]_B^+/\Omega \wedge_B [E_2]_B^+$. □

Lemma 3.4.6. *The inclusion of the full subcategory $_\rho\mathcal{H}_B^* \subset {_\rho\mathcal{H}_B}$ is strongly cofinal.*

Proof. For every object (E,F), there is a morphism $(E,F) \longrightarrow (E \oplus \underline{\mathbb{R}}, F \oplus \underline{\mathbb{R}})$. Therefore, we need to check the second axiom: Let $f_l = (i_{E,l}, i_{F,l}, \tilde{E}_l, \tilde{F}_l, \tau_l) : (E,F) \longrightarrow (E_i,F_i)$ $l = 1,2$ be two morphisms with $(E_i,F_i) \in {_\rho\mathcal{H}_B^*}$. It follows from Corollary 3.1.3 that there are morphisms $\hat{f}_l : (E_l,F_l) \longrightarrow (\hat{E},\hat{F})$, such that the diagram

$$\begin{array}{ccc} (E,F) & \xrightarrow{f_1} & (E_1,F_1) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \hat{f}_1} \\ (E_2,F_2) & \xrightarrow[\hat{f}_2]{} & (\hat{E},\hat{F}) \end{array} \tag{3.140}$$

commutes. Furthermore, with E_1 also $\hat{E}$ has a trivial real summand. □

In particular, the restriction of the functor π_Ω to the subcategory $_\rho\mathcal{H}_B^*$ admits a colimit which is canonically identified with $\operatorname{colim} \pi_\Omega$. After restriction to the subcategory $_\rho\mathcal{H}_B^*$, the functor π_Ω naturally takes values in Group. We use the same symbol for this induced functor:

$$\pi_\Omega : {_\rho\mathcal{H}_B^*} \longrightarrow \text{Group}\,. \tag{3.141}$$

It is this functor that we want to study in the sequel. Let $U : \text{Group} \longrightarrow \text{Set}$ be the forgetful functor that associates with a group its underlying set. To avoid confusion we will always write $U\pi_\Omega$ when we mean the Set-valued functor. Before going on let us repeat what we already know:

Corollary 3.4.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be two infinite dimensional ρ-bundles. Then there is a natural identification ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B = \mathrm{colim}(U\pi_\Omega)$.*

We now want to construct a colimit for the group valued functor π_Ω. Let $\mathcal{E},\mathcal{F}$ be two infinite dimensional ρ-bundles. Let ${}_\rho\mathcal{H}_B(\mathcal{E},\mathcal{F})^* \subset {}_\rho\mathcal{H}^*_B$ be the full subcategory whose objects (E,F) are subbundles $E \subset \mathcal{E}$ and $F \subset \mathcal{F}$, respectively.

Lemma 3.4.8. *The inclusion ${}_\rho\mathcal{H}_B(\mathcal{E},\mathcal{F})^* \subset {}_\rho\mathcal{H}^*_B$ is an equivalence of categories.*

Proof. We only need to prove that the inclusion functor is essentially surjective, because by definition it is bijective on morphisms: Let $(E,F) \in \mathrm{ob}\,{}_\rho\mathcal{H}_B$ be a pair of ρ-bundles. Choose embeddings $i_E : E \hookrightarrow \mathcal{E}$ and $i_F : F \hookrightarrow \mathcal{F}$ (possible by Proposition 2.8.5). They induce an isomorphism $f : (E,F) \xrightarrow{\cong} (i_E(E), i_F(F))$. The latter object is an object in the category ${}_\rho\mathcal{H}_B(\mathcal{E},\mathcal{F})^*$. □

Corollary 3.4.9. *Every functor $F : {}_\rho\mathcal{H}^*_B \longrightarrow \mathrm{Group}$ admits a colimit.*

Proof. The category ${}_\rho\mathcal{H}^*_B$ is equivalent to a small category. The statement follows from Proposition 2.9.3 and Lemma 2.9.5. □

In particular, the functor π_Ω admits a colimit. We put ${}_\rho\mathbb{P}_\Omega(B) := \mathrm{colim}\,\pi_\Omega$.

Corollary 3.4.10. *Let $\mathcal{E},\mathcal{F}$ be two infinite dimensional ρ-bundles. Then there is a natural map*

$$ {}_G[\mathcal{E},\mathcal{F}]^\Omega_B \longrightarrow {}_\rho\mathbb{P}_\Omega(B). \tag{3.142} $$

Proof. We have realized the set ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B$ as colimit of the functor $U\pi_\Omega$ and by definition ${}_\rho\mathbb{P}_\Omega(B) = \mathrm{colim}\,\pi_\Omega$. Hence there is a natural map ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B = \mathrm{colim}\,U\pi_\Omega \longrightarrow U\,\mathrm{colim}\,\pi_\Omega = {}_\rho\mathbb{P}_\Omega(B)$. □

Let $x \in K_\rho(B)$. We denote with ${}_\rho\mathcal{H}^*_B(x)$ the full subcategory of ${}_\rho\mathcal{H}^*_B$ whose objects (E,F) have the property that $[E]-[F] = x \in K_\rho(B)$.

Lemma 3.4.11. *The category ${}_\rho\mathcal{H}^*_B$ has a decomposition into connected components*

$$ {}_\rho\mathcal{H}^*_B = \coprod_{x\in K_\rho(B)} {}_\rho\mathcal{H}^*_B(x). \tag{3.143} $$

For every $x \in K_\rho(B)$, the category ${}_\rho\mathcal{H}^*_B(x)$ is filtered up to automorphisms.

Proof. It is an immediate consequence of Lemma 3.1.1 that ${}_\rho\mathcal{H}^*_B = \coprod_{x\in K_\rho(B)} {}_\rho\mathcal{H}^*_B(x)$ and that ${}_\rho\mathcal{H}^*_B(x)$ is strongly connected (compare also Corollary 3.1.3). Now let $f_0, f_1 : (E,F) \longrightarrow (E',F')$ be two morphisms. We write $f_l = (i_{E,l}, i_{F,l}, \tilde{E}_l, \tilde{F}_l, \tau_l)$. By definition $i_{E,0}(E) \oplus \tilde{E}_0 = E' = i_{E,1}(E) \oplus \tilde{E}_1$. Therefore we may choose a ρ-bundle G and an isomorphism $\varphi : \tilde{E}_0 \oplus G \xrightarrow{\cong} \tilde{E}_1 \oplus G$ (for instance $G = E$). Then we define automorphisms

1. $e := (i_{E,1} \circ i^{-1}_{E,0}) \oplus \varphi : E' \oplus G = i_{E,0}(E) \oplus \tilde{E}_0 \oplus G \xrightarrow{\cong} i_{E,1}(E) \oplus \tilde{E}_1 \oplus G = E' \oplus G$.

2. $f := (i_{F,1} \circ i^{-1}_{F,0}) \oplus \big((\tau_1 \oplus \mathrm{id}_G) \circ \varphi \circ (\tau_0^{-1} \oplus \mathrm{id}_G)\big) : F' \oplus G = i_{F,0}(F) \oplus \tilde{F}_0 \oplus G \xrightarrow{\cong} i_{F,1}(F) \oplus \tilde{F}_1 \oplus G = F' \oplus G$.

Let $i : (E',F') \longrightarrow (E' \oplus G, F' \oplus G)$ be the natural morphism. Then it is clear from the construction that $(e,f) \circ i \circ f_0 = i \circ f_1$. Therefore the category ${}_\rho\mathcal{H}^*_B(x)$ is filtered up to automorphisms. □

For every $x \in K_\rho(B)$, the functor π_Ω restricts to a functor $\pi_\Omega(x) : {}_\rho\mathcal{H}^*_B(x) \longrightarrow \mathrm{Group}$ whose colimit we denote by ${}_\rho\mathbb{P}_\Omega(B,x)$. By Lemma 2.9.21, there is a natural decomposition

$$ {}_\rho\mathbb{P}_\Omega(B) = \oplus_{x\in K_\rho(B)} \mathbb{P}_\Omega(B,x). \tag{3.144} $$

Furthermore, we obtain the following result:

Corollary 3.4.12. *Let $x \in K_\rho(B)$. Then the map ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B \longrightarrow {}_\rho\mathbb{P}_\Omega(B)$ induces a surjective map*

$$ {}_G[\mathcal{E},\mathcal{F}]^\Omega_B(x) \longrightarrow {}_\rho\mathbb{P}_\Omega(B,x). \tag{3.145}$$

Proof. The restriction of the map ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B \longrightarrow {}_\rho\mathbb{P}_\Omega(B,x)$ to the subset ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B(x) \subset {}_G[\mathcal{E},\mathcal{F}]^\Omega_B$ takes values in ${}_\rho\mathbb{P}_\Omega(B,x) \subset {}_\rho\mathbb{P}_\Omega(B)$. This restriction is given by the natural map

$$ {}_G[\mathcal{E},\mathcal{F}]^\Omega_B(x) = \operatorname{colim} U\pi_\Omega(x) \longrightarrow \operatorname{colim} \pi_\Omega(x) = {}_\rho\mathbb{P}_\Omega(B,x). \tag{3.146}$$

The category ${}_\rho\mathcal{H}^*_B(x)$ is filtered up to automorphisms, in particular it is strongly connected (Lemma 3.4.11). Therefore the map above is surjective (proof of Proposition 2.9.26). □

Proposition 3.4.13. *Assume that $K^{-1}_\rho(B) = 0$. Then the map*

$$ {}_G[\mathcal{E},\mathcal{F}]^\Omega_B(x) \longrightarrow {}_\rho\mathbb{P}_\Omega(B,x) \tag{3.147}$$

is a bijection.

Proof. We have ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B(x) = \operatorname{colim} U\pi_\Omega(x)$ and ${}_\rho\mathbb{P}_\Omega(B,x) = \operatorname{colim}\pi_\Omega(x)$. The category ${}_\rho\mathcal{H}^*_B(x)$ is filtered up to automorphisms by Lemma 3.4.11. We now prove that these automorphisms act stably trivially on the functor $\pi_\Omega(x)$: Let $a : (E,F) \longrightarrow (E,F)$ be an automorphism of an object $(E,F) \in \operatorname{ob} {}_\rho\mathcal{H}^*_B(x)$. The automorphism a is given by automorphisms $e : E \xrightarrow{\cong} E$ and $f : F \xrightarrow{\cong} F$. Since $K^{-1}(B) = 0$, there is a ρ-bundle G, such that both $e \oplus \mathrm{id}_G$ and $f \oplus \mathrm{id}_G$ are isotopic to the identity ([21, Lemma 3.7]). Let $i : (E,F) \longrightarrow (E \oplus G, F \oplus G)$ be the natural morphism. Then $\pi_\Omega(x)\,(i \circ a) = \pi_\Omega(x)\,(i)$. Therefore the automorphisms of ${}_\rho\mathcal{H}^*_B(x)$ act stably trivially on the functor $\pi_\Omega(x)$. By Lemma 2.9.19, the category ${}_\rho\mathcal{H}^*_B(x)$ is $\pi_\Omega(x)$-filtered. The statement then follows from Proposition 2.9.26. □

The group ${}_\rho\mathbb{P}_\Omega(B)$ is defined as a colimit and thus characterized by a universal property. Recall that ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B = \operatorname{colim} U\pi_\Omega$. We write

$$ p : U\pi_\Omega \longrightarrow {}_G[\mathcal{E},\mathcal{F}]^\Omega_B \tag{3.148}$$

for the corresponding natural transformation.

Corollary 3.4.14. *Let A be a group and $a : {}_G[\mathcal{E},\mathcal{F}]^\Omega_B \longrightarrow A$ a map, such that for every $(E,F) \in {}_\rho\mathcal{H}^*_B$, the map*

$$ a \circ p_{(E,F)} : \pi_\Omega(E,F) = {}_G\pi^0_B\left(E^+_B/\Omega, F^+_B\right) \longrightarrow A \tag{3.149}$$

is a morphism of groups. Then there is a unique morphism of groups $\tilde{a} : {}_\rho\mathbb{P}_\Omega(B) \longrightarrow A$, such that the diagram

$$\begin{array}{ccc} {}_G[\mathcal{E},\mathcal{F}]^\Omega_B & \longrightarrow & {}_\rho\mathbb{P}_\Omega(B) \\ & \searrow^{a} & \downarrow \tilde{a} \\ & & A \end{array} \tag{3.150}$$

commutes.

Proof. By the universal property of ${}_\rho\mathbb{P}_\Omega(B)$, there is a unique morphism $a : {}_\rho\mathbb{P}_\Omega(B) \longrightarrow A$, such that for every $(E,F) \in {}_\rho\mathcal{H}^*_B$, the diagram

$$\begin{array}{ccc} \pi_\Omega(E,F) & \longrightarrow & {}_\rho\mathbb{P}_\Omega(B) \\ & \searrow^{a \circ p_{(E,F)}} & \downarrow \tilde{a} \\ & & A \end{array} \tag{3.151}$$

commutes. But since ${}_G[\mathcal{E},\mathcal{F}]^\Omega_B = \operatorname{colim} U\pi_\Omega$, this is equivalent to the statement that the diagram

$$\begin{array}{ccc} {}_G[\mathcal{E},\mathcal{F}]^\Omega_B & \longrightarrow & {}_\rho\mathbb{P}_\Omega(B) \\ & \searrow^{a} & \downarrow \tilde{a} \\ & & A \end{array} \tag{3.152}$$

commutes. □

3.4.1 Framed cohomotopy groups

Throughout this section, we fix a subset $\rho_0 \subset \rho$ and we put $\rho_* = \rho - \rho_0$. For simplicity, we assume that $\mathbb{R} \in \rho_0$. Furthermore, we fix a ρ_0-Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and a l-adapted subbundle $V \subset \mathcal{F}_0$. We introduced the homotopy set ${}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}$ of l-framed Ω-maps. Furthermore, we associated with the data (l, V) a functor

$$\mathfrak{p}_\Omega^{l,V} = \mathfrak{p}_\Omega^{(l^{-1}(V),V)} : {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B \longrightarrow \text{Set}, \tag{3.153}$$

depending on the pair $(l^{-1}(V), V)$, and a natural transformation

$$p : \mathfrak{p}_\Omega^{l,V} \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}, \tag{3.154}$$

depending also on the embeddings $V \subset \mathcal{F}_0$ and $l^{-1}(V) \subset \mathcal{E}_0$. In Corollary 3.2.5, we proved that this natural transformation is a colimit.

Consider now the natural functor

$$I^{l,V} : {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B \longrightarrow {}_\rho\mathcal{H}_B^*, \ (W, (E, F)) \mapsto (l^{-1}(V) \oplus W \oplus E, \ V \oplus W \oplus F). \tag{3.155}$$

Recall that by definition $\mathfrak{p}_\Omega^{l,V} = \mathfrak{p}_\Omega \circ I^{l,V}$. It is therefore natural to put $\pi_\Omega^{l,V} := \pi_\Omega \circ I^{l,V}$. As before, we define the full subcategory ${}_{\rho_0}\mathcal{C}_B^*(\mathcal{E}_0) \times {}_{\rho_*}\mathcal{H}_B(\mathcal{E}_*, \mathcal{F}_*)$ of ${}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B$ to consist of those objects $(W, (E, F))$ with $W \subset \mathcal{E}_0$, $E \subset \mathcal{E}_*$, and $F \subset \mathcal{F}_*$.

Lemma 3.4.15. *The inclusion ${}_{\rho_0}\mathcal{C}_B^*(\mathcal{E}_0) \times {}_{\rho_*}\mathcal{H}_B(\mathcal{E}_*, \mathcal{F}_*) \longrightarrow {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B$ is an equivalence of categories.*

In particular, every functor $F : {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B \longrightarrow \text{Group}$ admits a colimit.

Proof. This is proven exactly as Lemma 3.4.8 and Corollary 3.4.9. □

We put ${}_\rho\mathbb{P}_\Omega^{l,V}(B) := \operatorname{colim} \pi_\Omega^{l,V}$. Our next step is to establish a map from the set of homotopy classes of framed maps ${}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}$ to the group ${}_\rho\mathbb{P}_\Omega^{l,V}(B)$. We do this by proving that ${}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}$ realizes the colimit of the functor $U\pi_\Omega^{l,V}$. This will be achieved by reducing to the unframed case.

Lemma 3.4.16. *There is a stably bijective natural transformation*

$$\tau : \mathfrak{p}_\Omega^{l,V} \longrightarrow U\pi_\Omega^{l,V}. \tag{3.156}$$

Proof. Recall that $\mathfrak{p}_\Omega^{l,V} = \mathfrak{p}_\Omega \circ I^{l,V}$. By Theorem 3.1.11, there is stably bijective natural transformation $+ : \mathfrak{p}_\Omega \longrightarrow \mathfrak{p}_\Omega^+$. The transformation $+$ induces a natural transformation $\mathfrak{p}_\Omega \circ I^{l,V} \longrightarrow \mathfrak{p}_\Omega^+ \circ I^{l,V}$. This natural transformation is still stably bijective: In fact the proof of Theorem 3.1.11 holds also in this case. The point is that in that proof we stabilize by adding a trivial real summand $\underline{\mathbb{R}}$ and this stabilization process is possible in the category ${}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B$.

By Lemma 3.4.2, there is an isomorphism of functors $c_\Omega : \mathfrak{p}_\Omega^+ \xrightarrow{\cong} U\pi_\Omega$. It induces an isomorphism of functors $\mathfrak{p}_\Omega^+ \circ I^{l,V} \xrightarrow{\cong} U\pi_\Omega \circ I^{l,V}$. By composition, we obtain a stably bijective natural transformation

$$\mathfrak{p}_\Omega^{l,V} = \mathfrak{p}_\Omega \circ I^{l,V} \longrightarrow U\pi_\Omega \circ I^{l,V} = U\pi_\Omega^{l,V}. \tag{3.157}$$

□

Corollary 3.4.17. *There is a natural bijection*

$${}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l} \xrightarrow{\hat{=}} \operatorname{colim} U\pi_\Omega^{l^{-1}(V),V}. \tag{3.158}$$

Proof. We have constructed a natural transformation $p : \mathfrak{p}_\Omega^{l^{-1}(V),V} \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l}$ and proven that it is a colimit (Theorem 3.2.4). By the previous lemma, there is a stably bijective natural transformation $\mathfrak{p}_\Omega^{l^{-1}(V),V} \longrightarrow U \circ \pi_\Omega^{l^{-1}(V),V}$. It induces a bijection between the respective colimits. Therefore:

$${}_G[\mathcal{E}, \mathcal{F}]_B^{\Omega,l} \xrightarrow{\hat{=}} \operatorname{colim} \mathfrak{p}_\Omega^{l^{-1}(V),V} \xrightarrow{\hat{=}} \operatorname{colim} U\pi_\Omega^{l^{-1}(V),V}. \tag{3.159}$$

□

Corollary 3.4.18. *There is a natural map*

$$ {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_\rho\mathbb{P}_\Omega^{V,l}(B). \tag{3.160}$$

Proof. This follows from the previous corollary as in the unframed case (Corollary 3.4.10). □

Remember that there is an index map

$$ \mathrm{ind}_\rho : {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega} \longrightarrow K_\rho(B). \tag{3.161}$$

Let now $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ be a l-framed Ω-map and let $[\mu]_l \in {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}$ be its l-framed homotopy class. Forgetting that it is l-framed, we obtain a homotopy class $[\mu] \in {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega}$. By definition, its index is of the form $\mathrm{ind}_\rho([\mu]) = (\mathrm{ind}_\rho\, l, y) \in K_{\rho_0}(B) \times K_{\rho_*}(B)$. Thus, we see that there is an induced index map

$$ \mathrm{ind}_{\rho_*} : {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow K_{\rho_*}(B). \tag{3.162}$$

Given $y \in K_{\rho_*}(B)$, we put ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}(y) := \mathrm{ind}_{\rho_*}^{-1}(y)$. There is a decomposition

$$ {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} = \coprod_{y\in K_{\rho_*}(B)} {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}(y). \tag{3.163}$$

As in the unframed case, we now give a functorial interpretation of this decomposition.

Lemma 3.4.19.

1. *There is a decomposition into connected components*

$$ {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B = \coprod_{y\in K_{\rho_*}(B)} {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B(y). \tag{3.164}$$

2. *The category ${}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B(y)$ is filtered up to automorphisms.*

Proof. Using the same argument as in the proof of Lemma 3.4.11 one proves that the category ${}_{\rho_0}\mathcal{C}_B^*$ is filtered up to automorphisms. Then it follows that ${}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B(y)$ is filtered up to automorphisms. The first statement follows immediately from the second and from Lemma 3.4.11. □

Corollary 3.4.20. *Let $y \in K_{\rho_*}(B)$. The functor $\pi_\Omega^{l,V}$ restricts to a functor*

$$ \pi_\Omega^{l,V}(y) : {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B(y) \longrightarrow \mathrm{Group}\,. \tag{3.165}$$

Furthermore,

1. $\operatorname{colim} \pi_\Omega^{l,V} = \oplus_{y\in K_{\rho_*}(B)} \operatorname{colim} \pi_\Omega^{l,V}(y)$*;*
2. ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}(y) = \operatorname{colim} U\pi_\Omega^{l,V}(y)$*.*

Proof. These are immediate consequences of Lemma 2.9.21 and Theorem 3.2.4. □

We put ${}_\rho\mathbb{P}_\Omega^{l,V}(B,y) := \operatorname{colim} \pi_\Omega^{l,V}(y)$. Then

$$ {}_\rho\mathbb{P}_\Omega^{l,V}(B) = \oplus_{y\in K_{\rho_*}(B)} {}_\rho\mathbb{P}_\Omega^{l,V}(B,y). \tag{3.166}$$

Similarly as in the unframed case we conclude:

Corollary 3.4.21. *Let $y \in K_{\rho_*}(B)$. The map ${}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l} \longrightarrow {}_\rho\mathbb{P}_\Omega^{l,V}(B)$ induces a surjective map*

$$ {}_G[\mathcal{E},\mathcal{F}]_B^{\Omega,l}(y) \longrightarrow {}_\rho\mathbb{P}_\Omega^{l,V}(B,y). \tag{3.167}$$

Proof. The proof is the same as for Corollary 3.4.12. □

Proposition 3.4.22. *Let $y \in K_{\rho_*}(B)$. Assume that $K^{-1}_{\rho_*}(B) = 0$. Then the map*

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \longrightarrow {}_\rho\mathbb{P}^{V,l}_\Omega(B, y) \tag{3.168}$$

is bijective.

Proof. As in the unframed case (Proposition 3.4.13) we claim that the automorphisms of the category ${}_{\rho_0}\mathcal{C}^*_B \times {}_{\rho_*}\mathcal{H}_B(y)$ act stably trivially on the functor $\pi^{l,V}_\Omega$. The same argument as in Proposition 3.4.13 proves that automorphisms of the form (id, a) with a an automorphism in ${}_{\rho_*}\mathcal{H}_B(y)$ act stably trivially on the functor $\pi^{l,V}_\Omega$. We need to prove that automorphisms of the form (a, id) with a an automorphism in ${}_{\rho_0}\mathcal{C}^*_B$ act stably trivially on $\pi^{l,V}_\Omega$. We do not loose any generality when we assume that $\rho_0 = \rho$. Therefore let $W \in \mathrm{ob}\, {}_{\rho_0}\mathcal{C}^*_B$ be a finite dimensional ρ_0-bundle and let $a : W \xrightarrow{\cong} W$ be an automorphism. Let $i : W \longrightarrow W \oplus W$ be the natural inclusion. We claim that $\pi^{l,V}_\Omega(i) = \pi^{l,V}_\Omega(i \circ a)$. To see this, we let $r_t : W \oplus W \xrightarrow{\cong} W \oplus W$ be the automorphism defined by the matrix $\begin{bmatrix} \cos t\ \mathrm{id}_W & -\sin t\ \mathrm{id}_W \\ \sin t\ \mathrm{id}_W & \cos t\ \mathrm{id}_W \end{bmatrix}$ for $t \in [0, \pi/2]$. Then put $a_t := r_t \circ (a \oplus \mathrm{id}_W) \circ r_t^{-1}$. Since a_t is a ρ_0-isotopy, we have $\pi^{l,V}_\Omega(a_0) = \pi^{l,V}_\Omega(a_{\pi/2})$. Now $a_0 = a \oplus \mathrm{id}_W$ and $a_{\pi/2} = \mathrm{id}_W \oplus a$. But observe that:

$$[a \oplus \mathrm{id}_W] \circ i = i \circ a : W \longrightarrow W \oplus W \text{ and } [\mathrm{id}_W \oplus a] \circ i = i : W \longrightarrow W \oplus W. \tag{3.169}$$

Therefore $\pi^{l,V}_\Omega(i \circ a) = \pi^{l,V}_\Omega(i)$ as claimed. □

The argument in the proof of the last proposition involving the rotation matrix is standard. See [13, Lemma II.3.2] and [34, Proposition 2.1].

The following statement has been obtained by Bauer ([5, Theorem 2.1]) in the case $\Omega = \emptyset$ and by Svarc in the case when additionally $G = \{e\}$ and $B = \{*\}$ (see [8, Theorem 5.3.20]).

Corollary 3.4.23. *When $\rho_0 = \rho$, then the map*

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B \longrightarrow {}_\rho\mathbb{P}^{V,l}_\Omega(B) \tag{3.170}$$

is a bijection.

Comparison of framed and unframed invariants

There is an obvious comparison map

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B. \tag{3.171}$$

We have proven in Proposition 3.2.3 that its image is the set $\mathrm{ind}^{-1}_\rho\left(\{\mathrm{ind}_{\rho_0} l\} \times K_{\rho_*}(B)\right)$. This amounts to saying that for any $y \in K_{\rho_*}(B)$, the induced map

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B\left((\mathrm{ind}_\rho l, y)\right) \tag{3.172}$$

is surjective.

Proposition 3.4.24. *Let $y \in K_{\rho_*}(B)$ and put $x := (\mathrm{ind}_\rho l, y) \in K_\rho(B)$.*

1. *There is a natural surjective morphism ${}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}_\Omega(B, x)$.*
2. *When $K^{-1}_{\rho_0}(B) = 0$, then the maps*

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \longrightarrow {}_G[\mathcal{E}, \mathcal{F}]^{\Omega}_B(x) \text{ and } {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}_\Omega(B, x) \tag{3.173}$$

are bijective.

Proof. By definition ${}_\rho\mathbb{P}^{l,V}_\Omega(B,y) = \operatorname{colim} \pi_\Omega\,(x) \circ I^{l,V}$ and ${}_\rho\mathbb{P}_\Omega(B,x) = \operatorname{colim} \pi_\Omega\,(x)$. Therefore there is an induced morphism

$$ {}_\rho\mathbb{P}^{l,V}_\Omega(B,y) = \operatorname{colim} \pi_\Omega\,(x) \circ I^{l,V} \longrightarrow \operatorname{colim} \pi_\Omega\,(x) = {}_\rho\mathbb{P}_\Omega(B,x). \tag{3.174} $$

It fits into a commutative diagram:

$$ \begin{array}{ccc} {}_G[\mathcal{E},\mathcal{F}]^{\Omega,l}_B(y) & \longrightarrow & {}_G[\mathcal{E},\mathcal{F}]_B(x) \\ \downarrow & & \downarrow \\ {}_\rho\mathbb{P}^{l,V}_\Omega(B,y) & \longrightarrow & {}_\rho\mathbb{P}_\Omega(B,x) \end{array} \tag{3.175} $$

The upper horizontal map is surjective by Proposition 3.2.3, and the two vertical maps are surjective by Corollary 3.4.12 and Corollary 3.4.21. Therefore the lower horizontal map is surjective too. (We will give another proof of this statement in Corollary 3.4.33.)

Now assume that $K^{-1}_{\rho_0}(B) = 0$. We prove the following statement: The functor $I^{l,V} : {}_{\rho_0}\mathcal{C}_B \times {}_{\rho_*}\mathcal{H}_B(y) \longrightarrow {}_\rho\mathcal{H}_B(x)$ is strongly $\pi_\Omega(x)$-cofinal. To simplify the notation we assume that $\rho_0 = \rho$. The general case does not present additional difficulties. We set $U := l^{-1}(V)$. Let $(U',V') \in {}_{\rho_0}\mathcal{H}_B(\operatorname{ind}_{\rho_0} l)$. Then $[U'] - [V'] = \operatorname{ind}_\rho(l) = [U] - [V]$. By Lemma 3.1.1, there is a diagram

$$ \begin{array}{ccccc} (U,V) & & & & (U',V') \\ & \searrow^{f} & & {}^{f'}\swarrow & \\ & & (U'',V'') & & \end{array} \tag{3.176} $$

in the category ${}_{\rho_0}\mathcal{H}_B(\operatorname{ind}_\rho(l))$. Let $f = (i_U, i_V, \tilde{U}, \tilde{V}, \tau)$. By Remark 3.1.2, the morphism f induces an isomorphism $\hat{f} : (U'',V'') \longrightarrow (U \oplus \tilde{U}, V \oplus \tilde{U})$. But clearly

$$ (U \oplus \tilde{U}, V \oplus \tilde{U}) = I^{l,V}(\tilde{U}). \tag{3.177} $$

Thus we obtain a morphism $\hat{f} \circ f' : (U',V') \longrightarrow I^{l,V}(\tilde{U})$. This proves that $I^{l,V}$ is weakly cofinal.

We still have to prove that for any two morphisms $f_j : (U',V') \longrightarrow I^{l,V}(W_j) = (U \oplus W_, V \oplus W_j)$ there is an object $W' \in \operatorname{ob}{}_{\rho_0}\mathcal{C}_B$ and there are morphisms $g_j : W_j \longrightarrow W'$, such that

$$ \pi_\Omega(x)\left(I^{l,V}(g_0) \circ f_0\right) = \pi_\Omega(x)\left(I^{l,V}(g_1) \circ f_1\right). \tag{3.178} $$

Because for any two objects W_0 and W_1, there are morphisms $W_0 \longrightarrow W_0 \oplus W_1 \longleftarrow W_1$, we can assume that $W_0 = W_1 = W$. As usual, we write $f_j = (i_{U',j}, i_{V',j}, \tilde{U}_j, \tilde{V}_j, \tau_j)$. We have seen in Remark 3.1.2 that the morphisms f_j naturally induce isomorphisms

$$ \hat{f}_j : (U \oplus W, V \oplus W) \longrightarrow (U' \oplus \tilde{U}_j, V' \oplus \tilde{U}_j). \tag{3.179} $$

From $[U'] + [\tilde{U}_0] = [U \oplus W] = [U'] + [\tilde{U}_1] \in K_{\rho_0}(B)$, it follows that $[\tilde{U}_0] = [\tilde{U}_1]$. Choose a ρ_0-bundle G and an isomorphism $\varphi : \tilde{U}_0 \oplus G \xrightarrow{\cong} \tilde{U}_1 \oplus G$. By construction, the following diagram commutes:

$$ \begin{array}{ccc} (U',V') & \xrightarrow{f_1} & (U \oplus W, V \oplus W) \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \hat{f}_1} \\ (U \oplus W, V \oplus W) & & (U' \oplus \tilde{U}_1, V' \oplus \tilde{U}_1) \\ \downarrow{\scriptstyle \hat{f}_0} & & \downarrow \\ (U' \oplus \tilde{U}_0, V' \oplus \tilde{U}_0) & & (U' \oplus \tilde{U}_1 \oplus G, V' \oplus \tilde{U}_1 \oplus G) \\ \downarrow & \nearrow_{\operatorname{id}_{(U',V')} \oplus \varphi} & \\ (U' \oplus \tilde{U}_0 \oplus G, V' \oplus \tilde{U}_0 \oplus G) & & \end{array} \tag{3.180} $$

The arrows without letter denote the natural morphisms. Furthermore, the diagram

$$\begin{array}{ccc} (U\oplus W, V\oplus W) & \longrightarrow & (U\oplus W\oplus G, V\oplus W\oplus G) \\ \downarrow \hat{f}_j & & \\ (U'\oplus \tilde{U}_j, V'\oplus \tilde{U}_j) & \swarrow \hat{f}_j\oplus \mathrm{id}_G & \\ \downarrow & & \\ (U'\oplus \tilde{U}_j\oplus G, V'\oplus \tilde{U}_j\oplus G) & & \end{array} \tag{3.181}$$

commutes for $j=0,1$. As a consequence, also the diagram

$$\begin{array}{ccc} (U', V') & \xrightarrow{f_1} & (U\oplus W, V\oplus W) \\ \downarrow f_0 & & \downarrow \\ (U\oplus W, V\oplus W) & & \\ \downarrow & & \\ (U\oplus W\oplus G, V\oplus W\oplus G) & \xrightarrow[\tilde{\varphi}]{} & (U\oplus W\oplus G, V\oplus W\oplus G) \end{array} \tag{3.182}$$

commutes. Here $\tilde{\varphi}$ is the automorphism defined as follows:

$$\tilde{\varphi} := \left(\hat{f}_1^{-1}\oplus \mathrm{id}_G\right)\circ (\mathrm{id}_{U',V'}\oplus\varphi)\circ\left(\hat{f}_0\oplus \mathrm{id}_G\right). \tag{3.183}$$

It consists of a pair of automorphisms $\tilde{\varphi}=(\tilde{\varphi}_0,\tilde{\varphi}_1)$ of $U\oplus W\oplus G$ and $V\oplus W\oplus G$, respectively. Now we use the assumption that $K^{-1}_{\rho_0}(B)=0$. It implies that there is a ρ_0-bundle H, such that both $\tilde{\varphi}_0\oplus \mathrm{id}_H$ and $\tilde{\varphi}_1\oplus \mathrm{id}_H$ are isotopic to the identity ([21, Lemma 3.7]). But this implies that $\pi_\Omega(x)(\varphi\oplus \mathrm{id}_H)=\pi_\Omega(x)(\mathrm{id})$.

As a consequence, the diagram

$$\begin{array}{ccc} (U', V') & \xrightarrow{f_1} & (U\oplus W, V\oplus W) \\ \downarrow f_0 & & \downarrow \\ (U\oplus W, V\oplus W) & \longrightarrow & (U\oplus W\oplus (G\oplus H), V\oplus W\oplus (G\oplus H)) \end{array} \tag{3.184}$$

is π_Ω-commutative. This proves that the functor $I^{l,V}$ is strongly $\pi_\Omega(x)$-cofinal and also $U\pi_\Omega(x)$-cofinal. As a consequence, it identifies $(U)\pi_\Omega(x)$-colimits. This proves the second statement. □

The groups ${}_\rho\mathbb{P}^{l,V}_\Omega(B)$ are characterized by a universal property similar to the unframed case.

Corollary 3.4.25. *Let A be a group and $a : {}_G[\mathcal{E},\mathcal{F}]^{\Omega,l}_B \longrightarrow A$ a map, such that for every $(W,(E,F))\in {}_{\rho_0}\mathcal{C}^*_B\times {}_{\rho_*}\mathcal{H}_B$, the map*

$$a\circ p_{(W,(E,F))} : \pi^{l,V}_\Omega(W,(E,F)) = {}_G\pi^0_B\left(\left[l^{-1}(V)\oplus W\oplus E\right]^+_B/\Omega, [V\oplus W\oplus F]^+_B\right)\longrightarrow A \tag{3.185}$$

is a morphism of groups. Then there is a unique morphism of groups $\tilde{a} : {}_\rho\mathbb{P}^{l,V}_\Omega(B)\longrightarrow A$, such that the diagram

$$\begin{array}{ccc} {}_G[\mathcal{E},\mathcal{F}]^{\Omega,l}_B & \longrightarrow & {}_\rho\mathbb{P}^{l,V}_\Omega(B) \\ & \searrow a & \downarrow \tilde{a} \\ & & A \end{array} \tag{3.186}$$

commutes.

Proof. The proof is the same as in the unframed case (Corollary 3.4.14). □

A different construction of colim π_Ω

We now explain a different description of the colimit colim π_Ω. It is closer to the construction used by Okonek and Teleman in [34].

Define a functor $S : {}_\rho\mathcal{C}_B \times {}_\rho\mathcal{H}^*_B \longrightarrow {}_\rho\mathcal{H}^*_B$ by putting $S(H,(E,F)) := (E \oplus H, F \oplus H)$. We define S on the level of morphisms in the obvious way.

Lemma 3.4.26. *The functor S identifies π_Ω-colimits.*

Proof. We show that the functor S induces an isomorphism of categories between the category of π_Ω-cocones and the category of $\pi_\Omega \circ S$-cocones. We need the following property of the functor π_Ω: Let $(H,(E,F)) \in {}_\rho\mathcal{C}_B \times {}_\rho\mathcal{H}^*_B$ be an object. Then $\pi_\Omega \circ S(H,(E,F)) = \pi_\Omega \circ S(0_B,(E \oplus H, F \oplus H)$. Let $i : (E,F) \longrightarrow (E \oplus H, F \oplus H)$ be the natural morphism. Then there are natural morphisms

- $f = (0, \mathrm{id}) : (0_B,(E \oplus H, F \oplus H)) \longrightarrow (H,(E \oplus H, F \oplus H))$;
- $g = (\mathrm{id}_H, i) : (H,(E,F)) \longrightarrow (H,(E \oplus H, F \oplus H))$.

We claim that $\pi_\Omega(S(f)) = \pi_\Omega(S(g)) : \pi_\Omega(E \oplus H, F \oplus H) \longrightarrow \pi_\Omega(E \oplus H \oplus H, F \oplus H \oplus H)$. The idea is rotating the two H components. Let $n : [E \oplus H]^+_B/\Omega \longrightarrow [F \oplus H]^+_B$ be a map. We write $n = n_F \wedge n_H$. We need to compare the following two maps:

1. $f_*n : [E \oplus H \oplus H]^+_B/\Omega \longrightarrow [F \oplus H \oplus H]^+_B, [e,h_1,h_2] \mapsto n_F(e,h_1) \wedge n_H(e,h_1) \wedge h_2$;
2. $g_*n : [E \oplus H \oplus H]^+_B/\Omega \longrightarrow [F \oplus H \oplus H]^+_B, [e,h_1,h_2] \mapsto n_F(e,h_2) \wedge h_1 \wedge n_H(e,h_2)$.

Let $r_t : [H \oplus H]^+_B \xrightarrow{\cong} [H \oplus H]^+_B$ be the isomorphism defined by the rotation matrix $\begin{bmatrix} \cos t \ \mathrm{id}_H & -\sin t \ \mathrm{id}_H \\ \sin t \ \mathrm{id}_H & \cos t \ \mathrm{id}_H \end{bmatrix}$. Then we consider the following homotopy:

$$n_t := \left(\mathrm{id}_{F^+_B} \wedge r_t\right) \circ f_*n \circ \left(\mathrm{id}_{E^+_B} \wedge r_t^{-1}\right) \text{ for } t \in [0, \pi/2]. \tag{3.187}$$

We have $n_0 = f_*n$ and $n_{\pi/2} = g_*n$, and therefore the homotopy classes of the maps f_*n and g_*n coincide.

Now we are able to define a push-forward functor

$$S_* : \mathrm{CC}(\pi_\Omega \circ S) \longrightarrow \mathrm{CC}(\pi_\Omega). \tag{3.188}$$

Let $a = \left(a_{(H,(E,F))} : \pi_\Omega(E \oplus H, F \oplus H) \longrightarrow A\right)_{H,(E,F)}$ be a $\pi_\Omega \circ S$-cocone. We are going to define a π_Ω-cocone S_*a. Let $(E,F) \in {}_\rho\mathcal{H}^*_B$. Then we put

$$(S_*a)_{(E,F)} := a_{(0,(E,F))} : \pi_\Omega(E,F) \longrightarrow A. \tag{3.189}$$

We claim that the functor S_* inverts the pull-back functor S^*. First we compare S^*S_*a to a: Let $(H,(E,F)) \in \mathrm{ob}\, {}_\rho\mathcal{C}_B \times {}_\rho\mathcal{H}^*_B$. By definition $(S^*S_*a)_{(H,(E,F))} = (S_*a)_{(E \oplus H, F \oplus H)} = a_{(0,(E \oplus H, F \oplus H))}$. Furthermore, by definition the diagrams

$$\begin{array}{ccc} \pi_\Omega(E \oplus H, E \oplus H) & \xrightarrow{\pi_\Omega(f)} & \pi_\Omega(E \oplus H \oplus H, F \oplus H \oplus H) \\ {\scriptstyle a_{(0,(E \oplus H, F \oplus H))}} \downarrow & \swarrow {\scriptstyle a_{(H,(E \oplus H, F \oplus H))}} & \\ A & & \end{array} \tag{3.190}$$

and

$$\begin{array}{ccc} \pi_\Omega(E \oplus H, E \oplus H) & \xrightarrow{\pi_\Omega(g)} & \pi_\Omega(E \oplus H \oplus H, F \oplus H \oplus H) \\ {\scriptstyle a_{(H,(E,F))}} \downarrow & \swarrow {\scriptstyle a_{(H,(E \oplus H, F \oplus H))}} & \\ A & & \end{array} \tag{3.191}$$

commute. Together with the equality $\pi_\Omega(f) = \pi_\Omega(g)$ this implies that $a_{(0,(E \oplus H, F \oplus H))} = a_{(H,(E,F))}$. Therefore $S^*S_*a = a$.

Now we start with a π_Ω-cocone $b = (b_{(E,F)})$ and we compare S_*S^*b with b. Let $(E,F) \in \mathrm{ob}\, {}_\rho\mathcal{H}^*_B$ be given. Then by definition $(S_*S^*b)_{(E,F)} = (S^*b)_{(0,(E,F))} = b_{(E,F)}$. Hence $S_*S^*b = b$. □

In the next step, we study the functor $\pi_\Omega \circ S : {}_\rho\mathcal{C}_B \times {}_\rho\mathcal{H}^*_B \longrightarrow {}_\rho\mathcal{H}^*_B$.

Proposition 3.4.27. *Let $(E, F) \in {}_\rho\mathcal{H}^*_B$. Then the induced functor*

$$\pi_\Omega \circ S_{(E,F)} : {}_\rho\mathcal{C}_B \longrightarrow \text{Group} \tag{3.192}$$

admits a colimit which is preserved by the forgetful functor.

Proof. The category ${}_\rho\mathcal{C}_B$ is equivalent to a small category. Therefore the functor admits a colimit. (See also the subsequent remark.) Furthermore, it is filtered up to automorphisms (see Lemma 3.4.19). We claim that the automorphisms of ${}_\rho\mathcal{C}_B$ act stably trivally on the functor $\pi_\Omega \circ S_{(E,F)}$. This is proven by the same argument as in the proof of Proposition 3.4.22. By Lemma 2.9.19, the category ${}_\rho\mathcal{C}_B$ is $\pi_\Omega \circ S_{(E,F)}$-filtered. The statement then follows from Corollary 2.9.24. □

Remark 3.4.28. We use the fact that the category ${}_\rho\mathcal{C}_B$ is equivalent to a small category to conclude that a colimit exists. Another approach is taken in [34]: Define a small category $\underline{\mathbb{N}}^{(\rho)}$ as follows: Its objects are families $(n_\chi)_{\chi\in\rho}$ of natural numbers $n_\chi \in \mathbb{N}$, all but finitely many zero. From an object (n_χ) to an object (n'_χ) there is by definition exactly one morphism when $n_\chi \leq n'_\chi$ for all $\chi \in \rho$, and none otherwise. This category is filtered. For each χ we fix an irreducible G-module W_χ with character χ. Mapping χ to $\underline{W_\chi}$ defines a functor

$$\theta^\rho_B : \underline{\mathbb{N}}^{(\rho)} \longrightarrow {}_\rho\mathcal{C}_B. \tag{3.193}$$

One can show that this functor is cofinal. Because the automorphisms of ${}_\rho\mathcal{C}_B$ act stably trivially on the functor $\pi_\Omega \circ S_{(E,F)}$ and because the category ${}_\rho\mathcal{C}_B$ is filtered up to automorphisms, the functor θ^ρ_B is strongly $\pi_\Omega \circ S_{(E,F)}$-cofinal. Therefore it identifies $\pi_\Omega \circ S_{(E,F)}$-colimits. This reduces the problem of constructing a colimit for a functor defined on ${}_\rho\mathcal{C}_B$ to constructing a colimit for a functor defined on a small and filtered category.

Proposition 3.4.29. *The functor $\pi_\Omega \circ S$ induces a functor*

$$\mathbb{p}_\Omega : {}_\rho\mathcal{H}^*_B \longrightarrow \text{Group},\ (E, F) \mapsto \operatorname{colim} \pi_\Omega \circ S_{(E,F)}. \tag{3.194}$$

This functor maps all morphisms to isomorphisms. In particular, it admits a colimit.

Proof. The functor $\pi_\Omega \circ S$ induces the functor $\mathbb{p}_\Omega$ by Proposition 2.9.20. We prove that this functor maps all morphisms to isomorphisms. Let $f = (i_E, i_F, \tilde{E}, \tilde{F}, \tau) : (E, F) \longrightarrow (E', F')$ be a morphism in the category ${}_\rho\mathcal{H}^*_B$. We show that $\mathbb{p}_\Omega(f)$ is an isomorphism. The morphism f induces a functor

$$f^* : (\pi_\Omega \circ S_{(E',F')} - \text{cocones}) \longrightarrow (\pi_\Omega \circ S_{(E,F)} - \text{cocones}). \tag{3.195}$$

This functor induces the morphism

$$\mathbb{p}_\Omega(f) : \mathbb{p}_\Omega(E, F) = \operatorname{colim} \pi_\Omega \circ S_{(E,F)} \longrightarrow \operatorname{colim} \pi_\Omega \circ S_{(E',F')} = \mathbb{p}_\Omega(E', F'). \tag{3.196}$$

We define an inverse functor

$$f_* : (\pi_\Omega \circ S_{(E,F)} - \text{cocones}) \longrightarrow (\pi_\Omega \circ S_{(E',F')} - \text{cocones}) \tag{3.197}$$

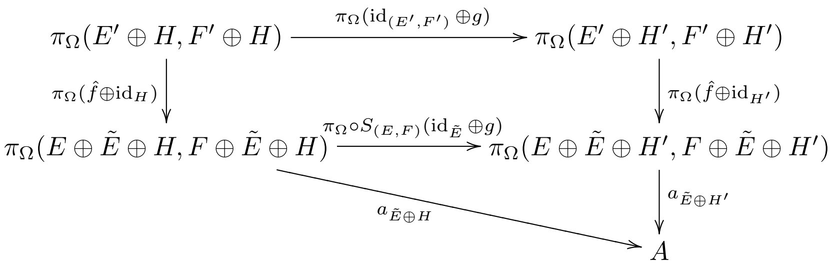
as follows: Let $a = (a_H : \pi_\Omega(E \oplus H, F \oplus H) \longrightarrow A)_H$ be a $\pi_\Omega \circ S_{(E,F)}$-cocone. We put

$$a'_H := a_{\tilde{E}\oplus H} \circ \pi_\Omega(\hat{f} \oplus \mathrm{id}_H) : \pi_\Omega(E' \oplus H, F' \oplus H) \longrightarrow \pi_\Omega(E \oplus \tilde{E} \oplus H, F \oplus \tilde{E} \oplus H) \longrightarrow A, \tag{3.198}$$

where $\hat{f} : (E', F') \longrightarrow (E\oplus\tilde{E}, F\oplus\tilde{E})$ is the isomorphism induced by f (Remark 3.1.2). Let $g : H \longrightarrow H'$ be a morphism in ${}_\rho\mathcal{C}_B$. The diagram

$$\begin{array}{ccc} (E' \oplus H, F' \oplus H) & \xrightarrow{\hat{f}\oplus\mathrm{id}_H} & (E \oplus \tilde{E} \oplus H, F \oplus \tilde{E} \oplus H) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{(E',F')}\oplus g} & & \Big\downarrow{\scriptstyle \mathrm{id}_{(E\oplus\tilde{E}, F\oplus\tilde{E})}\oplus g} \\ (E' \oplus H', F' \oplus H') & \xrightarrow[\hat{f}\oplus\mathrm{id}_{H'}]{} & (E \oplus \tilde{E} \oplus H', F \oplus \tilde{E} \oplus H') \end{array} \tag{3.199}$$

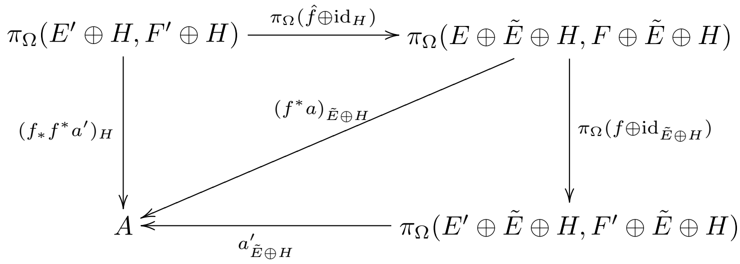
commutes. Observe that $\mathrm{id}_{(E\oplus\tilde{E},F\oplus\tilde{F})}\oplus g = S_{(E,F)}(\mathrm{id}_E \oplus g)$. Therefore also the following diagram commutes:



(3.200)

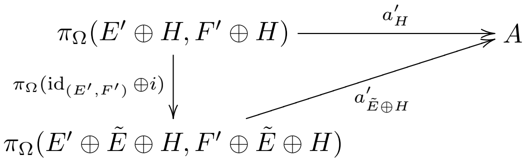
This shows that the family $a' := (a'_H)_H$ is indeed a $\pi_\Omega \circ S_{(E',F')}$-cocone. Now we prove that the two functors invert each other.

We start with a $\pi_\Omega \circ S_{(E',F')}$-cocone $a' = (a'_H)$. Then $(f_*f^*a')_H : \pi_\Omega(E'\oplus H, F'\oplus H) \longrightarrow A$ is defined by the following commutative diagram:



(3.201)

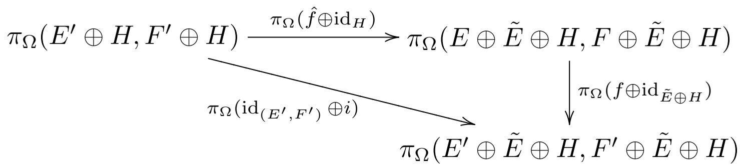
Let $i : H \longrightarrow \tilde{E}\oplus H$ denote the natural inclusion. Since a' is a cocone, the diagram



(3.202)

commutes.

The statement $(f_*f^*a')_H = a'_H$ will follow from the commutativity of the following diagram:



(3.203)

In diagram (3.201), we have seen that

$$(f_*f^*a')_H = a'_{\tilde{E}\oplus H} \circ \pi_\Omega(f\oplus\mathrm{id}_{\tilde{E}\oplus H}) \circ \pi_\Omega(\hat{f}\oplus\mathrm{id}_H). \tag{3.204}$$

Commutativity of diagram (3.203) means that

$$\pi_\Omega(f\oplus\mathrm{id}_{\tilde{E}\oplus H}) \circ \pi_\Omega(\hat{f}\oplus\mathrm{id}_H) = \pi_\Omega(\mathrm{id}_{(E',F')}\oplus i). \tag{3.205}$$

But then we can conclude that $(f_*f^*a')_H = a'_H$ from the equality $a_{\tilde{E}\oplus H}\circ\pi_\Omega(\mathrm{id}_{(E',F')}\oplus i) = a'_H$ expressed by the commutativity of diagram (3.202).

So we prove that diagram (3.203) commutes. Let us write $K := (f\oplus\mathrm{id}_{\tilde{E}\oplus H})\circ(\hat{f}\oplus\mathrm{id}_H)$ and $I := id_{(E',F')}\oplus i$ for the two morphisms $(E'\oplus H, F'\oplus H) \longrightarrow (E'\oplus\tilde{E}\oplus H, F'\oplus\tilde{E}\oplus H)$ we are considering.

Let $n : [E'\oplus H]^+_B/\Omega \longrightarrow [F'\oplus H]^+_B$ be a map. To describe the maps I_*n and K_*n we introduce the following additional isomorphisms:

- $T : F' \oplus H \oplus \tilde{E} \longrightarrow F' \oplus \tilde{E} \oplus H,\ \left(i_F(f) + \tilde{f}\right) + h + \tilde{e} \mapsto (i_F(f) + \tau(\tilde{e})) + \tau^{-1}(\tilde{f}) + h;$
- $Q : F' \oplus H \oplus \tilde{E} \longrightarrow F' \oplus \tilde{E} \oplus H,\ f' + h + \tilde{e} \mapsto f' + \tilde{e} + h.$

We now describe I_*n and K_*n:

- $I_*n : [E' \oplus \tilde{E} \oplus H]^+_B/\Omega \longrightarrow [F' \oplus \tilde{E} \oplus H]^+_B,\ [e' + \tilde{e} + h] \mapsto Q^+\big(n(e' + h) \wedge \tilde{e}\big).$
- $K_*n : [(i_E(E) \oplus \tilde{E}) \oplus \tilde{E} \oplus H]^+_B/\Omega \longrightarrow [F' \oplus \tilde{E} \oplus H]^+_B,\ [i_E(e) + \tilde{e}_1 + \tilde{e}_2 + h] \mapsto T^+\big(n(i_E(e) + \tilde{e}_2 + h) + \tilde{e}_1\big).$

We construct a homotopy between the two maps. For $t \in [0, \pi/2]$ let

- $e_t : [i_E(E) \oplus \tilde{E} \oplus H \oplus \tilde{E}]^+_B/\Omega \longrightarrow [i_E(E) \oplus \tilde{E} \oplus H \oplus \tilde{E}]^+_B$ be defined by the identity on the summands $i_E(E)$ and H and by the matrix $\begin{bmatrix} \cos t\ \mathrm{id}_{\tilde{E}} & \sin t\ \mathrm{id}_{\tilde{E}} \\ -\sin t\ \mathrm{id}_{\tilde{E}} & \cos t\ \mathrm{id}_{\tilde{E}} \end{bmatrix}$ on the summand $\tilde{E} \oplus \tilde{E}$;
- $f_t : [i_F(F) \oplus \tilde{F} \oplus H \oplus \tilde{E}]^+_B \longrightarrow [i_F(F) \oplus \tilde{F} \oplus H \oplus \tilde{E}]^+_B$ be defined by the identity on the summands $i_F(F)$ and H and by the matrix $\begin{bmatrix} \cos t\ \mathrm{id}_{\tilde{F}} & -\sin t\, \tau \\ \sin t\, \tau^{-1} & \cos t\ \mathrm{id}_{\tilde{E}} \end{bmatrix}$ on the summand $\tilde{F} \oplus \tilde{E}$.

Now we put $n_t := f_t \circ I_*n \circ e_t$. Then $n_0 = I_*n$ and $n_{\pi/2} = K_*n$. This proves that the diagram (3.203) commutes.

Now let $a = (a_H)_H$ be a cocone for the functor $\pi_\Omega \circ S_{(E,F)}$. We claim that

$$(f^* f_* a)_H = a_H : \pi_\Omega(E \oplus H, F \oplus H) \longrightarrow A. \tag{3.206}$$

The morphism $(f^* f_* a)_H$ is defined by the following commutative diagram:

$$\begin{array}{ccc} \pi_\Omega(E \oplus H, F \oplus H) & \xrightarrow{\pi_\Omega(f \oplus \mathrm{id}_H)} & \pi_\Omega(E' \oplus H, F' \oplus H) \\ \Big\downarrow{\scriptstyle (f^* f_*)_H} & & \Big\downarrow{\scriptstyle \pi_\Omega(\hat{f} \oplus \mathrm{id}_H)} \\ A & \xleftarrow[a_{\tilde{E} \oplus H}]{} & \pi_\Omega(E \oplus \tilde{E} \oplus H, F \oplus \tilde{E} \oplus H) \end{array} \tag{3.207}$$

Now observe that $\hat{f} \circ f = S_{(E,F)}(i) : (E \oplus H, F \oplus H) \longrightarrow (E \oplus \tilde{E} \oplus H, F \oplus \tilde{E} \oplus H)$, where $i : H \longrightarrow \tilde{E} \oplus H$ is the natural morphism. Since a is a cocone, the diagram

$$\begin{array}{ccc} \pi_\Omega(E \oplus H, F \oplus H) & \xrightarrow{\pi_\Omega \circ S_{(E,F)}(i)} & \pi_\Omega(E \oplus \tilde{E} \oplus H, F \oplus \tilde{E} \oplus H) \\ \Big\downarrow{\scriptstyle a_H} & \swarrow{\scriptstyle a_{\tilde{E} \oplus H}} & \\ A & & \end{array} \tag{3.208}$$

commutes. We conclude that $(f^* f_* a)_H = a_H$.

Now we apply Proposition 2.9.22 and conclude that the functor $\mathbb{p}_\Omega$ admits a colimit. □

Corollary 3.4.30. *The functor* $\pi_\Omega : {}_\rho\mathcal{H}^*_B \longrightarrow \mathrm{Group}$ *admits a colimit, which is naturally identified with* $\mathrm{colim}_G \mathbb{p}^\Omega_B$.

Proof. This is an immediate consequence of Proposition 3.4.29 and Proposition 2.9.20. □

Framed cohomotopy groups

We have given a second description of the groups ${}_\rho\mathbb{P}_\Omega(B)$. Now, we shortly explain how this description translates to the case of framed cohomotopy groups. As usual we fix a subset $\rho_0 \subset \rho$ and write $\rho_* := \rho - \rho_0$. Furthermore, we fix a ρ_0-Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and a l-adapted subbundle $V \subset \mathcal{F}_0$.

Consider the functors

- $S : {}_{\rho_*}\mathcal{C}^*_B \times {}_{\rho_*}\mathcal{H}_B \longrightarrow {}_{\rho_*}\mathcal{H}^*_B,\ (G,(E,F)) \mapsto (E \oplus G, F \oplus G).$
- $\Sigma : {}_{\rho}\mathcal{C}^*_B \times {}_{\rho_*}\mathcal{H}_B \longrightarrow {}_{\rho_0}\mathcal{C}^*_B \times {}_{\rho_*}\mathcal{H}_B, ; (G,(E,F)) \mapsto (G_0, (E \oplus G_*, F \oplus G_*).$

In the definition of the functor Σ, we used that every ρ-bundle G canonically splits as a direct sum $G = G_0 \oplus G_*$ of a ρ_0-bundle G_0 and a ρ_*-bundle G_*. This splitting defines an isomorphism of categories

$${}_{\rho}\mathcal{C}^*_B \cong {}_{\rho_0}\mathcal{C}^*_B \times {}_{\rho_*}\mathcal{C}_B. \tag{3.209}$$

Under this isomorphism the functor Σ corresponds to the functor $\mathrm{id} \times S$. Since the functor S identifies colimits, the same is true for the functor Σ. We now construct a colimit for the functor $\pi^{l,V}_\Omega \circ \Sigma$. This will be a straigthforward consequence from the unframed case.

Proposition 3.4.31. *Let $(E,F) \in {}_{\rho_*}\mathcal{H}_B$ be a fixed object. Then the induced functor*

$$\pi^{l,V}_\Omega \circ \Sigma_{(E,F)} : {}_{\rho}\mathcal{C}^*_B \longrightarrow \mathrm{Group} \tag{3.210}$$

admits a colimit which is preserved by the forgetful functor.

Proof. This statement reduces immediately to Proposition 3.4.31. Just observe the equality $\pi^{l,V}_\Omega \circ \Sigma_{(E,F)} = \pi_\Omega \circ S_{(l^{-1}(V)\oplus E, V \oplus F)}$. ☐

Proposition 3.4.32. *The functor $\pi^{l,V}_\Omega \circ \Sigma$ induces a functor*

$$\mathbb{p}^{l,V}_\Omega : {}_{\rho_*}\mathcal{H}^*_B \longrightarrow \mathrm{Group},\ (E,F) \mapsto \operatorname{colim} \pi^{l,V}_\Omega \circ \Sigma_{(E,F)}. \tag{3.211}$$

This functor admits a colimit, which is canonically identified with ${}_{\rho}\mathbb{P}^{l,V}_\Omega(B)$.

Proof. Again, it is an immediate consequence from Proposition 3.4.29 that $\mathbb{p}^{l,V}_\Omega(f)$ is an isomorphism for every morphism $f : (E,F) \longrightarrow (E',F')$ in the category ${}_{\rho_*}\mathcal{H}_B$. This implies that the functor $\mathbb{p}^{l,V}_\Omega$ admits a colimit and Proposition 2.9.22 provides us with a concrete description. By Proposition 2.9.20 this colimit is canonically a colimit for the functor $\pi^{l,V}_\Omega \circ \Sigma$. Because Σ identifies colimits it is canonically identified with $\operatorname{colim} \pi^{l,V}_\Omega = {}_{\rho}\mathbb{P}^{l,V}_\Omega(B)$. ☐

Corollary 3.4.33. *Let $y \in K_{\rho_*}(B)$ and set $x := (\mathrm{ind}_\rho l, y) \in K_\rho(B)$. The natural map*

$${}_{\rho}\mathbb{P}^{l,V}_\Omega(B,y) \longrightarrow {}_{\rho}\mathbb{P}_\Omega(B,x) \tag{3.212}$$

is surjective.

Proof. As we have done with the functors π_Ω and $\pi^{l,V}_\Omega$, we can also restrict the functors $\mathbb{p}_\Omega$ and $\mathbb{p}^{l,V}_\Omega$ to the categories ${}_{\rho}\mathcal{H}_B(x)$ and ${}_{\rho_*}\mathcal{H}_B(y)$ respectively. We denote these restricted functors with $\mathbb{p}_\Omega(x)$ and $\mathbb{p}^{l,V}_\Omega(y)$. Then

$${}_{\rho}\mathbb{P}_\Omega(B,x) = \operatorname{colim} \mathbb{p}_\Omega(x) \text{ and } {}_{\rho}\mathbb{P}^{l,V}_\Omega(B,y) = \operatorname{colim} \mathbb{p}^{l,V}_\Omega(y). \tag{3.213}$$

But the construction of $\operatorname{colim} {}_{\rho}\mathbb{p}^{l,V}_\Omega(y)$ in Proposition 2.9.22 is explicit: we take any object (E,F) and then construct the colimit as a quotient of $\mathbb{p}^{l,V}_\Omega(E,F) = \mathbb{p}_\Omega(l^{-1}(V) \oplus E, V \oplus F)$ by an equivalence relation constructed using the morphisms in the category ${}_{\rho_*}\mathcal{H}^*_B(y)$. In the same way we have a description of $\operatorname{colim} \pi_\Omega(x)$ as a quotient of $\mathbb{p}_\Omega(l^{-1}(V) \oplus E, V \oplus F)$ modulo an equivalence relation which takes into account all morphisms in the category ${}_{\rho}\mathcal{H}^*_B(x) = {}_{\rho_0}\mathcal{H}^*_B(\mathrm{ind}_\rho l) \times {}_{\rho_*}\mathcal{H}_B(y)$. Therefore the latter relation will in general be finer and the map $\operatorname{colim} \mathbb{p}^{l,V}_\Omega(y) \longrightarrow \operatorname{colim} \mathbb{p}_\Omega(x)$ is surjective. ☐

3.4.2 The Okonek-Teleman invariants

We start this section with explaining a third description of the colimit π_Ω. It is the direct adaption of the construction used by Okonek and Teleman in [34]. The existence of the colimits are guaranteed because all categories which appear are equivalent to small categories. As explained in Remark 3.4.28 Okonek and Teleman do not use this fact, but effectively construct the colimits using the techniques outlined in the previous section. We do not repeat these arguments here.

Let X and Y be fiberwise pointed G-spaces over B. Define a functor

$$ {}_G\pi^0_B(X,Y) : {}_\rho\mathcal{C}^*_B \longrightarrow \text{Group},\ H \mapsto {}_G\pi^0_B(X \wedge_B H^+_B, Y \wedge_B H^+_B). \tag{3.214}$$

We put ${}_G\alpha^0_B(X,Y) := \operatorname{colim} {}_G\pi^0_B(X,Y)$.

There is a natural bifunctor

$$ a_\Omega : {}_\rho\mathcal{C}^*_B \times {}_\rho\mathcal{H}_B \longrightarrow \text{Group},\ (H,(E,F)) \mapsto {}_G\pi^0_B(E^+_B/\Omega \wedge_B H^+_B, F^+_B \wedge_B H^+_B). \tag{3.215}$$

When we fix an object $(E,F) \in {}_\rho\mathcal{H}_B$, then

$$ {}_G\pi^0_B(E^+_B/\Omega, F^+_B) = a_\Omega(\circ,(E,F)) \text{ and } {}_G\alpha^0_B(E^+_B/\Omega, F^+_B) = \operatorname{colim} a_\Omega(\circ,(E,F)). \tag{3.216}$$

By Proposition 2.9.20, the functor a_Ω induces a functor

$$ \mathbb{a}_\Omega : {}_\rho\mathcal{H}_B \longrightarrow \text{Group},\ (E,F) \mapsto {}_G\alpha^0_B(E^+_B, F^+_B). \tag{3.217}$$

We want to identify the colimit of this functor with ${}_\rho\mathbb{P}_\Omega(B)$. The main point is the following:

Proposition 3.4.34. *There is a stably bijective natural transformation between the functor a_Ω and the functor $\pi_\Omega \circ S$.*

Proof. We define a natural transformation

$$ \tau : a_\Omega \longrightarrow \pi_\Omega \circ S \tag{3.218}$$

as follows: Let $(G,(E,F)) \in {}_\rho\mathcal{C}^*_B \times {}_\rho\mathcal{H}_B$. Then we define

$$ \tau_{G,(E,F)} : {}_G\pi^0_B\left(E^+_B/\Omega \wedge_B G^+_B, F^+_B \wedge_B G^+_B\right) \longrightarrow {}_G\pi^0_B\left([E\oplus G]^+_B/\Omega, [F\oplus G]^+_B\right). \tag{3.219}$$

by composition on the right with the contraction map

$$ c_{E\oplus G,G} : [E\oplus G]^+_B/\Omega \longrightarrow E^+_B/\Omega \wedge_B G^+_B \tag{3.220}$$

introduced in Lemma 3.4.5. (As always, we implicitely use the identification $[F\oplus G]^+_B = F^+_B \wedge_B G^+_B$.)

We now prove that this transformation is stably surjective: Let $\tilde{n} : [E\oplus G]^+_B/\Omega \longrightarrow [F\oplus G]^+_B$ be a map. Put $E' := E\oplus G$, $F' := F\oplus G$, and $H' := H$. Let $i : (E,F) \longrightarrow (E\oplus G, F\oplus G)$ be the inclusion morphism and put $I := (\mathrm{id}_H, i) = S(\mathrm{id}_H, i) : (H,(E,F)) \longrightarrow (H',(E'F'))$. We claim that $\pi_\Omega(I)([n])$ lies in the image of τ.

We start by describing $I_*n : [E\oplus G\oplus G]^+_B/\Omega \longrightarrow [F\oplus G\oplus G]^+_B$. It is given as

$$ I_*n([e\wedge g_1 \wedge g_2]) = n_F([e\wedge g_2]) \wedge g_1 \wedge n_G([e\wedge g_2]). \tag{3.221}$$

Here we use the notation $n = n_F \wedge n_G$.

Now we define a map

$$ \hat{n} : [E\oplus G]^+_B/\Omega \wedge_B G^+_B \longrightarrow F^+_B \wedge_B G^+_B \wedge_B G^+_B \tag{3.222}$$

by $\hat{n}([e\wedge g_1]\wedge g_2) := n_F([e\wedge g_1]) \wedge n_G([e\wedge g_1]) \wedge g_2$. We claim that $\bar{n} := \hat{n}\circ c_{E\oplus G\oplus G,G}$ is homotopic to I_*n. By definition

$$ \bar{n} : [E\oplus G\oplus G]^+_B/\Omega \longrightarrow [F\oplus G\oplus G]^+_B,\ [e\wedge g_1\wedge g_2] \mapsto n([e\wedge g_1]) \wedge g_2. \tag{3.223}$$

As before, we define $r_t : H\oplus H \longrightarrow H\oplus H$ by the matrix $r_t = \begin{bmatrix} \cos t\ \mathrm{id}_H & -\sin t\ \mathrm{id}_H \\ \sin t\ \mathrm{id}_H & \cos t\ \mathrm{id}_H \end{bmatrix}$ for $t \in [0,\pi/2]$. Furthermore, we define:

1. $e_t = [\mathrm{id}_E \oplus r_t^{-1}]^+ : [E \oplus H \oplus H]_B^+/\Omega \longrightarrow [E \oplus H \oplus H]_B^+/\Omega;$
2. $f_t := [\mathrm{id}_F \oplus r_t]^+ : [F \oplus H \oplus H]_B^+ \longrightarrow [F \oplus H \oplus H]_B^+.$

Observe that $e_0 = \mathrm{id}$ and $f_0 = \mathrm{id}$, and that

$$e_{\pi/2}([e \wedge h_1 \wedge h_2]) = [e \wedge h_2 \wedge (-h_1)] \text{ and } f_{\pi/2}([f \wedge h_1 \wedge h_2]) = [f \wedge (-h_2) \wedge h_1]. \tag{3.224}$$

We claim that

$$I_*\tilde{n} = f_{\pi/2} \circ \bar{n} \circ e_{\pi/2}. \tag{3.225}$$

Let $[e \wedge h_1 \wedge h_2] \in [E \oplus H \oplus H]_B^+/\Omega$. Then

$$\begin{aligned} f_{\pi/2} \circ \bar{n} \circ e_{\pi/2} ([e \wedge h_1 \wedge h_2]) &= f_{\pi_2} \left(n_F([e \wedge h_2]) \wedge n_G([e \wedge h_2] \wedge (-h_1)\right) \\ &= n_F([e \wedge h_2]) \wedge h_1 \wedge n_G([e \wedge h_2]) = I_*\tilde{n}([e \wedge g_1 \wedge g_2]). \end{aligned} \tag{3.226}$$

Therefore the family of maps $f_t \circ \bar{n} \circ e_t$ is a homotopy between $f_0 \circ \bar{n} \circ e_0 = \bar{n}$ and $f_{\pi/2} \circ \bar{n} \circ e_{\pi/2} = f_* n$. Hence

$$\tau_{H',(E',F')}\left([\hat{n}]\right) = [\bar{n}] = [I_* n] = \pi_\Omega \circ S(\mathrm{id}_H, i)\left([\tilde{n}]\right). \tag{3.227}$$

This proves that the transformation is stably surjective. Stable injectivity is proven using exactly the same construction. □

Corollary 3.4.35. *There is a natural isomorphism of groups*

$$\operatorname{colim} \mathbb{a}_\Omega \cong {}_\rho\mathbb{P}_\Omega(B). \tag{3.228}$$

Proof. By Proposition 2.9.20, the colimit of $\mathbb{a}_\Omega$ is canonically identified with the colimit of a_Ω. By Lemma 3.4.26, the colimit of a_Ω is canonically identified with the colimit of $a_\Omega \circ S$. Finally, the previous proposition together with Proposition 2.9.7 implies the statement. □

Now we put $\rho_0 := \{\mathbb{R}\} \subset \rho$. As always we write $\rho_* := \rho - \rho_0$. Furthermore, we assume that we are given a ρ_0-Fredholm morphism $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and a l-adapted subbundle $V \subset \mathcal{F}$. We set $U := l^{-1}(V)$.

We define a bifunctor

$$a_\Omega^{l,V} : {}_\rho\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B \longrightarrow \mathrm{Group},\ (H,(E,F)) \mapsto {}_G\pi_B^0(E_B^+/\Omega \wedge_B H_B^+ \wedge_B U_B^+, F_B^+ \wedge_B H_B^+ \wedge_B V_B^+). \tag{3.229}$$

Again by Proposition 2.9.20, this functor induces a functor

$$\mathbb{a}_\Omega^{l,V} : {}_{\rho_*}\mathcal{H}_B \longrightarrow \mathrm{Group},\ (E,F) \mapsto {}_G\alpha_B^0(E_B^+/\Omega \wedge_B U_B^+, F_B^+ \wedge_B V_B^+). \tag{3.230}$$

We want to identify the colimit of this functor with the group ${}_\rho\mathbb{P}_\Omega^{l,V}(B)$. We will use the following remark:

Remark 3.4.36. Let E be a ρ_*-bundle and W a ρ_0-bundle. Then there is a natural isomorphism

$$[E \oplus W]_B^+/\Omega \xrightarrow{\approx} E_B^+/\Omega \wedge_B W_B^+. \tag{3.231}$$

Proof. When $\Omega = \emptyset$, there is nothing to prove. Otherwise $G \in \Omega$. Since $\rho_0 = \{\mathbb{R}\}$, it follows that $[E \oplus W]_B^+(\Omega) = E_B^+(\Omega) \wedge W_B^+$. The statement then follows from Lemma 3.4.5. □

Proposition 3.4.37. *There is a natural isomorphism* $\operatorname{colim} \mathbb{a}_\Omega^{l,V} \cong {}_\rho\mathbb{P}_\Omega^{l,V}(B)$.

Proof. Recall that we have defined the functor

$$\Sigma : {}_\rho\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B \longrightarrow {}_{\rho_0}\mathcal{C}_B^* \times {}_{\rho_*}\mathcal{H}_B,\ (H,(E,F)) \mapsto (H_0,(E \oplus H_*, F \oplus H_*)) \tag{3.232}$$

and that it identifies colimits.

By Proposition 2.9.20, the colimit of $\mathbb{a}_\Omega^{l,V}$ is canonically identified with the colimit of $a_\Omega^{l,V}$. Using the same proof as in Proposition 3.4.34, one shows that the functor $a_\Omega^{l,V}$ and the functor $\pi_\Omega^{l,V} \circ \Sigma$ are stably bijective. (For the same argument to work one needs Remark 3.4.36.) Therefore the respective colimits are identified. Since Σ identifies colimits, the statement follows. □

Now we are able to relate our invariant to the invariant introduced by Okonek and Teleman in [34]. We specialize to the following situation: Let $G = S^1$, $\rho = \{\mathbb{R}, \mathrm{id}\}$, $\rho_0 = \{\mathbb{R}\}$, $\Omega = \{S^1\}$. Furthermore, we assume $\mathcal{E}_0$ and $\mathcal{F}_0$ to be trivialized and that $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ is a fiberwise constant linear embedding. We let $V \subset \mathcal{F}_0$ be a trivial closed complement of the image $l(\mathcal{E}_0)$ of fiberwise dimension $p \in \mathbb{N}$. Last, we fix an element $y \in K_{\rho_*}(B) = K(B)$.

Okonek and Teleman associate with an l-framed Ω-map $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ of index $\mathrm{ind}_{\rho_*} \mu = y$ and an orientation o of V, an invariant $\{\mu\} \in \alpha^{p-1}(y)$ ([34, Corollary 3.14]).

We recall their definition of the group $\alpha^{p-1}(y)$: It is defined as colimit of the following functor ([34, §2.3]):

$$ {}_{\rho_*}\mathcal{H}^*_B(y) \longrightarrow \text{Group}, \ (E, F) \mapsto {}_{S^1}\alpha^0_B \left(S(E)_{+B}, F^+_B \wedge_B [\underline{\mathbb{R}}^{p-1}]^+_B \right). \tag{3.233} $$

We need to reinterprete the space $S(E)_{+B}$. This is achieved by the following lemma:

Lemma 3.4.38. *Let E be a complex vector bundle. Then there is a canonical homeomorphism*

$$ E^+_B/_B \left(0_B \cup \infty_B \right) \xrightarrow{\approx} S(E)_{+B} \wedge_B \underline{\mathbb{R}}^+_B. \tag{3.234} $$

Proof. We can assume without loss of generality that $B = \{*\}$. Then there is a canonical homeomorphism

$$ S(E)_+ \wedge \mathbb{R}^+ \cong \left(S(E) \times [0,1] \right) / \left(S(E) \times \{0,1\} \right). \tag{3.235} $$

We define a map $f : S(E) \times [0,1] \longrightarrow D(E)$ by $f(e,t) := [te]$. Then $f(e,t) \in S(E) \cup \{0\}$ if and only if $t \in \{0,1\}$. Therefore the map f induces a bijective map

$$ \bar{f} : \left(S(E) \times [0,1] \right) / \left(S(E) \times \{0,1\} \right) \longrightarrow D(E) / \left(S(E) \cup \{0\} \right). \tag{3.236} $$

Because the space on the left is compact, the map is a homeomorphism. There are canonical homeomorphisms

1. $D(E) / \left(S(E) \cup \{0\} \right) \xrightarrow{\approx} \left(D(E)/S(E) \right) / \left(\{0\} \cup \{S(E)\} \right)$;
2. $\left(D(E)/S(E) \right) / \left(\{0\} \cup \{S(E)\} \right) \xrightarrow{\approx} E^+/\{0, \infty\}$.

Composing these four homeomorphisms, we obtain the homeomorphism

$$ E^+_B/_B \left(0_B \cup \infty_B \right) \xrightarrow{\approx} S(E)_{+B} \wedge_B \underline{\mathbb{R}}^+_B. \tag{3.237} $$

□

Notice that now a ρ_*-bundle is a complex vector bundle with standard S^1-action. Therefore $E^+_B(\Omega) = \infty_B \cup 0_B$. Thus we obtain a natural homeomorphism $E^+_B/\Omega \cong S(E)_{+B} \wedge_B [\underline{\mathbb{R}}]^+_B$. Together with the orientation o we obtain natural isomorphisms

$$ \begin{aligned} {}_{S^1}\alpha^0_B \left(S(E)_{+B}, F^+_B \wedge_B [\underline{\mathbb{R}}^{p-1}]^+_B \right) &\cong {}_{S^1}\alpha^0_B \left(S(E)_{+B} \wedge_B [\underline{\mathbb{R}}]^+_B, F^+_B \wedge_B [\underline{\mathbb{R}}^p]^+_B \right) \\ &\cong {}_{S^1}\alpha^0_B \left(E^+_B/\Omega, F^+_B \wedge_B [\underline{\mathbb{R}}^p]^+_B \right) \\ &\cong {}_{S^1}\alpha^0_B \left(E^+_B/\Omega, F^+_B \wedge_B V^+_B \right). \end{aligned} \tag{3.238} $$

Corollary 3.4.39. *There is a natural isomorphism*

$$ \alpha^{p-1}(y) \cong {}_\rho\mathbb{P}^{l,V}_\Omega(B, y). \tag{3.239} $$

Proof. This statement follows immediately from Proposition 3.4.37 after restricting the functors to the subcategory ${}_{\rho_*}\mathcal{H}_B(y)$. □

Corollary 3.4.40. *Assume that $K^{-1}(B) = 0$. Then there is a bijection*

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \xrightarrow{\hat{=}} \alpha^{p-1}(y). \tag{3.240} $$

Proof. In Corollary 3.4.39 we have identified $\alpha^{p-1}(y)$ with ${}_\rho\mathbb{P}^{l,V}_\Omega(B, y)$. It follows from Proposition 3.4.22, that the natural map

$$ {}_G[\mathcal{E}, \mathcal{F}]^{\Omega,l}_B(y) \longrightarrow {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \tag{3.241} $$

is a bijection. □

3.4.3 The Bauer-Furuta invariants

In [7], Bauer and Furuta consider the case $\Omega = \emptyset$ and $\rho_0 = \rho$. They fix a Fredholm morphism $d : \mathcal{E} \longrightarrow \mathcal{F}$, a d-adapted subbundle $F \subset \mathcal{F}$, and consider coercive maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$, which are compact perturbations of d. With such a map μ they associate an element $\{\mu\}$ in a group $\pi^0_{G,H}(B; \operatorname{ind} d)$, where H is an infinite dimensional ρ-Hilbert space.

Proposition 3.4.41. *Let H be a infinite dimensional ρ-Hilbert space. There is a natural isomorphism*

$$\pi^0_{G,H}(B; \operatorname{ind} d) \xrightarrow{\cong} {}_\rho\mathbb{P}^{d,F}_\emptyset(B). \tag{3.242}$$

Proof. This follows from the definition of the groups by Bauer and Furuta ([7, page 9]). □

3.4.4 A comparison map

We can compare Okonek and Teleman's with Bauer and Furuta's invariants in a special case. Assume that we are in the situation where the Okonek-Teleman invariants are defined: Let $G = S^1$, $\rho = \{\mathbb{R}, \mathrm{id}_{S^1}\}$, and $\rho_0 = \{\mathbb{R}\}$. Let $\mathcal{E}$ and $\mathcal{F}$ be two infinite dimensional ρ-bundles and assume that $\mathcal{E}_0$ and $\mathcal{F}_0$ are trivialized. Fix a fiberwise constant linear embedding $l : \mathcal{E}_0 \longrightarrow \mathcal{F}_0$ and let $V \subset \mathcal{F}_0$ a trivial closed complement of the image $l(\mathcal{E}_0)$ of fiberwise dimension $p \in \mathbb{N}$. Last, we fix an element $y \in K(B)$.

We let H be a fiber of $\mathcal{E}$ and we let $d : \mathcal{E} \longrightarrow \mathcal{F}$ be a ρ-Fredholm morphism with $d_0 = l$. Furthermore, we let $F \subset \mathcal{F}$ be a d-adapted subbundle with $F_0 = V$. (Recall that the bundle F has a canonical decomposition $F = F_0 \oplus F_*$ where F_0 is a ρ_0-bundle and F_* is a ρ_*-bundle.) Then:

Proposition 3.4.42. *Assume that $K^{-1}(B) = 0$. Then there is a natural morphism*

$$\alpha^{p-1}(y) \longrightarrow \pi^0_{G,H}(B; \operatorname{ind} d). \tag{3.243}$$

Proof. By Corollary 3.4.39 and Proposition 3.4.41, there are isomorphisms

$$\alpha^{p-1}(y) \xrightarrow{\cong} {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \text{ and } \pi^0_{G,H}(B; \operatorname{ind} d) \xrightarrow{\cong} {}_\rho\mathbb{P}^{d,F}_\emptyset(B). \tag{3.244}$$

First, observe that there is a natural map

$$f : {}_\rho\mathbb{P}^{d,F}_\emptyset(B) \longrightarrow {}_\rho\mathbb{P}^{l,V}_\emptyset(B, y), \tag{3.245}$$

which is an isomorphism since $K^{-1}(B) = 0$: This is proven exactly as Proposition 3.4.24. Furthermore, there is a natural map $g : {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}^{l,V}_\emptyset(B, y)$. The composition

$$f^{-1} \circ g : {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}^{d,F}_\emptyset(B), \tag{3.246}$$

together with the isomorphisms above defines the morphism $\alpha^{p-1}(y) \longrightarrow \pi^0_{G,H}(B; \operatorname{ind} d)$. □

The construction above makes use of the forgetful morphism $g : {}_\rho\mathbb{P}^{l,V}_\Omega(B, y) \longrightarrow {}_\rho\mathbb{P}^{l,V}_\emptyset(B, y)$. This morphism is induced as follows: Let $(E, F) \in {}_{\rho_*}\mathcal{H}_B$ and let $W \in {}_{\rho_0}\mathcal{C}^*_B$. Then there is a natural map

$${}_G\pi^0_B\left([l^{-1}(V) \oplus W \oplus E]^+_B/\Omega, [V \oplus W \oplus F]^+_B\right) \longrightarrow {}_G\pi^0_B\left([l^{-1}(V) \oplus W \oplus E]^+_B, [V \oplus W \oplus F]^+_B\right) \tag{3.247}$$

induced by the contraction map $c_\Omega : [l^{-1}(V) \oplus W \oplus E]^+_B \longrightarrow [l^{-1}(V) \oplus W \oplus E]^+_B/\Omega$. This defines a natural transformation of functors and thus the morphism g.

We have natural isomorphisms (Lemma 3.4.38 and Lemma 3.4.5):

1. $a : [l^{-1}(V) \oplus W \oplus E]^+_B/\Omega \xrightarrow{\approx} E^+_B/\Omega \wedge_B [l^{-1}(V) \oplus W]^+_B$;

2. $b : E^+_B/\Omega \xrightarrow{\approx} S(E)_{+B} \wedge_B [\mathbb{R}]^+_B$.

The fiberwise cofiber sequence

$$S(E)_{+B} \longrightarrow D(E)_{+B} \longrightarrow E_B^+. \tag{3.248}$$

induces a map

$$k : E_B^+ \longrightarrow S(E)_{+B} \wedge_B \underline{\mathbb{R}}_B^+, \tag{3.249}$$

well defined up to homotopy, and thus a well-defined map

$$k^* : {}_{S^1}\pi_B^0 \left(S(E)_{+B} \wedge_B \underline{\mathbb{R}}_B^+ \wedge_B Y, Z\right) \longrightarrow {}_{S^1}\pi_B^0 \left(E_B^+ \wedge_B Y, Z\right), \tag{3.250}$$

where $Y := [l^{-1}(V) \oplus W]_B^+$ and $Z := F_B^+ \wedge_B [W \oplus F]_B^+$.

Lemma 3.4.43. *Under the isomorphisms a and b, the map $-k^*$ corresponds to the map c_Ω^*.*

Proof. The map k is only well defined up to homotopy. We recall its description from [13, page 156-158]. See also [45, page 187-192].

Let $i : S(E)_{+B} \longrightarrow D(E)_{+B}$ be the inclusion. In the fiberwise cofiber sequence

$$S(E)_{+B} \xrightarrow{\ i\ } D(E)_{+B} \longrightarrow E_B^+\ , \tag{3.251}$$

we use the space $D(E)_{+B}/_B S(E)_{+B} = D(E)/_B S(E)$ as a model for E_B^+ (see Lemma 3.4.1).

Let $c(i) : D(E)_{+B} \longrightarrow C_B(i)$ be the fiberwise mapping cone ([13, p.156]) of i and let $c^2(i) := c(c(i)) : C_B(i) \longrightarrow C_B^2(i) := C_B(C_B(i))$ be the fiberwise mapping cone of $c(i)$. Then there are natural homotopy equivalences ([13, Lemma II.2.2])

$$\varphi : C_B(i) \longrightarrow D(E)/_B S(E) \text{ and } \psi : C_B^2(i)/_B C_B(i) \longrightarrow S(E)_{+B} \wedge_B \underline{\mathbb{R}}_B^+. \tag{3.252}$$

The map k is defined as a map, such that the diagram

$$\begin{array}{ccc} C_B(i) & \xrightarrow{c^2(i)} & C_B^2(i) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ E_B^+ & \xrightarrow[k]{} & S(E)_{+B} \wedge_B \underline{\mathbb{R}}_B^+ \end{array} \tag{3.253}$$

commutes up to homotopy. Now we assume that $B = \{*\}$ in order to lighten the notation. No generality is lost.

There is a natural homeomorphism $S(E)_+ \wedge \underline{\mathbb{R}}^+ \xrightarrow{\approx} (S(E) \times [0,1]) / (S(E) \times \{0,1\})$ (compare with the proof of Lemma 3.4.38). We will use this homeomorphism to identify the two spaces. Next, we describe the map

$$\psi \circ c^2(i) : C(i) \longrightarrow (S(E) \times [0,1]) / (S(E) \times \{0,1\}). \tag{3.254}$$

Recall that $C(i) = (S(E)_+ \times [0,1] \cup D(E)_+ \times \{1\}) / (S(E)_+ \times \{0\} \cup \{+\} \times [0,1])$. The map ψ is given by contracting $D(E)_+ \times \{1\} \cup \{+\} \times [0,1]$. Let $(e,t) \in C(i)$. Then

$$\psi(e,t) = \begin{cases} * & \text{when } t = 1, \\ (e,t) & \text{else.} \end{cases} \tag{3.255}$$

Now we describe a homotopy inverse of φ: Consider the retraction map

$$r : D(E) \times [0,1] \longrightarrow D(E) \times \{1\} \cup S(E) \times [0,1], \tag{3.256}$$

that is given by the projection from $(0,-1)$ (see [46, page 21]). Concretely, we have:

$$r(e,t) = \begin{cases} \left(\frac{2e}{t+1}, 1\right) & \text{when } |e| \leq \frac{t+1}{2}, \\ \left(\frac{e}{|e|}, -1 + \frac{t+1}{|e|}\right) & \text{else.} \end{cases} \tag{3.257}$$

A homotopy inverse to $\overline{\varphi}$ is given by composing r with the embedding $D(E) \xrightarrow{\approx} D(E) \times \{0\} \subset D(E) \times [0,1]$. To be precise, define $\tau : D(E)/S(E) \longrightarrow C(i)$ as follows:

$$\tau(e) := \begin{cases} (2e, 1) & \text{when } |e| \leq 1/2, \\ \left(\frac{e}{|e|}, -1 + \frac{1}{|e|}\right) & \text{else.} \end{cases} \tag{3.258}$$

Therefore the composite map $\psi \circ C^2(i) \circ \tau : D(E)/S(E) \longrightarrow (S(E) \times [0,1]) / (S(E) \times \{0,1\})$ is a model for k and we denote it with that symbol. We can now give a concrete formula: Let $e \in D(E)$. Then:

$$k(e) = \begin{cases} * & \text{when } |e| \leq 1/2, \\ \left(\frac{e}{|e|}, -1 + \frac{1}{|e|}\right) & \text{else.} \end{cases} \tag{3.259}$$

We let $S^1 = [0,1]/\{0,1\}$. Then we consider the involution $\iota : S^1 \longrightarrow S^1$, defined by $\iota(t) := \frac{2}{t+1} - 1$. It is a map of degree -1. Using ι we obtain a concrete model for the homotopy class $-[k]$: the map

$$\hat{k} : D(E)/S(E) \longrightarrow (S(E) \times [0,1]) / (S(E) \times \{0,1\}), \tag{3.260}$$

defined by

$$\hat{k}(e) := \begin{cases} * & \text{when } |e| \leq 1/2, \\ \left(\frac{e}{|e|}, \iota\left(-1 + \frac{1}{|e|}\right)\right) = \left(\frac{e}{|e|}, 2|e| - 1\right) & \text{else.} \end{cases} \tag{3.261}$$

For $s \in [0,1]$, we define $H_s : D(E)/S(E) \longrightarrow (S(E) \times [0,1]) / (S(E) \times \{0,1\})$ by

$$H_s(e) := \begin{cases} * & \text{when } |e| \leq \frac{s}{s+1}, \\ \left(\frac{e}{|e|}, |e|(s+1) - s\right) & \text{else.} \end{cases} \tag{3.262}$$

The family H_s defines a homotopy between $H_1 = \hat{k}$ and the map H_0. Recall that we are considering the contraction map

$$c_\Omega : E^+ \longrightarrow E^+/\Omega. \tag{3.263}$$

Let $\hat{c}_\Omega : D(E)/S(E) \longrightarrow (S(E) \times [0,1]) /_B (S(E) \times \{0,1\})$ be the map induced by identifying

$$E_B^+ \approx D(E)/S(E) \text{ and } E_B^+/\Omega \approx (S(E) \times [0,1]) /_B (S(E) \times \{0,1\}) \tag{3.264}$$

as discussed before. Then $\hat{c}_\Omega$ is given as a composition:

$$\hat{c}_\Omega : D(E)/S(E) \longrightarrow D(E)/ (S(E) \cup \{0\}) \xrightarrow{\approx} (S(E) \times [0,1]) /_B (S(E) \times \{0,1\}) \tag{3.265}$$

The first map of that composition is contraction, the second map the homoemorphism defined in Lemma 3.4.38.

For $e \in D(E)/S(E)$, we have:

$$\hat{c}_\Omega(e) = \begin{cases} * & \text{when } e = 0, \\ \left(\frac{e}{|e|}, |e|\right) & \text{else.} \end{cases} \tag{3.266}$$

This shows that $\hat{c}_\Omega = H_0$. Therefore H_s defines a homotopy between $\hat{k}$ and $\hat{c}_\Omega$. □

3.4.5 The result of Svarc

Now assume that $B = \{*\}$, $G = \{e\}$, $\rho_0 = \rho = \{\mathbb{R}\}$, and $\Omega = \emptyset$. Let $\mathcal{E}$ and $\mathcal{F}$ be separable Hilbert spaces and let $l : \mathcal{E} \longrightarrow \mathcal{F}$ be a linear Fredholm map of index $p \in \mathbb{Z}$. Let $V := \operatorname{coker} l$ and $U := \ker l$. We have seen that the canonical map

$${}_\rho\mathbb{P}_\Omega^{l,V}(B) \longrightarrow {}_\rho\mathbb{P}_\Omega^{l}(B) \tag{3.267}$$

is an isomorphism. But ${}_\rho\mathbb{P}^{l,V}_\Omega(B)$ is the colimit of the following functor:

$$\pi^{l,V}_\Omega : {}_\rho\mathcal{C}_B \longrightarrow \text{Group},\; W \mapsto \pi^0(U^+ \wedge_B W^+, V^+ \wedge_B W^+) \tag{3.268}$$

After trivializing $V \cong \mathbb{R}^n$ and $U \cong \mathbb{R}^m$, we see that this colimit is just the stable homotopy group of spheres $\pi^{\text{st}}_p(S^0)$. Let $[\mathcal{E},\mathcal{F}]^l$ be the set of homotopy classes of coercive maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$, which are compact perturbations of l and let $[\mathcal{E},\mathcal{F}](p)$ be the set of homotopy classes of coercive maps $\mu : \mathcal{E} \longrightarrow \mathcal{F}$ which are of the form $\mu = l' + c$ for some Fredholm morphism $l' : \mathcal{E} \longrightarrow \mathcal{F}$ of index p. Then:

Corollary 3.4.44. *There are bijections* $[\mathcal{E},\mathcal{F}]^l \xrightarrow{\hat{=}} [\mathcal{E},\mathcal{F}](p) \xrightarrow{\hat{=}} \pi^{\text{st}}_p(S^0)$.

Chapter 4

Cohomotopy invariants in gauge theoretical Gromov-Witten theory

In this chapter we introduce cohomotopy invariants in Abelian gauge theoretical Gromov-Witten theory. In [32], Okonek and Teleman introduced gauge theoretical Gromov-Witten invariants, generalizing the invariants defined by Cielebak, Gaio, Mundet, and Salamon (see [12] and the introduction). In the first section, we recall the setup as introduced by Okonek and Teleman, which leads to a certain type of symplectic vortex equations on a Riemann surface. The main data we need to fix consists in compact Lie groups $G \subset \hat{G}$ with G contained in the center of $\hat{G}$, a unitary representation $\rho : \hat{G} \longrightarrow U(V)$ of $\hat{G}$, and a proper $\hat{G}$-equivariant moment map $\mu : V \longrightarrow \mathfrak{g}^*$ for the G-action on the Hermitian vector space V. Furthermore, a Riemann surface Σ, a principal $\hat{G}$-bundle $\hat{P}$, and a connection a_0 in the $\hat{G}/G$-bundle $\hat{P}/G$. The vortex equations lead to the vortex map

$$v : \mathcal{A}_{a_0}(\hat{P}) \times A^0\left(\hat{P} \times_{\hat{G}} V\right) \longrightarrow A^{0,1}\left(\hat{P} \times_{\hat{G}} V\right) \times A^0(\Sigma, \mathfrak{g}). \tag{4.1}$$

This map is equivariant with respect to the natural $C^\infty(\Sigma, G)$-action. In the first section of this chapter we explain the setup leading to this map more in detail. In the second section, we present two examples. Then, in the third section, we explain how this map leads to the vortex map

$$\nu_k : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k, \tag{4.2}$$

where $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ are complex G-Hilbert bundles with typical fiber $A^0(\hat{E})_{k+1}$ and $A^{0,1}(\hat{E})_k$, respectively, and where $S_{k+1} \subset A^1(\Sigma, \mathfrak{g})_{k+1}$ is a closed complement of $\ker d$. The subscript k indicates Sobolev-completions with respect to the L^2_k-norm. The bundles are defined over a compact torus $B \cong \left(S^1\right)^{b_1(\Sigma) \dim G}$. The construction is analoguous to the construction of the monopole map outlined in [34, §3.4]. In the third section, we prove that the vortex map is Fredholm, i.e. that it is a compact perturbation of a fiberwise linear Fredholm map and that it is coercive. The fact that it is coercive is the main result; it depends essentially on the properness of the moment map. It generalizes compactness results obtained by Cieliebak, Gaio, and Salamon in [12, §3.6] for the case $G = \hat{G}$ and by Okonek and Teleman in [32, Proposition 2.12] for the case $G = S^1$, $\hat{G} = U(r)$, $V = \mathbb{C}^r$. Since ν_k is a Fredholm map, it defines a cohomotopy invariant. We discuss several aspects of this invariant in the fourth and final section.

4.1 The gauge theoretical moduli problem

We fix the following data:

- a finite dimensional Hermitian vector space (V, h);
- a compact Lie group $\hat{G}$ together with a unitary representation $\rho : \hat{G} \longrightarrow U(V)$;

- a closed subgroup $G \subset \hat{G}$, which lies in the center of $\hat{G}$;
- an invariant inner product on the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$, which we will use to identify $\hat{\mathfrak{g}}$ with its dual;

We put $G_0 := \hat{G}/G$ and we denote the orthogonal projection by $\pi_{\mathfrak{g}} : \hat{\mathfrak{g}} \longrightarrow \mathfrak{g}$.

The **discrete parameters** for our moduli problem are

- a compact and oriented smooth surface Σ;
- a $\hat{G}$-bundle $\hat{P}$ over Σ.

We put $P_0 := \hat{P}/G$ and write $\lambda_0 : \hat{P} \longrightarrow P_0$ for the natural projection map.
The **continuous parameters** for our moduli problem are

- a Riemannian metric g_0 on Σ;
- a $\hat{G}$-equivariant moment map $\mu : V \longrightarrow \mathfrak{g}^*$ for the G-action on V;
- a connection a_0 on P_0.

We let $\mathrm{Ad} : \hat{G} \longrightarrow \mathrm{Aut}(\hat{G})$ be the inner action of $\hat{G}$ on itself, and we write $\mathrm{ad} : \hat{G} \longrightarrow \mathrm{Aut}\,(\hat{\mathfrak{g}})\,,\ g \mapsto \mathrm{Ad}_*(g)$ for the adjoint representation of $\hat{G}$ on its Lie algebra $\hat{\mathfrak{g}}$.

We write $\hat{E} := \hat{P} \times_{\hat{G}} V$ for the Hermitian vector bundle associated with the $\hat{G}$-bundle $\hat{P}$ and the representation ρ.

Let $\mathcal{A}_{a_0}(\hat{P})$ denote the space of connections $\hat{a}$ in $\hat{P}$ that project onto a_0 via λ_0. Let $\mathrm{Aut}_{P_0}(\hat{P})$ denote the space of automorphisms of $\hat{P}$ that induce the identity on P_0. Note that

1. a section in $\hat{E}$ corresponds to a $\hat{G}$-equivariant map $\hat{P} \longrightarrow V$;
2. a connection $\hat{a}$ in $\hat{P}$ induces a Hermitian connection $d_{\hat{a}}$ in the vector bundle $\hat{E}$, and together with the complex structure on Σ determined by g_0, it induces a holomorphic structure $\bar{\partial}_{\hat{a}}$ on the vector bundle $\hat{E}$.

The next two statements are well known. We prove them for the reader's convenience. See for example [43, §3.2].

Lemma 4.1.1. *The gauge group* $\mathrm{Aut}(\hat{P})$ *is naturally identified with* $A^0(\Sigma, \hat{P} \times_{\mathrm{Ad}} \hat{G})$. *Under this identification, the subgroup* $\mathrm{Aut}_{P_0}(\hat{P})$ *corresponds to the global sections of the bundle* $\hat{P} \times_{\mathrm{Ad}} G$.

Proof. Let $u : \hat{P} \longrightarrow \hat{P}$ be a gauge transformation. Then a point $p \in \hat{P}$ will be mapped to a point in the same fiber, i.e. $u(p) = p\tilde{u}(p)$ with $\tilde{u}(p) \in \hat{G}$. This defines a smooth map $\tilde{u} : \Sigma \longrightarrow \hat{G}$. The equivariance of u translates to

$$p\tilde{u}(p)g = u(p)g = u(pg) = pg\tilde{u}(pg) \text{ for all } p \in \hat{P} \text{ and } g \in \hat{G}.$$

Hence $\tilde{u}(pg) = g^{-1}\tilde{u}(p)g$ for all $p \in \hat{P}$ and for all $g \in \hat{G}$. The subgroup $\mathrm{Aut}_{P_0}(\hat{P})$ consists by definition of those gauge transformation u, that make the diagram

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\lambda_0} & P_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \hat{P} & \xrightarrow[\lambda_0]{} & P_0 \end{array}$$

commutative. Let $\varepsilon_0 : \hat{G} \longrightarrow G_0$ be the projection. It follows that:

$$\lambda_0(p) = \lambda_0(u(p)) = \lambda_0(p\tilde{u}(p)) = \lambda_0(p)\varepsilon_0(\tilde{u}(p)) \text{ for all } p \in \hat{P}.$$

This is the case if and only if $\tilde{u}(p) \in G$ for all $p \in \hat{P}$. $\square$

Lemma 4.1.2. *The space of connections $\mathcal{A}_{a_0}(\hat{P})$ is an affine space with model vector space $A^1_{\mathrm{ad}}(\hat{P}, \mathfrak{g})$.*

Proof. Let $\hat{a}_0, \hat{a}_1 \in \mathcal{A}_{a_0}(\hat{P})$ be two connections with corresponding connection forms $\hat{\omega}_0, \hat{\omega}_1 \in A^1(\hat{P}, \mathfrak{g})$. Let $\omega_0 \in A^1_{\mathrm{ad}}(P_0, \mathfrak{g}_0)$ be the connection form of the connection $a_0 \in \mathcal{A}(P_0)$. Then by definition, for any point $p \in \hat{P}$ and for $i \in \{0, 1\}$, the diagram

$$\begin{array}{ccc} T_p\hat{P} & \xrightarrow{\hat{\omega}_i(p)} & \hat{\mathfrak{g}} \\ {\scriptstyle d\lambda_0(p)}\downarrow & & \downarrow{\scriptstyle d\pi} \\ T_{\lambda_0(p)}P_0 & \xrightarrow[\omega_0(\lambda_0(p))]{} & \mathfrak{g}_0 \end{array}$$

commutes. Therefore the form $\hat{\omega}_0(p) - \hat{\omega}_1(p)$ takes values in $\mathfrak{g}$. □

Since by assumption G is contained in the center of $\hat{G}$, there are natural isomorphisms $\hat{P} \times_{\mathrm{ad}} \mathfrak{g} \cong \Sigma \times \mathfrak{g}$ and $\hat{P} \times_{\mathrm{Ad}} G \cong \Sigma \times G$. In particular there are natural isomorphisms

$$\mathrm{Aut}_{P_0}(\hat{P}) \cong C^\infty(\Sigma, G) \text{ and } A^1_{\mathrm{ad}}(\hat{P}, \mathfrak{g}) \cong A^1(\Sigma, \mathfrak{g}). \tag{4.3}$$

Henceforth we will make these identifications without further notice.

The configuration space for the gauge theoretical moduli problem we are considering is the product $\mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E})$. We are interested in the moduli space of pairs $(\hat{a}, \varphi) \in \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E})$ that satisfy the following **vortex equations**:

$$\begin{aligned} \bar{\partial}_{\hat{a}}\varphi &= 0 \\ \pi_{\mathfrak{g}} * F_{\hat{a}} + \mu(\varphi) &= 0 \end{aligned} \tag{4.4}$$

The vortex equations define a map

$$v = (v^1, v^2) : \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E}) \longrightarrow A^{0,1}(\hat{E}) \times A^0(\Sigma, \mathfrak{g}). \tag{4.5}$$

We endow $A^0(\Sigma, \mathfrak{g})$ with the trivial $C^\infty(\Sigma, G)$-action. There are natural right actions of $C^\infty(\Sigma, G)$ on $\mathcal{A}_{a_0}(\hat{P})$ and on the spaces of sections $A^0(\hat{E})$ and $A^{0,1}(\hat{E})$.

Lemma 4.1.3. *The vortex map $v : \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E}) \longrightarrow A^{0,1}(\hat{E}) \times A^0(\Sigma, \mathfrak{g})$ is $C^\infty(\Sigma, G)$-equivariant.*

Proof. Let $u \in C^\infty(\Sigma, G)$ be a gauge transformation. Then for a pair $(\hat{a}, \varphi) \in \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E})$ we have

$$\begin{aligned} v^1\left((\hat{a}, \varphi).u\right) &= \bar{\partial}_{\hat{a}+u^{-1}du}(u^{-1}\varphi) = \bar{\partial}_{\hat{a}}(u^{-1}\varphi) + \rho_*(u^{-1}\bar{\partial}u)\left(u^{-1}\varphi\right) \\ &= u^{-1}\bar{\partial}_{\hat{a}}\varphi + \bar{\partial}(\rho(u^{-1}))\,(\varphi) + \rho_*(u^{-1}\bar{\partial}u)\left(u^{-1}\right)\varphi = u^{-1}\bar{\partial}_{\hat{a}}\varphi. \end{aligned} \tag{4.6}$$

This proves equivariance of v^1. On the other hand

$$v^2\left((\hat{a}, \varphi).u\right) = pr_{\mathfrak{g}} * F_{\hat{a}+u^{-1}du} + \mu(u^{-1}\varphi) = pr_{\mathfrak{g}} * F_{\hat{a}} + *d(u^{-1}du) + \mu(u^{-1}\varphi). \tag{4.7}$$

Invariance of v^2 follows from $d(u^{-1}du) = 0$ and the $\hat{G}$-equivariance of μ. □

Now we can formulate our moduli problem: Classify pairs $(\hat{a}, \varphi) \in \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E})$ that satisfy the vortex equations, up to the action of $C^\infty(\Sigma, G)$.

We now recall some facts about unitary actions which we will need later on.

Unitary actions

Let (V, h) be a finite dimensional unitary vector space. For $v \in V$, we write $v^* \in V^\vee$ for the adjoint of v, defined by $v^*(w) := h(w, v)$ for all $w \in W$. Furthermore, we write vw^* for the image of $v \otimes w^* \in V \otimes V^\vee$ in $\mathrm{End}(V)$ under the natural isomorphism $V \otimes V^\vee \cong \mathrm{End}(V)$.

Lemma 4.1.4. *Let (V, h) be a finite dimensional unitary vector space.*

1. *The Lie algebra of the Lie group $U(V)$ is $u(V) = \{A \in \mathrm{End}(V) \mid A + A^* = 0\}$;*
2. *The Lie algebra $u(V)$ carries a natural $U(V)$-invariant real inner product $\langle\ ,\ \rangle_{u(V)}$, given by*

$$\langle A, B\rangle_{u(V)} := \mathrm{tr}\,(AB^*) = -\,\mathrm{tr}(AB) \text{ for } A, B \in u(V) \tag{4.8}$$

The induced isomorphism $g : u(V) \longrightarrow u(V)^*$ is $U(V)$-equivariant, where the Lie group $U(V)$ acts on $u(V)$ via the adjoint and on $u(V)^*$ via the coadjoint action.

3. *For $A \in u(V)$ and $v \in V$, we have*

$$\langle A(iv), v\rangle_V = h\,(A(iv), v) = \langle A, -ivv^*\rangle_{u(V)}, \tag{4.9}$$

where $\langle\ ,\ \rangle_V := \mathrm{Re}\, h$ denotes the real inner product on V.

4. *The map $\mu_V : V \longrightarrow u(V),\ v \mapsto -\frac{i}{2}vv^*$ is a moment map for the $U(V)$-action of V.*

Proof. The first statement follows from differentiating the equation $UU^* = \mathrm{id}$ for $U \in U(V)$. The adjoint action of $U(V)$ on $u(V)$ is given by

$$\mathrm{ad}_U(A) = UAU^{-1} \text{ for } U \in U(V) \text{ and } A \in u(V), \tag{4.10}$$

whereas the coadjoint action of $U(V)$ on $u(V)^*$ is given by

$$\mathrm{ad}^*_U(\tau) = \tau \circ \mathrm{ad}_{U^{-1}} \text{ for } \tau \in u(V)^*. \tag{4.11}$$

We have to prove that for any $U \in U(V)$ the following diagram commutes:

$$\begin{array}{ccc} u(V)^* & \xrightarrow{\mathrm{ad}^*_U} & u(V)^* \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ u(V) & \xrightarrow[\mathrm{ad}_U]{} & u(V) \end{array} \tag{4.12}$$

Pick $A, B \in u(V)$ and $U \in U(V)$. We write $\langle\tau, B\rangle := \tau(B)$ for the evaluation of an element $\tau \in u(V)^*$ at the point $B \in u(V)$. Then:

$$\begin{aligned} \langle A, ad_{U^{-1}}B\rangle_{u(V)} &= \mathrm{tr}\left(A\,(U^*BU)^*\right) = \mathrm{tr}\,(AU^*B^*U) \\ &= \mathrm{tr}\,(UAU^*B^*UU^*) = \langle \mathrm{ad}_U A, B\rangle_{u(V)}. \end{aligned} \tag{4.13}$$

Therefore:

$$\langle \mathrm{ad}^*_U\,(g(A)), B\rangle = \langle g(A), \mathrm{ad}_{U^{-1}}\,(B)\rangle = \langle A, \mathrm{ad}_{U^{-1}}(B)\rangle_{u(V)} = \langle \mathrm{ad}_U A, B\rangle_{u(V)}. \tag{4.14}$$

Now we prove the third statement. Let $A \in u(V)$ and $v \in V$. Then:

$$\begin{aligned} \mathrm{im}\, h(A(iv), v) &= \frac{(h(A(iv), v) - h(v, A(iv)))}{2i} = \frac{(h(iv, A^*v) - h(v, A(iv)))}{2i} \\ &= \frac{(h(v, A^*v) + h(v, Av))}{2} = \frac{h(v, A^*v + Av)}{2} = 0. \end{aligned} \tag{4.15}$$

Therefore $\langle A(iv), v\rangle = h(A(iv), v) \in \mathbb{R}$. Let $v, w \in V$. We have $w^* = h(\circ, w) \in V^\vee$. The element $vw^* \in \mathrm{End}(V)$ denotes the image of $v \otimes w^*$ under the natural isomorphism $V \otimes V^\vee \cong \mathrm{End}(V)$. Then by definition $\mathrm{tr}(vw^*) = h(v, w)$ for all $v, w \in V$. It follows that:

$$\langle A(iv), v\rangle_V = h(A(iv), v) = \mathrm{tr}\,(iAvv^*) = \mathrm{tr}\,(A(-ivv^*)^*) = \langle A, -ivv^*\rangle. \tag{4.16}$$

To prove that $\mu_V : V \longrightarrow u(V)$ is a moment map for the $U(V)$-action on V, we have to verify two statements, namely

1. For each $A \in u(V)$, let $\mu_V^A : V \longrightarrow \mathbb{R},\ v \mapsto \langle \mu_V(v), A\rangle$ and let $A^\sharp$ be the vector field on V generated by A and the action of $U(V)$ on V. Then $d\mu_V^A = \iota_{A^\sharp}\omega$, where $\omega = -\operatorname{im} h$ is the symplectic form of V. (See [27, Chapter 8].)

2. The map μ_V is $U(V)$-equivariant.

The second statement is clearly true after what we have seen above. So let us prove the first claim. Let $A \in u(V)$. First note that for $v \in V$ we have $A^\sharp v = \frac{d}{dt}\big|_{t=0} \exp(tA)v = Av$. Hence for $v, w \in V$

$$\left(\iota_{A^\sharp}\omega\right)_v w = \omega_v\left(Av, w\right) = -\operatorname{im} h(Av, w). \tag{4.17}$$

Now we calculate μ_V^A:

$$\begin{aligned}\mu_V^A(v) = \langle A, \mu(v)\rangle_{u(V)} &= \left\langle A, -\frac{i}{2}vv^*\right\rangle_{u(V)} = \frac{1}{2}\left\langle A, -ivv^*\right\rangle_{u(V)} \\ &= \frac{1}{2}\left\langle A(iv), v\right\rangle_V = \frac{1}{2}h(A(iv), v) = \frac{1}{2}ih(Av, v).\end{aligned} \tag{4.18}$$

Because $h(A(iv), v) \in \mathbb{R}$, we conclude that $ih(Av, v) = -\operatorname{im} h(Av, v)$. Therefore:

$$\mu_V^A(v) = -\frac{1}{2}\operatorname{im} h(Av, v) \tag{4.19}$$

Let $w \in V$. We obtain:

$$\begin{aligned}d\mu_V^A(v)w &= \frac{d}{dt}\bigg|_{t=0}\left(-\frac{1}{2}\operatorname{im} h(A(v+tw), v+tw)\right) \\ &= -\frac{1}{2}\frac{d}{dt}\bigg|_{t=0}\operatorname{im}\big(h(Av, v) + th(Av, w) + th(Aw, v) + t^2h(Aw, w)\big) \\ &= -\frac{1}{2}\operatorname{im}\left(h(Av, w) + h(Aw, v)\right) = -\frac{1}{2}\operatorname{im}\left(h(Av, w) + h(w, -Av)\right) \\ &= -\frac{1}{2}\operatorname{im}\left(h(Av, w) - \overline{h(Av, w)}\right) = -\operatorname{im} h(Av, w).\end{aligned} \tag{4.20}$$

The equality $d\mu_V^A = \iota_{A^\sharp}\omega$ follows from the equations (4.17) and (4.20) . □

Corollary 4.1.5. *Let $\pi : u(V) \longrightarrow \mathfrak{g}$ be the adjoint of the $\rho_* : \mathfrak{g} \longrightarrow u(V)$. Then there exists $\tau \in \mathfrak{g}$, such that*

$$\mu = \pi \circ \mu_V + \tau. \tag{4.21}$$

Proof. Since $\mu_V : V \longrightarrow u(V)$ is a moment map for the $U(V)$-action on V, the composite map $\pi \circ \mu_V : V \longrightarrow \mathfrak{g}$ is a moment map for the G-action on V associated with the representation $\rho : G \longrightarrow U(V)$. Since G is Abelian, any two moment maps differ by a constant $\tau \in \mathfrak{g}$. □

The following observation will be an important ingredient for proving Theorem 4.4.1. It is due to Cieliebak, Gaio, and Salamon [12, Proof of Proposition 3.5].

Corollary 4.1.6. *Let $\tau \in \mathfrak{g}$, such that $\mu = \pi \circ \mu_V + \tau$. Then for any $v \in V$*

$$\langle \rho_*(\mu(v))(iv), v\rangle_V = 2\langle \mu(v), \mu(v) - \tau\rangle_{\mathfrak{g}}. \tag{4.22}$$

Proof. Let $v \in V$, then:

$$\begin{aligned}\langle \rho_*(\mu(v))(iv), v\rangle_V &= \langle \rho_*(\mu(v)), -ivv^*\rangle_{u(V)} = \langle \mu(v), \pi(-ivv^*)\rangle_{\mathfrak{g}} \\ &= \langle \mu(v), \pi(2\mu_V(v))\rangle_{\mathfrak{g}} = 2\langle \mu(v), \mu(v) - \tau\rangle_{\mathfrak{g}}.\end{aligned} \tag{4.23}$$

□

Lemma 4.1.7.

1. *There exist constants $C, D > 0$, such that*

$$|\mu(v)|_{\mathfrak{g}} \leq C|v|_V^2 + D \text{ for all } v \in V \tag{4.24}$$

2. *For all $v \in V$:* $\operatorname{tr}(\mu_V(v)) = -\frac{i}{2}|v|^2$.

Proof. The second statement follows immediately from the definition:

$$\operatorname{tr}(\mu_V(v)) = \operatorname{tr}\left(-\frac{i}{2}vv^*\right) = -\frac{i}{2}\operatorname{tr}(vv^*) = -\frac{i}{2}|v|^2. \tag{4.25}$$

Let $\tau \in \mathfrak{g}$, such that $\mu = \pi \circ \mu_V + \tau$. Let $v \in V$. Then:

$$|\mu(v)|_{\mathfrak{g}} \leq |\pi \circ \mu_V(v)|_{\mathfrak{g}} + |\tau|_{\mathfrak{g}} \leq ||\pi||\ |\mu_V(v)|_{u(V)} + |\tau|_{\mathfrak{g}}. \tag{4.26}$$

On the other hand

$$|\mu_V(v)|^2 = \operatorname{tr}(\mu_V(v)\mu_V(v)^*) = \operatorname{tr}\left(\frac{1}{4}vv^*vv^*\right). \tag{4.27}$$

The element vv^*vv^* denotes the endomorphism $(v \otimes v^*) \circ (v \otimes v^*) = |v|^2 v \otimes v^*$. Therefore

$$\operatorname{tr}\left(\frac{1}{4}vv^*vv^*\right) = \frac{1}{4}|v|^2\operatorname{tr}(vv^*) = \frac{|v|^4}{4}. \tag{4.28}$$

It follows that $|\mu(v)|_{\mathfrak{g}} \leq \frac{||\pi||}{2}|v|^2 + |\tau|_{\mathfrak{g}}$. □

The next proposition gives criteria for the properness of the moment map $\mu : V \longrightarrow \mathfrak{g}$.

Proposition 4.1.8. *Let $\mu : V \longrightarrow \mathfrak{g}$ be a moment map for the G-action on V. The following statements are equivalent:*

1. *The moment map μ is proper.*

2. *There exist constants $C, D > 0$, such that $|\mu(v)| \geq C|v|^2 - D$ for all $v \in V$.*

If there exists a constant $C > 0$, such that $|\pi(A)| \geq C|\operatorname{tr} A|$ for all $A \in u(V)$, then the moment map μ is proper.

Proof. Put $\mu_0 := \pi \circ \mu_V$. Since $\mu = \mu_0 + \tau$ for some $\tau \in \mathfrak{g}$, the moment map μ is proper if and only if μ_0 is proper.

Assume now that μ_0 is proper. If there is no constant $C > 0$, such that $|\mu_0(v)| \geq C|v|^2$ for all $v \in V$. Then for all $n \in \mathbb{N}_{>0}$ we find a vector $0 \neq v_n \in V$, with $|\mu_0(v_n)| < \frac{1}{n}|v_n|^2$. Observe that $\mu_V(\lambda v) = |\lambda|^2\mu_V(v)$ for all $\lambda \in \mathbb{C}$ and for all $v \in V$. Therefore,

$$\mu_0\left(\frac{v_n}{|v_n|}\right) = \frac{1}{|v_n|^2}|\mu_0(v_n)| < \frac{1}{n}. \tag{4.29}$$

Thus, the sequence $\mu_0(\frac{v_n}{|v_n|})$ converges to 0 as $n \to \infty$. We have $\frac{v_n}{|v_n|} \in S(V)$ for all $n \in \mathbb{N}$. Therefore, there is a subsequence $\frac{v_{n_k}}{|v_{n_k}|}$ converging to an element $\hat{v} \in S(V)$. By continuity, $\mu_0(\hat{v}) = 0$. For all $\lambda \in \mathbb{C}$, we have $\mu_0(\lambda\hat{v}) = |\lambda|^2\mu_0(\hat{v}) = 0$ and thus $\mathbb{C}\hat{v} \subset \mu_0^{-1}(0)$. This implies that $\mu_0^{-1}(0)$ is not compact, and thus μ_0 is not proper.

On the other hand, assume that there are constants $C, D > 0$, such that $|\mu(v)| \geq C|v|^2 - D$ for all $v \in V$. This inequality implies that preimages of bounded sets under μ are bounded and therefore that μ is proper.

We have shown that the first two statements are equivalent and that moreover, that μ_0 is proper if and only if for some $C > 0$ the inequality $|\mu_0(v)| \geq C|v|^2$ holds for all $v \in V$.

Assume that for some $C > 0$, the inequality $|\pi(A)| \geq C|\operatorname{tr}(A)|$ holds for all $A \in u(V)$. Then for all $v \in V$:

$$|\mu_0(v)| = |\pi(\mu_V(v))| \geq C|\operatorname{tr}(\mu_V(v)| = 2C|v|^2 \tag{4.30}$$

Therefore μ_0 is proper. □

4.2 Examples

We present two examples for the gauge theoretical moduli problem introduced in the previous section. See [33, §4] for a detailed discussion of more examples.

4.2.1 Projective spaces

Let $V := \mathbb{C}^r$ and let $G := S^1$ act on V via the representation $\rho_{can} : S^1 \longrightarrow U(r)$, $z \mapsto z\,\mathrm{id}$. The adjoint $\pi : u(r) \longrightarrow \mathfrak{g} = i\mathbb{R}$ of $(\rho_{can})_*$ is given by $\pi(A) = \mathrm{tr}(A)$ for $A \in u(r)$. Every moment map for this action is of the form

$$\mu_t(v) = -\frac{i}{2}|v|^2 + it \text{ for } v \in V, \tag{4.31}$$

for some parameter $t \in \mathbb{R}$. There are three different cases:

$$\mu_t^{-1}(0)/S^1 = \begin{cases} \emptyset & \text{if } t < 0 \\ \{*\} & \text{if } t = 0 \\ \mathbb{P}^{r-1}_{\mathbb{C}} & \text{if } t > 0. \end{cases}$$

To set up the gauge theoretical moduli problem we need to present S^1 as a subgroup of a compact group $\hat{G}$ that acts on $\mathbb{C}^r$ by a unitary representation $\rho : \hat{G} \longrightarrow U(r)$ with $\rho|_{S^1} = \rho_{can}$. Put $\hat{G} := S^1 \times U(r)$ and let $\rho : S^1 \times U(r) \longrightarrow U(r)$, $(z, U) \mapsto zU$. Then:

1. the discrete parameters consist of
   (a) compact and oriented smooth surface Σ;
   (b) a Hermitian line bundle $L \longrightarrow \Sigma$ and a Hermitian vector bundle $E \longrightarrow \Sigma$ of rank r.
2. the continuous parameters are given by
   (a) a Riemannian metric g_0 on Σ;
   (b) a constant $t \in \mathbb{R}$;
   (c) a Hermitian connection a_0 on E.

The moduli problem becomes then the following: Classify pairs $(\bar{\partial}_a, \varphi)$, consisting of a Hermitian connection a in L and of a section $\varphi : \Sigma \longrightarrow L \otimes E$, that verify the vortex equations:

- $\bar{\partial}_{a\otimes a_0}\varphi = 0$;
- $*F_a - \frac{i}{2}\varphi^*\varphi + it = 0$.

4.2.2 Toric varieties

We start with the data of an exact sequence of Lie groups

$$0 \longrightarrow G_w \longrightarrow \left(S^1\right)^r \xrightarrow{w} \left(S^1\right)^m \longrightarrow 1. \tag{4.32}$$

Let $w_* : (i\mathbb{R})^r \longrightarrow (i\mathbb{R})^m$ be its differential. It is given by a matrix $[e_{ij}]_{i=1}^{r}{}_{j=1}^{m}$. Let $\rho_w : G_w \longrightarrow U(r)$ be the representation given by the inclusions $G_w \subset \left(S^1\right)^r \hookrightarrow U(r)$, and let $\pi_w : u(r) \longrightarrow \mathfrak{g}_w$ be the adjoint of the differential $(\rho_w)_*$. Every moment map μ for the G_w-action on $\mathbb{C}^r$ is then of the form

$$\mu_t(v) = -\frac{i}{2}\pi_w\left(|v_1|^2, \ldots, |v_r|^2\right) + \tau \text{ for } v \in \mathbb{C}^r, \tag{4.33}$$

for some $\tau \in \mathfrak{g}_w$.

We put $G := G_w \subset \hat{G} := \left(S^1\right)^r$, and let $\rho : \hat{G} \longrightarrow U(r)$, $(z_i)_{i=1}^r \mapsto \mathrm{diag}(z_1, \ldots, z_r)$.

1. the discrete parameters consist of

(a) a compact and oriented smooth surface Σ;

(b) a family $(L_i)_{i=1}^r$ of Hermitian line bundles L_i.

We put $L_i^0 := \otimes_j L_j^{\otimes e_{ij}}$ for $i = 1, \ldots, m$.

2. the continuous parameters are given by

(a) a Riemannian metric g_0 on Σ;

(b) a constant $\tau \in \mathfrak{g}_\omega$;

(c) a family $(a_j^0)_{j=1}^m$ of Hermitian connections $a_j^0 \in \mathcal{A}(L_j^0)$.

The moduli problem becomes the following: Classify pairs (a, φ), consisting of a family $a = (a_i)_{i=1}^r$ of Hermitian connections a_i on L_i, such that $\otimes_j a_j^{\otimes e_{ij}} = a_i^0$, and of a family $\varphi = (\varphi_i)_{i=1}^r$ of sections $\varphi_i \in A^0(L_i)$ $(i = 1, \ldots, r)$, that verify the vortex equations:

- $\bar{\partial}_{a_i}\varphi_i = 0$ for $i = 1, \ldots, r$;
- $*F_{a_j} - \frac{i}{2}|\varphi_j|^2 + it = 0$ for $j = 1, \ldots, r$.

4.3 Reduction of the vortex map

In this section, we show how the vortex map $v : \mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E}) \longrightarrow A^{0,1}(\hat{E}) \times A^0(\Sigma, \mathfrak{g})$ induces a map $\nu_k : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$, where $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ are G-Hilbert bundles over a compact torus B, and where $S_{k+1} \subset A^1(\Sigma, \mathfrak{g})_{k+1}$ is a closed subspace.

Let g be any metric on Σ and denote by d_g the induced Levi-Civita connection on the cotangent bundle $T_\Sigma^\vee$. Let E be a Hermitian vector bundle over Σ and $d_A : A^0(E) \longrightarrow A^1(E)$ a unitary connection in E. The connection d_A on E and the Levi-Civita connection d_g on $T_\Sigma^\vee$ induce connections $d_g^{\wedge i} \otimes d_g^{\otimes l} \otimes d_A$ on the bundles $\Lambda^i T_\Sigma^\vee \otimes (T_\Sigma^\vee)^{\otimes l} \otimes E$ for $l \geq 0$. By composition of these operators, we obtain operators (see [15, page 37])

$$\nabla_A^{(k)} : A^i(E) \longrightarrow A^0(\Lambda^i T_\Sigma^\vee \otimes (T_\Sigma^\vee)^{\otimes k} \otimes E) \text{ for } k \geq 1. \tag{4.34}$$

These operators are used to define the Sobolev norms on the space of sections $A^0(E)$: let $\varphi \in A^0(E)$, then

$$||\varphi||_{L_k^p, A} := \left(\sum_{i=0}^k \int_\Sigma |\nabla_A^{(i)}\varphi|^p dvol_g \right)^{1/p}. \tag{4.35}$$

It is well known that the topology induced by these norms on $A^i(E)$ is independent of the choice of connection and metric. For $p = 2$, there is a scalar product $h_{k,A}$ that induces the norm $|| \, ||_{L_k^2, A}$. We work with the L_k^2-norm and write $|| \, ||_{L_k^2} := || \, ||_{L_k^2, A}$ when no confusion is possible. It is an important remark that the Hodge star operator is also an isometry with respect to the L_k^2-inner product. This is a consequence of the following lemma.

Lemma 4.3.1. *Let (X, g) be a compact, oriented Riemannian manifold of dimension n. Let E be a Hermitian vector bundle on X and $d_A : A^0(E) \longrightarrow A^1(E)$ a Hermitian connection in E. Let $* : A^i\left((T_X^\vee)^{\otimes l} \otimes E\right) \longrightarrow A^{n-i}\left((T_X^\vee)^{\otimes l} \otimes E\right)$ be the operator induced by the Hodge star operator on forms, and put*

$$\nabla_A^{i,l} := d_g^{\wedge i} \otimes d_g^{\otimes l} \otimes d_A : A^0\left(\Lambda^i T_X^\vee \otimes (T_X^\vee)^{\otimes l} \otimes E\right) \longrightarrow A^0\left(\Lambda^i T_X^\vee \otimes (T_X^\vee)^{\otimes^{l+1}} \otimes E\right). \tag{4.36}$$

Then:

$$\nabla_A^{n-i,l} \circ * = * \circ \nabla_A^{i,l}. \tag{4.37}$$

Proof. Let $\omega \otimes t \otimes s \in A^i(X) \otimes A^0\left((T_X^\vee)^{\otimes^l}\right) \otimes A^0(E)$. Then:

$$\begin{aligned}\nabla_A^{n-i,l} \circ * (\omega \otimes t \otimes s) &= \nabla_A^{n-i,l} (*\omega \otimes t \otimes s) \\ &= d_g^{\wedge(n-i)}(*\omega) \otimes t \otimes s + *\omega \otimes d_g^{\otimes l}(t) \otimes s + *\omega \times t \otimes d_A(s).\end{aligned} \tag{4.38}$$

On the other hand

$$\begin{aligned}* \circ \nabla_A^{i,l} (\omega \otimes t \otimes s) &= * \left(d_g^{\wedge i}(\omega) \otimes t \otimes s + \omega \otimes d_g^{\otimes l}(t) \otimes s + \omega \otimes t \otimes d_A(s)\right) \\ &= *d_g^{\wedge i}(\omega) \otimes t \otimes s + *\omega \otimes d_g^{\otimes l}(t) \otimes s + *\omega \otimes t \otimes d_A(s).\end{aligned} \tag{4.39}$$

Hence, it suffices to prove that $*d_g^{\wedge i}(\omega) = d_g^{\wedge n-i}(*\omega)$. For $\eta \in A^i(X)$, the defining equation for the Hodge star operator is:

$$\eta \wedge *\omega = \langle \eta, \omega \rangle dvol_g. \tag{4.40}$$

Applying $d_g^{\wedge n}$ to the left side of this equation yields

$$d_g^{\wedge n} (\eta \wedge *\omega) = d_g^{\wedge i}\eta \wedge *\omega + (-1)^i \eta \wedge d_g^{\wedge(n-i)}(*\omega). \tag{4.41}$$

Applying it to the right side and taking into accout that $d_g^{\wedge n}(dvol_g) = 0$, we obtain

$$\begin{aligned}d_g^{\wedge n} (\langle \eta, \omega \rangle dvol_g) &= \left(\langle d_g^{\wedge i}\eta, \omega \rangle + (-1)^i \langle \eta, d_g^{\wedge i}\omega \rangle\right) dvol_g \\ &= d_g^{\wedge i}\eta \wedge *\omega + (-1)^i \eta \wedge * \left(d_g^{\wedge i}\omega\right).\end{aligned} \tag{4.42}$$

Therefore $d_g^{\wedge i}(*\omega) = *(d_g^{\wedge i}\omega)$. □

Let us recall two main results of the theory of partial differential equations on manifolds. We cite them here for the reader's convenience. See [15, page 424], [35, §9], and [43, Theorems 10.4, 10.8, 10.12 and 10.13].

Let E be a Hermitian vector bundle on compact manifold X. Let $1 < p < \infty$ and $k \in \mathbb{N}$. With $L_k^p(E)$ we denote the topological vector space of L_k^p-sections in E.

Theorem 4.3.2 (Sobolev embedding). *Let E be a Hermitian vector bundle on a compact manifold X of dimension n. Let $C^r(E)$ denote the space of sections of class C^r in E. Let $1 < p < \infty$ and $k \in \mathbb{N}$.*

1. *If $k - n/p > r$, then the inclusion $\Gamma(E) \subset C^r(E)$ admits a continuous extension*

$$L_k^p(E) \hookrightarrow C^r(E).$$

2. *Let $l, q \in \mathbb{N}$, such that $k > l$ and $k - n/p \geq l - n/q$ hold. Then the inclusion $\Gamma(E) \subset L_l^q(E)$ admits a continuous extension*

$$L_k^p(E) \hookrightarrow L_l^q(E).$$

   If $k - n/p > l - n/q$, then this inclusion is compact.

Theorem 4.3.3 (Calderon-Zygmund). *Let E, F be Hermitian vector bundles on a compact Riemannian manifold X. Let $P : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator of order d. Let $1 < p < \infty$ and $k \in \mathbb{N}$.*

1. *There exists $C_k^p > 0$, such that for all $s \in L_{k+d}^p(E)$*

$$||s||_{L_{k+d}^p} \leq C_k^p \left(||s||_{L^p} + ||Ps||_{L_k^p}\right);$$

2. *Let L be a topological complement of $\ker P$ in $L_{k+d}^p(E)$. There exists a constant $C_{k,L}^p > 0$, such that*

$$||s||_{L_{k+d}^p} \leq C_{k,L}^p \left(||pr_{\ker P}(s)||_{L^p} + ||Ps||_{L_k^p}\right).$$

Theorem 4.3.4 (Sobolev multiplication). *Let E, F be Hermitian vector bundles on a compact surface Σ and let $k \geq 1$, $l \geq k$, and $l > 1$. The morphism $\Gamma(E) \times \Gamma(F) \longrightarrow \Gamma(E \otimes F)$ admits a continuous extension*

$$L^2_k(E) \times L^2_l(F) \longrightarrow L^2_k(E \otimes F).$$

Next we need the following inequality, resulting from a Weitzenböck type formula (see [23, Proposition 1.5, page 50]).

Lemma 4.3.5. *Let Σ be a Riemann surface, $E \longrightarrow \Sigma$ a Hermitian vector bundle with Hermitian metric h, A a Hermitian connection on E, and $\varphi \in A^0(E)$ a section. Then:*

$$i * \bar{\partial}\partial h(\varphi, \varphi) \leq h(i * F_A(\varphi), \varphi) + 2|h(i * \partial_A \bar{\partial}_A \varphi, \varphi)|.$$

Proof. Since A is a Hermitian connection, we have for all $\alpha \in A^i(E)$ and $\beta \in A^j(E)$, that

$$\partial h(\alpha, \beta) = h(\partial_A \alpha, \beta) + (-1)^i h(\alpha, \bar{\partial}_A \beta) \text{ and } \bar{\partial} h(\alpha, \beta) = h(\bar{\partial}_A \alpha, \beta) + (-1)^i h(\alpha, \partial_A \beta). \tag{4.43}$$

Therefore:

$$\begin{aligned}
\bar{\partial}\partial h(\varphi, \varphi) &= \bar{\partial} h(\partial_A \varphi, \varphi) + \bar{\partial} h(\varphi, \bar{\partial}_A \varphi) \\
&= h(\bar{\partial}_A \partial_A \varphi, \varphi) - h(\partial_A \varphi, \partial_A \varphi) + h(\bar{\partial}_A \varphi, \bar{\partial}_A \varphi) + h(\varphi, \partial_A \bar{\partial}_A \varphi) \\
&= h\bar{\partial}_A \partial_A \varphi, \varphi) + h(\partial_A \bar{\partial}_A \varphi, \varphi) - h(\partial_A \bar{\partial}_A \varphi, \varphi) \\
&- h(\partial_A \varphi, \partial_A \varphi) + h(\bar{\partial}_A \varphi, \bar{\partial}_A \varphi) + h(\varphi, \partial_A \bar{\partial}_A \varphi) \\
&= h(F_A(\varphi), \varphi) - h(\partial_A \varphi, \partial_A \varphi) + h(\bar{\partial}_A \varphi, \bar{\partial}_A \varphi) - h(\partial_A \bar{\partial}_A \varphi, \varphi) + h(\varphi, \partial_A \bar{\partial}_A \varphi).
\end{aligned} \tag{4.44}$$

We obtain the following formula:

$$\begin{aligned}
i * \bar{\partial}\partial h(\varphi, \varphi) =& i * h(F_A(\varphi), \varphi) - i * h(\partial_A \bar{\partial}_A \varphi, \varphi) + i * h(\varphi, \partial_A \bar{\partial}_A \varphi) \\
&- i * h(\partial_A \varphi, \partial_A \varphi) + i * h(\bar{\partial}_A \varphi, \bar{\partial}_A \varphi).
\end{aligned} \tag{4.45}$$

In the next step we proof that $i * h(\partial_A \varphi, \partial_A \varphi) \geq 0$ and $i * h(\bar{\partial}_A \varphi, \bar{\partial}_A \varphi) \leq 0$. This statement is local on Σ, so we choose a holomorphic coordinate $z = x + iy$ on Σ. Let $\gamma dx \wedge dy$ be the volume form with $\gamma : \Sigma \longrightarrow \mathbb{R}_{\geq 0}$. Note that $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$, and hence $*(dz \wedge d\bar{z}) = -2i\gamma^{-1}$. Let $\alpha = dz \otimes a$ be a $(1, 0)$ form, where a is a local section in E . Then

$$i * h(\alpha, \alpha) = i * h(a, a) dz \wedge d\bar{z} = 2g^{-1}|a|^2 \geq 0.$$

For a $(0, 1)$-form $\beta = d\bar{z} \otimes b$, where b is a local section in E, we obtain

$$i * h(\beta, \beta) = i * h(b, b) d\bar{z} \wedge dz = -2g^{-1}|b|^2 \leq 0.$$

Now the Lemma follows from the observation that

$$- h(i * \partial_A \bar{\partial}_A \varphi, \varphi) - h(\varphi, i * \partial_A \bar{\partial}_A \varphi) = -2 \operatorname{Re} h(i * \partial_A \bar{\partial}_A \varphi, \varphi). \tag{4.46}$$

□

Lemma 4.3.6. *Let E be a Hermitian vector bundle with metric h over a Riemann surface Σ. Let $\varphi \in A^0(E)$. Let $p \in \Sigma$ be a point, in which $h(\varphi, \varphi)$ attains its maximum. Then:*

$$0 \leq \left(i * \bar{\partial}\partial h(\varphi, \varphi)\right)(p).$$

Proof. Choose holomorphic coordinates $z = x + iy$ in a neighbourhood of p. Then for a function $f \in A^0(\Sigma)$, we obtain

$$i\bar{\partial}\partial f = i\frac{\partial^2 f}{\partial \bar{z} \partial z} d\bar{z} \wedge dz = \frac{i}{4}\Delta f(2idx \wedge dy) = -\frac{1}{2\tilde{g}} vol_g, \tag{4.47}$$

where $\gamma dx \wedge dy = vol_g$ is the volume form. Therefore:

$$i * \bar{\partial}\partial f = -\frac{1}{2\gamma}\Delta f. \tag{4.48}$$

Put $f := h(\varphi, \varphi)$. By assumption f has a maximum in p, and therefore the maximum principle [18, Theorem 2.2 on page 15] implies $\Delta f(x) \leq 0$. Thus $0 \leq \left(-\frac{1}{2\gamma}\Delta f\right)(p) = \left(i * \bar{\partial}\partial f\right)(p)$. □

Now we return to the study of the vortex map v.

Lemma 4.3.7. *Let $k > 1$. Then the map*

$$\mu : A^0(\hat{E}) \longrightarrow A^0(\Sigma, \mathfrak{g}),\ \varphi \mapsto \mu(\varphi) \tag{4.49}$$

admits a smooth extension

$$\mu : A^0(\hat{E})_k \longrightarrow A^0(\Sigma, \mathfrak{g})_k. \tag{4.50}$$

There are constants $C, D > 0$, such that $||\mu(\varphi)||_{L^2_k} \leq C||\varphi||^2_{L^2_k} + D$ for all $\varphi \in A^0(\hat{E})$.

Proof. Let $\tau \in \mathfrak{g}$, such that $\mu = \pi \circ \mu_V + \tau$. Then

$$\mu(\varphi) = (\pi \circ \mu_V)(\varphi) + \tau = \pi\left(-\frac{i}{2}\varphi\varphi^*\right) + \tau. \tag{4.51}$$

The map $\pi : u(V) \longrightarrow \mathfrak{g}$ is a linear map between finite dimensional vector spaces. Consider now the bilinear map

$$b : A^0(\hat{E}) \times A^0(\hat{E}) \longrightarrow A^0(\mathrm{End}(E)),\ (\varphi, \psi) \mapsto \varphi\psi^*. \tag{4.52}$$

By Theorem 4.3.4 there is a constant $C' > 0$, such that $||\varphi\psi^*||_{L^2_k} \leq C'||\varphi||_{L^2_k}||\psi||_{L^2_k}$ for all $\varphi, \psi \in A^0(\hat{E})$. It follows that the map b is a continuous and bilinear map and admits a continuous and bilinear extension. Therefore the associated map $A^0(\hat{E}) \longrightarrow A^0(\Sigma, \mathfrak{g})$, $\varphi \mapsto \pi\left(-\frac{i}{2}\varphi\varphi^*\right)$ admits a smooth extension and $||\mu(\varphi)||_{L^2_k} \leq ||\pi||C'/2||\varphi||^2_{L^2_k} + |\tau|$ for all $\varphi \in A^0(\hat{E})$. □

Proposition 4.3.8. *Assume that $k > 1$. Then the map v defines a smooth map*

$$v_k : \mathcal{A}_{a_0}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1} \longrightarrow \mathcal{A}^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k.$$

For $(\hat{a}, \varphi) \in \mathcal{A}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1}$ and for $(t, \psi) \in A^1(\Sigma, \mathfrak{g})_{k+1} \times A^0(\Sigma)_{k+1}$ we have

$$dv_k(\hat{a}, \varphi)(t, \psi) = \left(\rho_*(t)^{(0,1)}(\varphi) + \bar{\partial}_{\hat{a}}\psi, *dt + d\mu(\varphi)\psi\right).$$

Proof. Fix a connection $\hat{a}_0 \in \mathcal{A}_{a_0}(\hat{P})$, so that $\mathcal{A}_{a_0}(\hat{P}) = \hat{a}_0 + A^1(\Sigma, \mathfrak{g})$. Use this connection to define the norms on the space of sections $A^0(\hat{E})$ and $A^{0,1}(\hat{E})$ and also on $\mathcal{A}_{a_0}(\hat{P})$. Write $v^i := pr_i \circ v$ $(i = 1, 2)$ for the projections on the two factors of the target space of v. The map v^1 takes the form

$$v^1(\hat{a}_0 + t, \varphi) = \bar{\partial}_{\hat{a}_0+t}\varphi = \bar{\partial}_{\hat{a}_0}\varphi + \rho_*(t)^{0,1}(\varphi) \text{ for } t \in A^1(\Sigma, \mathfrak{g}) \text{ and } \varphi \in A^0(\hat{E}). \tag{4.53}$$

The map $\varphi \mapsto \bar{\partial}_{\hat{a}_0}\varphi$ is a continuous linear map, whereas the map $(t, \varphi) \mapsto \rho_*(t)^{0,1}(\varphi)$ is bilinear and by Theorem 4.3.4 continuous. Therefore v^1 is continuous and moreover, it induces on the completions a map $v^1_k : \mathcal{A}_{a_0}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1} \longrightarrow \mathcal{A}^{0,1}(\hat{E})_k$, which is the sum of a continuous linear and a continuous bilinear map and therefore smooth.

The second component v^2 has the form

$$v^2(\hat{a}_0 + t, \varphi) = pr_{\mathfrak{g}} * F_{\hat{a}_0} + *dt + \mu(\varphi). \tag{4.54}$$

The differential and the Hodge Star operator induce bounded linear operators $d : A^1(\Sigma, \mathfrak{g})_{k+1} \longrightarrow A^2(\Sigma, \mathfrak{g})_k$ and $* : A^2(\Sigma, \mathfrak{g})_k \longrightarrow A^0(\Sigma, \mathfrak{g})_k$ on the completions. By Lemma 4.3.7 the map $A^0(\hat{E}) \longrightarrow$

$A^0(\Sigma, \mathfrak{g}), \varphi \mapsto \mu(\varphi)$ induces a smooth map $A^0(E)_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_{k+1} \subset A^0(\Sigma, \mathfrak{g})_k$. Therefore v^2 induces a smooth map $v_k^2 : \mathcal{A}_{a_0}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k$.

Let $(\hat{a}, \varphi) \in \mathcal{A}_{a_0}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1}$ and $(t, \psi) \in A^1(\Sigma, \mathfrak{g})_{k+1} \times A^0(\hat{E})_{k+1}$. We calculate $dv_k(\hat{a}, \varphi)(t, \psi)$. First note that

$$\begin{aligned} dv_k^1(\hat{a}, \varphi)(t, 0) &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} v_k^1(\hat{a} + \lambda t, \varphi) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \bar{\partial}_{\hat{a}+\lambda t}\varphi \\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} \left(\bar{\partial}_{\hat{a}}\varphi + \lambda \rho_*(t)^{0,1}(\varphi)\right) \\ &= \rho_*(t)^{0,1}(\varphi). \end{aligned} \tag{4.55}$$

Second observe that v_k^1 is linear in the second argument and therefore

$$dv_k^1(\hat{a}, \varphi)(0, \psi) = \bar{\partial}_{\hat{a}}\psi. \tag{4.56}$$

The equations (4.55) and (4.56) together imply that

$$dv_k^1(\hat{a}, \varphi)(t, \psi) = \rho_*(t)^{(0,1)}(\varphi) + \bar{\partial}_{\hat{a}}\psi. \tag{4.57}$$

Now are left with the calculation of the differential of v_k^2. Again, first we differentiate in direction of a vector $t \in A^1(\Sigma, \mathfrak{g})_{k+1}$:

$$\begin{aligned} dv_k^2(\hat{a}, \varphi)(t, 0) &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} v_k^2(\hat{a} + \lambda t, \varphi) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} pr_{\mathfrak{g}} * F_{\hat{a}} + \lambda * dt + \mu(\varphi) \\ &= *dt. \end{aligned} \tag{4.58}$$

Then, we differentiate in direction of a vector $\psi \in A^0(\hat{E})_{k+1}$:

$$\begin{aligned} dv_k^2(\hat{a}, \varphi)(0, \psi) &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} v_k^2(\hat{a}, \varphi + \lambda\psi) = \left.\frac{d}{dt}\right|_{t=0} pr_{\mathfrak{g}} * F_{\hat{a}} + \mu(\varphi + \lambda\psi) \\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mu(\varphi + \lambda\psi) = d\mu(\varphi)\psi. \end{aligned} \tag{4.59}$$

From (4.58) and (4.59), we obtain $dv_k^2(\hat{a}, \varphi)(t, \psi) = *dt + d\mu(\varphi)(\psi)$, which completes the proof. □

Similar to [34, §3.4], we choose now L^2-closed linear subspaces $S \subset T \subset A^1(\Sigma, \mathfrak{g})$, such that

1. $A^1(\Sigma, \mathfrak{g}) = S \oplus \ker d = T \oplus \operatorname{im} d$.
2. The projections $p_S : A^1(\Sigma, \mathfrak{g}) \longrightarrow S$ and $p_T : A^1(\Sigma, \mathfrak{g}) \longrightarrow T$ are continuous in the L_k^2-topology for all $k \geq 0$.

We put $H := T \cap \ker d$. Furthermore we choose a T-affine subspace $\mathcal{A} \subset \mathcal{A}_{a_0}(\hat{P})$.

Remark 4.3.9. Let g be any metric on Σ. Then $S := d^* A^2(\Sigma, \mathfrak{g})$ and $T := \ker d^*$ satisfy these two conditions. The important point is that we do not need to use the metric g_0 on which the equations depend.

Remark 4.3.10. The morphism $H \subset \ker d \longrightarrow H_{dR}^1(\Sigma, \mathfrak{g})$ is an isomorphism of vector spaces.

Proof. Remember that we have $H = T \cap \ker d$ and $A^1(\Sigma, \mathfrak{g}) = T \oplus \operatorname{im} d$. We claim that this implies $\ker d = H \oplus \operatorname{im} d$. To see this, let $\omega \in \ker d \subset A^1(\Sigma, \mathfrak{g})$. Then $\omega = t + df$ for $t \in T$ and $f \in A^0(\Sigma, \mathfrak{g})$. Furthermore, $0 = d\omega = dt$. Thus $t \in T \cap \ker d = H$ and hence $\omega \in H \oplus \operatorname{im} d$. □

Let $u \in C^\infty(\Sigma, G)$. Then we write $u^{-1}du \in A^1(\Sigma, \mathfrak{g})$ for the $\mathfrak{g}$-valued 1-form on Σ, defined as follows:

$$\left(u^{-1}du\right)(x) = dL_{u(x)^{-1}}(u(x)) \circ du(x) : T_x\Sigma \longrightarrow \mathfrak{g}. \tag{4.60}$$

Put $\mathfrak{G} := \{u \in C^\infty(\Sigma, G) \,|\, u^{-1}du \in H\} \subset C^\infty(\Sigma, G)$. By definition, the actions of $C^\infty(\Sigma, G)$ on $\mathcal{A}_{a_0}(\hat{P}) \times A^0(\hat{E})$ and on $A^{0,1}(\hat{E})$ induce actions of $\mathfrak{G}$ on $\mathcal{A} \times A^0(\hat{E})$ and on $A^{0,1}(\hat{E})$.

The reason for making this reduction is provided by the following observation (see [15, §2.3.1] and [43, Proposition 6.10]):

Proposition 4.3.11.

1. *The subspace $\mathcal{A}$ intersects every $C^\infty(\Sigma, G)$-orbit in $\mathcal{A}_{a_0}(\hat{P})$.*

2. *Two connections in $\mathcal{A}$ are in the same $C^\infty(\Sigma, G)$-orbit if and only if they are in the same $\mathfrak{G}$-orbit.*

Proof. Fix a point $\hat{a}_0 \in \mathcal{A}$. Every connection $\hat{a} \in \mathcal{A}_{a_0}(\hat{P})$ is then of the form $\hat{a} = \hat{a}_0 + \omega$ for some $\omega = t + df \in A^1(\Sigma, \mathfrak{g}) = T \oplus dA^0(\Sigma, \mathfrak{g})$. Put $u := e^{-f}$. Then

$$\hat{a} + u^{-1}du = \hat{a}_0 + t + df - e^f e^{-f} df = \hat{a}_0 + t \in \mathcal{A}. \tag{4.61}$$

This proves the first statement.

Let $\hat{a}_1 \in \mathcal{A}$ be in the $C^\infty(\Sigma, G)$-orbit of $\hat{a}_0$. That means that $\hat{a}_1 = \hat{a}_0 + u^{-1}du$ for some $u \in C^\infty(\Sigma, G)$. Then clearly $u^{-1}du = \hat{a}_1 - \hat{a}_0 \in T$ and therefore $u \in \mathfrak{G}$. □

The subspaces S and T of $A^1(\Sigma, \mathfrak{g})$ canonically induce closed subspaces S_k and T_k in $A^1(\Sigma, \mathfrak{g})_k$. As a consequence of the next lemma, these subspaces satisfy

$$A^1(\Sigma, \mathfrak{g})_k = S_k \oplus \ker d = T_k \oplus \operatorname{im} d. \tag{4.62}$$

Note that the exterior differential $d : A^i(\Sigma, \mathfrak{g}) \longrightarrow A^{i+1}(\Sigma, \mathfrak{g})$ induces morphisms on the completions $d : A^i(\Sigma, \mathfrak{g})_{k+1} \longrightarrow A^{i+1}(\Sigma, \mathfrak{g})_k$, which we will denote by the same symbol.

Lemma 4.3.12. *Let V be a normed vector space, and denote its completion by $i : V \longrightarrow \bar{V}$. Let $A, B \subset V$ be a pair of closed subspaces, such that $V = A \oplus B$ and that the projections $p_A : V \longrightarrow A$ and $p_B : V \longrightarrow B$ are continuous. Then the spaces $\bar{A} := \overline{i(A)}^{\bar{V}}$ and $\bar{B} := \overline{i(B)}^{\bar{V}}$ are completions of A and B, respectively and*

$$\bar{V} = \bar{A} \oplus \bar{B}. \tag{4.63}$$

Proof. The first statement is clear. Let $\bar{v} \in \bar{A} \cap \bar{B}$. There exist sequences $a_n \in A$ and $b_n \in B$ that converge both to $\bar{v} \in \bar{V}$. Therefore, the sequence $a_n - b_n$ converges to 0 in V. By assumption, the projections $p_A : V \longrightarrow A$ and $p_B : V \longrightarrow B$ are continuous. Therefore $p_A(a_n - b_n) = a_n$ converges to 0, whence $\bar{v} = 0$.

Given any $\bar{v} \in \bar{V}$, we find a sequence $v_n \in V$ that converges to $\bar{v} \in \bar{V}$. Write $v_n = a_n + b_n$ with $a_n \in A$ and $b_n \in B$ uniquely determined by this equation. Continuity of the projections implies that both a_n and b_n are Cauchy sequences in A and B, respectively. Thus they converge to elements $\bar{a}$ and $\bar{b}$ in $\bar{A}$ and $\bar{B}$ with $\bar{a} + \bar{b} = \bar{v}$. □

Lemma 4.3.13. *Put $H_k := T_k \cap \ker d \subset A^1(\Sigma, \mathfrak{g})_k$ and $H := T \cap \ker d \subset A^1(\Sigma, \mathfrak{g})$. Then $H_k = H$.*

Proof. By definition we have the decompositions $A^1(\Sigma, \mathfrak{g}) = S \oplus \ker d = T \oplus \operatorname{im} d$ and $T = H \oplus S$. For for once we write $d_k : A^1(\Sigma, \mathfrak{g})_k \longrightarrow A^1(\Sigma, \mathfrak{g})_{k-1}$. We claim that

$$H_k = T_k \cap \ker d_k = \overline{T \cap \ker d} \subset A^1(\Sigma, \mathfrak{g})_k. \tag{4.64}$$

Since $T \cap \ker d \subset T_k \cap \ker d_k$, and because $T_k \cap \ker d_k \subset A^1(\Sigma, \mathfrak{g})_k$ is closed, the inclusion $\overline{T \cap \ker d} \subset H_k$ results.

So let $\hat{h} \in H_k = T_k \cap \ker d_k$. Then there are sequences $t_n \in T$ and $c_n \in \ker d$, such that $t_n \to \hat{h} \in T_k$ and $c_n \to \hat{h} \in \ker d_k$. We use now the decomposition $T = H \oplus S$ and write $t_n = h_n + s_n$ with $h_n \in H$ and $s_n \in S$. It follows that $h_n - c_n + s_n \to 0 \in A^1(\Sigma, \mathfrak{g})_k$. But $h_n - c_n \in \ker d$ and $s_n \in S$. Let $p_S : A^1(\Sigma, \mathfrak{g}) = S \oplus \ker d \longrightarrow S$ be the projection. Then $p_S(h_n - c_n + s_n) = s_n \to 0$. This implies that $h_n \to \hat{h}$ and thus that $\hat{h} \in \overline{H}$.Therefore $H_k = \overline{H} \subset A^1(\Sigma, \mathfrak{g})_k$. But Remark 4.3.10 implies that H is a finite dimensional. Therefore $\overline{H} = H$. □

Therefore, the subspace

$$H = T_k \cap \ker d \subset A^1(\Sigma, \mathfrak{g}) \subset A^1(\Sigma, \mathfrak{g})_k \tag{4.65}$$

is independent of k. Observe that Lemma 4.3.12 implies the following:

- $A^1(\Sigma, \mathfrak{g})_k = S_k \oplus \ker d = T_k \oplus \operatorname{im} d$;

- $T_k = H \oplus S_k$.

The gauge group

The goal of section is to study the gauge group $\mathfrak{G}$. We will show that it splits as $\mathfrak{G} = G \oplus \mathfrak{G}_0$, where $\mathfrak{G}_0$ is isomorphic to a lattice in the vector space H. This discussion is a straightforward generalization of the statements proven in [43, pages 44-48].

Let $\exp : \mathfrak{g} \longrightarrow G$ be the exponential map and put $\Gamma := \exp^{-1}(e)$. We recall that every connected compact Abelian Lie group is isomorphic to a torus and that the exponential map is its universal cover. See [10, Theorem 3.6 and its proof on page 25] for details. Therefore the subset $\Gamma \subset \mathfrak{g}$ is a lattice.

Remember that by the universal coefficient theorem [40, Theorem 5.3 on page 243], we have canonical isomorphisms

- $H^1(\Sigma, \mathfrak{g}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z}), \mathfrak{g})$;
- $H^1(\Sigma, \Gamma) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z}), \Gamma)$.

The de Rham theorem asserts that integration defines an isomorphism

$$H^1_{dR}(\Sigma, \mathfrak{g}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z}), \mathfrak{g}).$$

We write $H^1_{dR}(\Sigma, \Gamma) \subset H^1_{dR}(\Sigma, \mathfrak{g})$ for the preimage of $\mathrm{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z}), \Gamma) \cong H^1(\Sigma, \Gamma)$ under the de Rham isomorphism, and H_Γ for the preimage of $H^1_{dR}(\Sigma, \Gamma)$ in H under the canonical isomorphism $H \xrightarrow{\cong} H^1_{dR}(\Sigma, \mathfrak{g})$.

Proposition 4.3.14. *Let $\gamma : C^\infty(\Sigma, G) \longrightarrow A^1(\Sigma, \mathfrak{g})$ be defined by $\gamma(u) := u^{-1}du$ for $u \in C^\infty(\Sigma, G)$. The restriction of γ to $\mathfrak{G} \subset C^\infty(\Sigma, G)$ defines a morphism of groups $\gamma : \mathfrak{G} \longrightarrow H_\Gamma \subset A^1(\Sigma, \mathfrak{g})$. The natural sequence of groups*

$$0 \longrightarrow G \longrightarrow \mathfrak{G} \longrightarrow H_\Gamma \longrightarrow 0 \tag{4.66}$$

is exact. Evaluation at a point $x_0 \in \Sigma$ defines a section $ev(x_0) : \mathfrak{G} \longrightarrow G$ in this sequence and induces a splitting $\mathfrak{G} \cong G \oplus \mathfrak{G}_0$ with $\mathfrak{G}_0 := \ker ev(x_0) \cong H_\Gamma \subset H$.

Before proving this proposition, we need some preparations. First, we recall a result from topology:

Proposition 4.3.15. *The map*

$$[\Sigma, S^1] \longrightarrow H^1(\Sigma, \mathbb{Z}),\ [f] \mapsto H^1(f)\,(PD[*])$$

is a bijection.

Proof. This is an immediate consequence of [40, Theorem 8 on page 427]. □

Let $[\gamma] : C^\infty(\Sigma, G) \longrightarrow H^1_{dR}(\Sigma, \mathfrak{g})$ be defined by $[\gamma](u) := [u^{-1}du]_{dR}$.

Lemma 4.3.16. *The image of $[\gamma]$ coincides with $H^1_{dR}(\Sigma, \Gamma)$.*

Proof. Choose an isomorphism $G \xrightarrow{\cong} (S^1)^r \times A$, where $r \geq 0$ and A is a finite, Abelian group. It induces a commutative diagram

$$\begin{array}{ccc} C^\infty(\Sigma, G) & \xrightarrow{\cong} & C^\infty(\Sigma, S^1)^r \times A \\ \downarrow & & \downarrow \\ H^1_{dR}(\Sigma, \mathfrak{g}) & \xrightarrow{\cong} & H^1_{dR}(\Sigma, i\mathbb{R})^{\oplus r} \end{array}$$

This shows that we may assume w.l.o.g. that $G = S^1$. Now we apply the previous proposition. First note that under the de Rham isomorphism $H^1(S^1, \mathbb{Z}) \ni PD[*] \mapsto \omega_0 := \frac{1}{2\pi i}\frac{dz}{z} \in H^1_{dR}(S^1, \mathbb{Z})$. Hence, we obtain a surjective map

$$[\Sigma, S^1]_\infty \longrightarrow H^1_{dR}(\Sigma, 2\pi i\mathbb{Z}),\ [u] \mapsto 2\pi i u^*\omega_0,$$

where $[\Sigma, S^1]_\infty$ denotes the set of smooth homotopy classes of smooth maps from Σ to S^1. But $2\pi i u^*\omega_0 = u^{-1}du$. □

Now we can prove Proposition 4.3.14:

Proof. Let $\omega \in H_\Gamma$. We have already seen that there is a smooth function $u : \Sigma \longrightarrow G$, such that $[u^{-1}du]_{dR} = [\omega] \in H^1_{dR}(\Sigma, \mathfrak{g})$. Therefore $u^{-1}du = \omega + df$ for a function $f \in A^0(\Sigma, \mathfrak{g})$. Put $v := \exp(-f)u \in C^\infty(\Sigma, G)$. Exactness to the right follows then from

$$v^{-1}dv = e^f u^{-1}\left(e^{-f}du - ue^{-f}df\right) = u^{-1}du - df = \omega \in H_\Gamma.$$

To prove exactness in the middle, note that we have an isomorphism $H \xrightarrow{\cong} H^1_{dR}(\Sigma, \mathfrak{g})$. Since for any $u \in \mathfrak{G}$ we have $u^{-1}du \in H$, it follows that $u^{-1}du = 0$ iff u is constant. □

Corollary 4.3.17. *Let $k \geq 1$.*

1. *The group $\mathfrak{G}_0$ acts freely on the affine Banach space $\mathcal{A}_k$ with quotient*

$$\mathcal{A}^0_k := \mathcal{A}_k/\mathfrak{G}_0 \cong S_k \times H/H_\Gamma.$$

2. *The group S_k acts freely on $\mathcal{A}_k$ with quotient*

$$\bar{\mathcal{A}} := \mathcal{A}_k/S_k \cong H,$$

which is independent of k.

These group actions are compatible in the sense that the action of $\mathfrak{G}_0$ descends to an action on $\bar{\mathcal{A}}$ and the action of S_k descends to an action on $\mathcal{A}^0_k$. The respective quotients are canonically identified and independent of k:

$$B := \mathcal{A}^0_k/S_k = \bar{\mathcal{A}}/\mathfrak{G}_0.$$

There is a (non canonical) isomorphism $B \cong \left(S^1\right)^{b_1(\Sigma)\dim \mathfrak{g}}$.

Proof. The space of connections $\mathcal{A}_k$ is an affine space with model Banach space $T_k = H \oplus S_k$ and $\mathfrak{G}_0$ is canonically identified with the subspace $H_\Gamma \subset T_k$ in such a way that the action of $\mathfrak{G}_0$ corresponds to the affine action of H_Γ on $\mathcal{A}_k$. The group H_Γ is a lattice in the vector space H, which has dimension $\dim H = \dim H^1(\Sigma, \mathfrak{g}) = b_1(\Sigma)\dim \mathfrak{g}$. The remaining statements are then an immediate consequence. □

Lemma 4.3.18. *Let $\hat{a}_0, \hat{a}_1 \in \mathcal{A}_{a_0}(\hat{P})$ be two connections. Then:*

$$[pr_{\mathfrak{g}} F_{\hat{a}_0}] = [pr_{\mathfrak{g}} F_{\hat{a}_1}] \in H^2_{dR}(\Sigma, \mathfrak{g}).$$

Proof. Since $\mathcal{A}_{a_0}(\hat{P})$ is an $A^1(\Sigma, \mathfrak{g})$-affine space, we have $\hat{a}_0 - \hat{a}_1 = \omega$ for some $\omega \in A^1(\Sigma, \mathfrak{g})$. Hence $F_{\hat{a}_0} = F_{\hat{a}_1} + d\omega$, and also $pr_{\mathfrak{g}} F_{\hat{a}_0} = pr_{\mathfrak{g}} F_{\hat{a}_1} + d\omega$. □

We put $c(\hat{P}) := [pr_{\mathfrak{g}} F_{\hat{a}_0}] \in H^2_{dR}(\Sigma, \mathfrak{g})$. Now we pick a representative $\omega \in c(\hat{P})$. We define the space of ω-connections to be

$$\mathcal{A}(\omega) := \{\hat{a} \in \mathcal{A} \,|\, pr_{\mathfrak{g}} * F_{\hat{a}} = \omega\} \subset \mathcal{A}. \tag{4.67}$$

Proposition 4.3.19. *The space $\mathcal{A}(\omega) \subset \mathcal{A}$ is a H-affine subspace of $\mathcal{A}$. It defines a $\mathfrak{G}_0$-equivariant section in the S_k-bundle $\mathcal{A}_k \longrightarrow \bar{\mathcal{A}}$ which descends to a section in the S_k-bundle $\mathcal{A}^0_k \longrightarrow B$. In particular, the map $\mathcal{A}(\omega) \subset \mathcal{A}_k \longrightarrow \bar{\mathcal{A}}$ is an isomorphism and we obtain isomorphisms*

$$\mathcal{A}_k \cong \mathcal{A}(\omega) \times S_k, \;\; \mathcal{A}^0_k \cong B \times S_k. \tag{4.68}$$

Proof. Let $\hat{a}_i$ $(i = 0, 1)$ be two ω-connections. Then $\hat{a}_0 - \hat{a}_1 \in T = S \oplus H$. Let $s \in S$ and $h \in H$, such that $\hat{a}_0 - \hat{a}_1 = s + h$. Then by assumption

$$0 = pr_{\mathfrak{g}} * F_{\hat{a}_0} - pr_{\mathfrak{g}} * F_{\hat{a}_1} = *d(s + h) = *ds. \tag{4.69}$$

It follows that $ds = 0$. But $d : S \longrightarrow A^2(\Sigma, \mathfrak{g})$ is injective. Therefore $s = 0$. □

Let us return to the study of the map

$$v_k : \mathcal{A}_{a_0}(\hat{P})_{k+1} \times A^0(\hat{E})_{k+1} \longrightarrow A^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k. \tag{4.70}$$

By restricting v_k to the slice $\mathcal{A}_{k+1} \subset \mathcal{A}_{a_0}(\hat{P})_{k+1}$ and by using the identification $\mathcal{A}(\omega) \times S_{k+1} \cong \mathcal{A}_{k+1}$ of Proposition 4.3.19, we obtain an induced map, which we denote with the same symbol:

$$v_k : \mathcal{A}(\omega) \times A^0(\hat{E})_{k+1} \times S_{k+1} \longrightarrow A^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k. \tag{4.71}$$

The explicit formula is:

$$v_k(\hat{a}, \varphi, s) = \left(\bar{\partial}_{\hat{a}+s}\varphi, *\omega + *ds\right) \text{ for } (\hat{a}, \varphi, s) \in \mathcal{A}(\omega) \times A^0(\hat{E})_{k+1} \times S_{k+1}. \tag{4.72}$$

Recall that $\mathcal{A}(\omega) \cong \bar{\mathcal{A}} \longrightarrow B$ is a principal $\mathfrak{G}_0$-bundle. The map v_k is $\mathfrak{G}_0$-equivariant and hence descends to a morphism

$$\nu_k : \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^0(\hat{E})_{k+1} \times S_{k+1} \longrightarrow \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k. \tag{4.73}$$

Note that the choice of ω determines the isomorphism $\bar{\mathcal{A}} \cong \mathcal{A}(\omega) \subset \mathcal{A}$ and therefore the map ν_k depends on this choice. But the vector bundles $\mathcal{E}_{k+1} := \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^0(\hat{E})_{k+1}$ and $\mathcal{F}_k := \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^{0,1}(\hat{E})_k$ do not depend on ω.

In the following proposition, we produce fiberwise metrics on the bundles $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$. This is often overlooked in the literature.

Proposition 4.3.20. *Let $\alpha \in \{0, (0,1)\}$. There is a natural Hilbert metric h_ω on the vector bundle $\mathcal{A}(\omega) \times_{\mathfrak{G}_0} A^\alpha(\hat{E})_k$ over B, uniquely determined by*

$$h_\omega\left([\hat{a}, \varphi], [\hat{a}, \psi]\right) = h_{\hat{a},k}(\varphi, \psi) \text{ for } \hat{a} \in \mathcal{A}(\omega) \text{ and } \varphi, \psi \in A^\alpha(\hat{E}). \tag{4.74}$$

Proof. On the trivial bundle $\mathcal{A}(\omega) \times A^\alpha(\hat{E})$ over $\mathcal{A}(\omega)$, we define a fiberwise Hilbert metric by

$$h_\omega\left((\hat{a}, \varphi), (\hat{a}, \psi)\right) = h_{\hat{a},k}(\varphi, \psi) \text{ for } \hat{a} \in \mathcal{A}(\omega) \text{ and } \varphi, \psi \in A^\alpha(\hat{E}). \tag{4.75}$$

This metric is $\mathfrak{G}_0$-equivariant, since $d_{\hat{a}+g^{-1}dg} g^{-1}\varphi = g^{-1} d_{\hat{a}}\varphi$ for $\hat{a} \in \mathcal{A}(\omega), g \in \mathfrak{G}_0$, and $\varphi \in A^0(\hat{E})$ (see the proof of Lemma 4.1.3). Therefore h_ω descends to a metric on $\mathcal{A}(\omega) \times_{\mathfrak{G}_0} A^\alpha(\hat{E})$ over B. □

Therefore the metric h_ω induces a Hilbert metric h_ω on the vector bundles $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ via the isomorphism $\mathcal{A}(\omega) \cong \bar{\mathcal{A}}$ explained in Proposition 4.3.19. The Hilbert structure depends on the choice of ω.

Lemma 4.3.21. *The spaces $\mathcal{A}_k^0$ and B, and the vector bundles $\bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^\alpha(\hat{E})_k$ are up to canonical isomorphism independent of the choice of $\mathcal{A}$.*

Proof. Let $\mathcal{A}'$ be a different T-affine subspace of $\mathcal{A}_{a_0}(\hat{P})$. It follows from Proposition 4.3.11 that there exists a gauge transformation $u \in C^\infty(\Sigma, G)$, such that $\hat{a} + u^{-1}du \in \mathcal{A}'$ for all $\hat{a} \in \mathcal{A}$, and furthermore that any two choices $u, u' \in C^\infty(\Sigma, G)$ having this property differ by an element in G. Thus we obtain an isomorphism

$$\Psi : \mathcal{A}_k \longrightarrow \mathcal{A}'_k, \ \hat{a} \mapsto \hat{a} + u^{-1}du \tag{4.76}$$

that descends to an isomorphism

$$\mathcal{A}_k/\mathfrak{G}_0 \longrightarrow \mathcal{A}'_k/\mathfrak{G}_0$$

which does not depend on the choice of u. Therefore $\mathcal{A}_k/\mathfrak{G}_0$ is up to canonical isomorphism independent of the choice of $\mathcal{A}$. The isomorphism $\Psi : \mathcal{A}_k \longrightarrow \mathcal{A}'_k$ induces a $\mathfrak{G}_0$-equivariant isomorphism

$$\bar{\mathcal{A}} \times A^\alpha(\hat{E})_k \xrightarrow{\cong} \overline{\mathcal{A}}' \times A^\alpha(\hat{E})_k.$$

Since it is $\mathfrak{G}_0$-equivariant, it descends to an isomorphism

$$\bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^\alpha(\hat{E})_k \xrightarrow{\cong} \overline{\mathcal{A}}' \times_{\mathfrak{G}_0} A^\alpha(\hat{E})_k, \tag{4.77}$$

which does not depend on the choice of u. □

4.4 The Fredholm property of the vortex map

We have constructed a fiberwise map

$$\nu_k : \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^0(\hat{E})_{k+1} \times S_{k+1} \longrightarrow \bar{\mathcal{A}} \times_{\mathfrak{G}_0} \mathcal{A}^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k. \tag{4.78}$$

In this section we prove that this map ν_k is Fredholm provided that the moment map $\mu : V \longrightarrow \mathfrak{g}$ is proper. The main point is to prove that ν_k is coercive. This generalizes the compactness results obtained by by Cieliebak, Gaio, and Salamon in [12, §3.6] for the case $G = \hat{G}$ and by Okonek and Teleman in [32, Proposition 2.12] for the case $V = \mathbb{C}^r$, $G = S^1$, and $\hat{G} = U(r)$. The proof we give follows mainly the approach taken in [32], but we will use essentially Corollary 4.1.6, due to the authors of [12].

Let us first simplify the notation: we put $\mathcal{E} := \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^0(\hat{E})$ and $\mathcal{F} := \bar{\mathcal{A}} \times_{\mathfrak{G}_0} A^{0,1}(\hat{E})$. Now fix a point $b \in B$ in the base manifold and choose a connection $\hat{a} \in \mathcal{A}(\omega)$ that lies over b. By definition the trivializations of the fibers

$$A^0(\hat{E}) \xrightarrow{\cong} \mathcal{E}(b),\ \varphi \mapsto [\hat{a}, \varphi] \ \text{ and } \ A^{0,1}(\hat{E}) \xrightarrow{\cong} \mathcal{F}(b),\ \eta \mapsto [\hat{a}, \eta] \tag{4.79}$$

respect the L^2_k-inner product. Write

$$\nu_b : A^0(\hat{E}) \times S \longrightarrow A^{0,1}(\hat{E}) \times A^0(\Sigma, \mathfrak{g}) \tag{4.80}$$

for the morphism induced by the restriction of ν to the fibers over b and by the trivializations (4.79). Furthermore we write $\nu_b = (\nu_b^1, \nu_b^2)$ with

1. $\nu_b^1 : A^0(\hat{E}) \times S \longrightarrow A^{0,1}(\hat{E}),\ (\varphi, s) \mapsto \bar{\partial}_{\hat{a}+s}\varphi;$

2. $\nu_b^2 : A^0(\hat{E}) \times S \longrightarrow A^0(\Sigma, \mathfrak{g}),\ (\varphi, s) \mapsto *\omega + *ds.$

Recall that we assume the moment map μ to be proper (see also Proposition 4.1.8). Our next goal is to prove the following statement:

Theorem 4.4.1. *Let $k \geq 3$. There exists $P_b \in \mathbb{R}[X, Y]$, depending continuously on b, such that*

$$||s||_{L^2_{k+1}}, ||\varphi||_{L^2_{k+1}} \leq P_b\left(||\nu_b^1(\varphi, s)||_{L^2_k}, ||\nu_b^2(\varphi, s)||_{L^2_k}\right) \ \textit{for all } (\varphi, s) \in A^0(\hat{E}) \times S.$$

Lemma 4.4.2. *Let $k \geq 0$ and $1 < p < \infty$. There exists a constant $C > 0$, such that $||s||_{L^p_{k+1}} \leq C\,||ds||_{L^p_k}$ for all $s \in S$.*

Proof. We can assume w.l.o.g. that $\mathfrak{g} = \mathbb{R}$ and $S = d^*A^2(\Sigma)$. Notice first that the operator

$$d + d^* : A^1(\Sigma) \longrightarrow A^2(\Sigma) \oplus A^0(\Sigma)$$

is elliptic with kernel $\mathbb{H}^1(\Sigma)$. The L^2-orthogonal complement is $dA^0(\Sigma) \oplus d^*A^2(\Sigma)$. Its L^p_{k+1}-completion is a topological complement with respect to the L^p_{k+1}-norm. Therefore Theorem 4.3.3 tells us that there exists a constant $C > 0$, such that

$$||s||_{L^p_{k+1}} \leq C\left(||pr_{\ker(d+d^*)}(s)||_{L^p} + ||ds + d^*s||_{L^p_k}\right).$$

From $s \in d^*A^2(\Sigma)$ it follows that $pr_{\ker(d+d^*)}(s) = 0$ and that $d^*s = 0$. Hence the statement. □

Lemma 4.4.3. *Let $k > 2$. There exist constants $C, D > 0$, such that*

$$||\partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}\varphi||_{C^0} \leq C||\nu_b^1(s, \varphi)||_{L^2_k}\left(D + ||\nu_b^2(s, \varphi)||_{C^0} + ||\mu(\varphi)||_{C^0}\right) \ \textit{for all } (\varphi, s) \in A^0(\hat{E}) \times S.$$

Proof. First note that

$$\begin{aligned}||\partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}\varphi||_{C^0} &\leq ||\partial_{\hat{a}}\bar{\partial}_{\hat{a}+s}\varphi||_{C^0} + ||\rho_*(s)^{1,0}\wedge\bar{\partial}_{\hat{a}+s}\varphi||_{C^0}\\ &\leq ||\nu_b^1(s,\varphi)||_{C^1} + ||\rho_*(s)\wedge\nu_b^1(s,\varphi)||_{C^0}\\ &\leq ||\nu_b^1||_{C^1} + C||s||_{C^0}||\nu_b^1||_{C^0}.\end{aligned} \tag{4.81}$$

By assumption $k > 2$, hence by Theorem 4.3.2, there exists a constant $C' > 0$, such that

$$||\nu_b^1||_{C^i} \leq C'||\nu_b^1||_{L_k^2} \text{ for } i = 0, 1.$$

Now apply again Theorem 4.3.2, followed by Lemma 4.4.2, and finally use the compactness of Σ to obtain

$$||s||_{C^0} \leq C_1||s||_{L_1^3} \leq C_2||ds||_{L^3} \leq C_3||ds||_{C^0} = C_3||*ds||_{C^0}.$$

Hence

$$\begin{aligned}||s||_{C^0} \leq C_3||*ds||_{C^0} &= C_3||pr_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi) - pr_{\mathfrak{g}} * F_{\hat{a}} - \mu(\varphi)||_{C^0}\\ &\leq C_3\left(||\nu_b^2(s,\varphi)||_{C^0} + ||*\omega||_{C^0} + ||\mu(\varphi)||_{C^0}\right).\end{aligned} \tag{4.82}$$

□

Now we are in the position to prove Theorem 4.4.1:

Proof. Claim 1: There exist $C_1 > 0$, depending polynomially on $||\nu_b^2||_{L_2^2}$ and $C_2 > 0$, such that

$$h(i * F_{\hat{a}+s}(\varphi),\varphi) \leq C_1||\varphi||_{C^0}^2 - 2|\mu(\varphi)|^2 + C_2|\mu(\varphi)| \text{ for all } (s,\varphi) \in S \times A^0(\hat{E}). \tag{4.83}$$

Proof of claim 1: Let $\tau := \mu(0) \in \mathfrak{g}$. Then $\mu = \pi \circ \mu_V + \tau$ and by Lemma 4.1.6

$$h(\rho_*(\mu(\varphi))(i\varphi),\varphi) = 2\langle\mu(\varphi),\mu(\varphi) - \tau\rangle_{\mathfrak{g}}.$$

Then notice that $\pi_{\mathfrak{g}^\perp} * F_{\hat{a}+s}(i\varphi) = \pi_{\mathfrak{g}^\perp} * F_{\hat{a}}(i\varphi) = *F_{a_0}$. From Lemma 4.1.4 we derive the inequality

$$h(\rho_*(\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi))(i\varphi),\varphi) \leq ||\varphi||_{C^0}^2\ ||\rho_*(\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi))||_{C^0}. \tag{4.84}$$

Then:

$$\begin{aligned}h(i * F_{\hat{a}+s}(\varphi),\varphi) &= h(\rho_*(*F_{\hat{a}+s})(i\varphi),\varphi)\\ &= h(\rho_*(\pi_{\mathfrak{g}} * F_{\hat{a}+s})(i\varphi),\varphi) + h(\rho_*(\pi_{\mathfrak{g}^\perp} * F_{\hat{a}+s})(i\varphi),\varphi)\\ &= h(\rho_*(\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi))(i\varphi),\varphi) - h(\rho_*(\mu(\varphi))(i\varphi),\varphi) + h(\rho_*(\pi_{\mathfrak{g}^\perp} * F_{\hat{a}}(i\varphi),\varphi)\\ &\leq ||\varphi||_{C^0}^2\ ||\rho_*(\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi))||_{C^0} - 2\langle\mu(\varphi),\mu(\varphi) - \tau\rangle + ||\rho_*||\ ||*F_{\hat{a}}||_{C^0}||\varphi||_{C^0}^2\\ &\leq ||\varphi||_{C^0}^2\ ||\rho_*||\left\{||\nu_b^2(s,\varphi)||_{C^0} + ||*F_{a_0}||_{C^0}\right\} - 2|\mu(\varphi)|^2 + 2|\mu(\varphi)|\ |\tau|\end{aligned}$$

Last, by Theorem 4.3.2 there is a constant $C > 0$, such that $||\nu_b^2||_{C^0} \leq C||\nu_b^2||_{L_2^2}$.
Claim 2: There exist $C_3, C_4 > 0$, depending polynomially on $||\nu_b^i||_{L_3^2}$ $(i = 1, 2)$, such that

$$|h(i * \partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}\varphi,\varphi)| \leq C_3||\varphi||_{C^0}||\mu(\varphi)||_{C^0} + C_4||\varphi||_{C^0} \text{ for all } (\varphi, s) \in S \times A^0(\hat{E}). \tag{4.85}$$

Proof of claim 2: According to Lemma 4.4.3, we may choose $C, D > 0$, such that

$$\begin{aligned}||\partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}(\varphi)||_{C^0} &\leq C||\nu_b^1||_{L_3^2}\left(D + ||\nu_b^2||_{C^0} + ||\mu(\varphi)||_{C^0}\right)\\ &\leq C'||\nu_b^1||_{L_3^2}\left(D + ||\nu_b^2||_{L_k^2} + ||\mu(\varphi)||_{C^0}\right).\end{aligned} \tag{4.86}$$

We conclude that

$$\begin{aligned}|h(i * \partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}\varphi,\varphi)| &\leq ||\partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}(\varphi)||_{C^0}||\varphi||_{C^0}\\ &\leq C||\nu_b^1||_{L_3^2}||\mu(\varphi)||_{C^0}||\varphi||_{C^0} + C||\nu_b^1||_{L_3^2}\left(D + ||\nu_b^2||_{L_k^2}\right)||\varphi||_{C^0}\\ &= C_3||\varphi||_{C^0}||\mu(\varphi)||_{C^0} + C_4||\varphi||_{C^0}.\end{aligned}$$

Therefore inequality (4.85) holds as claimed.

Now let $(\varphi, s) \in A^0(\hat{E}) \times S$. Let $x \in \Sigma$ be a point in which the function $h(\varphi, \varphi)$ attains its maximum. Use Lemma 4.3.6, followed by Lemma 4.3.5, and then apply inequalities (4.83) and (4.85) to derive the following sequence of inequalities:

$$\begin{aligned}0 &\leq \left[i * \bar{\partial}\partial h(\varphi, \varphi)\right](x) \\ &\leq h(i * F_{\hat{a}+s}(\varphi), \varphi)(x) + 2|h(i * \partial_{\hat{a}+s}\bar{\partial}_{\hat{a}+s}\varphi, \varphi)(x)| \\ &\leq C_1||\varphi||^2_{C^0} - 2|\mu(\varphi(x))|^2 + ||\mu(\varphi)||_{C^0}C_2 + 2C_3||\varphi||_{C^0}||\mu(\varphi)||_{C^0} + 2C_4||\varphi||_{C^0}.\end{aligned}$$

It follows that

$$|\mu(\varphi(x))|^2 \leq D_1||\varphi||^2_{C^0} + D_2||\mu(\varphi)||_{C^0} \cdot ||\varphi||_{C^0} + D_3\left(||\mu(\varphi)||_{C^0} + ||\varphi||_{C^0}\right) \tag{4.87}$$

for $D_i > 0$ $(i = 1, 2, 3)$. Now we make use of Lemma 4.1.7 and Proposition 4.1.8:

1. $||\mu(\varphi)||_{C^0} \leq C||\varphi||^2_{C^0} + D$ for constants $C, D > 0$;
2. $|\mu(\varphi(x))| \geq C'||\varphi||^2_{C^0} - D'$ for constants $C', D' > 0$;

to conclude that $||\varphi||^4_{C^0} \leq P(||\varphi||_{C^0})$ for a polynomial $P \in \mathbb{R}[X]_{\leq 3}$, whose coefficients depend polynomially on $||\nu^i_b||_{L^2_3}$ $(i = 1, 2)$.

Thus for $(\varphi, s) \in A^0(\hat{E}) \times S$, there exists $\delta > 0$ depending polynomially on $||\nu^i_b(\varphi, s)||_{L^2_3}$ $(i = 1, 2)$, such that $||\varphi||_{C^0} < \delta$. The next step is to produce a bound for s in the L^2_1-norm. We apply Lemma 4.4.2:

$$\begin{aligned}||s||_{L^2_1} &\leq \tilde{C}||ds||_{L^2} = \tilde{C}||\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi) - \pi_{\mathfrak{g}} * F_{\hat{a}} - \mu(\varphi)||_{L^2} \\ &\leq \tilde{C}||\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi)||_{L^2} + ||\omega||_{L^2} + ||\mu(\varphi)||_{L^2} \\ &\leq \tilde{C}||\nu^2_b||_{L^2_3} + ||\omega||_{L^2} + \tilde{C}'||\mu(\varphi)||_{C^0} \\ &\leq \tilde{C}||\nu^2_b||_{L^2_3} + ||\omega||_{L^2} + \tilde{C}'\left(C||\varphi||^2_{C^0} + D\right) \\ &\leq \tilde{C}||\nu^2_b||_{L^2_3} + ||\omega||_{L^2} + \tilde{C}'\left(C\delta^2 + D\right)\end{aligned} \tag{4.88}$$

Hence we have a bound for $||s||_{L^2_1}$, depending polynomially on $||\nu^i_b||_{L^2_3}$ $(i = 1, 2)$. Now we do the same for φ:

$$\begin{aligned}||\varphi||_{L^2_1} &\leq C\left(||\varphi||_{L^2} + ||\bar{\partial}_{\hat{a}}\varphi||_{L^2}\right) \\ &\leq C\left(||\varphi||_{L^2} + ||\bar{\partial}_{\hat{a}+s}\varphi||_{L^2} + ||\rho_*(s)^{0,1} \otimes \varphi||_{L^2}\right) \\ &\leq C'\left(||\varphi||_{C^0} + ||\nu^1_b||_{L^2_1} + ||s||_{L^2}||\varphi||_{C^0}\right).\end{aligned} \tag{4.89}$$

Now we can complete the proof of the theorem by elliptic bootstrapping.
Claim 3: For any $1 \leq k' \leq k$, there is a polynomial $P_{k'} \in \mathbb{R}[X, Y, Z, W]$, such that

$$||\varphi||_{L^2_{k'+1}}, ||s||_{L^2_{k'+1}} \leq P_{k'}\left(||\varphi||_{L^2_{k'}}, ||s||_{L^2_{k'}}, ||\nu^1_b||_{L^2_k}, ||\nu^2_b||_{L^2_k}\right).$$

Proof of claim 3: Again, we first tackle the estimate for s: We start by proving that there are constants $C, D > 0$, depending polynomially on $||\nu^i_b||_{L^2_3}$, such that

$$||\mu(\varphi)||_{L^2_{k'}} \leq C||\varphi||^2_{L^2_{k'}} + D. \tag{4.90}$$

When $k' > 1$, then this follows from Lemma 4.3.7. The main point is that we use Theorem 4.3.4 to produce an estimate $||\varphi\varphi^*||_{L^2_{k'}} \leq C||\varphi||_{L^2_{k'}}||\varphi||_{L^2_{k'}}$. When $k' = 1$, then we use the estimate $||\varphi\varphi^*||_{L^2_1} \leq C||\varphi||_{C^0}||\varphi||_{L^2_1}$ which is derived in the same way. Since we have bounded $||\varphi||_{C^0}$ by a constant depending polynomially on $||\nu^i_b||_{L^2_3}$, we obtain the estimate (4.90). Using Lemma 4.4.2 we obtain:

$$\begin{aligned}
||s||_{L^2_{k'+1}} &\le C||ds||_{L^2_{k'}} \\
&\le ||\pi_{\mathfrak{g}} * F_{\hat{a}+s} + \mu(\varphi)||_{L^p_{k'}} + ||\omega||_{L^p_{k'}} + ||\mu(\varphi)||_{L^2_{k'}} \\
&\le ||\nu_b^2(s,\varphi)||_{L^2_k} + ||\omega||_{L^2_k} + C||\varphi||^2_{L^2_{k'}} + D.
\end{aligned}$$

Now we look at φ:

$$\begin{aligned}
||\varphi||_{L^2_{k'+1}} &\le C\left(||\varphi||_{L^2} + ||\bar{\partial}_{\hat{a}}\varphi||_{L^2_{k'}}\right) \\
&\le C\left(||\varphi||_{L^2} + ||\nu_b^1(s,\varphi)||_{L^2_{k'}} + ||\rho_*(s)\otimes\varphi||_{L^2_{k'}}\right) \\
&\le C\left(||\varphi||_{L^2} + ||\nu_b^1||_{L^2_k} + ||s||_{L^2_{k'+1}}||\varphi||_{L^2_{k'}}\right).
\end{aligned}$$

This completes the proof.

□

In the following theorem, we collect the properties of the vortex map. Recall that the moment map μ is assumed to be proper.

Theorem 4.4.4. *Let $k \ge 3$. The morphism $\nu_k : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$ satisfies the following properties:*

1. *It is G-equivariant with respect to the G-action on $\mathcal{E}_{k+1}$ and $\mathcal{F}_k$ induced by the representation $\rho : G \longrightarrow U(V)$ and with respect to the trivial G-action on S_{k+1} and $A^0(\Sigma, \mathfrak{g})_k$.*
2. *The fibrewise differential $d_b := d\left(\nu_k(b)\right)(b,0) : \mathcal{E}_{k+1}(b) \oplus S_{k+1} \longrightarrow \mathcal{F}_k(b) \oplus A^0(\Sigma, \mathfrak{g})_k$ is of the form $\delta_b \oplus l$, where δ_b and l are the following linear Fredholm operators:*
   (a) $\delta_b : \mathcal{E}_{k+1}(b) \longrightarrow \mathcal{F}_k(b),\ [\hat{a},\varphi] \mapsto \left[\hat{a}, \bar{\partial}_{\hat{a}}\varphi\right]$;
   (b) $l : S_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k,\ s \mapsto *ds.$
3. *The restriction of ν_k to the zero-section of $\mathcal{E}_{k+1}$ is of the following form:*
$$S_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k,\ s \mapsto *\omega + \mu(0) + l(s).$$
4. *The difference $c_k := \nu_k - d$ is compact, where d is fiberwise defined by d_b ($b \in B$).*
5. *There exists a polynomial $P \in \mathbb{R}[X]$, such that*
$$||(\varphi, s)||_{L^2_{k+1}} \le P\left(||\nu_k(\varphi,s)||_{L^2_k}\right) \text{ for all } (\varphi,s) \in \mathcal{E}_{k+1} \times S_{k+1}.$$

Proof. The first statement follows from Lemma 4.1.3. Let $b \in B$ be a fixed base point. Choose a connection $\hat{a} \in \mathcal{A}$, such that $\pi_{\mathfrak{g}} F_{\hat{a}} = \omega$ and that is mapped to b under the quotient map $\mathcal{A} \longrightarrow B$. As in (4.79), we trivialize the fibers $\mathcal{E}_{k+1}(b) \cong A^0(\hat{E})_{k+1}$ and $\mathcal{F}_k \cong A^{0,1}(\hat{E})$ as follows:

1. $A^0(\hat{E})_{k+1} \xrightarrow{\cong} \mathcal{E}_{k+1}(b),\ \varphi \mapsto [\hat{a},\varphi]$;
2. $A^{0,1}(\hat{E})_k \xrightarrow{\cong} \mathcal{F}_k(b),\ \psi \mapsto [\hat{a},\psi]$.

Under these trivializations, the restriction of the map ν_k to the fibers over b takes the following form:

$$\begin{aligned}
\nu_k(b) : A^0(\hat{E})_{k+1} \times S_{k+1} &\longrightarrow A^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k, \\
(\varphi, s) &\mapsto \left(\bar{\partial}_{\hat{a}}\varphi + \rho_*(s)^{0,1}(\varphi), *\omega + *ds + \mu(\varphi),\right)
\end{aligned} \tag{4.91}$$

As a consequence, the differential of $\nu_k(b)$ at a point $(\varphi_0, s_0) \in A^0(\hat{E})_{k+1} \times S_{k+1}$ is given by the following equation:

$$d(\nu_k(b))(\varphi_0, s_0,)(\varphi, s) = \left(\bar{\partial}_{\hat{a}}\varphi + \rho_*(s_0)^{0,1}(\varphi) + \rho_*(s)^{0,1}(\varphi_0), *ds + d\mu(\varphi_0)\varphi\right), \tag{4.92}$$

where $(\varphi, s) \in A^0(\hat{E})_{k+1} \times S_{k+1}$.

In particular, for $(\varphi_0, s_0) = (0, 0)$ and for $(\varphi, s) \in A^0(\hat{E})_{k+1} \times S_{k+1}$, we obtain:

$$d_b := d(\nu_k(b))(0,0)(\varphi, s) = \left(\bar{\partial}_{\hat{a}}\varphi,\ *ds + d\mu(0)\varphi\right) = \left(\bar{\partial}_{\hat{a}}\varphi, *ds\right). \tag{4.93}$$

Observe that the map $A^0(\hat{E})_{k+1} \longrightarrow A^{0,1}(\hat{E})_k,\ \varphi \mapsto \bar{\partial}_{\hat{a}}\varphi$ induces under the chosen trivializations for the fibers of $\mathcal{E}_{k+1}(b)$ and $\mathcal{F}_k(b)$ the morphism

$$\delta_b : \mathcal{E}_{k+1}(b) \longrightarrow \mathcal{F}_k(b),\ [\hat{a}, \varphi] \mapsto \left[\hat{a}, \bar{\partial}_{\hat{a}}\varphi\right]. \tag{4.94}$$

The morphism δ_b is an elliptic operator, and hence Fredholm. Consider the morphism $l : S_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k,\ s \mapsto *ds$: It follows from $A^1(\Sigma, \mathfrak{g})_{k+1} = S_{k+1} \oplus \ker d$ that $d : S_{k+1} \longrightarrow A^2(\Sigma, \mathfrak{g})_k$ is a closed embedding with image $\operatorname{im} d$. The operator $* : A^2(\Sigma, \mathfrak{g})_k \longrightarrow A^0(\Sigma, \mathfrak{g})_k$ is an isometry for the L^2_k-norm. Furthermore, $*d\left(A^1(\Sigma, \mathfrak{g})\right)_k = d^* A^1(\Sigma, \mathfrak{g})_k$ and $* \ker d^* = \ker d \subset A^0(\Sigma, \mathfrak{g})_k = \mathfrak{g}$. Therefore, l is a linear Fredholm map with with index $-\dim \mathfrak{g}$.

From (4.91) it is also clear that the restriction of ν_k to the zero-section of $\mathcal{E}_{k+1}$ has the form

$$S_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k,\ s \mapsto *\omega + l(s) + \mu(0). \tag{4.95}$$

Write $d : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$ for the fibrewise map defined by the family $(d_b)_{b \in B}$. Now consider the difference $c_k := \nu_k - d : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$. In the fiber over a point $b \in B$ and in the trivializations we made, it is given by

$$A^0(\hat{E})_{k+1} \times S_{k+1} \longrightarrow A^{0,1}(\hat{E})_k \times A^0(\Sigma, \mathfrak{g})_k,\ (\varphi, s) \mapsto (\rho_*(s)^{0,1}(\varphi),\ *\omega + \mu(\varphi)). \tag{4.96}$$

Therefore the invariant formula reads

$$c_k([\hat{a}, \varphi], s) = \left(\left[\hat{a}, \rho_*(s)^{0,1}(\varphi)\right], *\omega + \mu(\varphi)\right) \text{ for any } ([\hat{a}, \varphi], s) \in \mathcal{E}_{k+1} \times S_{k+1}. \tag{4.97}$$

It is clear from this description that the map c factors as $c : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_{k+1} \times A^0(\Sigma, \mathfrak{g})_{k+1} \hookrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$. The last inclusion is a compact embedding (Theorem 4.3.2), therefore c_k is compact.

The remaining statement has been proved in Theorem 4.4.1. □

4.5 The Cohomotopy Invariant

Let $\tilde{\rho} \subset \mathrm{Irr}(G, \mathbb{C}) \cup \{\mathbb{R}\}$ be the set consisting of $\mathbb{R}$ and of all irreducible representations of G that appear in the isotypical decomposition of the representation $\rho : G \longrightarrow U(V)$. Then the bundles $\mathcal{E}_{k+1} \times S_{k+1}$ and $\mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$ are $\tilde{\rho}$-bundles.

Let $d := (\delta, l) : \mathcal{E}_{k+1} \times S_{k+1} \longrightarrow \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$ be the Fredholm operator introduced in Theorem 4.4.4.

Corollary 4.5.1. *The map ν_k is a d-framed Fredholm map.*

Proof. This is an immediate consequence of Theorem 4.4.4. □

Until now we considered the vortex map ν_k as a Ω-Fredholm map with $\Omega = \emptyset$. In many interesting situations, the vortex map ν_k is a Ω-map for other choices of Ω. We now explain such a class of situations.

Let Ω_0 be the isotopy family of positive dimensional subgroups. Recall that $A^0(\Sigma, \mathfrak{g})_k = \mathfrak{g} \oplus \operatorname{im} d^*$. Let $p : A^0(\Sigma, \mathfrak{g})_k \longrightarrow \mathfrak{g}$ be the projection. In the following lemma, we will assume that $p(*\omega) + \mu(0) \neq 0$.

Lemma 4.5.2. *Assume that $p(*\omega) + \mu(0) \neq 0$. Then the image of $0_B \times S_{k+1} \subset \mathcal{E}_{k+1} \times S_{k+1}$ under the vortex map ν_k avoids a neighbourhood of zero in $0_{\mathcal{F}_k} \times A^0(\Sigma, \mathfrak{g})_k$.*

Proof. By Theorem 4.4.4, the restriction of the vortex map to the zero section in $\mathcal{E}_{k+1}$ has the following form:

$$\nu_k\,(0,s) = (0, *\omega + \mu(0) + l(s)) \text{ for } s \in S_{k+1}. \tag{4.98}$$

Now $l(s) \in \operatorname{im} d^*$ for all $s \in S_{k+1}$ and $p(*\omega) + \mu(0) \neq 0 \in \mathfrak{g}$. Therefore there is $\varepsilon > 0$, such that $|\nu_k(0,s)| > \varepsilon$ for all $s \in S_{k+1}$. $\square$

Recall that $V(\Omega_0)$ denotes the set of vectors $v \in V$, whose stabilizer G_v is positive dimensional.

Corollary 4.5.3. *Assume that $p\,(*\omega) + \mu(0) \neq 0$ and $V(\Omega_0) = \{0\}$. Then the map ν_k is an Ω_0-Fredholm map.*

Proof. It follows directly from the assumption, that only the 0-section in $A^0(\hat{E})$ has positive dimensional stabilizer. Therefore $(\mathcal{E}_{k+1} \times S_{k+1})\,(\Omega_0) = 0_B \times S_{k+1}$, and hence by Lemma 4.5.2 its image under ν_k avoids a neighbourhood of the 0-section in $\mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k$. $\square$

Let now Ω be an isotopy family of subgroups, such that ν_k is an Ω-Fredholm map. Then the homotopy class of ν_k defines elements

- $[\nu_k]_d \in {}_G[\mathcal{E}_{k+1} \times S_{k+1},\ \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k]_B^{\Omega,d}$.
- $[\nu_k]_l \in {}_G[\mathcal{E}_{k+1} \times S_{k+1},\ \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k]_B^{\Omega,l}$.
- $[\nu_k] \in {}_G[\mathcal{E}_{k+1} \times S_{k+1},\ \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k]_B^{\Omega}$.

The Fredholm operator $l : S_{k+1} \longrightarrow A^0(\Sigma, \mathfrak{g})_k$ is an embedding and its image has complement $\mathfrak{g}$. This natural choice allows us to realize the set of homotopy classes of l-framed Ω-maps as the colimit of the functor $\mathfrak{p}_\Omega^{l,\mathfrak{g}}$. Now, we make use of the natural identifications proven in Theorems 3.1.11 and 3.2.4 and of the natural maps explained in Corollaries 3.4.10 and 3.4.18. We obtain the following maps:

- ${}_G[\mathcal{E}_{k+1} \times S_{k+1},\ \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k]_B^{\Omega,l} = \operatorname{colim} \mathfrak{p}_\Omega^{l,\mathfrak{g}} \longrightarrow \operatorname{colim} \pi_\Omega^{l,\mathfrak{g}} = {}_{\tilde{\rho}}\mathbb{P}_\Omega^{l,\mathfrak{g}}(B)$;
- ${}_G[\mathcal{E}_{k+1} \times S_{k+1},\ \mathcal{F}_k \times A^0(\Sigma, \mathfrak{g})_k]_B^{\Omega} = \operatorname{colim} \mathfrak{p}_\Omega \longrightarrow \operatorname{colim} \pi_\Omega = {}_{\tilde{\rho}}\mathbb{P}_\Omega(B)$.

Thus, we obtain the **cohomotopy vortex invariants**

- $\{\nu_k\}_l \in {}_{\tilde{\rho}}\mathbb{P}_\Omega^{l,\mathfrak{g}}(B)$;
- $\{\nu_k\} \in {}_{\tilde{\rho}}\mathbb{P}_\Omega(B)$.

Bibliography

[1] M. Artin and B. Mazur. *Etale homotopy.* Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin, 1969.

[2] M. Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, 112:229–247, 1964.

[3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[4] S. Bauer. On connected sums of four dimensional manifolds. *preprint.* available at http://www.mathematik.uni-bielefeld.de/sfb343/preprints.

[5] S. Bauer. Refined Seiberg-Witten invariants. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 1–46. Kluwer/Plenum, New York, 2004.

[6] S. Bauer. A stable cohomotopy refinement of Seiberg-Witten invariants. II. *Invent. Math.*, 155(1):21–40, 2004.

[7] S. Bauer and M. Furuta. A stable cohomotopy refinement of Seiberg-Witten invariants. I. *Invent. Math.*, 155(1):1–19, 2004.

[8] M. S. Berger. *Nonlinearity and functional analysis.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977.

[9] N. Bourbaki. *General topology. Chapters 1–4.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1966 edition.

[10] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[11] K. Cieliebak, A. R. Gaio, I. Mundet i Riera, and D. A. Salamon. The symplectic vortex equations and invariants of Hamiltonian group actions. *J. Symplectic Geom.*, 1(3):543–645, 2002.

[12] K. Cieliebak, A. R. Gaio, and D. A. Salamon. J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions. *Internat. Math. Res. Notices*, (16):831–882, 2000.

[13] M. Crabb and I. James. *Fibrewise homotopy theory.* Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 1998.

[14] A. Deitmar and S. Echterhoff. *Principles of harmonic analysis.* Universitext. Springer, New York, 2009.

[15] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. , Oxford Science Publications.

[16] M. Furuta. Stable homotopy version of seiberg-witten invariant. *preprint.* available at http://www.mpim-bonn.mpg.de.

[17] M. Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. *Math. Res. Lett.*, 8(3):279–291, 2001.

[18] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[19] I. M. James. *Fibrewise topology*, volume 91 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1989.

[20] K. Jänich. Vektorraumbündel und der Raum der Fredholm-Operatoren. *Math. Ann.*, 161:129–142, 1965.

[21] M. Karoubi. *K-theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2008.

[22] A. W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1986.

[23] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan.* Princeton University Press, Princeton, NJ, 1987. Kano Memorial Lectures, 5.

[24] N. H. Kuiper. The homotopy type of the unitary group of Hilbert space. *Topology*, 3:19–30, 1964.

[25] S. Lang. *Differential and Riemannian manifolds*, volume 160 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, third edition, 1995.

[26] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[27] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[28] I. Mundet i Riera. *Yang-Mills-Higgs theory for symplectic fibrations.* PhD thesis, Universidad Autonoma de Madrid, 1999.

[29] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.

[30] I. Mundet i Riera. Hamiltonian Gromov-Witten invariants. *Topology*, 42(3):525–553, 2003.

[31] I. Mundet i Riera and G. Tian. A compactification of the moduli space of twisted holomorphic maps. *preprint*, 2004. arXiv:math/0404407v1 [math.SG].

[32] Ch. Okonek and A. Teleman. Gauge theoretical equivariant Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces. *Comm. Math. Phys.*, 227(3):551–585, 2002.

[33] Ch. Okonek and A. Teleman. Gauge theoretical Gromov-Witten invariants and virtual fundamental classes. In *The Fano Conference*, pages 591–623. Univ. Torino, Turin, 2004.

[34] Ch. Okonek and A. Teleman. Cohomotopy invariants and the universal cohomotopy invariant jump formula. *Journal of Mathematical Sciences*, to appear.

[35] R. S. Palais. *Foundations of global non-linear analysis.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[36] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[37] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[38] G. Segal. Equivariant contractibility of the general linear group of Hilbert space. *Bull. London Math. Soc.*, 1:329–331, 1969.

[39] G. Segal. Fredholm complexes. *Quart. J. Math. Oxford Ser. (2)*, 21:385–402, 1970.

[40] E. H. Spanier. *Algebraic topology.* Springer-Verlag, New York, 1981. Corrected reprint.

[41] N. Steenrod. *The Topology of Fibre Bundles.* Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[42] A. S. Švarc. On the homotopic topology of Banach spaces. *Dokl. Akad. Nauk SSSR*, 154:61–63, 1964.

[43] A. Teleman. *Introduction à la théorie de jauge.* available at http://www.cmi.univ-mrs.fr/˜teleman/documents/cours-sw.pdf.

[44] T. tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1987.

[45] T. tom Dieck. *Topologie.* de Gruyter Lehrbuch. [de Gruyter Textbook]. Walter de Gruyter & Co., Berlin, 1991.

[46] T. tom Dieck, K. H. Kamps, and D. Puppe. *Homotopietheorie.* Lecture Notes in Mathematics, Vol. 157. Springer-Verlag, Berlin, 1970.

[47] J. A. Wolf. *Harmonic analysis on commutative spaces*, volume 142 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2007.

[48] M. G. Zaĭdenberg, S. G. Kreĭn, P. A. Kučment, and A. A. Pankov. Banach bundles and linear operators. *Uspehi Mat. Nauk*, 30(5(185)):101–157, 1975.