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**The Distribution of Money Balances and the  
Non-Neutrality of Money**

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# The Distribution of Money Balances and the Non-Neutrality of Money\*

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## Abstract

Recent monetary models with explicit microfoundations are made tractable by assuming that agents have access to centralized markets after one round of decentralized trade. Given quasi-linear preferences, this makes the distribution of money degenerate – which keeps the models simple but precludes discussion of distributional effects of monetary policy. We generalize these models by assuming two rounds of trade before agents can readjust their money holdings to study a range of new distributional effects analytically. We show that unexpected, symmetric lump-sum money injections may increase short-run output and welfare, while asymmetric injections may increase long-run output and welfare.

Running Head: DISTRIBUTION AND NON-NEUTRALITY

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<sup>1</sup>The paper has benefitted from comments from Nobuhiro Kiyotaki, Guillaume Rocheteau, Carlo Strub, Randall Wright, Neil Wallace and participants at the monetary theory workshop at the Federal Reserve Bank of Cleveland, the EPRI/University of Western Ontario Money Conference, the SED-meeting in Paris and numerous university seminars. Please address correspondence to: Christopher Waller, Department of Economics and Econometrics, University of Notre Dame, 434 Flanner Hall, Notre Dame, IN 46556. Phone: 574-631-4963. Fax: 574-631-4783. E-mail: cwaller@nd.edu

# 1 Introduction

Following Lagos and Wright (2004), recent monetary models with explicit micro-foundations are made tractable by assuming that, between meetings of decentralized markets, agents have access to a centralized market that allows them to rebalance their money holdings. Given quasi-linear preferences, this makes the distribution of money degenerate – which keeps the models simple but precludes discussion of the distributional effects of policy. We generalize these models by assuming two rounds of trade before agents can rebalance their money holdings. This entails little loss in tractability, and allows us to study a wide range of new distributional effects related to monetary policy. We show that unexpected lump-sum money injections may increase short-run output and welfare. We also show that asymmetric money injections may increase long-run output and welfare.

The notion of long-run money neutrality but short-run non-neutrality dates back to Hume and has given rise to a large body of theoretical and empirical research (see Lucas, 1996). It is clear from this research that the answer one obtains hinges on the trading environment and the manner in which the money supply changes. Monetary injections can be non-neutral if prices are rigid, there are informational frictions, transfers are asymmetric, or there is a non-degenerate distribution of money. The problem with many of these models is that money is either assumed exogenously to be needed for trading or, if money is essential for trade, the models are so complex that they cannot be studied analytically.<sup>2</sup> We think that developing monetary models

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<sup>2</sup>Informational frictions generate non-neutralities for example in Lucas (1972), Katzman, Ken-

that allow us analytically to study non-neutrality and possible efficiency gains from monetary policy requires constructing models where money is essential. Wallace (1998) forcefully argues that we should construct models that explain *why* money is necessary and then proceed to study *how* monetary policy affects the economy. Our paper contributes to the growing literature in monetary economics that adopts this strategy.

In particular, our model is an attempt to combine the simplicity of the Lagos and Wright framework, as described above, with the ability to analyze distributional effects analytically that arise in much more complicated models.<sup>3</sup> The following results emerge. First, the Friedman rule, a policy that makes the expected return on money equal to the real interest rate, attains the first-best allocation. Moreover, at the Friedman rule random monetary injections are neutral. Under this rule, holding money is costless so agents are never cash constrained no matter what money shock prevails. This differs from Bewley's (1980) model because agents do not face an infinite sequence of random consumption opportunities (see Green and Zhou 2004, 

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nan, and Wallace (2004), and Wallace (1997). Examples where non-neutralities occur because of asymmetric injections are Kiyotaki and Wright (1993) type models, limited participation in financial markets (Lucas 1990, Fuerst 1992, Williamson 2004), segmented markets (Alvarez, Atkinson and Kehoe 2002), or overlapping generation models. Heterogeneity in money holdings play a role in for example in Scheinkman and Weiss (1986), Levine (1991), İmrohorğlu (1992), Camera and Corbae (1999), Molico (1999), Berentsen (2002), Deviatov and Wallace (2002), Zhu (2003), Berentsen, Camera and Waller (2004) and Bhattacharya, Haslag and Martin (forthcoming).

<sup>3</sup>İmrohorğlu (1992) and Molico (1999) for example solve numerically for the distribution of money holdings.

forthcoming). Our framework gives agents a chance to readjust their money balances after a finite number of trades. This has two consequences: it allows agents to undo their trading histories in finite time and it makes the future predictable. Consequently, in contrast to the Bewley model, the optimal quantity of money is finite, and agents at the Friedman rule are willing to hold it since holding money is costless.

Second, away from the Friedman rule, random monetary injections can be non-neutral even though all prices change proportionately. In particular, an unexpectedly high money growth rate can cause aggregate output to increase. These results occur even though injections are symmetric across agents and prices are fully flexible. We show that these non-neutralities only exist if the injections take place at a time when the distribution of real balances is not degenerate. Essentially what a high money injection provides is consumption insurance. This raises the question of whether monetary policy can be used to provide consumption insurance in a deterministic fashion. We therefore consider a deterministic version of the model where agents receive asymmetric transfers. We show that changes in the asymmetry of transfers have no real effects if the rate of inflation is low. In contrast, for high rates of inflation, such a change can raise aggregate output and welfare permanently.

The paper proceeds as follows. Section 2 describes the environment. In Section 3 we discuss the agents' decision problems and derive the equilibrium. In Section 4 we investigate the effects of asymmetric injections. The last section concludes. All proofs are in the appendix.

## 2 The Environment

The basic environment is that of Lagos and Wright (2004). Time is discrete and in each period a  $[0, 1]$  continuum of infinitely-lived agents trade on three Walrasian markets, that open and close sequentially. Only one market, denoted by  $j = 1, 2, 3$ , is open at any one time.<sup>4</sup>

One perishable good is produced and consumed by all agents. Before entering the first two markets an agent receives one of two equally probable consumption/production shocks. He may want to consume or produce but not both. As a result, there is an equal number of consumers and producers in each market. Agents get utility  $u(q)$  from consuming  $q > 0$  in the first two markets, where  $u'(q) > 0$ ,  $u'(0) = \infty$ ,  $u'(+\infty) = 0$ ,  $u''(q) < 0$  and  $u'''(q) \geq 0$ . In the last market all agents consume and produce, getting utility  $U(q)$  from consuming  $q$ , with  $U'(q) > 0$ ,  $U'(0) = \infty$ ,  $U'(+\infty) = 0$  and  $U''(q) < 0$ .<sup>5</sup> Production of  $q$  output generates disutility  $q$ . The discount factor across dates is  $\beta \in (0, 1)$ .

To rule out credit and motivate fiat monetary exchange, we assume anonymous trading, no record-keeping and no enforcement of contracts. This is sufficient to

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<sup>4</sup>In addition of having two trading rounds before the centralized market opens, our set-up departs from Lagos and Wright (2004) in two other dimensions. First, as in Rocheteau and Wright (2004) we assume that all markets are Walrasian rather than assuming bilateral bargaining, since, although one can get similar results under bargaining, this simplifies the presentation. Second, in place of random matching agents receive preference shocks at the beginning of the period that makes them either buyers or sellers, as in Arouba, Waller and Wright (2004).

<sup>5</sup>The difference in preferences over the good sold in the last market is a technical device we use to ensure a degenerate distribution of money holdings, at the beginning of a period.

make a medium of exchange essential for trade (Kocherlakota, 1998), and given that all goods are perishable, this role will be played by fiat money. In order to study the non-neutralities of money we want to see how unexpected changes of the money supply affect consumption and production. To this end, we assume that the law of motion of the money stock is  $M_t = z_t M_{t-1}$ , where  $z_t$  is a random variable such that

$$z_t = \begin{cases} z^H = \mu(1 + \varepsilon^H) & \text{with probability } \pi \\ z^L = \mu(1 - \varepsilon^L) & \text{with probability } 1 - \pi. \end{cases}$$

We assume  $\mu, \varepsilon^L, \varepsilon^H > 0$  and  $\pi = \frac{\varepsilon^L}{\varepsilon^H + \varepsilon^L}$  so that  $E(z_t) = \mu$ .

Money is injected via lump-sum transfers  $\tau_t = (z_t - 1)M_{t-1}$ , after the closing of market 1 in period  $t$ , but prior to the realization of individual trading shocks. In short, at the beginning of the second market one of two states, denoted  $i = H, L$ , can be realized. In one state money growth is high,  $z^H$ , in the other it is low,  $z^L$ .

We refer to  $M_{t-1}$  as the beginning-of-period money supply for date  $t$ . This is the money supply existing in the economy before the shock takes place. We refer to  $M_t$  as the money supply present in the market at the end of period  $t$ , after the shock is realized. This is the money stock available at the beginning of period  $t + 1$ .

## 2.1 Sequential market trades

In period  $t$ , let  $p_{j,t}$  be the nominal price in market  $j$ , and  $\phi_t = 1/p_{3,t}$  be the real price in the last market. We study equilibria where end-of-period real money balances are time-invariant

$$(1) \quad \phi_t M_t = \phi_{t+1} M_{t+1}.$$

We refer to this as a stationary equilibrium. For this reason we omit the time subscript when understood, and study a representative period working backwards from last to first market, within the period. In the steady state it then follows that prices change instantly and proportionately since

$$(2) \quad \frac{\phi^H}{\phi_{-1}} = \frac{1}{z^H} \text{ and } \frac{\phi^L}{\phi_{-1}} = \frac{1}{z^L}$$

where  $\phi^j$  is the price of money in state  $j = L, H$ .

We also have that average inflation is  $E\left(\frac{\phi_t}{\phi_{t+1}}\right) = E(z_t) = \mu$ , while the average gross real return on money is

$$R = E_t \left[ \frac{\phi_{t+1}}{\phi_t} \right] = E_t \left[ \frac{M_t}{M_{t+1}} \right] = \frac{1}{\mu} \frac{1 + \varepsilon_H - \varepsilon_L}{(1 + \varepsilon_H)(1 - \varepsilon_L)} \equiv \frac{1}{\gamma},$$

which is negatively associated with expected inflation  $\mu$ . In our analysis we will focus on  $\gamma$  rather than  $\mu$  since  $\gamma$  is proportional to  $\mu$ .

Note that the Friedman rule in this model corresponds to a policy that sets the expected return on money  $\frac{1}{\gamma}$  equal to the real interest rate  $\frac{1}{\beta}$ . This implies that with a stochastic policy the Friedman rule requires less deflation than in a deterministic model, i.e., it requires the average gross inflation rate  $\mu$  to be above  $\beta$ . To see this observe that  $\mu > \gamma$  since  $\frac{1 + \varepsilon_H - \varepsilon_L}{(1 + \varepsilon_H)(1 - \varepsilon_L)} > 1$ . In fact, the Friedman rule may require a positive average rate of inflation. For example, if  $\varepsilon = \varepsilon^H = \varepsilon^L$ , then one can show that the inflation under the Friedman rule is strictly positive if  $\varepsilon > \sqrt{1 - \beta}$ .

Let  $V_j(m_j)$  denote the expected value from trading in market  $j$  with  $m_j$  money. Let  $q_{jb}$  and  $q_{js}$  respectively denote the quantities bought or sold by an agent trading in market  $j$ . We let  $q_3^*$  satisfy  $U'(q_3^*) = 1$  and  $q^*$  satisfy  $u'(q^*) = 1$ .



### 2.1.1 The last market

In the last market agents can produce and consume. They choose how much to buy,  $q_{3b}$ , how much to sell,  $q_{3s}$ , and how much money to take into the next period,  $m_{1,+1}$ .

As a result, the representative agent's program is

$$\begin{aligned} V_3(m_3) &= \max_{q_{3b}, q_{3s}, m_{1,+1}} [U(q_{3b}) - q_{3s} + \beta V_1(m_{1,+1})] \\ \text{s.t. } & q_{3b} + \phi m_{1,+1} = q_{3s} + \phi m_3 \end{aligned}$$

Substituting for  $q_{3s}$  yields

$$(3) \quad V_3(m_3) = \phi m_3 + \max [U(q_{3b}) - q_{3b} - \phi m_{1,+1} + \beta V_1(m_{1,+1})]$$

where  $(q_{3b}, m_{1,+1})$  are choice variables, hence the conditions for maximization are

$$(4) \quad \begin{aligned} U'(q_{3b}) &= 1 \\ -\phi + \beta V_1'(m_{1,+1}) &= 0. \end{aligned}$$

The envelope condition is

$$(5) \quad V_3'(m_3) = \phi.$$

There are two key results. First, trades are always efficient in the last market, since  $q_{3b} = q_3^*$  always and for every agent. Second, and most importantly, the distribution of beginning-of-period money holdings is degenerate. This is because  $m_{1,+1}$  is chosen independently of  $m_3$ . It follows that in equilibrium everyone exits the last market with identical money holdings, regardless of how much money they brought into the last market. Those who bring excessive money into the last market, spend some on goods, while those with too little money sell output.<sup>6</sup> This feature of the Lagos and

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<sup>6</sup>Conditions need to be imposed to ensure  $q_{s3} \geq 0$ . See later.

Wright model makes the distribution of money degenerate at the beginning of market one.

### 2.1.2 The second market

Conditional on the realization of the shock  $z_t$ , an agent who has  $m_2$  money balances at the opening of the second market, at any date  $t$ , has expected lifetime utility

$$(6) \quad V_2(m_2) = \frac{1}{2} [u(q_{2b}) + V_3(m_2 - p_2 q_{2b})] + \frac{1}{2} [-q_{2s} + V_3(m_2 + p_2 q_{2s})].$$

Here  $p_2 q_{2b}$  is the amount of money spent when buying  $q_{2b}$  goods, and  $p_2 q_{2s}$  is the money received when selling  $q_{2s}$  goods.

The agent chooses quantities to buy and sell, taking the price  $p_2$  as given. Specifically, as a seller, the agent chooses  $q_{2s}$  to maximize  $-q_{2s} + V_3(m_2 + p_2 q_{2s})$ . This yields the first-order condition

$$(7) \quad p_2 V_3'(m_2 + p_2 q_{2s}) = 1 \Rightarrow p_2 = p_3 = \frac{1}{\phi}$$

where we have used (5). That is prices in the last two markets must be equal and are pinned down by the value of money in the last market. The intuition is that the seller can acquire a unit of money in the second or the third market and will do so at the lowest cost. Since sellers have linear production costs, if  $p_2 > p_3$  it is cheaper to acquire money in the second market and vice versa if  $p_2 < p_3$ . At price  $p_2 = p_3$  sellers are indifferent. This also implies that they are willing to supply all that is demanded, so the supply curve in the second market is flat.<sup>7</sup>

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<sup>7</sup>With a strictly convex cost function, the first-order condition is  $\frac{p_2}{c'(q_{2s})} = p_3$ . This would make the analysis more complicated but it would not change the results qualitatively.

As a buyer, the agent chooses  $q_{2b}$  to maximize his expected utility  $u(q_{2b}) + V_3(m_2 - p_2q_{2b})$ , given the constraint  $p_2q_{2b} \leq m_2$ . Letting  $\lambda_2 \geq 0$  be the multiplier on this constraint, the conditions for maximization are

$$(8) \quad \begin{aligned} u'(q_{2b}) &= p_2V_3'(m_2 - p_2q_{2b}) + p_2\lambda_2 \\ \lambda_2(m_2 - p_2q_{2b}) &= 0. \end{aligned}$$

We can now state

**Lemma 1** *Let  $m^* = q^*/\phi$ . In equilibrium, if*

(i)  $m_2 < m^*$  then  $\lambda_2 > 0$ ,  $q_{2b} = \phi m_2 < q^*$  and  $V_2(m_2)$  is strictly increasing and concave;

(ii)  $m_2 \geq m^*$  then  $\lambda_2 = 0$ ,  $q_{2b} = q^* \leq \phi m_2$  and  $V_2(m_2)$  is strictly increasing and linear.

The key implication is that trades in the second market are inefficient,  $q_{2b} < q^*$ , if the buyer is cash constrained,  $m_2 < m^*$ . Otherwise, they are efficient. To see why, in the appendix we show that if the constraint is binding,

$$(9) \quad V_2'(m_2) = \frac{\phi}{2} [u'(q_{2b}) + 1] > \phi$$

whereas if it is not binding

$$(10) \quad V_2'(m_2) = \phi.$$

Intuitively, if  $m_2 \geq m^*$ , then a buyer spends only part of his money and carries the rest into the last market. Thus, the marginal value of money for an agent entering market 2 with  $m_2 \geq m^*$  is simply  $\phi$ . The reason is, whether he ends up buying or

selling, the agent will not spend all his balances. If the agent enters market 2 with less than  $m^*$ , however, he is constrained as a buyer. Therefore, the marginal value of money is greater than  $\phi$  and has two components. With probability 1/2, the agent sells so he does not spend any money and values an extra dollar simply by  $\phi$ . With probability one half the agent buys, in which case an extra dollar buys  $\phi$  goods giving marginal utility  $u'(q_{2b})$ .

### 2.1.3 The first market

An agent starting a period with  $m_1$  money has expected lifetime utility

$$(11) \quad V_1(m_1) = \frac{1}{2} [u(q_{1b}) + EV_2(m_1 - p_1 q_{1b} + \tau)] + \frac{1}{2} [-q_{1s} + EV_2(m_1 + p_1 q_{1s} + \tau)]$$

where  $p_1 q_{1s}$  and  $p_1 q_{1b}$  are, respectively, the amounts of money received as a seller and spent as a buyer. Notice that agents take into account that they will receive a random nominal transfer  $\tau$  at the beginning of market 2.

As a seller, the agent chooses  $q_{1s}$  to maximize  $-q_{1s} + EV_2(m_1 + p_1 q_{1s})$ , taking the price  $p_1$  as given. This yields the first-order condition

$$(12) \quad p_1 EV_2'(m_1 + p_1 q_{1s} + \tau) = 1.$$

Production takes place until the expected marginal value of money,  $EV_2'(m_1 + p_1 q_{1s} + \tau)$ , equals its real price  $1/p_1$ . This money can be used to buy consumption in markets that open later.

As a buyer, the agent chooses  $q_{1b}$  to maximize  $u(q_{1b}) + EV_2(m_1 - p_1 q_{1b} + \tau)$  subject to the constraint  $p_1 q_{1b} \leq m_1$ . Letting  $\lambda_1 \geq 0$  be the multiplier on this

constraint, the conditions for maximization are

$$(13) \quad \begin{aligned} u'(q_{1b}) &= p_1 EV'_2(m_1 - p_1 q_{1b} + \tau) + p_1 \lambda_1 \\ \lambda_1(m_1 - p_1 q_{1b}) &= 0. \end{aligned}$$

We then have

**Lemma 2** *In equilibrium  $\lambda_1 = 0$ ,  $q_{1b} = q_{1s} = q_1 \leq q^*$ ,  $q_1 < m_1/p_1$ , and  $V_1(m_1)$  is strictly increasing and concave.*

The main implication of Lemma 2 is that  $q_1 < m_1/p_1$ . The marginal value of consuming even a little bit in the second market is very high for every agent, should a consumption opportunity arise. Consequently, agents always want to carry some cash into the second market.

We also have

$$(14) \quad V'_1(m_1) = \frac{1}{2p_1} [u'(q_{1b}) + 1]$$

that is, the marginal value of money at the opening of the first market is given by an expression similar to (9). The difference is that  $1/p_2$  is equal to  $\phi$  whereas  $1/p_1$  may not be. This possible price dispersion across markets plays a role in some of our results.

### 3 Equilibria

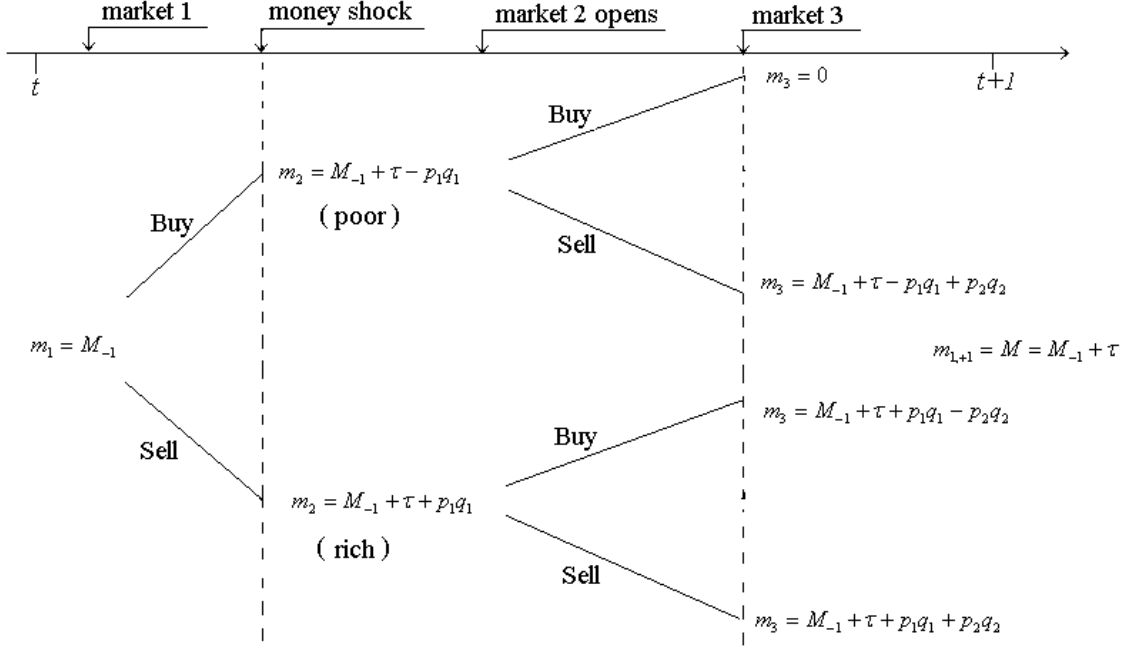
A key feature of our model is that the idiosyncratic consumption and production shocks generate intra-period heterogeneity in money balances. As we demonstrate

later, the existence of a non-degenerate distribution of money holdings is what opens the door to possible beneficial effects of money creation in the short-run.

More precisely, every agent enters a period with  $m_1 = M_{-1}$  money, i.e., the money stock from the prior period (see Figure 1). Then, agents are randomly divided into buyers and sellers in market 1. Buyers reduce their money holdings by  $p_1 q_1$  and sellers acquire  $p_1 q_1$ . Then the money injection occurs. Consequently, when the second market opens half of the agents will be ‘poor’, holding  $M_{-1} + \tau - p_1 q_1$  units of money, and half will be ‘rich’, holding  $M_{-1} + \tau + p_1 q_1$ . Then, agents will once more be divided into sellers and buyers. Since the marginal cost of production is constant in equilibrium sellers are indifferent to how much to produce. For simplicity, we assume that all sellers produce the same amount.<sup>8</sup> Therefore, when market 3 opens the support of the distribution of money will have four mass points. However, all agents leave market 3 with the same money holdings  $m_{1,+1} = M$ .

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<sup>8</sup>This indifference would vanish if we had increasing marginal cost. However, it would greatly complicate the analysis without changing the basic results.



Sequence of events

From what we have learned so far, consumption may differ across buyers only in the second market, due to heterogeneity in money holdings. Thus, let  $(q_2^p, q_2^r)$  and  $(\lambda_2^p, \lambda_2^r)$  denote the values of consumption and multipliers of, respectively, poor and rich buyers in market 2, contingent on the realization of state  $i = H, L$ . If, by a small abuse in notation, we let  $q_2 = (q_2^{pH}, q_2^{pL}, q_2^{rH}, q_2^{rL})$ ,  $\lambda_2 = (\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH}, \lambda_2^{rL})$ , and let  $\bar{m}_j$  denote the vector of possible money holdings at the opening of market  $j$  we can state the following

**Definition 1** A stationary monetary equilibrium is a list  $\{p_j, q_j, m_j\}_{j=1}^3$  and  $\{\lambda_1, \lambda_2\}$  that satisfy (1)-(4), (6)-(8), and (11)-(13).

In the proof of Proposition 1 we show that there exist critical values  $\gamma_1 \leq \gamma_2 \leq \gamma_3$  such that the following is true.

**Proposition 1** *A stationary monetary equilibrium exists only if  $\gamma \geq \beta$ . An equilibrium exists and is unique for  $\gamma \in (\beta, \gamma_2]$  where*

*i) for  $\gamma \in (\beta, \gamma_1]$ ,  $q_2^{pL} < q_2^{pH} = q^*$  and  $q_2^{rL} = q_2^{rH} = q^*$ ;*

*ii) for  $\gamma \in (\gamma_1, \gamma_2]$ ,  $q_2^{pL} < q_2^{pH} < q^*$  and  $q_2^{rL} = q_2^{rH} = q^*$ .*

*For  $\gamma > \gamma_2$  if an equilibrium exists then*

*iii) for  $\gamma \in (\gamma_2, \gamma_3]$ ,  $q_2^{pL} < q_2^{pH} < q^*$  and  $q_2^{rH} < q_2^{rL} = q^*$ ;*

*iv) for  $\gamma > \gamma_3$ ,  $q_2^{pL} < q_2^{pH} < q^*$  and  $q_2^{rH} < q_2^{rL} < q^*$ .*

Since  $\gamma$  is monotonically increasing in the expected gross inflation rate  $\mu$ , Proposition 1 also characterizes the monetary equilibrium as a function of  $\mu$ . When  $\gamma < \gamma_2$  we refer to this economy as the *low* inflation economy, and when  $\gamma \geq \gamma_2$  we call it the *high* inflation economy. In the low inflation economy rich buyers are never constrained. In the high inflation economy they can be constrained. Although we cannot prove existence of equilibrium for general utility functions when inflation is high, we can do so for particular functions.

**Corollary 1** *All quantities less than  $q^*$  are strictly decreasing in  $\gamma$  and approach  $q^*$  as  $\gamma \rightarrow \beta$ . Consequently, the Friedman rule attains the first-best allocation.*

Corollary 1 is a standard result. An increase in the money growth rate decreases the value of money which reduces consumption (e.g. see Lagos and Wright 2004 or Shi 1997). In contrast to Lagos and Wright (2004) we obtain the first-best under the Friedman rule because we have competitive markets and not Nash bargaining.

It is worthwhile to note that under the Friedman rule randomness of the monetary policy rule is completely irrelevant for the allocation in the first two markets. That



is the first-best is attained for any policy which involves  $\gamma \rightarrow \beta$ . To see this note that if the policy is deterministic, i.e.  $\varepsilon_H = \varepsilon_L = 0$ , then the Friedman rule requires that  $\mu \rightarrow \beta$ . If the policy is random, the rule requires that  $\mu \frac{(1+\varepsilon_H)(1-\varepsilon_L)}{1+\varepsilon_H-\varepsilon_L} \rightarrow \beta$ . In both cases, all buyers consume  $q^*$  in the first two markets. The intuition is that under the Friedman rule, the expected opportunity cost of holding money is zero so agents take enough money to buy the efficient quantity in both markets for all states.

**Proposition 2** *Shocks to the money supply are non-neutral in the short run.*

Proposition 2 summarizes the main result of our paper: monetary shocks have real effects for individual agents in all equilibria. The non-neutrality is a direct consequence of the distribution of money holdings. However, there is no persistence on quantities from these shocks. So all real effects are temporary. It can be shown that random injections in Lagos and Wright (2004) are neutral regardless of when they occur. The same is true in our model if they occur when the distribution is degenerate, i.e. in markets 1 or 3.

The intuition for this result is straightforward. After any shock to the money supply prices change proportionately. When money is higher than expected, the price increase reduces the real balances of every agent, acting as a *proportional* tax on their money holdings. Since in market 2 money holdings are heterogeneous, those who hold less cash are taxed less than those who hold more. This allows poor buyers—who are cash strapped—to increase their consumption in market 2 because the lump-sum transfer more than offsets the inflation tax. In contrast, rich buyers lose real wealth even after accounting for the lump-sum transfer. This does not affect their

consumption when inflation is low because in this case rich buyers are not constrained by their cash holdings. However, if inflation is high the inflation tax created by the surprise injection reduces the consumption of rich buyers because they are also cash constrained in market 2. Effectively, an unanticipated increase in the money stock redistributes real wealth from those with more to those with less through the price increase.

**Proposition 3** *Consider an unexpected increase in the money supply. For  $\gamma < \gamma_3$  aggregate output is higher than average. For  $\gamma \geq \gamma_3$  aggregate output is unaffected by the money supply shock.*

It is clear that aggregate output is increasing in the low inflation economy since rich buyers do not change their consumption while poor buyers consume more. In contrast, in the high inflation economy when all agents are constrained, i.e. case (iv) in Proposition 1, aggregate output is unaffected by the monetary shock since rich buyers reduce their consumption by the same amount as the poor buyers increase theirs.

In summary, the short-run non-neutralities of our model hinge on three key elements. First, monetary injections must be unanticipated. Second, such injections must take place when agents hold different amounts of money. Third, inflation cannot be out of hand, otherwise every buyer would be cash constrained and so aggregate output is unaffected. What our results do *not* hinge on are any price rigidities, information frictions, or asymmetric injections across agents. Similar effects are reported for example in Molico (1999). In contrast to his model, our results do not rely on

numerical simulations, but are derived analytically.

## 4 Consumption insurance

Clearly, the high money shock is welfare improving. It raises consumption of poor buyers without affecting the consumption of the rich buyers in the low inflation equilibrium. Although it lowers their consumption in the high inflation economy, it also increases the poor buyers' consumption by the same amount. Since the rich have a lower marginal utility of consumption than the poor there is still a potential for welfare gains from this redistribution. Unfortunately, the low money supply shock does just the opposite. So it is hard to imagine that these random injections improve welfare on average. From the perspective of the representative agent at the start of market 1, a high money shock acts like consumption insurance. This suggests that a scheme that transfers real balances from agents when they are rich to when they are poor in all periods would be welfare improving. In the following we explore this issue.

Let us assume that  $z^H = z^L = \gamma$  so that the money supply is deterministic. With this process the only possible equilibrium allocations are the ones described in (ii) and (iv) of Proposition 1. From (ii) in Proposition 1 rich buyers are unconstrained and from (iv) they are constrained.

Assume further that the perfectly anticipated lump-sum transfer received by agents depends on their trading state in the first market as follows. Each agent who drew a consumption opportunity in the first market gets the transfer  $\tau_t^p =$

$(\gamma - 1)xM_t$  and each agent who drew a production opportunity gets the transfer  $\tau_t^r = (\gamma - 1)(2 - x)M_t$  where  $x \in [0, 2]$ . This allows us to consider, for example, symmetric transfers ( $x = 1$ ) as in the previous section, transfers only to the poor ( $x = 2$ ) and transfers only to the rich ( $x = 0$ ).

Our new assumptions do not affect the equations in the second and last markets.

In the first market an agent with  $m_1$  units of money has expected lifetime utility

$$(15) \quad V_1(m_1) = \frac{1}{2} [u(q_{1b}) + V_2(m_1 - p_1 q_{1b} + \tau^p)] + \frac{1}{2} [-q_{1s} + V_2(m_1 + p_1 q_{1s} + \tau^r)]$$

The perfectly anticipated transfers  $\tau^p$  and  $\tau^r$  are new in equation (15). Accordingly, the first-order conditions of the sellers (12) and the buyers (13) have to be modified to take these new transfers into account.

In the following we analyze how a change in  $x$  affects steady state production and consumption in markets 1 and 2.

**Proposition 4** *In the low inflation economy, changes in  $x$  have no effect on individual or aggregate consumption in markets 1 and 2. In the high inflation economy if  $\gamma \neq 1$ , changes in  $x$  are non-neutral.*

Surprisingly, a change in  $x$  does not have any real effects when inflation is low. The only effect is that the steady state value of real money balances decreases. In contrast, if inflation is high changes in  $x$  are non-neutral.

What is the intuition for this result? Changes in  $x$  have different effects on the relative prices across markets. In the low inflation economy all prices are the same, i.e.  $p_1 = p_2 = p_3 = 1/\phi$ . Thus, any changes in  $x$  causes all prices to change proportionately. Since relative prices between markets 1 and 2 are unaffected, market

1 consumption does not change. But then, by the inter-market Euler equation, consumption of the poor buyers in market 2 cannot change. Finally, since the rich buyers continue to consume  $q^*$  in market 2, their consumption is not affected.

In the high inflation economy there is price dispersion across markets since  $p_1 < p_2 = p_3$ . The price in market 1 depends on the marginal value of money in market 2, which in contrast to the low inflation economy is non-linear in  $x$ . Then, changes in  $x$  change the relative price between markets 1 and 2. As a consequence, agents change their consumption patterns. More intuitively, an increase in  $x$  reduces the money holdings of rich agents in market 2 and, because they are cash constrained in this equilibrium, this increases their marginal value of money. Since they are the sellers in market 1, they choose to sell more in market 1 to acquire additional cash. By selling more, they lower  $p_1$  relative to  $p_2$  and so buyers find it optimal to consume more in market 1.

We define welfare as the life-time expected utility of a representative agent at the beginning of the period.

**Proposition 5** *For  $\gamma > 1$ , welfare is increasing in  $x$ . For  $\gamma < 1$ , if an equilibrium exists, welfare is decreasing in  $x$ .*

The reason welfare increases is that an increase in  $x$  provides consumption insurance in market 2 - agents give up consumption when they are rich but increase it when they are poor. Given our assumptions on preferences, this insurance lowers the expected marginal utility of consumption in market 2 which induces agents to increase consumption in market 1.

## 5 Conclusion

We have presented a framework in which a monetary expansion, while neutral in the long-run, can have beneficial effects in the short-run. The key feature of our model is that agents trade on a sequence of markets while being subject to idiosyncratic shocks. For this reason, there is equilibrium heterogeneity in money balances, so that one-time monetary transfers can be used to redistribute liquidity from rich to poor. Since an unexpectedly high money growth rate redirects consumption to those who most value it, welfare is positively affected.

The short-run non-neutralities of our model hinge on three elements. First, monetary injections must be unanticipated. Second, such injections must take place when agents hold different amounts of money. Third, average inflation cannot be too high, otherwise aggregate output is unaffected. What our results do *not* hinge on are any price rigidities, asymmetric information, or asymmetric injections across agents, devices that have been used in the literature to generate short-run non-neutralities of money. Finally, we show that by providing consumption insurance, fully anticipated asymmetric lump-sum transfers increase aggregate output and welfare in the high inflation economy. Surprisingly, such a scheme has no real effects when inflation is low.

## Appendix

**Proof of Lemma 1.** If the constraint is not binding then  $\lambda_2 = 0$ . Using (8) then  $u'(q_{2b}) = 1$ . Here trades are efficient. The buyer spends  $q^*/\phi$  money, and we let  $m^* = q^*/\phi$  denote money holdings such that the constraint on spending does not bind. If the constraint is binding,  $\lambda_2 > 0$ , then (8) implies  $u'(q_{2b}) = 1 + \frac{\lambda_2}{\phi}$  and  $p_2 q_{2b} = m_2$ . Here trades are inefficient. The buyer spends all his money,  $p_2 q_{2b} = m_2 < m^*$ , and consumes  $q_{2b} = m_2/p_2 < q^*$ .

To examine concavity of  $V_2$  differentiate (6) with respect to  $m_2$  to get

$$\begin{aligned} V_2'(m_2) = & \frac{1}{2} \left[ u'(q_{2b}) \frac{\partial q_{2b}}{\partial m_2} + V_3'(m_2 - p_2 q_{2b}) \left( 1 - p_2 \frac{\partial q_{2b}}{\partial m_2} \right) \right] \\ & + \frac{1}{2} \left[ -\frac{\partial q_{2s}}{\partial m_2} + V_3'(m_2 + p_2 q_{2s}) \left( 1 + p_2 \frac{\partial q_{2s}}{\partial m_2} \right) \right] \end{aligned}$$

Then (7), (8) and  $\phi = 1/p_2$  imply that

$$(16) \quad V_2'(m_2) = \frac{1}{2p_2} [u'(q_{2b}) + 1] + \frac{1}{2}\lambda_2 \left[ 1 - p_2 \frac{\partial q_{2b}}{\partial m_2} \right]$$

If  $\lambda_2 = 0$ , then  $u'(q_{2b}) = 1$  and  $V_2'(m_2) = \phi$ , so  $V_2(m_2)$  is linear in  $m_2$  for  $m \geq m^*$ .

If  $\lambda_2 > 0$ , then  $p_2 q_{2b} = m_2$ , which implies that  $1 - p_2 \frac{\partial q_{2b}}{\partial m_2} = 0$ . Hence,  $V_2'(m_2) = \phi \left[ \frac{u'(q_{2b})+1}{2} \right] > \phi$  since  $u'(q_{2b}) > 1$ . Note that  $V_2''(m_2) < 0$  because  $\frac{\partial q_{2b}}{\partial m_2} > 0$ , so that  $V_2(m_2)$  is concave  $\forall m_2 < m^*$ . ■

**Proof of Lemma 2.** First prove that  $\lambda_1 = 0$  always. Suppose  $\lambda_1 > 0$ . Then  $m_2 = 0$  and  $q_{2b} = 0$  implying  $u'(0) = 1 + \lambda_2/\phi$ , which is not possible since  $u'(0) = \infty$ . Thus  $\lambda_1 = 0$ , in which case (12)-(13) yield

$$(17) \quad u'(q_{1b}) = \frac{EV_2'(m_1 - p_1 q_{1b} + \tau)}{EV_2'(m_1 + p_1 q_{1s} + \tau)}.$$

If  $m_1 - p_1 q_{1b} < m^*$  then  $EV_2(m_1 - p_1 q_{1b} + \tau)$  is concave, hence  $u'(q_{1b}) > 1$  and  $q_{1b} < q^*$ . If  $m_1 - p_1 q_{1b} \geq m^*$  then both numerator and denominator are linear, hence  $u'(q_{1b}) = 1$  and  $q_{1b} = q^*$ . Hence,  $q_{1b} \leq q^*$ .

Differentiating (11) with respect to  $m_1$

$$V_1'(m_1) = \frac{1}{2} \left[ u'(q_{1b}) \frac{\partial q_{1b}}{\partial m_1} + EV_2'(m_1 - p_1 q_{1b} + \tau) \left( 1 - p_1 \frac{\partial q_{1b}}{\partial m_1} \right) \right] \\ + \frac{1}{2} \left[ -\frac{\partial q_{1s}}{\partial m_1} + EV_2'(m_1 + p_1 q_{1s} + \tau) \left( 1 + p_1 \frac{\partial q_{1s}}{\partial m_1} \right) \right]$$

Using (12) and (13) (for  $\lambda_1 = 0$ ) yields

$$V_1'(m_1) = \frac{1}{2p_1} [u'(q_{1b}) + 1]$$

Thus, if  $q_{1b} < q^*$  then  $V_1(m_1)$  is strictly increasing and concave.

Since everyone enters the first market with identical money balances, and there is an identical number of buyers and sellers, in equilibrium,  $q_{1b} = q_{1s} = q_1$ . ■

**Proof of Proposition 1.** The shock to the money supply is realized before the second market opens. Thus,  $p_2$  adjusts instantly and proportionately to the change in the money stock, and so does the expected value of  $\phi$ . Then (2) implies that

$$(18) \quad \frac{\phi^H}{E\phi} = k^H \equiv \frac{1 - \varepsilon^L}{1 + \varepsilon^H - \varepsilon^L} < 1$$

$$(19) \quad \frac{\phi^L}{E\phi} = k^L \equiv \frac{1 + \varepsilon^H}{1 + \varepsilon^H - \varepsilon^L} > 1$$

where  $k^i$  is the price of money in state  $i = L, H$  relative to the expected price. It does not depend on  $\mu$ . Note that  $k^H < k^L$ , so when the money shock is high the price of money is low.

Suppose first that  $\lambda_2^p = \lambda_2^r = 0$  for all states. Then  $q_{2b} = q^*$  for all agents in all states. Therefore as shown in Lemma 1 we have  $V_2'(m_2) = \phi$  and therefore



$EV'_2(m_2) = E\phi$ . Then from the first-order condition of the buyer in market 1 (13) we have  $u'(q_1)/p_1 = E\phi$  and from the first-order condition of the seller in market 1 (12) we have  $p_1 = 1/E\phi$ . Finally, (14) implies that the marginal value of money at the beginning of a period is equal to the expected value at the end of the period

$$V'_1(m_1) = E\phi$$

This condition says that if agents take a unit of money into the first market but do not intend to spend it in either the first or second markets, then the value of this extra unit of money is the goods it buys in the last market. Substituting this expression into (4), and backdating it, gives

$$(20) \quad \phi_{-1} \left[ \beta E \left( \frac{\phi}{\phi_{-1}} \right) - 1 \right] = \phi_{-1} (\beta/\gamma - 1) \leq 0.$$

For  $\beta/\gamma < 1$  this expression is negative implying  $m_1 = 0$  which cannot be an equilibrium. For  $\beta/\gamma > 1$  agents want to hold an infinite amount of money, since its rate of return is greater than the discount rate. This also cannot be an equilibrium. For  $\beta/\gamma = 1$ , there is an infinity of monetary equilibria, one for each value of  $\phi_{-1}$ .

Suppose  $\lambda_2^r > \lambda_2^p = 0$  in one or both states. From (8) this is a contradiction since  $m_2$  is larger for rich agents.

Now consider the remaining possibilities.

**Equilibrium 1:**  $\lambda_2^{pL} > 0$  and  $\lambda_2^{pH} = \lambda_2^{rL} = \lambda_2^{rH} = 0$ . In this case,  $q_{2b}^{pH} = q_{2b}^{rH} = q_{2b}^{rL} = q^*$  and  $q_{2b}^{pL} < q^*$ . First, we determine  $q_1$ . As shown in Lemma 1,  $V'_2(m_1 + p_1 q_{1s} + \tau) = \phi$ , thus  $EV'_2(m_1 + p_1 q_{1s} + \tau) = E\phi$ . Using (12), we have

$p_1 = 1/E\phi$ . Next, (14) implies that

$$2V_1'(m_1) = E\phi [u'(q_1) + 1]$$

Finally, (4) can be backdated to get

$$\frac{2\phi_{-1}}{\beta} = E\phi [u'(q_1) + 1].$$

Since  $1/\gamma = E\phi/\phi_{-1}$  then

$$(21) \quad u'(q_1) = 1 + 2 \left( \frac{\gamma - \beta}{\beta} \right)$$

Because of strict concavity of  $u(q)$  there is a unique value  $q_1$  that solves (21), and for  $\beta < \gamma$ ,  $q_1 < q^*$ . As  $\gamma \rightarrow \beta$ ,  $u'(q_1) \rightarrow 1$  and  $q_1 \rightarrow q^*$ .

Next we determine the real money balances  $\Omega$ . Using (17) and noting that  $q_{2b}^{pL} = \phi^L M^L - k^L q_1$  where  $\phi^L M^L = \Omega$  we get

$$(22) \quad 2u'(q_1) = (1 - \pi) k^L [u'(\Omega - k^L q_1) + 1] + 2\pi k^H$$

For a given value of  $q_1$ , it is straightforward to show that a unique value of  $\Omega$  exists.

It then follows that since the poor buyer spends all of his money in markets one and two, when the state is  $i = L$ , with  $M^L = z^L M_{-1}$  we have  $q_{2b}^{pL} = \Omega - k^L q_1$ . Then,

$$\phi^L = \frac{\Omega}{M^L}, \phi^H = \frac{\Omega}{M^H} \text{ and } 1/p_1 = (1 - \pi) \frac{\Omega}{M^L} + \pi \frac{\Omega}{M^H}.$$

Finally, for this equilibrium to exist it must be the case that

$$q_{2b}^{pL} = \Omega - k^L q_1 < q^* \text{ and } q_{2b}^{pH} = q^* \leq \Omega - k^H q_1$$

which implies

$$(23) \quad q^* + k^H q_1 < \Omega < q^* + k^L q_1.$$

Since  $q_{2b}^{pL} = \Omega - k^L q_1$  and  $\Omega < q^* + k^L q_1$ , then it follows that  $q_{2b}^{pL} < q^*$ . As  $\gamma \rightarrow \beta$ ,  $q_1 \rightarrow q^*$  and  $\Omega \rightarrow q^* (1 + k^L)$ . Since (22) yields

$$\frac{d\Omega}{dq_1} = k^L + \frac{2u''(q_1)}{(1 - \pi) u''(q_{2b}^{pL}) k^L} > k^L$$

as  $\gamma$  increases from  $\beta$ ,  $\Omega$  falls faster than the right-hand inequality in (23). For a sufficiently high value of  $\gamma$ , call it  $\gamma_1$ , the left-hand inequality will bind and beyond that will be violated. Hence, for  $\gamma \in (\beta, \gamma_1]$  this equilibrium exists. Note, if  $\varepsilon^L = \varepsilon^H = 0$  this equilibrium cannot exist for any  $\gamma$ .

**Equilibrium 2:  $\lambda_2^p > 0$  and  $\lambda_2^r = 0$  for both states.** In this case,  $q_{2b}^{rH} = q_{2b}^{rL} = q^*$ . Consequently, Lemma 1 implies that  $EV_2'(m_1 + p_1 q_{1s} + \tau) = E\phi$  and the first-order condition of the seller in market 1 implies that  $1/p_1 = E\phi$ .

By using the same procedure as before one can show that the solution for  $q_1$  is once again given by (21).

To find the real money balances  $\Omega$  use (17) to get

$$2u'(q_1) = (1 - \pi) k^L [u'(\Omega - k^L q_1) + 1] + \pi k^H [u'(\Omega - k^H q_1) + 1]$$

Again a unique value of  $\Omega$  exists. Using the solutions for  $\Omega$  and  $q_1$  we obtain

$$\begin{aligned} q_{2b}^{pH} &= \Omega - k^H q_1 > q_{2b}^{pL} = \Omega - k^L q_1 \\ \phi^L &= \frac{\Omega}{M^L}, \phi^H = \frac{\Omega}{M^H} \text{ and } 1/p_1 = (1 - \pi) \frac{\Omega}{M^L} + \pi \frac{\Omega}{M^H} \end{aligned}$$

For this equilibrium to exist we need that the poor buyers' money balances satisfy

$$q_{2b}^{pL} = \Omega - k^L q_1 < q^* \text{ and } q_{2b}^{pH} = \Omega - k^H q_1 < q^*$$

while the rich buyers' money balances satisfy

$$\Omega + k^L q_1 > q^* \text{ and } \Omega + k^H q_1 > q^*$$

Combining these two sets of inequalities, the sufficient condition for this equilibrium is

$$q^* - k^H q_1 < \Omega < q^* + k^H q_1$$

At  $\gamma_1$ , the right-hand inequality binds. As  $\gamma$  increases above  $\gamma_1$  once again  $\Omega$  falls faster than  $q_1$ . Finally at some  $\gamma_2 > \gamma_1$  the left-hand inequality binds. Thus for  $\gamma \in (\gamma_1, \gamma_2]$  this equilibrium exists.

**Equilibrium 3:**  $\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH} > 0$  and  $\lambda_2^{rL} = 0$ . In this case,  $q_{2b}^{rL} = q^*$  and  $q_{2b}^{rH}, q_{2b}^{pL}, q_{2b}^{pH} < q^*$ . Consequently, Lemma 1 implies that

$$V_2'(m_2) = \frac{1}{2p_2} [u'(q_{2b}^r) + 1]$$

Using (12) and (7) yields

$$(24) \quad \frac{1}{p_1} = \frac{\phi^H}{2k^H} \{ \pi k^H [u'(q_{2b}^{rH}) + 1] + 2(1 - \pi) k^L \}$$

Then, (14) implies that

$$V_1'(m_1) = \frac{\phi^H}{4k^H} \{ \pi k^H [u'(q_{2b}^{rH}) + 1] + 2(1 - \pi) k^L \} [u'(q_1) + 1]$$

Finally, (4) can be backdated to get

$$(25) \quad \frac{4\gamma}{\beta} = \{ \pi k^H [u'(\Omega + \phi^H p_1 q_1) + 1] + 2(1 - \pi) k^L \} [u'(q_1) + 1].$$

where  $q_{2b}^{rH} = \Omega + \phi^H p_1 q_1$ .

Then use (17) to get

$$(26) \quad 2u'(q_1) = \pi k^H [u'(\Omega - \phi^H p_1 q_1) + 1] + (1 - \pi) k^L \left[ u' \left( \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 \right) + 1 \right]$$

Then solving (24), (25) and (26) yields  $\phi^H p_1$ ,  $q_1$  and  $\Omega$ .

Using the solutions for  $\Omega$  and  $q_1$  we obtain

$$\begin{aligned} q_{2b}^{rH} &= \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 \\ \phi^L &= \frac{\Omega}{M^L}, \phi^H = \frac{\Omega}{M^H} \end{aligned}$$

For this equilibrium to exist we need that the poor buyers' money balances satisfy

$$q_{2b}^{pL} = \Omega - \frac{z^H}{z^L} \phi^H p_1 < q^* \text{ and } q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 < q^*$$

while the rich buyers' money balances satisfy

$$\Omega + \phi^H p_1 q_1 < q^* < \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1$$

Combining these two sets of inequalities, the sufficient condition for this equilibrium

is

$$q^* - \frac{k^L}{k^H} \phi^H p_1 q_1 < \Omega < q^* - \phi^H p_1 q_1$$

At  $\gamma_2$ , the right-hand inequality binds. As  $\gamma$  increases above  $\gamma_2$  once again  $\Omega$  falls faster than  $\phi^H p_1 q_1$ . Finally at some  $\gamma_3 > \gamma_2$  the left-hand inequality binds. Thus for

$\gamma \in (\gamma_2, \gamma_3]$  this equilibrium exists if a solution to (24), (25) and (26) exists. Note, if

$\varepsilon^L = \varepsilon^H = 0$  this equilibrium cannot exist for any  $\gamma > \beta$ .

**Equilibrium 4:**  $\lambda_2^{pH}, \lambda_2^{pL}, \lambda_2^{rH}, \lambda_2^{rL} > 0$ . In this case,  $q_{2b}^{pH}, q_{2b}^{pL}, q_{2b}^{rH}, q_{2b}^{rL} < q^*$ . Consequently, Lemma 1 implies that

$$V_2'(m_2) = \frac{1}{2p_2} [u'(q_{2b}^r) + 1]$$

Using (12) and (7) yields

$$(27) \quad \frac{1}{p_1} = \frac{\phi^H}{2k^H} \{ \pi k^H [u'(q_{2b}^{rH}) + 1] + (1 - \pi) k^L [u'(q_{2b}^{rL}) + 1] \}$$

Then, (14) implies that

$$V_1'(m_1) = \frac{\phi^H}{4k^H} \{ \pi k^H [u'(q_{2b}^{rH}) + 1] + (1 - \pi) k^L [u'(q_{2b}^{rL}) + 1] \} [u'(q_1) + 1]$$

Finally, (4) can be backdated to get

$$(28) \quad \frac{4\gamma}{\beta} = \{ \pi k^H [u'(q_{2b}^{rH}) + 1] + (1 - \pi) k^L [u'(q_{2b}^{rL}) + 1] \} [u'(q_1) + 1]$$

where  $q_{2b}^{rH} = \Omega + \phi^H p_1 q_1$  and  $q_{2b}^{rL} = \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1$ .

Use (17) to get

$$(29) \quad u'(q_1) = \frac{\pi k^H}{2} [u'(\Omega - \phi^H p_1 q_1) + 1] + \frac{(1 - \pi) k^L}{2} \left[ u' \left( \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 \right) + 1 \right]$$

Then solving (27), (28) and (29) yields  $\phi^H p_1, q_1$  and  $\Omega$ .

Using the solutions for  $\Omega$  and  $q_1$  we obtain

$$\begin{aligned} q_{2b}^{rL} &= \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1, & q_{2b}^{rH} &= \Omega + \phi^H p_1 q_1, \\ q_{2b}^{pH} &= \Omega - \phi^H p_1 q_1, & q_{2b}^{pL} &= \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 \\ \phi^L &= \frac{\Omega}{ML}, & \phi^H &= \frac{\Omega}{MH} \end{aligned}$$

From these solutions we get  $q_{2b}^{rL} > q_{2b}^{rH} > q_{2b}^{pH} > q_{2b}^{pL}$ .

For this equilibrium to exist we need that the rich buyers' money balances satisfy

$$q_{2b}^{rL} = \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1 < q^*$$

respectively

$$\Omega < q^* - \frac{k^L}{k^H} \phi^H p_1 q_1$$

As indicated above this inequality binds at  $\gamma_3$ . As  $\gamma$  increases above  $\gamma_3$  once again  $\Omega$  falls faster than  $\phi^H p_1 q_1$ . Thus for  $\gamma \geq \gamma_3$  this equilibrium exists if a solution to (27), (28) and (29) exists.

Finally, it is straightforward to show that  $q_3 = q_3^*$  in all states and in all periods.

To ensure that the richest agents have non-negative production we need to impose that  $q_3^* \geq 2q^*$ . This requires scaling of  $U(q)$  such that this condition holds. ■

**Proof of Proposition 3.** Average aggregate output in market 2 is

$$\pi \left( q_{2b}^{pL} + q_{2b}^{rL} \right) + (1 - \pi) \left( q_{2b}^{pH} + q_{2b}^{rH} \right).$$

Consider all possible equilibria.

In case (i) of Proposition 1 we have  $q_{2b}^{pL} < q_{2b}^{rL} = q_{2b}^{pH} = q_{2b}^{rH} = q^*$ . If an unanticipated increase in money takes place aggregate output realized is  $q_{2b}^{pH} + q_{2b}^{rH} = 2q^*$ , which is higher than average aggregate output since

$$\pi \left( q_{2b}^{pL} + q^* \right) + (1 - \pi) 2q^* < 2q^*.$$

In case (ii) of Proposition 1 we know have  $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rL} = q_{2b}^{rH} = q^*$ . In state  $i = H$  aggregate output is  $q_{2b}^{pH} + q^* > q_{2b}^{pL} + q^*$ . Thus

$$\pi \left( q_{2b}^{pL} + q^* \right) + (1 - \pi) (q_{2b}^{pH} + q^*) < q_{2b}^{pH} + q^*$$

i.e. aggregate output is larger than average aggregate output if  $i = H$ .

In case (iii) of Proposition 1 we have  $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rH} < q_{2b}^{rL} = q^*$ . We also know that

$$q_{2b}^{rH} = \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{z^H}{z^L} \phi^H p_1 q_1$$

Thus, in state  $i = H$  aggregate output is  $q_{2b}^{pH} + q_{2b}^{rH} = 2\Omega$ . In state  $i = L$  we have

$$q_{2b}^{pL} + q_{2b}^{rL} = \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 + q^* < \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 + \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1 = 2\Omega$$

since  $q^* < \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1$  in this equilibrium. Thus average aggregate output is

$$\pi \left( q_{2b}^{pL} + q_{2b}^{rL} \right) + (1 - \pi) 2\Omega < 2\Omega.$$

i.e. aggregate output is larger than average aggregate output if  $i = H$ .

In case (iv) of Proposition 1 we have  $q_{2b}^{pL} < q_{2b}^{pH} < q_{2b}^{rH} < q_{2b}^{rL} < q^*$ . We also know that

$$q_{2b}^{rL} = \Omega + \frac{k^L}{k^H} \phi^H p_1 q_1 > q_{2b}^{rH} = \Omega + \phi^H p_1 q_1 > q_{2b}^{pH} = \Omega - \phi^H p_1 q_1 > q_{2b}^{pL} = \Omega - \frac{k^L}{k^H} \phi^H p_1 q_1 .$$

Thus, in state  $i = H$  aggregate output is  $q_{2b}^{pH} + q_{2b}^{rH} = 2\Omega$ . In state  $i = L$  aggregate output is  $q_{2b}^{pL} + q_{2b}^{rL} = 2\Omega$ . Thus aggregate output is independent across states and it is equal to average aggregate output. ■

**Proof of Proposition 4.** In the low inflation economy,  $p_1 = p_2 = 1/\phi$  and the following equations determine  $q_1$  and  $q_{2b}^p$  :

$$u'(q_1) = 1 + 2 \frac{\gamma - \beta}{\beta} \quad \text{and} \quad u'(q_1) = \frac{u'(q_{2b}^p) + 1}{2}.$$

Since neither of these expressions depend on  $x$ , the quantities  $q_1$  and  $q_{2b}^p$  are unaffected by a change in  $x$ . For a poor buyer we have  $p_2 q_{2b}^p = M_{-1} - p_1 q_1 + \tau^p$ , respectively,



$q_{2b}^p + q_1 = \phi(M_{-1} + \tau^p)$ . If the quantities bought by a poor buyer in markets one and two do not change then  $\phi(M_{-1} + \tau^p)$  must remain the same. Thus,

$$\frac{\partial \phi}{\partial x} = -\frac{\phi(\mu - 1)}{1 + (\mu - 1)x} < 0 \quad \text{and} \quad \frac{\partial \phi M}{\partial x} < 0.$$

The proof for the high inflation economy is by contradiction. In this equilibrium both buyers spend all of their money in market two. This implies, noting that  $\phi = 1/p_2$ , the budget constraints satisfy

$$\begin{aligned} \phi [M_{-1} + x(\gamma - 1)M_{-1}] &= \phi p_1 q_1 + q_{2b}^p \\ \phi [M_{-1} + (2 - x)(\gamma - 1)M_{-1}] &= -\phi p_1 q_1 + q_{2b}^r \end{aligned}$$

Now add and subtract  $\phi(1 - x)(\gamma - 1)M_{-1}$  on the left hand side of the first constraint and rewrite the second to get

$$(30) \quad \phi [M - (1 - x)(\gamma - 1)M_{-1}] = \phi p_1 q_1 + q_{2b}^p$$

$$(31) \quad \phi [M + (1 - x)(\gamma - 1)M_{-1}] = -\phi p_1 q_1 + q_{2b}^r$$

Now conjecture that a change in  $x$  leaves the quantities unchanged. Then it must also leave  $\phi p_1$  unaffected since

$$\frac{1}{\phi p_1} = \frac{V_2'(m_1 + p_1 q_{1s} + \tau)}{\phi} = \frac{1}{2} [u'(q_{2b}^r) + 1]$$

Totally differentiate (30) and (31) holding the right hand sides constant to get

$$\begin{aligned} \frac{d\phi}{dx} &= -\frac{\phi(\gamma - 1)M_{-1}}{[M - (1 - x)(\gamma - 1)M_{-1}]} = -\frac{\phi(\gamma - 1)}{[\gamma - (1 - x)(\gamma - 1)]} \\ \frac{d\phi}{dx} &= \frac{\phi(\gamma - 1)M_{-1}}{[M + (1 - x)(\gamma - 1)M_{-1}]} = \frac{\phi(\gamma - 1)}{[\gamma + (1 - x)(\gamma - 1)]} \end{aligned}$$

Clearly these expressions are unequal for  $\gamma \neq 1$ . As a result, the quantities must change. ■

**Proof of Proposition 5.** Life-time expected utility of the representative agent equals

$$(32) \quad \mathcal{W}(1 - \beta) = \frac{1}{2} [u(q_1) - q_1] + \frac{1}{4} [u(q_{2b}^r) - q_{2b}^r] + \frac{1}{4} [u(q_{2b}^p) - q_{2b}^p] \\ + U(q_3^*) - q_3^*$$

Differentiating with respect to  $x$  yields

$$(33) \quad \frac{\partial \mathcal{W}(1 - \beta)}{\partial x} = \frac{1}{2} [u'(q_1) - 1] \frac{\partial q_1}{\partial x} + \frac{1}{4} [u'(q_{2b}^r) - 1] \frac{\partial q_{2b}^r}{\partial x} + \frac{1}{4} [u'(q_{2b}^p) - 1] \frac{\partial q_{2b}^p}{\partial x}$$

In this equilibrium all of the quantities are less than  $q^*$  so the bracketed terms are all positive. We also know  $q_{2b}^p + q_{2b}^r = \phi M$  so

$$\frac{\partial q_{2b}^r}{\partial x} = M \frac{\partial \phi}{\partial x} - \frac{\partial q_{2b}^p}{\partial x}$$

Substitute in to obtain

$$\frac{\partial \mathcal{W}(1 - \beta)}{\partial x} = \frac{1}{2} [u'(q_1) - 1] \frac{\partial q_1}{\partial x} + \frac{1}{4} [u'(q_{2b}^p) - u'(q_{2b}^r)] \frac{\partial q_{2b}^p}{\partial x} \\ + \frac{1}{4} [u'(q_{2b}^r) - 1] M \frac{\partial \phi}{\partial x}$$

To determine  $\frac{dq_1}{dx}$ ,  $\frac{\partial q_{2b}^p}{\partial x}$ , and  $\frac{d\phi}{dx}$  note that (27), (28), and (29) must hold with  $k^H = k^L = 1$  and quantities constant across states. Totally differentiate the resulting expressions and evaluate the derivatives at  $x = 1$ . Then solve for  $\frac{dq_1}{dx}$ ,  $\frac{d\phi}{dx}$ , and  $\frac{\partial q_{2b}^p}{\partial x}$  to get

$$\frac{dq_1}{dx} = \frac{1}{D} (\gamma - 1) p_1 \phi^2 [1 + u'(q_1)] u''(q_{2b}^r) u''(q_{2b}^p) \\ \frac{d\phi}{dx} = \frac{1}{D} \left( \frac{\gamma - 1}{\gamma} \right) \phi u''(q_1) [u''(q_{2b}^p) - u''(q_{2b}^r)]$$

where

$$D = u''(q_1) u''(q_{2b}^p) + u''(q_1) u''(q_{2b}^r) + p_1^2 \phi^2 u''(q_{2b}^r) u''(q_{2b}^p) [1 + u'(q_1) + q_1 u''(q_1)]$$

and we have set  $M = 1$  for simplicity. A sufficient condition for  $D > 0$  is  $1 + u'(q_1) + q_1 u''(q_1) \geq 0$ . This condition is satisfied for any CRRA utility function if the degree of risk aversion is less or equal to 1.

We then have  $\frac{dq_1}{dx} > 0$  if  $\gamma > 1$ . Moreover, if  $\frac{dq_1}{dx} > 0$ , then  $\frac{dq_{2b}^r}{dx} < 0$  and  $\frac{dq_{2b}^p}{dx} > 0$ . Thus, the first two terms of (33) are strictly positive. Since  $u'''(\cdot) \geq 0$  we have  $[u''(q_{2b}^p) - u''(q_{2b}^r)] \leq 0$  which implies that  $\frac{\partial \phi}{\partial x} > 0$  if  $\gamma > 1$ . Consequently,  $\frac{\partial \mathcal{W}(1-\beta)}{\partial x} > 0$  if  $\gamma > 1$ . It is straightforward to show that  $\frac{\partial \mathcal{W}(1-\beta)}{\partial x} < 0$  if  $\gamma < 1$ . ■

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