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Abstract: Risk aversion is traditionally defined in the context of lotteries over monetary payoffs. This paper extends the notion of risk aversion to a more general setup where outcomes (consequences) may not be measurable in monetary terms and people may have fuzzy preferences over lotteries, i.e. they may choose in a probabilistic manner. The paper considers comparative risk aversion within neoclassical expected utility theory, a constant error/tremble model and a strong utility model of probabilistic choice (which includes the Fechner model and the Luce choice model as special cases). The paper also provides a new definition of relative riskiness of lotteries.

Keywords: risk aversion, more risk averse than, riskiness, probabilistic choice, expected utility theory, Fechner model, Luce choice model

JEL classification codes: D00, D80, D81

Risk Aversion

Risk aversion is traditionally defined in the context of lotteries over monetary payoffs (Pratt, 1964). However, one can also consider risk aversion when the outcomes of risky lotteries may not be measurable in monetary terms. For example, people can be risk averse or risk prone when driving their car, choosing a medical treatment, selecting a holiday destination, deciding to marry or to divorce etc. This paper extends the notion of risk aversion to decision problems where outcomes (consequences) may not be measurable in monetary terms. Epstein (1999) defines uncertainty aversion when the outcome set is arbitrary rather than Euclidean but people have a unique preference ordering over uncertain alternatives.

Numerous empirical studies show that people generally have fuzzy preferences over lotteries, i.e. they choose in a probabilistic manner (e.g. Camerer, 1989; Hey and Orme, 1994; Loomes and Sugden, 1998). Therefore, this paper also extends the notion of risk aversion to allow for the possibility of fuzzy preferences. Hilton (1989) and Wilcox (2008) define risk aversion in the context of lotteries over monetary payoffs when people choose probabilistically between lotteries.

The paper is organized as follows. Section 1 defines comparative risk aversion in the context of an arbitrary outcome set. Section 2 considers the implications of this definition for neoclassical expected utility theory. Section 3 extends the notion of risk aversion to a more general setup where people have fuzzy preferences over lotteries. Section 4 analyses risk aversion within two well-known models of probabilistic choice (a constant error/tremble model and a strong utility model). Section 5 defines absolute risk aversion and relative riskiness of lotteries. Section 6 concludes.

1. Comparative Risk Aversion

Let X be a finite nonempty set of outcomes (consequences). We will treat X as an arbitrary abstract set so that an element $x \in X$ can be a monetary payoff, a consumption bundle, a health state, marriage or divorce, birth of a child, the afterlife etc. A lottery $L: X \rightarrow [0,1]$ is a probability distribution on X, i.e. it delivers an outcome $x \in X$ with a probability $L(x) \in [0,1]$ and $\sum_{x \in X} L(x)=1$. A degenerate lottery that yields one outcome $x \in X$ with probability one is denoted by $(x,1)$. The set of all lotteries is denoted by $\mathcal{L}$.

In this and the next section we consider a "traditional" decision maker who has a unique binary preference relation $\succsim$ on $\mathcal{L}$. As customary, we will use the sign $\succ$ to denote the asymmetric component of $\succsim$, and the sign $\sim$ to denote the symmetric component of $\succsim$. We will consider two individuals: an individual ♀ characterized by a preference relation $\succsim_{♀}$ and an individual ♂ characterized by a preference relation $\succsim_{♂}$.

Definition 1 An individual ♀ is unambiguously *more risk averse than* an individual ♂ if $(x,1) \succsim_{♂} L$ implies $(x,1) \succsim_{♀} L$ for all $x \in X$ and all $L \in \mathcal{L}$ and there exists at least one degenerate lottery $(x,1) \in \mathcal{L}$ and one lottery $L \in \mathcal{L}$ such that $(x,1) \sim_{♂} L$ and $(x,1) \succ_{♀} L$.

According to Definition 1, a more risk averse individual weakly prefers a degenerate lottery over another lottery whenever a less risk averse individual does so as well. This definition of the more-risk-averse-than relation between individuals is very general. Specifically, we do not require that lottery outcomes are measurable in real numbers. We also do not require that individual preferences over lotteries are represented by a specific decision theory (e.g. expected utility theory).

Definition 1 immediately implies the following result. If an individual ♀ is more risk averse than an individual ♂, or vice versa, then $(x,1) \succsim_{♀} (y,1)$ if and only if

$(x,1) \succsim_{♂} (y,1)$ for all $x,y \in X$. This implication of our definition is quite intuitive. We can unambiguously rank two individuals in terms of their risk preferences only if they have identical preferences over riskless alternatives (degenerate lotteries). If the two individuals do not have the same preferences in choice under certainty, one of them may choose a specific degenerate lottery because it is her most preferred alternative and not because she is averse to risk. Therefore, to have a meaningful concept of comparative risk aversion, we need to consider individuals with identical preferences over the set of riskless consequences.

2. Risk Aversion in Expected Utility Theory

Let us now apply the concept of comparative risk aversion in the context of expected utility theory (von Neumann and Morgenstern, 1944). In expected utility theory there exists an utility function $u:X\rightarrow\mathbb{R}$ that is unique up to a positive linear transformation, such that

(1) $$S \succsim R \quad \text{if and only if} \quad \sum_{x\in X} S(x)u(x) \geq \sum_{x\in X} R(x)u(x),$$

for any two lotteries S, $R \in \mathcal{L}$. Formula (1) simply states that a lottery S is weakly preferred over a lottery R if and only if the expected utility of S is greater than or equal to the expected utility of R.

As we already discussed above, for comparing risk aversion across individuals we need to consider people with identical preferences over the set of riskless outcomes. We will say that two individuals ♀ and ♂ have ordinally equivalent utility functions when $u_{♀}(x)\geq u_{♀}(y)$ if and only if $u_{♂}(x)\geq u_{♂}(y)$ for any two outcomes x, $y \in X$.

Proposition 1 An expected utility maximizer ♀ with utility function $u_{♀}:X\rightarrow\mathbb{R}$ is more risk averse than an expected utility maximizer ♂ with ordinally equivalent utility function $u_{♂}:X\rightarrow\mathbb{R}$ if and only if

(2) $$\frac{u_{♀}(y)-u_{♀}(x)}{u_{♀}(z)-u_{♀}(y)} \geq \frac{u_{♂}(y)-u_{♂}(x)}{u_{♂}(z)-u_{♂}(y)},$$

for any $x, y, z \in X$ such that $u_{♀}(x) < u_{♀}(y) < u_{♀}(z)$ and there exists at least one triple of outcomes $\{x, y, z\} \subset X$ for which inequality (2) holds with strict inequality.

Proof is presented in the Appendix.

Proposition 1 can be interpreted in the following way. We can use an index $I_{♀}(x, y, z) \equiv (u_{♀}(y)-u_{♀}(x))/(u_{♀}(z)-u_{♀}(y))$ to measure the risk aversion of an expected utility maximizer ♀ in the context of outcomes $x, y, z \in X$ such that $u_{♀}(x) < u_{♀}(y) < u_{♀}(z)$. Similar to the Arrow-Pratt coefficient of absolute risk aversion for lotteries over monetary payoffs, the index $I_{♀}(x, y, z)$ captures only local risk aversion. Therefore, for an expected utility maximizer ♀ to be unambiguously more risk averse than an expected utility maximizer ♂, we need to have $I_{♀}(x, y, z) \geq I_{♂}(x, y, z)$ for all triples of outcomes $\{x, y, z\} \subset X$ such that $u_{♀}(x) < u_{♀}(y) < u_{♀}(z)$. Note that an index $I_{♀}(x, y, z)$ is an adequate measure of local risk aversion because it is invariant to positive linear transformations of the utility function $u_{♀}:X \rightarrow \mathbb{R}$.

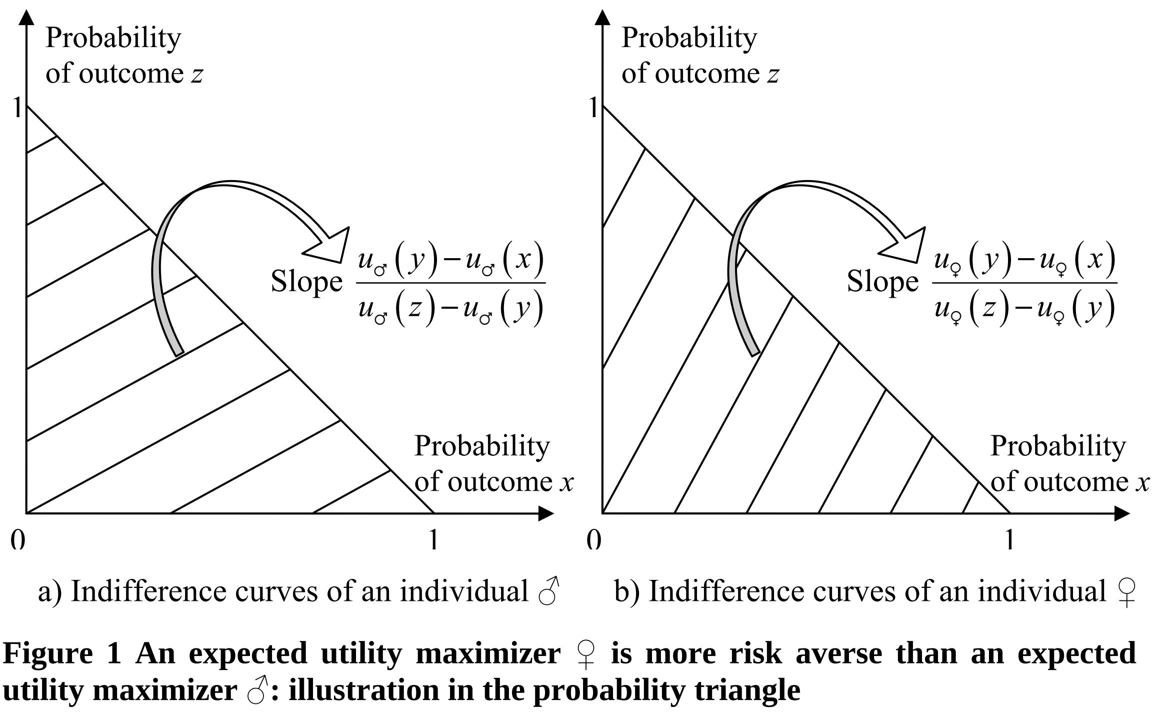

Figure 1 An expected utility maximizer ♀ is more risk averse than an expected utility maximizer ♂: illustration in the probability triangle

Proposition 1 can be conveniently illustrated inside the probability triangle (e.g. Machina, 1982). When the outcome set X has only three elements ($X=\{x, y, z\}$), the set of all lotteries $\mathcal{L}$ can be represented as a rectangular triangle. By convention, the probability of the outcome that yields the lowest utility (x) is shown on the horizontal axis and the probability of the outcome that yields the highest utility (z) is shown on the vertical axis. Indifference curves inside the probability triangle represent the set of all lotteries that yield the same expected utility. Specifically, indifference curves of an expected utility maximizer with utility function $u(.)$ are straight parallel lines with a positive slope $(u(y)-u(x))/(u(z)-u(y))$. Thus, if an expected utility maximizer ♀ is more risk averse than an expected utility maximizer ♂, the indifference curves of individual ♀ have a greater slope inside the probability triangle (indifference curves are steeper).

3. Probabilistic Risk Aversion

Numerous experimental studies find that binary choice under risk is generally probabilistic in nature (e.g. Camerer, 1989; Hey and Orme, 1994; Loomes and Sugden, 1998). In this section we will extend Definition 1 to a more general setup where people may choose in a probabilistic manner. There are several alternative explanations why people make inconsistent choices under risk when decision problems are repeated within a short period of time (e.g. Loomes and Sugden, 1995). For example, people may have multiple preference relations on $\mathcal{L}$ or they may make random errors.

We will now assume that the primitive of choice is a binary choice probability function $P{:}\mathcal{L}\times\mathcal{L}\rightarrow[0,1]$, which is also known as a fuzzy preference relation (e.g. Zimmerman et al., 1984). The notation $P(S,R)$ represents the probability that an individual chooses lottery $S \in \mathcal{L}$ over lottery $R \in \mathcal{L}$ in a direct binary choice. For any two lotteries $S, R \in \mathcal{L}$, $S \neq R$, the probability $P(S,R)$ is observable from the relative frequency with which an individual chooses S when asked to choose repeatedly between S and R. We will consider two individuals: an individual ♀ characterized by a binary choice probability function $P_{♀}(.,.)$ and an individual ♂ characterized by a binary choice probability function $P_{♂}(.,.)$.

Definition 1a An individual ♀ is probabilistically *more risk averse than* an individual ♂ if $P_{♀}((x,1),L) \geq P_{♂}((x,1),L)$ for all $x \in X$ and all $L \in \mathcal{L}$ and there exists at least one degenerate lottery $(x,1) \in \mathcal{L}$ and one lottery $L \in \mathcal{L}$ such that $P_{♀}((x,1),L) > P_{♂}((x,1),L)$.

According to Definition 1a, a more risk averse individual is always at least as likely to choose a degenerate lottery over a risky lottery as a less risk averse individual. This definition of the more-risk-averse-than relation between individuals is very general. As before, we do not restrict lottery outcomes to be measurable in real numbers. We also do not require that fuzzy preferences over lotteries are represented by a specific model of probabilistic choice. Thus, we can apply Definition 1a to very distinct models of probabilistic choice, e.g. when people have multiple preference relations on $\mathcal{L}$ (Loomes and Sugden, 1995) or when people have a unique preference relation on $\mathcal{L}$ but they make random errors (Fechner, 1860; Hey and Orme, 1994; Blavatskyy, 2007).

If lottery L in Definition 1a is another degenerate lottery $(y,1)$, $y \in X$, then we arrive at the following result. If an individual ♀ is more risk averse than an individual ♂, or vice versa, then $P_{♀}((x,1), (y,1)) = P_{♂}((x,1), (y,1))$ for all $x,y \in X$. In other words, we can unambiguously rank two individuals in terms of their risk attitudes only if they choose in identical manner between riskless alternatives (degenerate lotteries). If this is not the case, heterogeneous risk attitudes are confounded with heterogonous tastes over riskless outcomes and we cannot make a clear comparison of individuals in terms or relative risk aversion.

4. Risk Aversion in Models of Probabilistic Choice

In this section we consider the implications of Definition 1a for several well-known models of probabilistic choice. Arguably, the simplest model of probabilistic choice is the constant error/tremble model of Harless and Camerer (1994). In this model, an individual has a unique preference relation on $\mathcal{L}$ but she does not always choose the preferred lottery. With a constant probability $\tau\in[0,1]$ a tremble occurs and the individual chooses the less preferred lottery. Specifically, in a constant error/tremble model there exists an utility function $u{:}X\to\mathbb{R}$ that is unique up to a linear transformation, such that

$$(3) \qquad P(S,R) = 0.5 + (0.5 - \tau)\ \mathrm{sign}\big(\textstyle\sum_{x\in X} S(x)u(x) - \sum_{x\in X} R(x)u(x) \big),$$

for any two lotteries S, $R\in\mathcal{L}$ and a probability $\tau\in[0,1]$. Formula (3) states that a lottery S is chosen over a lottery R with probability $1\text{-}\tau$ if the expected utility of S is greater than the expected utility of R; with probability 0.5—if the expected utilities of lotteries S and R are exactly equal; and with probability τ—if the expected utility of S is less than the expected utility of R. The result of Proposition 1 can be extended to a constant error/tremble model of probabilistic choice.

Proposition 2 If individual choices are represented by a constant error/tremble model (3) then an individual ♀ with utility function $u_{♀}{:}X\to\mathbb{R}$ and the probability of a tremble $\tau_{♀}$ is more risk averse than an individual ♂ with ordinally equivalent utility function $u_{♂}{:}X\to\mathbb{R}$ and the probability of a tremble $\tau_{♂} = \tau_{♀}$ if and only if

$$(4) \qquad \frac{u_{♀}(y)-u_{♀}(x)}{u_{♀}(z)-u_{♀}(y)} \geq \frac{u_{♂}(y)-u_{♂}(x)}{u_{♂}(z)-u_{♂}(y)},$$

for any $x, y, z \in X$ such that $u_{♀}(x) < u_{♀}(y) < u_{♀}(z)$ and there exists at least one triple of outcomes $\{x, y, z\} \subset X$ for which inequality (2) holds with strict inequality.

Proof is analogous to the proof of Proposition 1.

Not all models of probabilistic choice allow for an unambiguous ranking of individuals in terms of their risk preferences. For example, consider a strong utility model (e.g. Luce and Suppes, 1965). In a strong utility model there exists an utility function u:$X\rightarrow\mathbb{R}$ that is unique up to a positive linear transformation, and a strictly increasing function φ:$\mathbb{R}\rightarrow[0,1]$, which satisfies $\varphi(v)+\varphi(-v)=1$ for all $v\in\mathbb{R}$, such that

$$(5) \qquad P(S,R) = \varphi \left(\sum_{x\in X} S(x)u(x) - \sum_{x\in X} R(x)u(x) \right)$$

for any two lotteries S, $R\in\mathcal{L}$. If the function $\varphi(.)$ is the cumulative distribution function of a normal distribution with zero mean and constant standard deviation, model (5) becomes the Fechner model of random errors (Fechner, 1860; Hey and Orme, 1994). If the function $\varphi(.)$ is the cumulative distribution function of the logistic distribution: $\varphi(v)=1/(1+\exp(-\lambda v))$, where $\lambda>0$ is constant, model (5) becomes the Luce choice model (Luce, 1959). Blavatskyy (2008) provides axiomatic characterization of model (5).

Proposition 3 If individual choices are represented by a strong utility model (5) then it is impossible to find two individuals such that one of them is probabilistically more risk averse than the other.

Proof is presented in the Appendix.

In a recent study, Wilcox (2008) discusses the failure of a strong utility model to rank individuals in terms of their risk preferences (in the context of lotteries over monetary outcomes).

5. Absolute Risk Aversion and Relative Riskiness

So far we considered only comparative risk aversion. To measure absolute risk aversion, we need to fix one binary choice probability function $P_{RN} : \mathcal{L} \times \mathcal{L} \rightarrow [0,1]$. An individual is called *risk neutral* if she has the binary choice probability function $P_{RN}(.,.)$. An individual is called *risk averse* if she is more risk averse (according to Definition 1a) than the risk neutral individual. Similarly, an individual is called *risk seeking* or *risk loving* if the risk neutral individual is more risk averse than this individual. Notice that the concept of absolute risk aversion depends on an ad hoc selection of a risk neutral binary choice probability function $P_{RN}(.,.)$. This is similar to our temperature measurement that also requires an ad hoc selection of zero temperature (e.g. the triple point of water in the Celsius scale or absolute zero in the Kelvin scale).

The concept of comparative risk aversion can be also used to define the relative riskiness of lotteries. In a sense, we are now looking at the other side of a coin. We ask the question: when is one lottery riskier than the other so that more risk averse people dislike it? For expositional clarity, let us first define relative riskiness when people have a unique rational preference relation $\succsim$ on the set of lotteries $\mathcal{L}$.

One way to define relative riskiness is the following. A lottery $R \in \mathcal{L}$ is riskier than a lottery $S \in \mathcal{L}$ if $S \succsim_{♂} R$ implies $S \succsim_{♀} R$ for any two individuals ♀ and ♂ such that ♀ is more risk averse than ♂, and $S \succsim_{♂} R$ implies $S \succ_{♀} R$ for at least one such pair of individuals. Aumann and Serrano (2007) use a similar definition of relative riskiness in the context of lotteries over monetary outcomes.

However, there may exist two lotteries S, $R \in \mathcal{L}$ such that every individual strictly prefers S over R (in this case we say that lottery S dominates lottery R). For such pair of lotteries, a more risk averse individual would always strictly prefer S over R.

However, this strong preference is not related to the relative riskiness of the two lotteries. It simply reflects the fact that *S* is relatively better than *R*. To distinguish between relative riskiness and relative attractiveness of lotteries, we use the following definition.

Definition 2 A lottery $R \in \mathcal{L}$ is riskier than a lottery $S \in \mathcal{L}$ if $S \succsim_{♂} R$ implies $S \succsim_{♀} R$ for any two individuals ♀ and ♂ such that ♀ is more risk averse than ♂, and there exists at least one such pair of individuals for whom we have $S \sim_{♂} R$ and $S \succ_{♀} R$.

Definition 2 is more general than the traditional definitions of relative riskiness in terms of second-order stochastic dominance or mean-preserving spreads (Rothschild and Stiglitz, 1970). First of all, Definition 2 does not require lottery outcomes to be measurable in real numbers. Second, traditional definitions of relative riskiness are equivalent to Definition 2 with an additional restriction that an individual ♂ is risk-neutral. Thus, the traditional definitions of relative riskiness apply only to a subset of lotteries $\mathcal{L}' \subset \mathcal{L}$ such that for any $S, R \in \mathcal{L}'$ we have $S \sim_{RN} R$ (when lottery outcomes are monetary, this condition simply means that *S* and *R* have the same expected value). In contrast, Definition 2 imposes a partial ordering in terms of relative riskiness on a significantly larger set of lotteries.

Definition 2 defines riskiness as the attribute of lotteries that risk averse people dislike. If we make additional assumptions about individual preferences over lotteries, we can define riskiness in terms of objective characteristics of lotteries without referring to subjective preferences. For example, let us consider individual preferences that are represented by expected utility theory. As we already discussed above, comparative risk aversion is well-defined under expected utility theory only if people have ordinally equivalent utility functions.

Proposition 4 For a group of expected utility maximizers that have utility functions ordinally equivalent to a function $u{:}X\rightarrow\mathbb{R}$, a lottery $R\in\mathcal{L}$ is riskier than another lottery $S\in\mathcal{L}$ if there exists an outcome $y\in X$ such that $\sum_{x\in X\mid u(x)\leq v}S(x)\leq\sum_{x\in X\mid u(x)\leq v}R(x)$ for any $v<u(y)$ and $\sum_{x\in X\mid u(x)\leq w}S(x)\geq\sum_{x\in X\mid u(x)\leq w}R(x)$ for any $w>u(y)$ and there exists at least one $v<u(y)$ and one $w>u(y)$ such that both inequalities hold as strict inequalities.

Proof is presented in the Appendix.

Proposition 4 can be conveniently illustrated inside the probability triangle when the outcome set is $X=\{x, y, z\}$. As it is conventional, we plot the probability of the outcome x (z) that yields the lowest (highest) utility on the horizontal (vertical) axis. Figure 2 shows the set $\{R\in\mathcal{L}|R(x)>L(x),R(z)>L(z)\}$ of all lotteries that are riskier than an arbitrary selected lottery L and the set of all lotteries $\{S\in\mathcal{L}|S(x)<L(x),S(z)<L(z)\}$ that are less risky than L. Figure 2 adheres to informal usage of the concept of relative riskiness. Many studies refer to the lotteries located in the north-eastern direction as "riskier" and to the lotteries located in the south-western direction—as "safer" but do not provide a formal definition of relative riskiness (e.g. Camerer, 1989; Loomes and Sugden, 1998).

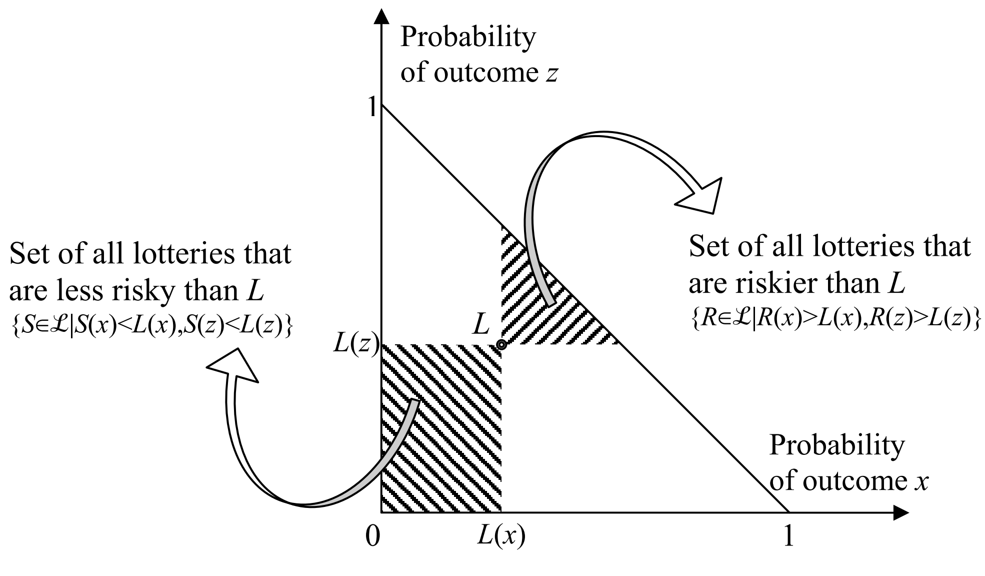

Figure 2 The set of all lotteries that are riskier than lottery *L* and the set of all lotteries that are less risky than lottery *L*: illustration in the probability triangle

We can extend Definition 2 to a more general setup where people have fuzzy preferences captured by a binary choice probability function $P:\mathcal{L}\times\mathcal{L} \rightarrow [0,1]$.

Definition 2a A lottery $R\in\mathcal{L}$ is riskier than a lottery $S\in\mathcal{L}$ if $P_{♀}(S,R)\geq P_{♂}(S,R)$ for any two individuals ♀ and ♂ such that ♀ is more risk averse than ♂, and there exists at least one such pair of individuals for whom we have $P_{♀}(S,R)>P_{♂}(S,R)$.

6. Conclusion

Risk aversion is a fundamental concept in many fields of economics. However, it is traditionally defined only in the context of lotteries over monetary payoffs. This paper extends the definition of risk aversion to a more general setup where lottery outcomes are not necessarily measurable in real numbers and people do not necessarily have a unique preference relation over risky lotteries, i.e. they may choose in a probabilistic manner.

We show that in neoclassical expected utility theory risk aversion can be captured by a simple index of local risk aversion (the slope of indifference curves inside the "local" probability triangle). The same result holds for a constant error/ tremble model of probabilistic choice. However, not all models of probabilistic choice allow for an unambiguous ranking of individuals in terms of their risk preferences. In particular, we prove an impossibility theorem for a strong utility model of probabilistic choice (which includes the Fechner model and the Luce choice model as special cases).

Finally, we show that the definition of comparative risk aversion can be used to define a related concept of the relative riskiness of lotteries. Our proposed definition of relative riskiness generalizes traditional definitions (second-order stochastic dominance, mean preserving spreads) to a larger class of lotteries (that may differ in expected value or may yield non-monetary payoffs). The proposed definition adheres to the informal usage of the concept of relative riskiness in the literature.

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Appendix

Proof of Proposition 1.

Let us first prove the necessity of condition (2). If an individual ♀ is more risk averse than an individual ♂ then there exists at least one triple of outcomes $\{x, y, z\}\subset X$ such that $u_♀(x) < u_♀(y) < u_♀(z)$. Otherwise, an utility function $u_♀:X\rightarrow\mathbb{R}$ maps all outcomes to only one or two real numbers and an ordinally equivalent utility function $u_♂:X\rightarrow\mathbb{R}$ does the same. Hence, utility function $u_♂:X\rightarrow\mathbb{R}$ is a linear transformation of utility function $u_♀:X\rightarrow\mathbb{R}$ and both individuals have the same binary preference relation on $\mathcal{L}$ i.e. an individual ♀ cannot be more risk averse than an individual ♂.

For any triple $\{x, y, z\}\subset X$ such that $u_♀(x) < u_♀(y) < u_♀(z)$ we can construct a lottery L that yields an outcome x with a probability $1\text{-}q$ and an outcome z with a probability $q\in(0,1)$. An expected utility maximizer ♂ prefers a degenerate lottery that yields outcome y for sure over lottery L if $u_♂(y) \geq (1\text{-}q)u_♂(x) + qu_♂(z)$. We can rearrange this condition into

$$\frac{u_♂(y)-u_♂(x)}{u_♂(z)-u_♂(y)} \geq \frac{q}{1-q}. \qquad (6)$$

Similarly, an expected utility maximizer ♀ prefers a degenerate lottery that yields outcome y for sure over lottery L if

$$\frac{u_♀(y)-u_♀(x)}{u_♀(z)-u_♀(y)} \geq \frac{q}{1-q}. \qquad (7)$$

If the left-hand-side of (7) is strictly less than the left-hand-side of (6), then we can find a probability $q\in(0,1)$ sufficiently close to one such that inequality (6) holds but inequality (7) does not hold. However, this contradicts to our premise that an individual ♀ is more risk averse than an individual ♂ (we found a degenerate lottery and a risky lottery L such that an individual ♂ prefers the degenerate lottery over L but an individual ♀ does not). Hence, the left-hand-side of (7) should be greater than or equal to the left-hand-side of (6) for any triple $\{x, y, z\}\subset X$ such that $u_♀(x) < u_♀(y) < u_♀(z)$.

Let us now prove that condition (2) is sufficient for characterizing an individual ♀ as more risk averse. An individual ♂ prefers a degenerate lottery that yields an outcome $y \in X$ for sure over an arbitrary lottery $L \in \mathcal{L}$ if

$$(8) \qquad u_♂(y) \geq \sum_{x\in X} L(x)u_♂(x).$$

If $u_♂(y) = \max_{x\in X} u_♂(x)$ then condition (8) is satisfied for any lottery L. Since individuals ♀ and ♂ have ordinally equivalent utility functions, it must be also the case that $u_♀(y) = \max_{x\in X} u_♀(x)$ so that $u_♀(y) \geq \sum_{x\in X} L(x)u_♀(x)$. Thus, an individual ♀ also prefers a degenerate lottery that yields y for sure over any lottery L.

If $u_♂(y) = \min_{x\in X} u_♂(x)$ then condition (8) can be satisfied only if lottery L yields the lowest possible expected utility $u_♂(y)$. Since individuals ♀ and ♂ have ordinally equivalent utility functions, it must be also the case that $u_♀(y) = \min_{x\in X} u_♀(x)= =\sum_{x\in X} L(x)u_♀(x)$. Thus, in this case, if an individual ♂ weakly prefers a degenerate lottery that yields y for sure over lottery L, an individual ♀ does so as well.

If $u_♂(y) \neq \max_{x\in X} u_♂(x)$ and $u_♂(y) \neq \min_{x\in X} u_♂(x)$ then it is possible to find an outcome $w \in X$ that has the highest utility $u_♂(w)$ such that $u_♂(w) < u_♂(y)$. Similarly, it is possible to find an outcome $z \in X$ that has the lowest utility $u_♂(z)$ so that $u_♂(z)>u_♂(y)$. For convenience, let us introduce the following notation. For an arbitrary lottery $L \in \mathcal{L}$ let $q(w)$ denote the cumulative probability of all outcomes that have the utility of $u_♂(w)$, i.e. $q(w) \equiv \sum_{x\in X|\ u_♂(x)=u_♂(w)} L(x)$. Similarly, let us define $q(y) \equiv \sum_{x\in X|\ u_♂(x)=u_♂(y)} L(x)$ and $q(z) \equiv$

$\sum_{x\in X|\ u_♂(x)=u_♂(z)} L(x)$. An individual ♂ prefers a degenerate lottery that yields an outcome y for sure over an arbitrary lottery $L \in \mathcal{L}$ if

$$(9)\quad \begin{aligned} u_♂(y) \geq \sum\nolimits_{x\in X|u_♂(x)<u_♂(w)} L(x)u_♂(x) + q(w)u_♂(w) + q(y)u_♂(y) + \\ + q(z)u_♂(z) + \sum\nolimits_{x\in X|u_♂(x)>u_♂(z)} L(x)u_♂(x). \end{aligned}$$

We can rearrange inequality (9) into the following condition

$$(10)\quad \begin{aligned} \frac{u_♂(y)-u_♂(w)}{u_♂(z)-u_♂(w)}\left(1-q(y)\right) + \sum\nolimits_{x\in X|u_♂(x)<u_♂(w)} L(x)\frac{u_♂(w)-u_♂(x)}{u_♂(z)-u_♂(w)} - \\ -\sum\nolimits_{x\in X|u_♂(x)>u_♂(z)} L(x)\frac{u_♂(x)-u_♂(w)}{u_♂(z)-u_♂(w)} \geq q(z). \end{aligned}$$

If condition (2) holds then we have $\frac{u_♀(w)-u_♀(x)}{u_♀(z)-u_♀(w)} \geq \frac{u_♂(w)-u_♂(x)}{u_♂(z)-u_♂(w)}$ for any outcome $x\in X$ such that $u_♂(x)<u_♂(w)$. Condition (2) also implies that $\frac{u_♀(y)-u_♀(w)}{u_♀(z)-u_♀(w)} \geq \frac{u_♂(y)-u_♂(w)}{u_♂(z)-u_♂(w)}$ and $-\frac{u_♀(x)-u_♀(w)}{u_♀(z)-u_♀(w)} \geq -\frac{u_♀(x)-u_♀(w)}{u_♂(z)-u_♂(w)}$ for any outcome $x\in X$ such that $u_♂(x)>u_♂(z)$. Using these results and inequality (10), we can write

$$(11)\quad \begin{aligned} \frac{u_♀(y)-u_♀(w)}{u_♀(z)-u_♀(w)}\left(1-q(y)\right) + \sum\nolimits_{x\in X|u_♂(x)<u_♂(w)} L(x)\frac{u_♀(w)-u_♀(x)}{u_♀(z)-u_♀(w)} - \\ -\sum\nolimits_{x\in X|u_♂(x)>u_♂(z)} L(x)\frac{u_♀(x)-u_♀(w)}{u_♀(z)-u_♀(w)} \geq q(z). \end{aligned}$$

Finally, inequality (11) can be rewritten as $u_♀(y) \geq \sum_{x\in X} L(x)u_♀(x)$. To sum up, if an individual ♂ prefers a degenerate lottery that yields an arbitrary outcome y for sure over an arbitrary lottery L and condition (2) holds, then an individual ♀ also prefers the degenerate lottery that yields outcome y for sure over lottery L. Thus, according to Definition 1, an individual ♀ is more risk averse than an individual ♂. *Q.E.D.*

Proof of Proposition 3.

Suppose that there is an individual ♀, characterized by utility function $u_♀:X \to \mathbb{R}$ and function $\varphi_♀:\mathbb{R} \to [0,1]$, who is more risk averse than another individual ♂, characterized by utility function $u_♂:X \to \mathbb{R}$ and function $\varphi_♂:\mathbb{R} \to [0,1]$. First, let us prove that there is a constant k such that $\varphi_♀(v) = \varphi_♂(kv)$ for all $v \in \mathbb{R}$.

Let $y \in X$ be an outcome such that $u_♂(y) = \min_{x\in X} u_♂(x)$ and let $z \in X$ be an outcome such that $u_♂(z) = \max_{x\in X} u_♂(x)$. Note that if $u_♂(y) = u_♂(z)$ then an individual ♂ chooses with probabilities 50%-50% between any two lotteries (including degenerate lotteries). Since both individuals should have identical binary choice probabilities in choice under certainty, it follows that an individual ♀ also chooses with probabilities 50%-50% between any two degenerate lotteries. This implies that function $u_♀:X \to \mathbb{R}$ maps all outcomes to just one number and individual ♀ also chooses with probabilities 50%-50% between any two risky lotteries i.e. she cannot be more risk averse than an individual ♂. Therefore, we need only to consider the case when $u_♂(y) > u_♂(z)$.

Let us consider a degenerate lottery $(y,1)$ that yields outcome y for sure and a risky lottery L that yields outcome y with probability $1\text{-}q$ and outcome z with probability $q\in[0,1]$. An individual ♂ chooses $(y,1)$ over L with a probability $\varphi_♂(-q(u_♂(z)-u_♂(y)))$. An individual ♀ chooses $(y,1)$ over L with a probability $\varphi_♀(-q(u_♀(z)- u_♀(y)))$. If an individual ♀ is more risk averse than and individual ♂ we must have

$$(12) \qquad \varphi_♀(-q(u_♀(z)- u_♀(y))) \geq \varphi_♂(-q(u_♂(z)-u_♂(y))).$$

Since $\varphi_♀(-v) = 1 - \varphi_♀(v)$ and $\varphi_♂(-v) = 1 - \varphi_♂(v)$ for any $v \in \mathbb{R}$ we can rewrite (12) as

$$(13) \qquad \varphi_♂(q(u_♂(z)-u_♂(y))) \geq \varphi_♀(q(u_♀(z)- u_♀(y))).$$

Let us consider a degenerate lottery $(z,1)$ that yields outcome z for sure and a risky lottery L' that yields outcome y with probability q and outcome z with probability

1-q. An individual ♂ chooses (z,1) over L' with a probability $\varphi_{♂}(q(u_{♂}(z)-u_{♂}(y)))$. An individual ♀ chooses (z,1) over L' with a probability $\varphi_{♀}(q(u_{♀}(z)- u_{♀}(y)))$. If an individual ♀ is more risk averse than and individual ♂ we must have

(14) $$\varphi_{♀}(q(u_{♀}(z)- u_{♀}(y))) \geq \varphi_{♂}(q(u_{♂}(z)-u_{♂}(y))).$$

Inequalities (13) and (14) can hold simultaneously only if

(15) $$\varphi_{♀}(q(u_{♀}(z)- u_{♀}(y))) = \varphi_{♂}(q(u_{♂}(z)-u_{♂}(y)))$$

for any $q\in[0,1]$. Using a substitution of variables $q = v/(u_{♀}(z)- u_{♀}(y))$ in equation (15) we arrive at $\varphi_{♀}(v) = \varphi_{♂}(kv)$, were $k = (u_{♂}(z)-u_{♂}(y))/(u_{♀}(z)- u_{♀}(y)) >0$ is constant.

Let us now prove that utility function $u_{♀}(.)$ is a positive linear transformation of utility function $u_{♂}(.)$ i.e. there are constants a>0 and b such that $u_{♀}(x) = au_{♂}(x) + b$ for all $x\in X$. An individual ♂ chooses a degenerate lottery (x,1) over a degenerate lottery (y,1) with a probability $\varphi_{♂}(u_{♂}(x)-u_{♂}(y))$. An individual ♀ chooses (x,1) over (y,1) with a probability $\varphi_{♀}(u_{♀}(x)- u_{♀}(y))$. If the two individuals can be ranked in terms of their risk preferences, we must have

(16) $$\varphi_{♀}(u_{♀}(x)- u_{♀}(y)) = \varphi_{♂}(u_{♂}(x)-u_{♂}(y)).$$

We already established that $\varphi_{♀}(u_{♀}(x)- u_{♀}(y)) = \varphi_{♂}(k(u_{♀}(x)- u_{♀}(y)))$. Plugging this result into (16) we receive

(17) $$\varphi_{♂}(k(u_{♀}(x)- u_{♀}(y))) = \varphi_{♂}(u_{♂}(x)-u_{♂}(y)).$$

Since function $\varphi_{♂}:\mathbb{R}\rightarrow[0,1]$ is strictly increasing, equation (17) holds only if

(18) $$k(u_{♀}(x)- u_{♀}(y)) = u_{♂}(x)-u_{♂}(y).$$

Rearranging (18) we obtain $u_{♀}(x) = u_{♂}(x)/k + u_{♀}(y) - u_{♂}(y)/k$ i.e. utility function $u_{♀}(.)$ is a positive linear transformation of utility function $u_{♂}(.)$. Thus, individuals ♀ and ♂ choose in an identical manner between any two lotteries. In other words, an individual ♀ cannot be more risk averse than an individual ♂. *Q.E.D.*

Proof of Proposition 4.

Let $U \subset \mathbb{R}$ be the range (the image of the domain) of an utility function $u{:}X \rightarrow \mathbb{R}$. Since the outcome set X is a finite nonempty set, U must be also a finite nonempty set of real numbers. Thus, we can number the elements of U so that $u^i < u^j$ whenever $i<j$ for any $u^i, u^j \in U$ and any $i, j \in \{1,\ldots,|U|\}$. For any lottery $S \in \mathcal{L}$ and any number $u^i \in U$ let $\acute{S}_i$ denote the cumulative probability of all outcomes that yield utility u^i, i.e. $\acute{S}_i \equiv \sum_{x\in X|\, u(x)=u^i} S(x)$.

Let us now consider an expected utility maximizer ♀ with utility function $u_♀{:}X \rightarrow \mathbb{R}$ who is more risk averse than an expected utility maximizer ♂ with ordinally equivalent utility function $u_♂{:}X \rightarrow \mathbb{R}$. An individual ♂ weakly prefers lottery S over lottery R if and only if $\sum_{x\in X} [S(x)\text{-}R(x)]u_♂(x) \geq 0$, which is equivalent to:

(19) $$\sum_{i=1}^{|U|} \left[\acute{S}_i - \acute{R}_i \right] u_♂^i \geq 0 .$$

Inequality (19) can be then rearranged into:

(20) $$u_♂^k + \sum_{j=1}^{k-1} \left(\sum_{i=1}^{j} \left[\acute{R}_i - \acute{S}_i \right] \right) \left(u_♂^{j+1} - u_♂^j \right) + \sum_{j=k+1}^{|U|} \left(\sum_{i=j}^{|U|} \left[\acute{S}_i - \acute{R}_i \right] \right) \left(u_♂^j - u_♂^{j-1} \right) \geq 0 ,$$

for any $k \in \{2,\ldots,|U|\text{-}1\}$. Furthermore, we can rewrite inequality (20) as follows:

(21) $$\begin{aligned} &\left(\ldots \left(\left[\acute{R}_1 - \acute{S}_1 \right] \frac{u_♂^2 - u_♂^1}{u_♂^3 - u_♂^2} + \left[\acute{R}_1 + \acute{R}_2 - \acute{S}_1 + \acute{S}_2 \right] \right) \frac{u_♂^3 - u_♂^2}{u_♂^4 - u_♂^3} + \ldots + \sum_{i=1}^{k-1} \left[\acute{R}_i - \acute{S}_i \right] \right) \times \\ &\times \frac{u_♂^k - u_♂^{k-1}}{u_♂^{k+1} - u_♂^k} + \left(\ldots \left(\left[\acute{S}_{|U|} - \acute{R}_{|U|} \right] \frac{u_♂^{|U|} - u_♂^{|U|-1}}{u_♂^{|U|-1} - u_♂^{|U|-2}} + \left[\acute{S}_{|U|} + \acute{S}_{|U|-1} - \acute{R}_{|U|} - \acute{R}_{|U|-1} \right] \right) \times \right. \\ &\left. \times \frac{u_♂^{|U|-1} - u_♂^{|U|-2}}{u_♂^{|U|-2} - u_♂^{|U|-3}} + \ldots + \sum_{i=k+1}^{|U|} \left[\acute{S}_i - \acute{R}_i \right] \right) + \frac{u_♂^k}{u_♂^{k+1} - u_♂^k} \geq 0. \end{aligned}$$

According to Proposition 1, if an expected utility maximizer ♀ is more risk averse than an expected utility maximizer ♂ then we must have:

(22) $$\frac{u_♀^j - u_♀^i}{u_♀^m - u_♀^j} \geq \frac{u_♂^j - u_♂^i}{u_♂^m - u_♂^j}$$

for any i, j, $m \in \{1,\ldots,|U|\}$ such that $i < j < m$. Note that if there exists a number $k \in \{2,\ldots,|U|\text{-}1\}$ such that $\sum_{i=1}^{j}\left[\acute{R}_i - \acute{S}_i\right] \geq 0$ for any $j \in \{1,\ldots,k\text{-}1\}$ and $\sum_{i=j}^{|U|}\left[\acute{S}_i - \acute{R}_i\right] \leq 0$ for any $j \in \{k+1,\ldots, |U|\}$ then we can use inequality (22) to rewrite (21) as:

$$\begin{aligned}(23)\quad &\left(\ldots\left(\left[\acute{R}_1 - \acute{S}_1\right]\frac{u_{\text{♀}}^2 - u_{\text{♀}}^1}{u_{\text{♀}}^3 - u_{\text{♀}}^2} + \left[\acute{R}_1 + \acute{R}_2 - \acute{S}_1 + \acute{S}_2\right]\right)\frac{u_{\text{♀}}^3 - u_{\text{♀}}^2}{u_{\text{♀}}^4 - u_{\text{♀}}^3} + \ldots + \sum_{i=1}^{k-1}\left[\acute{R}_i - \acute{S}_i\right]\right)\times \\ &\times\frac{u_{\text{♀}}^k - u_{\text{♀}}^{k-1}}{u_{\text{♀}}^{k+1} - u_{\text{♀}}^k} + \left(\ldots\left(\left[\acute{S}_{|U|} - \acute{R}_{|U|}\right]\frac{u_{\text{♀}}^{|U|} - u_{\text{♀}}^{|U|-1}}{u_{\text{♀}}^{|U|-1} - u_{\text{♀}}^{|U|-2}} + \left[\acute{S}_{|U|} + \acute{S}_{|U|-1} - \acute{R}_{|U|} - \acute{R}_{|U|-1}\right]\right)\times \\ &\times\frac{u_{\text{♀}}^{|U|-1} - u_{\text{♀}}^{|U|-2}}{u_{\text{♀}}^{|U|-2} - u_{\text{♀}}^{|U|-3}} + \ldots + \sum_{i=k+1}^{|U|}\left[\acute{S}_i - \acute{R}_i\right]\right) + \frac{u_{\text{♀}}^k}{u_{\text{♀}}^{k+1} - u_{\text{♀}}^k} \geq 0.\end{aligned}$$

Finally, we can rearrange (23) into $\sum_{i=1}^{|U|}\left[\acute{S}_i - \acute{R}_i\right]u_{\text{♀}}^i \geq 0$, which is equivalent to:

$$(24)\qquad \textstyle\sum_{x \in X}\,[S(x)\text{-}R(x)]\; u_{\text{♀}}(x) \geq 0.$$

To summarize, we showed that a more risk averse expected utility maximizer ♀ weakly prefers lottery *S* over lottery *R* whenever a less risk averse expected utility maximizer ♂ weakly prefers *S* over *R*, if there exists a number $k \in \{2,\ldots,|U|\text{-}1\}$ such that $\sum_{i=1}^{j}\left[\acute{R}_i - \acute{S}_i\right] \geq 0$ for any $j \in \{1,\ldots,k\text{-}1\}$ and $\sum_{i=j}^{|U|}\left[\acute{S}_i - \acute{R}_i\right] \leq 0$ for any $j \in \{k+1,\ldots, |U|\}$.

Note that if $\sum_{i=j}^{|U|}\left[\acute{S}_i - \acute{R}_i\right] = 0$ for all $j \in \{k+1,\ldots, |U|\}$ then the left-hand side of (21) is strictly positive, i.e. we cannot find an individual ♂ who is exactly indifferent between *S* and *R*. In this case, according to Definition 2, *S* cannot be riskier than *R*. Therefore, we must have $\sum_{i=j}^{|U|}\left[\acute{S}_i - \acute{R}_i\right] < 0$ for at least one $j \in \{k+1,\ldots, |U|\}$. By a similar argument, we must also have $\sum_{i=1}^{j}\left[\acute{R}_i - \acute{S}_i\right] > 0$ for at least one $j \in \{1,\ldots,k\text{-}1\}$.

Hence, lottery *R* is riskier than lottery *S* if there exists an outcome $y \in X$ such that $\sum_{x \in X \mid u(x) \leq v} S(x) \leq \sum_{x \in X \mid u(x) \leq v} R(x)$ for any $v < u(y)$ and $\sum_{x \in X \mid u(x) \leq w} S(x) \geq \sum_{x \in X \mid u(x) \leq w} R(x)$ for any $w > u(y)$ and there exists at least one $v < u(y)$ and one $w > u(y)$ such that both inequalities hold as strict inequalities. *Q.E.D.*