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Cournot Games with Biconcave Demand

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Abstract. Biconcavity is a simple condition on inverse demand that corresponds to the ordinary concept of concavity after simultaneous parameterized transformations of price and quantity. The notion is employed here in the framework of the homogeneous-good Cournot model with potentially heterogeneous firms. The analysis leads to unified conditions, respectively, for the existence of a pure-strategy equilibrium via nonincreasing best-response selections, for existence via quasiconcavity, and for uniqueness of the equilibrium. The usefulness of the generalizations is illustrated in cases where inverse demand is either "nearly linear" or isoelastic. It is also shown that commonly made assumptions regarding large outputs are often redundant.

Keywords. Cournot games; existence and uniqueness of a pure-strategy Nash equilibrium; generalized concavity; supermodularity.

JEL-Codes. C72 - Noncooperative games; L13 - Oligopoly and other imperfect markets; C62 - Existence and stability conditions of equilibrium.

1. PRELIMINARIES

1.1. *Introduction*

This paper employs expanded notions of concavity to review the main conditions for existence and uniqueness of a pure-strategy Nash equilibrium in Cournot's (1838) homogeneous-good oligopoly with potentially heterogeneous firms.[1] Central to the approach is a family of monotone transformations given by $\varphi_\alpha(x) = x^\alpha/\alpha$ if $\alpha \neq 0$ and by $\varphi_\alpha(x) = \ln(x)$ if $\alpha = 0$. An inverse demand function $P = P(Q)$ is then called (α, β)-*biconcave* if P becomes concave (in the interval where inverse demand is positive) after transforming the price scale by φ_α and, simultaneously, the quantity scale by φ_β, where $\alpha, \beta \in \mathbb{R}$.[2]

Many of the concavity assumptions used in the literature can be expressed in terms of biconcavity. Concavity of inverse demand, as assumed by Szidarovszky and Yakowitz (1977, 1982), corresponds to $(1, 1)$-biconcavity. Selten (1970) and Murphy et al. (1982), respectively, impose concavity conditions on industry revenues that correspond to strict and non-strict variants of $(1, -1)$-biconcavity. Novshek's (1985) marginal revenue condition corresponds to $(1, 0)$-biconcavity. Amir's (1996) log-concavity of inverse demand corresponds to $(0, 1)$-biconcavity. Last but not least, Deneckere and Kovenock (1999) use a condition on direct demand that corresponds to a strict variant of $1/P$ being convex, i.e., to $(-1, 1)$-biconcavity.

Thus, the notion of biconcavity provides a simple framework for organizing the main conditions in the literature.[3]

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[1] Vives (1999) offers an excellent introduction to the Cournot model. Conditions for the existence and uniqueness of a pure-strategy Nash equilibrium in markets with identical firms have been derived by McManus (1962, 1964), Roberts and Sonnenschein (1976), and Amir and Lambson (2000).

[2] Thus, P is (α, β)-biconcave if $\varphi_\alpha \circ P \circ \varphi_\beta^{-1}$ is concave, where φ_β^{-1} is the inverse of φ_β.

[3] Given this perspective, it is natural to seek unified conditions. However, having a well-rounded theory is desirable also because the Cournot model features prominently in some broader classes of games, such as games with strategic complementarities (Milgrom and Roberts, 1990; Vives, 1990), surplus sharing games (Watts, 1996), and aggregative games with strategic substitutes (Dubey et al., 2006; Jensen, 2010).

The analysis reviews conditions in three areas. A first topic is equilibrium existence in the tradition of Novshek's (1985) landmark fixed-point argument, i.e., via the availability of nonincreasing best-response selections. Novshek assumed that each firm's marginal revenue is declining in the aggregate output of its competitors. However, as pointed out by Amir (1996), log-concavity of inverse demand is likewise a condition that guarantees the availability of nonincreasing best-response selections. While a certain consolidation of these two conditions for existence can be achieved by considering monotone transformations of the profit functions (Amir, 2005), the present paper will instead follow Amir's (1996) initial approach, which considers monotone transformations of the revenue function. This has some advantages. Specifically, as we show, cross-partial conditions can be replaced by simpler biconcavity conditions, and cost functions may be general (i.e., nondecreasing and lower semi-continuous), rather than linear. Moreover, exploiting the intuitive interpretation of biconcavity, assumptions for large outputs turn out to be redundant.

The second topic of the paper is equilibrium existence via quasiconcavity or even concavity of the profit functions, in the tradition of Friedman (1971) and Okuguchi (1976). In this case, we consider a smooth model with or without capacity constraints. Quasiconcavity of profits is established then via a simple second-order condition, where we employ an argument used by Vives (1999) in a related exercise for logconcave inverse demand.[4] While this approach does impose restrictions on costs, it leads to additional conditions for existence in cases where the availability of monotone best-response selections cannot be ascertained, i.e., when inverse demand satisfies only relatively weak forms of biconcavity. We also show that straightforward variants of such conditions ensure that profit functions are either strictly quasiconcave or strongly pseudoconcave (in the relevant domains). These ancillary results prove useful both

[4]See Vives (1999, Ch. 4, Note 16). We adapt the proof by allowing for a wider class of biconcavity conditions and by using a different second-order condition.

for the discussion of examples and for the later analysis of uniqueness.

The third and final topic is, consequently, the uniqueness of the pure-strategy equilibrium, both in games admitting nonincreasing best-response selections and in games with quasiconcave profit functions. Intuitively, the assumption of biconcavity is employed here to ensure the famous "necessary and sufficient" conditions that result from the index approach to uniqueness (Kolstad and Mathiesen, 1987). For convenience, however, the formal analysis will be based directly upon Selten's (1970) "backward mapping" approach and its subsequent developments by Szidarovszky and Yakowitz (1977) and Gaudet and Salant (1991).[5] Extending arguments of Deneckere and Kovenock (1999), we find a single additional condition,

$$(\alpha+\beta)P' - C_i'' < 0 \qquad (i = 1, ..., N), \tag{1}$$

that implies the uniqueness of the pure-strategy equilibrium in the smooth model with or without capacity constraints (here C_i denotes, of course, firm i's cost function). In fact, as will become clear, variants of condition (1) allow to consolidate a large variety of uniqueness conditions.

Quite obviously, the present analysis draws heavily upon a strand of literature that has emphasized the role of expanded notions of concavity for economic theory in general, and for the analysis of imperfect competition in particular. Most notably, Caplin and Nalebuff (1991) defined ρ-concavity via parameterized transformations of the quantity variable, and thereby introduced the notion of generalized concavity (together with the Prékopa-Borell theorem) to the economics literature. More closely related to the present analysis is Anderson and Renault (2003), who apply generalized concavity to derive efficiency and surplus bounds in the Cournot framework. Other related applications include price discrimination (Cowan, 2007, 2012;

[5] A useful discussion of this approach can be found in Friedman (1982).

Aguirre et al., 2010) and hazard-rate conditions (Ewerhart, 2013). In contrast to all those contributions, however, the present analysis employs *simultaneous parameterized* transformations of price and quantity.

The rest of the paper is structured as follows. The following two subsections introduce the notion of biconcavity and the set-up. Section 2 derives conditions for existence via nonincreasing best-response selections. Conditions for quasiconcave payoffs are stated in Section 3. Section 4 deals with uniqueness. Section 5 concludes. All proofs can be found in an Appendix.

1.2. *Biconcavity*

This subsection introduces the notion of biconcavity more formally, and derives some of its elementary properties.[6]

Consider the parameterized family of transformations $\{\varphi_\alpha\}_{\alpha \in \mathbb{R}}$ defined in the Introduction. Given arbitrary parameters $\alpha, \beta \in \mathbb{R}$, an (inverse demand) function $P = P(Q) \geq 0$, possibly unbounded at $Q = 0$, will be called (α, β)-*biconcave* if the domain $I_P = \{Q > 0 : P(Q) > 0\}$ is an interval and $\varphi_\alpha(P(Q))$ is a concave function of $\varphi_\beta(Q)$ over the domain where $Q \in I_P$. Clearly, the condition that I_P is an interval holds trivially when P is nonincreasing, which will be assumed essentially everywhere in the paper.

The following useful result extends a well-known ranking property of ρ-concavity (cf. Caplin and Nalebuff, 1991) to the case of simultaneous parameterized transformations.

Lemma 1.1. *Let $\alpha', \beta' \in \mathbb{R}$ with $\alpha' \leq \alpha$ and $\beta' \leq \beta$. If P is nonincreasing and (α, β)-biconcave, then P is also (α', β')-biconcave.*

For example, $(1, 1)$-biconcavity is more stringent than $(1, 0)$-biconcavity, which in

[6]The definition given below is based upon an extension briefly mentioned in Avriel (1972).

turn is more stringent than $(0, 0)$-biconcavity. The property captured by Lemma 1.1 is intuitive because a lower value of either α or β makes it easier for the transformed function to be concave. It is essential here, however, that P is nonincreasing. Without this assumption, the ranking result regarding β would not hold in general.[7]

The following immediate property of biconcavity translates conditions on direct demand $D = D(p) \geq 0$, possibly unbounded at $p = 0$, into conditions on inverse demand (and vice versa), in the spirit of Deneckere and Kovenock (1999).

Lemma 1.2. *Let $P = P(Q)$ and $D = D(p)$ be continuous and nonincreasing, with $D(P(Q)) = Q$ over I_P. Then P is (α, β)-biconcave if and only if D is (β, α)-biconcave.*

For example, $(0, 1)$-biconcavity of direct demand corresponds to $(1, 0)$-biconcavity of inverse demand, etc.

Finally, it is often convenient to work with the following second-order characterization of biconcavity.

Lemma 1.3. *Assume that P is nonincreasing, and twice differentiable on I_P. Then P is (α, β)-biconcave if and only if $\Delta^P_{\alpha,\beta} \leq 0$ holds on I_P, where*

$$\Delta^P_{\alpha,\beta}(Q) = (\alpha - 1)QP'(Q)^2 + QP(Q)P''(Q) + (1 - \beta)P(Q)P'(Q). \tag{2}$$

Intuitively, the criterion captured by Lemma 1.3 puts a bound on a weighted sum of the elasticity, $e_P = -QP'/P$, and the curvature, $e_{P'} = -QP''/P'$, of inverse demand. Indeed, if $P' < 0$, then condition (2) is easily seen to be equivalent to the inequality $(\alpha - 1)e_P + e_{P'} \leq 1 - \beta$.

The lemma above is straightforward to apply. E.g., linear inverse demand, $P(Q) = \max\{1-Q; 0\}$, is (α, β)-biconcave if and only if $\alpha \leq 1$ and $\beta \leq 1$. For another example,

[7] Indeed, if inverse demand were to be upward-sloping, e.g., due to general equilibrium effects, then applying a concave transformation to the quantity scale would make the transformed function more convex rather than more concave.

isoelastic inverse demand, defined through $P(Q) = Q^{-\eta}$ for $\eta > 0$, is (α, β)-biconcave if and only if $\alpha\eta + \beta \leq 0$. Further examples will be given in Section 2.

1.3. *Set-up*

The following set-up will be used throughout the paper. There is an industry composed of $N \geq 2$ firms. Each firm $i = 1, ..., N$ produces a quantity $q_i \in T_i$ of the homogeneous good, where $T_i \subseteq \mathbb{R}_+$ denotes the set of output levels that are technologically feasible for firm i. Aggregate output $Q = \sum_{i=1}^{N} q_i$ determines *inverse demand* $P(Q) \geq 0$.[8] Firm i's profit is $\Pi_i(q_i, Q_{-i}) = R(q_i, Q_{-i}) - C_i(q_i)$, where $Q_{-i} = \sum_{j \neq i} q_j$ is the joint output of firm i's competitors, $R = R(q_i, Q_{-i}) \equiv q_i P(q_i + Q_{-i})$ is the revenue function, and $C_i = C_i(q_i)$ is firm i's cost function. Firm i's *best-response correspondence* $\widehat{r}_i$ is given by

$$\widehat{r}_i(Q_{-i}) = \{q_i \in T_i : \Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widetilde{q}_i, Q_{-i}) \text{ for all } \widetilde{q}_i \in T_i\}, \tag{3}$$

where $Q_{-i} \geq 0$. Should $\widehat{r}_i(Q_{-i})$ be a singleton for a range of $Q_{-i} \geq 0$, then the *best-response function* that maps Q_{-i} to the unique element of $\widehat{r}_i(Q_{-i})$ will be denoted by $r_i = r_i(Q_{-i})$. A *pure-strategy Nash equilibrium* is a vector $(q_1, ..., q_N) \in T_1 \times ... \times T_N$ such that $q_i \in \widehat{r}_i(Q_{-i})$ for $i = 1, ..., N$.

2. EXISTENCE VIA NONINCREASING BEST-RESPONSE SELECTIONS

2.1. *Existence theorem*

This section deals with the issue of existence when firms are not necessarily symmetric and profit functions are not necessarily quasiconcave. As already mentioned in the Introduction, Novshek (1985) observed for this case that, if marginal revenues are nonincreasing in rivals' aggregate output, then a firm's best-response correspondence satisfies a downward monotonicity property that can be exploited to prove

[8] In all what follows, P may be infinite at $Q = 0$ provided that $\lim_{Q \to 0, Q > 0} QP(Q) = 0$.

existence. Following this route, the first existence result of the present paper provides conditions ensuring that a firm's *smallest* best response is well-defined and nonincreasing in rivals' aggregate output. The monotonicity property is established here using the ordinal variant of supermodularity (Milgrom and Shannon, 1994). More specifically, the proof of the theorem below extends Amir's (1996) intuitive argument for log-concave inverse demand functions by showing that an entire family of biconcavity conditions implies the crucial dual single-crossing condition for general cost specifications.

The following theorem is the first main existence result of the present paper.

Theorem 2.1. *Assume that P is continuous, nonincreasing, non-constant, and $(\alpha, 1-\alpha)$-biconcave for some $\alpha \in [0,1]$. Assume also that T_i is nonempty and closed, and that C_i is lower semi-continuous and nondecreasing, for $i = 1, ..., N$. Then, a pure-strategy Nash equilibrium exists.*

2.2. *Discussion*

Theorem 2.1 embeds the two main conditions for existence via nonincreasing best-response selections. Indeed, the second-order characterization of $(\alpha, 1-\alpha)$-biconcavity reduces to Novshek's (1985) marginal revenue condition

$$P'(Q) + QP''(Q) \leq 0 \qquad\qquad (Q \in I_P) \tag{4}$$

at $\alpha = 1$, and to Amir's (1996) log-concavity assumption

$$P(Q)P''(Q) - P'(Q)^2 \leq 0 \qquad\qquad (Q \in I_P) \tag{5}$$

at $\alpha = 0$. As discussed in the Introduction, the theorem above may be seen as convexifying these two conditions.[9]

[9] Obviously, Theorem 2.1 also accounts for the fact that convexity of choice sets is not essential for equilibrium existence via monotone best-response selections (cf. Dubey et al., 2006).

The additional generality achieved by Theorem 2.1 might even be of some applied value, as the following example with "nearly linear" demand suggests.

Example 2.2. Consider an inverse demand function P given by $P(Q) = (1 - Q^\delta)^{1/\gamma}$ if $Q \leq 1$ and by $P(Q) = 0$ otherwise, where $\gamma \approx 1$ and $\delta \approx 1$. A straightforward calculation shows that

$$\Delta^P_{\alpha,\beta}(Q) = \frac{\delta}{\gamma^2} Q^{\delta-1}(1 - Q^\delta)^{\frac{2-2\gamma}{\gamma}} \left\{(\alpha - \gamma)\delta Q^\delta + (\beta - \delta)\gamma(1 - Q^\delta)\right\}. \qquad (6)$$

Since the expression in the curly brackets is linear in Q^δ, it suffices to check the sign of $\Delta^P_{\alpha,\beta}(Q)$ for $Q \to 0$ and for $Q \to 1$. It follows that P is (α, β)-biconcave if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$.

The point of this example is that if γ and δ are marginally smaller than unity, then inverse demand becomes practically indistinguishable from the linear specification, yet neither (4) nor (5) holds. In contrast, all biconcavity conditions corresponding to values of α with $1 - \delta \leq \alpha \leq \gamma$ are satisfied.

2.3. *Large outputs*

Theorem 2.1 does away with the commonly made assumption that output levels above some threshold are suboptimal. For intuition, consider an inverse demand function P that is (α, β)-biconcave for some $\alpha, \beta \in \mathbb{R}$. If P is nonincreasing and non-constant, the same is true for the transformed function, so the graph with transformed scales has a negative slope somewhere. Provided $\alpha > 0$ and $\beta > 0$, concavity implies that the market price reaches zero at some finite $Q_0 > 0$. Hence, given that costs are nondecreasing, a firm has never a strict incentive to operate at an output level of Q_0 or higher. More generally, as shown in the Appendix, the game is effectively compact provided $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$, which strictly includes the cases considered in Theorem 2.1. In particular, assumptions for large outputs made in prior work are seen to be redundant.[10]

2.4. *Other values of α and β*

It is immediate that Theorem 2.1 applies more generally when P is (α, β)-biconcave with $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta \geq 1$. However, if any of these three constraints is marginally relaxed, keeping the respective other two, then best-response correspondences need not allow a nonincreasing selection, and an equilibrium may fail to exist. The following example establishes these facts for the case in which the constraint $\alpha + \beta \geq 1$ is relaxed.

Example 2.3. Consider an inverse demand function P given by $P(Q) = (1 - Q^{\delta})^{1/\gamma}$ if $Q \leq 1$ and by $P(Q) = 0$ otherwise, where $\gamma > 0$, $\delta > 0$, and $\gamma + \delta < 1$. Suppose initially that costs are zero. Then, because P is (γ, δ)-biconcave and $P' < 0$, profits are strongly pseudoconcave in the range where $q_i + Q_{-i} \in (0, 1)$.[11] The monopoly output in this market is given by $Q^M \equiv r_i(0) = (\frac{\gamma}{\gamma+\delta})^{1/\delta}$. Implicit differentiation of the first-order condition at $Q_{-i} = 0$ shows that $r_i'(0) = \frac{1-\gamma-\delta}{\gamma+\delta} > 0$, so that r_i is indeed locally upward-sloping. Moreover, Q^M is a "potentially optimal output" in the sense of Novshek's (1985, Theorem 4) necessary conditions for existence, whereas the marginal revenue condition fails to hold at Q^M. Therefore, there exists an integer $N \geq 2$ as well as nondecreasing, lower semi-continuous cost functions $C_1, ..., C_N$ such that the market with inverse demand P does not possess a pure-strategy Nash equilibrium.

Similar examples of non-existence may be constructed if one of the other two constraints is relaxed.[12] Thus, for general cost specifications, the parameter restrictions

[10] Biconcavity has also implications for small output levels. Given the same restrictions on the values of α and β as before, the biconcavity assumption implies $\lim_{Q \to 0, Q > 0} QP(Q) = 0$.

[11] This can be verified using Theorem 3.4 below.

[12] Here is a brief outline of these examples. When the constraint $\alpha \geq 0$ is relaxed, one considers a market with $P(Q) = (1 + Q^{\delta})^{1/\gamma}$, where $\gamma < 0$, $|\gamma|$ small, and $\delta \geq 1$. Then, with zero costs, $r_i'(Q_{-i}) > 0$ for large Q_{-i}. Similarly, when the constraint $\beta \geq 0$ is relaxed, one considers an inverse demand function P given by $P(Q) = (Q^{\delta} - 1)^{1/\gamma}$ if $Q \leq 1$ and by $P(Q) = 0$ otherwise, where $\gamma \geq 1$,

in Theorem 2.1 are indeed just as tight as possible.

3. EXISTENCE VIA QUASICONCAVE PROFITS

3.1. *Another existence theorem*

This section considers environments in which a firm's profit function is quasiconcave in own output. The property is of interest, in particular, because it ensures the existence of a pure-strategy equilibrium when profit functions are continuous and effective choice sets are non-empty compact intervals. However, since quasiconcavity is neither necessary nor sufficient for the availability of nonincreasing best-response selections, the analysis leads to conditions for existence that differ from (but overlap with) the conditions considered in the previous section.

For convenience, the subsequent discussion will focus on the smooth case, as captured by the following assumption.

Assumption 3.1. *P is continuous and nonincreasing on $\mathbb{R}_+$, as well as twice continuously differentiable on I_P; for any $i = 1, ..., N$, either $T_i = \mathbb{R}_+$ or $T_i = [0, k_i]$ with $0 \leq k_i < \infty$, and C_i is nondecreasing and twice continuously differentiable over T_i.*[13]

The next assumption captures the effective compactness of the Cournot game.

Assumption 3.2. *There is a finite $\overline{Q} > 0$ such that for any $i = 1, ..., N$, any $Q_{-i} \geq 0$, and any $q_i > \overline{Q}$, there is some $\widetilde{q}_i \leq \overline{Q}$ such that $\Pi_i(\widetilde{q}_i, Q_{-i}) \geq \Pi_i(q_i, Q_{-i})$.*

Of course, this assumption is required only if at least one firm has unbounded capacity and inverse demand is everywhere positive. Even then, as explained in Section 2, the assumption will often be redundant.

$\delta < 0$, and $|\delta|$ small. Then, with constant marginal costs c_i, one finds that $r_i'(0) > 0$ for sufficiently large c_i. Note also that, as a consequence of Lemma 1.1, there are no other cases to be considered.

[13] In particular, at $q_i = 0$, the first two directional derivatives of C_i exist and are finite, and similarly at $q_i = k_i$ if T_i is bounded. As before, P may be unbounded at $Q = 0$ provided that $\lim_{Q \to 0, Q > 0} QP(Q) = 0$.

The following result provides biconcavity conditions sufficient for a firm's profit function to be quasiconcave in own output. Thereby, a second main existence result is obtained.[14]

Theorem 3.3. *Impose Assumptions 3.1 and 3.2. Let* $\alpha \leq 1$ *and* $\beta \leq 1$ *be such that (i)* $\Delta^P_{\alpha,\beta} \leq 0$, *and (ii)* $(\alpha+\beta)P' - C_i'' \leq 0$ *for any* $i = 1, ..., N$. *Then, a pure-strategy Nash equilibrium exists.*

This theorem obviously subsumes a variety of known conditions for quasiconcavity and existence.

It will also be noted that the inequalities required in Theorem 3.3 are weak, which is a departure from the strict second-order conditions commonly employed in the smooth model. Indeed, our proof uses $\partial\Pi_i/\partial q_i > 0 \implies \partial^2\Pi_i/\partial q_i^2 \leq 0$ as a condition sufficient for quasiconcavity over an open interval. While intuitive, this condition does not appear to be widely known, so that a self-contained proof will be given in the Appendix.[15]

3.2. *Strong pseudoconcavity*

By strengthening the assumptions of Theorem 3.3 somewhat, one may ensure that profits are strictly quasiconcave over the interval where the market price is positive, or even strongly pseudoconcave over the interval where both market price and industry output are positive.[16]

Theorem 3.4. *Under the assumptions of the previous theorem, suppose that either*

[14] Here and in the sequel, obvious constraints on Q, q_i, and Q_{-i} will be omitted. E.g., the use of the derivative of P is meant to indicate a restriction to $Q \in I_P$, etc.

[15] To be sure, we remind the reader that the condition $\partial\Pi_i/\partial q_i = 0 \implies \partial^2\Pi_i/\partial q_i^2 \leq 0$ is not sufficient for quasiconcavity.

[16] A twice continuously differentiable function $f = f(x)$ is strongly pseudoconcave over an open interval X if and only if $f'(x) = 0$ implies $f''(x) < 0$. When X has a non-empty boundary, then strong pseudoconcavity requires in addition that, if the directional derivative is zero at a boundary point, then f decreases quadratically in a neighborhood in the direction of the derivative. See Diewert et al. (1981) for further details.

inequality (i) holds strictly with $\alpha+\beta<2$ or that inequality (ii) holds strictly for any $i=1,...,N$. Then, for any $Q_{-i}\geq 0$ with $P(Q_{-i})>0$, the function $\Pi_i(.,Q_{-i})$ is strictly quasiconcave over the interval where $P(Q)>0$, and even strongly pseudoconcave over the interval where $Q\in I_P$.

Thus, under the assumptions of Theorem 3.4, the best-response function $r_i(Q_{-i})$ is well-defined in the range where $P(Q_{-i})>0$. Moreover, the first-order condition holding with equality at some $q_i\in T_i$ with $q_i+Q_{-i}\in I_P$ is sufficient for a unique global maximum at q_i, and the second-order condition is then satisfied at q_i with strict inequality.

4. UNIQUENESS

4.1. *Conditions for uniqueness*

This section derives biconcavity conditions sufficient for the existence of a unique pure-strategy Nash equilibrium. The assumptions of smoothness and effective compactness from the previous section will be kept. Note, however, that smoothness is no longer assumed for convenience only.[17] The following additional assumption will be imposed.

Assumption 4.1. *For any $(q_1,...,q_N)\in T_1\times...\times T_N$ with $P(Q)=0$, there is some $i=1,...,N$ such that $C_i(q_i)>C_i(0)$.*

The sole purpose of this assumption is it to exclude the possibility of pathological equilibria in which the market price is zero, yet any individual firm is unable to generate a positive price by reducing its output.

The following result is the main uniqueness theorem of the present paper.

[17]Differentiability of inverse demand is needed, in fact, to avoid multiple equilibria. See Szidarovszky and Yakowitz (1982).

Theorem 4.2. *Impose Assumptions 3.1, 3.2, and 4.1. Assume that P is (α, β)-biconcave with $0 \leq \alpha \leq 1$ and $\alpha + \beta \leq 1$. Assume also that*

$$(\alpha + \beta)P' - C_i'' < 0 \qquad (i = 1, ..., N). \tag{7}$$

Then, there is precisely one pure-strategy Nash equilibrium. Moreover, condition (7) may be replaced by a weak inequality (simultaneously for all $i = 1, ..., N$) provided that $\Delta^P_{\alpha,\beta} < 0$ and $\alpha + \beta < 1$.

It is important to acknowledge that, under the conditions of the theorem, necessarily $P' - C_i'' < 0$.[18] In particular, in the range where the market price is positive, best-response functions have a slope strictly exceeding -1, so that multiple equilibria with inactive firms indeed cannot occur.

4.2. *Discussion*

The theorem above offers a unifying perspective on numerous sufficient conditions for uniqueness that have been used in the literature.[19]

Theorem 4.2 also adds some flexibility to existing conditions, as the following example illustrates.

Example 4.3. Consider isoelastic inverse demand $P(Q) = Q^{-\eta}$, with $0 < \eta < 1$, and assume finite capacities $k_i > 0$, for $i = 1, ..., N$. Note that the condition for small output levels holds, i.e., $\lim_{Q \to 0, Q > 0} QP(Q) = 0$. Given that P is (α, β)-biconcave if and only if $\alpha\eta + \beta \leq 0$, the tightest condition available from Theorem 4.2 is $(1 - \eta)P' - C_i'' < 0$. Thus, cost functions may be strictly concave within capacity constraints, whereas existing conditions would all require convex costs.[20]

[18] This is obvious if condition (7) holds strictly. Otherwise, i.e., if merely $(\alpha + \beta)P' - C_i'' \leq 0$, one notes that $\Delta^P_{\alpha,\beta} < 0$ implies $P' < 0$ over I_P, so that $P' - C_i'' < 0$ follows from $\alpha + \beta < 1$.

[19] Some of those conditions are listed and discussed more thoroughly in the working paper version of this paper.

[20] This type of example might prove useful in applications of quantity competition in which the assumption of strategic substitutes would be too restrictive, as in Bulow et al. (1985), while increasing returns to scale cannot be ruled out a priori.

5. CONCLUSION

This paper has used expanded notions of concavity to review conditions for existence and uniqueness of a pure-strategy Nash equilibrium in the homogeneous-good Cournot model with potentially heterogeneous firms. While a number of potentially useful generalizations and simplifications have been obtained, the most immediate benefit of the approach is probably its unifying character. In particular, conditions on inverse and direct demand have been integrated in a natural way, which addresses a concluding request in Deneckere and Kovenock (1999).

Further research is desirable. For example, the theorem of Nishimura and Friedman (1981) has not been reviewed here. McLennan et al. (2011) manage to subsume that result and Novshek's (1985) existence theorem in the duopoly case, yet the general relationship still seems to be unexplored. Further, as the discussion in Section 4 has shown, there is a lack of conditions (on the primitives of the model) that imply uniqueness even if profit functions are not quasiconcave. Last but not least, further applications of biconcavity appear desirable, both within the framework of the Cournot model and beyond.

APPENDIX: PROOFS

Proof of Lemma 1.1. For $x, \widehat{x} > 0$, $\lambda \in [0,1]$, and $\rho \in \mathbb{R}$, write $M_\rho(x, \widehat{x}, \lambda) = \varphi_\rho^{-1}((1-\lambda)\varphi_\rho(x) + \lambda\varphi_\rho(\widehat{x}))$, where φ_ρ^{-1} is the inverse of φ_ρ. Then, by definition, P is (α, β)-biconcave if and only if $M_\alpha(P(Q), P(\widehat{Q}), \lambda) \leq P(M_\beta(Q, \widehat{Q}, \lambda))$ for all $Q, \widehat{Q} \in I_P$ and all $\lambda \in [0,1]$. Since $M_\rho(x, \widehat{x}, \lambda)$ is nondecreasing in ρ, the condition of (α, β)-biconcavity becomes more stringent as α increases, and if P is nonincreasing, also as β increases. $\square$

Proof of Lemma 1.2. If P is (α, β)-biconcave, then the function that maps $\varphi_\beta(Q)$ to $\varphi_\alpha(P(Q))$ is concave (in the interval where $Q \in I_P$). Since P is necessarily strictly declining on I_P, also the function that maps $\varphi_\alpha(P(Q))$ to $\varphi_\beta(Q)$ is concave. Substituting $P(Q)$ by p, and Q by $D(p)$, shows that D is (β,α)-biconcave. The converse is similar. $\square$

Proof of Lemma 1.3. The function that maps $\varphi_\beta(Q)$ to $\varphi_\alpha(P(Q))$ is concave over the interval where $Q \in I_P$ if and only if

$$\frac{d\varphi_\alpha(P(Q))}{d\varphi_\beta(Q)} = \frac{\varphi'_\alpha(P(Q))P'(Q)}{\varphi'_\beta(Q)} \tag{8}$$

is nonincreasing over I_P. Differentiating (8) with respect to Q leads to (2). $\square$

Proof of Theorem 2.1. By Lemma A.1 below, w.l.o.g., $T_i \subseteq [0, \overline{Q}]$ for $i = 1, ..., N$, where $\overline{Q} > 0$ is finite. Since $\Pi_i(., Q_{-i})$ is u.s.c. for any $Q_{-i} \geq 0$, the minimum best response, $\min \widehat{r}_i$, is well-defined. Take $\widehat{Q}_{-i} > Q_{-i}$, and suppose $\widehat{q}_i \equiv \min \widehat{r}_i(\widehat{Q}_{-i}) > \min \widehat{r}_i(Q_{-i}) \equiv q_i$. Since $q_i \in \widehat{r}_i(Q_{-i})$, it follows that $\Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widehat{q}_i, Q_{-i})$. Moreover, $P(\widehat{q}_i + \widehat{Q}_{-i}) > 0$ because $\widehat{q}_i > 0$ is a minimum best response. Thus, by Lemma A.2, $\Pi_i(q_i, \widehat{Q}_{-i}) \geq \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$, contradicting $q_i < \widehat{q}_i$. Thus, $\min \widehat{r}_i$ is nonincreasing. But $\widehat{r}_i$ is u.h.c. because $\Pi_i(q_i, Q_{-i})$ is both u.s.c. in q_i for any Q_{-i}, and continuous in Q_{-i} for any $q_i \in T_i$. Existence follows now from Kukushkin (1994). $\square$

The lemma below is used to verify the effective compactness of the Cournot game.

Lemma A.1. *Assume that P is nonincreasing, non-constant, and (α, β)-biconcave for some $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta > 0$. Then there is a finite $\overline{Q} > 0$ such that $R(., Q_{-i})$ is nonincreasing in the interval $[\overline{Q}; \infty)$ for any $Q_{-i} \geq 0$.*

Proof. The case where $\alpha > 0$ and $\beta > 0$ has been dealt with in Section 2. The case where $\alpha > 0$ and $\beta = 0$ is similar. Consider now $\alpha = 0$. One may clearly assume w.l.o.g. that $P > 0$. Then, for almost any $Q \geq 0$,

$$\frac{\partial \ln R(q_i, Q_{-i})}{\partial\varphi_\beta(q_i)} = \frac{\partial \ln P(Q)}{\partial\varphi_\beta(Q)}\frac{\partial\varphi_\beta(Q)}{\partial\varphi_\beta(q_i)} + \frac{\partial \ln q_i}{\partial\varphi_\beta(q_i)}. \tag{9}$$

Since P is nonincreasing and non-constant, $\frac{\partial \ln P(Q)}{\partial \varphi_\beta(Q)}|_{Q=Q^\#} \equiv s < 0$ for some $Q^\# \geq 0$. But P is $(0,\beta)$-biconcave, hence $\frac{\partial \ln P(Q)}{\partial \varphi_\beta(Q)} \leq s$ for almost any $Q \geq Q^\#$. Note also that $\frac{\partial \varphi_\beta(Q)}{\partial \varphi_\beta(q_i)} = (1+\frac{Q_{-i}}{q_i})^\beta \geq 1$, and that $\frac{\partial \ln q_i}{\partial \varphi_\beta(q_i)} = q_i^{-\beta} < |s|$ for all sufficiently large q_i. Thus, (9) is negative for almost any sufficiently large q_i, regardless of Q_{-i}. $\square$

The following lemma establishes the dual single-crossing property of Cournot profits.

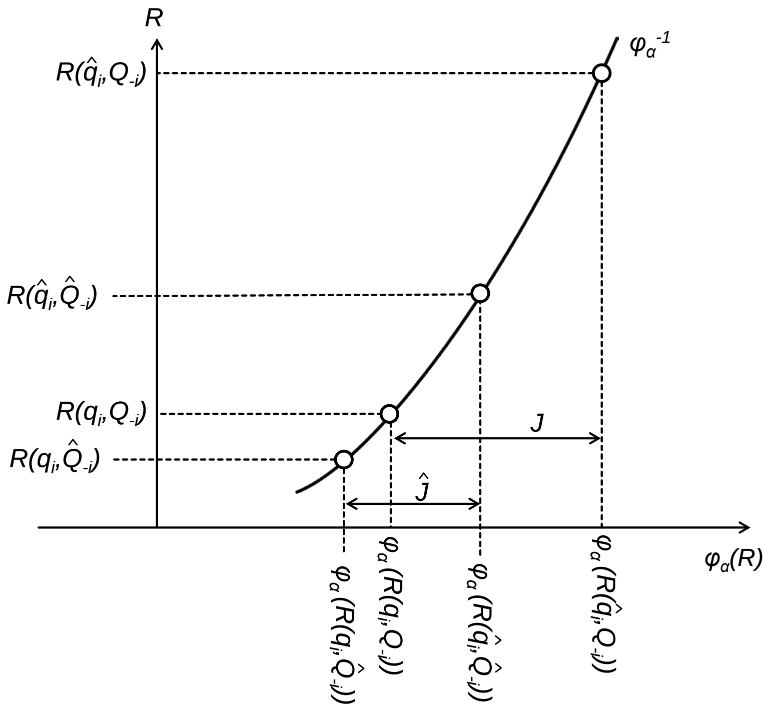

Figure 1: Extending Amir's (1996) key argument.

Lemma A.2. *Let P be nonincreasing and $(\alpha, 1-\alpha)$-biconcave for some $\alpha \in [0,1]$. Assume also that C_i is nondecreasing. Then, for any $\widehat{q}_i > q_i$ and $\widehat{Q}_{-i} > Q_{-i}$ such that $P(\widehat{q}_i+\widehat{Q}_{-i}) > 0$ and $\Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widehat{q}_i, Q_{-i})$, it follows that $\Pi_i(q_i, \widehat{Q}_{-i}) \geq \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$.*

Proof. Suppose $\Pi_i(q_i, \widehat{Q}_{-i}) < \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$. Then, $R(q_i, \widehat{Q}_{-i}) < R(\widehat{q}_i, \widehat{Q}_{-i})$, and the interval $\widehat{J} = [\varphi_\alpha(R(q_i, \widehat{Q}_{-i})), \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i}))]$ is non-degenerate. By Lemma A.3 below, $J = [\varphi_\alpha(R(q_i, Q_{-i})), \varphi_\alpha(R(\widehat{q}_i, Q_{-i}))]$ is at least as wide as $\widehat{J}$. Moreover, the left endpoint of J weakly exceeds the left endpoint of $\widehat{J}$, as in Figure 1. Applying the convex inverse φ_α^{-1} to J and $\widehat{J}$ yields $R(\widehat{q}_i, Q_{-i}) - R(q_i, Q_{-i}) \geq R(\widehat{q}_i, \widehat{Q}_{-i}) - R(q_i, \widehat{Q}_{-i})$. Hence, $\Pi_i(\widehat{q}_i, Q_{-i}) - \Pi_i(q_i, Q_{-i}) \geq \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) - \Pi_i(q_i, \widehat{Q}_{-i}) > 0$, a contradiction. □

The next lemma extends an argument in Novshek (1985) and enters the proof above.

Lemma A.3. *Let P be nonincreasing and $(\alpha, 1-\alpha)$-biconcave for some $\alpha \geq 0$. Then for any $\widehat{q}_i > q_i$ and $\widehat{Q}_{-i} > Q_{-i}$ such that $P(\widehat{q}_i + \widehat{Q}_{-i}) > 0$,*

$$\varphi_\alpha(R(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) \geq \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, \widehat{Q}_{-i})). \tag{10}$$

Proof. By Lemma A.4 below, $P(q_i + \widetilde{Q}_{-i})$ is $(\alpha, 1-\alpha)$-biconcave in q_i, for any $\widetilde{Q}_{-i} \in [Q_{-i}, \widehat{Q}_{-i}]$. Therefore, for almost any $\widetilde{Q}_{-i} \in [Q_{-i}, \widehat{Q}_{-i}]$, the inequality

$$\frac{\partial \varphi_\alpha(P(q_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(q_i)} \geq \frac{\partial \varphi_\alpha(P(\widehat{q}_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(\widehat{q}_i)} \tag{11}$$

is well-defined and holds. Using (8) and the functional form of φ_α,

$$\frac{\partial \varphi_\alpha(P(q_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(q_i)} = \frac{\varphi_\alpha'(P(q_i + \widetilde{Q}_{-i}))P'(q_i + \widetilde{Q}_{-i})}{\varphi_{1-\alpha}'(q_i)} = \frac{\partial \varphi_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}}. \tag{12}$$

Integrating over the interval $[Q_{-i}, \widehat{Q}_{-i}]$ yields

$$\varphi_\alpha(R(q_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) = \int_{Q_{-i}}^{\widehat{Q}_{-i}} \frac{\partial \varphi_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}} d\widetilde{Q}_{-i}. \tag{13}$$

Since (12) and (13) hold likewise with q_i replaced by $\widehat{q}_i$, inequality (10) follows. □

The next lemma, used in the proof above, generalizes a result in Murphy et al. (1982).

Lemma A.4. *Assume that P is $(\alpha, 1-\alpha)$-biconcave and nonincreasing, for $\alpha \geq 0$. Then $P(q_i + Q_{-i})$ is $(\alpha, 1-\alpha)$-biconcave in q_i, for any $Q_{-i} \geq 0$.*

Proof. Suppose $\varphi_\alpha(P(Q))$ is concave and nonincreasing in $\varphi_{1-\alpha}(Q)$ over the domain where $Q \in I_P$. Using Lemma 1.3, $\varphi_{1-\alpha}(Q) \equiv \varphi_{1-\alpha}(q_i + Q_{-i})$ is easily seen to be convex in $\varphi_{1-\alpha}(q_i)$ if $Q_{-i} \geq 0$. Hence, $\varphi_\alpha(P(q_i + Q_{-i}))$ is concave in $\varphi_{1-\alpha}(q_i)$ over the domain where $Q \in I_P$. □

Proof of Theorem 3.3. To apply Lemma A.5 below, suppose that $\partial\Pi_i(q_i, Q_{-i})/\partial q_i > 0$, where $Q \in I_P$. Then, $q_iP'(Q) + P(Q) > 0$ and by Lemma A.6, inequality (14) holds. Using $\beta \leq 1$ and $q_i/Q \leq 1$ yields $q_iP''(Q) + P'(Q)(2 - \alpha - \beta) \leq 0$. Adding $P'(Q)(\alpha+\beta) - C_i''(q_i) \leq 0$, one obtains $\partial^2\Pi_i(q_i, Q_{-i})/\partial q_i^2 \leq 0$. Thus, $\Pi_i(., Q_{-i})$ is quasiconcave over the domain where $Q \in I_P$. Since $C_i' \geq 0$, and by continuity, $\Pi_i(., Q_{-i})$ is quasiconcave over the whole of T_i. Existence now follows from Assumptions 3.1 and 3.2. $\square$

The proof of the following lemma is adapted from Diewert et al. (1981).

Lemma A.5. *Assume that $f = f(x)$ is twice continuously differentiable on an open interval $X \subseteq \mathbb{R}$. Then f is quasiconcave over X if $f'(x) > 0$ implies $f''(x) \leq 0$.*

Proof. Suppose f is not quasiconcave. Then, there are $x_1 < x_* < x_2$ such that $f(x_*) < \min\{f(x_1), f(x_2)\}$. Take some $\tilde{x}_1 \in (x_1, x_*)$ with $f'(\tilde{x}_1) < 0$, and some $\tilde{x}_2 \in (x_*, x_2)$ with $f'(\tilde{x}_2) > 0$. Denote by x_0 the largest element in the interval $(\tilde{x}_1, \tilde{x}_2)$ such that $f'(x_0) = 0$. By Taylor's theorem, there is some $x^* \in (x_0, \tilde{x}_2)$ with $f(\tilde{x}_2) = f(x_0) + f'(x_0)(\tilde{x}_2 - x_0) + (1/2)f''(x^*)(\tilde{x}_2 - x_0)^2$. Using $f'(x_0) = 0$ and $f(\tilde{x}_2) > f(x_0)$ shows $f''(x^*) > 0$. Yet $x_0 < x^* < \tilde{x}_2$ implies $f'(x^*) > 0$. $\square$

The following lemma is needed for Theorems 3.3, 3.4, and 4.2.

Lemma A.6. *Let $Q \in I_P$, and assume that $\Delta^P_{\alpha,\beta}(Q) \leq 0$, where $\alpha \leq 1$ and $\beta \in \mathbb{R}$. Then, $q_iP'(Q) + P(Q) \geq 0$ implies*

$$q_iP''(Q) + P'(Q) \leq (\alpha - \frac{q_i}{Q}(1 - \beta))P'(Q). \tag{14}$$

Proof. To obtain (14), one multiplies $q_iP'(Q) + P(Q) \geq 0$ through with $(1 - \alpha)P'(Q) \leq 0$, and subsequently adds $(q_i/Q)\Delta^P_{\alpha,\beta}(Q) \leq 0$. $\square$

Proof of Theorem 3.4. Assume first $\Delta^P_{\alpha,\beta} < 0$ with $\alpha + \beta < 2$. Let $q_i \in T_i$ such that $Q \in I_P$, and suppose $\partial\Pi_i(q_i, Q_{-i})/\partial q_i = 0$. Then, for $q_i > 0$, the proof of Theorem

3.3 shows that $\partial^2\Pi_i(q_i,Q_{-i})/\partial q_i^2 < 0$. For $q_i = 0$, the second-order condition is $2P'(Q_{-i}) - C_i''(0) < 0$, which follows from $(\alpha+\beta)P' - C_i'' \leq 0$ and $\alpha+\beta < 2$ because $\Delta^P_{\alpha,\beta} < 0$ implies $P' < 0$ over I_P. Thus, $\Pi_i(.,Q_{-i})$ is strongly pseudoconcave over the range where $Q \in I_P$, and by continuity, strictly quasiconcave over the range where $P(Q) > 0$. The case where $(\alpha+\beta)P' - C_i'' < 0$ is analogous. □

Proof of Theorem 4.2. Existence follows from Theorem 3.3. As for uniqueness, note first that $P(Q) > 0$ in any equilibrium, by Assumption 4.1. Assume next that $q_i > 0$ for some firm i in an equilibrium $(q_1,...,q_N)$. Then, by strict quasi-concavity, $\Pi_i(q_i,0) \geq \Pi_i(q_i,Q_{-i}) > \Pi_i(0,Q_{-i}) = \Pi_i(0,0)$, so that $Q=0$ is not a second equilibrium. Consider, finally, $\chi(Q) = \sum_{i=1}^N \chi_i(Q)$, where $\chi_i(Q)$ is defined in Lemma A.7 below. Since $\chi(Q) = Q$ holds in any equilibrium, it suffices to show that the right-derivative of χ, denoted by $D^+\chi$, satisfies $D^+\chi < 1$. Write $B(Q) = \{i : D^+\chi_i(Q) \neq 0\}$. Then,

$$D^+\chi(Q) = \sum_{i=1}^N D^+\chi_i(Q) = \sum_{i\in B(Q)} \frac{q_iP''(Q)+P'(Q)}{C_i''(q_i)-P'(Q)}. \tag{15}$$

Note that $q_iP'(Q)+P(Q) \geq 0$ for any $i=1,...,N$. Indeed, if $q_iP'(Q)+P(Q) < 0$, then $\chi_i(Q) = 0$, which would imply $P(Q) < 0$. Hence, by Lemma A.6,

$$D^+\chi(Q) \leq \sum_{i\in B(Q)} \frac{(\alpha - \frac{q_i}{Q}(1-\beta))P'(Q)}{C_i''(q_i)-P'(Q)} \leq \sum_{i\in \widetilde{B}(Q)} \frac{(\alpha - \frac{q_i}{Q}(1-\beta))P'(Q)}{C_i''(q_i)-P'(Q)}, \tag{16}$$

where $\widetilde{B}(Q) = \{i \in B(Q) : \alpha - \frac{q_i}{Q}(1-\beta) < 0\}$. If now either $P'(Q) = 0$ or $\widetilde{B}(Q) = \emptyset$, then (16) implies $D^+\chi(Q) \leq 0$. Otherwise, i.e., if $P'(Q) < 0$ and $|\widetilde{B}(Q)| \geq 1$, then necessarily $\alpha+\beta < 1$, and hence,

$$D^+\chi(Q) < \sum_{i\in\widetilde{B}(Q)} \frac{-\alpha + \frac{q_i}{Q}(1-\beta)}{1-\alpha-\beta} \leq \frac{1-\alpha|\widetilde{B}(Q)|-\beta}{1-\alpha-\beta} \leq 1. \tag{17}$$

Moreover, if (7) holds merely as a weak inequality yet $\Delta^P_{\alpha,\beta} < 0$, then inequality (14) in Lemma A.6 becomes strict for any $i \in \widetilde{B}(Q)$. Thus, $D^+\chi < 1$ in any case, and there is precisely one equilibrium. □

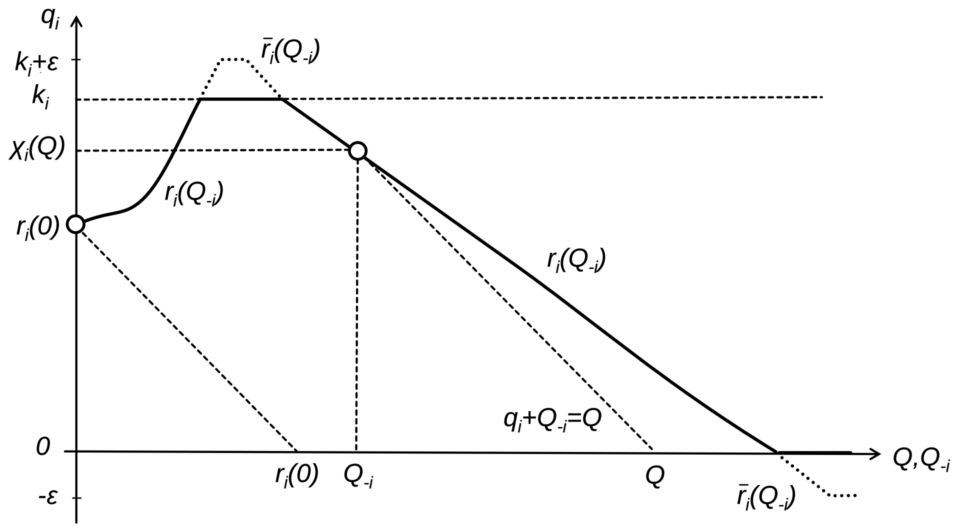

Figure 2: The slope of the best-response function strictly exceeds -1.

The following lemma is needed for the argument above. See also Figure 2.

Lemma A.7. *Impose the assumptions of Theorem 4.2. Then, for any $Q \in I_P$, the equation $q_i = r_i(Q - q_i)$ has a unique solution $q_i \equiv \chi_i(Q) \in [0; Q]$ if $Q \geq r_i(0)$, and no solution if $Q < r_i(0)$. Moreover,*

$$D^+\chi_i(Q) = \frac{q_i P''(Q) + P'(Q)}{C_i''(q_i) - P'(Q)} \mathcal{I}_{M_i}(Q), \tag{18}$$

where $\mathcal{I}_{M_i}$ is the indicator function of a measurable set $M_i \subseteq I_P$.

Proof. Let $\overline{\Pi}_i(q_i, Q_{-i}) = q_i P(q_i + Q_{-i}) - \Gamma_i(q_i)$, where Γ_i is twice continuously differentiable over $\mathbb{R}$, and $\Gamma_i(q_i) = C_i(q_i)$ over T_i. By Theorem 3.4, for any $Q_{-i} \in I_P$, the function $\overline{\Pi}_i(., Q_{-i})$ is strongly pseudoconcave over the subinterval of T_i where $P(q_i + Q_{-i}) > 0$. Hence, for any $Q_{-i}^0 \in I_P$, there is some $\varepsilon > 0$, and a neighborhood $\mathcal{U}$ of Q_{-i}^0 such that $\overline{\Pi}_i(., Q_{-i})$ is strongly pseudoconcave over the corresponding subinterval of $T_i^\varepsilon = [-\varepsilon, \infty)$ if $k_i = \infty$, and of $T_i^\varepsilon = [-\varepsilon, k_i + \varepsilon]$ if $k_i < \infty$, for any $Q_{-i} \in \mathcal{U}$. By making $\varepsilon > 0$ sufficiently small, $\overline{r}_i(Q_{-i}) = \arg\max_{q_i \in T_i^\varepsilon} \overline{\Pi}_i(q_i, Q_{-i})$ is well-defined on any given compact subset of I_P. Since, locally, either $r_i(Q_{-i}) = \max\{0; \overline{r}_i(Q_{-i})\}$

or $r_i(Q_{-i}) = \min\{\overline{r}_i(Q_{-i}); k_i\}$,

$$D^+ r_i(Q_{-i}) = -\frac{P'(Q) + q_i P''(Q)}{2P'(Q) + q_i P''(Q) - C_i''(q_i)} \mathcal{I}_{M_i^0}(Q_{-i}) \tag{19}$$

for some measurable set $M_i^0 \subseteq I_P$. Now $P' - C_i'' < 0$ implies $D^+ r_i > -1$. Thus, $\psi_i(Q_{-i}) \equiv Q_{-i} + r_i(Q_{-i})$ is continuous and strictly increasing, with $\psi_i(0) = r_i(0)$ and $\psi_i(Q) \geq Q$, proving the first assertion. As $D^+\psi_i = 1 + D^+ r_i > 0$, the directional version of the inverse function theorem implies $D^+(\psi_i^{-1}) = 1/(1 + D^+ r_i)$. Hence, $D^+\chi_i(Q) = D^+ r_i(Q)/(1 + D^+ r_i(Q))$, and (18) holds with $M_i = \psi_i(M_i^0)$. $\square$

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