Institute for Empirical Research in Economics
University of Zurich

Working Paper Series
ISSN 1424-0459

Working Paper No. 427

Cournot Oligopoly and Concavo-Concave Demand

Christian Ewerhart

September 2009

Cournot Oligopoly and Concavo-Concave Demand

Christian Ewerhart*

September 3, 2009

*) University of Zurich; postal address: Chair for Information Economics and Contract Theory, Winterthurerstrasse 30, CH-8006 Zurich, Switzerland; e-mail: christian.ewerhart@iew.uzh.ch; phone: 41-44-6343733; fax: 41-44-6344978.

Cournot Oligopoly and Concavo-Concave Demand

Abstract. The N-firm Cournot model with general technologies is reviewed to derive generalized and unified conditions for existence of a pure strategy Nash equilibrium. Tight conditions are formulated alternatively (i) in terms of concavity of two-sided transforms of inverse demand, or (ii) as linear constraints on the elasticities of inverse demand and its first derivative. These conditions hold, in particular, if a firm's marginal revenue decreases in other firms' aggregate output, or if inverse demand is logconcave. The analysis relies on lattice-theoretic methods, engaging both cardinal and ordinal notions of supermodularity. As a byproduct, a powerful test for strict quasiconcavity is obtained.

Keywords. Cournot competition; existence of Nash equilibrium; concavity of demand; supermodular games; strict quasiconcavity.

JEL-Codes. L13 - Oligopoly and other imperfect markets; C72 - Noncooperative games; C62 - Existence and stability conditions of equilibrium.

1. Introduction

Given that the prediction of the Cournot model is widely acknowledged as a useful benchmark in the analysis of antitrust regulation, it is a striking fact that the conditions under which a pure strategy Nash equilibrium exists are still not fully understood. Existence is known to depend on characteristics of the market demand, on restrictions imposed by technology, and even on the question of equal access to technology (cf., e.g., Vives, 1999). But it is certainly desirable to fully clarify how existence depends on these primitives of the environment. The present paper develops a new notion, referred to as biconcavity of demand, and applies it to derive unified and generalized conditions for equilibrium existence in an oligopolistic market for a single commodity. The analysis investigates the usefulness of the new concept also for a number of issues other than equilibrium existence.

In the literature, there are essentially two main types of theorems that ascertain existence from conditions on the primitives of the model. A first approach, associated with contributions by McManus (1962, 1964) and Roberts and Sonnenschein (1976), allows for general demand functions and convex technologies. Existence can be shown under these conditions for a finite number of identical firms.[1] Amir and Laibson (2000) have rebuilt and generalized this theory for symmetric firms using the theory of supermodular games and Tarski's fixed point theorem. A second approach, developed by Novshek (1985) and Bamon and Frayssé (1985), applies to any finite collection of firms and arbitrary technologies, imposing instead restrictions on

[1] Kukushkin (1993) relaxes the assumption of identical technologies to incorporate firm-specific capacity constraints.

market demand. This approach has its origins in an observation by Selten (1970).[2] The state of the art is Amir's (1996) comprehensive derivation of results from the perspective of the theory of supermodular games.[3]

Despite this impressive progress, the inquiry into the existence question is by no means settled. In particular, in the second approach that allows for general technologies, two genuinely different conditions coexist in the literature. The so-called *cardinal test* evaluates the sign of a certain cross-partial derivative of the firm's profit function. If this sign is nonpositive, then Cournot profits are submodular in own output and rivals' joint output, so that an equilibrium can be shown to exist. The so-called *ordinal test* merely requires that inverse demand is logconcave. In that case, profits are logsubmodular, which likewise helps existence. In an important contribution, Amir (2005) has pointed out that neither of these tests is more stringent than the respective other. Moreover, other tests work only in special cases. Thus, it appears that two conditions of equal standing individually ensure existence for general technologies, which is puzzling.

In this paper, we introduce a new concept, biconcavity of demand, and use it to formulate generalized conditions for existence of a pure-strategy Nash equilibrium in Cournot games. Biconcavity is an intuitive notion. It requires that the inverse demand function is concave following monotonous transformations of the scales for both price and quantity. Examples of bicon-

[2] While Selten (1970) conceptualized the important "Einpassungsfunktion" and proved existence using Brouwer's fixed point theorem, the general result obtained by Novshek (1985) and Bamon and Frayssé (1985) requires an additional argument, which was distilled into a fixed-point theorem by Kukushkin (1994).

[3] A third, complementary approach assumes profits to be quasiconcave. See below and Section 6.

cavity have been used in the literature, but the general notion is apparently new.[4] Biconcavity seems to be just the right concept to formulate conditions on demand that imply monotonicity properties for the best response. In the smooth case with downward-sloping inverse demand, we obtain as main condition for existence that

$$1 + \varepsilon_{p'} + \beta(|\varepsilon_p| - 1) \geqq 0 \tag{1}$$

for some parameter $\beta \in [0; 1]$, where ε_p and $\varepsilon_{p'}$ denote the elasticities of inverse market demand and of its first derivative. We also prove a result to the effect that this linear bound cannot be tightened further.

To derive the general existence theorem, which does not impose smoothness, methods developed by Novshek (1985) and Amir (1996) are combined and enhanced. First, we show that biconcavity of demand implies that suitably transformed revenues have decreasing differences in the relevant domain. Then, a new lattice-theoretic argument is used to prove that this property of transformed revenues is sufficient for a dual single-crossing property of the firm's profits. Finally, invoking a generalized theorem of the maximum, condition (1) is found to imply that best-response correspondences allow a nonincreasing selection. By the virtue of the Bamon-Frayssé/Novshek/Kukushkin fixed point theorem, this ensures existence of a Cournot-Nash equilibrium in pure strategies. As a matter of fact, the resulting existence theorem contains virtually all earlier results for general technologies as special cases, and even makes an additional step ahead.

[4] Specifically, in the context of Bertrand competition, Caplin and Nalebuff (1991b) consider a direct demand function that is logconcave in the log-transformed price. See also Section 2.

The theoretical development allows a number of extensions, some of which may be unexpected. In particular, we construct an "atypical" oligopoly with strategic substitutes that exhibits local complementarities once linear costs are introduced. As will be argued, this possibility implies that an increase in the corporate tax rate might affect interfirm competition in a potentially important way. Further extensions deal with global strategic complements, strict quasiconcavity, technological indivisibilities, and equilibrium uniqueness. The result on quasiconcavity deserves special mentioning because this finding, too, unifies various conditions that can be found in the literature.

The rest of the paper is structured as follows. Section 2 introduces our assumption of biconcave demand, and explains why it implies decreasing differences of suitably transformed Cournot revenues. The existence theorem is stated and discussed in Section 3. We elaborate on the tightness of the biconcavity condition in Section 4. Section 5 uses an example to illustrate the mechanics behind the existence result. Further extensions can be found in Section 6. Most of the proofs have been collected in the Appendix.

2. Biconcavity

In this section, we introduce the notion of biconcavity, which will be the main technical tool in the subsequent analysis of the Cournot model. Biconcavity is a property of real-valued functions that is more flexible than the standard definition of concavity. In particular, new classes of demand functions can be described in a convenient way. We first define biconcavity, and discuss its elementary properties. Following the general definition, an important lemma is presented that links biconcavity of demand to decreasing differences of suitably transformed revenues.

Definition 2.1. Let $\alpha, \beta \in \mathbb{R}$. For $\alpha \neq 0$, $\beta \neq 0$, a function f defined on the positive reals is called (α, β)-*biconcave* if for all $x > 0, \widehat{x} > 0$ such that $f(x) > 0, f(\widehat{x}) > 0$, the inequality

$$f(\{\lambda x^\beta + (1-\lambda)\widehat{x}^\beta\}^{1/\beta}) \geqq \{\lambda f(x)^\alpha + (1-\lambda) f(\widehat{x})^\alpha\}^{1/\alpha} \tag{2}$$

is satisfied for all $0 \leqq \lambda \leqq 1$. For $\alpha = 0$ or $\beta = 0$, the respective weighted sum in (2) is replaced by the corresponding weighted harmonic mean.[5]

For α and β positive, the definition says that $f(x)^\alpha$ is concave in x^β in the (necessarily convex) domain where f is positive. This follows as a result of the substitution $z = x^\beta$. More generally, for $\alpha, \beta \neq 0$, the condition for (α, β)-biconcavity of f is that the domain where f is positive is convex, and that $f(x)^\alpha/\alpha$ is concave in x^β/β in that domain.

If f is both (α, β)-biconcave and weakly decreasing, then f is also (α', β')-biconcave for any $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$. Indeed, as in Caplin and Nalebuff (1991a), the Hölder mean on the right-hand side of (2) and likewise the argument on the left-hand side is weakly increasing in α and β, respectively, which proves the claim. It is essential here that f is nonincreasing. Without this assumption, it is not generally true that higher values for β correspond to more stringent variants of biconcavity.

Obviously, $(1,1)$-biconcavity corresponds to the standard definition of concavity, while $(0,1)$-biconcavity is logconcavity of f. More generally, $\beta = 1$ corresponds to the univariate case of generalized concavity (Caplin and Nale-

[5]I.e., at $\alpha = 0$, the right-hand side of (2) is set to $f(x)^\lambda f(\widehat{x})^{1-\lambda}$. At $\beta = 0$, the argument in the left-hand side of (2) is set to $x^\lambda \widehat{x}^{1-\lambda}$.

buff, 1991a). To see that biconcavity is more flexible than generalized concavity, one checks that $(0,0)$-biconcavity corresponds to $f(x)$ being logconcave in $\ln(x)$, but this condition, as noted by Caplin and Nalebuff (1991b), does not correspond to a generalized concavity condition.

When f is smooth, a simple condition involving first and second derivatives characterizes biconcavity.

Lemma 2.2. *Consider a function f defined on the positive reals that is twice differentiable when positive. For $\alpha, \beta \in \mathbb{R}$, the function f is (α, β)-biconcave if and only if the support set $I_f = \{x > 0 : f(x) > 0\}$ is an interval, and the inequality*

$$\Delta^f_{\alpha,\beta}(x) = (\alpha - 1)f'(x)^2 x + f(x)f''(x)x + (1-\beta)f(x)f'(x) \leqq 0 \qquad (3)$$

holds for all $x \in I_f$.

Proof. See the Appendix. □

Of particular importance for equilibrium existence is the class of (α, β)-biconcave inverse demand functions with $\beta = 1 - \alpha$. As the next result will show, these demand functions have the useful property that suitably transformed revenues exhibit decreasing differences, which will be a key step in the proof of the existence result. For revenues, we will use the notation $f_*(y, Y) = yf(y + Y)$. Write $\varphi_\alpha : \mathbb{R}_{++} \to \mathbb{R}$ for the mapping $x \mapsto x^\alpha/\alpha$ if $\alpha \neq 0$, and φ_0 for the logarithm.

Lemma 2.3. *Let f be a nonincreasing function that is $(\alpha, 1-\alpha)$-biconcave for some $\alpha \geqq 0$. Then for any $\widehat{y} > y \geqq 0$ and for any $\widehat{Y} > Y \geqq 0$ such that both $f_*(\widehat{y}, \widehat{Y}) > 0$ and $f_*(y, Y) > 0$, transformed revenues $\varphi_\alpha \circ f_*$ exhibit* *decreasing differences, i.e.,*

$$\varphi_\alpha(f_*(y,Y)) - \varphi_\alpha(f_*(y,\widehat{Y})) \leqq \varphi_\alpha(f_*(\widehat{y},Y)) - \varphi_\alpha(f_*(\widehat{y},\widehat{Y})) \tag{4}$$

is satisfied.[6]

Proof. See Appendix. □

To get an intuition for this result, recall the fact that if inverse demand is actually concave and declining over the relevant domain, then the drop in price resulting from a given increase in rivals' output will be more pronounced as own output increases. Lemma 2.3 follows a similar logic. Under the assumptions made, the loss in (transformed) revenues resulting from a given increase in rivals' output will be larger as own output increases. However, to take account of the quantity dimension of revenues, concavity of demand needs to be replaced by the biconcavity assumption.

3. Existence theorem

We consider an industry composed of $N \geqq 1$ firms producing a homogeneous good. Firm i produces the good in quantity $q_i \geqq 0$. Its cost function is $C_i(q_i)$, defined for all $q_i \geqq 0$. Denote *inverse demand* by $p(Q) \in \mathbb{R}_+$, where $Q = \sum_{i=1}^N q_i$ is aggregate output. Firm i's profit is

$$\Pi_i(q_i, Q_{-i}) = p(q_i + Q_{-i})q_i - C_i(q_i), \tag{5}$$

where $Q_{-i} = \sum\limits_{j \neq i} q_j$ is the joint output of firm i's competitors. Denote by

$$r_i(Q_{-i}) = \{q_i \geqq 0 : \Pi_i(q_i, Q_{-i}) \geqq \Pi_i(\widetilde{q}_i, Q_{-i}) \text{ for all } \widetilde{q}_i \geqq 0\} \tag{6}$$

[6] Note that inequality (4) is well-defined because I_f is convex.

the set of firm i's *best responses* to an aggregate rival output $Q_{-i} \geqq 0$. A *Cournot-Nash equilibrium* is a vector of quantities $(q_1, ..., q_N)$ such that $q_i \in r_i(Q_{-i})$ for $i = 1, ..., N$.

The following lattice-theoretic result is the key to the subsequently stated existence theorem. The result says that provided suitably transformed revenues exhibit decreasing differences, and costs are nondecreasing, profits satisfy the dual single-crossing property.

Lemma 3.1. *Let $\alpha \in [0;1]$. Take some $\widehat{q}_i > q_i \geqq 0$, $\widehat{Q}_{-i} > Q_{-i} \geqq 0$, such that $p(\widehat{q}_i + \widehat{Q}_{-i}) > 0$. Assume that p is nonincreasing, and that C_i is nondecreasing. Assume also that $\varphi_\alpha \circ R_i$ exhibits decreasing differences, i.e.,*

$$\varphi_\alpha(R_i(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R_i(q_i, Q_{-i})) \geqq \varphi_\alpha(R_i(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R_i(q_i, \widehat{Q}_{-i})). \quad (7)$$

Then Π_i satisfies the dual single-crossing property, i.e.,

$$\Pi_i(q_i, Q_{-i}) - \Pi_i(\widehat{q}_i, Q_{-i}) \geqq 0 \Rightarrow \Pi_i(q_i, \widehat{Q}_{-i}) - \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) \geqq 0. \quad (8)$$

If in addition, p is strictly decreasing when positive, $\alpha < 1$, and C_i is strictly increasing, then Π_i satisfies the strict single-crossing property, i.e., (8) with a strict inequality in the concluding inequality.

Proof. See the Appendix. □

To understand why Lemma 3.1 holds, it is important to acknowledge that there are two possible regimes. In the first, marginal revenues are zero or negative when the output of other firms is low. Under this regime, the fact that revenues are "submodularizable" directly implies that marginal revenues remain zero or negative when the output of other firms increases. Clearly then, a fortiori, marginal profits must be zero or negative when rivals' joint output is high, as desired. In the second regime, marginal revenues are positive when rivals' joint output is low. But then, since revenues are declining in the other firms' output, and since in addition, concavely transformed revenues exhibit decreasing differences, marginal revenues must be smaller when the output of other firms increases. Thus, under either regime, an expansion of output cannot become attractive just because competing firms increase their output.

Combining Lemmas 2.2 and 3.1, we arrive at the following existence theorem, which is the main result of this paper.

Theorem 3.2. *Given a market for a single homogeneous good with inverse demand* p*, and* N *firms with cost functions* $C_1, ..., C_N$*, if*

1. there exists an output level $\overline{Q} \geqq 0$ *such that* $Qp(Q) \leqq C_i(Q) - C_i(0)$ *for all* $Q > \overline{Q}$ *and for all* $i = 1, ..., N$*,*

2. the function p *is nonincreasing, left-continuous with* $\lim_{Q \to 0} Qp(Q) = 0$*, as well as* $(\alpha, 1 - \alpha)$*-biconcave for some* $\alpha \in [0; 1]$*, and*

3. for $i = 1, ..., N$*, the cost function* $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ *is left-continuous and nondecreasing,*

then each firm i*'s best-response correspondence* r_i *allows a nonincreasing selection. Moreover, a Cournot-Nash equilibrium exists.*

Proof. See the Appendix. □

At the time of writing, Theorem 3.2 is the most general existence result for Cournot games with general technologies. To start with, Amir's (1996) Theorem 3.1 corresponds to $\alpha = 0$. Here, the transformation φ_α is the logarithm, while $\varphi_{1-\alpha}$ is the identity mapping, so that inverse demand is assumed logconcave. It will be noted that weak monotonicity of inverse demand and costs is sufficient for existence.[7] Another special case is Novshek's (1985) Theorem 3. To see this, assume for the moment that p is twice continuously differentiable when positive. For $\alpha = 1$, the smooth condition for biconcavity (Lemma 2.2) requires that the range of quantities where prices are positive is an interval where

$$p''(Q)Q + p'(Q) \leqq 0, \tag{9}$$

which follows from Novshek's conditions. Moreover, Theorem 3.2 confirms the intuition that smoothness is indeed dispensable for the existence assertion. The nondifferentiable variant of the marginal revenue condition (9), as suggested informally by Novshek, would then require that $p(Q)$ be concave in $\ln(Q)$ at positive prices. These cases are not exhaustive, as will be shown in the next section.

What is the economic content of the biconcavity assumption? In the smooth case with downwards-sloping positive inverse demand, Lemma 2.2 shows that the condition can be reformulated into

$$1 + \varepsilon_{p'} + \beta(|\varepsilon_p| - 1) \geqq 0 \tag{10}$$

for $\beta \in [0; 1]$, where $\varepsilon_p = Qp'(Q)/p(Q)$ and $\varepsilon_{p'} = Qp''(Q)/p'(Q)$ denote the elasticities of inverse demand and its first derivative. We are going to argue that condition (10) captures the intuition that a firm's marginal revenue at a given output level may actually be increasing in the aggregate output of the other firms, but only if this marginal revenue is negative, which rules out a best response.

[7] Moreover, for logconcave prices and linear costs, the proof goes through without monotonicity of demand, generalizing Amir's (1996) Theorem 4.2 to asymmetric firms.

For a *heuristic* derivation that supports this interpretation, we start from Novshek' (1985) result that a downwards-sloping inverse demand function p does not guarantee existence for general (nondecreasing) cost functions if the marginal revenue condition fails at a "potential optimal output." So consider nonnegative output levels q_i for firm i and Q_{-i} for the rivals of firm i such that the marginal revenue condition fails, i.e., assume

$$p'(q_i + Q_{-i}) + q_i p''(q_i + Q_{-i}) > 0. \tag{11}$$

Rewriting (11) in terms of elasticities yields the less familiar inequality

$$1 + \varepsilon_{p'} < -\frac{Q_{-i}}{q_i}, \tag{12}$$

The marginal revenue condition (9) ensures that the left-hand side of this inequality is nonnegative, so that Novshek's construction of counterexamples is not feasible. To obtain a tighter condition, note that non-existence may occur if together with (12), we have that q_i is potentially (i.e., for a suitable nondecreasing cost function), a best response to Q_{-i}, i.e., if

$$p(q_i + Q_{-i})q_i > p(q_i' + Q_{-i})q_i' \qquad (q_i' < q_i). \tag{13}$$

To fix ideas, assume that revenues are strictly quasiconcave in own output. Then condition (13) is actually equivalent to revenues being nondecreasing in own output, i.e.,

$$p(q_i + Q_{-i}) + q_i p'(q_i + Q_{-i}) \geqq 0. \tag{14}$$

Equivalently,

$$|\varepsilon_p| - 1 \leqq \frac{Q_{-i}}{q_i}. \tag{15}$$

Clearly, conditions (12) and (15) cannot be satisfied simultaneously if (10) holds for some $\beta \in [0;1]$. Thus, as would follow also more directly from Theorem 3.2, the biconcavity assumption implies that the marginal revenue condition is satisfied at any "potential optimal output," precluding Novshek's construction of counterexamples.[8]

We conclude this section with a few remarks on the remaining assumptions made in Theorem 3.2. We first note that the particular form of the upper boundary condition (1) in the statement of the theorem is not crucial. A more restrictive requirement would be that the price function reaches zero at some finite output level. Note also that the assumption of $(\alpha, 1-\alpha)$-biconcavity implies continuity of inverse demand in the interior of the domain where $p > 0$. Therefore, the left-continuity assumption on the price function p matters only if p jumps to zero. Without left-continuity at such a jump, the firm's problem of setting an optimal output level may not allow a solution, and an equilibrium need not exist. Finally, unbounded prices are not generally inconsistent with existence.

4. Tightness

This section discusses the tightness of the biconcavity assumption. It is shown first that none of the values for α listed in Theorem 3.2 can be dropped without losing generality. We then investigate if biconcavity properties marginally weaker than those required above would yield the same conclusions.

We come to the first result. Provided that costs are strictly increasing, there is a continuum of pairwise independent biconcavity conditions, para-

[8] It does not seem feasible to use this heuristic derivation to strengthen the existence result.

meterized by α. In particular, Theorem 3.2 is more general than the union of existing results.

Proposition 4.1. *For each $\alpha \in [0;1]$, there exists an inverse demand function p_α such that (i) for any number of firms $N \geqq 1$ and for any specification of strictly increasing and left-continuous cost functions $C_1, ..., C_N$, the assumptions of Theorem 3.2 hold, and (ii) for any $\widehat{\alpha} \in \mathbb{R}$, the function p_α is $(\widehat{\alpha}, 1-\widehat{\alpha})$-biconcave if and only $\widehat{\alpha} = \alpha$.*

Proof. See the Appendix. □

Next, to explore the tightness of the biconcavity assumption made in Theorem 3.2, we investigate examples of demand function that satisfy weaker requirements. It turns out that a further relaxation of the concavity conditions within the class of biconcavity conditions is not feasible without losing the property of strategic substitutes.

Theorem 4.2. *Let α, β be parameters such that at least one of the inequalities $\alpha + \beta < 1$, $\alpha < 0$, or $\beta < 0$ holds. Then, for any given $N \geqq 2$, there exists an inverse demand function p and cost functions $C_1, ..., C_N$, in general depending on α, β, such that*

1. there exists an output level $\overline{Q} \geqq 0$ such that $Qp(Q) \leqq C_i(Q) - C_i(0)$ for all $Q > \overline{Q}$ and for all $i = 1, ..., N$,

2. the function p is nonincreasing, left-continuous with $\lim_{Q \to 0} Qp(Q) = 0$, as well as (α, β)-biconcave, and

3. for $i = 1, ..., N$, the cost function $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ is left-continuous and nondecreasing,

yet r_i does not allow a nonincreasing selection for any $i = 1, .., N$.

Proof. See the Appendix. □

To construct these counterexamples, one starts from inverse demand functions that are just somewhat too convex to satisfy the assumptions of the existence theorem. Then, cost functions are carefully chosen such that the best-response correspondence is single valued, and such that the best response is strictly increasing when rivals' joint output is not too large.

5. An "atypical" case with strategic substitutes

Besides allowing to relax conditions for equilibrium existence, the notion of biconcavity can be used to identify examples of Cournot games with new properties. In this section, we show that a costless Cournot game with strategic substitutes may exhibit (locally) strategic complementarities once linear costs are introduced. This possibility may be unexpected because one might suspect that the presence of costs would create, if anything, an asymmetry *in favor of* nonincreasing best-responses. However, as the example proves, this need not be the case when strategic substitutes are based on the ordinal notion of submodularity. The example is economically relevant because it shows that even linear taxes, which have a similar effect as linear costs, might affect strategic interaction in a potentially important way.

We come to our example. It will be shown that ordinal submodularity is not stable under the introduction of linear costs.

Example 5.1. Consider inverse demand $p(Q) = \sqrt{(1-Q)/Q}$ for $0 < Q \leqq 1$ and $p(Q) = 0$ for $Q > 1$. The function p is $(\alpha, 1-\alpha)$-biconcave for $\alpha = 2$. Hence, by Lemma 2.3, squared revenues $R_i(q_i, Q_{-i})^2$ are submodular at positive prices. The firm's objective function is strictly concave in own output provided $Q \leqq 1$. So the best response with zero variable costs is nonincreasing. However, with positive variable costs, this need not be true anymore. Since revenues are not submodular everywhere when increasing in own output, it is feasible to construct a counterexample. For instance, for $q_i = 0.1$, $\widehat{q}_i = 0.5$, $Q_{-i} = 0$, $\widehat{Q}_{-i} = 0.1$, and costs $C_i(q_i) = 0.5 \cdot q_i$, the single-crossing condition is violated. Moreover, as a numerical calculation shows, the best response with costs satisfies $r_i(Q_{-i}) \approx 0.276 < 0.286 \approx r_i(\widehat{Q}_{-i})$, i.e., it is not weakly decreasing.

While illustrating the proof of Theorem 4.2, the example given here suggests also an extension of Theorem 3.2 to parameter values satisfying $\alpha > 1$. However, as should be clear from the previous section, the expansion of the parameter range comes at the cost of an additional assumption. Indeed, because the transformation φ_α is convex, the appropriate condition on inverse demand is here that each firm i's net-of-variable-cost demand $\widehat{p}_i = p - c_i$ must be $(\alpha_i, 1-\alpha_i)$-biconcave for some $\alpha_i > 1$.

Proposition 5.2. *Consider an N-firm Cournot oligopoly. Assume that for each $i = 1, ..., N$, there is an $\alpha_i > 1$ such that $\widehat{p}_i$ is $(\alpha_i, 1-\alpha_i)$-biconcave. Then the quantity-setting game is ordinally submodular at positive prices. In particular, with the boundary condition (1) from Theorem 3.2 imposed, a Cournot-Nash equilibrium exists.*

Proof. The ordinal submodularity of profits at positive prices follows immediately from Lemma 2.3. A straightforward adaptation of the proof of Theorem 3.2 proves then the existence claim. □

6. Further extensions

This section deals with extensions concerning global complementarities, strict quasiconcavity, technological indivisibilities, and equilibrium uniqueness.

Global complementarities. An approach alternative to the one used so far is to identify Cournot games with global strategic complementarities, and to prove existence via Tarski's fixed point theorem.[9] In this subsection, we will generalize an existence result in this vein that was hitherto restricted to logconvex price functions.

For parameters $\alpha, \beta \in \mathbb{R}$, inverse demand p will be called (α, β)*-biconvex* if the support set $I_p = \{Q > 0 : p(Q) > 0\}$ is an interval, and $\varphi_\alpha(p(Q))$ is convex in $\varphi_\beta(Q)$ for $Q \in I_p$. Logconvexity is equivalent to $(0, 1)$-biconvexity, yet global complementarities result for any $(\alpha, 1-\alpha)$-biconvex inverse demand, where $\alpha \leqq 0$.

Proposition 6.1. *Consider an N-firm oligopoly with exogenous capacities $\overline{q}_1, ..., \overline{q}_N$, and for each firm $i = 1, ..., N$, a linear cost functions $C_i(q_i) = c_i q_i$ for $q_i \in [0; \overline{q}_i]$, where $c_i \geqq 0$. Assume that for each $i = 1, .., N$, there is an $\alpha_i \leqq 0$ such that $\widehat{p}_i$ is $(\alpha_i, 1-\alpha_i)$-biconvex on $[0, \overline{q}_1 + ... + \overline{q}_N]$. Then the quantity-setting game is ordinally supermodular at positive prices. Moreover, a Cournot-Nash equilibrium exists.*

Proof. To prove the claim, note that φ_α-transformed revenues are supermodular in (q_i, Q_{-i}) provided $\varphi_\alpha(p(q_i + Q_{-i}))$ is convex in $\varphi_{1-\alpha}(q_i)$. The assertion follows now from Lemma B.2 in the Appendix. The existence follows from Tarski's fixed point theorem, applied to nondecreasing best-response

[9] Ordinally supermodular games have many useful properties which are not reviewed here. See Milgrom and Shannon (1994).

selections which are constructed as in the proof of Theorem 3.2. □

The result above generalizes Amir's (1996) Theorem 3.2, which corresponds to the case $\alpha = 0$. The generalization is strict, as the following example shows.

Example 6.2. Consider the inverse demand function $p(Q) = (1 + Q^{1-\alpha})^{1/\alpha}$ for $\alpha < 0$. It is straightforward to verify that p is not logconvex. However, p is $(\alpha, 1-\alpha)$-biconvex.

Global complementarities obtain also in duopoly games satisfying the generalized assumptions of either Theorem 3.2 or Proposition 5.2, provided the order on either firm's strategy space is reversed.

Strict quasiconcavity. The literature on the Cournot game offers a diversity of criteria for strict quasiconcavity. One test checks if $Qp(Q)$ is strictly concave and p is strictly decreasing. With convex costs, this ensures strict quasiconcavity (cf. Murphy et al., 1982). Another test checks the inequality $pp'' - 2p'^2 < 0$ (cf. Amir, 1996). This test works in the smooth model provided p is nonincreasing, and provided costs are both nondecreasing and convex. For nonconvex technologies, the requirement on the demand function will generally depend on the shape of the cost function. One important differentiable test allowing for this possibility requires logconcavity of p and $p' - C_i'' < 0$ (cf. Vives, 1999).

Maybe interestingly, the notion of biconcavity can be used to generalize and unify all these criteria for the smooth model, integrating not only the slope of the demand function, but also its curvature.

Proposition 6.3. *Consider firm i's problem, keeping $Q_{-i} \geqq 0$ fixed. Let p* *be twice continuously differentiable when positive, and nonincreasing. Furthermore, let C_i be twice continuously differentiable, nondecreasing, and in fact strictly increasing unless $p > 0$. For some $\alpha \leqq 1$, $\beta \leqq 1$, assume that (i) $\Delta^p_{\alpha,\beta} \leqq 0$ and (ii) $(\alpha + \beta)p' - C_i'' \leqq 0$ at positive prices and quantities. If in addition, at least one of the inequalities (i) and (ii) holds strictly, then $\Pi_i(q_i, Q_{-i})$ is strictly quasiconcave in $q_i \geqq 0$.*

Proof. See the Appendix. □

The following example illustrates the use of the criterion.

Example 6.4. Consider the inverse demand function p of Example 6.2. One can check that $Qp(Q)$ is not concave, that p is not logconcave, and that $pp'' - 2p'^2 < 0$ does not hold. However, p is $(\alpha, 1-\alpha)$-biconcave and strictly decreasing, therefore $(\alpha, 1)$-biconcave since $\alpha < 0$. Thus, from Proposition 6.3, the firm's problem is strictly quasiconcave provided that $\alpha \in (-1; 0)$.

Indivisibilities. So far, we assumed that technologies do not exhibit indivisibilities. However, neither Kukushkin's nor Tarski's fixed point theorem requires choice sets to be convex. Moreover, the monotonicity of a best response selection, the completeness of the strategy lattice, and upper hemicontinuity of best response correspondences are all inherited to arbitrary compact subsets of the strategy space. As a consequence, the existence assertions contained in Theorem 3.2, Proposition 5.2, as well as Proposition 6.1 all hold when firms' choices are restricted to closed subsets. In particular, the existence is ensured for any discretization of the quantity scale. This generalizes results obtained by Dubey et al. (2006).

Implications for uniqueness. Finally, the methods developed in this paper can be used to strengthen various existing uniqueness results.[10]

Proposition 6.6. *Assume that p is once differentiable when positive with $p' < 0$, and $(\alpha, 1-\alpha)$-biconcave, for some $\alpha \in [0;1]$. Assume either*

(i) $\alpha < 1$, and C_i is strictly increasing and convex for $i = 1, ..., N$, or

(ii) both p and the C_i's are twice continuously differentiable with $-p' + C_i'' > 0$ at positive prices.

Then there is at most one Cournot-Nash equilibrium.

Proof. Impose first condition (i). It follows then from the second part of Lemma 3.1 that *any* selection from r_i must be nonincreasing. Moreover, a standard supermodularity argument shows that the slope of the best response is strictly larger than -1. The assertion then follows from Theorem 2.8 in Vives (1999). Impose now condition (ii). Then, by Proposition 6.3 above, r_i is single-valued. The argument now proceeds as in the first case. $\square$

In particular, in a smooth Cournot model, sufficient conditions for equilibrium uniqueness and strict quasiconcavity are that $p' < 0$, the function p is $(\alpha, 1-\alpha)$-biconcave for some $\alpha \in [0;1]$, and $p' - C_i'' < 0$ for all i. This extends a well-known criterion (cf. Vives, 1999).

Appendix: Proofs

This appendix contains two auxiliary results and proofs of results stated in the body of the paper.

Lemma A.1. *Let $Y \geqq 0$. Then for any $\alpha \leqq 1$ [$\alpha \geqq 1$], the translation mapping $\tau_Y : y \mapsto y + Y$, defined on $\mathbb{R}_+$, is (α, α)–biconvex [(α, α)–biconcave].*

[10] See, in particular, Szidarovszky and Yakowitz (1982), Murphy et al. (1982), and Amir (1996).

Proof. Using the smooth criterion for biconcavity, Lemma 2.2, τ_Y is (α,α)-biconcave [biconvex] if and only if $(1-\alpha)Y \leqq 0$ $[\geqq 0]$. Thus, if $\alpha \leqq 1$, then τ_Y is (α,α)-biconvex, while for $\alpha \geqq 1$, the function τ_Y is (α,α)-biconcave. □

Lemma A.2. *Assume that a function f is $(\alpha, 1-\alpha)$-biconcave [$(\alpha,1-\alpha)$-biconvex] and nonincreasing, for $\alpha \geqq 0$ [for $\alpha \leqq 0$]. Then $f(y+Y)$ is $(\alpha,1-\alpha)$-biconcave [$(\alpha,1-\alpha)$-biconvex] in y for any $Y \geqq 0$.*

Proof. Assume first that f is $(\alpha, 1-\alpha)$-biconcave with $\alpha \geqq 0$. Then $\varphi_\alpha(f(x))$ is concave and nonincreasing in $\varphi_{1-\alpha}(x)$ in the interval where $f > 0$. Substituting x by $y+Y$, it follows trivially that $\varphi_\alpha(f(y+Y))$ is concave and nonincreasing in $\varphi_{1-\alpha}(y+Y)$ in the interval where $f > 0$. But by Lemma A.1, $\varphi_{1-\alpha}(y+Y)$ is convex in $\varphi_{1-\alpha}(y)$ provided $Y \geqq 0$. Since any composition of a concave and nonincreasing function with a convex function is concave, $\varphi_\alpha(f(y+Y))$ must be concave in $\varphi_{1-\alpha}(y)$ in the interval where $f > 0$. Hence, $f(y+Y)$ is indeed $(\alpha, 1-\alpha)$-biconcave in y. The assertion for biconvex functions is proved in an analogous way. □

Proof of Lemma 2.2. Write $\varphi = \varphi_\alpha$ and $\psi = \varphi_\beta$. By the standard differential condition, the function $\varphi(f(x))$ is concave as a function of $\psi(x)$ if and only if

$$\frac{d\varphi(f(x))}{d\psi(x)} = \frac{\varphi'(f(x))f'(x)}{\psi'(x)} \tag{16}$$

is nonincreasing in $\psi(x)$, or what is equivalent, nonincreasing in x. Differentiating the right-hand side of (16) with respect to x delivers that f is (α,β)-biconcave if and only if

$$\psi'(x)\varphi''(f(x))f'(x)^2 + \psi'(x)\varphi'(f(x))f''(x) - \varphi'(f(x))f'(x)\psi''(x) \leqq 0. \tag{17}$$

Replacing the derivatives of φ and ψ by the respective explicit expressions, and subsequently multiplying through with $f(x)^{2-\alpha}x^{2-\beta}$ delivers (3). This proves the assertion. $\square$

Proof of Lemma 2.3. Fix $\widetilde{Y} \geqq 0$ for the moment. Then, by Lemma A.2, $f(y+\widetilde{Y})$ is $(\alpha, 1-\alpha)$-biconcave in y, i.e., $\varphi_\alpha(f(y+\widetilde{Y}))$ is concave in $\varphi_{1-\alpha}(y)$ in the interval where $f > 0$. Since any concave function is differentiable almost everywhere, with a derivative that is weakly decreasing,

$$\frac{d\varphi_\alpha(f(y+\widetilde{Y}))}{d\varphi_{1-\alpha}(y)} = \frac{\varphi'_\alpha(f(y+\widetilde{Y}))f'(y+\widetilde{Y})}{\varphi'_{1-\alpha}(y)} \tag{18}$$

is both a.e. well-defined and nonincreasing in $\varphi_{1-\alpha}(y)$, or equivalently, in y. Recalling the fact that a concave function of one variable, defined on an open domain, is absolutely continuous on any compact subinterval of its domain, it is easy to check that $\varphi_\alpha(f_*(y,.))$ is absolutely continuous on $[Y, \widehat{Y}]$. Hence,

$$\varphi_\alpha(f_*(y,\widehat{Y})) - \varphi_\alpha(f_*(y,Y)) = \int_Y^{\widehat{Y}} \varphi'_\alpha(f_*(y,\widetilde{Y}))yf'(y+\widetilde{Y})d\widetilde{Y} \tag{19}$$

$$= \int_Y^{\widehat{Y}} \frac{\varphi'_\alpha(f(y+\widetilde{Y}))f'(y+\widetilde{Y})}{\varphi'_{1-\alpha}(y)}d\widetilde{Y} \tag{20}$$

is nonincreasing in y. This proves the lemma. $\square$

Proof of Lemma 3.1. To prove the dual single-crossing property, assume that $\Pi_i(q_i, Q_{-i}) \geqq \Pi_i(\widehat{q}_i, Q_{-i})$. There are two cases. Assume first that, in fact, $R_i(q_i, Q_{-i}) \geqq R_i(\widehat{q}_i, Q_{-i})$. Write $\varphi = \varphi_\alpha$. Clearly, since $\varphi \circ R_i$ is submodular and φ is strictly increasing, R_i satisfies the dual single-crossing property (cf. Milgrom and Shannon, 1994). Hence, $R_i(q_i, \widehat{Q}_{-i}) \geqq R_i(\widehat{q}_i, \widehat{Q}_{-i})$. Since costs are nondecreasing, $\Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) \geqq \Pi_i(q_i, \widehat{Q}_{-i})$, as desired. Assume now that $R_i(q_i, Q_{-i}) < R_i(\widehat{q}_i, Q_{-i})$. We will show that in this case,

$$R_i(\widehat{q}_i, \widehat{Q}_{-i}) - R_i(q_i, \widehat{Q}_{-i}) \leqq R_i(\widehat{q}_i, Q_{-i}) - R_i(q_i, Q_{-i}), \tag{21}$$

which is sufficient for the dual single-crossing property. For this, write $u = \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i}))$ and

$$v = \varphi(R_i(q_i, Q_{-i})) - \varphi(R_i(\widehat{q}_i, Q_{-i})) + \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i})). \tag{22}$$

As φ is strictly increasing, it follows from the case assumption that $u > v$. Moreover, since p in nonincreasing,

$$R_i(\widehat{q}_i, Q_{-i}) = \widehat{q}_i p(\widehat{q}_i + Q_{-i}) \geqq \widehat{q}_i p(\widehat{q}_i + \widehat{Q}_{-i}) = R_i(\widehat{q}_i, \widehat{Q}_{-i}), \tag{23}$$

which implies that $w \equiv \varphi(R_i(\widehat{q}_i, Q_{-i})) - \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i})) \geqq 0$. Denote by φ^{-1} the inverse function of φ. Since φ^{-1} is convex, $\varphi^{-1}(\widetilde{u} + \widetilde{w})$ is supermodular in $(\widetilde{u}, \widetilde{w})$. Hence,

$$\varphi^{-1}(u) - \varphi^{-1}(v) \leqq \varphi^{-1}(u + w) - \varphi^{-1}(v + w). \tag{24}$$

Plugging the expressions for u, v, and w into (24) and simplifying yields

$$\begin{aligned} &\varphi^{-1}\{\varphi(R_i(q_i, Q_{-i})) - \varphi(R_i(\widehat{q}_i, Q_{-i})) + \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i}))\} \\ &\geqq R_i(\widehat{q}_i, \widehat{Q}_{-i}) - R_i(\widehat{q}_i, Q_{-i}) + R_i(q_i, Q_{-i}). \end{aligned} \tag{25}$$

On the other hand, from the submodularity property of $\varphi \circ R_i$,

$$\varphi(R_i(q_i, \widehat{Q}_{-i})) \geqq \varphi(R_i(q_i, Q_{-i})) - \varphi(R_i(\widehat{q}_i, Q_{-i})) + \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i})), \tag{26}$$

or equivalently,

$$R_i(q_i, \widehat{Q}_{-i}) \geqq \varphi^{-1}\{\varphi(R_i(q_i, Q_{-i})) - \varphi(R_i(\widehat{q}_i, Q_{-i})) + \varphi(R_i(\widehat{q}_i, \widehat{Q}_{-i}))\}. \tag{27}$$

Combining (25) with (27), one arrives at (21), as desired. To prove the second assertion, note that in the first case considered above, $\Pi_i(q_i, \widehat{Q}_{-i}) >$

$\Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$ because costs are strictly increasing. In the second case, $w > 0$ because p is strictly decreasing when positive, and φ^{-1} is strictly convex, so again, the second inequality in (8) is strict. This proves the second assertion, whence the lemma. $\square$

Proof of Theorem 3.2. By assumption, each firm's effective choice set is compact. Take arbitrary quantities $\widehat{Q}_{-i} > Q_{-i} \geqq 0$. Clearly, the profit function is upper semicontinuous in own output, for any given level of rivals' output. Since any u.s.c. function on a compact set attains a maximum value and the set of maximizers is compact, $r_i(\widehat{Q}_{-i})$ and $r_i(Q_{-i})$ are compact and non-empty. Take $\widehat{q}_i = \min r_i(\widehat{Q}_{-i})$ and $q_i = \min r_i(Q_{-i})$. We will show that $\widehat{q}_i \leqq q_i$. To provoke a contradiction, assume $\widehat{q}_i > q_i$. Then, clearly, $\Pi_i(q_i, \widehat{Q}_{-i}) < \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$. Write $R_i(q_i, Q_{-i}) = q_i p(q_i + Q_{-i})$. By assumption, there exists a parameter $\alpha \in [0; 1]$ such that p is $(\alpha, 1-\alpha)$-biconcave. Since a positive production level leading to a zero market price is strictly dominated, $R_i(\widehat{q}_i, \widehat{Q}_{-i}) > 0$ and $R_i(q_i, Q_{-i}) > 0$. Lemma 2.3 implies

$$\varphi_\alpha(R_i(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R_i(q_i, Q_{-i})) \geqq \varphi_\alpha(R_i(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R_i(q_i, \widehat{Q}_{-i})). \quad (28)$$

Using Lemma 3.1,

$$\Pi_i(q_i, Q_{-i}) - \Pi_i(\widehat{q}_i, Q_{-i}) \geqq 0 \Rightarrow \Pi_i(q_i, \widehat{Q}_{-i}) - \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) \geqq 0. \quad (29)$$

Hence, $\Pi_i(q_i, Q_{-i}) < \Pi_i(\widehat{q}_i, Q_{-i})$, which contradicts the maximum property of q_i. Consequently, $q_i \geqq \widehat{q}_i$, as desired. Thus, taking the minimum within each best-response set, a selection ρ_i is constructed that is nonincreasing in rivals' output. This proves the first assertion. For the second assertion, one notes that, since C_i is left-continuous and nondecreasing, C_i is lower semi-continuous. Moreover, either R_i is continuous in (q_i, Q_{-i}), viz. when p is continuous at positive quantities, or R_i is continuous in (q_i, Q_{-i}) in the closed domain where $p(q_i + Q_{-i}) > 0$, viz. when p jumps to zero. In either case, profits are quasi-transfer upper continuous in (q_i, Q_{-i}) with respect to $[0; \overline{Q}]$. To see why, assume that

$$\Pi_i(\widehat{q}_i, Q_{-i}) > \Pi_i(q_i, Q_{-i}), \tag{30}$$

and consider a small perturbation of (q_i, Q_{-i}). If costs jump upwards through the perturbation, inequality (30) still holds. On the other hand, if revenues drop to zero through the perturbation, then necessarily $R_i(\widehat{q}_i, Q_{-i}) > 0$, hence $\widehat{q}_i > 0$. Since revenues are continuous in the domain of positive prices, we may replace $\widehat{q}_i$ by a smaller value so that (30) still holds. This shows that Π_i is indeed quasi-transfer upper continuous. It follows now from Theorem 3 in Tian and Zhou (1995) that r_i has a closed graph when firms are restricted to choose a quantity from the compact interval $[0; \overline{Q}]$. Consequently, the assumptions of Kukushkin's (1994) fixed point theorem are satisfied in the restricted game. But a choice above $\overline{Q}$ can never increase profits. Thus, an equilibrium exists also in the unconstrained game. □

Proof of Proposition 4.1. At $\alpha \in (0; 1)$, choose $p(Q) = (1 - Q^{1-\alpha})^{1/\alpha}$ for $Q \leqq 1$, and $= 0$ for $Q > 1$. The test

$$\Delta^p_{\widehat{\alpha},1-\widehat{\alpha}}(Q) = (\widehat{\alpha} - \alpha)(Q^{1-\alpha} - \alpha)\frac{(1 - Q^{1-\alpha})^{2-2/\alpha}}{\alpha^2 Q^\alpha} \leqq 0, \tag{31}$$

holds for all $Q \in (0; 1)$ if and only if $\widehat{\alpha} = \alpha$. At $\alpha = 0$, choose $p(Q) = \exp(-Q)$. Here, $\Delta^p_{\widehat{\alpha},1-\widehat{\alpha}}(Q) = \widehat{\alpha}(Q - 1)\exp(-Q)^2$, which is globally nonpositive if and only if $\widehat{\alpha} = 0$. At $\alpha = 1$, choose $p(Q) = -\ln(Q)$ for $Q \leqq 1$, and

$= 0$ for $Q > 1$. Also in this case, the test

$$\Delta^p_{\widehat{\alpha},1-\widehat{\alpha}}(Q) = -(1-\widehat{\alpha})\frac{1+\ln(Q)}{Q} \leqq 0 \tag{32}$$

holds for all $Q \in (0;1)$ if and only if $\widehat{\alpha} = 1$. Clearly, for all $\alpha \in [0;1]$, with specifications of demand as made above, the assumptions of Theorem 3.2 hold for any specification of costs $C_1, ..., C_N$, provided these cost functions are strictly increasing and left-continuous. Hence, the assertion. □

Proof of Theorem 4.2. Consider first the case $\alpha < 1$, $\beta < 1$. Then, by assumption, $\alpha + \beta < 1$. Furthermore, since (α, β)-biconcavity is more stringent for higher values of α and β (cf. Section 2), we may assume without loss of generality that $\alpha > 0$ and $\beta > 0$. Consider $p(Q) = (1 - Q^\beta)^{1/\alpha}$ for $Q < 1$ and $p(Q) = 0$ for $Q \geqq 0$. One can check that $\Delta^p_{\alpha,\beta}(Q) \equiv 0$. Hence, p is (α, β)-biconcave. Define $C_i(q_i) \equiv 0$ for $i = 1, ..., N$. It follows from Proposition 6.3 below that firm i's problem is strictly quasiconcave. Moreover, an implicit differentiation of the first-order condition $q_i p' + p = 0$ shows that the slope of the best response at $Q_{-i} = 0$ is given by

$$\left.\frac{dq_i}{dQ_{-i}}\right|_{Q_{-i}=0} = -\frac{q_i p''(q_i) + p'(q_i)}{q_i p''(q_i) + 2p'(q_i)} = \frac{1-\alpha-\beta}{\alpha+\beta} > 0. \tag{33}$$

where the second equality follows from $\Delta^p_{\alpha,\beta}(Q) \equiv 0$ and the first-order condition. The remaining conditions can be easily checked. This proves the assertion in the case $\alpha < 1$, $\beta < 1$. Assume now $\alpha \geqq 1$. Then, by assumption, $\beta < 0$. Consider $p(Q) = (Q^\beta - 1)^{1/\alpha}$ for $Q < 1$ and $p(Q) = 0$ for $Q \geqq 0$. Without loss of generality, $\alpha + \beta > 1$, so that $\lim_{Q \to 0} Qp(Q) = 0$. Assuming strict quasiconcavity for the moment, generalizing (33) to the case of nonzero smooth costs yields

$$\left.\frac{dq_i}{dQ_{-i}}\right|_{Q_{-i}=0} = \frac{1-\alpha-\beta+(\alpha-1)C_i'(q_i)/p(q_i)}{\alpha+\beta+(\alpha-1)C_i'(q_i)/p(q_i)-C_i''(q_i)/p'(q_i)}. \tag{34}$$

Choose cost functions

$$C_i(q_i) = \int_0^{q_i} (\widetilde{q}_i^{\beta} - 1)^{1/\alpha} d\widetilde{q}_i \tag{35}$$

for $i = 1, .., N$. Clearly, $C_i'(q_i) = p(q_i)$, and $C_i''(q_i) = p'(q_i)$. By Proposition 6.3, the firm's problem is indeed stricty quasiconcave. Hence,

$$\left.\frac{dq_i}{dQ_{-i}}\right|_{Q_{-i}=0} = -\frac{\beta}{2(\alpha-1)+\beta}. \tag{36}$$

Without loss of generality, $\beta < 0$ can be chosen close to zero such that $2(\alpha-1)+\beta$ is positive. But then, (36) is positive. This proves the assertion in the case $\alpha \geqq 1$. Assume, finally, $\beta \geqq 1$. Then, by assumption, $\alpha < 0$. Hence, without loss of generality, $\alpha + \beta > 1$. Consider $p(Q) = (1 + Q^{\beta})^{1/\alpha}$ for $Q \geqq 0$. For $\varepsilon > 0$ small, choose cost functions

$$C_i(q_i) = (\alpha+\beta-\varepsilon) \int_0^{q_i} (1 + \widetilde{q}_i^{\beta})^{1/\alpha} d\widetilde{q}_i \tag{37}$$

for $i = 1, .., N$. Clearly, $C_i'(q_i) = (\alpha+\beta-\varepsilon)p(q_i)$, and $C_i''(q_i) = (\alpha+\beta-\varepsilon)p'(q_i)$. In particular, firm i's problem is strictly quasiconcave. Moreover,

$$\left.\frac{dq_i}{dQ_{-i}}\right|_{Q_{-i}=0} = \frac{1-\alpha-\beta+(\alpha-1)(\alpha+\beta-\varepsilon)}{\alpha+\beta+(\alpha-2)(\alpha+\beta-\varepsilon)}. \tag{38}$$

For ε small enough, the denominator is negative, so that (38) is positive also in this case. Since all possible cases have been covered, this proves the assertion. □

Proof of Proposition 6.3. It is shown first that $\Pi_i(., Q_{-i})$ is strictly quasiconcave in the largest open interval $\widetilde{I}$ where $q_i > 0$ and $p > 0$. If $\widetilde{I}$ is empty, there is nothing to show, so assume $\widetilde{I} \neq \varnothing$. It suffices to show that $\partial^2\Pi_i/\partial q_i^2 < 0$ at any $q_i \in \widetilde{I}$ with $\partial\Pi_i/\partial q_i = 0$ (see, e.g., Diewert et al., 1981). So take $q_i \in \widetilde{I}$ with

$$0 = q_i p'(Q) + p(Q) - C_i'(q_i), \tag{39}$$

where we write Q for $q_i + Q_{-i}$. With $p(Q) > 0$ satisfied within $\widetilde{I}$, it follows from the assumption that

$$0 \geqq (\alpha - 1)p'(Q)^2 Q + p(Q)p''(Q)Q + (1 - \beta)p(Q)p'(Q). \tag{40}$$

Multiplying (39) through with $(1-\alpha)p'(Q)$, and (40) with $q_i/Q > 0$, a simple addition yields

$$0 \geqq -(1-\alpha)p'(Q)C_i'(q_i) + p(Q)(q_i p''(Q) + p'(Q)\{(1-\alpha) + \frac{q_i}{Q}(1-\beta)\}). \tag{41}$$

For $\alpha \leqq 1$, the first term is nonnegative. Hence, noting that $q_i/Q \leqq 1$, and exploiting the remaining assumptions, one finds the second-order condition $0 > p''(Q)q_i + 2p'(Q) - C_i''(q_i)$. Thus, Π_i is indeed strictly quasiconcave on $\widetilde{I}$. Since Π_i is continuous in $q_i = 0$, the function Π_i must also be strictly quasiconcave on $\widetilde{I} \cup \{0\}$. This proves the assertion in the case where $p > 0$. If, however, p reaches zero at a finite output level, costs must be strictly increasing by assumption, so again $\Pi_i(., Q_{-i})$ is strictly quasiconcave for $q_i \geqq 0$. $\square$

References

Amir, R., 1996, Cournot oligopoly and the theory of supermodular games, Games and Economic Behavior 15, 132-148.

Amir, R., 2005, Ordinal versus cardinal complementarity: The case of Cournot oligopoly, Games and Economic Behavior 53, 1-14.

Amir, R., Lambson, V., 2000, On the effects of entry in Cournot games, Review of Economic Studies 67, 235-254.

Bamon, R., Frayssé, J., 1985, Existence of Cournot equilibrium in large markets, Econometrica 53, 587-597.

Caplin, B., Nalebuff, C., 1991a, Aggregation and social choice: a mean voter theorem, Econometrica 59, 1-23.

Caplin, B., Nalebuff, C., 1991b, Aggregation and imperfect competition: On the existence of equilibrium, Econometrica 59, 25-59.

Diewert, W., Avriel, M., Zang, I., 1981, Nine kinds of quasiconcavity and concavity, Journal of Economic Theory 25, 397-420.

Dubey, P., Haimanko, O., Zapechelnyuk, B., 2006, Strategic complements and substitutes, and potential games, Games and Economic Behavior 54, 77-94.

Kukushkin, N., 1993, Cournot equilibrium with 'almost' identical convex costs, mimeo, Russian Academy of Sciences.

Kukushkin, N., 1994, A fixed-point theorem for decreasing mappings, Economics Letters 46, 23-26.

McManus, M., 1962, Number and size in Cournot oligopoly, Yorkshire Bulletin of Social and Economic Research, 14.

McManus, M., 1964, Equilibrium, numbers and size in Cournot oligopoly, Yorkshire Bulletin of Social and Economic Research, 16.

Milgrom, P., Shannon, D., 1994, Monotone comparative statics, Econometrica 62, 157-180.

Murphy, F., Sherali, H., Soyster, B., 1982, A mathematical programming approach for determining oligopolistic market equilibrium, Mathematical Programming 24, 92-106.

Novshek, W., 1985, On the existence of Cournot equilibrium, Review of Economic Studies 52, 85-98.

Roberts, J., Sonnenschein, H., 1976, On the existence of Cournot equilibrium without concave profit functions, Journal of Economic Theory 13, 112-117.

Selten, R., 1970, Preispolitik der Mehrproduktenunternehmung in der statischen Theorie, Springer, Berlin.

Szidarovszky, F., Yakowitz, S., 1982, Contributions to Cournot oligopoly theory, Journal of Economic Theory 28, 51-70.

Tian, G., Zhou J., 1995, Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization, Journal of Mathematical Economics 24, 281-303.

Vives, A., 1999, Oligopoly Pricing, MIT Press, Cambridge.