



Institute for Empirical Research in Economics  
University of Zurich

Working Paper Series  
ISSN 1424-0459

---

Working Paper No. 225

**Nonparametric Estimation of the Time-varying Sharpe Ratio  
in Dynamic Asset Pricing Models**

Peter Woehrmann, Willi Semmler and Martin Lettau

January 2005

---

# Nonparametric Estimation of the Time-varying Sharpe Ratio in Dynamic Asset Pricing Models\*

Peter Woehrmann,<sup>†</sup> Willi Semmler,<sup>‡</sup> and Martin Lettau<sup>§</sup>

September 2004

## Abstract

Economic research of the last decade linking macroeconomic fundamentals to asset prices has revealed evidence that standard intertemporal asset pricing theory is not successful in explaining (unconditional) first moments of asset market characteristics such as the risk-free interest rate, equity premium and the Sharpe-ratio. Subsequent empirical research has pursued the question whether those characteristics of asset markets are time varying and, in particular, varying over the business cycle. Recently intertemporal asset pricing models have been employed to replicate those time varying characteristics. The aim of our contribution is (1) to relax some of the assumptions that previous work has imposed on underlying economic and financial variables, (2) to extend the solution technique of Marcet and Den Haan (1990) for those models by nonparametric expectations and (3) to propose a new estimation procedure based on the above solution technique. To allow for nonparametric expectations in the expectations approach for numerically solving the intertemporal economic model we employ the Local Linear Maps (LLMs) of Ritter, Martinetz and Schulten (1992) to approximate conditional expectations in the Euler equation. In our estimation approach based on nonparametric expectations we are able to use full structural information and, consequently, Monte Carlo simulations show that our estimations are less biased than the widely applied GMM procedure. Based on quarterly U.S. data we also empirically estimate structural parameters of the model and explore its time varying asset price characteristics for two types of preferences, power utility and habit persistence. We in particular focus on the Sharpe-ratio and

---

\*We would like to thank participants of the North American Winter Meeting of the Econometric Society in Boston, 2000, in particular, John Heaton for very helpful comments. Further we are grateful to seminar participants of Cambridge University, U.K., New York University, New York, Bielefeld University and participants of the Annual Conference of the Society for Computation in Economics and Finance, Barcelona. We are also grateful for comments by two referees of the Journal. Financial support by the national centre of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged.

<sup>†</sup>Institute for Empirical Research in Economics, University of Zurich, Blumlisalpstr. 10, 8006 Zurich, Switzerland.

<sup>‡</sup>Center for Empirical Macroeconomics, Bielefeld and New School University, New York

<sup>§</sup>Stern School of Business, New York University.

find indication that the model is able to capture the time variation of the Sharpe-ratio.

# 1 Introduction

Economic research in the past has attempted to link macroeconomic fundamentals to asset prices in the context of intertemporal models. The intertemporal asset pricing literature has relied either on models of a pure exchange economy such as Lucas (1978) and Breeden (1979) or on the stochastic growth model with production as in Brock and Mirman (1972) and Kydland and Prescott (1982). These models are referred to as the consumption based CAPM and the stochastic growth model of Real Business Cycle (RBC) type respectively. In the pure exchange model asset prices are computed in an economy where there is an exogenous dividend stream for a representative agent. Given the observed low variability in consumption it has been shown that the risk-free interest rate is too high and the mean equity premium as well as the Sharpe-ratio, a measure of the risk-return trade-off, too low. These phenomena are referred to as the interest rate puzzle, the equity premium puzzle and the Sharpe-ratio puzzle, respectively. For a survey on these problems, see e.g., Mehra and Prescott (1985), Kocherlakota (1996) and Cochrane (2001).

Among others, Rouwenhorst (1994) and Lettau and Uhlig (1997a) have argued that it is crucial how consumption is modeled. In models with production, e.g., the production and investment based Capital Asset Pricing Model by Cochrane (1991, 1996) or the stochastic growth model the fundamental shock is to the production function of firms and consumption is not an exogenous process as consumers can optimize their consumption path in response to production shocks. They thus can smooth consumption via savings and labor input if the latter is in the model. If consumption is modeled as a choice variable and endogenous the intertemporal marginal rate of substitution<sup>1</sup> may become even less variable and asset market facts are even harder to match.<sup>2</sup>

In order to allow to match asset price characteristics with data economic research has extended standard intertemporal models. Those extensions include the use of different utility functions, in particular habit formation,<sup>3</sup> see

---

<sup>1</sup>This is also referred to as stochastic discount factor or pricing kernel.

<sup>2</sup>See e.g. Rouwenhorst (1994), Lettau (1998), Lettau and Uhlig (1997a), and Lettau, Gong and Semmler (2001).

<sup>3</sup>Note, that path dependence of consumption choices in habit formation models imply the possibility of negative marginal utility of consumption and equivalently (implausible) negative Arrow-Debreu prices – these may be prohibited by imposing rather strong

e.g. Heaton (1993, 1995), Campbell and Cochrane (1999) and Boldrin, Christiano and Fisher (1997, 2001), consider incomplete markets, see e.g. Telmer (1993), Heaton and Lucas (1996), Luttmer (1996) and Lucas (1994), introduce heterogenous agents as in Constantinides and Duffie (1996), or replace the stochastic discount factor with a nonparametric function as in Chapman (1997)<sup>4</sup>. Other approaches, for example, have focused on the variation of the dividend stream rather than on the discount factor to explain the asset price characteristics, see e.g. Bansal and Yaron (2000). Although some progress has been made to match asset price characteristics with the data none of the models is able to resolve all the puzzles at once.

In this paper we investigate whether the dynamic stochastic growth model is able to replicate time variation in asset price characteristics, in particular the countercyclical movement of the Sharpe-ratio over the business cycle. Since some of the models we consider are nonlinear, we derive an inference scheme that considers the original nonlinear first-order conditions of the intertemporal models. To be able to spell out time series behavior of asset market facts of intertemporal asset pricing models empirically we develop computational efficient estimation strategies based on explicit numerical solutions of the nonlinear first-order conditions using the full structure of the model. Therefore, we extend the expectations approach of Den Haan and Marcet (1990) to incorporating nonparametric expectations in our numerical solution method<sup>5</sup> and show how estimation schemes are obtained. Based on Monte Carlo simulations we show its dominance over the standard GMM approach in terms of small sample performance which is crucial for empirical economics.

We consider two specification of the stochastic growth model. First, we study the standard case using time-separable CRRA preferences. Second, we solve the model for habit formation preferences using Campbell and Cochrane's (1999) specification. Our main findings are as follows. The Sharpe ratio in to model with CRRA preferences is low and does not vary much over time. In contrast, the model with Campbell-Cochrane preferences is able to generate a higher Sharpe ratio as well as significant time-variation

---

assumptions regarding to distributions of asset returns, see Chapman (1998) for details.

<sup>4</sup>Bansal, Hsieh and Viswanathan (1993) and Bansal and Viswanathan (1993) use a similar approach to estimate the consumption based capital asset pricing model.

<sup>5</sup>This is strongly supported by Kuan and White (1994), Brown and Withney (1998) and Chen and White (1998) since it is likely to end up with incorrect belief equilibria if incorrect parameterizations are applied.

in the risk-return tradeoff.

The remainder is organized as follows. Section 2 reviews some empirical evidence of time varying asset market characteristics as well as a method of how to capture them. We obtain the behavior of the Sharpe-ratio over the business cycle from a discrete-time stochastic volatility model. In section 3 we spell out asset market characteristics of the intertemporal business cycle model without imposing distributional assumptions on underlying variables. Section 4 discusses recent inference schemes for the structural parameters in dynamic economic model. Here we also present our new method. Section 5 provides results of the Monte Carlo study of the performance of various estimation procedures. In section 6 we present empirical results of the significance of our method applied to U.S. data and present Monte Carlo simulations on the Sharpe-ratio for both types of utility functions, power utility and habit persistence. Section 7 concludes the paper. The appendix explains the LLM procedure and discusses some problems pertaining to the GMM estimation .

## 2 Time-Varying Stock Market Characteristics

It has been a tradition in modeling asset prices to contrast historical time series with those generated from the models. Models are required to match statistical regularities of actual time series in terms of the first and second moments. As aforementioned most researchers have focused on the (unconditional) mean and variance of asset price characteristics and attempted to match the risk-free rate, the equity premium and the Sharpe-ratio of the data<sup>6</sup> with those of the model. As also stated above, recent empirical research moved a step further and has stated that asset market characteristics are time-varying. Empirical studies reveal conditional mean and conditional variance in stock return time series, see e.g. the vast literature on stochastic volatility and GARCH models, see Gouriéroux (1997) for surveys. It is stressed that conditional mean and variance change over the business cycle and are linked to variables representing real activity. The main finding, for

---

<sup>6</sup>The size of the empirical risk-free rate, equity premium and Sharpe-ratio for U.S. time series data for the time period 1947.1-1993.3 are reported in Lettau, Gong and Semmler (2001).

example, by Schwert (1989, 1990) and Hamilton and Lin (1996) is that equity returns are more volatile during recession periods.<sup>7</sup> Whitelaw (1997) and Lettau and Ludvigson (2002) estimate time-varying Sharpe-ratios that vary countercyclically with regard to the business cycle.

Further indication on the time-varying Sharpe-ratio for U.S. data is reported in Figure 1, respectively.<sup>8</sup> To obtain the time-varying Sharpe-ratio we follow Härdle and Tsybakov (1997) in estimating a nonparametric univariate stochastic volatility model where conditional mean and variance of excess returns,  $R_t^e$ , are unknown functions of past returns,

$$R_t^e = \mu(R_t^e | R^e(t-1)) + \sigma(R_t^e | R^e(t-1))\varepsilon_t$$

with  $R^e(t) = (R_t^e, \dots, R_{t-3}^e)$  and  $\varepsilon_t \sim N(0, 1)$ . In particular, we estimate

$$\mu(R_t^e | R^e(t-1)) = E [R_t^e | R^e(t-1)] = f(R^e(t-1); \theta_{\mu(R^e)}),$$

and

$$\sigma_t(R_t^e | R^e(t-1)) = \sqrt{E [(R_t^e)^2 | R^e(t-1)] - E [R_t^e | R^e(t-1)]^2}$$

with

$$E [(R_t^e)^2 | R^e(t-1)] = E [(R_t^e)^2 | R^e(t-1)] = f(R^e(t-1); \theta_{\mu(R^e)^2}). \quad (1)$$

Function  $f$  is implemented by the use of Local Linear Maps (LLM) of Ritter et al. (1992) as described in appendix 1. In our application, we use 5 local mappings. The lag length is chosen according to Schwert (1989, 1990). As Figure 1 shows there is indication for the Sharpe-ratio as a reward-to-risk measure, to move-countercyclically over the business cycle.

---

<sup>7</sup>From the additional findings of countercyclical behaviour of volatility of short-term interest rates and yields on corporate bonds, relating them to the growth rate of industrial production, Schwert (1989, 1990) concludes that the variation in stock return volatility is only partly due to changes in leverage, dividend yields and macroeconomic variables.

<sup>8</sup>The time series for equity prices (S&P 500) and three month treasury bonds to compute the time varying Sharpe-ratio are from Citibase (1999).

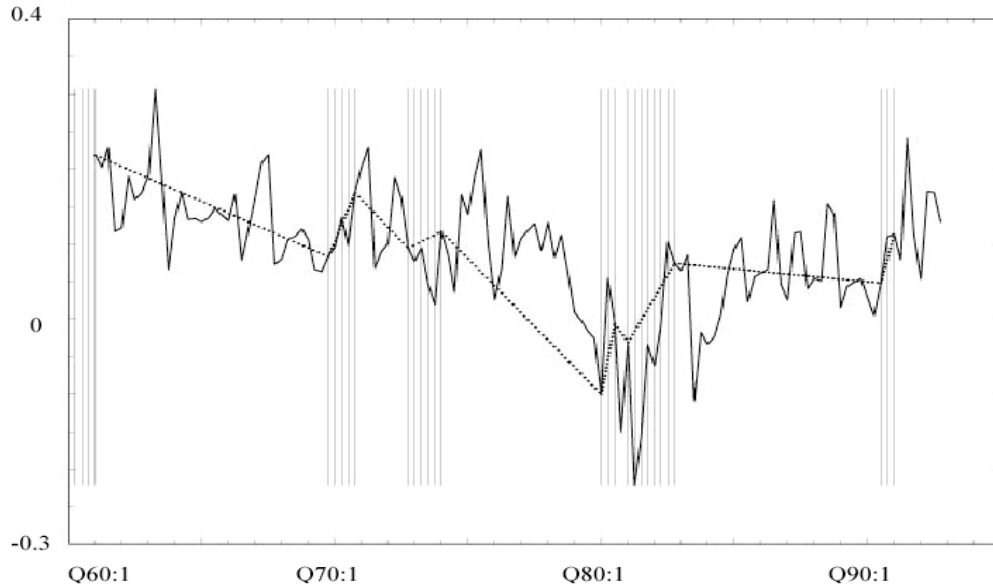


Figure 1: Sharpe ratio of the S&P500 in a stochastic volatility model.

The bold line represents estimation of the Sharpe-ratio. Vertical lines indicate quarters in recessions defined by NBER. Following the dotted (support) line it can be seen that the Sharpe-ratio increases during recessions and decreases in business cycle up-swings.

### 3 The Procedure for Solving the Euler Equation

In the subsequent numerical study we use two variants of utility functions. As first variant of a utility function we take power utility, as usually taken in the stochastic growth model of RBC type. Since we are concerned here not with the unconditional mean of the equity premium and Sharpe-ratio but rather with their time variation it suffices, for our purpose, to take the stochastic growth model with power utility as starting point.

Yet, because of the failure of the power utility model in empirically matching the unconditional mean of equity premium and the Sharpe ratio, in recent studies preferences with habit formation have been employed to study the equity premium and the Sharpe-ratio, see Campbell and Cochrane (2001),



Cochrane (2001, ch. 21) and Boldrin, Christiano and Fisher (2001). In a second variant we thus explore the asset price implications of a model with habit formation. Introducing habit formation into stochastic growth models does not only have implications for asset price characteristics but also for consumption, output, investment and employment. We are here concerned only the asset price characteristics and leave aside the implications for the real variables.<sup>9</sup>

We start with the time-separable CRRA case and extend the approach to allow for habit formation below. In the baseline stochastic growth model of RBC type with constant labor supply the representative agent is assumed to choose consumption,  $C_t$ ,  $t = 1, 2, \dots$ , so as to maximize current and discounted future utilities (using discount factor  $\beta \in [0, 1]$ ) arising from consumption. The model variant with power utility can be stated as

$$\max_{\{C_t\}} E_t \sum_{\tau=0}^{\infty} \beta^\tau \left[ \frac{C_{t+\tau}^{1-\gamma} - 1}{1-\gamma} \right],$$

subject to

$$K_{t+1} = (1 - \rho)K_t + Y_t - C_t \tag{2}$$

$$Y_t = A_t K_t^\alpha \tag{3}$$

In the context of this model business cycles are then assumed to be driven by an exogenous stochastic technology shock,  $A_t$ ,  $t = 1, 2, \dots$ , following the autoregressive process

$$\ln A_t = \phi \ln A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \tag{4}$$

with persistence  $\phi \in [0, 1]$ . Power utility function with constant relative risk aversion  $\gamma \in R^+$  is a common choice for the utility function.

In contrast to pure exchange economies the stochastic growth model allows for saving by introducing capital stock  $K_t$ ,  $t = 1, 2, \dots$ . Yet, the choice of optimal policies,  $(C_t, K_t)$ ,  $t = 1, 2, \dots$ , is constrained by the typical budgeted equation (2) where capital stock is decreased by consumption and depreciation, denoted by  $\rho \in [0, 1]$ , and is increased by output,  $Y_t$ ,  $t = 1, 2, \dots$ , obtained from the Cobb–Douglas production function (3).

---

<sup>9</sup>For a detailed study to what extent the habit formation model matches the latter characteristics, see Boldrin, Christiano and Fisher (2001).

The Euler equation derived from the first order condition of this intertemporal optimization problem with power utility reads

$$1 = E_t [\mathcal{M}_{t+1} R_{t+1}] \quad (5)$$

with stochastic discount factor  $M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$  and gross return on capital  $R_{t+1} = \alpha A_{t+1} K_t^{\alpha-1} + 1 - \rho$ . Note, that the Euler equation can only be solved analytically for  $\gamma = 1$  and full depreciation, i.e.  $\rho = 1$ . Otherwise numerical solution techniques have to be applied.

From the above outlined baseline model one can spell out the following asset market implications. From the Euler equation (5) follows that (maximal) Sharpe-ratio can be obtained from the derivation of volatility bounds in Hansen and Jagannathan (1991) as

$$\delta_t^{\max} = \frac{\sigma_t [\mathcal{M}_t]}{E_t [\mathcal{M}_t]}. \quad (6)$$

In recent research asset market characteristics of intertemporal models are mostly derived under the crucial assumption of jointly log-normally distributed asset prices and consumption.<sup>10</sup> In the framework of the baseline RBC model, this implies a time invariant equity premium and Sharpe ratio.

In order to evaluate (6) without imposing distributional assumptions on consumption and the constancy of the equity premium and Sharpe-ratio we aim to determine  $E_t [\mathcal{M}_t]$  and  $\sigma_t [\mathcal{M}_t]$  via the in section 2 described multivariate version of the nonparametric stochastic volatility model of Härdle and Tsybakov (1997), where, now in the present case, today's expectations are determined nonparametrically based on the relevant observable state of the economy: present capital stock and technology shock.<sup>11</sup> Expectations of the stochastic discount factor are obtained by

$$E_t [\mathcal{M}_{t+1}] = E_t [\mathcal{M}_{t+1} | K_t, A_t] = f(K_t, A_t; \theta_{\mathcal{M}}), \quad (7)$$

where  $f$  is again implemented by nonparametric regression via the Local Linear Maps of Ritter et al. (1992) described in appendix 1. To proceed in this way is in line with Den Haan and Marcet (1990) and Duffy and McNelis (1997) who determine expectations in the Euler equation and model the

<sup>10</sup>See Campbell, Lo and MacKinlay (1997).

<sup>11</sup>Note, that the utility function in our model is time separable.

stochastic discount factor of the first order conditions of the stochastic growth model based on capital stock and the technology shock as conditional variables. Application of nonparametric expectations is recommended by Kuan and White (1994), Brown and Withney (1998) and Chen and White (1998).<sup>12</sup>

The standard deviation of the stochastic discount factor (SDF) is estimated by

$$\sigma_t(\mathcal{M}_t) = \sqrt{E_t[\mathcal{M}_{t+1}^2] - E_t[\mathcal{M}_{t+1}]^2}. \quad (8)$$

Therefore, expectations of the squared SDF are also determined nonparametrically via

$$E_t[\mathcal{M}_{t+1}^2] = E[\mathcal{M}_{t+1}^2 | K_t, A_t] = f(K_t, A_t; \theta_{\mathcal{M}^2}). \quad (9)$$

Function  $f$  is implemented by the LLM of Ritter et al. (1992) as described in the appendix 1. Here again we use 5 local mappings.

There are numerous numerical solution techniques that have recently been employed to solve the stochastic growth model. We will provide a short description of our application of nonparametric methods to approximate conditional expectations of the Euler equation.

The aim of most numerical solution methods is to obtain the control variable  $C$  in feedback form from the state variables  $K$  and  $A$ . Early numerical solution techniques mostly use linearization techniques, neglecting higher order terms in the Taylor series.<sup>13</sup> To spell out the solution more accurately recently algorithms have been employed that use advanced nonlinear or nonparametric estimation methods.<sup>14</sup> Along the line of Den Haan and Marcet (1990) and Duffy and McNelis (1997) we model conditional expectations of

---

<sup>12</sup>Yet, we want to note, however, as a referee has pointed out the Den Haan and Marcet (1990) procedure may become nonparametric as the order of polynomial increases.

<sup>13</sup>For a survey of linearization techniques see Taylor and Uhlig (1990). Examples are the log-linear version of Campbell (1994), see also Lettau and Uhlig (1997) and Lettau, Gong and Semmler (1997), LQ-approximation of Tauchen (1990), the Chow method using the Lagrangian multiplier approach (Chow (1991, 1993), Semmler and Gong (1996)).

<sup>14</sup>See, for example, the approximation of conditional expectations in the Euler equation in Den Haan and Marcet (1990), the method of finite elements in McGratten (1996), projection methods in Judd (1992), Judd and Gaspar (1996), genetic programming in Schmertmann (1996) and methods that approximate iteratively the value function of the dynamic programming formulation of the optimization problem, see Sieveking and Semmler (1997) and Santos and Vigo-Aguiar (1998).

the Euler equation using nonparametric regression in the aforementioned variant of self-organizing maps provided by Ritter et al (1992).

The basic idea of the expectations approach introduced by Den Haan and Marcet (1990) is that the expectational part of the Euler equation (5) can be modeled as a function of the observable variables  $K$  and  $A$ , parameterized in  $\theta \in R^k$ ,

$$\psi : \mathbb{R}^2 \rightarrow \mathbb{R}, E_t [C_{t+1}^{-\gamma} R_{t+1}] = \psi(K_{t-1}, A_t; \theta). \quad (10)$$

Then the Euler equation reads

$$C_t^{-\gamma} = \beta\psi(K_{t-1}, A_t; \theta). \quad (11)$$

Den Haan and Marcet (1990) and Duffy and McNelis (1997) parameterize conditional expectations by polynomial and logistic functions, respectively. Here, we use the LLM provided by Ritter et al. (1992) as a more powerful nonparametric function approximator to capture possibly nonlinear dynamics. Hence the function  $\psi$  may be estimated on the basis of LLM using the following fixed-point iteration, suggested by Marcet and Den Haan (1990).

Having generated technology shocks,  $A$ , via (4) an initial sequence of control variables,  $(C, K)$ , has to be computed. A randomly drawn initial parameter set  $\theta^{(0)}$  can be employed in  $\psi_0$ . Alternatively, sequences of  $(C, K)$  may be taken from solutions of this procedure for less general functions, e.g. polynomial regression. Then the fixed-point iteration is formalized through

$$\Phi : \mathbb{R}^m \rightarrow \mathbb{R}, \theta^{(i)} = \Phi(\theta^{(i-1)}) = (1 - \lambda)\theta^{(i-1)} + \lambda\hat{\theta}^{(i-1)}, i = 1, 2, \dots \quad (12)$$

with  $\hat{\theta}^{(i-1)} = \operatorname{argmin}_{\theta} \|C_{t+1}^{-\gamma} R_{t+1} - \psi(K_{t-1}, A_t; \theta)\|$  and adaption rate  $\lambda$ . In each iteration the sequence  $(C, K)$  is updated by (11) and (2). If the rational expectations equilibrium of the model is stable under learning, the parameters will converge, provided  $\lambda \in (0, 1]$  is small enough.

Employing the assumptions of high complexity of a function such as  $\psi$  and suitable choice of  $\lambda \in (0, 1]$  Marcet and Marshall (1994) use the results of Ljung (1977) to show local convergence for  $\theta_0 \in \Theta$  to  $\theta^*$ , i.e.

$$\lim_{i \rightarrow \infty} \|\theta_i - \theta^*\| = 0. \quad (13)$$

We apply the aforementioned nonparametric functional form to model  $\psi$ .

Next we introduce the model variant with habit formation in the utility function. For this type of preferences it has been shown that with the assumption of log-normal distribution of asset prices and consumption habit formation alone may not improve much the unconditional mean of the equity premium and the Sharpe-ratio. This at least holds for models with production, see Boldrin, Christiano and Fisher (2001) and Jerman (1998). Yet, the time varying characteristics of the equity premium and Sharpe ratio may improve, in particular the expected Sharpe-ratio may be non-constant and behave countercyclically. Since those results are obtained under the assumption of exogenously given consumption-stream and log-normal distribution of asset prices and consumption – from which we depart in our numerical procedure – we are interested what the asset price implications are for a model that drop the above two assumptions by solving numerically a stochastic growth model with production. We consider the habit preferences proposed in Campbell and Cochrane (1999) since that specification is successful in replicating a wide range of asset pricing facts in an exchange economy.

The representative household has a utility function that not only depends on current consumption but also on the habit  $X$ :

$$U(C, X) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}. \quad (14)$$

Campbell and Cochrane (1999) define the surplus consumption ratio as  $S_t = (C_t - X_t)/C_t$  and specify an autoregressive process for the log of the surplus ratio:

$$s_{t+1} = (1 - \phi_H)\bar{s} + \phi_H s_t + \lambda(s_t)(c_{t+1} - c_t - g). \quad (15)$$

The impact of consumption on habit may be state dependent, described by

$$\lambda(s_t) = \frac{1}{\bar{S}} \sqrt{1 - 2(s_t - \bar{s})} - 1, \quad \bar{S} = \sigma_c \sqrt{\frac{\gamma}{1 - \phi_H}} \quad (16)$$

The above utility function (14) provides us with a stochastic discount factor incorporating habit formation such as

$$\mathcal{M}_{t+1} = \beta \left( \frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\gamma}. \quad (17)$$

In order to solve the Euler equation numerically for the model variant with habit formation we employ the discount factor (17) in the Euler equation (5)

instead of the discount factor  $\mathcal{M}_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$  arising from the power utility model. Yet since the gross return on capital are the same, it remains the same in the Euler equation (5).

The numerical solution of the model variant with habit formation follows the steps as indicated in equs. (10)-(13) except habit formation (14) is employed as underlying utility function and the stochastic discount factor (17) is used instead of the discount factor derived from power utility. When the discount factor (17) is used in (5) it is assumed that habit consumption moves forward as given in equ. (15). Then  $X_t$  and  $S_t$ , and thus  $C_t$ , from  $(C_t S_t)^\gamma$ , can be obtained so that the capital stock next period is determined by the budget equation (2). The remainder of the algorithm (12)-(13) remains the same as for the power utility model.

## 4 Estimation Procedures

In recent years there have been efforts undertaken to estimate intertemporal asset pricing models. Next we present some econometric results concerning the estimation of the stochastic growth model. In the literature mostly the baseline version of the RBC model using power utility function has been the underlying model. In general econometric methodologies different from those employed in early empirical studies of static beta pricing theories have been considered. While testing hypothesis of beta pricing theories requires methods from time series and cross-sectional analysis, empirical tests of the validity of first-order conditions arising in intertemporal models are faced with moment restrictions on functions of random variables. In particular, these conditions involve conditional expectations of a function  $f : R^m \rightarrow R$  of realizations of some stochastic vector process  $x_t = (x_{1,t}, x_{2,t}, \dots, x_{m,t})$ ,  $t = 1, 2, \dots, T$  of random variables  $X$  and a parameter vector  $\theta$  describing agents' tastes,

$$E_t [f(x_t, \theta)] = E [f(x_t, \theta) | \Omega_t] = 1, \quad t = 1, 2, \dots, T \quad (18)$$

with information  $\Omega_t$  available in  $t$ . Typically,  $f$  is the product of asset returns and the stochastic discount factor depending on consumption, risk aversion and the discount factor.

In the case of linearized models efficient and analytically tractable standard inference schemes are available.<sup>15</sup> Estimating the parameters involved in

---

<sup>15</sup>See, for example, Hansen and Sargent (1980), Chow (1991,1993) and improved versions

the original nonlinear first-order conditions, however, turns out to be more difficult. In principle, there are three types of estimation strategies. It is worth summarizing them briefly:

1. Application of the Generalized Method of Moments (GMM) introduced by Hansen (1982).<sup>16</sup> It does not require the solution of first-order conditions, but may be inefficient, as frequently mentioned, due to omitting structural information of the model. Simulations in the next section demonstrate that this approach is biased by assuming ergodicity of  $f(x_t, \theta)$ . If one uses conditioning information via instrumental variables, as outlined by Hansen and Singleton (1982), in our simulations biases are significantly reduced. It is also quite important to note that the orthogonality condition tested in this approach is an implication of the moment condition but not an equivalent statement. A more detailed discussion of this issue is provided in appendix 2.
2. Inference about structural parameters based on numerical solutions of first-order conditions. These methods are designed to be efficient, but they turn out to be computationally intractable and are associated with weak consistency results. Examples are the indirect inference approach of Gourieroux, Monfort and Renault (1993) and the maximum likelihood approach of Miranda and Rui (1997) who require the crucial assumption that asset returns follow a first order Markov process and further use a finite approximation to an infinite optimization problem via truncation.
3. Inspired by the parameterized expectations approach of Den Haan and Marcet (1990) to solve rational expectations models numerically, our approximation method of solving the Euler equation, as discussed in section 3, applies a computational tractable inference scheme for the structural parameters that is efficient and consistent. Although it does require numerical solutions, no structural information is omitted. Our

---

by Semmler and Gong (1996), Smith (1993), Campbell (1994), Lettau, Gong and Semmler (1997) and Chow and Kwan (1998).

<sup>16</sup>GMM has frequently been employed to test the Consumption based Capital Asset Pricing Model. Applications of GMM estimation to the nonlinear Euler equation in the first-order conditions of the stochastic growth model of RBC type can be found in Christiano and Eichenbaum (1992) or Fève and Langot (1994).

nonparametric method solves the rational expectations model numerically but also delivers an estimation method for the above discussed intertemporal model.

The estimation method of case 3., for the case of a power utility function, can be derived as follows. Measuring the exogenous sequence of technology shocks,  $A$ , by the Solow residual the set of parameters to be estimated reduces to  $\varphi = (\beta, \gamma, \rho)$ . We start by considering actual time series of consumption and capital stock, denoted by  $C^*$  and  $K^*$ , respectively, as the outcome of the representative agent's optimization problem of the stochastic growth model of RBC type. Assume that in equilibrium,  $(C^*, K^*)$ , the fixed-point algorithm (12) with  $\lambda = 1$  exhibits stability for true parameters  $\varphi^*$  and

$$\|C^* - C(\varphi^*)\| < \|C^* - C(\varphi')\|, \varphi^* \neq \varphi', \quad (19)$$

where  $C(\varphi)$  results from applying one step of (12) based on solution  $(C^*, K^*)$  and  $\varphi$ .<sup>17</sup> Thus, we can estimate the structural parameters of the baseline stochastic growth model by

$$\hat{\varphi} = \operatorname{argmin}_{\varphi} \|C^* - C(\varphi, \hat{\theta})\| \quad (20)$$

with  $\hat{\theta} = \operatorname{argmin}_{\theta} \|\beta C_{t+1}^{*-\gamma} R_{t+1}^* - \psi(K_{t-1}^*, A_t^*; \theta)\|$  and  $(C(\varphi, \hat{\theta}), K(\varphi, \hat{\theta}))$  resulting from (11) and (2). This minimization can be solved by standard nonlinear optimization routines. A well known example is the Newton algorithm which has been used here and which were sufficient in our case.<sup>18</sup>

To use full structural information of the stochastic growth model of RBC type  $\varphi$  may be estimated as follows:

$$\hat{\varphi} = \operatorname{argmin}_{\varphi} \|K^* - Y + LK + LC(\varphi, \hat{\theta})\|, \quad (21)$$

where  $L$  denotes the lag-operator. Results of Ljung (1977) apply directly to show convergence, efficiency and asymptotic normality.

<sup>17</sup>This can be shown numerically.

<sup>18</sup>If this optimization problem is not tractable by the Newton method, alternatively, other search algorithms such as Simulated Annealing, see Semmler and Gong (1996) or Tabu Search, developed to overcome the problems associated with the standard algorithm, could be applied.



## 5 Tests of the Estimation Procedures

To evaluate the performance of GMM and the nonparametric estimation method to estimate the parameters of the stochastic growth model as above discussed we proceed in two steps.

1. Numerical solution of the stochastic growth model using the expectations approach of Den Haan and Marcet (1990) extended by nonparametric expectations given the structural parameters  $\bar{\varphi}$ .
2. Employing GMM and nonparametric method to estimate structural parameters,  $\hat{\varphi}$ , based on simulated time series.

In our numerical investigations we employ the calibration parameters from Den Haan and Marcet (1994), reported in Table 1.

Table 1: Parameters used in simulations of the RBC model.

$\beta$	$\gamma$	$\rho$	$\alpha$	$\phi$	$\sigma_\varepsilon$
<i>.95</i>	<i>.5</i>	<i>0.975</i>	<i>0.33</i>	<i>0.95</i>	<i>0.1</i>

In order to approximate conditional expectations in the Euler equation (5) we implement the fixed–point iteration (12) to obtain sequences of  $C_t$ ,  $K_t$ ,  $t = 1, \dots, T$  with  $T = 200$  as follows:

Applying  $\bar{\varphi}$  the stochastic shock,  $(A_t)$ ,  $t = 1, \dots, T$ , is generated following (4). An initial sequence,  $(C_t^0, K_t^0)$ ,  $t = 1, \dots, T$ , is obtained from the solution of a first–order polynomial function applied to  $\psi$ .<sup>19</sup> Here, LLM with 5 reference vectors is set up to approximate conditional expectations in (5).

The time series simulated using estimated  $\psi$  serve as a basis for various tests of simulation accuracy as described in Taylor and Uhlig (1990).

Having generated  $(C_t, K_t)$ ,  $t = 1, \dots, T$ , we are able to evaluate the performance of different estimation schemes proposed for the stochastic growth model. We perform Monte Carlo experiments with 1000 replications. The box plots in Figure 2 show that  $\beta$  is estimated quite accurately by all estimation procedures under consideration.

<sup>19</sup>Results of this iterative estimation procedure are provided in Den Haan and Marcet (1994).

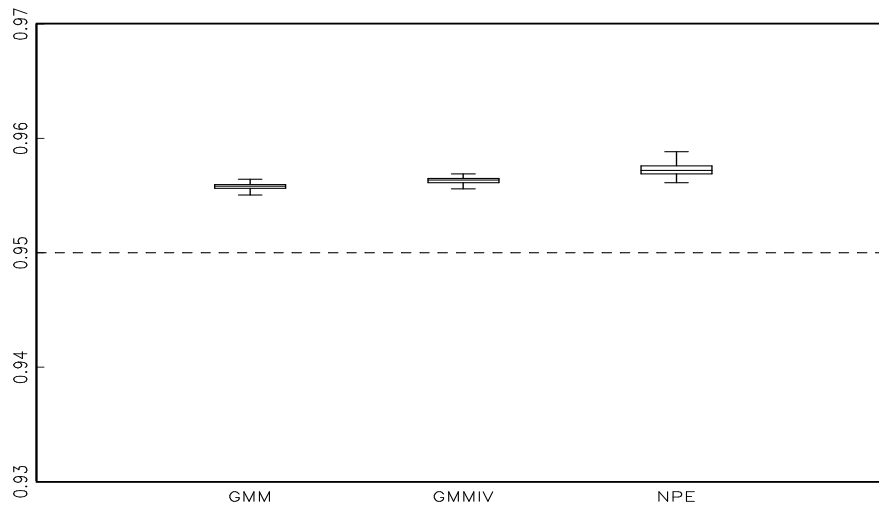


Figure 2: Monte Carlo results for estimating  $\beta$ .

The figure shows box plots of Monte Carlo results for estimates of  $\beta$  based on GMM, GMM with instrumental variables and our nonparametric expectations approach.

Estimation of  $\gamma$ , however, turns out to be more difficult. The bias of GMM estimation is large as can be observed from Figure 3.

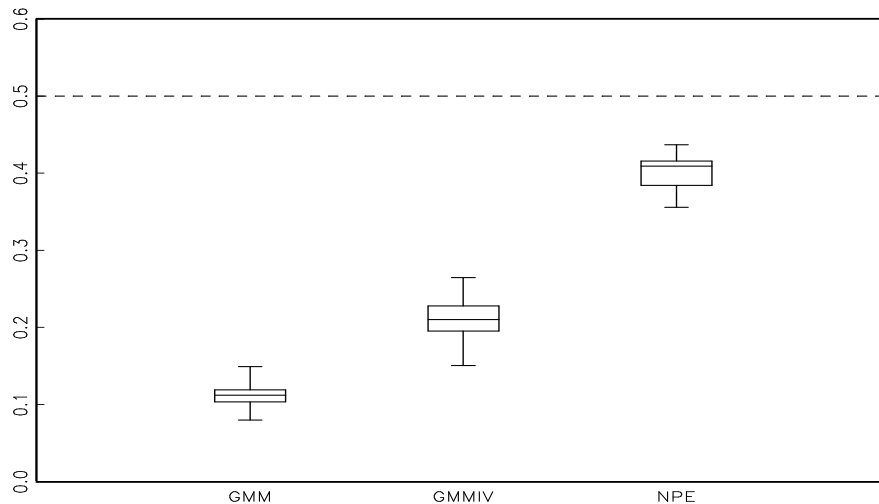


Figure 3: Monte Carlo results for estimating  $\gamma$ .

The figure shows box plots of Monte Carlo results for estimates of  $\gamma$  based on GMM, GMM with instrumental variables and our nonparametric expectations approach.

Our nonparametric estimation scheme performs significantly better, but the bias does not vanish. Numerical studies of Tauchen (1986), Christiano and Den Haan (1996), Hansen, Heaton and Yaron (1996) and Smith (1999) also report poor small sample performance of GMM estimation of intertemporal models.

## 6 Empirical Results on Asset Market Characteristics

We estimate the risk aversion coefficient  $\gamma$  and the discount factor  $\beta$  using quarterly data from 1960:1 to 1993:4.<sup>20</sup> The depreciation parameter  $\rho$  and the Cobb-Douglas parameter  $\alpha$  are fixed at 0.975 and 0.33, respectively.

<sup>20</sup>Data are taken from Citibase (1995). Note that we employ data for the range 1960.1 to 1993.4. Since most RBC studies use this time period for matching the model to the data.

Technology shocks are measured by the Solow residual with respect to a Cobb–Douglas production function with capital share  $\alpha = 0.33$ .

Since the behavior of the real variables are well understood, we will focus on the asset pricing implication, in particular on the Sharpe ratio. For the case with power utility we follow the discussion in the previous section and apply our nonparametric inference scheme to estimate  $\varphi$ . Convergence of nonlinear least squares via Newton algorithm applied to (21) based on empirical time series is obtained and leads to the parameter estimation as reported in Table 1. Since we have reasonable a priori knowledge concerning depreciation rate and capital share these parameters are fixed<sup>21</sup>, indicated by bars.

Table 2: *Parameter estimates of the RBC model, power utility.*

$\hat{\gamma}$	(std. dev.)	$\hat{\beta}$	(std. dev.)	$\bar{\rho}$	$\bar{\alpha}$
0.7131	(0.0465)	0.9548	(0.0279)	0.9750	0.3300

To explore the time-varying asset market characteristics for the implications of the power utility model we employ the above discussed stochastic volatility model of Härdle and Tsybakov (1997) to achieve conditional expectations and variances of the stochastic discount factor based on lagged capital stock and technology shock as described in section 3. Hence, time-varying stylized facts such as the maximal Sharpe-ratio can be computed by (6).

---

<sup>21</sup>Note that the technology parameters, the depreciation rate and the capital share, are fixed here, in order to compute the technology shocks.

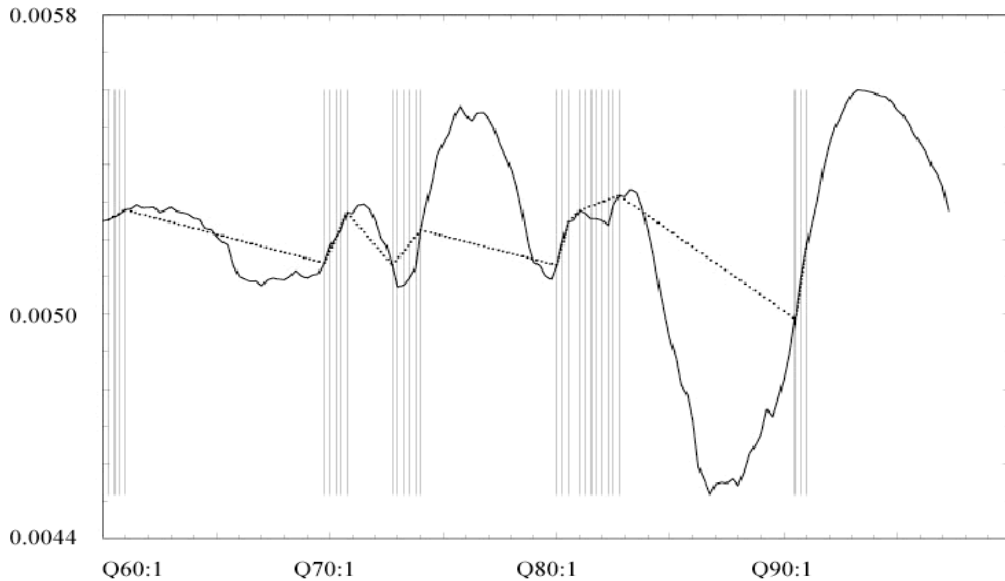


Figure 4: Maximal Sharpe ratio in the estimated RBC model, power utility

Figure 4 shows estimates of the Sharpe-ratio in the RBC model with power utility. The bold line represents estimations of the Sharpe-ratio,  $\hat{\delta}_t$ . Vertical lines indicate quarters in recessions defined by NBER. Following the dotted (support) line it can be seen that the Sharpe ratio increases during recessions and decreases in business cycle up-swings. As hypothesized in related literature and using a stochastic volatility model as discussed in section 2, there is some indication for countercyclical movement of the Sharpe-ratio, but the absolute level of the Sharpe ratio is small and the variations around the mean are relatively minor. As indicated by the broken line, the expected excess returns relative to risk may increase in recessions, i.e., in periods of low economic activity and decreases in periods with high level of economic activity, and decrease in upswings with high level of economic activity. Thus, the Sharpe-ratio appears to move countercyclically.

Next we report results from the solution of the Euler equation using our above discussed numerical solution for the model variant with habit formation. The parameter estimates of  $\gamma$  and  $\beta$  of the preferences (14), incorporating habit formation are estimated by applying the estimation procedure 3 of sect. 4. The results are reported in Table 3 where again the technology parameters were pre-fixed. The standard errors for  $\gamma$  and  $\beta$  were obtained

by Monte Carlo simulations with 1000 replications where the starting values for the LLM parameters were chosen from normal distribution. Thus for the numerical method to compute the time varying Sharpe-ratio the parameter of table 3 have been used.

Table 3: Parameter estimates of the RBC model, habit formation

$\hat{\gamma}$	(std. dev.)	$\hat{\beta}$	(std. dev.)	$\bar{\rho}$	$\bar{\alpha}$
2.160	0.0230	0.9400	0.0018	0.9750	0.3300

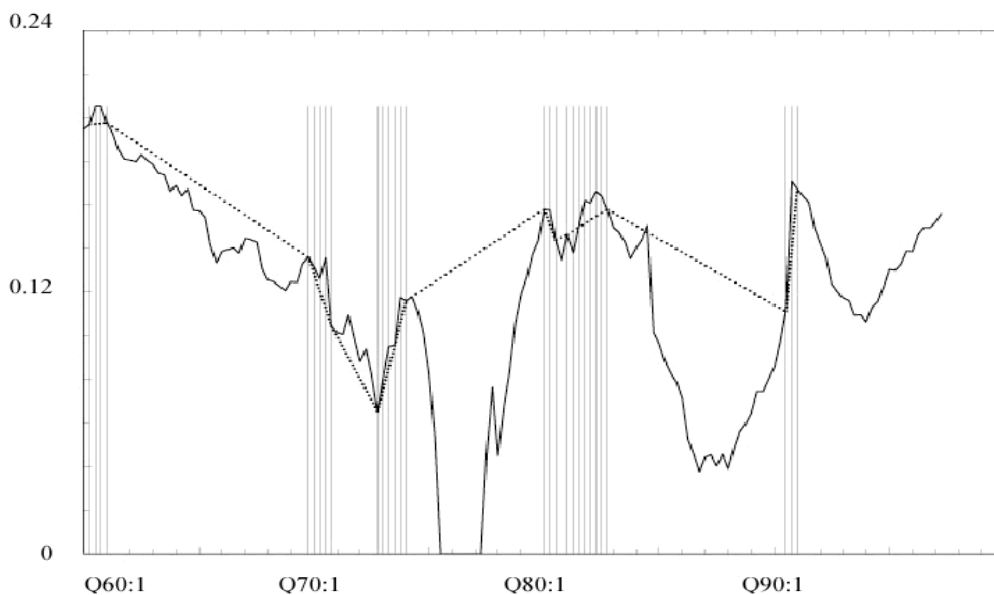


Figure 5: Maximal Sharpe ratio of the RBC model, habit formation

Figure 5 represents the time series of the estimate the Sharpe-ratio for the model variant with habit formation. As expected, the unconditional mean of the Sharpe-ratio is greater than for the model variant with power utility. More interestingly, the Sharpe ratio in the habit model displays significant time-variation. In particular, the Sharpe ratio is strongly countercyclical. Wit the exception of the the recession in the early 70s, the Sharpe ratio rose

in recessions while in declined in boom times. Moreover, it rose before most recessions suggesting that the Sharpe ratio is a leading indicator. We also find that the Sharpe ratio is positively autocorrelated as shown in Table 4.

Indeed, one possible way to measure whether our two model variants exhibit a time varying Sharpe-ratio is to compute the auto-correlation coefficient of the time series of the Sharpe-ratio.

On the other hand, we also want to know whether the Sharpe-ratio moves countercyclically. In order to test the countercyclical movement of the Sharpe-ratio we estimated the contemporaneous correlation between the estimated time series of the Sharpe-ratio and a (quarterly) production index.<sup>22</sup>

Table 4: Time variation of the SR and its correlation with the business cycle

power utility			
mean	std. dev.	autocorr	correl.coeff.
0.00514	0.000268	0.920	-0.210
habit formation			
mean	std. dev.	autocorr.	correl.coeff.
0.189	0.386	0.730	-0.180

Table 4 presents results from Monte Carlo simulations with 1000 replications for both the model variants with power utility and with habit formation. For both cases the unconditional mean, the standard deviation, the auto-correlation (autocorr.) and correlation coefficients (correl.coeff.) with the production index are reported in Table 4.

First, note that the SR is on average higher in the model with habit formation, which is not surprising since risk aversion is higher than in the model with power preferences. The SR in the power case is very low, but varies somewhat around the low mean. In contrast, the SR in the habit model is much more volatile. In both cases the auto-correlation is high. Moreover, the significant and negative correlation of the Sharpe-ratio with the production index – as indicator for the business cycle – demonstrates that

---

<sup>22</sup>The quarterly production index is taken from Hamilton and Lin (1996) where a monthly data set is used which is transformed into a quarterly series which we used as a business cycle indicator.

Sharpe-ratio moves countercyclically for both model variants. The highly serially correlated countercyclical movement of the Sharpe ratio is consistent with the empirical studies cited in the introduction.

## 7 Conclusions

The aim of this work was to explain asset market characteristics that have been found in a variety of empirical studies in different types of frameworks. We have here studied asset market characteristics in the framework of an intertemporal asset pricing model. We were particularly interested in studying time-varying asset market characteristics. For the purpose of this paper it was sufficient to employ the standard stochastic growth model. We have solved and estimated the baseline model nonparametrically and find indication that it is capable of capturing countercyclical movement of the Sharpe-ratio over the business cycle for both the model variant with power utility as well as habit formation. Yet, as we have mentioned, we were not aiming at capturing the high (unconditional) risk-free rate, equity premium and Sharpe-ratio. Of course, the next step would be to pursue our study of time-varying asset market characteristics by allowing for an extended intertemporal model that admits different types of preferences, technology shocks with greater variance and adjustment costs of capital. This might be helpful to also match the mean of the risk-free rate, equity premium and the Sharpe-ratio. An appropriate starting point for such a study is the recent paper by Boldrin, Christiano and Fisher (2001).



# Appendices

## 7.1 Appendix 1: Nonparametric Local Linear Maps

In empirical finance nonparametric methods to estimate conditional mean and variance of time series are now widely applied. Therefore, nonparametric methods, either global or local techniques are used. Well known examples are neural networks or kernel regression.<sup>23</sup> Local techniques offer the advantage that they only consume a small amount of computation time, and, at the same time, they are capable of modeling complex time series. Furthermore, behavior of agents may not be the same in different economic conditions,<sup>24</sup> and may, therefore, be well described by state dependent functions of local nonparametric techniques.

In this work we decide to implement with the Local Linear Maps (LLMs) of Ritter et al. (1992). LLMs have been proposed independently by Stokro, Umberger and Hertz (1990) as a generalization of the widely used technique of Moody and Darken (1989). It is a variant of self-organizing neural networks and improves drastically convergence properties of standard neural networks such as multilayer perceptron with backpropagation while it is capable of modeling complex structures as, for example, generated by chaotic maps. Subsequently, we provide a short description of the LLM.

To approximate an unknown functional relationship between variables  $x \in R$  and  $y \in R$ , on the basis of data  $y_t$ , and  $x_t$ ,  $t = 1, 2, \dots$ ,

$$f : y_t = f(x_t), R^m \rightarrow R,$$

one first has to specify a function  $\psi(x_t, \phi)$  parameterized in  $\phi$  that represents a class of functions including  $f$ .<sup>25</sup> Then an estimation procedure has to be designed to obtain  $\phi$  so as to minimize expectations of the expected loss function  $L$ , the so called risk function of Vapnik (1992),

$$R(\phi) = \int_0^\infty L(y; \psi(x, \phi)) dP(x, y)$$

---

<sup>23</sup>For a detailed discussion see Härdle, Lütkepohl and Chen (1997).

<sup>24</sup>E.g., many studies come to the conclusion that risk aversion varies over the business cycle.

<sup>25</sup>We call a regression function nonparametric if it cannot be characterized by specific distributions.

with  $L(y, \psi(x, \phi)) = \|y - \psi(x, \phi)\|$ , joint probability  $P(x, y)$  and  $\|\cdot\|$  denoting the  $l_2$ -norm. As  $P(x, y)$  is not known it is suggested to minimize the empirical risk function

$$R_{\text{emp}} = T^{-1} \sum_{t=1}^T L(y_t, \psi(x_t, \phi))$$

based on observations  $x_t, y_t, t = 1, \dots, T$ .

Here, we choose LLMs to implement  $\psi(\cdot)$ , i.e., we use  $n$  linear maps that are used locally in input space. In particular, LLMs are built up by  $n$  units,  $r = 1, \dots, n$ , representing regions of the linear maps. Each unit consists of a vector in the input space,  $w_r \in R^m$ , the so called reference vector, a vector in the output space,  $v_r \in R$ , and a coefficient matrix  $A_r \in R \times R^m$ . Parameters  $w_r, v_r$  and  $A_r$  may be summarized in  $\theta$ . The output of an LLM for an input vector  $x \in R^m$  is then computed as

$$\hat{y}_t = f_{LLM}(x_t|\theta) = v_s + A_s(x_t - w_s)$$

with

$$s = \operatorname{argmin}_r \|x_t - w_r\|.$$

The vector  $x_t$  is processed by the linear map associated with the nearest unit in input space. Note, that reformulating  $f_{LLM}$  by

$$\hat{y}_t = f_{LLM}(x_t|\theta) = \alpha_s + \beta_s x_t, \quad \alpha_s = v_s - A_s w_s, \quad \beta_s = A_s$$

offers an expression familiar to econometricians.

An appropriate adaptive estimation scheme for parameters  $A, w$  and  $v$  is provided by Ritter et al. (1992),

$$\begin{aligned} \Delta w_s &= \epsilon_w (x_t - w_s), \\ \Delta v_s &= \epsilon_v (y_t - x_t) + A_s \Delta w_s, \\ \Delta A_s &= \epsilon_A d_s^{-2} (y_t - x_t) (x_t - w_s)' \end{aligned}$$

with  $d_s = \|\tilde{y} - w_s\|$  and learning rates  $\epsilon_w, \epsilon_v$  and  $\epsilon_A$ .<sup>26</sup> Convergence of  $(w, v, A)$  to its equilibrium state  $(w^*, v^*, A^*)$  is proved for similar learning schemes in Ritter and Schulten (1989) using the Fokker–Planck equation approach.

---

<sup>26</sup>Note, that initial values for parameters are chosen randomly.

To show approximation and generalization ability of the technique described above, in Woehrmann (2001) this technique is applied to recover complex time series such as logistic map and the Mackey-Glass equation with encouraging results.

## 7.2 Appendix 2: On GMM with Instrumental Variables

In their influential contribution Hansen and Singleton (1982) propose a test for nonlinear rational expectations asset pricing models. It is an instrumental variables approach to generalized method of moments of Hansen (1982) based on the implication of the models' Euler equations that the product of stochastic discount factor and asset return is orthogonal to any variable in the information set. However, we would like to point out that rational expectations are not guaranteed by the proposed algorithm in a number of empirical applications, e.g. Bansal, Hsieh (1993), Bansal and Viswanathan (1993) and Chapman (1997), where no constants are included in information sets and pricing kernels consist of universal function approximators such as polynomials or neural networks. In those cases testing nonlinear rational expectations asset pricing models based on Hansen and Singleton (1982) may be inconsistent with the models' first-order conditions in the sense that the proposed inference scheme does not ensure expectations in the Euler equations holding unconditionally. This issue is discussed subsequently.

First-order conditions of widely investigated nonlinear rational expectations asset pricing models, such as described in section 3, involve conditional expectations of a function  $f : R^m \rightarrow R$  of realizations of some stochastic vector process  $x_t = (x_{1,t}, x_{2,t}, \dots, x_{m,t})$ ,  $t = 1, 2, \dots, T$ , of economic and financial random variables  $X \in R^m$  and a parameter vector  $\theta \in \Theta \subset R^k$  describing agents' tastes and production technology,

$$E_t [f(x_{t+1}, \theta)] = E [f(x_{t+1}, \theta) | \mathcal{I}_t] = 0, \quad t = 1, 2, \dots, T, \quad (22)$$

where expectations are built upon the information set  $I_t = (I_{1,t}, \dots, I_{n,t})$ . Typically,  $f$  is the product of asset returns and the stochastic discount factor depending on consumption, risk aversion and the discount factor.

To test the Euler equation (5) empirically using conditional information based on the instrumental variable approach to GMM of Hansen and Singleton (1982) has become a common procedure. Therefore, it is tested whether  $f(x_t, \theta)$  and any element in the information set  $I_t$ , are orthogonal, i.e.

$$E[f(x_{t+1}, \theta) \otimes \mathcal{I}_t] = 0, \quad (23)$$

which is an implication of (5). Suppose  $\exists \theta^* \in \Theta$  such that

$$\lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T f(x_{t+1}, \theta^*) \otimes \mathcal{I}_t = 0 \quad (24)$$

it is possible to test (23) empirically by determining parameters  $\hat{\theta}$  that minimize sample means  $\bar{f}_t(\theta) \equiv T^{-1} \sum_{t=1}^T f(x_t, \theta) \otimes I_t$  through minimizing the quadratic form

$$\hat{\theta} = \theta \in \Theta \operatorname{argmin} \bar{f}_T(\theta)' \Omega \bar{f}_T(\theta) \quad (25)$$

with a symmetric, positive definite matrix of weights  $\Omega$ . The weighting matrix derived in Hansen and Singleton (1982) allows for an estimator based on a local optimization scheme such as the Newton algorithm,  $\hat{\theta}$ , that is consistent and asymptotically efficient, i.e. has minimal asymptotic covariance matrix.<sup>27</sup>

Hansen and Singleton (1982) state that (23) is an implication of (5), i.e.

$$E_t[f(x_{t+1})] = 0 \Rightarrow E[f(x_{t+1}, \theta) \otimes \mathcal{I}_t] = 0$$

should hold.<sup>28</sup> Furthermore, it follows straight forward that expectations in the Euler equation (5) hold unconditionally. Thus,  $E_t[f(x_{t+1})] = 0$  should imply

$$E[f(x_{t+1}, \theta) \otimes \mathcal{I}_t] = E[f(x_{t+1}, \theta)] = 0.$$

However, if no constant is included in the information set there may be functions  $f$  having  $E[f(x_t, \theta) I_{i,t}] = 0$  for  $i = 1, 2, \dots, n$ , and  $E[f(x_t, \theta)] \neq 0$  with

---

<sup>27</sup>Small sample performance, however, is not satisfactory as pointed out by Tauchen (1986). To overcome this problem Kitamura and Stutzer (1997) and independently Imbens, Johnson and Spady (1998) improved GMM inspired by principles of information theory.

<sup>28</sup>Note that  $E$  without index  $t$  indicates the sample mean.

1.  $E [I_{i,t}] \neq 0, E [f(x_t, \theta)] E [I_{i,t}] = -\text{COV} [f(x_t, \theta), I_{i,t}] \neq 0$ , or
2.  $E [I_{i,t}] = 0, E [f(x_t, \theta)] E [I_{i,t}] = -\text{COV} [f(x_t, \theta), I_{i,t}] = 0$ ,

for  $i = 1, 2, \dots, n$ .<sup>29</sup> It follows that the objective function in (25) could be zero although the sample version of the Euler equation (22) does not hold unconditionally. Note that the objective function in (25) in combination with a constant in the information set forces  $E [f(x_t, \theta)] = 0$  since the covariance of  $f$  and a constant is zero.

One could conjecture that the simple parameterized form of  $f$  in intertemporal asset prices models may not lead to functional forms such that cases 1. and 2. hold for realizations. This justifies the empirical test in Hansen and Singleton (1982) –and other studies– where no constants are included in information sets.

However, recently, pricing kernels arising from the consumption based capital asset pricing model or the baseline real business cycle model have been replaced with universal function approximators such as polynomials or neural networks to obtain smaller pricing errors, see, e.g., Bansal and Viswanathan (1993), Bansal, Hsieh and Viswanathan (1993) and Chapman (1997).<sup>30</sup> Since any function can be approximated by those pricing kernels driving the objective function in (25) to zero based on a (finite) sample is not a difficult task. But as information sets in those studies do not include constants, functions  $f$  may have been found that satisfy cases 1. or 2. for  $i = 1, 2, \dots, n$ , and thus do not guarantee rational expectations as discussed above. Furthermore, the Bansal, Hsieh and Viswanathan (1993) and Chapman (1997) do not report out-of-sample performance although nonparametric asset pricing models are exposed to the danger of overfitting.

We would like to conclude that it remains to re-check whether the Euler equation holds unconditionally in those nonparametric asset pricing models and, in addition, out-of-sample performance should be investigated. Further we would like to mention that including a constant in the information set permits the Euler equation to hold unconditionally.

---

<sup>29</sup>Note that  $E [f(x_t, \theta)I_{i,t}] = E [f(x_t, \theta)] E [I_{i,t}] + \text{COV}[f(x_t, \theta), I_{i,t}]$ ,  $i = 1, \dots, n$ .

<sup>30</sup>One should not argue that the equity premium puzzle is solved because this nonparametric approach is purely data driven and does not deliver explanation from an economic point of view.

## Literature

Boldrin, M., L. J. Christiano and J. D. M. Fisher (1997), Habit Persistence and Mean Asset Returns in an Exchange Economy, *Macroeconomic Dynamics* 1, pp. 312–332.

Boldrin, M., L. J. Christiano and J. D. M. Fisher (2001), Habit Persistence, Asset Returns and the Business Cycle, *American Economic Review*, vol. 91, 1, pp. 149–166 .

Brandt, M. (1999), Estimating Portfolio and Consumption Choice: A Conditional Euler Equations Approach, *Journal of Finance*, forthcoming.

Breeden, D. T. (1979), An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities, *Journal of Financial Economics* 7, pp. 231–262.

Brock, W. A. and L. J. Mirman (1972), Optimal Economic Growth and Uncertainty: The Discounted Case, *Journal of Economic Theory* 4, pp. 479–513.

Brown, B. W. and W. K. Newey (1998), Efficient Semiparametric Estimation of Expectations, *Econometrica* 66, pp. 453–464.

Campbell, J. Y. (1994), Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model, *Journal of Monetary Economics* 33, pp. 463–506.

Campbell, J. Y. and J. H. Cochrane (1999), By Force of Habit: A Consumption-based Explanation of Aggregate Stock Market Behavior, *Journal of Political Economy* x, pp. x.

Campbell, J. Y. and J. H. Cochrane (1999), Explaining the poor performance of consumption-based asset pricing models, Working Paper, Harvard University.

Campbell, J. Y., A. W. Lo and C. MacKinlay (1997), *The Econometrics*

of Financial Markets, Princeton University Press, Princeton.

Campbell, J. Y. and L. M. Viceira (1998), Consumption and Portfolio Decisions When Expected Returns are Time Varying, forthcoming in Quarterly Journal of Economics.

Chapman, D. A. (1997), Approximating the Asset Pricing Kernel, Journal of Finance 52, pp. 1383–1410.

Chapman, D. A. (1998), Habit Formation and Aggregate Consumption, Econometrica 66, pp. 1223–1230.

Chen, X. and H. White (1998), Nonparametric adaptive learning with feedback, Journal of Economic Theory 82, pp. 190–222.

Chow, G. C. (1991), Dynamic Optimization Without Dynamic Programming, Working Paper 361, Princeton University.

Chow, G. C. (1993), Statistical Estimation and Testing of a Real Business Cycle Model, Working Paper, Princeton University.

Chow, G. C. and Y. K. Kwan (1998), How the Basic RBC Model Fails to Explain US Time Series, Journal of Monetary Economics 41, pp. 301–318.

Christiano, L. J. and W. J. Den Haan (1996), Small-sample properties of GMM for business-cycle analysis, Journal of Business & Economic Statistics 14, pp. 309–280.

Christiano, L. J. and Eichenbaum (1992), Current Real Business Cycle Theories and Aggregate Labor Market Fluctuation, American Economic Review, pp. 431–472.

Cochrane, J. H. (1991), Production-based Asset Pricing and the Link between Stock Returns and Economic Fluctuations, Journal of Finance 46, pp. 209–237.

- Cochrane, J. H. (1996), A Cross-Sectional Test of an Investment Based Asset Pricing Model, *Journal of Political Economy* 104, pp. 572–621.
- Constantinides, G. M. and D. Duffie (1996), Asset Pricing with Heterogeneous Consumers, *Journal of Political Economy* 104, pp. 219–240.
- Den Haan, W. J. and A. Marcet (1990), Solving the Stochastic Growth Model by Parameterizing Expectations, *Journal of Business & Economic Statistics* 8, pp. 31–34.
- Den Haan, W. J. and A. Marcet (1994), Accuracy in Simulations, *Review of Economic Studies* 61, pp. 3–17.
- Duffie, D. and K. J. Singleton (1993), Simulated Moments Estimation of Markov Models of Asset Prices, *Econometrica* 61, pp. 929–952.
- Duffy, J. and P. D. McNelis (1997), Approximating and Simulating the Stochastic Growth Model: Parameterized Expectations, Neural Nets, and the Genetic Algorithm, Working Paper, University of Pittsburgh.
- Fève, P. and F. Langot, Unemployment and the Business Cycle in a Small Open Economy: G.M.M. Estimation and Testing with French Data, *Journal of Applied Econometrics* 9, pp. 1609–1639.
- Gouriéroux, C. (1997), *ARCH Models and Financial Applications*, Springer, New York et al.
- Gouriéroux, C., A. Monfort and E. Renault (1993), Indirect Inference, *Journal of Applied Econometrics* 8, pp. 85–118.
- Hamilton, J. D. and G. Lin (1996), Stock Market Volatility and the Business Cycle, *Journal of Applied Econometrics* 11, pp. 573–593.
- Hansen, L. (1982), Large Sample Properties of Generalized Method of Moments Estimators, *Econometrica* 50, pp. 1029–1054.



Hansen, L. P., J. Heaton and A. Yaron (1996), Finite-sample properties of some alternative GMM estimators, *Journal of Business & Economic Statistics* 14, pp. 262–280.

Hansen, L. P. and Ravi Jagannathan (1991), Implications of Security Market Data for Models of Dynamic Economies, *Journal of Political Economy* 99, pp. 225–262.

Hansen, L. P. and T. J. Sargent (1980), Formulating and Estimating Dynamic Linear Rational Expectations Models, *Journal of Economic Dynamics and Control* 2, pp. 7–46.

Hansen, L. P. and K. J. Singleton (1982), Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models, *Econometrica* 50, pp. 1269–1286.

Härdle, W. and Tsybakov (1997), Local Polynomial Estimators of the Volatility Function in Nonparametric Autoregression, *Journal of Econometrics* 81, pp. 223–242.

Heaton, J. (1993), The Interaction between Time–Nonseparable Preferences and Time Aggregation, *Econometrica* 61, pp. 353–85.

Heaton, J. (1995), An Empirical Investigation of Asset Pricing with Temporally Dependent Preference Specifications, *Econometrica* 63, pp. 681–717.

Heaton, J. and D. J. Lucas (1996), Evaluating the Effects of Incomplete Markets of Risk Sharing and Asset Pricing, *Journal of Political Economy* 104, pp. 443–487.

Jermann, U. J. (1998), Asset Pricing in Production Economies, *Journal of Monetary Economics* 41, pp. 257–275.

Judd, K. L. (1992), Projection Methods for Solving Aggregate Growth Models, *Journal of Economic Theory* 58, pp. 410–452.

Judd, K. J. and Jess Gaspar (1996), Solving Large Scale Rational Expectations Models, Working Paper, Hoover Institution, Stanford University.

Kandel, S. and R. F. Stambaugh (1990), Expectations and volatility of consumption and asset returns, *Review of Financial Studies* 3, pp. 207–232.

Kandel, S. and R. F. Stambaugh (1990), Asset returns and intertemporal preferences, *Journal of Monetary Economics* 27, pp. 39–71.

Kocherlakota, N. R. (1996), The Equity Premium: It's Still a Puzzle, *Journal of Economic Literature* 34, pp. 42–71.

Kuan, C.–M. and H. White (1994), Adaptive learning with nonlinear dynamics driven by dependent processes, *Econometrica* 62, pp. 1087–1114.

Kushner, H. J. and D. S. Clark (1978), *Stochastic Approximation Methods for Constrained and Unconstrained Systems*, Springer–Verlag, New York et al.

Kydland, F. E. and E. C. Prescott (1982), Time to Build and Aggregate Fluctuations, *Econometrica* 50, pp. 1345–1370.

Lettau, M. (1998), Inspecting the Mechanism: The Determination of Asset Prices in the Real Business Cycle Model, Working Paper, Humboldt University of Berlin.

Lettau, M., G. Gong and W. Semmler (2001), Statistical Estimation and Moment Evaluation of a Stochastic Growth Model with Asset Market Restrictions, *Journal of Economic Organization and Behavior*, vol. 44, pp. 85–103.

Lettau, M. and H. Uhlig (1997a), Asset Prices and Business Cycles: Successes and Pitfalls of the General Equilibrium Approach, CEPR Working Paper, Tilburg University.

Lettau, M. and H. Uhlig (1997b), Preferences, Consumption Smoothing, and

Risk Premia, CEPR Working Paper, Tilburg University.

Ljung, L. (1977), Analysis of Recursive Stochastic Algorithms, IEEE Transactions on Automatic Control AC-22, pp. 551–575.

Lucas, D. J. (1994), Asset Pricing with Undiversifiable Income Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle, Journal of Monetary Economics 34, pp. 325–341.

Lucas, R., Jr. (1978), Asset Prices in an Exchange Economy, Econometrica 46, pp. 1429–1446.

Luttmer, E. G. J. (1996), Asset Pricing in Economies with Frictions, Econometrica 64, pp. 1439–1467.

Marcet, A. and D. A. Marshall (1994), Solving Nonlinear Rational Expectations Models by Parameterized Expectations: Convergence to Stationary Solutions, Working Paper, University of Pompeu Fabra.

Marshall, D. A. and N. G. Parekh (1998), Can Costs of Consumption Adjustment Explain Asset Pricing Puzzles?, forthcoming in Journal of Finance.

McGrattan, E. R. (1996), Solving the Stochastic Growth Model with a Finite Element Method, Journal of Economic Dynamics and Control 20, pp. 19–42.

Mehra, R. and E. C. Prescott (1985), The Equity Premium: A Puzzle, Journal of Monetary Economics 15, pp. 1455–161.

Miranda, M. J. and X. Rui (1997), Maximum Likelihood Estimation of the Nonlinear Rational Expectations Asset Pricing Model, Journal of Economic Dynamics and Control 21, pp. 1493–1510.

Ritter, H., T. Martinetz and K. Schulten (1991), Introduction to the Theory of Self-organizing maps, Addison Wesley, New York.

Ritter, H. and K. Schulten (1989), Convergence properties of Kohonen's

topology conserving maps: Fluctuations, stability and dimension selection, *Biological Cybernetics* 60, pp. 59–71.

Rouwenhorst, K. G. (1995), Asset pricing implications of equilibrium business cycle models, ..., pp. 295–330.

Santos, M. S. and Vigo-Aguiar, Jesús (1998), Analysis of a Numerical Dynamic Programming Algorithm Applied to Economic Models, *Econometrica* 66, pp. 409–426.

Schmertmann, C. P. (1996), Functional Search in Economics Using Genetic Programming, *Computational Economics* 9, pp. 275–298.

Schwert, G. W. (1989), Why Does Stock Market Volatility Change Over Time?, *Journal of Finance* 44, pp. 1115–1153.

Schwert, G. W. (1990), Stock returns and real activity: A century of evidence, *Journal of Finance* 45, pp. 1237–1257.

Semmler, W. and G. Gong (1996), Estimating Parameters of Real Business Cycle Models, *Journal of Economic Behavior & Organization* 30, pp. 301–325.

Sieveking, M. and W. Semmler (1997), The Present Value of Resources with Large Discount Rates.

Smith, A. A. (1993), Estimating Nonlinear Time-Series Models Using Simulated Vector Autoregressions, *Journal of Applied Econometrics* 8, pp. S63–S84.

Smith, D. C. (1999), Finite sample properties of tests of the Epstein–Zin asset pricing model, *Journal of econometrics* 93, pp. 113–148.

Tauchen, G. (1986), Statistical Properties of Generalized Method-of-Moments Estimates of Structural Parameters Obtained From Financial Market Data, *Journal of Business and Economic Statistics* 4, pp. 397–417.

Tauchen, G. (1990), Solving the Stochastic Growth Model by Using Quadra-

ture Methods and Value–function Iterations, *Journal of Business & Economic Statistics* 8, pp. 49–51.

Taylor, J. B. and H. Uhlig (1990), Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods, *Journal of Business & Economic Statistics* 8, pp. 1–17.

Telmer, C. I. (1993), Asset–pricing Puzzles and Incomplete Markets, *Journal of Finance* 48, pp. 1803–1832.

Veronesi, P. (1999), Stock market overreaction to bad news in good times: A rational expectations equilibrium, *Review of Financial Studies* 12, pp. 975–1007.

Whitelaw, R. F. (1997), Time–varying Sharpe ratios and market timing, Working Paper, New York University.

Wöhrmann, P. (1999), Nonparametric Inference Based on Local Linear Maps, Working Paper, University of Bielefeld.

Wöhrmann, P. (1999), On Testing Implications of Dynamic Asset Pricing Theories Using GMM with Instrumental Variables, Working Paper, University of Bielefeld.