



Institute for Empirical Research in Economics
University of Zurich

Working Paper Series
ISSN 1424-0459

Working Paper No. 213

**Second Order Stochastic Dominance, Reward-Risk Portfolio
Selection and the CAPM**

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January 2005

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First Draft: 4th August 2004

This Version: 23rd December 2004

*Financial support from the National Center of Competence in Research “Financial Valuation and Risk Management” (NCCR-FINRISK), Tinbergen Institute (TI), Erasmus Research Institute of Management (ERIM) and the Erasmus Center of Financial Research (ECFR) is gratefully acknowledged. We received useful comments from Thorsten Hens, Olivier Scaillet, Fabio Trojani, Pim van Vliet and Philippe Versijp. An earlier version of the paper was presented at the NCCR-FINRISK Research Day in Berne. We thank the participants for their helpful comments and suggestions. Pim van Vliet is credited for making the data available.

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Abstract

Starting from the reward-risk model for portfolio selection introduced in De Giorgi (2004), we derive the reward-risk Capital Asset Pricing Model (CAPM) analogously to the classical mean-variance CAPM. The reward-risk portfolio selection arises from an axiomatic definition of reward and risk measures based on few basic principles, including consistency with second order stochastic dominance. With complete markets, we show that at any financial market equilibrium, investors' optimal allocations are comonotonic and therefore the capital market equilibrium model can be reduced to a representative investor model. Moreover, the pricing kernel is an explicitly given, monotone function of the market portfolio return, corresponding to the increments of the distortion function characterizing the representative investor's risk perceptions. Finally, an empirical application shows that the reward-risk CAPM better captures the cross-section of US stock returns than the mean-variance CAPM does.

Keywords: stochastic dominance, mean-risk models, portfolio optimization, CAPM.

JEL Classification: G11, D81.

1 Introduction

The Portfolio Theory of Markowitz (1952) evaluates investments in term of their means and variances. In this context, if investors agree on the assets' return distributions and the risk free asset exists, the mean-variance Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966) arises as the equilibrium model. From a theoretical point of view, variance is a debatable measure of risk. First, it is a symmetric measure on the space of random variables and treats positive and negative deviations from the mean in the same way, while investors typically assign a higher weight to negative deviations than to positive ones (due to decreasing risk aversion). Second, there exists a plenty of empirical evidence to suggest that the return distribution of many assets exhibits fat tails and hence variance fails to fully describe the risk of extreme losses. Also, the empirical problems of the mean-variance CAPM in capturing the cross-sectional pattern of stock returns may reflect the problems of variance in capturing risk.

Several alternative risk measures have been proposed in order to better capture investors' risk perception, including semivariance (Markowitz 1959, Ogryczak and Ruszczyński 1997), general lower partial moments (Jean 1975, Bawa 1975, Unser 2000), value-at-risk (Jorion 1997), expected shortfall (Acerbi and Tasche 2002, Rockafellar and Uryasev 2002). Equilibrium capital-market models based on these risk measures have also been developed, for instance the mean-semivariance CAPM by Hogan and Warren (1974) and Bawa and Lindenberg (1977). Interestingly, this particular model seems to better capture the cross-section of stock returns than the mean-variance model (see for instance Harlow and Rao 1989, Post and Van Vliet 2004).

Unfortunately, economic theory gives only minimal guidance in selecting a specific risk measure. One alternative approach is to use the general rules of stochastic dominance (see for example Post 2003, Kuosmanen 2004). These rules however generally do not imply a specific risk measure. Artzner, Delbaen, Eber, and Heath (1997, 1999), being concerned with *banking regulation*, proposed an axiomatic definition of risk measures based on four properties (positive sub-additivity, monotonicity, translation invariance and homogeneity) and introduced the concept of coherent measure of risk, that strongly influenced the way of thinking at risk measurement and risk management.

De Giorgi (2004) extends this approach to a general risk-reward framework for portfolio selection, based on consistency with second order stochastic dominance, in addition to other basic properties. From Rothschild and Stiglitz (1971) we know that an investment opportunity dominates another by second order stochastic dominance, if and only if it is preferred by all risk averse expected utility maximizers. Hence, the consistency with the second order stochastic dominance is the minimal requirement on the reward-risk model in order to preserve the preference relations shared by *all* risk averse, expected utility maximizers. De Giorgi (2004) also gives a unique characterization of reward measures (the mean) and suggests a class of risk measures for portfolio selection, that are related to the Choquet Expected Utility Theory.¹ In this setup, a risk measure arises from a convex distortion of

the physical survival distribution function and the portfolio decision problem can be solved by means of linear quintile regression (Bassett, Koenker, and Kordas 2004).

In this study we derive the “reward-risk CAPM”, that is, the financial market equilibrium model associated with the general risk-reward framework. The model basically is a general model of a complete market with risk averse investors. Interestingly, the model implies that every investor’s optimal allocation is comonotonic to the market portfolio, analogously to the Tobin separation theorem in the classical mean-variance approach. Comonotonicity implies that the preferences of the different investors must be very similar in order to achieve market equilibrium. This in turn reduces the model to a *representative investor model*; aggregate demand and market equilibrium can be described as the outcome of the optimization problem of an individual investor.² Equilibrium is characterized by a pricing kernel that is a decreasing function of market portfolio return. The exact shape of the kernel is determined by the distortion function that describes the risk attitude of the representative investor and the relationship is explicitly given.

An empirical study compares the empirical performance of the reward-risk CAPM with that of the mean-variance CAPM for US stock market data. The results suggest that the reward-risk CAPM better captures the cross-section of stock returns than the mean-variance CAPM does. Interestingly, the results support the mean-semivariance CAPM, suggesting that semivariance is a better measure for investment risk than variance. This result is quite surprising, because our general reward-risk model rests on substantially more general assumptions than the mean-semivariance model does.

The remainder of the paper is organized as follows: in Section 2 we present the reward-risk portfolio selection problem and we derive the reward-risk Capital Asset Pricing Model. In Section 3, we test empirically the market efficiency based on the reward-risk CAPM. Section 4 concludes.

2 The model

We consider a two-periods exchange economy. The model setup follows Duffie (1988). Let $\Omega = \{1, \dots, S\}$ denote the state of nature at the final period T .³ $\mathcal{F} = 2^\Omega$ is the power algebra on Ω , i.e. the set of all possible events arising from Ω . Uncertainty is modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the probability measure \mathbb{P} on Ω satisfies $p_s = \mathbb{P}[\{s\}] > 0$ for all $s = 1, \dots, S$, i.e. every state of the world has strictly positive probability to occur.

The space \mathcal{G} of real-valued measurable functions is endowed with the scalar product $X \cdot Y = \sum_{s=1}^S X(s)Y(s)p_s = \mathbb{E}_{\mathbb{P}}[XY]$.

There are $K + 1$ assets with payoffs A_k . The asset 0 is the risk-free asset with payoff $A_0 = \mathbf{1}$. The supply of risky assets is exogenously given and denoted by $\bar{\theta}_k > 0$ ($k = 1, \dots, K$), while the risk-free asset is in elastic supply with exogenously given price $\frac{1}{1+r}$, where $r > 0$ is the risk-free rate of return. The marketed subspace \mathcal{X} is the span of $(A_k)_{k=0,1,\dots,K}$. Without loss of generality, we assume that no asset is redundant, i.e. $\dim(\mathcal{X}) = K + 1$, where obviously

$K + 1 \leq S$ and $K + 1 = S$ if markets are complete. The market portfolio is the sum of all available risky assets, i.e. $\tilde{\omega} = \sum_{k=1}^K A_k \bar{\theta}_k$.

There are $i = 1, \dots, I$ investors, initially endowed with wealth $w^i > 0$. The numbers θ_k^i denote the amount of security k held by investor i , q_k denotes the k -th security price. Thus, when trading these securities, the investor can attain the payoff $X = \sum_{k=0}^K A_k \theta_k^i \in \mathcal{X}$ where θ^i satisfies the budget restriction (i.e. $q(X) = \sum_{k=0}^K q_k \theta_k^i \leq w^i$). We denote by \mathcal{B}^i the subset of \mathcal{X} , such that $X \in \mathcal{B}^i$ is budget-feasible for investor i , i.e. $\mathcal{B}^i = \{X \in \mathcal{X} | q(X) \leq w^i\}$. Note that \mathcal{B}^i is a convex set.

Investors evaluate portfolio payoffs according to a risk-reward pair (μ, ρ^i) , where $\mu(X) = \mathbb{E}[X]$ and $\rho^i : \mathcal{G} \rightarrow \mathbb{R}$ is a risk measure as defined in De Giorgi (2004, Definition 3.2). The measure ρ^i satisfies the following four properties:

- Convexity: $\rho^i(\alpha X + (1 - \alpha)Y) \leq \alpha \rho^i(X) + (1 - \alpha) \rho^i(Y)$ for all $\alpha \in [0, 1]$, $X, Y \in \mathcal{G}$;
- Zero payoff condition: $\rho^i(0) = 0$;
- Translation invariance: $\rho^i(X + \alpha) = \rho^i(X)$ for all $X \in \mathcal{G}$, $\alpha \in \mathbb{R}$;
- Isotonicity w.r.t. second order stochastic dominance: $X \succ_{SSD} Y \Rightarrow \rho^i(X) \leq \rho^i(Y)$.

The convexity ensures the diversification effect. In fact, when convexity is not satisfied for some $X, Y \in \mathcal{G}$ and $\alpha \in (0, 1)$, then one could split the portfolio $\alpha X + (1 - \alpha)Y$ in two parts, hold α times the position X and $1 - \alpha$ times the position Y and consequently reduce the risk. Similar arguments are provided by Artzner, Delbaen, Eber, and Heath (1997, 1999). The *zero payoff condition* is a natural assumption, which states that “no investment” means no risk. The *risk-free condition* says that adding a risk free position to the portfolio does not change the risk! This is different as in the definition of coherent risk measure introduced by Artzner, Delbaen, Eber, and Heath (1999), where the authors interpret a risk measure as the minimal extra cash one should add to his risky position and allocate “prudently”, to make the investment acceptable. Dealing with portfolio selection, we suggest that the contribution of a risk free position to the portfolio should be captured by the reward measure and not by the risk measure. Finally the *isotonicity with respect to second order stochastic dominance* ensures that an investment which is preferred to another one by all risk averse rational expected utility maximizers, is at most so risky as the dominated investment.

The measure ρ^i arises from an axiomatic definition of risk measures based on the four properties listed above: A well known example is the expected shortfall (see Bertsimas, Lauprete, and Samatov 2004). The variance, value-at-risk and, coherent risk measures instead, do not satisfies our axiomatic definition, as demonstrated in De Giorgi (2004).

Note that we do not impose the same measure of risk for all investors. In fact, investors’ perception of risk can differ and thus also the way of measuring it (Weber and Milliman 1997), as long as the four properties above are satisfied.

The portfolio payoff $X = \sum_{k=0}^K A_k \theta_k^i \in \mathcal{X}$ for investor i is said to be (μ, ρ^i) -efficient iff

- (i) $q(X) \leq w^i$ (budget feasible), and
- (ii) $\nexists Y \in \mathcal{X}$ such that $q(Y) \leq w^i$ and one of the following two statements is satisfied
 - (a) $\rho^i(X) > \rho^i(Y)$ and $\mu(X) = \mu(Y)$ or,
 - (b) $\rho^i(X) = \rho^i(Y)$ and $\mu(X) < \mu(Y)$.

From De Giorgi (2004, Theorem 2.1), $X \in \mathcal{X}$ is (μ, ρ^i) -efficient iff X is budget feasible and uniquely minimizes the function $\mathcal{R}^i = \xi^i \rho^i - \mu^i$ over \mathcal{B}^i , for some $\xi^i > 0$. Moreover, there exists a convex, non-decreasing function g^i on $[0, 1]$, with $g^i(0) = 0$ and $g^i(1) = 1$ such that

$$\mathcal{R}^i(X) = - \int_{-\infty}^0 (g^i(\bar{F}_X(x)) - 1) dx - \int_0^{\infty} g^i(\bar{F}_X(x)) dx \quad (2.1)$$

and \bar{F}_X is the cumulative distribution function of X under \mathbb{P} . The convex function g^i is called distortion and uniquely characterizes the investor's risk preferences. In particular, due to the convexity of g^i , investors put more weights to the negative outcomes of X . For example, if investors only weight the outcome of X that are smaller than a given $\alpha\%$ -quintile, then the corresponding measure of risk is the expected-shortfall. Moreover, the function \mathcal{R}^i corresponds to a Choquet integral and is used in the Choquet Expected Utility Theory of Schmeidler (1989). Therefore, the investor's portfolio choice problem is:

$$\min_{X \in \mathcal{B}^i} \mathcal{R}^i(X), \quad (2.2)$$

or equivalently

$$\max_{X \in \mathcal{B}^i} -\mathcal{R}^i(X). \quad (2.3)$$

We introduce the following definition:

Definition 2.1 (Financial market equilibrium). *Given a risk-free rate r , a financial market equilibrium consists of a price vector $\hat{q} \in \mathbb{R}^{K+1}$ with $\hat{q}_0 = \frac{1}{1+r}$ and allocations $\hat{X}^i \in \mathcal{X}$ for $i = 1, \dots, I$, such that*

- (i) \hat{X}^i maximizes $-\mathcal{R}^i$ over \mathcal{B}^i (investors' portfolio choice), and
- (ii) $\exists \alpha_0 \in \mathbb{R}$ such that $\alpha_0 \mathbf{1} + \sum_{i=1}^I \hat{X}^i = \tilde{\omega}$ (markets clear).

Instead of using this last Definition directly, we first impose some restrictions on the equilibrium prices \hat{q} . Note that the goal function \mathcal{R}^i is strictly monotone and therefore a necessary condition for the portfolio decision problem (2.3) to have a solution (and thus for condition (i) in Definition 2.1) is arbitrage opportunities do not exist. Therefore, market equilibrium requires the following condition on prices:

$$\mathcal{G}_+ \cap \{X \in \mathcal{X} | q(X) \leq 0\} = \{0\}, \quad (2.4)$$

where \mathcal{G}_+ is the subset of elements in \mathcal{G} with non-negative outcomes in all states of nature. Equation (2.4) means that every non-negative, non-zero payoff must have a strictly positive price. If equation (2.4) is violated for some price vector q , investors can obtain, at zero costs, a payoff that is non-negative in all states and strictly positive in at least one state. Therefore, they can infinitely increase their objective functions and no optimal solution to their investment problem exist. A price vector $q \in \mathbb{R}^{K+1}$ such that equation (2.4) is satisfied, is said to be *arbitrage free for the marketed subspace \mathcal{X}* .

In order to discuss the equilibrium outcome of our model, suppose that q is an arbitrage free price vector for the marketed subspace \mathcal{X} . In this case, there exists a security $\ell \in \mathcal{X}$, $\mathbb{E}_{\mathbb{P}}[\ell] = 1$ such that

$$q(X) = \frac{1}{1+r} \ell \cdot X \quad (2.5)$$

for all $X \in \mathcal{X}$ and $q(X) \geq 0$. ℓ is called the pricing portfolio (Duffie 1988) or ideal security (Magill and Quinzii 1996). Using the pricing portfolio ℓ we can rewrite the budget set as $\mathcal{B}^i = \{X \in \mathcal{X} \mid \ell \cdot X \leq (1+r)w^i\}$ and the no-arbitrage decision problem of investor i is given by

$$\max_{X \in \mathcal{X}} -\mathcal{R}^i(X), \ell \cdot X \leq (1+r)w^i. \quad (2.6)$$

An equivalent definition of financial market equilibria is now the following:

Definition 2.2. *Given a risk-free rate r , a financial market equilibrium consists of a price vector $\hat{\ell} \in \mathcal{X}$ and allocations $\hat{X}^i \in \mathcal{X}$ for $i = 1, \dots, I$, such that*

- (i) \hat{X}^i maximizes $-\mathcal{R}^i$ subject to $\hat{\ell} \cdot X \leq (1+r)w^i$ for $i = 1, \dots, I$, and
- (ii) $\exists \alpha_0 \in \mathbb{R}$ such that $\alpha_0 \mathbf{1} + \sum_{i=1}^I \hat{X}^i = \tilde{\omega}$.

We now come back to the individual portfolio choice of equations (2.2) and (2.3). We rewrite the function \mathcal{R}^i using its integral representation (2.1). Consider $X \in \mathcal{G}$ and take a permutation ζ of $\Omega = \{1, \dots, S\}$ such that $X(\zeta(1)) \leq X(\zeta(2)) \leq \dots \leq X(\zeta(S))$. Then

$$\mathcal{R}^i(X) = -X(\zeta(1)) - \sum_{s=1}^{S-1} g^i \left(1 - \sum_{l=1}^s p_{\zeta(l)} \right) [X(\zeta(s+1)) - X(\zeta(s))].$$

Let

$$\begin{aligned} q_{\zeta(1)}^i &= 1 - g^i(1 - p_{\zeta(1)}), \\ q_{\zeta(s)}^i &= g^i \left(1 - \sum_{l=1}^{s-1} p_{\zeta(l)} \right) - g^i \left(1 - \sum_{l=1}^s p_{\zeta(l)} \right), \quad \text{for } s = 2, \dots, S. \end{aligned}$$

Note that $q_{\zeta(s)}^i \geq 0$ since g is non-decreasing, $\sum_{s=1}^S q_{\zeta(s)}^i = 1$, and $q_{\zeta(1)}^i \geq q_{\zeta(2)}^i \geq \dots \geq q_{\zeta(S)}^i$ since g^i is convex. Moreover,

$$\mathcal{R}^i(X) = - \sum_{s=1}^S q_{\zeta(s)}^i X(\zeta(s)) = - \sum_{s=1}^S \frac{q_{\zeta(s)}^i}{\sum_{l: X(\zeta(l))=X(\zeta(s))} p_{\zeta(l)}} p_{\zeta(s)} X(\zeta(s)).$$

Note that

$$\frac{\sum_{l: X(\zeta(l))=X(\zeta(s))} q_{\zeta(l)}^i}{\sum_{l: X(\zeta(l))=X(\zeta(s))} p_{\zeta(l)}} = \frac{g^i(\mathbb{P}[X \geq X(\zeta(s))]) - g^i(\mathbb{P}[X > X(\zeta(s))])}{\mathbb{P}[X = X(\zeta(s))]} = f_X^i(X(\zeta(s)))$$

where

$$f_X^i(x) = \frac{g^i(\mathbb{P}[X \geq x]) - g^i(\mathbb{P}[X > x])}{\mathbb{P}[X = x]}$$

is a positive, non-increasing function of x , since g^i is non-decreasing and convex. Moreover, by definition, $f_X^i(X) \in \mathcal{G}$ with

$$\mathbb{E}_{\mathbb{P}}[f_X^i(X)] = 1, \quad f_X^i(X) \geq 0 \quad \text{and} \quad \mathcal{R}^i(X) = -\mathbb{E}_{\mathbb{P}}[f_X^i(X) X] = -f_X^i(X) \cdot X.$$

Thus, the vector $f_X^i(X) \in \mathcal{G}$ is a probability measure on (Ω, \mathcal{F}) and the functional \mathcal{R}^i is the negative expectation with respect to $f_X^i(X) \in \mathcal{G}$. Similar results are given by Carlier and Dana (2003) for non atomic spaces. The optimization problem (2.6) can be rewritten as

$$\max_{X \in \mathcal{X}, \lambda^i} f_X^i(X) \cdot X - \lambda^i (\ell \cdot X - (1+r)w^i) = \max_{X \in \mathcal{X}, \lambda^i} (f_X^i(X) - \lambda^i \ell) \cdot X + \lambda^i (1+r)w^i. \quad (2.7)$$

where λ^i is the Lagrange multiplier. Let $\mathcal{L}^i(X, \lambda^i) = (f_X^i(X) - \lambda^i \ell) \cdot X + \lambda^i (1+r)w^i$ be the Lagrange function. We are now able to prove the main result of this section. The following relationship between any efficient allocation \hat{X}^i and the pricing portfolio is satisfied.

Theorem 2.1. *Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$, then*

$$f_{\hat{X}^i}^i(\hat{X}^i)_{\parallel} = \ell$$

and $\ell \cdot \hat{X}^i = (1+r)w^i$ for all $i = 1, \dots, I$, where for $Y \in \mathcal{G}$, $Y = Y_{\perp} + Y_{\parallel}$ is the unique orthogonal decomposition of Y with respect to \mathcal{X} , i.e. $Y_{\perp} \perp \mathcal{X}$ and $Y_{\parallel} \in \mathcal{X}$.

Proof. (i) We prove: $\ell \cdot \hat{X}^i = (1+r)w^i$.

Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$. Since the function $-\mathcal{R}^i(X)$ is strictly monotone and the risk-less asset exists, \hat{X}^i must satisfy the budget restriction with equality, i.e. $\ell \cdot \hat{X}^i = (1+r)w^i$.

(ii) We prove: $f_{\hat{X}^i}^i(\hat{X}^i)_\parallel = \ell$.

Let $Z \in \mathcal{X}$ such that $\ell \cdot Z = 0$ (i.e. $Z \in \text{span}(\ell)^\perp \cap \mathcal{X}$) and $Y^i = \hat{X}^i + \epsilon Z$ for $\epsilon > 0$. Then $Y_i \in \mathcal{X} \cap \mathcal{B}^i$ and

$$\begin{aligned} \mathcal{L}^i(Y^i, \lambda^i) &= (f_{Y^i}^i(Y^i) - \lambda^i \ell) \cdot Y^i + \lambda^i (1+r) w^i \\ &= f_{Y^i}^i(Y^i) \cdot Y^i - \lambda^i \ell \cdot \hat{X}^i + \lambda^i (1+r) w^i \\ &= \left(f_{Y^i}^i(Y^i) - f_{\hat{X}^i}^i(\hat{X}^i) \right) \cdot Y^i + \epsilon f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z + \mathcal{L}^i(\hat{X}^i, \lambda^i). \end{aligned}$$

Let ζ be a permutation of Ω such that $\hat{X}^i(\zeta(1)) \leq \hat{X}^i(\zeta(2)) \leq \dots \leq \hat{X}^i(\zeta(S))$. Without loss of generality, for $\epsilon > 0$ small enough, $Y^i(\zeta(1)) \leq Y^i(\zeta(2)) \leq \dots \leq Y^i(\zeta(S))$. In fact, if for some $s \in \{1, \dots, S-1\}$, $\hat{X}^i(\zeta(s)) = \hat{X}^i(\zeta(s+1))$ and $\hat{Y}^i(\zeta(s)) > \hat{Y}^i(\zeta(s+1))$, then we take the permutation ζ of Ω such that $\zeta(l) = \zeta(l)$ for all $l \neq s, s+1$ and $\tilde{\zeta}(s) = \zeta(s+1)$, $\tilde{\zeta}(s+1) = \zeta(s)$. Then $(f_{Y^i}^i(Y^i) - f_{\hat{X}^i}^i(\hat{X}^i)) \cdot Y_i = -\sum_{s=1}^S (q_{\zeta(s)} - q_{\tilde{\zeta}(s)}) Y_i(\zeta(s)) = 0$ and thus

$$\mathcal{L}^i(Y^i, \lambda^i) = \epsilon f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z + \mathcal{L}^i(\hat{X}^i, \lambda^i).$$

Therefore $f_{\hat{X}^i}^i(\hat{X}^i) \cdot Z = 0$, else either $Y_i = \hat{X}^i + \epsilon Z$ or $Y_i = \hat{X}^i - \epsilon Z$ contradicts the optimality of \hat{X}^i .

Let now decompose $f_{\hat{X}^i}^i(X^i)$ as $f_{\hat{X}^i}^i(X^i) = f_{\hat{X}^i}^i(X^i)_\parallel + f_{\hat{X}^i}^i(X^i)_\perp$, where $f_{\hat{X}^i}^i(X^i)_\parallel \in \mathcal{X}$ and $f_{\hat{X}^i}^i(X^i)_\perp \perp \mathcal{X}$. Let $Z \in \text{span}(\ell)^\perp$. Then $0 = \ell \cdot Z = \ell \cdot (Z_\perp + Z_\parallel) = \ell \cdot Z_\parallel$, therefore $Z_\parallel \in \text{span}(\ell)^\perp \cap \mathcal{X}$. From the previous result it follows

$$0 = f_{\hat{X}^i}^i(X^i) \cdot Z_\parallel = f_{\hat{X}^i}^i(X^i)_\parallel \cdot Z_\parallel = f_{\hat{X}^i}^i(X^i)_\parallel \cdot Z.$$

Since this is true for all $Z \in \text{span}(\ell)^\perp$, it follows that $f_{\hat{X}^i}^i(X^i)_\parallel \in \text{span}(\ell)$ and therefore it exists $\hat{\alpha}^i \in \mathbb{R}$ such that $f_{\hat{X}^i}^i(\hat{X}^i)_\parallel = \hat{\alpha}^i \ell$. Since $\mathbf{1} \in \mathcal{X}$, $0 = f_{\hat{X}^i}^i(\hat{X}^i)_\perp \cdot \mathbf{1} = \mathbb{E}_\mathbb{P}[f_{\hat{X}^i}^i(\hat{X}^i)_\perp]$ and thus

$$1 = \mathbb{E}_\mathbb{P}[f_{\hat{X}^i}^i(\hat{X}^i)] = \mathbb{E}_\mathbb{P}[f_{\hat{X}^i}^i(\hat{X}^i)_\perp] + \mathbb{E}_\mathbb{P}[f_{\hat{X}^i}^i(\hat{X}^i)_\parallel] = \mathbb{E}_\mathbb{P}[f_{\hat{X}^i}^i(\hat{X}^i)_\parallel] = \hat{\alpha}^i \mathbb{E}_\mathbb{P}[\ell] = \hat{\alpha}^i.$$

This completes the proof. □

In our model, the vector $f_{\hat{X}^i}^i(\hat{X}^i)$ defines the marginal rate of substitution of investor i which is strictly related to its risk perception described by g^i . The theorem states that the component of the marginal rate of substitution that belongs to the marketed subspace, must be collinear to the pricing portfolio at any optimal allocation \hat{X}^i . Now, we restrict ourself to the case of complete markets. Then $f_{\hat{X}^i}^i(\hat{X}^i) \in \mathcal{X}$ and thus $f_{\hat{X}^i}^i(\hat{X}^i)_\perp = 0$ for all i . Therefore,

$f_{\hat{X}^i}^i(\hat{X}^i) = f_{\hat{X}^i}^i(\hat{X}^i)_\parallel$. With incomplete markets, the same is true if we assume that $f_{\hat{X}^i}^i(\hat{X}^i)$ is in the marketed subspace. From the previous theorem, we immediately obtain the following result on the optimal allocations \hat{X}^i , $i = 1, \dots, I$.

Theorem 2.2. *Let $\hat{X}^i \in \arg \max_{X \in \mathcal{X}} -\mathcal{R}^i(X)$, s.t. $\ell \cdot X \leq (1+r)w^i$ for $i = 1, \dots, I$. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K+1 = S$, i.e. markets are complete, the optimal payoffs $\hat{X}^1, \dots, \hat{X}^I$ are comonotonic, i.e. for all $s, s' \in \Omega$ and $i, j \in \{1, \dots, I\}$ we have $(\hat{X}^i(s) - \hat{X}^i(s'))(\hat{X}^j(s) - \hat{X}^j(s')) \geq 0$ and the inequality is strict if $\hat{X}^i(s) \neq \hat{X}^i(s')$ for some i .*

Proof. From Theorem 2.1, $f_{\hat{X}^i}^i(\hat{X}^i) = \ell$ for $i = 1, \dots, I$. The functions $f_{\hat{X}^i}^i$ are strictly decreasing, since the g^i 's are strictly convex. Suppose now that for $s, s' \in \Omega$, $\hat{X}^i(s) \geq \hat{X}^i(s')$. Then $f_{\hat{X}^i}^i(\hat{X}^i(s)) \leq f_{\hat{X}^i}^i(\hat{X}^i(s'))$, i.e. $\ell(s) \leq \ell(s')$. Thus, $f_{\hat{X}^j}^j(\hat{X}^j(s)) \leq f_{\hat{X}^j}^j(\hat{X}^j(s'))$ and therefore also $\hat{X}^j(s) \geq \hat{X}^j(s')$ for any $j \in \{1, \dots, I\}$. Moreover, the inequality is strict for \hat{X}^j if it is for \hat{X}^i . \square

The theorem states that investors' optimal allocations are comonotonic, i.e. cannot be used as a hedge of each other. This is a generalization of the well-known Tobin Separation Principle in the classical mean-variance model. Comonotonicity immediately implies the following property of the pricing portfolio $\hat{\ell}$ at any financial market equilibrium:

Theorem 2.3. *Let $(\hat{\ell}, \hat{X}^1, \dots, \hat{X}^I)$ be a financial market equilibrium. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K+1 = S$, i.e. markets are complete, there exists a strictly decreasing function f such that $f(\tilde{\omega}) = \ell$ and $f(\tilde{\omega}) = f_{\tilde{\omega}}^i(\tilde{\omega})$ for all $i = 1, \dots, I$. The functions $f^i(\cdot)$ are given by:*

$$f_X^i(x) = \frac{g^i(\mathbb{P}[X \geq x]) - g^i(\mathbb{P}[X > x])}{\mathbb{P}[X = x]}. \quad (2.8)$$

Proof. From the previous theorem, we have that all optimal payoffs \hat{X}^i ($i = 1, \dots, I$) are comonotonic and therefore also the sums $\sum_{i=1}^I \hat{X}^i$ and $\sum_{i=1}^I \hat{X}^i + \alpha \mathbf{1}$ for all $\alpha \in \mathbb{R}$. By definition of financial market equilibrium, we find $\alpha_0 \in \mathbb{R}$ such that $\tilde{\omega} = \sum_{i=1}^I \hat{X}^i + \alpha_0 \mathbf{1}$. Therefore, $\tilde{\omega}$ is also comonotonic to $\hat{X}^1, \dots, \hat{X}^I$ and thus $f_{\tilde{\omega}}^i(\tilde{\omega}) = f_{\hat{X}^i}^i(\hat{X}^i) = \ell$. Take $f = f_{\tilde{\omega}}^i$ for some $i = 1, \dots, I$, where f^i are given as in equation (2.8). \square

At any financial market equilibrium, the pricing kernel $\hat{\ell}$ is a monotone function of the aggregate market return. The exact shape of this function is given by equation (2.8) and directly relates to the investors' perception of risk through the distortion g^i . It also follows that a *necessary* condition for the existence of financial market equilibria is that investors have homogeneous risk perceptions. Obviously, this result is not true in a General Equilibrium Model, as shown for example in Hens and Pilgram (2003, Chapter 2). In our model this is due to the comonotonicity of investors' optimal allocations, which implies the relationship

$f_{\hat{X}^i}^i(\hat{X}^i) = f_{\hat{X}^j}^j(\hat{X}^j)$ (for all $i, j \in \{1, \dots, I\}$) holding at any financial market equilibrium. If comonotonicity is violated, then for any given pricing kernel there exists at least one investor with a marginal rate of substitution that is not collinear to the pricing kernel. Consequently, since markets are complete, for this investor there exists an allocation that is orthogonal to the pricing kernel, but not to the vector of marginal rates of substitution, thus a zero-costs allocation that further increase his or her utility, i.e. an arbitrage opportunity. This contradicts the existence of equilibria. Therefore, the investors' distortions g^i must be identical at the survival probabilities $\bar{F}_{\hat{X}^i}(\hat{X}^i(s)) = \bar{F}_{\hat{X}^j}(\hat{X}^j(s))$, and thus investors' risk perceptions must correspond. This result also reduces the equilibrium model just described to a *representative investor model*, where market equilibrium can be fully described as the outcome of a single investor optimization problem. Consequently, the market portfolio is second order stochastic dominance efficient (see also Post 2003). We can state the “reward-risk” Security Market Line Theorem.

Corollary 2.1. *Let $(\hat{l}, \hat{X}^1, \dots, \hat{X}^I)$ be a financial market equilibrium and $q(\tilde{\omega}) > 0$. Suppose that the corresponding distortions g^i are strictly convex for all $i = 1, \dots, I$. Then if $K + 1 = S$, i.e. markets are complete, for all $X \in \mathcal{X}$:*

$$\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}})(R_X - r)] = 0, \quad (2.9)$$

where $R_X = \frac{X - q(X)}{q(X)}$ and $R_{\tilde{\omega}} = \frac{\tilde{\omega} - q(\tilde{\omega})}{q(\tilde{\omega})}$. Therefore For $X \in \mathcal{X}$

$$\mathbb{E}_{\mathbb{P}}[R_X] - r = \frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]} (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r). \quad (2.10)$$

Proof. (i) $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic.

Since $q(\tilde{\omega}) > 0$, then if for $s, s' \in \Omega$, $\tilde{\omega}(s) \geq \tilde{\omega}(s')$ then $R_{\tilde{\omega}}(s) \geq R_{\tilde{\omega}}(s')$. Thus, $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic.

(ii) Since $\tilde{\omega}$ and $R_{\tilde{\omega}}$ are comonotonic, then $f(\tilde{\omega}) = f(R_{\tilde{\omega}})$, where f is defined as in the proof of the previous Corollary. Moreover, for $X \in \mathcal{X}$

$$\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_X] = f(\tilde{\omega}) \cdot \left(\frac{X - q(X)}{q(X)} \right) = \frac{1}{q(X)} f(\tilde{\omega}) \cdot X - 1 = (1 + r) - 1 = r.$$

Therefore, for $X \in \mathcal{X}$

$$\begin{aligned} & (r - \mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}]) (\mathbb{E}_{\mathbb{P}}[R_X] - r) = (r - \mathbb{E}_{\mathbb{P}}[R_X]) (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\ \Rightarrow & (\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_{\tilde{\omega}}] - \mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}]) (\mathbb{E}_{\mathbb{P}}[R_X] - r) = (\mathbb{E}_{\mathbb{P}}[f(R_{\tilde{\omega}}) R_X] - \mathbb{E}_{\mathbb{P}}[R_X]) (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\ \Rightarrow & \text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}] (\mathbb{E}_{\mathbb{P}}[R_X] - r) = \text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X] (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r) \\ \Rightarrow & \mathbb{E}_{\mathbb{P}}[R_X] - r = \frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]} (\mathbb{E}_{\mathbb{P}}[R_{\tilde{\omega}}] - r). \end{aligned}$$

□

We call the factor $\frac{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_X]}{\text{cov}_{\mathbb{P}}[f(R_{\tilde{\omega}}), R_{\tilde{\omega}}]}$, the reward-risk beta.

3 Empirical analysis

This section empirically tests the equilibrium condition (2.9) and compares the empirical performance of the reward-risk CAPM with that of the mean-variance CAPM.

3.1 Transformation function

Our analysis will focus on the simple one-parameter distortion function

$$g(x) = 1 - (1 - x)^\gamma, \tag{3.11}$$

where $0 < \gamma < 1$. This transformation function has a compelling interpretation. If $x = \mathbb{P}[R \geq r^*]$, then $g(x)$ is the probability that the maximum of γ independent draws of R exceeds the critical value r^* , that is $g(x) = \mathbb{P}[\max_{i=1, \dots, \gamma} R_i \geq r^*]^4$. Thus, γ is a natural measure of optimism and pessimism (see Bassett, Koenker, and Kordas 2004); $\gamma > 1$ reflects optimistic view and $0 < \gamma < 1$ a pessimistic view. Indeed, for any integer $\gamma \geq 1$, applying the distortion function g to the true distribution, investors evaluate portfolio payoffs as they were distributed as the maximum of γ independent draws from the original distribution, suggesting that their belief are more optimistic than the true distribution. Instead, if $\gamma < 1$, investors' perception is more pessimistic.

We also considered a set of alternative distortion functions. The alternatives yield either comparable results (for example $g(x) = -\frac{1}{\gamma} \log(-(1 - \exp(-\gamma))x + 1)$) or a worst fit ($g(x) = x^\gamma$). We also used the second-order stochastic dominance tests of Post (2003) and Kuosmanen (2004), which entirely avoid the specification of the transformation function. Interestingly, these tests yield results that are very comparable with the results for the distortion function (3.11). These findings suggest that optimization across all possible transformation functions does not materially change our conclusions, supporting the choice of the one-parameter transformation (3.11).

3.2 Empirical methodology

In practice, we cannot directly check the equilibrium condition (2.9), because the ex ante return distribution of the assets is unknown. However, we can estimate the return distribution using time-series return observations and employ statistical tests to determine if the equilibrium condition is violated to a significant degree. Throughout the text, we will represent the observations for the risky assets by the matrix $\mathbf{R} = (\mathbf{r}_1, \dots, \mathbf{r}_t, \dots, \mathbf{r}_\tau)$, where $\mathbf{r}_t = (r_{1,t}, \dots, r_{K,t})'$ be the observation of the risky assets' returns $(R_1, \dots, R_K)'$, $R_k = \frac{A_k - q_k}{q_k}$, at time $t = 1, \dots, \tau$. Also, we will use $\mathbf{r}_{\tilde{\omega}} = (r_{\tilde{\omega},1}, \dots, r_{\tilde{\omega},\tau})'$ for the observed returns of the market portfolio.

The function $F : \mathbb{R}^K \rightarrow [0, 1]^K$ is the empirical multivariate cumulative distribution

function of $(R_1, \dots, R_K)'$ for the observations $(\mathbf{r}_t)_{t=1, \dots, \tau}$, i.e. for $\mathbf{r}^* \in \mathbb{R}^K$

$$F(\mathbf{r}^*) = \frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{\mathbf{r}_t \leq \mathbf{r}^*},$$

where $\mathbf{1}_{\mathbf{r}_t \leq \mathbf{r}^*}$ is the vector $(\mathbf{1}_{r_{1,t} \leq r_1^*}, \dots, \mathbf{1}_{r_{K,t} \leq r_K^*})'$. Similarly, we can construct the following univariate empirical cumulative distribution function for the market portfolio return, i.e. for $r^* \in \mathbb{R}$:

$$G(r^*) = \frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{r_{\bar{\omega},t} \leq r^*}.$$

As discussed above, we will use the distortion function (3.11) in our analysis. Applying this distortion to the empirical distribution G of the market portfolio return, we obtain the following reward-risk pricing kernel:

$$f_{R_{\bar{\omega}}}(r^*) = \frac{g\left(\frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{r_{\bar{\omega},t} \geq r^*}\right) - g\left(\frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{r_{\bar{\omega},t} > r^*}\right)}{\frac{1}{\tau} \sum_{t=1}^{\tau} \mathbf{1}_{r_{\bar{\omega},t} = r^*}}, \quad (3.12)$$

where $r^* \in \{r_{\bar{\omega},1}, \dots, r_{\bar{\omega},\tau}\}$. In this section, we will use $\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}}) = (f_{R_{\bar{\omega}}}(r_{\bar{\omega},1}), \dots, f_{R_{\bar{\omega}}}(r_{\bar{\omega},\tau}))$. The empirical deviations from the equilibrium condition (2.9), also known as pricing errors or alphas, are defined as

$$\hat{\boldsymbol{\alpha}} = \mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}})(\mathbf{R}' - \mathbf{e}_{\tau} \mathbf{e}'_{\tau} r) \quad (3.13)$$

where \mathbf{e}_{τ} is a $(1 \times \tau)$ unity vector. Alternatively, we can write the alphas in the following manner:

$$\hat{\boldsymbol{\alpha}} = (\hat{\boldsymbol{\mu}} - \mathbf{e}_{\tau} r) - \hat{\boldsymbol{\beta}}_{\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}})} (\hat{\boldsymbol{\mu}}_{\bar{\omega}} - r) \quad (3.14)$$

where $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_K)'$ and $\hat{\boldsymbol{\mu}}_{\bar{\omega}}$ are the sample means of the individual assets and the market portfolio respectively and

$$\hat{\boldsymbol{\beta}}_{\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}})} = \frac{(\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}}) - \mathbf{e}'_{\tau})(\mathbf{R}' - \mathbf{e}_{\tau} \hat{\boldsymbol{\mu}}')}{(\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}}) - \mathbf{e}'_{\tau})'(\mathbf{r}'_{\bar{\omega}} - \mathbf{e}_{\tau} \hat{\boldsymbol{\mu}}_{\bar{\omega}})} \quad (3.15)$$

are the reward-risk betas.

We now turn to the issue of statistical inference about the equilibrium condition (2.9) based on the estimated alphas. Under the null hypothesis, the alpha have means $\mathbb{E}[\hat{\boldsymbol{\alpha}}] = \mathbf{0}_K$. The covariance matrix $\boldsymbol{\Omega} = \mathbb{E}[\hat{\boldsymbol{\alpha}} \hat{\boldsymbol{\alpha}}']$ of the alphas can be estimated in a consistent manner by⁵

$$\hat{\boldsymbol{\Omega}} = \frac{1}{\tau} (\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}}) \odot \mathbf{R}) (\mathbf{f}_{R_{\bar{\omega}}}(\mathbf{r}_{\bar{\omega}}) \odot \mathbf{R})'. \quad (3.16)$$

In the spirit of the generalized Methods of Moments, we can use the following test statistic to aggregate the individual alphas⁶:

$$JT = \tau \hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Omega}}^{-1} \hat{\boldsymbol{\alpha}}. \quad (3.17)$$

Assuming that the observations are serially independently and identically distributed (iid) random draws, the test statistic obeys an asymptotic chi-squared distribution with $K - 1$ degrees of freedom⁷. We will compare the empirical performance of our reward-risk CAPM with that of the standard mean-variance CAPM. The mean-variance CAPM can be represented by the linear pricing kernel $f(R_{\bar{\omega}}) = a + b R_{\bar{\omega}}$ with $a = 1 + \frac{\hat{\mu}_{\bar{\omega}}^2}{\hat{\sigma}_{\bar{\omega}}^2}$ and $b = -\frac{\hat{\mu}_{\bar{\omega}}}{\hat{\sigma}_{\bar{\omega}}^2}$, where $\hat{\sigma}_{\bar{\omega}}^2$ is the sample variance of the market portfolio return⁸. We can then apply the same methodology as for the reward-risk CAPM. For the mean-variance CAPM, this methodology comes very close to the Gibbons, Ross and Shanken (1989) methodology. The difference is that GRS assume that the returns obey a multivariate normal distribution and hence are able to use the small sample distribution rather than the asymptotic distribution. Since we use large samples, both methodologies yield identical results in our study.

3.3 Data sets

In our analysis, we will employ benchmark portfolios that are formed on market beta and price momentum. Beta-sorted portfolios have been used extensively to test the mean-variance CAPM; see Black, Jensen, and Scholes (1972), Friend and Blume (1973), Fama and MacBeth (1973), Reinganum (1981) and Fama and French (1992), among others. The empirical results suggest that the mean-variance CAPM is violated, because the difference between the means of low-beta stocks and high-beta stocks is too small relative to the difference between the betas. In other words, by buying low-beta stocks and selling high-beta stocks, we can “beat the market” (achieve a higher Sharpe-ratio than the market portfolio).

More recently, much research attention has been focused on momentum portfolios (see for example Jagadeesh and Titman 1993, 2001). The empirical results suggest that the mean-variance CAPM is severely violated, because the short-term loser stocks have low means and high betas, while the short-term winner stocks have high means and low betas. Thus, we can beat the market by buying short-term winners and selling short-term losers. Part of this effect can be attributed to the high turnover and correspondingly high transactions costs of momentum strategies (see for instance Lesmond, Schill, and Zhou 2004). Thus, capital market models that assume a perfect capital market (including the mean-variance and reward-risk models) cannot be expected to completely explain the returns of momentum strategies. Still, momentum portfolios are an interesting test case for comparing the mean-variance and reward-risk models, because the returns to momentum strategies generally are characterized by asymmetry and hence the two models can be expected to yield different results⁹.

Our analysis uses monthly stock returns (including dividends and capital gains) for the period from January 1931 to December 2002 from the Center for Research in Security Prices (CRSP) at the University of Chicago. We select ordinary common US stocks listed on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and NASDAQ markets. We exclude ADRs, REITs, closed-end-funds, units of beneficial interest, and

foreign stocks. Stocks are required to have 60 months of prior return data available and information about the market capitalization at formation date; the past returns are needed for calculating beta and the market capitalization is required for constructing value-weighted portfolios. Thus, to be included at December 1930 a stock must have trading information since January 1926 and a (positive) market capitalization for December 1930. A stock is excluded from the analysis if there is no more return information available and the delisting return or partial monthly return provided by CRSP is then used for the last observation. On average 1,854 stocks are included in the portfolios, starting with 373 (in December 1930) and ending with 3,730 (in December 2002) after reaching a maximum of 3,907 (January 1999).

From the selected stocks, we form our beta portfolios and momentum portfolios. We sort the individual stocks into ten deciles based on a given stock characteristic (beta or momentum) and compute value-weighted portfolio returns of all stocks in each decile. Beta portfolios are based on historical prior 60-month market beta. These portfolios are formed at the end of each year, starting in December 1930, 60 months after January 1926, the start of the CRSP reporting for individual stocks. Momentum portfolios are based on the price performance during the period from 12 months ago to one month ago (past 12-1 returns), similar to Fama and French (1996). For these portfolios, formation takes place on a monthly basis rather than annually. The market index is a value-weighted average of all US stocks included in this study. The one-month US Treasury bill is obtained from Ibbotson Associates. Table I gives descriptive statistics for the excess returns of the benchmark portfolios and the market portfolio.

[Table 1 about here.]

3.4 Empirical results

As discussed in Section 3.2, we fix the risk aversion parameter γ such that the market portfolio alpha equals zero. Since both data sets use the same market portfolio (the CRSP all-share index) and the same sample period (January 1931 to December 2002), both data sets yield the same parameter value; $\gamma = 0.877$. Panel A of Figure 1 shows the associated distortion function $g(x)$. Clearly, the distortion function is convex, reflecting the risk aversion of the representative investor ($\gamma < 1$); the representative investor has a pessimistic view of the probabilities. Panel B shows the reward-risk pricing kernel $f_{\mathbb{R}_{\tilde{\omega}}}(R_{\tilde{\omega}})$. The pricing kernel is a parameterized function of the probabilities and depends indirectly on the market portfolio return via the link between returns and probabilities. Due to this indirect link, the pricing kernel has a “twisting” shape when plotted against the returns. Still, the pricing kernel is approximately linear for losses and approximately constant for gains. Interestingly, the mean-semivariance CAPM of Hogan and Warren (1974) and Bawa and Lindenberg (1977) predicts this shape for the pricing kernel; this model replaces variance with semivariance and yields a piecewise linear pricing kernel that is linear for losses and constant for gains.

[Figure 1 about here.]

The value for the risk aversion parameter γ is fixed without reference to the benchmark portfolios and it is not clear a priori that this value yields a good empirical fit for the benchmark portfolios. Table 2 shows the empirical fit for the mean-variance and reward-risk models. Confirming known results, we see a strong “beta effect”. Given the equity premium of 0.67% per month, the spread of the means (0.28% per month) is too small to be consistent with the spread of the mean-variance betas (0.77). This translates to large alphas, ranking from 0.180 for the lowest-beta stocks to -0.274 for the highest-beta stocks. The “momentum effect” is also clearly present. The short-term losers have a low mean (0.014) and a high mean-variance beta (-1.57), yielding an alpha of -1.031. By contrast, the short-term winners have a high mean (1.331) and a low mean-variance beta (1.002), which translates to an alpha of 0.662.

Interestingly, the reward-risk model gives a better fit than the mean-variance model. For both the beta portfolios and the momentum portfolios, the reward-risk betas are more in line with the means than the mean-variance betas are. For example, the beta of the lowest-beta portfolio increases from 0.48 to 0.49, while the beta of the highest-beta portfolio decreases from 1.31 to 1.26, reducing the beta spread from 0.83 to 0.77. The narrowing in the beta spread is translated into generally lower alphas. The overall JT statistic decreases from 13.55 to 11.87 and the associated p -value increases from 0.14 to 0.22.

For the momentum portfolios, the improvements are even greater. The beta spread increases from -0.55 to -0.35 and the overall JT statistic decreases from 61.57 to 50.59. However, the improvements do not suffice to yield a positive mean-beta relationship, and the reward-risk model still has to be convincingly rejected (p -value=0.00). Again, part of the momentum profits may be attributable to transaction costs and hence we do not expect any perfect-market model to rationalize the entire momentum effect. Still, the substantial reduction in the alphas suggests that the reward-risk model better captures the cross-section of returns than the mean-variance model does.

Since the reward-risk pricing kernel is similar to the two-piece pricing kernel of the mean-semivariance model, it is not surprising that these results are very similar to those of Post and Van Vliet (2004), who compared the mean-variance model and the mean-semivariance model.

[Table 2 about here.]

As a final illustration of the improvements from using the reward-risk model, Figure 2 shows the actual and predicted means for the beta portfolios and the momentum portfolios. We clearly see the improved fit for the beta portfolios due to the narrowing of the beta spread. We also see the improvements for momentum portfolios, despite the poor fit of both models (a flat/negative mean-beta relationship).

[Figure 2 about here.]

Despite the improved fit for beta and momentum portfolios, we stress that we have considered only one particular distortion function (10) and we have considered only beta and momentum portfolios. Further research could focus on a more rigorous comparison of the mean-variance and reward-risk models.

4 Conclusion

The “reward-risk CAPM” is the capital market equilibrium model associated with the general reward-risk portfolio model by De Giorgi (2004). Instead of arguing in favor of one particular risk measure, this model is based on general axioms for investors’ preferences, including isotonicity with respect to the second order stochastic dominance. Our analysis builds on three pillars.

First, we establish that a necessary condition for the existence of market equilibrium in complete markets with risk averse investors is that investors’ optimal allocations are comonotonic. If this is not the case, then for any given pricing kernel, there exists at least one investor with the marginal rate of substitution that is not collinear to the pricing kernel. Consequently, since markets are complete, for this investor there exist zero-costs allocations that increase her utility, thus arbitrage opportunities. In turn, comonotonicity implies that investors’ risk profiles are identical and, the market equilibrium and aggregate demand must be described with a representative agent model. This result is quite surprising: Capital market equilibria in complete markets with reward-risk investors requires homogenous preferences.

Second, we derive the pricing kernel as an explicitly given monotone decreasing function of market portfolio return, depending on the representative agent’s risk perception through his probability distortion function. The pricing kernel formulation also allows for the familiar formulation in terms of a trade-off between expected return and systematic risk, where systematic risk is measured by means of our risk-reward beta.

Third, we empirically compare the mean-variance CAPM with the “reward-risk” CAPM. For illustrative purposes, we impose one particular shape for the probability distortion function characterizing the representative investor’s risk profile and, we use canonical portfolios formed on market beta and price momentum to test market portfolio efficiency. Interestingly, the pricing kernel arising from the reward-risk analysis is similar to that obtained in the mean-semivariance equilibrium model. Moreover, the reward-risk model significantly improves the fit relative to the classical mean-variance model, because the reward-risk betas exhibit less dispersion than the standard betas.

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Notes

¹The Choquet Expected Utility Theory of Schmeidler (1989) generalizes the Expected Utility Theory by assuming that the independence axiom of von Neumann and Morgenstern (1944) only holds for comonotonic outcomes. Two random outcomes $X, Y \in \mathcal{G}$ are said to be comonotonic, if $X(s) \geq X(s') \Rightarrow Y(s) \geq Y(s')$ for all states of nature $s, s' \in \Omega$. Schmeidler (1989) shows that preferences that satisfy comonotonic independence, together with monotonicity, continuity (Archimedian axiom) and non-degeneracy, can be uniquely represented as expectation of a concave utility index with respect to a non-additive probability (called capacity). The motivation for the Choquet Expected Utility Theory comes from the observation by the Ellsberg (1961) experiment, that decision makers' behavior is inconsistent with the independence axiom of the classical Expected Utility Theory.

²The notion of “representative investor” considered here is that of “demand aggregation”, as defined by Rubinstein (1974). This is a stronger concept of “representative investor” than the “in-sample” notion of Huang and Litzenberger (1988), where the *given* equilibrium point is equivalently described as the solution of a single investor optimization problem. The stronger notion that we use allows “out-of-sample” predictions.

³For the sake of simplicity, we consider a finite number of possible state of nature S .

⁴For any integer $\gamma > 0$, this can be demonstrated by the means of the following chain of equalities:

$$\begin{aligned}\mathbb{P}\left[\max_{i=1,\dots,\gamma} R_i \geq r^*\right] &= 1 - \mathbb{P}\left[\max_{i=1,\dots,\gamma} R_i < r^*\right] = 1 - \mathbb{P}\left[R_i < r^*, i = 1, \dots, \gamma\right] \\ &= 1 - \mathbb{P}\left[R_i < r^*\right]^\gamma = 1 - (1 - \mathbb{P}\left[R_i \geq r^*\right])^\gamma = 1 - (1 - x)^\gamma = g(x).\end{aligned}$$

⁵Slightly departing from the conventional notation, the Hadamard product operator \odot is used here for element-by-element multiplication of the given column vector $\mathbf{f}_{R_\omega}(\mathbf{r}_\omega)$ with every column of the matrix \mathbf{R} .

⁶Strictly speaking, our statistical methodology applies only when the kernel is exogenously given. However, the distortion parameter gamma is determined to set the sample pricing error of the market portfolio equal to zero and thus depends on the return data. Still, the univariate return distribution of the market portfolio, which determines gamma, is far less sensitive to sampling variation than the multivariate return distribution of the ten benchmark portfolios, which determines the pricing errors. Indeed, a bootstrapping exercise revealed that gamma is not materially affected by sampling variation. For this reason, we treat the kernel as exogenously given in our analysis. We thank Fabio Trojani for pointing out this potential problem.

⁷The reduction of one degree of freedom occurs due to the restriction that the alpha of the market portfolio should equal zero. Thus, in the case of a single risky asset ($K = 1$), the market portfolio is fully efficient and $JT = 0$ by construction. More generally, for K assets, the test statistics behaves as the sum of squares of $K - 1$ contemporaneously identically distributed and independent random variables.

⁸This function is found by selecting the parameters a and b such that the sample mean of $f_{R_{\bar{\omega}}}(R_{\bar{\omega}})$ equals unity and the sample alpha of the market portfolio equals zero.

⁹Under a normal distribution, all representative agent models that are consistent second-order stochastic dominance, including the reward-risk model, yield the same prediction as the mean-variance model.

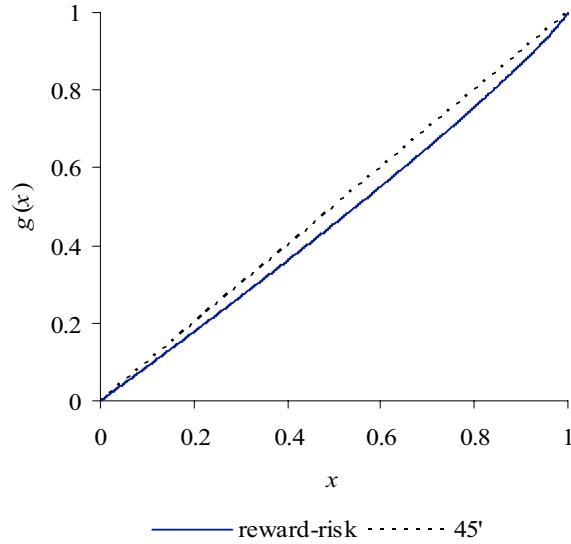
		Mean	Stdev.	Skewness	Kurtosis	Min	Max
	Market	0.668	5.467	0.457	8.909	-28.59	38.57
Beta Portfolios	Low	0.601	4.257	-0.151	6.136	-23.51	25.93
	2	0.620	4.643	0.399	7.954	-23.00	34.53
	3	0.666	5.231	0.549	9.399	-25.76	39.45
	4	0.634	5.486	0.562	8.919	-28.65	40.97
	5	0.852	6.337	0.813	10.182	-29.60	52.58
	6	0.721	6.770	0.771	10.408	-36.07	49.72
	7	0.766	7.189	0.773	10.489	-33.19	58.67
	8	0.772	8.219	1.446	14.445	-34.39	69.63
	9	0.853	9.153	1.483	14.728	-37.26	80.11
	High	0.885	10.386	1.191	10.980	-37.91	86.49
Momentum Portfolios	Loser	0.014	10.200	1.923	16.170	-43.17	94.38
	2	0.411	8.607	2.025	21.104	-43.37	84.03
	3	0.444	7.243	1.656	18.435	-38.02	65.34
	4	0.618	6.616	1.414	14.977	-32.11	55.17
	5	0.570	6.320	1.342	16.800	-33.08	57.66
	6	0.602	5.977	0.663	11.134	-31.37	49.58
	7	0.715	5.687	0.008	6.746	-31.68	35.05
	8	0.921	5.673	0.404	6.349	-24.83	40.56
	9	0.939	5.961	-0.170	4.437	-28.78	38.43
	Winner	1.331	6.696	-0.281	2.212	-28.21	32.44

Table 1: The table shows descriptive statistics for the monthly return of the value-weighted CRSP all-share market portfolio and the ten beta portfolios and the ten momentum portfolios. The sample period is from January 1931 to December 2002 ($\tau = 864$). Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill.

		Mean-Variance CAPM		Reward-Risk CAPM	
		$\hat{\alpha}^i$	$\hat{\beta}^i$	$\hat{\alpha}^i$	$\hat{\beta}^i$
Beta Portfolios	Low	0.180	0.476	0.166	0.491
	2	0.107	0.579	0.109	0.577
	3	0.071	0.672	0.082	0.660
	4	0.000	0.716	0.009	0.706
	5	0.115	0.833	0.135	0.810
	6	-0.068	0.892	-0.052	0.874
	7	-0.072	0.947	-0.057	0.930
	8	-0.178	1.073	-0.132	1.021
	9	-0.188	1.176	-0.136	1.118
	High	-0.274	1.310	-0.233	1.263
		<i>JT</i>	<i>p</i> -value	<i>JT</i>	<i>p</i> -value
		13.546	0.139	11.872	0.221
		Mean-Variance CAPM		Reward-Risk CAPM	
		$\hat{\alpha}^i$	$\hat{\beta}^i$	$\hat{\alpha}^i$	$\hat{\beta}^i$
Momentum Portfolios	Loser	-1.031	1.566	-0.952	1.446
	2	-0.508	1.377	-0.437	1.271
	3	-0.359	1.203	-0.309	1.129
	4	-0.132	1.123	-0.089	1.059
	5	-0.155	1.087	-0.123	1.037
	6	-0.093	1.042	-0.085	1.029
	7	0.053	0.992	0.038	1.014
	8	0.268	0.978	0.267	0.980
	9	0.273	0.998	0.240	1.048
	Winner	0.662	1.002	0.599	1.097
		<i>JT</i>	<i>p</i> -value	<i>JT</i>	<i>p</i> -value
		61.567	0.000	50.530	0.000

Table 2: This table shows the empirical results for the mean-variance and reward-risk model for the beta portfolios and momentum portfolios. The results include the alpha ($\hat{\alpha}_i$) and the beta ($\hat{\beta}_i$) of each portfolio as well as the overall *JT* statistic and the associated *p*-value. the sample period is from January 1031 to December 2002 ($\tau = 864$).

Panel A. Distortion function $g(x)$



Panel B: Pricing kernel $f_{R_{\bar{\omega}}}(\cdot)$

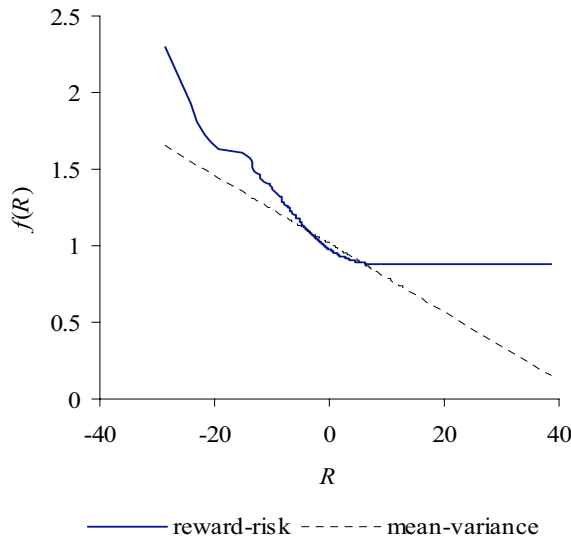
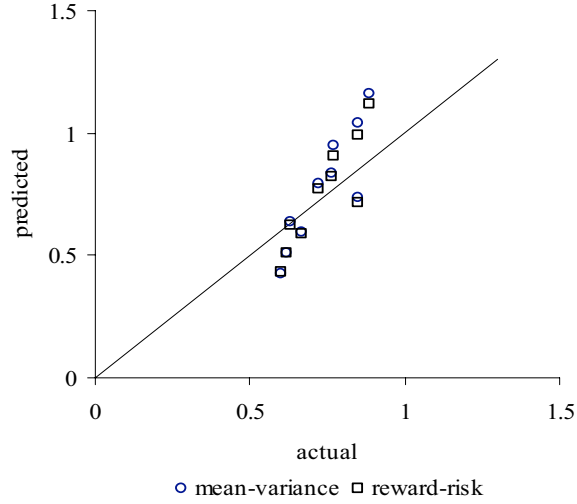


Figure 1: The figure shows the distortion function $g(x)$ (Panel A) and the pricing kernel $f_{R_{\bar{\omega}}}(\cdot)$ (Panel B) used in the empirical analysis of the beta data set and the momentum data set. The underlying value $\gamma = 08.77$ is selected such that the alpha of the market portfolio equals zero. The distortion function and pricing kernel apply for both data sets, because both data sets use the same market portfolio (the CRSP all-share index) and the same sample period (January 1931-December 2002). In Panel A, a 45° line is added for ease of interpretation. In Panel B, the linear mean-variance kernel is added.

Panel A: Beta portfolios



Panel B: Momentum portfolios

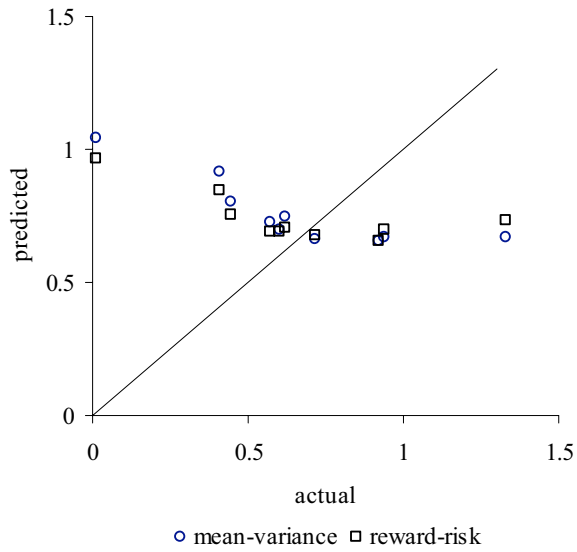


Figure 2: The figure shows the actual means of the beta portfolio (Panel A) and the momentum portfolios (Panel B) and the predicted means of the mean-variance model (the dots) and the reward-risk model (the boxes). The predicted mean of a portfolio equals the actual mean (see Table 1) minus the alphas (see Table 2). The parameters of both models are fixed by setting the alpha of the market portfolio in the sample period (January 1931-December 2002) equal to zero. In both panels, a 45° line is added for ease of interpretation.