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EFFECTIVITY OF BRAUER–MANIN OBSTRUCTIONS ON SURFACES

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Abstract. We study Brauer–Manin obstructions to the Hasse principle and to weak approximation on algebraic surfaces over number fields.

1. Introduction

Let X be a smooth projective variety over a number field k. An important area of research concerns the behavior of the set of k-rational points $X(k)$. One of the major open problems is the decidability problem for $X(k) \neq \emptyset$. An obvious necessary condition is the existence of points over all completions k_v of k; this can be effectively tested given defining equations of X. One says that X *satisfies the Hasse principle* when

$$X(k) \neq \emptyset \Leftrightarrow X(k_v) \neq \emptyset \ \forall v. \tag{1.1}$$

One well-studied obstruction to this is the *Brauer–Manin obstruction* [Man71]. It has proved remarkably useful in explaining counterexamples to the Hasse principle, especially on curves [Sto07] and geometrically rational surfaces [CSS87]; see also [Sko01]. Although there are counterexamples not explained by the Brauer–Manin obstruction [Sko99], [Po10], there remains a wide class of algebraic varieties for which the sufficiency of the Brauer–Manin obstruction is a subject of active research. This includes $K3$ surfaces, studied for instance in [Swi00], [Wit04], [HS05], [SS05], [Bri06], [Ie], [HVV].

We recall, that an element $\alpha \in \mathrm{Br}(X)$ cuts out a subspace

$$X(\mathbb{A}_k)^\alpha \subseteq X(\mathbb{A}_k)$$

of the adelic space, defined as the set of all $(x_v) \in X(\mathbb{A}_k)$ satisfying

$$\sum_v \mathrm{inv}_v(\alpha(x_v)) = 0.$$

Here, inv_v is the local invariant of the restriction of α to a k_v-point, taking its value in $\mathbb{Q}/\mathbb{Z}$. By the exact sequence of class field theory

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \stackrel{\mathrm{inv}}{\to} \mathbb{Q}/\mathbb{Z} \to 0$$

(here inv is the sum of inv_v), we have

$$X(k) \subseteq X(\mathbb{A}_k)^\alpha.$$

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Therefore, for any subset $\mathrm{B} \subseteq \mathrm{Br}(X)$ we have

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{B}} := \bigcap_{\alpha \in \mathrm{B}} X(\mathbb{A}_k)^{\alpha}.$$

A natural goal is to be able to compute the space $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ *effectively.* By this we mean, to give an algorithm, for which there is an *a priori* bound on the running time, in terms of the input data (e.g., the defining equations of X). The existence of such an effective algorithm was proved for geometrically rational surfaces in [KT08]. Here we prove the following result.

Theorem 1. *Let X be a smooth projective geometrically irreducible surface over a number field k, given by a system of homogeneous polynomial equations. Assume that the geometric Picard group $\mathrm{Pic}(X_{\bar{k}})$ is torsion free and generated by finitely many divisors, each with a given set of defining equations. Then for each positive integer n there exists an effective description of a space $X_n \subseteq X(\mathbb{A}_k)$ which satisfies*

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subseteq X_n \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)[n]},$$

where $\mathrm{Br}(X)[n] \subseteq \mathrm{Br}(X)$ denotes the n-torsion subgroup. In particular, $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ is effectively computable provided that $|\,\mathrm{Br}(X)/\,\mathrm{Br}(k)|$ can be bounded effectively.

For instance, in the case of a diagonal quartic surface over $\mathbb{Q}$ there is an effective bound on $|\,\mathrm{Br}(X)/\,\mathrm{Br}(\mathbb{Q})|$ due to Ieronymou, Skorobogatov, and Zarhin [ISZ].

While it is not known how to compute $\mathrm{Pic}(X_{\bar{k}})$ effectively, in general, there is a method of computation involving reduction modulo primes used by van Luijk [vL07]; further examples can be found in [EJ08] and [HVV].

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2. Picard schemes

Let $X \to S$ be a finite-type morphism of locally Noetherian schemes. We recall that the functor associating to an S-scheme T the group

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X \times_S T)/\,\mathrm{Pic}(T)$$

is known as the *relative Picard functor*. It restricts to a sheaf on the étale site S_{et} when S is a nonsingular curve over an algebraically closed field, by Tsen's theorem. See [Kle05].

We use $\mathrm{Br}(X)$ to denote the cohomological Brauer group of a Noetherian scheme X, i.e., the torsion subgroup of the étale cohomology group $H^2(X, \mathbb{G}_m)$. When X is regular, $H^2(X, \mathbb{G}_m)$ is itself a torsion group. By Gabber's theorem, if X admits an ample invertible sheaf then $\mathrm{Br}(X)$ is also equal to the Azumaya Brauer group, i.e., the equivalence classes of sheaves of Azumaya algebras on X. For background on the Brauer group, the reader is referred to [Gro68], and for a proof of Gabber's theorem, see [dJ05].

Let S be a nonsingular irreducible curve over an algebraically closed field, and let $f : X \to S$ be a smooth projective morphism of relative dimension 1 with connected fibers. Then the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_* \mathbb{G}_m) \Longrightarrow H^{p+q}(X, \mathbb{G}_m)$$

gives, by [Gro68, Cor. III.3.2], an isomorphism

$$\mathrm{Br}(X) \xrightarrow{\sim} H^1(S, \mathrm{Pic}_{X/S}). \tag{2.1}$$

Furthermore, we have an exact sequence

$$0 \to \mathrm{Pic}^0_{X/S} \to \mathrm{Pic}_{X/S} \to \mathbb{Z} \to 0$$

of sheaves (on S_{et}) hence an exact sequence

$$0 \to \mathbb{Z}/d\mathbb{Z} \to H^1(S, \mathrm{Pic}^0_{X/S}) \to H^1(S, \mathrm{Pic}_{X/S}) \to 0, \tag{2.2}$$

where d is the gcd of the relative degrees of all multisections of f. Now assume that the algebraically closed base field has characteristic not dividing n. Then we have the exact sequence of sheaves

$$0 \to \mathrm{Pic}_{X/S}[n] \to \mathrm{Pic}^0_{X/S} \xrightarrow{n\cdot} \mathrm{Pic}^0_{X/S} \to 0$$

(exactness on the right follows by [EGAIV, 21.9.12] and [Kle05, Prop. 9.5.19]) from which the long exact sequence in cohomology gives a surjective homomorphism

$$H^1(S, \mathrm{Pic}_{X/S}[n]) \to H^1(S, \mathrm{Pic}^0_{X/S})[n]. \tag{2.3}$$

Lemma 2. *Let K be a field, and let D be a geometrically irreducible smooth projective curve over K. Let n be a positive integer, not divisible by* $\mathrm{char}(K)$. *Let C be a nonempty open subset of D, with $Y := D \smallsetminus C$ nonempty. The inclusions will be denoted $i\colon Y \to D$ and $j\colon C \to D$.*

(i) *We have $R^1 j_* \mu_n = i_*(\mathbb{Z}/n\mathbb{Z})$.*

(ii) *For a tuple of integers $(a_y)_{y\in Y}$ with reductions $(\bar a_y)$ modulo n, we have $(\bar a_y)$ in the image of the map*

$$H^1(C, \mu_n) \to H^0(D, R^1 j_* \mu_n) = \bigoplus_{y\in Y} \mathbb{Z}/n\mathbb{Z}$$

coming from the Leray spectral sequence if and only if there exists a divisor δ on D with $n\delta \sim \sum a_y[y]$, where $\sim$ denotes linear equivalence of divisors.

Proof. The Leray spectral sequence gives

$$0 \to H^1(D, \mu_n) \to H^1(C, \mu_n) \to H^0(D, R^1 j_* \mu_n) \xrightarrow{d_2^{0,1}} H^2(D, \mu_n) \to H^2(C, \mu_n). \tag{2.4}$$

For (i), by standard spectral sequences we have $R^1 j_* \mu_n = i_* \underline{H}^2_Y(\mu_n)$ (cf. [Mil80, proof of Thm. VI.5.1]). So we are reduced to a local computation, and we may therefore assume that D is affine and Y consists of a single point which is a principal Cartier divisor on D. By the Kummer sequence and injectivity of $\mathrm{Br}(C) \to \mathrm{Br}(D)$ the right-hand map in (2.4) is injective, while the left-hand map has cokernel cyclic of order n. (Such an isomorphism exists generally for regular codimension 1 complements, see [SGA4, (XIX.3.3)].)

For the "if" direction of (ii), we take $r \in K(D)^*$ to be a rational function whose divisor is $-n\delta + \sum a_y[y]$. Then adjoining $r^{1/n}$ to the function field of D yields an element of $H^1(C, \mu_n)$ whose image in $H^0(D, R^1 j_* \mu_n)$ is $(\bar a_y)$ by the isomorphism in (i). For the "only if" direction, an element of $H^1(C, \mu_n)$ gives rise by the Kummer exact sequence to a divisor δ on C and $r \in K(C)^*$ by which $n\delta \sim 0$ on C. Then $n\delta \sim \sum b_y[y]$ on D, for some integers b_y, and the given element of $H^1(C, \mu_n)$ maps by $d_2^{0,1}$ to $(\bar b_y)$. This means that $a_y \equiv b_y \bmod n$ for all $y \in Y$, and we easily obtain δ' on D with $n\delta' \sim \sum a_y[y]$. □

3. Brauer groups

We start with some general results about cocycles in étale cohomology.

Lemma 3. *Let X be a Noetherian scheme, union of open subschemes X_1 and X_2, and let G be an abelian étale sheaf. Suppose given étale covers $Y_i \to X_i$ and Čech cocycles $\beta_i \in Z^2(Y_i \to X_i, G)$ for $i = 1, 2$. With $X_{12} = X_1 \cap X_2$ and $Y_{12} = Y_1 \times_X Y_2$, we suppose further that a cochain $\delta \in C^1(Y_{12} \to X_{12}, G)$ is given, satisfying*

$$\frac{\delta(y_1, y_1', y_2, y_2')\delta(y_1', y_1'', y_2', y_2'')}{\delta(y_1, y_1'', y_2, y_2'')} = \frac{\beta_1(y_1, y_1', y_1'')}{\beta_2(y_2, y_2', y_2'')}$$

for $(y_1, y_2, y_1', y_2', y_1'', y_2'') \in Y_{12} \times_X Y_{12} \times_X Y_{12}$. Then we have $\beta \in Z^2(Y_1 \amalg Y_2 \to X, G)$, given by

$$\begin{aligned}
(y_1, y_1', y_1'') &\mapsto \beta_1(y_1, y_1', y_1''),\\
(y_1, y_1', y_2'') &\mapsto \delta(y_1, y_1', y_2'', y_2'')\beta_2(y_2'', y_2'', y_2''),\\
(y_1, y_2', y_1'') &\mapsto \delta(y_1, y_1'', y_2', y_2')^{-1}\beta_2(y_2', y_2', y_2')^{-1},\\
(y_1, y_2', y_2'') &\mapsto \delta(y_1, y_1, y_2', y_2'')^{-1}\beta_1(y_1, y_1, y_1),\\
(y_2, y_1', y_1'') &\mapsto \delta(y_1', y_1'', y_2, y_2)\beta_2(y_2, y_2, y_2),\\
(y_2, y_1', y_2'') &\mapsto \delta(y_1', y_1', y_2, y_2'')\beta_1(y_1', y_1', y_1')^{-1},\\
(y_2, y_2', y_1'') &\mapsto \delta(y_1'', y_1'', y_2, y_2')^{-1}\beta_1(y_1'', y_1'', y_1''),\\
(y_2, y_2', y_2'') &\mapsto \beta_2(y_2, y_2', y_2''),
\end{aligned}$$

whose class restricts to the class of β_i in $H^2(X_i, G)$ for $i = 1, 2$.

Proof. This is just a portion of the Mayer-Vietoris sequence, written out explicitly in terms of cocycles. ☐

The following two results are based on the existence of Zariski local trivializations of 1-cocycle with values in $\mathbb{G}_m$. Such trivializations exist effectively when the 1-cocycle is effectively presented, say on a scheme of finite type over a number field.

Lemma 4. *Let k be a number field, X a finite-type scheme over k, $Y \to X$ and $Z \to X$ finite-type étale covers, and $Y \to Z$ a morphism over X. Suppose that $\beta \in Z^2(Z \to X, \mathbb{G}_m)$ and $\delta \in C^1(Y \to X, \mathbb{G}_m)$ are given, so that the restriction of β by $Y \to Z$ is equal to the coboundary of δ. Then we may effectively produce a Zariski open covering $Z = \bigcup_{i=1}^N Z_i$ for some N and a 1-cochain for $\coprod_{i=1}^N Z_i \to X$ whose coboundary is equal to the restriction of β by $\coprod_{i=1}^N Z_i \to Z$.*

Proof. Replacing Y by $Y \times_X Z$ and using that the restriction maps on the level of Čech cocycles corresponding to $Y \times_X Z \to Y \to Z$ and $Y \times_X Z \to Z$ differ by an explicit coboundary (cf. [Mil80, Lem. III.2.1]), we are reduced to the case that $Y \to Z$ is also a covering.

Then we have the 1-cocycle for $Y \to Z$

$$(y, y') \mapsto \frac{\beta(z, z, z)}{\delta(y, y')},$$

for $(y, y') \in Y \times_Z Y$ over z. This may be trivialized effectively on a Zariski open neighborhood of any point of Z, so we may effectively obtain a refinement of Z to a

Zariski open covering and functions ε_i satisfying

$$\frac{\varepsilon_i(y')}{\varepsilon_i(y)} = \frac{\beta(z,z,z)}{\delta(y,y')}$$

for all i and $(y,y') \in Y \times_Z Y$ over $z \in Z_i$. It follows that for all i and j, and $(y,y') \in Y \times_X Y$ over $(z,z') \in Z \times_X Z$ with $z \in Z_i$, $z' \in Z_j$, the function

$$\frac{\varepsilon_j(y')}{\varepsilon_i(y)}\delta(y,y')$$

depends only on (z,z'), hence we obtain $\delta_0 \in C^1(\coprod_{i=1}^N Z_i \to X)$ satisfying

$$\frac{\varepsilon_j(y')}{\varepsilon_i(y)}\delta(y,y') = \delta_0(z,z').$$

The conclusion follows immediately from this formula. □

Lemma 5. *Let X be a smooth finite-type scheme over a number field k, let $Z \to X$ be a finite-type étale covering, and let $Y \to Z$ be a finite-type étale morphism with dense image. Let $\beta \in Z^2(Z \to X, \mathbb{G}_m)$ be given, along with $\delta \in \mathcal{O}^*_{Y\times_X Y}$ satisfying*

$$\delta(y,y')\delta(y',y'')/\delta(y,y'') = \beta(z,z',z''),$$

for all $(y,y',y'') \in Y \times_X Y \times_X Y$ over $(z,z',z'') \in Z \times_X Z \times_X Z$. Then there exists, effectively, a Zariski open covering $(Z_i)_{1\le i\le N}$ of Z (for some N) and a 1-cocycle for $\coprod Z_i \to X$ whose coboundary is the restriction of β by $\coprod Z_i \to Z$.

Proof. Let X_0 denote the image of the composite morphism $Y \to X$, and Z_0 the pre-image of X_0 in Z. By Lemma 4 (or rather its proof) there exists a Zariski open covering of Z_0 of the form $(Z_0 \cap Z_i)_{1\le i\le N}$ for some Zariski open covering (Z_i) of Z (the 1-cocycle mentioned in the proof determines a line bundle on Z_0, which can be extended to a line bundle on Z, since Z is smooth) and a 1-cochain for $\coprod Z_i \cap Z_0 \to X_0$ whose coboundary is the restriction of β. Using the fact that divisors on smooth schemes are locally principal (and effectively so, e.g., see [KT08, §7]) and [Mil80, Exa. III.2.22], we see that after further refinement of (Z_i) the 1-cochain extends to a 1-cochain for $\coprod Z_i \to X$. □

Let X be a regular Noetherian scheme of dimension 2. It is known [Gro68, Cor. II.2.2] that for any element $\alpha \in \mathrm{Br}(X)$ of the (cohomological) Brauer group there exists a sheaf of Azumaya algebras on X having class equal to α.

Lemma 6. *Let X be a smooth projective surface over a number field k, $\widehat{X} \subset X$ an open subscheme whose complement has codimension 2, and $\alpha \in \mathrm{Br}(X)$ an element whose restriction over $\widehat{X}$ is represented by a 2-cocycle $\hat\beta$, relative to some finite-type étale cover $\pi\colon \widehat{Y} \to \widehat{X}$. We suppose that X, $\widehat{X}$, $\widehat{Y}$, π, and $\hat\beta$ are given by explicit equations. Then there is an effective procedure to produce a sheaf of Azumaya algebras on X representing the class α.*

Note, by purity for the Brauer group [Gro68, Thm. III.6.1], we have $\mathrm{Br}(\widehat{X}) = \mathrm{Br}(X)$, so α is uniquely determined by the cocycle $\hat\beta$.

Proof. Take $V \subset \widehat{Y}$ nonempty open such that $\psi_0 = \pi|_V$ is a finite étale covering of some open subscheme of $\widehat{X}$. Let $\psi\colon \widehat{W} \to \widehat{X}$ be the normalization of $\widehat{X}$ in $(\psi_0)_*\mathcal{O}_V$. Shrinking $\widehat{X}$ (and maintaining that its complement in X has codimension 2) we may suppose that $\widehat{W}$ is smooth. By the universal property of the normalization ([EGAII, 6.3.9]) there is a (unique) lift $\widehat{Y} \to \widehat{W}$ of π. Consider the element of $Z^2(\widehat{Y} \times_{\widehat{X}} \widehat{W} \to \widehat{W}, \mathbb{G}_m)$ obtained by restricting $\hat{\beta}$. The further restriction to $Z^2(\widehat{Y} \times_{\widehat{X}} \widehat{Y} \to \widehat{Y}, \mathbb{G}_m)$ is (explicitly) a coboundary, we apply Lemma 5 and observe that by the proof, from the fact that $\widehat{W} \to \widehat{X}$ is finite and hence universally closed, the Zariski refinement may be taken to come from a Zariski refinement of $\widehat{Y}$, i.e., we obtain $\hat{\gamma} \in C^1(\coprod \widehat{Y}_i \times_{\widehat{X}} \widehat{W} \to \widehat{W}, \mathbb{G}_m)$ whose coboundary is the restriction of $\hat{\beta}$. Using the flatness of $\widehat{W} \to \widehat{X}$, we may regard $\hat{\gamma}$ as patching data for a sheaf of Azumaya algebras over $\widehat{X}$ as in [Mil80, Prop. IV.2.11], whose class in the Brauer group is that of $\hat{\beta}$. Pushforward via $\widehat{X} \to X$ may be computed by making an arbitrary extension as a coherent sheaf, and forming the double dual. This is then a sheaf of Azumaya algebras on X by [Gro68, Thm. I.5.1(ii)]. □

4. Proof of Theorem 1

The proof of Theorem 1 is carried out in several steps.

Step 1. (Proposition 7) We obtain a nonempty open subscheme X° of X, a finite Galois extension K of k, and a sequence of elements

$$(\alpha_1, \dots, \alpha_N) \subset \mathrm{Br}(X^\circ_K)$$

for some N which generate a subgroup of $\mathrm{Br}(X^\circ_{\bar{k}})$ containing $\mathrm{Br}(X_{\bar{k}})[n]$. We obtain an étale covering $Y^\circ \to X^\circ$, such that each α_i is given by an explicit 2-cocycle for the étale cover $Y^\circ_K \to X^\circ_K$.

Step 2. (Proposition 9) Given $\alpha \in \mathrm{Br}(X^\circ_K)$ defined by an explicit cocycle, we provide an effective procedure to test whether α vanishes in $\mathrm{Br}(X_{\bar{k}})$, and in case of vanishing, to produce a 1-cochain lift of the cocycle, defined over some effective extension of K. We use this procedure in two ways.

- (i) By repeating Step 1 with another open subscheme $\widetilde{X}^\circ$, with $X \smallsetminus (X^\circ \cup \widetilde{X}^\circ)$ of codimension 2 (or empty), to identify the geometrically *unramified* Brauer group elements, i.e., those in the image of $\mathrm{Br}(X_K) \to \mathrm{Br}(X^\circ_K)$ after possibly extending K.
- (ii) To identify those α such that α and ${}^g\alpha$ have the same image in $\mathrm{Br}(X_{\bar{k}})$ for all $g \in \mathrm{Gal}(K/k)$. Again after possibly extending K (remaining finite Galois over k), we may suppose that all such α satisfy $\alpha = {}^g\alpha$ in $\mathrm{Br}(X_K)$ for all $g \in \mathrm{Gal}(K/k)$.

The result is a sequence of elements

$$(\alpha'_1, \dots, \alpha'_M) \subset \mathrm{Br}(X_K)[n]^{\mathrm{Gal}(K/k)},$$

each given by a cocycle over X°_K as well as one over $\widetilde{X}^\circ_K$, generating $\mathrm{Br}(X_{\bar{k}})[n]^{\mathrm{Gal}(\bar{k}/k)}$.

Step 3. (Proposition 10) Combine the data from the Galois invariance of the α'_i and the alternate representation over $\widetilde{X}^\circ_K$ to obtain cocycle representatives of each α'_i defined over the complement of a codimension 2 subset of X, as well as cochains there that encode the Galois invariance.

Step 4. (Proposition 11) For every Galois-invariant n-torsion element of $\mathrm{Br}(X_{\bar{k}})$, with representing cocycle defined over K obtained in Step 3, compute the obstruction to the existence of an element of $\mathrm{Br}(X)$ having the same image class in $\mathrm{Br}(X_{\bar{k}})$. When the obstruction vanishes, produce a cocycle representative of such an element of $\mathrm{Br}(X)$, defined over the complement of a codimension 2 subset of X. Each such element of $\mathrm{Br}(X)$ will be unique up to an element of $\ker(\mathrm{Br}(X) \to \mathrm{Br}(X_K))$, the algebraic part of the Brauer group, which has been treated in [KT08].

Step 5. From the cocycle representatives of elements of $\mathrm{Br}(X)$ obtained in Step 4, produce sheaves of Azumaya algebras defined globally on X (Lemma 6).

Step 6. Compute local invariants. A sheaf of Azumaya algebras may be effectively converted to a collection of representing 2-cocycles, each for a finite étale covering of some U_i with (U_i) a Zariski covering of X ([Gro68, Thm. I.5.1(iii), 5.10]). Then we are reduced to the local analysis described in [KT08, §9].

5. Generators of $\mathrm{Br}(X_{\bar{k}})[n]$ by fibrations

For the first step, we produce generators of the n-torsion in the Brauer group of $\overline{X} := X_{\bar{k}}$. Starting from $X \subset \mathbb{P}^N$, a general projection to $\mathbb{P}^1$ yields, after replacing X by its blow-up at finitely many points, a fibration

$$f\colon X \to \mathbb{P}^1 \tag{5.1}$$

with geometrically connected fibers. By removing the exceptional divisors from the codimension 2 complement in Step 5 and viewing it as a codimension 2 complement of X, the proof of Theorem 1 is reduced to the case that f as in (5.1) exists.

Notice that, given a finite set of divisors on X, (5.1) may be chosen so that each of these divisors maps dominantly to $\mathbb{P}^1$.

That the n-torsion in the Brauer group of a smooth projective surface over $\bar{k}$ may be computed using a fibration is standard. We include a sketch of a proof, for completeness.

Proposition 7. *Let X be a smooth projective geometrically irreducible surface over a number field k, and let $f\colon X \to \mathbb{P}^1$ be a nonconstant morphism with connected geometric fibers, both given by explicit equations. Let n be a given positive integer. Then there exist, effectively:*

(i) *a finite Galois extension K of k,*
(ii) *a nonempty open subset $S \subset \mathbb{P}^1$,*
(iii) *an étale covering $S' \to S$,*
(iv) *2-cocycles of rational functions for the covering $X_K \times_{\mathbb{P}^1_K} S'_K \to X_K \times_{\mathbb{P}^1_K} S_K$,*

such that $\mathrm{Br}(X_{\bar{k}} \times_{\mathbb{P}^1_{\bar{k}}} S_{\bar{k}})[n]$ is spanned by the classes of the 2-cocycles, base-extended to $\bar{k}$.

Proof. We let $S \subset \mathbb{P}^1$ denote the maximal subset over which f is smooth, and $X^\circ = f^{-1}(S)$. By the exact sequences of Section 2, it suffices to carry out following tasks (perhaps for a larger value of n):

(1) Compute $H^1(S_{\bar{k}}, \mathrm{Pic}_{X^\circ_{\bar{k}}/S_{\bar{k}}}[n])$ by means of cocycles.
(2) Find divisors on $X_{\bar{k}}$ whose classes in $\mathrm{Pic}(X^\circ_{\bar{k}}/S_{\bar{k}})$ represent the elements appearing in these cocycles.
(3) Find explicit 2-cocycle representatives of elements of $\mathrm{Br}(X^\circ_{\bar{k}})$ which correspond to these elements by the isomorphism (2.1).

The field K is an explicit suitable extension, over which the steps are carried out. Step (1) is clear, since there is an explicit finite étale covering $C \to S$ trivializing $\mathrm{Pic}_{X^\circ/S}[n]$. Then there is a finite étale covering $S' \to C$, with $S'_{\bar{k}} \to C_{\bar{k}}$ a product of cyclic étale degree n covers, such that $S'_{\bar{k}} \to S_{\bar{k}}$ trivializes $H^1(S_{\bar{k}}, \mathrm{Pic}_{X^\circ_{\bar{k}}/S_{\bar{k}}}[n])$. (The proof of Lemma 2 provides an effective procedure to compute S', using effective Jacobian arithmetic.) Step (2) can be carried out effectively as described in [KT08, §4], using an effective version of Tsen's theorem (for the function field, this is standard, see e.g. [Pr], then apply Lemma 5). On the level of cocycles, the Leray spectral sequence (2.1) gives rise to a 3-cocycle, and Step (3) can be carried out as soon as this is represented as a coboundary, which we have again possibly after making a Zariski refinement of S' (cf. [Mil80, Exa. III.2.22(d)]). An explicit description of the procedure to produce the 3-cocycle using the Leray spectral sequence may be found in [KT08, Prop. 6.1]. $\square$

6. Relations among generators

In this section we show how to compare elements of the Brauer group of a Zariski open subset of a smooth projective surface $\overline{X}$ over $\bar{k}$, under the assumption that the geometric Picard group $\mathrm{Pic}(\overline{X})$ is finitely generated, and $\overline{X}$ as well as a finite set of divisors generating $\mathrm{Pic}(\overline{X})$ are explicitly given. The method goes back to Brauer [Bra28], with refinements in [Bra32].

Lemma 8. *Let X° be a smooth quasi-projective geometrically irreducible surface over a number field k, $Z^\circ \to X^\circ$ a finite étale morphism, and $\beta \in Z^2(Z^\circ \to X^\circ, \mathbb{G}_m)$ a Čech cocycle representative of an element $\alpha \in \mathrm{Br}(X^\circ)$. We suppose X°, Z° and β are given by explicit equations, respectively functions. Let n be a given positive integer, and $\gamma \in C^1(Z^\circ \to X^\circ, \mathbb{G}_m)$ a Čech cochain whose coboundary is equal to $n \cdot \beta$. We suppose that k contains the n-roots of unity, and that an identification $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$ is fixed. Then there exists, effectively, a finite group G, a finite étale morphism $Y^\circ \to Z^\circ$, a G-torsor structure on $Y^\circ \to X^\circ$, a central extension of finite groups*

$$1 \to \mathbb{Z}/n\mathbb{Z} \to H \to G \to 1, \tag{6.1}$$

and a 1-cochain $\delta \in C^1(Y^\circ \to X^\circ, \mathbb{G}_m)$ such that image in $\mathrm{Br}(X^\circ) = H^2(X^\circ, \mathbb{G}_m)$ of the induced element of $H^2(X^\circ, \mathbb{Z}/n\mathbb{Z}) \simeq H^2(X^\circ, \mu_n)$ is equal to α, and the coboundary δ is the difference between the latter and the given β refined by $Y^\circ \to Z^\circ$.

Proof. There exists a finite étale cover Y° of Z°, such that the restriction of γ to $Z^\circ \times_{X^\circ} Z^\circ$ is an nth power. It follows that the restriction of β differs by a coboundary from an element of the image of $Z^2(Y^\circ \to X^\circ, \mu_n)$. These can be produced explicitly. Upon further refinement of Y°, we may suppose that Y° is irreducible, $Y^\circ \to X^\circ$ is a Galois G-covering for some finite group G, and then the cocycle condition is precisely the condition to be a 2-cocycle for the group cohomology of G with values in $\mathbb{Z}/n\mathbb{Z}$ (with trivial G-action on $\mathbb{Z}/n\mathbb{Z}$). This gives us (6.1). $\square$

Proposition 9. *Let X be a smooth projective geometrically irreducible surface over a number field k with finitely generated geometric Picard group $\mathrm{Pic}(X_{\bar{k}})$. Let X° be a nonempty open subscheme, $\pi\colon Y^\circ \to X^\circ$ an étale cover, and $\beta \in Z^2(Y^\circ \to X^\circ, \mathbb{G}_m)$ a Čech cocycle representative of an element $\alpha \in \mathrm{Br}(X^\circ)$. Let n be a given positive integer, and $\gamma \in C^1(Y^\circ \to X^\circ, \mathbb{G}_m)$ a Čech cochain whose coboundary is equal to $n \cdot \beta$. We suppose X, a finite set of divisors generating $\mathrm{Pic}(X_{\bar{k}})$, X°, Y°, π, β,* *and γ are given by explicit equations, respectively functions. Then there exists an effective procedure to determine whether $\alpha_{\bar{k}} = 0$ in $\mathrm{Br}(X^\circ_{\bar{k}})$, and in case $\alpha_{\bar{k}} = 0$, to produce a finite extension K of k, a Zariski open covering (Y_i°) of Y°, and a 1-cochain $\delta \in C^1(\coprod (Y_i^\circ)_K \to X^\circ_K, \mathbb{G}_m)$, whose coboundary is equal to the base-extension to K of the refinement of β by $\coprod Y_i^\circ \to Y^\circ$.*

Proof. It suffices to prove the result after an effective shrinking of X° and extension of the base field, by Lemma 5 (we note that the Zariski open subsets that are produced in the proof may be taken to be Galois invariant) and, by Lemma 4, after a refinement of the given cover. So we may suppose that π is finite, $\mathrm{Pic}(X^\circ_{\bar{k}}) = 0$, the field k contains the nth roots of unity (with a fixed identification $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$), and the cocycle β takes its values in μ_n (Lemma 8) and is the universal one for a G-torsor structure on $Y^\circ \to X^\circ$ and an extension (6.1). Without loss of generality, Y° is geometrically irreducible, and the class of the extension in $H^2(G, \mathbb{Z}/n\mathbb{Z})$ (group cohomology for $\mathbb{Z}/n\mathbb{Z}$ with trivial G-action) is not annihilated by any positive integer smaller than n. It follows from $\mathrm{Pic}(X^\circ_{\bar{k}}) = 0$ that $\alpha_{\bar{k}} = 0$ in $\mathrm{Br}(X^\circ_{\bar{k}})$ if and only if the class of β is 0 in $H^2(X^\circ_{\bar{k}}, \mu_n)$.

The Leray spectral sequence gives rise to an exact sequence

$$0 \to H^1(G, \mu_n) \to H^1(X^\circ_{\bar{k}}, \mu_n) \to H^1(Y^\circ_{\bar{k}}, \mu_n)^G \to H^2(G, \mu_n) \to H^2(X^\circ_{\bar{k}}, \mu_n).$$

It follows that the class of β is 0 in $\mathrm{Br}(X^\circ_{\bar{k}})$ if and only if there exists an irreducible finite étale covering $\overline{V}^\circ$ of $\overline{Y}^\circ := Y^\circ_{\bar{k}}$, cyclic of degree n, admitting a structure of H-torsor over $\overline{X}^\circ := X^\circ_{\bar{k}}$ compatible with the G-torsor structure on $\overline{Y}^\circ$. This can be tested, provided that we can explicitly generate all degree n cyclic étale coverings of $\overline{Y}^\circ$. If we have such a covering, we take K so that the covering and H-torsor structure are defined over K, then the restriction of β to the covering is explicitly a coboundary.

Choose an explicit fibration $\tau\colon \overline{Y}^\circ \to \mathbb{P}^1$. Since we may shrink $\overline{Y}^\circ$, we may replace $\overline{Y}^\circ$ by the preimage of Zariski open $T \subsetneq \mathbb{P}^1$, chosen so that the geometric fibers are complements of exactly some number ℓ of distinct points in a smooth irreducible curve of some genus g, these ℓ points being the fibers of a finite étale cover of T.

By the Leray spectral sequence, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(T)^*/(\mathcal{O}(T)^*)^n & \longrightarrow & H^1(\overline{Y}^\circ, \mu_n) & \longrightarrow & H^0(T, R^1\tau_*\mu_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{k}(T)^*/(\bar{k}(T)^*)^n & \longrightarrow & H^1(\overline{Y}^\circ_{\bar{k}(T)}, \mu_n) & \longrightarrow & H^0(\bar{k}(T), R^1(\tau_{\bar{k}(T)})_*\mu_n) & \longrightarrow & 0
\end{array}$$

where we have used $H^2(T, \mu_n) = H^2(\bar{k}(T), \mu_n) = 0$. Arguing as in [Mil80, proof of Lemma III.3.15] we see that $R^1\tau_*\mu_n$ is a locally constant torsion sheaf with finite fibers. It follows that the right-hand vertical map is an isomorphism. By the snake lemma, the leftmost two vertical maps are injective and have isomorphic cokernels.

Let $\overline{Y}_{\bar{k}(T)}$ be a smooth projective curve containing $\overline{Y}^\circ_{\bar{k}(T)}$ as an open subscheme. We can find generators of $H^1(\overline{Y}_{\bar{k}(T)}, \mu_n)$, modulo $H^1(\bar{k}(T), \mu_n) = \bar{k}(T)^*/(\bar{k}(T)^*)^n$ by computing the $\bar{k}(T)$-rational n-torsion points of the Jacobian and (using effective Tsen's theorem) lifting these to divisor representatives. Lemma 2 supplies additional generators of $H^1(\overline{Y}^\circ_{\bar{k}(T)}, \mu_n)$: for elements of $\bigoplus \mathbb{Z}/n\mathbb{Z}$ (sum over points of $\overline{Y}_{\bar{k}(T)} \smallsetminus \overline{Y}^\circ_{\bar{k}(T)}$) of weighted (by degree) sum 0, we test whether a fiber of a multiplication by n map of Jacobians has a $\bar{k}(T)$-rational point (and again use effective Tsen's theorem to produce divisor representatives). Each generator of $H^1(\overline{Y}^\circ_{\bar k(T)},\mu_n)$, modulo $H^1(\bar k(T),\mu_n)$, may be effectively lifted to $H^1(\overline{Y}^\circ,\mu_n)$ by a diagram chase, using the isomorphism of the cokernels of left two vertical morphisms in the diagram. □

7. Galois invariants in $\mathrm{Br}(X_{\bar k})[n]$

In this section, we focus on the problem of deciding whether a Galois invariant element in $\mathrm{Br}(X_{\bar k})[n]$ lies in the image in $\mathrm{Br}(X_{\bar k})$ of an element of $\mathrm{Br}(X_K)^{\mathrm{Gal}(K/k)}$. Concretely, this amounts to adjusting elements of $\mathrm{Br}(X_K)$ by elements of $\mathrm{Br}(K)$, when possible, so that they become $\mathrm{Gal}(K/k)$-invariant. This will be done effectively.

Proposition 10. *Let X be a smooth projective geometrically irreducible surface over a number field k. Let K be a finite Galois extension of k. Let X° and $\widetilde{X}^\circ$ be open subschemes whose union is the complement of a subset that has codimension 2 (or is empty), $Y^\circ\to X^\circ$ and $\widetilde{Y}^\circ\to\widetilde{X}^\circ$ étale coverings, $\beta\in Z^2(Y^\circ_K\to X^\circ_K,\mathbb{G}_m)$ and $\tilde\beta\in Z^2(\widetilde{Y}^\circ_K\to\widetilde{X}^\circ_K,\mathbb{G}_m)$ cocycles, and $\delta_g\in C^1(Y^\circ_K\to X^\circ_K,\mathbb{G}_m)$ having coboundary $\beta-{}^g\beta$ for every $g\in\mathrm{Gal}(K/k)$. Assume that β and $\tilde\beta$ give rise to the same class in $\mathrm{Br}((X^\circ\cap\widetilde{X}^\circ)_{\bar k})$. Then we may effectively produce an open subscheme $\widehat{X}\subset X$, containing X°, whose complement has codimension 2 (or is empty), an étale cover $\widehat{Y}\to\widehat{X}$, a finite extension L of K, Galois over k, a cocycle $\hat\beta\in Z^2(\widehat{Y}\to\widehat{X},\mathbb{G}_m)$ giving rise to the same class as β in $\mathrm{Br}(X^\circ_{\bar k})$, and cochain $\hat\delta_g\in C^1(\widehat{Y}\to\widehat{X},\mathbb{G}_m)$ having coboundary $\hat\beta-{}^g\hat\beta$, for all $g\in\mathrm{Gal}(L/k)$.*

Proof. Let ξ_1, …, ξ_N denote the codimension 1 generic points of $Y^\circ\times_X\widetilde{Y}^\circ$ whose image in $\widetilde{X}^\circ$ is one of the generic points in X of the codimension 1 irreducible components of $X\smallsetminus X^\circ$. We may apply Proposition 9 to the covering $Y^\circ\times_X\widetilde{Y}^\circ\to X^\circ\cap\widetilde{X}^\circ$ and insist that one of the open sets that is produced, in addition to being Galois invariant, contains all the points above ξ_1, …, ξ_N. (The field that emerges, enlarged if necessary, is taken as the field L mentioned in the statement.) Call the open set U. We replace $\widetilde{X}^\circ$ with the complement of the closure of the image of the complement of U in $Y^\circ_L\times_{X_L}\widetilde{Y}^\circ_L$, and restrict $\widetilde{Y}^\circ$ accordingly. Now we have $U=Y^\circ_L\times_{X_L}\widetilde{Y}^\circ_L$, so we may apply Lemma 3 to produce $\hat\beta\in Z^2(Y^\circ_L\amalg\widetilde{Y}^\circ_L\to X^\circ_L\cup\widetilde{X}^\circ_L,\mathbb{G}_m)$. We apply Lemma 5 to produce $\hat\delta_g$, which involves replacing Y° and $\widetilde{Y}^\circ$ by Zariski covers. □

Proposition 11. *Let X be a smooth projective geometrically irreducible variety over a number field k, given by explicit equations, let K be a finite Galois extension of k, and assume that $\mathrm{Pic}(X_{\bar k})$ is torsion-free, generated by finitely many explicitly given divisors, defined over K. Let $\alpha\in\mathrm{Br}(X_{\bar k})$ be given by means of a cocycle representative $\beta\in Z^2(\widehat{Y}_K\to\widehat{X}_K,\mathbb{G}_m)$, where $\widehat{Y}_K\to\widehat{X}_K$ is an étale cover, with $\widehat{X}$ an open subscheme of X whose complement has codimension at least 2 (or is empty). Assume given $\delta^{(g)}\in C^1(\widehat{Y}_K\to\widehat{X}_K,\mathbb{G}_m)$, having coboundary $\beta-{}^g\beta$, for every $g\in\mathrm{Gal}(K/k)$. Then there exists an effective computable obstruction in $H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K))$ to the existence of $\alpha_0\in\mathrm{Br}(X)$ such that α_0 and α have the same image in $\mathrm{Br}(X_{\bar k})$. When the obstruction class vanishes, we can effectively construct a cocycle representative of $\alpha_0|_{\widehat{X}}$ in $Z^2(\widehat{Y}_K\to\widehat{X},\mathbb{G}_m)$ for some $\alpha_0\in\mathrm{Br}(X)$ satisfying $(\alpha_0)_K=\alpha$.*

Proof. By the Leray spectral sequence, we have an exact sequence

$$\mathrm{Br}(X)\to\ker\Big(\mathrm{Br}(X_K)^{\mathrm{Gal(K/k)}}\to H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K))\Big)\to H^3(\mathrm{Gal}(K/k),K^*).$$

Also note that the nontriviality in $H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K))$ implies the nontriviality in $H^2(\mathrm{Gal}(L/k),\mathrm{Pic}(X_L))$ for any finite extension L of K, Galois over k, by the Hochschild-Serre spectral sequence

$$0 = H^1(\mathrm{Gal}(L/K),\mathrm{Pic}(X_L))^{\mathrm{Gal}(K/k)} \to H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K)) \to H^2(\mathrm{Gal}(L/k),\mathrm{Pic}(X_L)).$$

The hypothesis concerning $\delta^{(g)}$ may be written

$$\frac{\delta^{(g)}(y,y')\delta^{(g)}(y',y'')}{\delta^{(g)}(y,y'')} = \frac{\beta(y,y',y'')}{{}^g\beta(y,y',y'')} \tag{7.1}$$

and implies that

$$\frac{\delta^{(g)}\,{}^g\delta^{(g')}}{\delta^{(gg')}} \in Z^1(\widehat{Y}_K \to \widehat{X}_K,\mathbb{G}_m) \tag{7.2}$$

for every g, $g' \in \mathrm{Gal}(K/k)$. Arguments as in [KT08, §6] show that (7.2) gives the obstruction class in $H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K))$. Of course, each cocycle (7.2) may be explicitly represented by a divisor, whose class in $\mathrm{Pic}(X_K)$ is then readily computed.

Assuming that the obstruction class in $H^2(\mathrm{Gal}(K/k),\mathrm{Pic}(X_K))$ vanishes, each $\delta^{(g)}$ may be modified by a cocycle so that each element (7.2) is a coboundary, i.e., so that there exist $\varepsilon^{(g,g')} \in \mathcal{O}^*_{\widehat{Y}_K}$ satisfying

$$\frac{\varepsilon^{(g,g')}(y')}{\varepsilon^{(g,g')}(y)} = \frac{\delta^{(g)}(y,y')^g\delta^{(g')}(y,y')}{\delta^{(gg')}(y,y')}. \tag{7.3}$$

In this case the divisor representative of (7.2) is a principal divisor, hence the divisor associated to an effectively computable rational function.

Combining (7.1) and (7.3), we have

$$\frac{\varepsilon^{(g,g')}(y)\varepsilon^{(gg',g'')}(y)}{\varepsilon^{(g,g'g'')}(y)^g\varepsilon^{(g',g'')}(y)} = \frac{\varepsilon^{(g,g')}(y')\varepsilon^{(gg',g'')}(y')}{\varepsilon^{(g,g'g'')}(y')^g\varepsilon^{(g',g'')}(y')},$$

hence

$$\varepsilon^{(g,g')}\varepsilon^{(gg',g'')}/(\varepsilon^{(g,g'g'')}\,{}^g\varepsilon^{(g',g'')}) \in \mathcal{O}^*_{\widehat{X}_K}, \tag{7.4}$$

i.e., is a constant function, for every g, g', $g'' \in \mathrm{Gal}(K/k)$. The rest of the argument is similar to [KT08, Prop. 6.3]. The constants (7.4) determine a class in $H^3(\mathrm{Gal}(K/k),K^*)$, which may be effectively tested for vanishing. In case of nonvanishing a further finite extension may be effectively computed, which kills this class. In case of vanishing, a 2-cochain lift is effectively produced. Modifying $\varepsilon^{(g,g')}$, then, yields

$$\varepsilon^{(g,g')}(y)\varepsilon^{(gg',g'')}(y) = \varepsilon^{(g,g'g'')}(y)^g\varepsilon^{(g',g'')}(y) \tag{7.5}$$

Now if we set

$$\beta^{(g,g')}(y,y',y'') = \frac{\beta(y,',y'')\varepsilon^{(g,g')}(y'')}{\delta^{(g)}(y',y'')}$$

then we have

$$\beta^{(g,g')}(y,y',y'')\beta^{(gg',g'')}(y,y'',y''') = \beta^{(g,g'g'')}(y,y',y''')^g\beta^{(g',g'')}(y',y'',y'''),$$

i.e., we have an element of $Z^2(\widehat{Y}_K \to \widehat{X},\mathbb{G}_m)$ determining an element $\alpha_0 \in H^2(X,\mathbb{G}_m)$. The restriction to $\widehat{X}_K$ is defined by the cocycle $\beta^{(e,e)}$, which is equal to β, up to coboundary. □

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