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Keywords   Tradeability · Liquidity · Option pricing

JEL Classification   G13

1 Introduction

The recent financial crisis has highlighted the importance of asset liquidity for inves-
tors. For many assets the costs of trading were enormous, and some assets could not
be traded at all for several months. But even in less turbulent times, asset liquidity is
highly relevant for investors: "Investors want three things from the markets: liquidity,

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liquidity, and liquidity."[1] Therefore, it comes as no surprise that investors demand a price discount for illiquid assets. A vast body of empirical literature documents such a price discount with Amihud and Mendelson (1986, 1989) being the seminal papers.

Most theoretical models capture the illiquidity of an asset by the costs associated with trading the asset. Examples include the general equilibrium models of Amihud and Mendelson (1986), Vayanos (1998), and Acharya and Pedersen (2005).[2] In these models price discounts compensate investors for the higher trading costs of less liquid stocks. In a similar way, the partial equilibrium models of Kempf and Uhrig-Homburg (2000) and Grinblatt (2002) assume that the liquid asset provides a continuous additional cash flow as compared to the illiquid one. Therefore, the illiquid asset trades at a discount. None of these papers looks at the impact of non-tradeability on asset prices, the focus of our paper.

Temporary non-tradeability is an important issue for investors since many assets cannot always be traded. For example, many small stocks are traded only once a day during a batch auction. There are lock-up periods for company insiders after an IPO to avoid flooding the market with their shares. CEOs are often compensated by shares of the company, but not allowed to sell the stocks for a while to make the incentives long-term. Restricted stocks, i.e. privately placed stocks issued by companies under SEC's Rule 144, cannot be sold by the investors within two years after purchase. In other asset classes like private equity and venture capital trading restrictions are often even more severe and the non-trading periods longer. As these examples show, temporary non-tradeability is a frequently observed phenomenon which can have strong price implications. For example, Silber (1991) reports that restricted stocks trade at a discount of 33%. Brenner et al. (2001) find that nontradable options are priced about 21% less than the exchange-traded options.

Given the importance of temporary non-tradeability, surprisingly little theoretical work is done on the impact of non-tradeability on asset prices. A possible reason is that investors do not care about trading in most asset pricing models. But if investors do not care about trading, tradeability has no value for them. As Longstaff (2009) points out, the key challenge is to model the incentive to trade. In the pioneer work, the partial equilibrium model of Longstaff (1995), the incentive to trade comes from the (obviously unrealistic) assumption that the investor has perfect foresight with respect to future prices and, therefore, knows when the price is most favorable. Consequently, she can trade at a better price when the asset is liquid (i.e. can be traded continuously) as compared to the case in which the asset is illiquid (i.e. cannot be traded for some time). Therefore, the illiquid asset trades at a discount as compared to the otherwise identical liquid one. Longstaff (1995) adopts the model to determine the price discount for an illiquid stock and Koziol and Sauerbier (2007) modify the approach to analyze the price discount of illiquid bonds. Both models show that the value of tradeability increases with the length of the non-trading period and the volatility of the asset price. Longstaff (2009) gives up the assumption of perfect foresight and develops a model where the incentive to trade results from the heterogeneity of the investors

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[1] Handa and Schwartz (1996, p. 44).

[2] Amihud et al. (2005) provide an excellent survey on the literature.

with respect to their patience. In this model, the impatient trader tilts her portfolio towards the always tradable asset and away from the other asset which is restricted with respect to trading. As a consequence, the equilibrium price of the restricted asset is below the price of the unrestricted asset and the discount increases with the length of the non-trading period.[3]

In this paper, we propose a new partial equilibrium model to analyze the impact of non-tradeability on asset prices. In our model, the incentive to trade arises from the ability of traders to exploit a temporary pricing inefficiency in the stock market. Our core assumption is that the stock price can deviate temporarily from its fundamental value—a phenomenon which is well-documented over both short- and long-term horizons.[4] The basic structure of our model is as follows. We model the asset price as a mean reverting process around the fair asset value which itself follows a random walk. This modeling captures the idea that the stock price is linked to its fair value, but might temporarily deviate from it. The investor has an incentive to trade because trading allows her to exploit the temporary mispricing of the stock. In our model, there are two assets which differ only with respect to the number of trading dates. The illiquid stock can be traded only at a fixed trading date whereas the investor can choose the optimal trading date of the liquid stock. Therefore, the investor is better off when the asset is liquid, tradeability is valuable, and its value is given by the difference in the expected trading revenues in the liquid and the illiquid asset, respectively.

Our model delivers four main insights: (1) The higher the pricing efficiency of a market, the greater the value of tradeability. To clarify the intuition, let us assume that there is no pricing efficiency at all, i.e. the stock price is not linked to its fundamental value. Then the investor has no incentive to trade and liquidity has no value. In contrast, if the pricing efficiency is high, the investor has a strong incentive to trade because she knows that an observed mispricing will disappear quickly. Therefore, tradeability is very valuable. (2) The value of tradeability increases with uncertainty, no matter whether it results from noise trading or from news about the fundamental value. Both, noise trading in the stock and new information about the fair value increase the probability of large mispricings in the future which would lead to large profits when trading the liquid asset. (3) The longer the non-trading period of the illiquid asset, the greater the value of tradeability. This result is intuitive and consistent with the outcomes of all models cited above. (4) The value of tradeability increases with the number of possible trading dates of the liquid asset. This is sensible since a higher number of trading dates increases the flexibility of the investor holding the liquid stock.

Our paper is organized as follows. In Sect. 2.1 we develop a basic model to calculate the value of tradeability. In this model, the fair value of the stock is constant and the liquid stock can only be traded at two possible trading dates. In Sect. 2.2 we extend this model by allowing the liquid asset to be traded at n possible trading dates. Here we also solve the model when n goes to infinity, i.e. the liquid asset can be traded

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[3] This result holds for the baseline calibration of the model, but not for arbitrary parameter combinations.

[4] For example, Chordia et al. (2005) analyze the speed of convergence towards efficiency. They find that stock prices can be predicted up to an hour. Seminal papers on long-term stock price predictability are DeBondt and Thaler (1985), Fama and French (1988), and Poterba and Summers (1988). See also the survey of DeBondt and Thaler (1989).

continuously. In Sect. 2.3 we analyze how information about the fair value influences the value of tradeability. We conclude in Sect. 3.

2 Pricing tradeability

To calculate the value of tradeability we consider an investor who holds one stock at time 0.[5] The investor searches for the optimal strategy to sell this stock. To keep the model simple, we do not allow the investor to carry out further trades. However, the model can be extended in this direction like, for example, in Brennan and Schwartz (1990) who allow the arbitrage trader to build up and unwind her arbitrage positions several times before maturity.

The investor has to sell the stock no later than time T. If the stock is illiquid, she can sell it only at time T. However, she can sell the stock at time $\tau \leq T$ if the stock is liquid. She will do so if the stock is sufficiently overpriced in τ. Thus, in our model, the incentive to trade arises from the fact that the stock price S deviates temporarily from its fair value θ. We assume that the fair value changes randomly over time:

$$d\theta_t = \gamma d B_t \tag{1}$$

$(B_t, t \geq 0)$ is a Brownian motion under the historical probability P and γ determines the fundamental information flow. The stock price fluctuates around the fair value according to the following Ornstein–Uhlenbeck process:

$$dS_t = k(\theta_t - S_t)dt + \sigma dW_t \tag{2}$$

Equation (2) shows that the stock price S is pulled towards its fundamental value θ. The speed of adjustment is characterized by the parameter k which measures the pricing efficiency of the market. The higher k, the more quickly the stock price moves towards its fundamental value. σ measures the extent of noise traders. The larger σ, the more the stock price can temporarily deviate from the fundamental value. The parameters k and σ are assumed to be positive and constant. $(W_t, t \geq 0)$ is a Brownian motion under the historical probability P and independent from $(B_t, t \geq 0)$. Equation (2) implies that there is a non-zero probability that the stock price becomes negative. However, given the continuity of the price process and the fact that we look at fairly short non-trading periods, this probability is negligible as shown in "Appendix 1".

Given this general structure of the model, the investor determines her optimal selling strategy. The value of tradeability is determined by the difference between the expected payoff of the optimal trading strategy when the stock is liquid and the expected payoff of the trading strategy when the stock is illiquid.[6]

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[5] Since we focus solely on liquidity in the sense of tradeability, we use the terms liquidity and tradeability interchangeably throughout the rest of the paper.

[6] To keep the exposition of the model simple, we ignore discounting effects. This assumption is reasonable given that we analyze short non-trading periods. Obviously, discounting terms could be easily added.

2.1 Basic model

In the basic version of our model we make two additional simplifying assumptions which allow us to solve the model analytically: (1) The investor can sell the liquid stock only at times $t = 0$ or T. (2) The fundamental value of the stock, θ, is constant over time.

The value of tradeability at time 0 is determined by the difference between the expected payoffs of the optimal trading strategies when holding a liquid stock and an illiquid stock, respectively. When holding an illiquid stock, the investor has to sell at T and the expected payoff is $E_0(S_T)$. In contrast, the investor has the flexibility to sell in 0 or in T when holding a liquid stock. She sells the liquid stock at time 0 if the stock price at that time is higher than the expected stock price for time T. Thus, the expected payoff from her optimal trading strategy is

$$S_0 1_{S_0 > E_0(S_T)} + E_0(S_T 1_{S_0 \leqslant E_0(S_T)}), \tag{3}$$

and the value of liquidity is given as:

$$VL(S_0, T) = (S_0 - E_0(S_T)) 1_{S_0 > E_0(S_T)} \tag{4}$$

Given (2), the expected stock price is:

$$E_0(S_T) = \theta - (\theta - S_0)e^{-kT} \tag{5}$$

Based on (5) one can easily show that the critical stock price above which the investor sells the stock is equal to the fundamental value θ:

$$1_{S_0 > E_0(S_T)} = 1_{S_0 > \theta} \tag{6}$$

This is a sensible strategy since the investor expects the price to decline if the stock is currently overpriced, i.e. $S_0 > \theta$. Given (5) and (6), the pricing formula (4) can be written as:

$$VL(S_0, T) = (1 - e^{-kT})(S_0 - \theta) 1_{S_0 > \theta} \tag{7}$$

Equation (7) shows that liquidity can be interpreted as a call on the stock with strike price θ and instantaneous maturity. The value of liquidity is positive only if the current stock price (underlying) is above the fair value (strike price), i.e. if the stock is overpriced. Only in this case is it optimal for the investor to sell the stock at time 0 making liquidity valuable for the investor. The value of liquidity then increases linearly in the level of the overpricing, $S_0 - \theta$, and the slope, $1 - e^{-kT}$, depends on the pricing efficiency of the market, k, and on the length of the non-trading period of the illiquid stock, T. Both parameters have a positive and concave impact on the value of liquidity.

Thus, the simple version of our model predicts that the longer the non-trading period T of the illiquid stock, the greater the value of liquidity. This is sensible since

the length of the non-trading period captures the difference in liquidity in our model. The more the liquid and illiquid assets differ with respect to their tradeability, the more valuable liquidity is.

Furthermore, (7) implies that the value of liquidity is higher, the more quickly a temporary mispricing disappears, i.e. the higher the pricing efficiency k of the market. The economic intuition is as follows: If the investor holds the liquid stock, she can exploit the overpricing $S_0 - \theta$ at time 0. However, the proportion e^{-kT} of this overpricing is expected to exist until time T and can be exploited by the investor even when holding the illiquid stock. The value of liquidity is given by the difference in trading profits: the proportion $1 - e^{-kT}$ of the current overpricing $S_0 - \theta$, i.e., the overpricing which is not expected to disappear by T. The higher the pricing efficiency k of the market, the more the overpricing is expected to disappear before the illiquid stock can be traded and, consequently, the more valuable liquidity is. Equation (7) highlights the crucial role of our assumption that the stock price is linked to its fair value. It translates the differences in tradeability (as captured by T) into differences in trading profits. Even if the two assets differ dramatically with respect to tradeability, i.e. T is large, liquidity would have no value for $k = 0$. In this case, the stock price would follow a random walk and the investor would not expect to make an additional profit from selling the liquid asset before T. Thus, liquidity has no value for her.

2.2 Multiple interim trading dates

We now give up the first simplifying assumption and extend our model to several interim trading dates. This gives the investor more flexibility when selling the liquid stock and thus increases the difference in tradeability between the liquid and the illiquid stock. In this setting, the value of liquidity depends not only on the current mispricing of the stock, but also on the future mispricing. Consequently, the value of liquidity will now depend on the amount of noise trading, σ, since noise trading makes mispricing occur in the future.

As a starting point, we allow the investor to sell the liquid stock not only at times $t = 0$ and T, but also at time $\tau = \frac{T}{2}$. In this model, the value of liquidity is given as

$$VL(S_0, T) = E_0(1_{S_0 > S_0^*}(S_0 - S_T)) + E_0(1_{S_0 < S_0^*}(S_\tau 1_{S_\tau > \theta} + S_T 1_{S_\tau \leqslant \theta}) - S_T), (8)$$

where S_0^* is the critical selling price at time 0 above which the investor sells the stock. If the stock price is equal to the critical price, the investor is indifferent between selling the stock at time 0 and waiting to sell the stock later. The expected future selling price is determined by the expected selling prices at times τ and T. Thus, the critical selling price at time $t = 0$ has to satisfy

$$S_0^* = E_0(S_\tau 1_{S_\tau > \theta} \mid S_0 = S_0^*) + E_0(S_T 1_{S_\tau \leqslant \theta} \mid S_0 = S_0^*). \qquad (9)$$

If S_τ is smaller than θ, the expected value of S_T is larger than S_τ since the process of the stock price is mean-reverting. Therefore,

$$
\begin{aligned}
S_0^* &> E_0(S_\tau 1_{S_\tau > \theta} \mid S_0 = S_0^*) + E_0(S_\tau 1_{S_\tau \leq \theta} \mid S_0 = S_0^*) \\
&> E_0(S_\tau \mid S_0 = S_0^*)
\end{aligned}
\tag{10}
$$

which is true if and only if $S_0^* > \theta$. This shows that the critical stock price S_0^* is above the fair value θ. If the stock price is above its fair value but below the critical stock price, $S_0^* > S_0 > \theta$, the investor does not sell the stock although it is overpriced and, therefore, expected to decrease. Since there are noise traders in the market, it is optimal for the investor to wait due to the chance of exploiting a larger overpricing later. Hence, the time value of the option to sell the stock is strictly positive and it is not optimal to exercise the option by selling the stock. In "Appendix 2" we show that the critical stock price S_0^* has to satisfy

$$
\begin{aligned}
S_0^* &= e^{-k\Delta t}\left(\theta - (\theta - S_0^*)e^{-k\Delta t}\right) \\
&\quad + \left[(\theta - e^{-k\Delta t}(\theta - S_0^*))N(b^*) + e^{-k\Delta t}\frac{\sqrt{f(\Delta t)}}{\sqrt{2\pi}}e^{-(b^*)^2/2}\right](1 - e^{-k\Delta t}) \\
&\quad + \theta(1 - e^{-k\Delta t})N(-b^*) \\
&= e^{-k\Delta t}\left(\theta - (\theta - S_0^*)e^{-k\Delta t}\right) \\
&\quad + \left[\theta - e^{-k\Delta t}(\theta - S_0^*)N(b^*) + e^{-k\Delta t}\frac{\sqrt{f(\Delta t)}}{\sqrt{2\pi}}e^{-(b^*)^2/2}\right](1 - e^{-k\Delta t})
\end{aligned}
\tag{11}
$$

with Δt being the time between two possible trading dates of the liquid stock and

$$
b^* = (S_0^* - \theta)/\sqrt{f(\Delta t)}
\tag{12}
$$

$$
f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1).
\tag{13}
$$

As shown in "Appendix 3", the value of liquidity can be written as follows:

$$
\begin{aligned}
VL(S_0, T) &= 1_{S_0 > S_0^*}(1 - e^{-kT})(S_0 - \theta) \\
&\quad + 1_{S_0 < S_0^*}(1 - e^{-k\Delta t})\left(e^{-k\Delta t}(S_0 - \theta)N(b) + \frac{e^{-k\Delta t}}{\sqrt{2\pi}}\sqrt{f(\Delta t)}e^{-b^2/2}\right)
\end{aligned}
\tag{14}
$$

The value of liquidity is determined by the additional profit of the investor given that it is optimal to sell at time 0 [first line of Eq. (14)] or to sell at time τ [second line of Eq. (14)]. If it is optimal to sell at time 0, the investor makes the same profit as in the basic model. However, if $S_0 < S_0^*$, the investor is better off by selling the stock not at time 0, but by waiting until time τ or T.

Figure 1 shows the value of liquidity in the model with two interim trading dates. The figure is based on the following parameters: The non-trading period is 1 day,
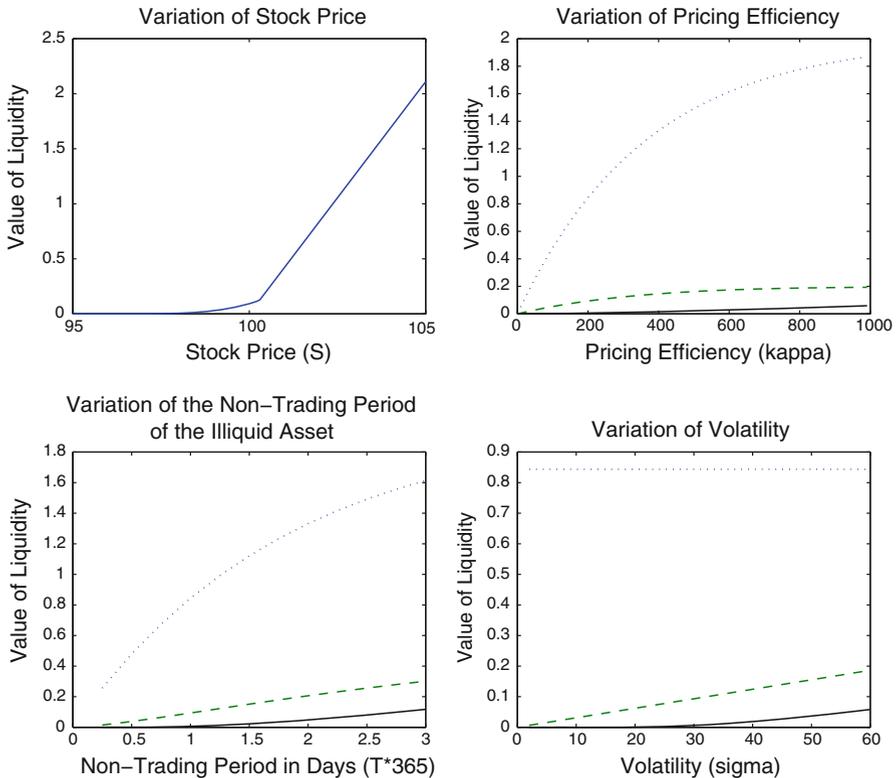
Fig. 1 Impact of stock price, pricing efficiency, length of the non-trading period, and stock price volatility on the value of liquidity. The figures show the value of liquidity for varying values of S_0 (*upper left* figure), k (*upper right* figure), T (*bottom left* figure), and σ (*bottom right* figure) when the liquid stock can be traded at times 0, τ, or T. The fair value of the stock is constant and chosen as $\theta = 100$. In the upper left figure, the pricing efficiency is $k = 200$, the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$, and the volatility is $\sigma = 30$. In all remaining figures we analyze the cases of $S_0 = 102$ (*dotted line*), $S_0 = 100$ (*dashed line*), and $S_0 = 98$ (*solid line*). The remaining model parameters are chosen as follows: in the *upper right* figure, the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$ and the volatility is $\sigma = 30$. In the *bottom left* figure, the pricing efficiency is $k = 200$ and the volatility is $\sigma = 30$. In the *bottom right* figure the pricing efficiency is $k = 200$ and the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$

i.e. $T = 1/365$. The mean reversion parameter is $k = 200$ which implies that the half-life of the mispricing is 1.26 days. The noise trading parameter is $\sigma = 30$ which corresponds to a stock volatility of about 30%. The upper two figures show how the value of liquidity depends on the stock price S_0 and on the pricing efficiency k, respectively. The bottom figures show the impact of the length of the non-trading period (measured in days) and the volatility of the stock price on the value of liquidity, respectively.

The value of liquidity shows the same characteristics as in the basic model. It increases with the stock price S_0 (for a given fair value), with the pricing efficiency k of the market, and with the length T of the non-trading period of the illiquid stock.

The impact of k and T on the value of liquidity is positive and concave, no matter whether the stock is currently overpriced, fairly priced, or underpriced.

In the basic model, the investor could sell the liquid stock early only at time 0, i.e. her option to sell early expires at time 0 and, consequently, the option value does only depend on the mispricing at this time, but not on the future mispricing. Therefore, noise trading which makes future mispricing occur does not matter and σ does not influence the option price. In the extended model, the option to sell the liquid stock early expires at time τ and, therefore, future mispricing and stock price volatility σ matter for the value of the option. As shown in the bottom right figure, the value of liquidity increases with stock price volatility. Evidently, the relation is strongest when the stock is currently fair priced. In this case, the investor benefits most from noise trading due to the asymmetric nature of the option. The value of liquidity rises linearly with volatility from almost zero to approximately 0.2%.

We now generalize the model by allowing the investor to sell the stock at times $0, \Delta t, \ldots, n\Delta t = T$ and denote the stock price at time $(n - i)\Delta t$ as S_{n-i}. By generalizing Eqs. (45)–(47) in "Appendix 3", the value of liquidity can be calculated as

$$
\begin{aligned}
VL_n(S_0, T) = {} & 1_{S_0 > S_0^*}(1 - e^{-kT})(S_0 - \theta) + 1_{S_0 < S_0^*}(1 - e^{-k(T - \Delta t)})E_0((S_1 - \theta)1_{S_1 > S_1^*}) \\
& + (1 - e^{-k(T - 2\Delta t)})E_0((S_2 - \theta)1_{S_1 < S_1^*}1_{S_2 > S_2^*}) \\
& + \cdots + (1 - e^{-k(T - i\Delta t)})E_0((S_i - \theta)1_{S_1 < S_1^*}\ldots 1_{S_{i-1} < S_{i-1}^*}1_{S_i > S_i^*}) + \cdots \\
& + (1 - e^{-k(T - (n-1)\Delta t)})E_0((S_{n-1} - \theta)1_{S_1 < S_1^*}\ldots 1_{S_{n-2} < S_{n-2}^*}1_{S_{n-1} > \theta})
\end{aligned}
\tag{15}
$$

where the critical levels are computed recursively. The critical value at the last interim trading date is given by the fundamental value of the stock

$$
S_{n-1}^* = \theta
\tag{16}
$$

and S_{n-2}^* is the smallest x such that:

$$
x = E_{n-2}(S_{n-1}1_{S_{n-1} > \theta} \mid S_{n-2} = x) + E_{n-2}(S_T 1_{S_{n-1} \leqslant \theta} \mid S_{n-2} = x)
\tag{17}
$$

More generally, S_{n-i}^* has to satisfy:

$$
\begin{aligned}
x = {} & E_{n-i}(S_{n-i+1}1_{S_{n-i+1} > S_{n-i+1}^*} \mid S_{n-i} = x) \\
& + E_{n-i}(S_{n-i+2}1_{S_{n-i+1} < S_{n-i+1}^*}1_{S_{n-i+2} > S_{n-i+2}^*} \mid S_{n-i} = x) + \cdots \\
& + E_{n-i}(S_n 1_{S_{n-i+1} < S_{n-i+1}^*}1_{S_{n-i+2} < S_{n-i+2}^*}\ldots 1_{S_{n-1} < \theta} \mid S_{n-i} = x)
\end{aligned}
\tag{18}
$$

To solve the valuation problem, we first compute critical stock prices recursively using Eq. (18). Then we calculate the expectations on the right-hand side of Eq. (15). This delivers the value of liquidity for n trading dates.

When the investor can trade continuously, i.e. the number of interim trading dates tends toward infinity, generalizing Eq. (15) leads to

$$VL(S_0, T) = \lim_{n \to +\infty} VL_n(S_0, T) = \sup_{\tau \in \Gamma} E_0((1 - e^{-k(T-\tau)})(S_\tau - \theta)) \quad (19)$$

where Γ is the set of stopping times with values in $[0, T]$. The value of liquidity can now be calculated as the price of an American option and we have to solve a free boundary problem. In general, there is no exact analytical solution for pricing American options. Instead, one can use the solution (15)–(18) of the discrete version of the model and calculate the value of liquidity for a sufficiently high number of trading dates. Alternatively, one can search for an analytical approximation of the solution of the continuous version of the model. We follow the second path since it delivers closed form results which can be economically interpreted. The value of liquidity can be approximated as follows:

$$VL(S_0, T) \simeq \sup_{C} \int_0^T (1 - e^{-k(T-t)})(K(t) - \theta) p(T_k \in dt) \quad (20)$$

As shown in "Appendix 4", the density of the first passage time T_k of the Ornstein–Uhlenbeck process S at the boundary $K(\cdot)$ is

$$p(T_k \in dt) = e^{2kt} \sigma^2 \frac{|C - S_0|}{\sqrt{2\pi f(t)^3}} e^{-\frac{(C - S_0 + \alpha f(t))^2}{2f(t)}} dt \quad (21)$$

with

$$f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1) \quad (22)$$

$$\alpha = \frac{2k(\theta - C)}{\sigma^2(e^{2kT} - 1)} \quad (23)$$

$$K(t) = e^{-kt}(C - \theta)\left(1 - \frac{e^{2kt} - 1}{e^{2kT} - 1}\right) + \theta \quad (24)$$

where C is a free parameter.

The structure of the pricing formula is very similar to the structure of the pricing formula in our earlier models. The value of liquidity is equal to the value of an American call option with a payoff equal to $(1 - e^{-k(T-\tau)})(K(\tau) - \theta)$ at the optimal exercise time τ, i.e. when exercised at the first passage time of the process S at the boundary K.

As shown in Fig. 2, the value of liquidity depends on S_0, k, T, and σ in the same way as in the earlier models: It increases in the overpricing of the stock, in the pricing efficiency of the market, in the length of the non-trading period of the illiquid stock, and in the amount of noise trading.

In Fig. 3 we show how the value of liquidity depends on the number of interim trading dates. We plot the value of liquidity for varying S_0 for the basic model (trading
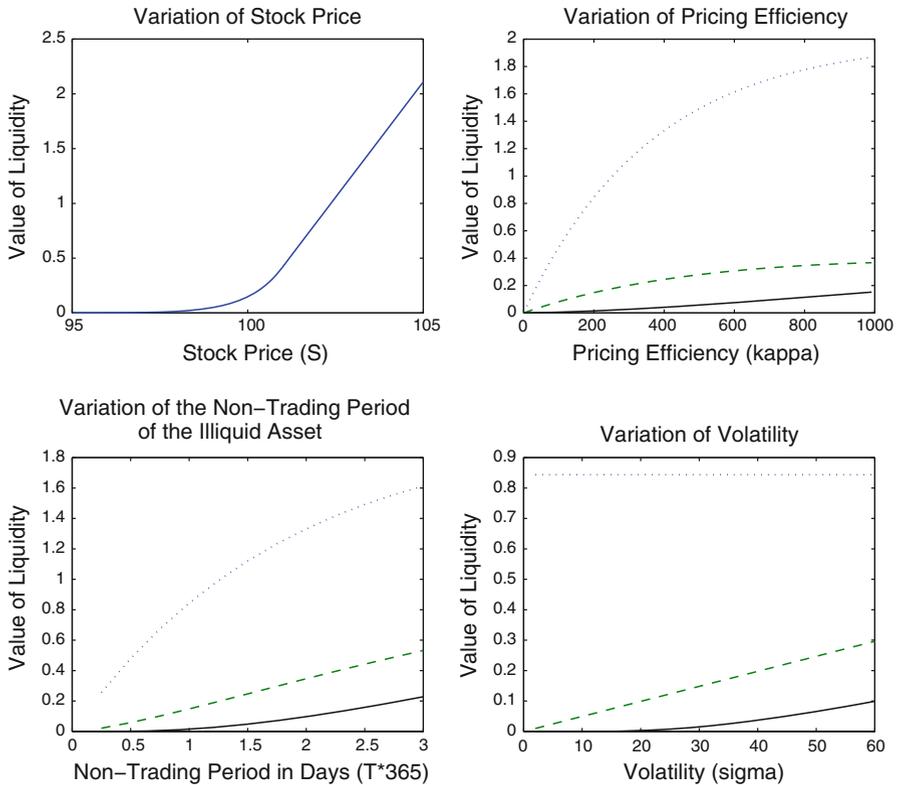
Fig. 2 Impact of stock price, pricing efficiency, length of the non-trading period, and stock price volatility on the value of liquidity. The figures show the value of liquidity for varying values of S_0 (*upper left* figure), k (*upper right* figure), T (*bottom left* figure), and σ when the liquid stock can be traded continuously. The fair value of the stock is constant and chosen as $\theta = 100$. In the upper left figure, the pricing efficiency is $k = 200$, the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$, and the volatility is $\sigma = 30$. In all remaining figures we analyze the cases of $S_0 = 102$ (*dotted line*), $S_0 = 100$ (*dashed line*), and $S_0 = 98$ (*solid line*). The remaining model parameters are chosen as follows: In the *upper right* figure, the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$ and the volatility is $\sigma = 30$. In the *bottom left* figure, the pricing efficiency is $k = 200$ and the volatility is $\sigma = 30$. In the *bottom right* figure the pricing efficiency is $k = 200$ and the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$

at times 0 and T), the model with trading at times 0, $\tau = \frac{T}{2}$, and T, the model with fifteen interim dates and the model with continuous trading of the liquid stock.

Figure 3 clearly shows that the value of liquidity increases with the number of trading dates. This is sensible since a larger number of trading dates results in higher trading flexibility of the liquid stock. This effect is most pronounced when the stock is currently fairly priced, i.e. $S_0 = \theta = 100$. In this case the investor benefits most from the increased flexibility. The figure also shows that the value of liquidity converges to (20) when the number of interim trading dates increases. In Fig. 3 one can hardly see any difference between the value of liquidity with fifteen interim trading dates and the value of liquidity with continuous trading. For $S_0 = \theta = 100$, the value of liquidity is 0.1454 in the first case, and 0.1482 in the second case. Thus, the

Variation of Stock Price
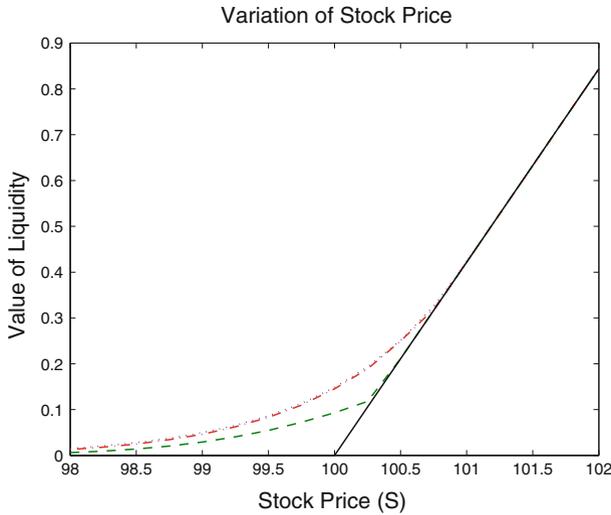
Fig. 3 Impact of the stock price on the value of liquidity as a function of the number of interim trading dates. The figure shows the value of liquidity for varying values of S_0 in four cases. (1) The liquid stock can be traded at times 0 and T (*solid line*). (2) The liquid stock can be traded at times 0, $\tau = T/2$, and T (*dashed line*). (3) The liquid stock can be traded at fifteen interim trading dates (*dashed dotted line*). (4) The liquid stock can be traded continuously (*dotted line*). The fair value of the stock is constant and chosen as $\theta = 100$. The pricing efficiency is $k = 200$, the non-trading period of the illiquid stock is 1 day, i.e. $T = 1/365$, and the volatility is $\sigma = 30$

difference is only -0.0028 which is about -2% of the value of liquidity. For 100 interim trading dates (not shown in Fig. 3), the difference is even smaller and amounts to only -0.0005 which equals about -0.29%. Thus, our analytical approximation of the continuous model and the numerical solution of the discrete model converge well for a high number of trading dates highlighting the accuracy of the analytical approximation (20)–(24).

2.3 Random fundamental value

We now give up the assumption that the fundamental value θ of the stock is constant and develop our full model. The fair value evolves according to Eq. (1) and the stock price according to Eq. (2). The investor can sell the liquid stock at any time $0 \leq \tau \leq T$, but the illiquid stock only at T. In this model, the dynamics of the stock price can be written as:

$$dS_t = k(\theta_0 + \gamma B_t - S_t)dt + \sigma dW_t \tag{25}$$

Obviously, the stock price S evolves no longer according to an Ornstein–Uhlenbeck process. Therefore, we define a modified stock price

$$\hat{S}_t = S_t - \gamma B_t \tag{26}$$

which is the stock price corrected for the impact of the innovations of the fair value. The modified stock price follows again an Ornstein–Uhlenbeck process with a constant fundamental value θ_0 and a volatility $\sqrt{\sigma^2 + \gamma^2}$:

$$d\hat{S}_t = k(\theta_0 - \hat{S}_t)dt + \sigma dW_t - \gamma dB_t \qquad (27)$$

To solve the model as in the previous section, we characterize the optimal exercise boundary in this new setting. In "Appendix 5" we show that the optimal exercise boundary S^* of the process S is given by the sum of the boundary $\hat{S}^*$ of the Ornstein–Uhlenbeck process $\hat{S}$ and γB_t:

$$S_t^* = \hat{S}_t^* + \gamma B_t \qquad (28)$$

Therefore, the optimal exercise boundary can be approximated by the sum of γB_t and the function $K(t)$ given in Eq. (24). In Sect. 2.2 we showed that $K(t)$ is an accurate approximation for a given volatility σ. Since the volatility is now $\sqrt{\sigma^2 + \gamma^2}$, we need to check whether the approximation remains accurate for larger values of volatility. To do so, we calculate the difference in the value of liquidity between the discrete model with 100 interim trading dates and our analytical approximation for volatility levels between $\sigma = 10$ and $\sigma = 100$. When the stock price and the fundamental value are both 100 as before, the difference (-0.29%) between the model value and the approximation value is the same (up to an accuracy level of 10^{-10}) for all levels of volatility. This suggests that the quality of our analytical approximation does not depend on the volatility of the stock price, i.e. it is the same in Sects. 2.2 and 2.3. Consequently, we can calculate the value of liquidity as in Eqs. (20)–(24), but now based on the new Ornstein–Uhlenbeck process $\{\hat{S}_t; t \geq 0\}$.

This suggests that allowing the fair value to change randomly does not change our earlier result qualitatively. Using a fair value that is random leads to an increase in volatility ($\sqrt{\sigma^2 + \gamma^2}$ instead of σ) and, consequently, to an increase in the value of liquidity. Thus, we get the final prediction of our model: The more information about the fair value flows into the market, the more valuable liquidity is. Thus, uncertainty increases the value of liquidity, no matter whether it results from noise trading or from fundamental news.

3 Conclusions

Tradeability of assets is crucial for investors. If is it lacking, investors might encounter serious difficulties in their risk management as seen during the recent financial crisis. Yet, even in more modest financial situations, tradeability is important for investors since it allows them to adjust their portfolio when needed. Trading needs make tradeability valuable.

In this paper, we determine the value of tradeability using an option pricing approach. In our model, tradeability is valuable since it allows the investor to exploit temporary mispricing of stocks. We assume that the stock price is linked to its fair value, but may temporarily deviate from it. The liquid asset allows the investor to

exploit the mispricings, the illiquid one does not. The larger this mispricing, the more valuable tradeability is. The second main factor driving the value of tradeability is the difference in liquidity between the liquid and the illiquid stock. The longer the illiquid stock cannot be traded and the more trading dates the liquid stock offers, the more valuable tradeability is. However, differences in liquidity translate only to a positive value of tradeability if the mispricing is expected to disappear over time. Otherwise, the investor cannot predict future price changes and, therefore, cannot expect to make a higher trading profit when holding the liquid stock. Hence, our model predicts that the value of tradeability increases with the strength of the pricing efficiency of the market. The more quickly a mispricing disappears, the higher the incentive to trade, and the higher the value of tradeability. Finally, we show that the value of tradeability increases with uncertainty, no matter whether the uncertainty results from noise trading or from news about the fundamental value. Both effects increase the probability of large temporary mispricings in the future and, thus, the value of liquidity.

In our view, the most striking result of our model is that the value of tradeability is especially high in markets with high pricing efficiency and strong information flow. Testing these predictions of our model empirically is an interesting avenue of further research.

Appendix 1

We first assume that the fair value of the stock is positive and constant, i.e. $\theta > 0$. Let us define the following stopping time

$$T'_k = \inf\{t \geq 0; S < K(t)\}, \tag{29}$$

where the function $K(\cdot)$ (see also "Appendix 4") is given by

$$K(t) = (\alpha f(t) + \beta + S_0 - \theta) e^{-kt} + \theta. \tag{30}$$

with $f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1)$. The function $K(\cdot)$ is close to zero and positive. The two parameters α and β which characterize this function are defined as follows:

$$K(0) = \beta + S_0 = C = 10^{-4} \tag{31}$$
$$K(T) = (\alpha f(T) + C - \theta) e^{-kT} + \theta = C = 10^{-4} \tag{32}$$

Therefore

$$\alpha = \frac{\theta - C}{f(T)}(1 - e^{kT}). \tag{33}$$

It can be easily shown that the function $K(\cdot)$ is positive on $[0, T]$:

$$K(t) = \left(\frac{\theta - C}{f(T)} (1 - e^{kT}) f(t) + C - \theta \right) e^{-kt} + \theta \tag{34}$$

From the definition of the function $f(\cdot)$ it follows that

$$K(t) = (\theta - C) \left(\frac{1 - e^{2kt}}{e^{kT} + 1} - 1 \right) e^{-kt} + \theta \tag{35}$$

and

$$K(t) = (\theta - C) \left(\frac{-(e^{kT} + e^{2kt}) e^{-kt} + e^{kT} + 1}{e^{kT} + 1} \right) + C. \tag{36}$$

The numerator of this ratio is positive on $[0, T]$.

Let us now define the first passage time of the process S at level 0:

$$T_0 = \inf \{t \geq 0; S_t < 0\} \tag{37}$$

Since the function $K(\cdot)$ is positive, it follows:

$$p(T_0 < T) < p(T_k' < T) = \int_0^T p(T_k' \in dt) \tag{38}$$

From Eq. (21) the density is obtained:

$$p(T_k' \in dt) = e^{2kt} \sigma^2 \frac{|C - S_0|}{\sqrt{2\pi f(t)^3}} e^{-\frac{(C - S_0 + \alpha f(t))^2}{2f(t)}} dt \tag{39}$$

where α is given by Eq. (32).

With our set of parameters ($\theta = 100$, $S_0 = 100$, $\sigma = 30$, $k = 200$, $C = 10^{-4}$), the resulting probability is smaller than 10^{-100} even for a non-trading period of one year. Thus, the probability of reaching a negative stock price is negligible.

We now turn to the case that the fundamental value of the stock is stochastic with $\theta_0 > 0$. In Sect. 2.3 we show that this case is equivalent to the case of a constant fundamental value with an increased volatility ($\sqrt{\sigma^2 + \gamma^2}$ instead of σ). Assuming reasonable values for the volatility of the fundamental value (e.g. $\gamma = 30$) the probability of reaching a negative stock price is still negligible. Even for an non-trading period of one year it is smaller than 10^{-100}.

Appendix 2

To solve Eq. (9) for the critical stock price, we have to calculate the expected profit of the investor when she decides not to sell at time 0. This is given by the expected stock

price at time τ given she sells in τ (first term on the right hand side of Eq. (9)) plus the expected stock price in T given she waits until T (second term on the right hand side of Eq. (9)). We now calculate the first term on the right hand side of Eq. (9). Given the dynamics of the stock price in Eq. (2), the probability of S_τ being larger than the fair value θ is

$$p(S_\tau > \theta) = p(\theta - e^{-k\Delta t}(\theta - S_0 - Z_{f(\Delta t)}) > \theta) \tag{40}$$

with $f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1)$. $\{Z_{f(t)}, t \geq 0\}$ is a time-changed Brownian motion. Rewriting yields

$$p(S_\tau > \theta) = p\left(-\frac{Z_{f(\Delta t)}}{\sqrt{f(\Delta t)}} \leq \frac{(S_0 - \theta)}{\sqrt{f(\Delta t)}}\right)$$
$$= N(b) \tag{41}$$

with $b = (S_0 - \theta)/\sqrt{f(\Delta t)}$ and N being the cumulative density of a standard normal variable. Therefore, the expected stock price conditional on being above the fair value is

$$E_0(S_\tau 1_{S_\tau > \theta}) = E_0\left((\theta - e^{-k\Delta t}(\theta - S_0 - Z_{f(\Delta t)}))1_{\left(-\frac{Z_{f(\Delta t)}}{\sqrt{f(\Delta t)}} \leq b\right)}\right)$$
$$= (\theta - e^{-k\Delta t}(\theta - S_0))N(b) + e^{-k\Delta t}\sqrt{f(\Delta t)}E_0\left(\frac{Z_{f(\Delta t)}}{\sqrt{f(\Delta t)}}1_{\left(-\frac{Z_{f(\Delta t)}}{\sqrt{f(\Delta t)}} \leq b\right)}\right)$$
$$= (\theta - e^{-k\Delta t}(\theta - S_0))N(b) - \frac{e^{-k\Delta t}\sqrt{f(\Delta t)}}{\sqrt{2\pi}}\int_{-\infty}^{b} x e^{-\frac{x^2}{2}} dx$$
$$= (\theta - e^{-k\Delta t}(\theta - S_0))N(b) + \frac{e^{-k\Delta t}\sqrt{f(\Delta t)}}{\sqrt{2\pi}}e^{-\frac{b^2}{2}}. \tag{42}$$

We now turn to the second term on the right hand side of Eq. (9).

$$E_0(S_T 1_{S_\tau \leq \theta}) = E_0(1_{S_\tau \leq \theta}(\theta - (\theta - S_\tau)e^{-k\Delta t}))$$
$$= e^{-k\Delta t}E(1_{S_\tau \leq \theta}S_\tau) + \theta(1 - e^{-k\Delta t})p(S_\tau < \theta) \tag{43}$$

Putting (43) into (9) and rearranging the terms yields

$$S_0^* = e^{-k\Delta t}E(S_\tau \mid S_0 = S_0^*) + (1 - e^{-k\Delta t})E(1_{S_\tau > \theta}S_\tau \mid S_0 = S_0^*)$$
$$+ \theta(1 - e^{-k\Delta t})p(S_\tau < \theta \mid S_0 = S_0^*) \tag{44}$$

We use (5) for the calculation of the first term, (42) for the second term, and (41) for the final term of (44). This gives us Eq. (11).

Appendix 3

Equation ([8](#)) can be written as

$$VL(S_0, T) = 1_{S_0 \geq S_0^*} E_0(S_0 - S_T) + 1_{S_0 < S_0^*} E_0((S_\tau - S_T)1_{S_\tau > \theta}). \tag{45}$$

Given the dynamics of the stock price, the first expectation on the right hand side is

$$E_0(S_0 - S_T) = (1 - e^{-kT})(S_0 - \theta). \tag{46}$$

The second expectation is

$$
\begin{aligned}
E_0\left(E_\tau\left((S_\tau - S_T)1_{S_\tau > \theta}\right)\right) &= E_0\left(\left(S_\tau - (\theta - e^{-k\Delta t}(\theta - S_\tau - (Z_{f(T)} - Z_{f(\tau)})))1_{S_\tau > \theta}\right)\right) \\
&= E_0\left((S_\tau - \theta)(1 - e^{-k\Delta t})1_{S_\tau > \theta}\right) \tag{47}
\end{aligned}
$$

and by relying on "Appendix 2" we get

$$
\begin{aligned}
E_0\left(E_\tau\left((S_\tau - S_T)1_{S_\tau > \theta}\right)\right) &= -\theta(1 - e^{-k\Delta t})N(b) \\
&\quad + (1 - e^{-k\Delta t})\left((\theta - e^{-k\Delta t}(\theta - S_0))N(b) + \frac{e^{-k\Delta t}\sqrt{f(\Delta t)}}{\sqrt{2\pi}}e^{-b^2/2}\right) \\
&= (1 - e^{-k\Delta t})\left(e^{-k\Delta t}(S_0 - \theta)N(b) + \frac{e^{-k\Delta t}\sqrt{f(\Delta t)}}{\sqrt{2\pi}}e^{-b^2/2}\right) \tag{48}
\end{aligned}
$$

with $f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1)$ and $\{Z_{f(t)}, t \geq 0\}$ being a time-changed Brownian motion. This leads to Eq. ([14](#)).

Appendix 4

The stochastic differential ([2](#)) has the solution

$$S_t = \theta - e^{-kt}(\theta - S_0 - Z_{f(t)}), \tag{49}$$

with $f(t) = \frac{\sigma^2}{2k}(e^{2kt} - 1)$ and $\{Z_{f(t)}, t \geq 0\}$ being a time changed Brownian motion. We define the stopping time

$$T_k = \inf\{t \geq 0; S_t \geq K(t)\}, \tag{50}$$

where $K(t)$ is an exercise boundary. Using ([49](#)) the stopping time can be written as:

$$
\begin{aligned}
T_k &= \inf\{t \geq 0; \theta - e^{-kt}(\theta - S_0 - Z_{f(t)}) \geq K(t)\}, \\
&= \inf\{t \geq 0; \theta - S_0 - Z_{f(t)} \leq (\theta - K(t))e^{kt}\} \\
&= \inf\{t \geq 0; Z_{f(t)} \geq (K(t) - \theta)e^{kt} + \theta - S_0\} \tag{51}
\end{aligned}
$$

Let us now assume that the boundary $K(\cdot)$ satisfies

$$(K(t) - \theta)e^{kt} + \theta - S_0 = \alpha f(t) + \beta \tag{52}$$

where α and β are two parameters. (52) implies

$$K(t) = (\alpha f(t) + \beta + S_0 - \theta) e^{-kt} + \theta. \tag{53}$$

This assumption allows us to derive analytically the density $p(T_k \in dt)$ of the first passage time of the Ornstein–Uhlenbeck process S at this boundary. Using (52) the stopping time (51) can be written as

$$T_k = \inf \left\{ t \geq 0; Z_{f(t)} \geq \alpha f(t) + \beta \right\} \tag{54}$$

with the boundary condition $K(T) = \theta$. Applying this boundary condition to (53) yields

$$\alpha = \frac{\theta - S_0 - \beta}{f(T)} \tag{55}$$

$$= \frac{2k(\theta - C)}{\sigma^2(e^{2kT} - 1)} \tag{56}$$

with $C = S_0 + \beta$. Using (56), the boundary (53) can be written as:

$$K(t) = \left(\frac{2k(\theta - C)}{\sigma^2(e^{2kT} - 1)} \frac{\sigma^2(e^{2kt} - 1)}{2k} + C - \theta \right) e^{-kt} + \theta \tag{57}$$

which leads to Eq. (24) of the main text. The density of T_k can be written as

$$\begin{aligned} p(T_k \in dt) &= \frac{dp(T_k \leq t)}{dt} \\ &= \frac{dp(f(T_k) \leq f(t))}{dt} \\ &= \frac{df(t)}{dt} \frac{dp(f(T_k) \leq f(t))}{df(t)} \end{aligned} \tag{58}$$

because $\frac{df(t)}{dt}$ is positive. Therefore

$$p(T_k \in dt) = \frac{df(t)}{dt} g(f(t))dt, \tag{59}$$

where g is the density of the first passage time of a drifted Brownian motion $\{Z_t - \alpha t, t \geq 0\}$ at level β. This leads to Eq. (21).

Appendix 5

In this appendix we prove Eq. (28) of the main text. The discrete time version of this equation is

$$S^*_{n-i} = \hat{S}^*_{n-i} + \gamma B_{n-i}, \quad i \in \{1, 2, \ldots, n\} \tag{60}$$

where S^*_{n-i} and $\hat{S}^*_{n-i}$ are the exercise boundaries for the price processes S and $\hat{S}$ and B_{n-i} the Brownian motion at time $(n-i)\Delta t$, respectively. n is defined by $n\Delta t = T$. We proceed recursively when proving (60).

For $i = 1$, the proof is straightforward. At the last interim trading date, $(n-1)\Delta t = T - \Delta t$, the stock will be sold if and only if its value exceeds the current fundamental value $\theta_0 + \gamma B_{n-1}$. Therefore, the optimal exercise boundary is given as

$$S^*_{n-1} = \theta_0 + \gamma B_{n-1}. \tag{61}$$

According to Eq. (16), θ_0 is the exercise boundary at the last interim trading date if the fundamental value takes a constant value θ_0 (as it is the case for $\hat{S}$ according to Eq. (27)). Thus, Eq. (60) holds for $i = 1$.

For $i = 2$, the exercise boundary S^*_{n-2} is characterized by the equality between the profit if the stock is sold instantaneously and the expected profit if the stock is sold later. Therefore, S^*_{n-2} satisfies the following equation:

$$x = E_{n-2}(S_{n-1} 1_{S_{n-1} > S^*_{n-1}} \mid S_{n-2} = x) + E_{n-2}(S_n 1_{S_{n-1} \leqslant S^*_{n-1}} \mid S_{n-2} = x)$$

This general characterization of an exercise boundary holds true whatever the dynamics of the fundamental value is, i.e. it holds for a constant fundamental value as well as for a stochastic one. Since the exercise boundary S^*_{n-1} is given by (61), S^*_{n-2} satisfies

$$x = E_{n-2}(S_{n-1} 1_{S_{n-1} > \theta_0 + \gamma B_{n-1}} \mid S_{n-2} = x) + E_{n-2}(S_n 1_{S_{n-1} \leqslant \theta_0 + \gamma B_{n-1}} \mid S_{n-2} = x).$$

Expanding this equation leads to

$$\begin{aligned}
x = {} & E_{n-2}((S_{n-1} - \gamma B_{n-1}) 1_{S_{n-1} - \gamma B_{n-1} > \theta_0} \mid S_{n-2} = x) \\
& + E_{n-2}((S_n - \gamma B_n) 1_{S_{n-1} - \gamma B_{n-1} \leqslant \theta_0} \mid S_{n-2} = x) \\
& + \gamma E_{n-2}((B_n - B_{n-1}) 1_{S_{n-1} - \gamma B_{n-1} \leqslant \theta_0} \mid S_{n-2} = x) \\
& + \gamma E_{n-2}((B_{n-1} - B_{n-2}) \mid S_{n-2} = x) + \gamma B_{n-2}
\end{aligned}$$

The last two expectations are zero since the increments of the Brownian motion are normally distributed with mean zero:

$$\begin{aligned}
& \gamma E_{n-2}((B_n - B_{n-1}) 1_{S_{n-1} - \gamma B_{n-1} \leqslant \theta_0} \mid S_{n-2} = x) + \gamma E_{n-2}((B_{n-1} - B_{n-2}) \mid S_{n-2} = x) \\
& = \gamma E_{n-2}(1_{S_{n-1} - \gamma B_{n-1} \leqslant \theta_0} E_{n-1}(B_n - B_{n-1}) \mid S_{n-2} = x) + \gamma E_{n-2}((B_{n-1} - B_{n-2}) \mid S_{n-2} = x) \\
& = 0
\end{aligned}$$

Using the definition (26) of the process $\hat{S}$ we can rewrite the optimal boundary condition as

$$S^*_{n-2} = E_{n-2}(\hat{S}_{n-1} 1_{\hat{S}_{n-1} > \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2})$$
$$+ E_{n-2}(\hat{S}_n 1_{\hat{S}_{n-1} \leqslant \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2}) + \gamma B_{n-2} \quad (62)$$

If

$$\hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2} < \hat{S}^*_{n-2}, \quad (63)$$

it is optimal to wait before selling the stock with price $\hat{S}$ and dynamics given by Eq. (27). This implies

$$E_{n-2}(\hat{S}_{n-1} 1_{\hat{S}_{n-1} > \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2})$$
$$+ E_{n-2}(\hat{S}_n 1_{\hat{S}_{n-1} \leqslant \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2}) > \hat{S}_{n-2}$$

and given Eq. (62):

$$S^*_{n-2} - \gamma B_{n-2} > \hat{S}_{n-2} \quad (64)$$

Equations (63) and (64) are incompatible, i.e. $\hat{S}_{n-2}$ cannot be strictly smaller than $\hat{S}^*_{n-2} - \gamma B_{n-2}$.
Conversely if

$$\hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2} > \hat{S}^*_{n-2}, \quad (65)$$

it is optimal not to wait, but to sell the stock with price $\hat{S}$ and dynamics given by Eq. (27). This implies

$$E_{n-2}(\hat{S}_{n-1} 1_{\hat{S}_{n-1} > \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2})$$
$$+ E_{n-2}(\hat{S}_n 1_{\hat{S}_{n-1} \leqslant \theta_0} \mid \hat{S}_{n-2} = S^*_{n-2} - \gamma B_{n-2}) < \hat{S}_{n-2}.$$

and given Eq. (62):

$$S^*_{n-2} - \gamma B_{n-2} < \hat{S}_{n-2} \quad (66)$$

Equations (65) and (66) are incompatible, i.e. $\hat{S}_{n-2}$ cannot be strictly larger than $\hat{S}^*_{n-2} - \gamma B_{n-2}$. Since $\hat{S}_{n-2}$ can neither be strictly smaller nor larger than $\hat{S}^*_{n-2} - \gamma B_{n-2}$, we have proved

$$S^*_{n-2} - \gamma B_{n-2} = \hat{S}^*_{n-2}. \quad (67)$$

We now turn to the general case and prove that for any i in $\{1, 2, \ldots, n-1\}$

$$S^*_{n-i-1} = \hat{S}^*_{n-i-1} + \gamma B_{n-i-1} \tag{68}$$

if the following conditions are satisfied:

$$S^*_{n-j} = \hat{S}^*_{n-j} + \gamma B_{n-j}, \, j \in \{1, i\} \tag{69}$$

We proceed along the same lines as above. According to Eq. (18), S^*_{n-i-1} satisfies

$$
\begin{aligned}
x &= E_{n-i-1}(S_{n-i} 1_{S_{n-i} > S^*_{n-i}} \mid S_{n-i-1} = x) \\
&\quad + E_{n-i-1}(S_{n-i+1} 1_{S_{n-i} < S^*_{n-i}} 1_{S_{n-i+1} > S^*_{n-i+1}} \mid S_{n-i-1} = x) + \cdots \\
&\quad + E_{n-i-1}(S_n 1_{S_{n-i} < S^*_{n-i}} 1_{S_{n-i+1} < S^*_{n-i+1}} \cdots 1_{S_{n-1} < S^*_{n-1}} \mid S_{n-i-1} = x)
\end{aligned}
$$

Given (69) we can rewrite this equation as

$$
\begin{aligned}
x &= E_{n-i-1}(S_{n-i} 1_{S_{n-i} > \hat{S}^*_{n-i} + \gamma B_{n-i}} \mid S_{n-i-1} = x) \tag{70} \\
&\quad + E_{n-i-1}(S_{n-i+1} 1_{S_{n-i} < \hat{S}^*_{n-i} + \gamma B_{n-i}} 1_{S_{n-i+1} > \hat{S}^*_{n-i+1} + \gamma B_{n-i+1}} \mid S_{n-i-1} = x) + \cdots \\
&\quad + E_{n-i-1}(S_n 1_{S_{n-i} < \hat{S}^*_{n-i} + \gamma B_{n-i}} 1_{S_{n-i+1} < \hat{S}^*_{n-i+1} + \gamma B_{n-i+1}} \cdots 1_{S_{n-1} < \theta_0 + \gamma B_{n-1}} \mid S_{n-i-1} = x)
\end{aligned}
$$

and expand it to

$$
\begin{aligned}
S^*_{n-i-1} &= E_{n-i-1}((S_{n-i} - \gamma B_{n-i}) 1_{S_{n-i} - \gamma B_{n-i} > \hat{S}^*_{n-i}} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
&\quad + E_{n-i-1}((S_{n-i+1} - \gamma B_{n-i+1}) 1_{S_{n-i} - \gamma B_{n-i} < \hat{S}^*_{n-i}} 1_{S_{n-i+1} - \gamma B_{n-i+1} > \hat{S}^*_{n-i+1}} \\
&\quad \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) + \cdots \\
&\quad + E_{n-i-1}((S_n - \gamma B_n) 1_{S_{n-i} - \gamma B_{n-i} < \hat{S}^*_{n-i}} 1_{S_{n-i+1} - \gamma B_{n-i+1} < \hat{S}^*_{n-i+1}} \cdots \\
&\quad 1_{S_{n-1} - \gamma B_{n-1} < \theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
&\quad + \gamma E_{n-i-1}((B_n - B_{n-1}) 1_{S_{n-i} - \gamma B_{n-i} < \hat{S}^*_{n-i}} 1_{S_{n-i+1} - \gamma B_{n-i+1} < \hat{S}^*_{n-i+1}} \cdots \\
&\quad 1_{S_{n-1} - \gamma B_{n-1} < \theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
&\quad + \gamma E_{n-i-1}((B_{n-1} - B_{n-2}) 1_{S_{n-i} - \gamma B_{n-i} < \hat{S}^*_{n-i}} 1_{S_{n-i+1} - \gamma B_{n-i+1} < \hat{S}^*_{n-i+1}} \cdots \\
&\quad 1_{S_{n-2} - \gamma B_{n-2} < \theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) + \cdots \\
&\quad + \gamma E_{n-i-1}((B_{n-i} - B_{n-i-1}) \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
&\quad + \gamma B_{n-i-1}. \tag{71}
\end{aligned}
$$

Since the increments of the Brownian motion are normally distributed with a mean equal to zero, according to the law of iterative expections the equation simplifies to

$$
\begin{aligned}
S^*_{n-i-1} = {} & E_{n-i-1}((S_{n-i} - \gamma B_{n-i})1_{S_{n-i}-\gamma B_{n-i}>\hat{S}^*_{n-i}} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
& E_{n-i-1}((S_{n-i+1} - \gamma B_{n-i+1})1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}>\hat{S}^*_{n-i+1}} \\
& \mid \hat{S}_{n-i-1} = S^*_{n-i-1}) - \gamma B_{n-i-1} + \cdots \\
& + E_{n-i-1}((S_n - \gamma B_n)1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}<\hat{S}^*_{n-i+1}}\cdots \\
& 1_{S_{n-1}-\gamma B_{n-1}<\theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
& + \gamma B_{n-i-1}.
\end{aligned}
\tag{72}
$$

If

$$
\hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1} < \hat{S}^*_{n-i-1},
\tag{73}
$$

it is optimal to wait before selling the stock with price $\hat{S}$ and dynamics given by Eq. (27). This implies

$$
\begin{aligned}
& E_{n-i-1}((S_{n-i} - \gamma B_{n-i})1_{S_{n-i}-\gamma B_{n-i}>\hat{S}^*_{n-i}} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
& E_{n-i-1}((S_{n-i+1} - \gamma B_{n-(n-i+1)})1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}>\hat{S}^*_{n-i+1}} \\
& \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) + \cdots \\
& + E_{n-i-1}((S_n - \gamma B_n)1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}<\hat{S}^*_{n-i+1}}\cdots \\
& 1_{S_{n-1}-\gamma B_{n-1}<\theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) \\
& \quad > \hat{S}_{n-i-1}
\end{aligned}
\tag{74}
$$

which leads in combination with Eq. (72) to

$$
S^*_{n-i-1} - \gamma B_{n-i-1} > \hat{S}_{n-i-1}.
\tag{75}
$$

Equations (73) and (75) are incompatible, i.e. $\hat{S}_{n-i-1}$ cannot be smaller than $\hat{S}^*_{n-i-1}$. Conversely if

$$
\hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1} > \hat{S}^*_{n-i-1},
\tag{76}
$$

it is optimal not to wait, but to sell the stock. This implies

$$E_{n-i-1}((S_{n-i} - \gamma B_{n-i})1_{S_{n-i}-\gamma B_{n-i}>\hat{S}^*_{n-i}} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1})$$

$$E_{n-i-1}((S_{n-i+1} - \gamma B_{n-i+1})1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}>\hat{S}^*_{n-i+1}}$$

$$\mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1}) + \cdots$$

$$+E_{n-i-1}((S_n - \gamma B_n)1_{S_{n-i}-\gamma B_{n-i}<\hat{S}^*_{n-i}}1_{S_{n-i+1}-\gamma B_{n-i+1}<\hat{S}^*_{n-i+1}}\cdots$$

$$1_{S_{n-1}-\gamma B_{n-1}<\theta_0} \mid \hat{S}_{n-i-1} = S^*_{n-i-1} - \gamma B_{n-i-1})$$

$$< \hat{S}_{n-i-1} \tag{77}$$

which leads in combination with Eq. (72) to

$$S^*_{n-i-1} - \gamma B_{n-i-1} < \hat{S}_{n-i-1}. \tag{78}$$

Equations (76) and (78) are incompatible, i.e. $\hat{S}_{n-i-1}$ cannot be larger than $\hat{S}^*_{n-i-1}$. Since $\hat{S}_{n-i-1}$ can also not be smaller than $\hat{S}^*_{n-i-1}$, Eq. (60) is proved.

References

Acharya, V., & Pedersen, L. (2005). Asset pricing with liquidity risk. *Journal of Financial Economics, 77*(2), 375–410.

Amihud, Y., & Mendelson, H. (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics, 17*(2), 223–249.

Amihud, Y., & Mendelson, H. (1989). The effects of beta, bid-ask spread, residual risk, and size on stock returns. *The Journal of Finance, 44*(2), 479–486.

Amihud, Y., Mendelson, H., & Pedersen, L. (2005). Liquidity and asset prices. *Foundations and Trends in Finance, 1*(4), 269–364.

Brennan, M., & Schwartz, E. (1990). Arbitrage in stock index futures. *Journal of Business, 63*(1), S7–S31.

Brenner, M., Eldor, R., & Hauser, S. (2001). The price of options illiquidity. *The Journal of Finance, 56*(2), 789–805.

Chordia, T., Roll, R., & Subrahmanyam, A. (2005). Evidence of speed of convergence to market efficiency. *Journal of Financial Economics, 76*(2), 271–292.

DeBondt, W., & Thaler, R. (1985). Does the stock market overreact? *The Journal of Finance, 40*(3), 793–805.

DeBondt, W., & Thaler, R. (1989). Anomalies. A mean-reverting walk down Wall Street. *Journal of Economic Perspectives, 3*(1), 189–202.

Fama, E., & French, K. (1988). Permanent and temporary components of stock prices. *Journal of Political Economy, 96*(2), 246–273.

Grinblatt, M. (2002). *An analytic solution for interest rate SWAP spreads*. Working Paper, Yale ICF.

Handa, P., & Schwartz, R. (1996). How best to supply liquidity to a securities market. *Journal of Portfolio Management, 22*(2), 44–51.

Kempf, A., & Uhrig-Homburg, M. (2000). Liquidity and its impact on bond prices. *Schmalenbach Business Review, 52*(1), 26–44.

Koziol, C., & Sauerbier, P. (2007). Valuation of bond illiquidity: An option-theoretical approach. *Journal of Fixed Income, 16*(4), 81–107.

Longstaff, F. (1995). How much can marketability affect security values. *The Journal of Finance, 50*(5), 1767–1774.

Longstaff, F. (2009). Portfolio claustrophobia: Asset pricing in markets with illiquid assets. *American Economic Review, 99*(4), 1119–1144.

Poterba, J., & Summers, L. (1988). Mean reversion in stock prices—evidence and implications. *Journal of Financial Economics, 22*(1), 27–59.

Silber, W. (1991). Discounts on restricted stock: The impact of illiquidity on stock prices. *Financial Analysts Journal, 47*(4), 60–64.

Vayanos, D. (1998). Transaction costs and asset prices: A dynamic equilibrium model. *Review of Financial Studies, 11*(1), 1–58.