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Abstract

This paper studies the validity of the Friedman rule in a search model with divisible money and divisible goods where the terms of trades are determined endogenously. We show that ex post bargaining generates a holdup problem similar to the one emphasized in the labour-market literature. Buyers cannot obtain the full return that an additional unit of money provides to the match, which makes the purchasing power of money inefficiently low in equilibrium. Consequently, even though the Friedman rule maximizes the purchasing power of money, it fails to generate the first-best allocation of resources unless buyers have all the bargaining power.

Keywords : Money, Search, Friedman rule

J.E.L. classification : E00, D83, E52

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1 Introduction

This paper studies the validity of the Friedman rule in search-theoretic models with fully divisible money that abstract from nondegenerate distributions of money holdings. In the absence of distorting taxes and search externalities, these models generate contradictory conclusions regarding the ability of the Friedman rule to guarantee an efficient allocation of resources. In Shi (1997, 1999, 2001) and in Berentsen and Rocheteau (2000) the Friedman rule generates the first-best allocation of resources, whereas in Rauch (2000) and Lagos and Wright (2001) the Friedman rule only generates a second-best allocation unless buyers have all the bargaining power.[1]

In order to shed light on what the Friedman rule can accomplish in a search environment with fully divisible money, we modify Shi's (2001) framework in the following way. As in Shi (2001), we assume that the economy is populated by households consisting of a large number of members that pool together their money holdings after trading, which renders the distribution of money holdings degenerate.[2] In contrast to Shi (2001), but as in Rauch (2000), we assume that bargaining strategies are determined ex post (after the matches have been formed). In comparison with Rauch (2000), our analysis has three distinctive features. First, we consider alternating offer bargaining games to determine the quantities traded in each match. The sequential bargaining procedure, in contrast to the axiomatic approach, reveals how the agents' behavior in out-of-equilibrium matches prevents the economy from attaining the first best even under the Friedman rule and why the purchasing power of money is inefficiently low. Second, we consider different bargaining powers for buyers and sellers, which allows us to analyze how the purchasing power of money depends on the bargaining power of the buyers. Third, we allow for money hoarding by letting the households choose the amount of money their members carry to the search market. This eliminates monetary equilibria, where the gross growth rate of the money supply is smaller than the discount factor.

[1] We restrict our attention to search models with fully divisible money that abstract from distributional issues because it is well known that with nondegenerate distributions expansionary monetary policies can be beneficial (Deviatov and Wallace 2001; Molico 1997). For a survey on the optimum quantity of money, see Woodford (1990).

[2] The large-household assumption, extending a similar one in Lucas (1990), avoids difficulties that arise in models with a nondegenerate distribution of money holdings, and so allows for a tractable analysis of money growth and inflation. In a series of papers, Shi (1997, 1999, 2001) has explored the use of this assumption in search monetary models.

The following results emerge from our paper. First, we show that the frameworks of Shi (1997, 1999, 2001) and Lagos and Wright (2001) are equivalent with respect to their closed-form solutions. This result is noteworthy, because Lagos and Wright (2001) and Shi (1999, 2001) use different formalization strategies to obtain a degenerate distribution of money holdings.[3] Second, and more importantly, we show that the different results in the literature are due to the presence of one term in the envelope condition, which reflects how agents expect other agents to react in response to an out-of-equilibrium change of their money holdings.

The different results with respect to the efficiency of the monetary equilibria under the Friedman rule reported above are due to the different timing of the decisions of the models. All models have in common that in each period the households make two decisions: they choose with how much money their members enter the search market, and what trading strategies their members apply in search market. In Shi (1999, 2001), households first simultaneously endow their members with money *and* instruct them which offers to make and which offers to accept (the trading strategies) in the search market. Then, the household members are matched and they carry out the trading strategies. Because the households choose the trading strategies and the money holdings of their members simultaneously before the members are matched, when a buyer and a seller meet and the buyer's money holdings differs from what is expected in equilibrium, the seller's trading strategy is left unchanged.[4]

In contrast, in Rauch (2000) and in Lagos and Wright (2001), the bargaining strategies are determined ex post (after the members are matched). That is, all households first simultaneously endow their members with money. Then, the members enter the search market, where they are matched. Once matched, the households choose the bargaining strategies for each match. Consequently, the bargaining strategies are match-specific, i.e., they take into account the money holdings of the buyer in a match.[5] In the symmetric

[3] Lagos and Wright (2001) assume that, after the random-matching market closes, a Walrasian market opens in which agents can trade a homogeneous good for money. They show that if the traders have quasi-linear preferences for this good in equilibrium all agents leave the Walrasian market with the same amount of money.

[4] In this sense, the sellers are like vending machines that are programmed in advance to accept any offer that gives them their reservation value.

[5] This decision structure is particular natural in Lagos and Wright (2001), because their economy is populated by individuals and not by large families as in Rauch (2000) or Shi (2001). These individuals cannot commit to a trading strategy ex ante so that it is natural to assume that the agents determine

equilibrium, the money holdings of all agents in the market are equal (the distribution of money holdings is degenerate), and so all agents make the same equilibrium offers and have the same reservation values. Nevertheless, a seller's offer and reservation value in an out-of-equilibrium match differ from the offer and the reservation value in equilibrium.

The different reservation values in out-of-equilibrium matches are the reason why in Rauch (2000) and in Lagos and Wright (2001) the economy cannot attain the first best even under the Friedman rule. Because of this change, a buyer who brings an additional unit of money into a match cannot extract the whole surplus that this unit provides to the match, which lowers the marginal value of money. In this sense, money is an asset whose holder — the buyer — is not able to capture its entire return. As pointed out by Acemoglu and Shimer (1999), this holdup inefficiency is very common in search models with ex post bargaining. Because the Friedman rule only compensates for the time impatience of agents, correction of this holdup inefficiency requires a *higher* rate of deflation than the rate of time preference, which is inconsistent with the existence of a monetary equilibrium.

The paper is organized as follows. Section 2 presents the model and its main assumptions. Section 3 derives the symmetric equilibrium. The Friedman rule is discussed in Section 4. Section 5 concludes.

2 The model

2.1 The environment

The environment is similar to that of Shi (1999, 2001). Time is discrete. The economy consists of a continuum of infinite-lived households that specialize in consumption and production. There are $H \geq 3$ types of goods and H types of households. Households are uniformly distributed among types. A household of type k produces good k and consumes good $k+1$ (modulo H). Denote by $z \equiv \frac{1}{H}$ the single-coincidence-of-wants probability.

Each household consists of a continuum of members, normalized to one, who carry out different tasks but regard the household's utility as the common objective. We focus on a representative household, which we call household h. Decision variables of household h are denoted by lowercase letters. Capital letters denote other households' variables.

their bargaining strategies only after they have been matched. In contrast, the large household construct acts as a commitment device because the household members are programmed to carry out the trading instructions that have been given to them by their households before the matches have been formed.

Furthermore, variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1.

At the beginning of each period, household h holds m units of money. These m units of money can be either hoarded or distributed across members in order to be spent on the search market. Let $y \leq m$ be the number of units that each member carries in a match.[6] Once members are matched, the household instructs them on how to bargain. Since goods and money are perfectly divisible, agents can exchange any quantity of money and goods they wish, provided that the traded quantity of money does not exceed the money holdings of the buyer in the match. After trading, household members consume the acquired goods and, then, they return home, where they pool together their receipts of money. At the end of a period, each household receives a lump-sum transfer of money τ, which can be negative. The gross growth rate of the money supply is $\gamma = M_{+1}/M$, where M is the money supply per household in period t.

For each member of household h, consumption of q units of good $h+1$ provides utility $u(q)$, where $u(.)$ is a twice continuously differentiable function with $u(0) = 0$, $u'(q) > 0$, $u'(0) = \infty$, and $u''(q) < 0$. Production of q units of good h provides disutility $c(q)$, where $c(.)$ is a twice continuously differentiable function with $c(0) = 0$, $c'(q) > 0$, $c'(0) = 0$, and $c''(q) > 0$. We assume that there exists $q^* < +\infty$ that satisfies $u'(q^*) = c'(q^*)$, and we add the following restriction:[7]

$$\lim_{q \to 0} \frac{[u'(q)]^2}{\left\{c''(q) - \frac{u''(q)}{u'(q)} c'(q)\right\} [u(q) - c(q)]} = +\infty \tag{1}$$

All through the paper we restrict our attention to values of q such that $u(q) - c(q) \geq 0$. Finally, the utility of the household is defined as the discounted sum of the consumption utilities of all its members minus their production costs. The discount factor for the household is $\beta \in (0, 1)$.

If $V(m)$ denotes the lifetime expected utility of an household endowed with m units of money, the marginal value of money is $\omega = \beta V'(m_{+1})$. Because the wealth of the household

[6] In search models with divisible money it is usualy assumed that agents bring all their money holdings into the matches, i.e., $y = m$. By introducing an explicit hoarding decision (choice of y), we show that no monetary equilibrium exists if the rate of growth of the money supply is smaller than what is prescribed by the Friedman rule ($\gamma < \beta$).

[7] This restriction is a sufficient condition to guarantee the existence of a monetary equilibrium. It is satisfied, for example, for $u(q) = q^a$ and $c(q) = q^b$ with $0 < a < 1 < b$.

is private information, we assume that the marginal value of money of a household is not observable too. Nevertheless, in a symmetric equilibrium all households know that they are equal, and consequently it is reasonable to assume that in the bargaining they attribute to each other the value Ω that prevails in equilibrium. In contrast, there is no obvious way in dealing with the issue of which value for the marginal utility of money h's partners will attribute to h if h deviates from its equilibrium strategy, for instance by accumulating more money. Throughout the paper we assume that in out-of-equilibrium matches the bargaining partners still attribute to each other the value Ω that prevails in equilibrium.[8]

2.2 Bargaining

Each agent in the market is endowed with money and with a production opportunity, and therefore each agent is a potential buyer and a potential seller. When two agents meet, the match is either a single-coincidence meeting, where one agent (the buyer) is a consumer of the good produced by the other agent (the seller), or a no-coincidence meeting. Terms of trade are determined in alternating-offer bargaining games. In contrast to Shi (1999, 2001), households determine their bargaining strategies ex post (after the matches have been formed). Consequently, the bargaining strategies take into account the specific level of money holdings of the trading partners of their members in the matches. For this to be feasible, we assume that in a match the level of money holdings of each trader is common knowledge.

Without loss of generality, we consider the bargaining between agent i, who is a representative member of household h, and a randomly chosen agent of another household, whom we will call agent j. Agent i's decision variables are denoted by lowercase letters, whereas decision variables of agent j are denoted by capital letters. Each period is divided into a large number of subperiods of length Δ. If, in a given subperiod, it is agent i's turn to make an offer and if agent j has rejected the offer, in the following subperiod agent j makes a counteroffer. There is an exogenous risk of a breakdown of the negotiation each time an offer has been rejected. This breakdown risk differs for sellers and buyers. If the seller has rejected the buyer offer, the breakdown probability is $\theta\Delta$ with $0 < \theta \leq 1$. If the buyer has rejected the seller offer, the breakdown probability is $(1-\theta)\,\Delta$. Because

[8] This assumption is innocuous under symmetric Nash bargaining (see Rauch (2000)). It has also no consequences in the framework of Lagos and Wright (2001), because they assume that agents can trade money at a given price ω.

the length of a subperiod is small, $\theta\Delta$ and $(1-\theta)\Delta$ are assumed to be smaller than one. Finally, we assume that members of the representative household h make always the first offer. This simplifies the exposition, but does not affect the results, because we will consider the bargaining game when Δ goes to zero, where the first-mover advantage vanishes.

Assume first that agent i is the buyer. Then agent i proposes the offer $\left(q^b, x^b\right)$, where q^b is the quantity of goods produced by his partner in exchange for x^b units of money. If seller j accepts the offer, the acquired money x^b will add to j's household at the beginning of the next period, whose value today is Ωx^b.[9] Any optimal offer must make seller j indifferent between accepting and rejecting the offer:

$$-c(q^b) + x^b\Omega = R^s \tag{2}$$

where R^s is the reservation value of seller j.

Assume now that agent i is the seller. To be optimal, seller i's offer (q^s, x^s) must satisfy

$$u(q^s) - x^s\Omega = R^b \tag{3}$$

where R^b is the reservation value of buyer j.

The reservation values of sellers and buyers are endogenous and satisfy

$$R^s = (1-\theta\Delta)\left[-c(Q^s) + X^s\Omega\right] \tag{4}$$

$$R^b = (1-(1-\theta)\Delta)\left[u(Q^b) - X^b\Omega\right] \tag{5}$$

According to (4), if with probability $1-\theta\Delta$ there is no breakdown of the negotiation after a seller has rejected a buyer offer, the seller makes the counteroffer (Q^s, X^s). The reservation value of a buyer (5) has a similar interpretation.

Denote by R_y^s the partial derivative of seller j's reservation value with respect to buyer i's money holdings y. From (4), it satisfies

$$R_y^s = (1-\theta\Delta)\left[-c'(Q^s)\frac{\partial Q^s}{\partial y} + \Omega\frac{\partial X^s}{\partial y}\right] \tag{6}$$

[9] To see why, suppose that the measure of a member is μ. Then for the household, the value of x additional units of money received by a member is $\beta\left[V\left(m_{+1}+x\mu\right)-V(m_{+1})\right]$. To express the value of x additional units of money for a member, we must multiply this quantity by the scale factor $1/\mu$. Because members are atomistic, we let $\mu\to 0$ to get $\lim_{\mu\to 0}\beta[V\left(m_{+1}+x\mu\right)-V(m_{+1})]/\mu = x\beta V'(m_{+1}) = x\omega$. Thus, from the point of view of the household, $x\omega$ is a member's indirect utility of receiving x units of money in a match.

We will see that if the constraint on money holdings of buyer i binds, then $R_y^s > 0$. Thus, if a buyer arrives with one unit of money more than what is expected in equilibrium, the seller increases his reservation value. In contrast, if agent i is the seller in a match, the reservation value of agent j is independent of i's money holdings, because agent j is the buyer. Hence, $R_y^b = 0$.

Each household member is constrained by the level of his money holdings. Accordingly,

$$x^b \leq y \tag{7}$$

Moreover, a seller of household h cannot ask for more money than what a buyer of another household has, which implies that

$$x^s \leq Y \tag{8}$$

3 Symmetric Monetary Equilibrium

Now we can describe household h's choice problem. We have already placed the household h in a symmetric environment, where all his trading partners hold the same amount of money Y. Denote by $\lambda \in \mathbb{R}^+$ the Lagrange multiplier associated with constraint (7), by $\pi \in \mathbb{R}^+$ the Lagrange multiplier associated with constraint (8), and by $\phi \in \mathbb{R}^+$ the Lagrange multiplier associated with the constraint $y \leq m$. Taking the bargaining strategies of other households and the distribution of money holdings as given, in each period the household chooses $\left(m_{+1}, y, q^b, x^b, q^s, x^s\right)$ to solve the following dynamic programming problem:

$$\begin{aligned} V(m) = \max_{q^b,q^s,x^b,x^s,y,m_{+1}} & \left[zu(q^b) - zc(q^s) + z\lambda\left(y - x^b\right)\right. \\ & \left. + z\pi\left(Y - x^s\right) + \phi\left(m - y\right) + \beta V(m_{+1})\right] \end{aligned} \tag{9}$$

subject to the constraints (2), (3), and

$$m_{+1} - m = \tau + zx^s - zx^b \tag{10}$$

The first two terms in (9) are the utility of consumption and the disutility of production. Equality (10) describes the law of motion of the household's money balances. The first term on the right-hand side is the lump-sum transfer of currency that the household receives each period. The second term specifies sellers' money receipts when selling goods, and the third term specifies buyers' expenses when spending money for goods.

3.1 Bargaining outcome

The offers $\left(q^b, x^b\right)$ and (q^s, x^s) satisfy the following conditions:

$$u'(q^b) = \frac{\lambda + \omega}{\Omega} c'(q^b) \tag{11}$$

$$c'(q^s) = \frac{\omega - \pi}{\Omega} u'(q^s) \tag{12}$$

$$\lambda\left(x^b - m\right) = 0 \tag{13}$$

$$\pi\left(x^s - M\right) = 0 \tag{14}$$

Equations (11) and (12) are the first-order conditions with respect to q^b and q^s, respectively. Equations (13) and (14) are the Kuhn-Tucker conditions for the inequalities (7) and (8), respectively. Note that the first-best allocation requires $q = q^*$, where q^* satisfies $u'(q^*) = c'(q^*)$. This condition maximizes the total surplus in each match and the utility of the representative household.

Let us first consider the bargaining solution in a single-coincidence meeting between a buyer i from household h, who holds y units of money, and a seller j from some other household. Assume further that both agents attribute to his opponent in the bargaining game the same economy-wide average value Ω for the marginal utility of money, which is the common value for all households at the symmetric monetary equilibrium that we will focus on later. Then, if we let $\Delta \to 0$, and if the constraint (7) on i's money holdings is binding ($\lambda > 0$), we have $X^s = x^b = y$ and $Q^s = q^b = q\left(y\right)$. From (2)-(5), we obtain:

$$\frac{-c(q^b) + c(Q^s)}{u(Q^s) - u(q^b)} = \frac{\theta\left[-c(Q^s) + y\Omega\right]}{(1-\theta)\left[u(q^b) - y\Omega\right]}$$

Using the fact that q^b approaches Q^s as Δ goes to zero, one can show that $q\left(y\right)$ satisfies:[10]

$$\frac{c'(q)}{u'(q)} = \frac{\theta\left[-c(q) + y\Omega\right]}{(1-\theta)\left[u(q) - y\Omega\right]} \tag{15}$$

From (15), one can check that $q(y) < q^*$.

If the constraint on i's money holdings is not binding ($\lambda = 0$), then $Q^s = q^b = q^*$, and $X^s = x^b = x \leq y$, where x satisfies

$$u(q^*) - x\Omega = \theta\left[u(q^*) - c(q^*)\right] \tag{16}$$

[10] For more details, see Shi (2001) and Berentsen and Rocheteau (2000). Note that given the real value of money holdings, $y\Omega$, the terms of trade (q, x) given by (15) and (16) maximize the asymmetric Nash product $\left[u(q) - x\Omega\right]^{\theta}\left[-c(q) + x\Omega\right]^{1-\theta}$ subject to $x\Omega \leq y\Omega$.

Let us next consider the bargaining solution in a single-coincidence meeting between a seller i from household h and a buyer j from some other household, who holds Y units of money. Then, if we let $\Delta \to 0$, and if the constraint on j's money holdings is binding ($\pi > 0$), then $X^b = x^s = Y$ and $Q^b = q^s = q(Y) < q^*$, where $q(Y)$ satisfies (15) when y in equation (15) is replaced by Y. If the constraint on j's money holdings is not binding ($\pi = 0$), then $q^s = Q^b = q^*$, and $x^s = X^b = x \leq Y$, where x satisfies (16).

Thus, in a match between a buyer of household h and a seller of some other household, if the constraint on the money holdings of the buyer is binding, then the bargaining solution $(q(y), x(y))$ is a function of the level of the buyer's money holdings in the match y. In contrast, in a match between a seller of household h and a buyer of some other household, if the constraint on the money holdings of the buyer is binding, then the bargaining solution $(q(Y), x(Y))$ is independent of the seller's money holdings in the match y.

3.2 Incentive to spend money

To determine the amount of money the household disburses to its members, differentiate (9) with respect to y to get

$$z(\lambda + \omega)\left(1 - \frac{R_y^s}{\Omega}\right) - \phi - \omega z \leq 0 \tag{17}$$

In a monetary equilibrium $y > 0$, and consequently (17) holds with equality. Because the Lagrange multiplier ϕ is non-negative, in a monetary equilibrium equality (17) implies that

$$(\lambda + \omega)\left(1 - \frac{R_y^s}{\Omega}\right) \geq \omega \tag{18}$$

From (6), when Δ goes to zero, the change in the seller's reservation value R_y^s is given by

$$R_y^s = -c'(Q^s)\frac{\partial Q^s}{\partial y} + \frac{\partial X^s}{\partial y}\omega$$

We will demonstrate latter that in a monetary equilibrium the constraints (7) and (8) are in fact binding, which implies that $(Q^s, X^s) = (q(y), y)$, where $q(y)$ satisfies (15).[11] In this case

$$R_y^s = -c'(q)q'(y) + \omega \tag{19}$$

[11] If they were not binding, we would have $(Q^s, X^s) = (q^*, x)$, where q^* is the efficient quantity and $x < y$. Consequently, $R_y^s = 0$. Note also that R_y^s is not continuous at $q = q^*$. We therefore adopt the convention that R_y^s is equal to its left derivative at this point, that is, $R_y^s\big|_{q=q^*} = \lim_{q \to q^{*-}} R_y^s$.

Replace $\lambda + \omega$ in (18) by its expression given in (11), and use (19) to get

$$u'(q)q'(y) - \omega \geq 0 \tag{20}$$

Inequality (20) states that the consumption utility the household gets from spending its last unit of money, $u'(q)q'(y)$, must not be smaller than the indirect utility of hoarding the money, ω. Note that the left-hand side of (20) is a buyer's marginal surplus S_B' from spending an additional unit of money, where $S_B = u(q) - \omega y$ is the surplus of a buyer in a match. Thus, inequality (18) states that the household is only willing to marginally increase the amount of money that its members carry in matches if it does not decrease a buyer's surplus in a match.

To derive $q'(y)$, differentiate (15) with respect to y to get[12]

$$\frac{1}{\omega}q'(y) = \frac{c'(q)(1-\theta) + u'(q)\theta}{c''(q)(1-\theta)\left[u(q) - y\omega\right] + c'(q)u'(q) - u''(q)\theta\left[-c(q) + y\omega\right]} \tag{21}$$

From (15), the surpluses of buyers and sellers satisfy

$$\begin{aligned} u(q) - y\omega &= \Theta(q)\left[u(q) - c(q)\right] && (22) \\ -c(q) + y\omega &= (1 - \Theta(q))\left[u(q) - c(q)\right] && (23) \end{aligned}$$

where $\Theta(q) = \frac{\theta u'(q)}{\theta u'(q) + (1-\theta)c'(q)}$ is a buyer's share of the total surplus in a match. Note that $\Theta(q) > \theta$ if the trade is constrained by the buyer's money holdings. Thus, the division of the total surplus is determined by how severely the trade is constrained by the buyer's real money balance, and hence influenced by monetary policy. When money growth obeys the Friedman rule, the money constraint does not bind and so $\Theta(q) = \theta$.

Combining (21), (22), and (23), we obtain

$$\frac{1}{\omega}q'(y) = \frac{\left[c'(q)(1-\theta) + u'(q)\theta\right]^2}{(1-\theta)\theta\left[c''(q)u'(q) - u''(q)c'(q)\right]\left[u(q) - c(q)\right] + c'(q)u'(q)\left[\theta u'(q) + (1-\theta)c'(q)\right]} \tag{24}$$

From (19) and (24) we get

$$\frac{R_y^s}{\omega} = (1-\theta)\varphi(q,\theta) \tag{25}$$

where

$$\varphi(q,\theta) = \frac{c'(q)\left[c'(q)(1-\theta) + u'(q)\theta\right]\left[u'(q) - c'(q)\right] + \theta\left[c''(q)u'(q) - u''(q)c'(q)\right]\left[u(q) - c(q)\right]}{(1-\theta)\theta\left[c''(q)u'(q) - u''(q)c'(q)\right]\left[u(q) - c(q)\right] + c'(q)u'(q)\left[\theta u'(q) + (1-\theta)c'(q)\right]}$$

[12] To compute this derivative, we use the assumption discussed above that in the bargaining the traders attribute to each other the economy-wide average value Ω for the indirect marginal utility of money.

The quantity R_y^s/ω can be interpreted as a corrected measure of the seller's bargaining power. At the symmetric equilibrium $\omega = \Omega$. Replacing R_y^s/ω in (18) by its expression given in (25) and λ/ω by its expression given in (11), we obtain

$$u'(q)\left[u'(q) - c'(q)\right]\left[\theta u'(q) + (1-\theta)c'(q)\right] \geq (1-\theta)\left[c''(q)u'(q) - u''(q)c'(q)\right]\left[u(q) - c(q)\right] \tag{26}$$

Inequality (26) gives all the pairs (θ, q) for which .the households wish to spend part or all of their money units $(y > 0)$. If $q > q^*$, then $u'(q) - c'(q) < 0$, and consequently equation (26) cannot hold for any value of θ. If we denote by $\widehat{q}$ the maximum value of q satisfying (26) and $u(q) - c(q) \geq 0$, it can be shown that $\widehat{q} < q^*$ for all $\theta < 1$.

3.3 The real value of money

Differentiating (9) with respect to m gives the following envelope condition:

$$\frac{\omega_{-1}}{\beta} = \phi + \omega \tag{27}$$

Using the fact that in a monetary equilibrium (17) holds with equality, (27) can be rewritten as

$$\frac{\omega_{-1}}{\beta} = z\left(\lambda + \omega\right)\left(1 - \frac{R_y^s}{\Omega}\right) + (1-z)\omega \tag{28}$$

Replacing λ by its expression given in (11) and taking into account that in a symmetric equilibrium $\omega = \Omega$ and $q^b = q$, the envelope condition can be rewritten to get

$$\omega_{-1} = \beta\left\{z\omega\frac{u'(q)}{c'(q)}\left(1 - \frac{R_y^s}{\omega}\right) + (1-z)\omega\right\} \tag{29}$$

This equation can be interpreted in the same way as an asset pricing equation. The left-hand side of (29) is the value in terms of utility of an additional unit of money at the end of the previous period. The right-hand side is the discounted value of holding this unit in the current period before the traders are matched. In the current period, with probability z a member is in a match where he can buy consumption goods. He buys the goods if their consumption utility $(1 - R_y^s/\omega)[\omega/c'(q)]u'(q)$ is larger than the indirect utility of hoarding the money ω. With probability $1 - z$, a member has no opportunity to spend it, and consequently it is saved, which yields indirect utility ω. The term R_y^s/ω can be interpreted as a markup imposed by the seller.

By using $\gamma = m/m_{-1}$, the envelope conditions (29) can be transformed to display the evolution of the real value of money holdings $m\omega$:[13]

$$\frac{(m\omega)_{-1}}{m\omega} = \frac{\beta}{\gamma}\left\{z\frac{u'(q)}{c'(q)}\left(1-\frac{R_y^s}{\omega}\right)+(1-z)\right\} \tag{30}$$

Note that $\phi \geq 0$ is equivalent to $\frac{u'(q)}{c'(q)}\left(1-\frac{R_y^s}{\omega}\right) \geq 1$. As a consequence, if $\gamma < \beta$ there exists no steady-state equilibrium. Indeed, if $\gamma < \beta$, the right-hand side of (30) is strictly greater than one, and consequently $(m\omega)_{-1}/m\omega \geq \beta/\gamma > 1$, which implies that the real value of money converges to 0.

In the following, we will focus our attention on equilibria, where the real value of money holdings $(m\omega)$ is stationary. Furthermore, because there is no equilibrium when $\gamma < \beta$ and a continuum of equilibria with money hoarding when $\gamma = \beta$, we restrict the rate of growth of the money supply to be larger than β. In a steady-state monetary equilibrium, the real value of money holdings is constant, which, from (15), implies that q is constant too. Hence, (30) can be written as follows:

$$\frac{\gamma-\beta}{z\beta} = \frac{u'(q)}{c'(q)}\left(1-\frac{R_y^s}{\Omega}\right)-1 \tag{31}$$

The envelope conditions in Berentsen and Rocheteau (2000), Lagos and Wright (2001), Rauch (2000), and Shi (2001) differ only in the specification of the holdup term R_y^s in (31). As discussed in the introduction, in Berentsen and Rocheteau (2000) and in Shi (2001), the households at the beginning of the period choose simultaneously the trading strategies and the level of money holdings of their members before the matches are formed. Thus, when they choose their money holdings they take the trading strategies and in particular the reservation values of all other households as given. Consequently, $R_y^s = 0$ so that the envelope condition (31) reduces to

$$\frac{\gamma-\beta}{z\beta} = \frac{u'(q)}{c'(q)}-1 \tag{32}$$

[13]Equation (30) implies that there are trajectories where $m\omega$ diverges to infinity. For example, suppose that $[u'(q)/c'(q)]\left(1-R_y^s/\omega\right)=1$ and $\gamma > \beta$. The envelope condition (30) reduces to $(m\omega)_{-1}/m\omega = \beta/\gamma$, and the real value of money holdings diverges to infinity. In such equilibria, members trade the quantity $\widehat{q}$ that satisfies $[u'(q)/c'(q)]\left(1-R_y^s/\omega\right)=1$, and each household hoards an ever increasing amount of money. In the same way, if $\gamma = \beta$, there exists a continuum of steady-state monetary equilibria with identical terms of trade and such that $[u'(q)/c'(q)]\left(1-R_y^s/\omega\right)=1$, i.e., $q=\widehat{q}$. These equilibria only differ in their stationary value of $m\omega$.

When the trading strategies are determined after the members are matched as in Lagos and Wright (2001) and Rauch (2000), we obtain from (25) and (31)

$$\frac{\gamma - \beta}{z\beta} = \Psi(q) - 1 \tag{33}$$

where

$$\Psi(q) = \frac{u'(q)\left[\theta u'(q) + (1-\theta)c'(q)\right]^2}{(1-\theta)\theta\left\{c''(q)u'(q) - u''(q)c'(q)\right\}\left[u(q) - c(q)\right] + c'(q)u'(q)\left[\theta u'(q) + (1-\theta)c'(q)\right]}$$

Equation (33) equals the envelope condition in Lagos and Wright (2001). This result is of particular interest because our model differs in two respects from Lagos and Wright (2001). First, we use a different formalization devise to render the distribution of money holdings degenerate (the large household). Second, we consider a sequential bargaining game and not the axiomatic Nash bargaining solution as they do. Furthermore, for $\theta = \frac{1}{2}$ and $u(q) = q$ equation (33) replicates the envelope condition of Rauch (2000, eq. (25)), who also imposes the axiomatic symmetric Nash bargaining solution.

4 The Friedman rule

The Friedman rule requires to deflate the nominal stock of money approximately at the rate of time preference. Because the utility of a representative household is $z\left[u(q) - c(q)\right]$ and because q cannot be larger than q^* according to (26), the Friedman rule holds in our model if the monetary policy that consists in setting $\gamma = \beta$ asymptotically maximizes the purchasing power of money by bringing q as close as possible to q^*.

In Shi (2001) and Berentsen and Rocheteau (2000) the envelope condition is (32) which implies that the Friedman rule holds for any value of θ. Furthermore, q approaches q^* as γ tends to β. Thus, Shi (2001) and Berentsen and Rocheteau (2000) the Friedman rule guarantees the first-best allocation. The reason for this result is that a buyer can get the full return on its marginal unit of money, because of the inability of sellers to capture part of the surplus that an additional unit of money generates for the match.

In the following we consider the validity of the Friedman rule when the bargaining strategies are determined ex post (after the matches have been formed) as in Rauch's (2000) and Lagos and Wright's (2001). For this purpose, we use equation (33) to define a symmetric monetary steady-state equilibrium.

Definition 1 *For all $\gamma > \beta$, a symmetric monetary steady-state equilibrium is a $q > 0$ satisfying equation (33).*

Proposition 1 *If $\theta = 1$, for all $\gamma > \beta$ a unique symmetric monetary steady-state equilibrium exists and $\lim_{\gamma \to \beta} q = q^*$. If $\theta \in (0, 1)$, for all $\gamma > \beta$ there exists a symmetric monetary steady-state equilibrium such that $\lim_{\gamma \to \beta} q = \hat{q}(\theta) < q^*$, where $\hat{q}(\theta)$ is the maximum value of q satisfying (26).*

Proof. Assume first that $\theta = 1$. The envelope condition is identical to (32). The function $\Psi(q)$ is strictly decreasing, $\Psi(0) = +\infty$, and $\Psi(q^*) = 1$. Therefore, for all $\gamma > \beta$ there is a unique monetary equilibrium. Furthermore, q is a decreasing function of γ, and $\lim_{\gamma \to \beta} q = q^*$.

Assume now that $\theta < 1$. Then $\Psi(q)$ is continuous for all $q \in (0, q^*]$, $\Psi(0) = +\infty$, and $\Psi(q^*) < 1$. Let $\widehat{q}(\theta)$ be the maximum value of q such that (26) is satisfied. As previously indicated, if $\theta < 1$ then $\widehat{q} < q^*$. Because $\Psi(q) > 1$ is equivalent to the condition (26), there is no monetary equilibrium with $\gamma > \beta$ such that $q \geq \widehat{q}$. Furthermore, for all $q < \widehat{q}$, there exists a $\gamma > \beta$ such that q sustains a monetary equilibrium. Finally, from the continuity of $\Psi(.)$ we deduce that there exists a monetary equilibrium such that $\lim_{\gamma \to \beta} q = \widehat{q} < q^*$.■

Proposition 1 confirms that the optimal monetary policy is the Friedman rule. This result is robust, because it is shared by all search models with divisible money, divisible goods, and a degenerate distribution of money holdings. Thus, in the absence of distributional effects such as those studied by Camera and Corbae (1999), Deviatov and Wallace (2001), and Molico (1997) and in the absence of search externalities as studied in Berentsen, Rocheteau, and Shi (2001), there is no welfare gain by choosing a rate of growth of the money supply larger than what is prescribed by the Friedman rule.

Nonetheless, according to Proposition 1, for $\theta < 1$, the Friedman rule fails to generate the efficient quantity of trade in each match. This inefficiency arises because at the time of production agents cannot contract with their future trading partners how much consumption goods they will receive in return for the acquired money. Accordingly, when buyers spend an additional unit of money, the cost of acquiring the unit is sunk, and consequently they are not able to appropriate the full return of a marginal unit of money, unless they have all the bargaining power. This inefficiency is nothing else than the *holdup problem*, which arises in environments with bargaining and incomplete contracts. If buyers cannot get the full return of an additional unit of money, they reduce their initial investment, that is, they produce less to obtain the unit. This inefficiency cannot be corrected by the Friedman rule: The Friedman rule only corrects inefficiencies that are associated with the fact that agents discount future utilities.

Correction of the holdup inefficiency would require a higher rate of deflation than what is prescribed by the Friedman rule, which is inconsistent with the existence of a monetary equilibrium in a model where agents are allowed to hoard money. Such a policy would be feasible only if the households were constrained to bring all their money holdings into the matches. This can explain why the first-best allocation can be restored in models with indivisible money when agents use lotteries to determine the terms of trade even though a holdup problem similar to the one emphasized in this paper exists (e.g. Berentsen, Molico, and Wright 1999), because with indivisible money buyers cannot hoard money, that is, they are technically forced to bring all their money holdings into a match. If we had constrained the household to redistribute all its money across members at the beginning of each period, i.e., $y = m$, then the optimal gross growth rate of the money supply γ^* would have been smaller than the discount factor β for all $\theta \in (0, 1)$ and would have guaranteed the first-best allocation.

5 Conclusion

This paper has studied the validity of the Friedman-rule search-theoretic models with divisible money, divisible goods, and a degenerate distribution of money holdings. The following results have emerged from our analysis. First, we have shown that the different formalization devices proposed by Shi (1999, 2001) and by Lagos and Wright (2001) are equivalent with respect to their closed-form solutions. Second, in the absence of search externalities and distributional issues, the Friedman rule is the optimal monetary policy. Third, the different results with respect to the efficiency of the monetary equilibrium under the Friedman rule reported in the search literature of divisible money have their origin in the different decision structures of the models. If, as in Shi (1999, 2001) and in Berentsen and Rocheteau (2000), the households determine their bargaining strategies ex ante (before the matches have been formed), the Friedman rule implements the first best. In contrast, if, as in Lagos and Wright (2001) and in Rauch (2000), the households determine them ex post, the Friedman rule does not guarantee the first best, unless buyers have all the bargaining power. The reason for this inefficiency is a holdup problem. A buyer who holds more money than what is expected in equilibrium cannot appropriate the entire surplus that this additional money provides to a match. In this sense, money is an asset whose holder — the buyer — is not able to capture its entire return, which results in an inefficiently low purchasing power of money even under the Friedman rule.

Literature

Acemoglu, Daron and Robert Shimer (1999). *Holdups and Efficiency with Search Frictions*, International Economic Review 40, 827–849.

Berentsen, Aleksander, Miguel Molico, and Randall Wright (2002). *Indivisibilities, Lotteries, and Monetary Exchange.* Journal of Economic Theory, 107, p. 70 - 94.

Berentsen, A. and Rocheteau, G. (2002), *On the Efficiency of Monetary Exchange: How Divisibility of Money Matters.* Journal of Monetary Economics, 49:8; November 2002, 1621-49.

Berentsen, A. and Rocheteau, G. (2003), *Money and the Gains from Trade*, International Economic Review, Vol. 44, No. 1, February 2003.

Berentsen, Aleksander, Guillaume Rocheteau, and Shouyong Shi (2001). *Hosios Meets Friedman: Efficiency in Search Models of Money.* Mimeo.

Camera, Gabriele and Dean Corbae (1999), *Money and Price Dispersion*, International Economic Review 40, 985–1008.

Deviatov Alexei and Neil Wallace (2001). *Another Example in which Lump-Sum Money Creation is Beneficial*, Advances in Macroeconomics 1.

Friedman, M. (1969). *The Optimum Quantity of Money.* In The Optimum Quantity of Money and Other Essays, Aldine, Chicago, 1–50.

Lagos, Ricardo and Randall Wright (2001). *A Unified Framework for Monetary Theory and Policy Analysis.* Mimeo.

Lucas, Robert (1990). Liquidity and interest rates. Journal of Economic Theory 50, 237–264.

Molico, Miguel (1997). *The Distribution of Money and Prices in Search Equilibrium.* Ph.D. Dissertation, The University of Pennsylvania.

Rauch, Bernhard (2000). *A Divisible Search Model of Fiat Money: A Comment.* Econometrica 68, 149–156.

Shi, Shouyong (1997). *A Divisible Search Model of Fiat Money.* Econometrica 65, 75–102.

—— (1999). *Search, Inflation and Capital Accumulation.* Journal of Monetary Economics 44, 81–103.

—— (2001). *Liquidity, Bargaining, and Multiple Equilibria in a Search Monetary Model*, Annals of Economics and Finance 2, 191–217.

Woodford, Michael (1990). *The Optimum Quantity of Money.* In Handbook of Monetary Economics, Vol. 2, Ed. B. M. Friedman and F. H. Hahn, 1067–1152.