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Random Dynamical Systems in Economics

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Abstract

This paper surveys recent advances in the application of random dynamical systems theory in economics. It illustrates the usefulness of this framework for modeling and analysis of economic phenomena with stochastic components, mainly focusing on stochastic dynamic models in economic growth. The paper also highlights some directions for further applications and interdisciplinary research on random dynamical systems.

1 Introduction

In this survey, I discuss and illustrate the role of dynamical systems theory in economics which I believe to be more than a fashionable trend in economic thought. In particular I am concerned with the potential impact and usefulness of the theory of random dynamical systems for economic modeling and economic analysis. The theory and application of random dynamical systems is at the cutting edge of research in both mathematics and economics. Right from the start I would like to point out that this survey does not reflect any common point of view in economics because, on the one hand, no coherent view exists and, on the other hand, this paper is biased by my view of the role of mathematics in social sciences. Before further outlining the goal of this paper, I give a brief review of the history and the current state of dynamic models in economic theory.

Economic modeling is strongly influenced by theories that are based on concepts that can be traced back to the Newtonian approach to mechanics. The influential nineteenth century economists Walras, Pareto, Marshall, Jevons, and Edgeworth introduced a rigorous formulation of “the mechanic of utility and self-interest” (Jevons) which transferred the ideas of classical mechanics of gravitating systems into economics. They focused exclusively on a state of rest; a market equilibrium in which individual plans are mutually compatible and the utility of all agents is maximal given mutually imposed constraints. Their approach culminated in the general equilibrium theory, a centerpiece in economic theory, see Arrow [5], Debreu [18], and also Arrow and Hahn [6].

Research focused on atemporal aspects at first; leaving out the dynamics of the actual market process. In the 1920s, dynamic economics came into being

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when economists first employed differential and difference equations in macro-economic models of business cycles (merely to study out-of-equilibrium adjustment dynamics).¹ However, the clarification of terminology due to Frisch [24]² makes clear that there is an ambiguous relation of these early approaches to dynamical systems theory. He proposed to use the terms *static* and *dynamic* to characterize the structural relations of variables that make up the model. In Frisch’s definition, a structural relation is called static, if all variables refer to the same time; otherwise the relation is called dynamic. He suggested to classify a model as dynamic if it contains at least one dynamic structural relation.³ Down to the present day, this terminology is generally accepted in economics.⁴ This discussion makes clear why general equilibrium theory became a dynamic theory – in the sense of Frisch – after “time” had been added to the description of goods and services.

It is important to emphasize that dynamic economic models – in the above sense – are, in general, not related to dynamical systems theory. From a mathematical point of view, a dynamic economic model can be described exclusively by implicit equations, relating states and variables across different periods in time. An equilibrium in such a model is a time-path of allocations of commodities and prices. There is, in general, neither a “direction of evolution” in such models nor an “initial state” that can be varied on some state space. It makes therefore no sense to talk about the time-path of the economy out of such an equilibrium; it is simply not defined. Thus one cannot come up with an equivalent description of the model in the framework of dynamical systems in general. The derivation of a dynamical systems description of an economic model requires that (1) a state space can be defined (in particular, initial states can be identified), (2) all implicit equations in the economic model have solutions for different initial states and variables (from an open set, for instance, to permit stability analysis), and (3) these solutions exhibit a suitable time-structure in the dependence on states and variables.

We use the following terminology in this paper. An economic model is called *explicit*, if it can be described by a dynamical system; otherwise the model is called *implicit*. The notion dynamical system is understood here as a generic term for non-autonomous as well as random dynamical systems, see Arnold [4], and Crauel and Gundlach [16]. We propose the usage of this terminology because a dynamical systems representation provides an explicit description of the

¹Pioneering work is due to Frisch, Hicks, Kaldor, Kalecki, Goodwin, Samuelson, and others.

²His paper was motivated by a discussion between Breit, Frisch, F. G. Koopmans, Marschak, and Tinbergen at a meeting of the Econometric Society in September 1935. Ragnar Frisch was awarded the first Nobel price in Economics (jointly with Tinbergen) in 1969.

³This characterization of dynamic models expresses how Frisch’s view is perceived by most economists, see e.g. Niehans [40, Chap. 29]. In my opinion, the notion of a dynamic model, as described in Frisch [24], is very much in line with dynamical systems theory.

⁴John R. Hicks, Economic Nobel price laureate in 1972 (jointly with Kenneth J. Arrow), commented on this terminology in *Value and Capital*, 1939, Chapter IX “The Method of Analysis”: “The definition of economic dynamics (that much controverted term) which I have in mind here is this. I call Economic Statics those parts of economic theory where we do not trouble about dating; Economic Dynamics those parts where every quantity must be dated.”

evolution of an economy⁵ in time. On the one hand, restricting the class of employed models to those that are explicit, forces one to abandon some standard dynamic economic models. On the other hand, one gains the concepts, methods, and paradigms⁶ of dynamical systems theory which have proved seminal to many areas of science.

The goal of this paper is twofold. First, I want to illustrate that it is worth to give more thought to the application of explicit models in economics. In my view, explicit economic models should be employed to study a boarder range of economic problems than only short-run phenomena such as tâtonnement processes or business cycles. Random dynamical systems theory enables us in particular to analyze the (global) stability properties of economic systems, taking into account e.g. exogenous perturbations, uncertainty, and repeated (random) interaction as well as time-dependent environments. This cannot consistently be carried out in dynamic economic models in general. Economists who pursue the application of dynamical systems theory and employ explicit models are ultimately required to rethink and alter the modeling of economic processes, including the time-structure of actions and decisions as well as the behavior of individuals. However, the analysis of many issues that are central in economics often requires concepts and methods which are nonexistent in dynamical systems theory at the time of this writing. The second goal of this paper is therefore to point out some open mathematical problems to motivate further research in the field of dynamical systems.

The remainder of the paper is organized in the following fashion. Section 2 surveys literature on explicit models. Random dynamical systems theory is briefly reviewed in Section 3. Section 4 presents recent research on stochastic explicit models in economic growth. Finally, Section 5 points out open problems and directions for future research.

2 Toward Explicit Models

Although implicit models seem to be pervasive in economic modeling, there are many examples of explicit models. For instance, explicit economic models are ubiquitous in macroeconomics. To give a flavor of the diverse macro applications, an incomplete list of references, in which mainly deterministic difference and differential equations are employed, follows. For neoclassical growth theory, see e.g. the original papers by Solow [49], and Swan [52], and the respective chapters in the textbooks by Aghion and Howitt [1] and Barro and Sala-i-Martin [8]. For intertemporal macroeconomics in the overlapping generations framework, see e.g. the pioneering work by Diamond [19] and the textbook by Azariadis [7]

⁵For instance, the evolution of prices, allocations of goods and services, and aggregate quantities such as gross domestic product, as well as flows of commodities or money.

⁶It is important to emphasize that there are critical assessments, for instance, of the concept of causation, e.g. by David Hume. Blaug [10, p. 6], “For Hume, what is called *causation* is nothing but the constant conjunction of two events, that happen to be contiguous in time and space, the event that is prior in time being labeled the “cause” of the latter event labeled “effect,” although there is actually no necessary connection between them.”

who thoroughly applies dynamical systems theory⁷. Further, for intertemporal macroeconomics with representative agent models, deterministic as well as stochastic (mainly in real business cycle theory), see e.g. Stokey, Lucas, and Prescott [51] and Cooley [15] (and the references in both books), and the pioneering work by Ramsey [43]. For Keynesian (nonmarket-clearing) models, e.g. Flaschel, Franke, and Semmler [22]. To close the list, let me mention the numerous mathematics-oriented books which focus on general nonlinear phenomena in economics by Brock and Malliaris [11], Chiarella [13], Day [17], Gandolfo [25], Lorenz [33], Puu [42], and others.

Stochastic explicit models, which are the main focus of this paper, have attracted less attention so far; albeit the need to take into account randomly fluctuating variables in economic modeling has been acknowledged long time ago, Frisch [23, 24]. Let me point out some recent developments in stochastic explicit models of economic growth. Descriptive stochastic growth models present the most well-known class of stochastic explicit models. Mirman [35, 36] studied these models from a Markovian point of view. Schenk–Hoppé and Schmalfuß [48] extended the analysis to sample-path stability by applying random dynamical systems theory. Random dynamical systems have further been employed in models with consumer optimization. Explicit business cycle models with overlapping generations are studied e.g. in Peter and Schenk–Hoppé [41] and, in combination with technical progress, in Schenk–Hoppé [44]. Optimal policies in stochastic growth models with an infinitely-lived representative agent also give rise to explicit stochastic models, see e.g. Stokey, Lucas, and Prescott [51], and Montrucchio and Privileggi [39]. The latter fall within the class of stochastic control models, see e.g. Arkin and Evstigneev [3]. We discuss these three different types of growth models in detail in Section 4.

From the many other areas of economics in which stochastic explicit models are (though not generally) applied, we outline recent research in evolutionary economics. It is a very active field and, in my opinion, represents a promising approach to the understanding of many economic phenomena. Evolutionary economics builds on the Darwinian approach to evolutionary biology⁸, see Hodgson [29] and Witt [56]. Market processes with the introduction and disappearance of products, technologies, and firms, can be considered as an instance of “economic natural selection.” The self-transformation of the economy through the generation and dissemination of novelty can be taken as an evolutionary process⁹. The evolutionary approach in economics inherits a non-aprioristic view and, therefore, is strongly linked with explicit models of dynamic stochastic phenomena. Evolutionary game theory, in particular, has recently experienced a boost of contributions in which stochastic explicit models are applied. Among

⁷Azariadis [7, p. xii] in his foreword: “Dynamical systems have spread so widely into macroeconomics that vector fields and phase diagrams are on the verge of displacing the familiar supply-demand schedules and Hicksian crosses of static macroeconomics.”

⁸See e.g. Maturana and Varela [34] for a different approach to evolutionary biology.

⁹I will not try to assess whether a metaphorical use of the notions of evolutionary biology in economics is sufficient or whether economics should be interpreted as the theory of all kinds of human behavior and thus the concept of evolution is directly applicable. The interested reader is referred to Witt [55, 57].

many others, Kandori et al. [31], Young [58], Vega–Redondo [53], and Alós–Ferrer et al. [2]. However, this approach is not without problems, see e.g. Bergin and Lipman [9] and Schenk–Hoppé [45]. Further evolutionary approaches, often in the framework of explicit models, are pursued in behavioral and evolutionary finance in the study of competition of trading strategies and herd behavior of investors.

While most of the above-mentioned literature is concerned with explicit models from the outset, other approaches have been revised to meet the empirically observed sequential structure of trade (as well as the amounts of commodities traded) and (possibly) incorrect forecasts of agents. For instance, general equilibrium theory, which is closely-knit with implicit models, has been challenged on grounds of its unrealistic assumptions of correct forecasting of future events by agents and the mutual compatibility of their plans. These criticisms have triggered the emergence of temporary general equilibrium theory. The theory of temporary equilibrium under uncertainty is comprehensively summarized in Grandmont [27].¹⁰ The evolution of an economy is described as a sequence of equilibria in which compatibility of agents’ plans in the short-run is achieved by price or rationing mechanisms. The models on stochastic economic growth discussed in the following fall within this category.

3 Random dynamical systems

We briefly review the framework used in our study of dynamic economic models which fall within the category of explicit models. We restrict ourselves to models in discrete time. The reader is referred to the monograph by Arnold [4] for the general theory and any additional information.

The model of the stochastic perturbation is given by a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, i.e. a probability space together with a measurable and measurably invertible map $\theta : \Omega \rightarrow \Omega$ such that $\theta\mathbb{P} = \mathbb{P}$. We call the system ergodic, if \mathbb{P} is ergodic with respect to θ .

Consider the random difference equation $x_{t+1} = h(\theta^t\omega, x_t)$ on some space $X \subset \mathbb{R}^d$; where the map $h(\omega, \cdot) =: h(\omega) : X \rightarrow X$ is assumed to be measurable and measurably invertible. X is equipped with the trace- σ -algebra. Define,

$$\varphi(t, \omega, x) = \begin{cases} h(\theta^{t-1}\omega) \circ \dots \circ h(\omega)x & \text{for } t \geq 1 \\ x & \text{for } t = 0 \\ h(\theta^t\omega)^{-1} \circ \dots \circ h(\theta^{-1}\omega)^{-1}x & \text{for } t \leq -1 \end{cases} \quad (1)$$

$\varphi(t, \omega, x)$ is the state of the stochastic system (described by the random map h) at time t which has been started at $x_0 = x$ under the perturbation determined by ω . $\theta^t = \theta \circ \dots \circ \theta$. In particular, $\varphi(1, \omega, \cdot) \equiv h(\omega)$. h is called the *generator* of φ . If $h(\omega) : X \rightarrow X$ is measurable but not invertible, then only the first

¹⁰The first sentence in Grandmont [27]: “Traditional general equilibrium theory [Arrow and Hahn (1971), Debreu (1959)] has for a long time studied an economic model which is essentially static, where the agents’ expectations are self-fulfilling, and where the equilibrium is achieved solely by the price system.”

and the second equation in (1) hold. In particular, the sample path $\varphi(t, \omega, x)$ is only defined for $t \geq 0$. Let $\mathbb{T} = \mathbb{Z}$ ($\mathbb{T} = \mathbb{N}$), if the generator h is invertible (non-invertible).

The family of maps $\varphi(t, \omega, x)$ is called a *random dynamical system*. That is, $\varphi : \mathbb{T} \times \Omega \times X \rightarrow X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ is a $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X)$, $\mathcal{B}(X)$ -measurable mapping such that $\varphi(0, \omega) = \text{id}_X$ and $\varphi(s+t, \omega) = \varphi(t, \theta^s \omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}$, and $\omega \in \Omega$. These properties replace the flow property of a deterministic dynamical system that is generated by the iteration of a map. Obviously, $\varphi(t, \omega)$ inherits the regularities (such as continuity or smoothness) of h for $t \geq 0$ and of h^{-1} for $t \leq 0$. The stochastic perturbation is defined for two-sided time $\mathbb{T} = \mathbb{Z}$ in both invertible and non-invertible case.

We need the following key concept in the analysis of our models.

Definition 3.1 *A random fixed point of a random dynamical system φ on X is a random variable $x^* : \Omega \rightarrow X$ such that almost surely¹¹*

$$x^*(\theta\omega) = \varphi(1, \omega, x^*(\omega)) := h(\omega, x^*(\omega)). \quad (2)$$

Equation (2) implies $x^*(\theta^{t+1}\omega) = h(\theta^t\omega, x^*(\theta^t\omega)) = \varphi(t+1, \omega, x^*(\omega))$ for all t , i.e. a random fixed point is a stationary solution of the random difference equation. If the perturbation is trivial, then above definition coincides with the notion of a deterministic steady state. Determining a random fixed point is equivalent to solving a (typically infinite) number of coupled equations. The coupling enters through the operator θ appearing on the left-hand side of (2).

4 Economic Growth

This section surveys recent progress in the application of random dynamical systems theory in stochastic economic growth. This framework is used in the derivation of stochastic explicit models as well as in their analysis. In the models presented here, randomness enters through the stochastic perturbations of the fundamentals of the economy (such as technology and preferences).

Consider an economy that consists of many identical households and firms. Therefore, agents take prices as given when making consumption, investment, or production decisions. There is a single homogeneous good in the economy which can be either consumed or used as capital input in production. Two factors, capital and labor, are needed in the production process. The technology is described by the production function

$$Y_t = F(K_t, L_t, z_t, a_t)$$

K_t and L_t are capital resp. labor input at time t ; z_t and a_t present a measure of productivity resp. of the state of technical progress. L_t , z_t , and a_t are

¹¹In the context of random dynamical systems, we will use the notion *almost surely* (abbreviated a.s.) in the following (non-standard) sense. A statement holds a.s. if there exists a θ -invariant set $\Omega' \subset \Omega$ ($\theta\Omega' = \Omega'$) of full \mathbb{P} -measure ($\mathbb{P}(\Omega') = 1$) such that this statement holds true for all $\omega \in \Omega'$. For instance, the statement of the ergodic theorem can be understood in this sense, see Arnold [4, App. A.1].

random variables. Given z_t and a_t , Y_t is the aggregate output at time t if K_t units of capital and L_t units of labor are employed in production. We assume that for each fixed (z_t, a_t) , the production function is neoclassical and linear homogeneous.

A production function is neoclassical if it exhibits positive and diminishing marginal products with respect to each input, i.e.

$$\frac{\partial F}{\partial K} > 0, \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial L^2} < 0$$

Linear homogeneity means that $\lambda F(K, L, z, a) = F(\lambda K, \lambda L, z, a)$ for all $\lambda > 0$.

We restrict our analysis to a representative firm. Assuming that the economy is closed, i.e. the endowment of capital at the beginning of period $t + 1$ is equal to the resources not consumed in the preceding period, the stochastic law of motion for the aggregate capital stock is given by

$$K_{t+1} = F(K_t, L_t, z_t, a_t) + (1 - \delta_t)K_t - C_t \quad (3)$$

where δ_t is the random rate of depreciation of capital at time t and C_t is aggregate consumption at time t . In (3), we implicitly assume that total investment equals the savings of households. The price of the consumption good is taken as numeraire.

Firms determine their demand for capital and labor maximizing profits in each period. Assuming perfectly competitive markets, capital and labor earn their marginal products in all states of nature, i.e.

$$r_t = F_1(K_t, L_t, z_t, a_t) \quad (4)$$

$$w_t = F_2(K_t, L_t, z_t, a_t) \quad (5)$$

where the random variables r_t and w_t denote real interest rate and real wage, respectively. We use an integer subscript to denote partial derivative with respect to the corresponding argument. Since F exhibits constant returns to scale, competitive payments to factors exhaust the total output in all states of nature: $Y_t = r_t K_t + w_t L_t$.

The decision of households concerns consumption in each period and supply of labor and capital. Their behavior depends crucially on the framework used. We discuss a descriptive growth model in which the decision of households is not modeled (Section 4.1), an overlapping generations model in which households live for two periods (Section 4.2), and the optimal growth model with an infinitely-lived representative agent (Section 4.3). We will mainly focus on the stability analysis of these models.

Here we treat technical progress as an exogenously given process; postponing some remarks on future research on explicit endogenous growth models to the last section. Therefore, no research and development sector is modeled.

4.1 A descriptive growth model

We follow the approach by Solow [49] and Swan [52] and replace the consumption-saving decision of households by a stylized description of empirically ob-

servable aggregate behavior of households, i.e. the decisions of households are treated as a black box. While this approach is empirically justifiable, it leaves little room for policy analysis. Our discussion mainly follows Schenk–Hoppé and Schmalfuß [48].

In the descriptive growth model, we assume that the behavior of households is described by the consumption of a fraction $1 - s_t$ of the total output in each period. Moreover, we assume that households do not have disutility from work and inelastically supply their total endowment of labor.

We make the following specific assumptions on the model.

Assumption 4.1 *The production functions is given by*

$$F(K, L, z, a) = z \tilde{F}(K, aL) \quad (6)$$

where \tilde{F} is neoclassical and exhibits constant returns to scale. That is, technical progress is labor-augmenting (Harrod neutral) and the production shock enters multiplicatively.

The evolution of the efficient labor supply, $a_t L_t$, is given by $a_{t+1} L_{t+1} = (1 + n_t) a_t L_t$; and the exogenous variable $(z_t, n_t, \delta_t, s_t)$ is described by an ergodic process.

The assumption on the consumption behavior of households is compatible with the assumption that capital is irreversible, i.e. the output in each period can be consumed while the non-depreciated capital cannot be consumed.

Define the capital per efficient unit of labor $k_t = K_t / (a_t L_t)$, henceforth called capital intensity. Under assumption 4.1, (3) yields the following stochastic law for the capital intensity,

$$k_{t+1} = \frac{K_{t+1}}{a_{t+1} L_{t+1}} = \frac{(1 - \delta_t) K_t + s_t z_t \tilde{F}(K_t, a_t L_t)}{(1 + n_t) a_t L_t} = \frac{(1 - \delta_t) k_t + s_t z_t f(k_t)}{1 + n_t}$$

with $f(k) := \tilde{F}(k, 1)$. $f(k)$ is a neoclassical production function.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic dynamical system. Further, let δ , ξ , and n be random variables such that $\delta(\theta^t \omega)$ is the rate of depreciation, $\xi(\theta^t \omega) f(k_t)$ is the invested share of output, and $n(\theta^t \omega)$ is the rate of growth of efficient labor.

For a given initial state k_0 of the capital intensity, the stochastic evolution of the capital intensity is given by

$$k_{t+1} = \frac{(1 - \delta(\theta^t \omega)) k_t + \xi(\theta^t \omega) f(k_t)}{1 + n(\theta^t \omega)} \quad (7)$$

The random difference equation (7) is henceforth called the *stochastic Solow model*. It generates a random dynamical system on \mathbb{R}_+ under appropriate assumptions on the parameters.

Mirman [35, 36] studied the existence and uniqueness of Markov equilibria (i.e. stationary probability measures) in growth models of this type for Markovian perturbations. However, it is well-known that the dynamical behavior of random dynamical systems cannot be captured with this approach in general.

The next theorem shows that the dynamics and, in particular, the long-run behavior of all sample-paths of the stochastic Solow model is uniquely determined by a random fixed point. The result is due to Schenk–Hoppé and Schmalfuß [48]. The result is further applied in a study of the Golden Rule saving rate in Schenk–Hoppé [46].

Theorem 4.2 *Assume that $\delta(\omega) \in [\delta_{min}, \delta_{max}] \subset [0, 1]$, $n(\omega) \in [n_{min}, n_{max}] \subset] - 1, \infty[$, and $\xi(\omega) \in [\xi_{min}, \infty[\subset]0, \infty[$ with $\mathbb{E}\xi < \infty$. Assume further that f is non-negative, increasing, strictly concave, and continuously differentiable.*

Suppose that

- (i) $\delta_{max} + n_{max} > 0$;
- (ii) $0 \leq \lim_{k \rightarrow \infty} f'(k) < \frac{\delta_{max} + n_{max}}{\xi_{min}} < \lim_{k \rightarrow 0} f'(k) \leq \infty$; and
- (iii) $\mathbb{E} \log \frac{1 - \delta(\omega) + \xi(\omega)f'(\bar{k})}{1 + n(\omega)} < 0$,

where $\bar{k} := \bar{k}(\delta_{max}, n_{max}, \xi_{min})$ is the positive fixed point of the deterministic Solow model with respective parameters, i.e. (7) with $\delta(\omega) \equiv \delta_{max}$, $n(\omega) \equiv n_{max}$, and $\xi(\omega) \equiv \xi_{min}$. \bar{k} is well-defined and unique by the assumptions on f and conditions (i) and (ii).

Then there exists a unique positive random fixed point k^ for the stochastic Solow model (7). k^* is asymptotically stable, measurable with respect to the past, and globally attracting on \mathbb{R}_{++} , i.e. $|\varphi(t, \omega, k) - k^*(\theta^t \omega)| \rightarrow 0$ as $t \rightarrow \infty$ a.s. for all $k > 0$. Therefore, the long-run behavior of all sample-paths is uniquely determined by the random fixed point k^* .*

If $f(0) > 0$, then no condition on $\lim_{k \rightarrow 0} f'(k)$ is needed and k^ is even globally attracting on \mathbb{R}_+ .*

Condition (ii) of Theorem 4.2, so-called Inada condition, provides bounds on the marginal product of the technology as the capital intensity tends to zero resp. infinity. The bounds on the stochastic parameters and conditions (i) and (ii) ensure that the interval $[\bar{k}, \infty[$ is forward-invariant for the random dynamical system, i.e. $\varphi(t, \omega, k) \in [\bar{k}, \infty[$ for all $t \geq 0$ and all $k \geq \bar{k}$. Condition (iii) requires the generator to be contracting on average, i.e. in the mean. The main ingredient in the proof of this result is an extension of the Banach fixed point theorem for random dynamical systems due to Schmalfuß.

The main assertion of the above result is that, for any initial capital intensity k , the sample-path $\varphi(t, \omega, k)$ asymptotically moves jointly with the path of the random fixed point $k^*(\theta^t \omega)$, i.e. $\lim_{t \rightarrow \infty} |\varphi(t, \omega, k) - k^*(\theta^t \omega)| = 0$. In particular, the long-run behavior of the capital intensity is independent of the initial state. The convergence of two sample-paths is actually exponentially fast with rate given by (iii). The theorem further ensures that for any economy in which production is impossible without capital, i.e. $f(0) = 0$, the state of no capital is not a poverty trap. Any supply of outside capital, no matter how small, results

in positive capital intensities for all future times. Moreover, it can be proved that for an i.i.d. process $(\delta(\omega), n(\omega), \xi(\omega))$, the probability measure $\rho(B) := k^* \mathbb{P}(B)$, $B \in \mathcal{B}(\mathbb{R}_{++})$, is a (unique) Markov equilibrium.

Theorem 4.2 also renders the well-known result for the existence of a unique globally asymptotically stable fixed point in the deterministic Solow model, see e.g. Barro and Sala-i-Martin [8, Chap. 1].

It can further be proved that the random fixed point is tempered, this yields that the growth rate of the capital intensity, $\gamma_{k^*}(\omega) := (k^*(\theta\omega) - k^*(\omega))/k^*(\omega)$, is tempered and that, by stationarity of k^* , $\mathbb{E} \log(1 + \gamma_{k^*}(\omega)) = 0$, if $\log k^*$ is integrable. This result resembles the well-known result for the deterministic case in which capital per efficient unit of labor eventually is constant (and thus the growth rate is zero) due to the diminishing marginal product of the neoclassical production function. Of course, on the aggregate level, capital stock, consumption, and output grow perpetually. Notably, in the stochastic case the mean value of the growth rate $\mathbb{E} \gamma_{k^*}(\omega)$ can be distinct from zero.

In summary, Theorem 4.2 provides a full description of the dynamics of the stochastic Solow model, which is achieved by the application of random dynamical systems theory.

Example. To illustrate the applicability of the theorem, let us give an example with a standard production function. Using Jensen's inequality and the fact that $n(\omega) \geq n_{min}$, it is straightforward to prove that the contraction condition (iii) of Theorem 4.2 follows from

$$f'(\bar{k}(\delta_{max}, n_{max}, \xi_{min})) < \frac{\mathbb{E} \delta(\omega) + n_{min}}{\mathbb{E} \xi(\omega)} \quad (8)$$

Consider the Cobb-Douglas production function $f(k) = k^\alpha$ with $0 < \alpha < 1$. The Inada condition (ii) of Theorem 4.2 is satisfied for all α . Condition (8) (and thus condition (iii) of Theorem 4.2) is fulfilled, if

$$\alpha < \frac{\xi_{min}}{\mathbb{E} \xi} \frac{\mathbb{E} \delta(\omega) + n_{min}}{\delta_{max} + n_{max}} \quad (9)$$

This can be seen as follows. One has that $\bar{k} = ((\delta_{max} + n_{max})/\xi_{min})^{\frac{1}{\alpha-1}}$ and thus $f'(\bar{k}) = \alpha (\delta_{max} + n_{max})/\xi_{min}$. Inserting the last term into (8) immediately yields the above condition on α . If the stochastic variables are constant, then (9) reduces to the redundant condition $\alpha < 1$.

4.2 An overlapping generations model

The following model of a stochastic economy originates from the Diamond [19] overlapping generations model. The presentation follows Schenk-Hoppé [44] and Peter and Schenk-Hoppé [41]. Related work on the existence and uniqueness of rational expectations equilibria in a simpler framework is due to Wang [54]. Similar models without production, i.e. exchange economies with stochastic endowments, have been studied by Grandmont and Hildenbrand [28] and Spear and Srivastava [50] from the Markovian point of view.

Individuals are identical within as well as across time and live for two consecutive periods. Each young individual inelastically supplies one unit of labor and earns labor income w_t which she divides between first-period consumption c_t^1 and saving s_t ,

$$c_t^1 = w_t - s_t \quad (10)$$

In the second period of her life she retires and dissaves, i.e. she consumes the savings s_t plus the accrued interest,

$$c_{t+1}^2 = (1 - \delta + \tilde{r}_{t+1}) s_t \quad (11)$$

where $\delta \in [0, 1]$ is the rate of capital depreciation and \tilde{r}_{t+1} denotes the next period's real interest rate. Futures markets are incomplete, i.e. \tilde{r}_{t+1} is a random variable with realization in \mathbb{R}_+ .¹²

In this economy, only the young supply labor and only the old supply capital. Total consumption in (3) is given by $C_t \equiv L_t c_t^1 + L_{t-1} c_t^2$ where c_t^1 resp. c_t^2 is consumption per capita of the young resp. old generation in period t . L_t is the number of young individuals in period t as well as the number of old individuals in period $t + 1$

We make the following assumptions on the model.

Assumption 4.3 *The lifetime preferences of an individual born in period t are described by a time-additive, state-dependent expected utility function¹³*

$$u(c_t^1) + \int_{\mathbb{R}_+} v(c_{t+1}^2, \tilde{r}_{t+1}) \mu(d\tilde{r}_{t+1}) \quad (12)$$

where the functions $u(\cdot)$ and $v(\cdot, \tilde{r}_{t+1})$ (for all $\tilde{r}_{t+1} \in \mathbb{R}_+$) are twice continuously differentiable, strictly increasing, and strictly concave. μ describes the subjective expectations concerning the distribution of the next period's real interest rate.

The exogenous variable (L_t, z_t, a_t) is a Markov processes with values in \mathbb{R}_{++}^3 .

The behavior of households is described by the saving decision in the first period which maximizes the lifetime utility function (12) subject to the budget constraints (10) and (11), i.e.

$$s_t^* = \operatorname{argmax}_{0 \leq s_t \leq w_t} \left\{ u(w_t - s_t) + \int_{\mathbb{R}_+} v((1 - \delta + \tilde{r}_{t+1}) s_t, \tilde{r}_{t+1}) \mu(d\tilde{r}_{t+1}) \right\} \quad (13)$$

The optimal saving decision s_t^* is uniquely determined by the first-order condition

$$u_1(w_t - s_t^*) = \int_{\mathbb{R}_+} (1 - \delta + \tilde{r}_{t+1}) v_1((1 - \delta + \tilde{r}_{t+1}) s_t^*, \tilde{r}_{t+1}) \mu(d\tilde{r}_{t+1}) \quad (14)$$

¹²Note that wage and interest rate are normalized by the price of the consumption good in the respective period t . The young individual is uncertain about the future price of capital.

¹³The reader may consult e.g. Karni and Schmeidler [32] for a treatment of state-dependent expected utility theory. The dependence of utility on next period's real interest rate can be interpreted as money illusion.

assuming that the integrals on the right-hand side of (13) and (14) are finite for each fixed s_t resp. s_t^* . The saving function $s_t^* = s(w_t, \mu)$ depends only on the current income w_t of the young household and on the probability measure μ .

We suppose that the individual's saving decision is consistent with rational expectations. The solution of (14) determines the saving of each young individual which yields a total supply of capital $K_{t+1} = L_t s(w_t, \mu)$ in the next period $t + 1$; this relation can also be derived from (3). According to (4), the real interest rate r_{t+1} is given by the random variable $F_1(L_t s(w_t, \mu), L_{t+1}, z_{t+1}, a_{t+1})$ with distribution

$$F_1(L_t s(w_t, \mu), \cdot) P(L_t, z_t, a_t | \cdot) \quad (15)$$

where $P(L_t, z_t, a_t | \cdot)$ denotes the transition probability of the exogenous Markov process (L_t, z_t, a_t) .

Definition 4.4 *Let $(K_t, L_t, z_t, a_t) \in \mathbb{R}_+ \times \mathbb{R}_{++}^3$ be a current state of the economy. The saving behavior is consistent with rational expectations, if the subjective expectation $\mu \in \text{Prob}(\mathbb{R}_+)$ satisfies*

$$s(w_t, \mu) = s(w_t, F_1(L_t s(w_t, \mu), \cdot) P(L_t, z_t, a_t | \cdot)) \quad (16)$$

where $w_t = w(K_t, L_t, z_t, a_t)$ is the real wage.

A probability measure μ solving the functional equation (16) depends on the state of the economy in general. In Definition 4.4 it is required that the young individual makes her saving decision “as if” her subjective expectation μ coincides with the actual distribution of next period's real interest rate. If (16) possesses a solution for each state (K_t, L_t, z_t, a_t) , then the individual's saving behavior is consistent with rational expectations along each sample path of the capital stock

$$K_{t+1} = L_t s(w_t, \mu(K_t, L_t, z_t, a_t)) \quad (17)$$

The sequence of random variables K_{t+1} together with the solution of (16) is also called a rational, or self-fulfilling expectations equilibrium, cf. Grandmont [26].

Suppose there exists a solution to the functional equation (16) for each (K_t, L_t, z_t, a_t) . Then we have a complete specification of an overlapping generations economy with stochastic production, stochastic technical progress, and rational expectations. The law of motion is given by (17) together with the Markov process governing (L_t, z_t, a_t) .

4.2.1 A closed-form solution in the absence of technical progress

To further elaborate the above approach, we give an example in which a closed-form solution can be derived under the assumption of no technical progress. We follow the study of business cycles in a stochastic overlapping generations economy due to Peter and Schenk–Hoppé [41]. The assumption of absence of technical progress makes sense, for instance, when dealing with stylized business

cycle facts which are associated with detrended time-series of economic growth data.

Consider the production function $F(K_t, L_t, z_t, a_t) = A z_t L_t \log(1 + K_t/L_t)$ with $A > 1$. The output is independent of the state of technology. The per capita production function is therefore given by

$$f(k_t, z_t) = A z_t \log(1 + k_t) \quad (18)$$

The lifetime preferences of an individual born in period t are specified by (12) with $u(c) = \log c$, and $v(c, \tilde{r}_{t+1}) = (1 - \delta + \tilde{r}_{t+1}) \log c$.

Under these assumptions, (16) can be solved and the evolution of the capital intensity, which is derived from (17), is described by

$$k_{t+1} = \frac{s(w(k_t, z_t), R(w(k_t, z_t), z_t))}{1 + n} = \frac{R(w(k_t, z_t), z_t)}{1 + R(w(k_t, z_t), z_t)} \frac{w(k_t, z_t)}{1 + n} \quad (19)$$

where the mean-return-on-capital function is given by

$$R(w_t, z_t) = \frac{-b(w_t, z_t) + \sqrt{b(w_t, z_t)^2 + 4a(w_t)c(z_t)}}{2a(w_t)} \quad (20)$$

with $a(w_t) \equiv 1 + n + w_t$, $b(w_t, z_t) \equiv (1 + n)(1 - \mathbb{A}\mathbb{E}(z_{t+1} | z_t)) - (1 - \delta)a(w_t)$, $c(z_t) \equiv (1 + n)(\mathbb{A}\mathbb{E}(z_{t+1} | z_t) + 1 - \delta)$, $\mathbb{E}(z_{t+1} | z_t) \equiv \int_{\mathcal{Z}} z_{t+1} P(z_t, dz_{t+1})$, and $w_t \equiv w(k_t, z_t)$.

Peter and Schenk–Hoppé [41] provide an analytical study of the random dynamical system generated by (19) under the assumption of no population growth, $L_t \equiv \text{const.}$, full depreciation of capital, $\delta = 1$, and an i.i.d. technology shock with unity mean, i.e. $\mathbb{E}(z_{t+1} | z_t) = 1$. In this case, (20) becomes

$$k_{t+1} = \left[1 + \frac{2(1 + w(k_t, z_t))}{A - 1 + \sqrt{(A + 1)^2 + 4Aw(k_t, z_t)}} \right]^{-1} w(k_t, z_t) \quad (21)$$

where $w(k_t, z_t)$ is defined according to (5) and (18). For fixed z_t , the right-hand side of (21) is an S-shaped function of k_t . In [41], the authors prove existence of a stable random fixed point for bounded production shocks and sufficiently large A . The main economic result is that this model exhibits business cycle phenomena very much in line with empirically observed, so-called stylized, business cycle facts. Moreover, it is proved that the state of no capital is a poverty trap. Only sufficiently high outside investment can move the economy permanently away from the zero capital state.

The main advantage of this approach, compared to the analysis of real business cycle models, is that one avoids a linear approximation around some deterministic “steady state.” This “stochastic linearization” method is commonly employed in economic analysis albeit its validity has not yet been established rigorously.

4.2.2 Example with technical progress

The idea to decompose data on the GDP (gross domestic product) of countries into a trend and a cyclic component, which are thought of as long-term and short-term variations of GDP, has created two distinct fields of economics: economic growth theory and business cycle theory. However, the exclusive study of either phenomenon ignores any possible interdependence. Recent research attempts to reconcile these approaches. The following example contributes to this direction of research. In the model discussed here, the trend is attributable to the state of the variable a_t whereas the cyclic part of a time-series corresponds to variations in z_t . In contrast to the previous example, we allow for technical progress and therefore have to dispense with a closed-form solution. The model is due to Schenk–Hoppé [44].

The following assumptions on utility and production functions are standard in real business cycle theory except for allowing state-dependent preferences. The technology is described by the Cobb–Douglas production function

$$F(K_t, L_t, z_t, a_t) = z_t K_t^\alpha (a_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1 \quad (22)$$

As in the descriptive growth model, technical progress is labor-augmenting and the production shock enters multiplicatively. The lifetime preferences of an individual born in period t are specified by (12) with $u(c) = c^{1-\sigma}/(1-\sigma)$, $0 < \sigma \neq 1$, and $v(c, \tilde{r}_{t+1}) = \beta(1-\delta + \tilde{r}_{t+1})^\gamma u(c)$ with $\beta > 0$, and $\gamma \in \mathbb{R}$.

It is straightforward to check that the optimal saving decision, which is characterized by (14), is given by

$$s_t^* = s(w_t, \mu) = \frac{w_t}{1 + [\beta \mathbb{E}_\mu (1 - \delta + \tilde{r}_{t+1})^{1+\gamma-\sigma}]^{-1/\sigma}} \quad (23)$$

In order to simplify the implicit condition for rational saving behavior in Definition 4.4, we make the following

Assumption 4.5 *Let $\gamma = \sigma$.*

Under this assumption, the optimal saving function depends only on the current total income w_t , and on the subjective mean of the real net return on next period's capital $R^\mu \equiv 1 - \delta + \mathbb{E}^\mu \tilde{r}_{t+1}$. It turns out that equation (16), the condition for a saving behavior which is consistent with rational expectations, is satisfied if for a given state of the economy (K_t, L_t, z_t, a_t) , the mean R^μ of the subjective probability measure μ satisfies

$$R^\mu = 1 - \delta + \alpha \left(\frac{L_t w_t}{1 + (\beta R^\mu)^{-1/\sigma}} \right)^{\alpha-1} \mathbb{E}_{P(L_t, z_t, a_t|\cdot)} (z_{t+1} (a_{t+1} L_{t+1})^{1-\alpha}) \quad (24)$$

where $w_t = w(K_t, L_t, z_t, a_t)$ is real wage. It is proved in Schenk–Hoppé [44] that for each state of the economy $(K_t, L_t, z_t, a_t) \in \mathbb{R}_{++}^4$ there exists a unique solution to (24). Notably no assumption on the correlation of z_t and a_t has been made.

Despite the occurrence of the implicit condition for R^μ , we found that the stochastic overlapping generations economy with technical progress and production shocks possesses a description as an explicit model: (23) together with the unique solution to (24) and (17) defines a random dynamical system on \mathbb{R}_+ . Of course in further studies one has to resort to numerical methods. In Schenk–Hoppé [44], the above explicit model is applied to illustrate the shortcomings of standard detrending methods in real business cycle theory.

4.3 Optimal stochastic growth model

In the optimal stochastic growth model, an infinitely-lived representative agent makes a consumption-investment decision in each period. His decision is constrained by the evolution of the capital stock which is subject to production shocks. The optimal policy of the agent gives rise to a stochastic explicit model. Pioneering work for the one-sector model is due to Brock and Mirman [12]. We follow Montrucchio and Privileggi [39]. A corresponding result for the deterministic case is due to Mitra and Sorger [38].

Let z_t , $t \geq 0$, be a Markov process on some measurable space (Z, \mathcal{Z}) with transition probability $P(z_t, B)$. Let (Z^t, \mathcal{Z}^t) the measurable space associated to sequences of length t , and denote by $\mu^t(z_0, \cdot)$ the corresponding probability measure for the process started in z_0 . Assume that the state variable (e.g. the stock of different commodities) take values in some compact convex space $X \subset \mathbb{R}^m$. $(S, \mathcal{S}) := (X \times Z, \mathcal{B}(X) \otimes \mathcal{Z})$ denotes the state space of the system. The dynamic constraint is a measurable set $D \subset X \times X \times Z$ such that all z -sections D_z are convex. For each $(x, z) \in S$, denote the set of feasible actions by $\Gamma(x, z) = \{y \in X \mid (x, y, z) \in D\}$. For instance, $\Gamma(x, z)$ can be interpreted as the set of consumption bundles for given input x and production shock z . The return function $U : D \rightarrow \mathbb{R}$ is assumed to be measurable, bounded, and its z -sections $U_z : D_z \rightarrow \mathbb{R}$ are concave. The discount rate $\beta \in (0, 1)$.

A feasible plan from an initial state (x_0, z_0) is a value $\pi_0 \in X$ and a sequence of \mathcal{Z}^t -measurable functions $\pi_t : Z^t \rightarrow X$, $t \geq 0$ such that $\pi_t \in \Gamma(\pi_{t-1}, z_t)$, $\mu^t(z_0, \cdot)$ -a.s., for all $t \geq 1$. Let $\Pi(s_0)$ denote the set of all plans that are feasible from s_0 . $\Pi(s_0)$ is assumed to be non-empty throughout the following.

The optimization problem of the representative agent is given by

$$\sup_{\pi \in \Pi(s_0)} \left(U(x_0, \pi_0, z_0) + \mathbb{E} \sum_{t=1}^{\infty} \beta^t U(\pi_{t-1}, \pi_t, z_t) \right) \quad (25)$$

A measurable function $g : X \times Z \rightarrow X$, with $g(x, z) \in \Gamma(x, z)$ is called an *optimal policy*, if for any $s_0 = (x_0, z_0) \in S$ the plan $\pi^* = \{\pi_t^* \mid t \geq 0\}$ generated by g (i.e. $\pi_0^* = g(x_0, z_0)$ and $\pi_{t+1}^* = g(\pi_t^*, z_{t+1})$ for $t > 0$) attains the supremum in (25).

An optimal policy g defines a random difference equation on the space X , where the perturbation is given by the Markov process z_t . Therefore, it generates a random dynamical system on X .

Montrucchio and Privileggi [39] prove the indeterminacy of optimal policies, i.e. “anything goes.” They show that many policy functions can be rationalized by an optimization problem of the form (25). Let $g : S \rightarrow X$ be any function such that $(x, g(x, z), z) \in D$ (spaces as introduced above) and, further (1) for each $z \in Z$, $g(\cdot, z)$ is differentiable over an open set containing X and there is a finite constant k_1 with $k_1 = \sup_{(x,z) \in S} \|D_1 g(x, z)\|$, and (2) there is a finite constant k_2 such that $\|D_1 g(x, z) - D_1 g(y, z)\| \leq k_2 \|x - y\|$, for all $x, y \in X$. Define $k_0 = \max_{x,y \in X} \|x - y\|$. Then for any discount factor $0 < \beta < (k_1 + \sqrt{k_0 k_2})^{-2}$, there exists a function U_β with the above properties such that g is an optimal policy for the corresponding optimization problem (25).

5 On future research

Dynamic economic modeling comprises optimizing behavior of agents and (temporary) equilibrium conditions. This interplay of implicit conditions raises difficulties in finding a representation as an explicit model. However, if the modeling is guided by the framework of random dynamical systems at the beginning of any abstraction and not as a mere ex-post effort, explicit models can be achieved, as illustrated by the examples. Here we showed exemplarily the usefulness of this approach in the (global) stability analysis of random fixed points in stochastic growth models. The results on the dynamic long-run behavior are more general and complete than those obtained using Markov equilibria. However, economic modeling with random dynamical systems is apparently in its infancy. Supposedly this is due to the circumstances that (1) economists are not sufficiently familiar with the required mathematics, (2) mathematicians have not enough experience in economic modeling, and (3) many tools for the analysis of such models are the subject of recent research. The specific problems I will draw attention to are thus subjects of interdisciplinary research.

Having dealt with stochastic growth models, I first mention the problem of determining explicit *endogenous* growth models in which technical progress is the result of economic activities such as research and development. I am convinced that for some aspects an evolutionary approach is particularly appropriate. For instance, while the number of innovations is strongly correlated with the amount of investment, the innovation as such is random and so is its impact on existent markets. Other aspects certainly call for a modeling via random dynamical systems in connection with control theory; an example is Conway and Schenk–Hoppé’s [14] study of endogenous growth and ecological policy.

The dynamics in the stochastic optimal growth model has been thoroughly analyzed from a Markovian point of view, see e.g. Brock and Mirman [12], and Hopenhayn and Prescott [30]. Moreover, Duffie et al. [20] studied the existence of Markov equilibria in a more general class of models. However, a corresponding analysis in the random dynamical systems framework is still lacking.

A different subject concerns the problem of coexistence of strategic and competitive behavior in an economy. In the framework of an optimal stochastic growth model with two goods and two countries, Mirman and Schenk–

Hoppé [37] study the impact of financial markets on investment behavior. In their model, consumers are price-takers while a social planner in each country strategically decides on aggregate consumption for the domestic good. This presents certainly only a first step.

Let me turn to the mathematical issues. In random dynamical systems theory, the perturbation is a stationary (or even ergodic) process. In economics, however, it is an empirical fact that many randomly fluctuating variables are nonstationary. Providing tools for this class of models is certainly a formidable task for mathematicians. The main reward is that the study of nonlinear phenomena with nonstationary fluctuations becomes possible without resorting to the application of filters or other approximations (which is often not justified on grounds of economic theory). As a first step, Schenk–Hoppé [47], extending work due to Schmalfuß, proved a Banach fixed point theorem for random dynamical systems with nonstationary perturbations.

Let me close this incomplete list by mentioning set-valued random dynamical systems which are studied by Evstigneev and various coauthors, see e.g. Evstigneev and Taksar [21] (and references therein). In economic contexts, such multivalued operators have been studied in connection with stochastic versions of the von Neumann-Gale model of economic growth and have recently been applied in mathematical finance. However, this approach seems to be suitable for many more interesting problems in finance and growth.

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