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WAVENUMBER EXPLICIT CONVERGENCE ANALYSIS FOR GALERKIN DISCRETIZATIONS OF THE HELMHOLTZ EQUATION*

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(*Dedicated to Ivo Babuška on the occasion of his* 85*th birthday*)

Abstract. We develop a stability and convergence theory for a class of highly indefinite elliptic boundary value problems (bvps) by considering the Helmholtz equation at high wavenumber k as our model problem. The key element in this theory is a novel k-explicit regularity theory for Helmholtz bvps that is based on decomposing the solution into two parts: the first part has the Sobolev regularity properties expected of second order elliptic PDEs but features k-independent regularity constants; the second part is an analytic function for which k-explicit bounds for all derivatives are given. This decomposition is worked out in detail for several types of bvps, namely, the Helmholtz equation in bounded smooth domains or convex polygonal domains with Robin boundary conditions and in exterior domains with Dirichlet boundary conditions. We present an error analysis for the classical *hp*-version of the finite element method (*hp*-FEM) where the dependence on the mesh width h, the approximation order p, and the wavenumber k is given explicitly. In particular, under the assumption that the solution operator for Helmholtz problems is polynomially bounded in k, it is shown that quasi optimality is obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $\mathrm{O}(\log k)$.

Key words. Helmholtz equation at high wavenumber, stability, convergence, *hp*-finite elements

AMS subject classifications. 35J05, 65N12, 65N30

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1. Introduction.

We analyze the Galerkin method applied to a class of highly indefinite boundary value problems (bvps), which arise, for example, when electromagnetic or acoustic scattering problems are modeled in the frequency domain. As our model problem we consider the Helmholtz equation at high wavenumbers k.

For low order h-version finite element methods, it is well known that unique solvability of the discrete problem is guaranteed only under very restrictive stability conditions. More precisely, rigorous results of the type going back to [6] require the dimension N of, e.g., a $\mathcal{P}_1$ finite element space to satisfy $N \gtrsim k^{2d}$, where $d \in \{1, 2, 3\}$ denotes the spatial dimension. In the present paper, we demonstrate that it is possible to ensure stability and quasi optimality under the substantially relaxed condition $N \gtrsim k^d$. A different way of stating this result is that quasi optimality of a piecewise polynomial-based finite element method (FEM) can be achieved in a setting where (on average) the number of degrees of freedom per wavelength is independent of k. At first glance, this lack of "pollution" seems to contradict the results of [8], where it is proved that, for any (even generalized) FEM, $N \gtrsim k^d$ is not a sufficient condition to guarantee quasi optimality in general. However, in [8] only polynomial approximations of *fixed* order were considered, and a key result of the present paper is that the polynomial order must be chosen in a wavenumber-dependent way in order to obtain optimal stability conditions.

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This quasi optimality result hinges on two observations. First, as is typical of problems satisfying a Gårding inequality, the proof of quasi optimality of Galerkin methods for Helmholtz problems can be reduced to the question of how well certain adjoint problems may be approximated from the ansatz space. This has been exploited, for example, in [9, 43, 49, 59, 60]. Second, approximability questions are closely related to regularity issues. The heart of the present paper, therefore, is new k-explicit regularity results for solutions of Helmholtz bvps (and their adjoints). These regularity assertions take the form of a decomposition of the solution into a highly oscillatory but analytic part $u_{\mathcal{A}}$ and an "elliptic" part u_{H^2} with k-independent regularity properties. We develop this new regularity theory for three cases, namely, the Helmholtz equation

(I) in bounded domains in $\mathbb{R}^d$ ($d \in \{2,3\}$) with analytic boundary and Robin boundary conditions,
(II) in convex two-dimensional polygons with Robin boundary conditions,
(III) in exterior domains in $\mathbb{R}^d$ ($d \in \{2,3\}$) with analytic boundaries and Dirichlet boundary conditions.

The regularity theory of the present paper is based on [49], where the simpler case of a full space problem is considered. Each of the above three cases (I)–(III) represents a characteristic class of problems whose features distinguish it from the other ones and [49]. In contrast to the full space problem of [49], the presence of *boundaries* in case (I) mandates the introduction of suitable extension operators for the definition of a stable splitting of the Helmholtz solution into an elliptic and an analytic, highly oscillatory part *in lieu* of the simple Fourier-based frequency filter used in [49]. For polygonal domains as in model problem (II) the highly oscillatory part has characteristic conical singularities requiring the use of *weighted* Sobolev spaces for an adequate description of high regularity. Finally, case (III) is a model problem for highly oscillatory scattering problems in *unbounded* exterior domains. Here, we consider Dirichlet boundary conditions to illustrate that our theory can also handle essential boundary conditions.

Our decomposition results (Theorems 4.10, 4.20) and hence our stability analysis (Theorem 5.8) rely on norm bounds for the *continuous* solution operator for the corresponding Helmholtz problem. It is known that the stability properties of Helmholtz bvps depend strongly on the type of boundary condition and the geometry. Regarding the influence of the geometry, the case of star-shaped geometries is probably best understood; for example, for cases (I) and (II), the norm of the solution operator is bounded uniformly in k, as was shown in [43, Prop. 8.1.4] for $d = 2$ and subsequently for $d = 3$ in [18]. Uniform-in-k bounds were established in [31] for star-shaped domains and certain boundary conditions of mixed type. Also for star-shaped domains, [16] established bounds for case (III) that are uniform in k. Helmholtz problems in more complex geometries can exhibit trapping or near-trapping, which typically results in large norm bounds. For example, [11] exhibits an exterior Dirichlet problem and a sequence of frequencies $(k_m)_{m=0}^{\infty}$ tending to infinity, for which the norm of the solution operator grows exponentially in k_m.

In principle, the regularity theory developed in the present paper merely requires the continuous solution operator to be bounded (with possibly arbitrary k-dependence). Nevertheless, we formulate all our decomposition results under the additional assumption that the solution operator is polynomially bounded in k (see Assumptions 4.8 and 4.18). Our main motivation for this restriction is that for this class of Helmholtz problems, a satisfactory approximation and stability theory for high order methods can be developed. Indeed, for the three examples listed above, the "elliptic" part u_{H^2} is an H^2-function whose H^2-norm can be bounded uniformly in k, and therefore the approximation theory for this contribution is well understood. The analytic part $u_{\mathcal{A}}$ depends critically on the wavenumber; however, it is its smoothness that can be exploited in high order numerical schemes. We illustrate this point for the *hp*-version of the finite element method (*hp*-FEM) by showing for the cases of domains with analytic boundary (i.e., cases (I) and (III)) that the condition

$$\frac{kh}{p} \text{ small} \quad \text{together with} \quad p \geq C \log k \tag{1.1}$$

suffices to ensure quasi optimality of the Galerkin method. Here, h stands for the mesh size and p for the order of the method. For case (II) of polygonal domains, the condition (1.1) is modified in the sense that appropriate geometric mesh refinement is required in small neighborhoods of the vertices. While the regularity theory for the three cases (I)–(III) is based on the assumption of polynomial bounds for the solution operator, we mention that [20] shows this assumption for cases (I) and (II), so that these two cases are fully covered by the present theory.

Discretizations of Helmholtz bvps have been studied considerably in the past decades with the ambitious goal of controlling the notorious pollution and dispersion phenomena and, more generally, improving the performance of numerical schemes for large k. One line of studies is based on variational formulations other than the classical Galerkin methods. These include stabilized methods such as the Galerkin least squares method [29, 30], the work [7], and discontinuous Galerkin methods (see [23] and references therein). A second line of methods bases the numerical scheme on nonstandard ansatz functions. In a Galerkin setting, this idea has been pursued in the partition of unity methods/generalized FEM by several authors, e.g., in [5, 37, 38, 43, 47, 55, 56, 64]. A variety of other methods have been proposed that use systems of functions related to the Helmholtz equation elementwise and enforce the jump across element boundaries in a weak sense. This can be done by least squares techniques (see [10, 40, 52, 57, 63] and references therein), by Lagrange multiplier techniques as in the discontinuous enrichment method [21, 22, 65], or by discontinuous Galerkin (DG)-type methods. Of this last class, an early representative is the ultra weak variational formulation (see [14, 15, 34, 42]), although its connection with DG methods was not fully realized until [13] and [24]. The convergence theory for DG-type couplings of plane wave–based methods has significantly matured in recent times [13, 24, 32, 33, 50]. We mention also [51] in connection with DG-type couplings.

Within the broad field of numerical analysis of the Helmholtz equation, the present paper is most closely connected with earlier work on pollution and dispersion effects [2, 3, 4, 19, 26, 27, 28, 35, 36, 54]. A key result of the detailed analyses of [2, 3, 4, 19, 35, 36] on *structured*, translation invariant meshes is that high order methods have less dispersion error and are less prone to the above-mentioned pollution effect than lower order methods. One outcome of the present paper is that the same conclusion holds true for the Galerkin method on *unstructured* meshes. We point out, however, that our analysis differs significantly from the earlier work on pollution, since powerful tools such as discrete Green's functions, discrete Fourier analysis, and Bloch waves are not available for unstructured meshes.

The paper is structured as follows. In section 2, we formulate the model Helmholtz problems (I)–(III) and the corresponding abstract Galerkin discretizations. In section 3, we briefly recapitulate the general convergence theory in which stability and convergence follows from approximability of certain adjoint problems (Theorem 3.2).

In section 4, which is at the heart of the paper, we present the decomposition of the solution of Helmholtz bvps into an analytic part and a part with finite Sobolev regularity, as discussed above (Theorems 4.10, 4.20). The ability to decompose the solution of Helmholtz bpvs in this way appears to be a general feature of this problem class. Indeed, in our earlier work [49] we studied the simpler case of a full space problem and derived an analogous decomposition. Furthermore, similar decompositions have been developed in [41, 46] for the solutions of some boundary integral formulations of scattering problems. Finally, as an application of the abstract convergence theory of section 3 and the regularity theory of section 4, we study in section 5 the hp-FEM applied to the model problems (I)–(III). We show that the scale resolution condition (1.1) ensures stability and quasi optimality (Theorem 5.8). By appropriately selecting p and the mesh, it is possible to obtain discrete stability and quasi optimality with a fixed number of degrees of freedom per wavelength (Remark 5.9). We mention that the scale resolution condition (1.1) is ultimately an outcome of the regularity theory of section 4; therefore it may not come as a surprise that it ensured stability and quasi optimality of the hp-FEM in the earlier work [49] and of the hp-boundary element method in [41].

1.1. Function spaces and notation. We employ standard notation concerning Sobolev spaces [1]. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and $k > 0$ we introduce the following k-dependent norms:

$$\|u\|^2_{\mathcal{H},\Omega} := k^2 \|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)}, \tag{1.2a}$$

$$\|u\|^2_{1/2,\mathcal{H},\partial\Omega} := |u|^2_{H^{1/2}(\partial\Omega)} + k\|u\|^2_{L^2(\partial\Omega)}, \tag{1.2b}$$

$$\|u\|^2_{3/2,\mathcal{H},\partial\Omega} := k^{-2}|u|^2_{H^{3/2}(\partial\Omega)} + \|u\|^2_{1/2,\mathcal{H},\partial\Omega}; \tag{1.2c}$$

here, the norm (1.2c) will be employed only for smooth $\partial\Omega$ so that it is indeed well defined. If the domain Ω is clear from the context, we write $\|\cdot\|_{\mathcal{H}}$, short for $\|\cdot\|_{\mathcal{H},\Omega}$.

A large part of the analysis will be concerned with domains with analytic boundary or convex polygons. For ease of future reference we therefore introduce the following.

Assumption 1.1. $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded Lipschitz domain. **Either** it has an analytic boundary **or** it is a convex polygon in $\mathbb{R}^2$ with vertices A_j, $j = 1, \ldots, J$.

For domains Ω that have a smooth boundary or are polygonal (not necessarily convex), we introduce the following shorthand:

$$H^{1/2}_{pw}(\partial\Omega) := \begin{cases} \left\{g \in L^2(\partial\Omega) : g \text{ is edgewise in } H^{1/2}\right\} & \text{if } \partial\Omega \text{ is a polygon,} \\ H^{1/2}(\partial\Omega) & \text{if } \partial\Omega \text{ is smooth.} \end{cases} \tag{1.3}$$

Furthermore, for domains satisfying Assumption 1.1, we require spaces of analytic functions, specifically, the countably normed spaces introduced in [44]. These function spaces are defined with the aid of weight functions $\Phi_{p,\vec{\beta},k}$ that we now define. For $\beta \in [0, 1)$, $p \in \mathbb{N}_0$, and $k > 0$ we set

$$\Phi_{p,\beta,k}(x) = \min\left\{1, \frac{|x|}{\min\left\{1, \frac{|p|+1}{k+1}\right\}}\right\}^{p+\beta}.$$

For a polygon Ω with vertices A_j, $j = 1, \ldots, J$, and given $\overrightarrow{\beta} \in [0,1)^J$, we define

$$\Phi_{p,\overrightarrow{\beta},k}(x) = \prod_{j=1}^{J} \Phi_{p,\beta_j,k}(x - A_j). \tag{1.4}$$

If $\Omega \subset \mathbb{R}^d$ is not a polygon, then we set, for all p and any $\overrightarrow{\beta}$,

$$\Phi_{p,\overrightarrow{\beta},k}(x) :\equiv 1. \tag{1.5}$$

We use the symbol ∇^n to denote derivatives of order n; more precisely, for a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we write $|\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2$.

Definition 1.2. *Given C_u, γ, $k > 0$, and $\overrightarrow{\beta}$ we set*

$$\begin{aligned} \mathcal{B}_{\overrightarrow{\beta},k}(C_u,\gamma) := \{ & u \in H^1(\Omega) \,|\, \|u\|_{\mathcal{H},\Omega} \le C_u k \quad \wedge \\ & \|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} u\|_{L^2(\Omega)} \le C_u (\gamma \max\{p,k\})^{p+2} \quad \forall p \in \mathbb{N}_0 \}, \end{aligned} \tag{1.6}$$

where the weight functions $\Phi_{p,\overrightarrow{\beta},k}$ are given by (1.4) *for polygonal Ω and by* (1.5) *otherwise.*

Since we will prove approximation theorems for functions in the unit ball $\mathcal{H}_{\text{ell}}$ in $H^2(\Omega)$ and for functions in the subset $\mathcal{H}_{\text{osc}}(\gamma, k)$ of $\mathcal{B}_{\overrightarrow{\beta},k}(C_u,\gamma)$ obtained by the scaling condition $C_u = 1$, we introduce these spaces now:

$$\mathcal{H}_{\text{ell}} := \left\{ v \in H^2(\Omega) : \|v\|_{H^2(\Omega)} \le 1 \right\}, \qquad \mathcal{H}_{\text{osc}}(\gamma,k) := \mathcal{B}_{\overrightarrow{\beta},k}(1,\gamma). \tag{1.7}$$

We close with some general comments on constants: $C > 0$ denotes a generic constant that may have different values in different occurrences. However, C is always independent of critical parameters such as k, p, h, q (which will be introduced in what follows) and functions appearing in the estimates. We write $A \lesssim B$ to denote $A \le CB$, where C is a generic constant. We write $A \sim B$ if $A \lesssim B$ together with $B \lesssim A$.

2. Model Helmholtz problems and their discretization. We start by introducing the three model problems that will be analyzed in the paper. Throughout this paper, a standing assumption on the wavenumber is

$$k \ge k_0 > 0. \tag{2.1}$$

2.1. Robin boundary conditions for a bounded domain.

2.1.1. The continuous problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain. The model problem with Robin boundary conditions is

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} - \mathrm{i}\, k u = g \quad \text{on } \partial\Omega. \tag{2.2}$$

The weak form of (2.2) is

(2.3)

$$\text{Find } u \in H^1(\Omega): \quad \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} - \mathrm{i} \int_{\partial\Omega} k u \overline{v} = \int_\Omega f \overline{v} + \int_{\partial\Omega} g \overline{v} \quad \forall v \in H^1(\Omega).$$

Proposition 2.1 (see [43, Prop. 8.1.3]). *Let Ω be a bounded Lipschitz domain. Then,* (2.2) *is uniquely solvable for all $f \in (H^1(\Omega))'$, $g \in H^{-1/2}(\Gamma)$ and the solution depends continuously on the data.*

2.1.2. Abstract Galerkin discretization. Given a finite-dimensional subspace $S \subset H^1(\Omega)$, the conforming Galerkin discretization of (2.3) reads

(2.4)
$$\text{Find } u_S \in S: \qquad \int_\Omega \nabla u_S \cdot \nabla \overline{v} - k^2 u_S \overline{v} - \mathrm{i} \int_{\partial\Omega} k u_S \overline{v} = \int_\Omega f \overline{v} + \int_{\partial\Omega} g \overline{v} \qquad \forall v \in S.$$

2.2. Dirichlet boundary conditions for an exterior domain. For the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, we denote its exterior Ω^c by $\Omega^c := \mathbb{R}^d \backslash \overline{\Omega}$. For $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_R$ for some ball B_R of radius R, we consider the exterior Dirichlet problem with Sommerfeld radiation condition given by

$$-\Delta u - k^2 u = f \quad \text{in } \Omega^c, \qquad u|_{\partial\Omega} = g, \tag{2.5a}$$
$$\left| \frac{\partial u}{\partial r} - \mathrm{i}\, k u \right| = o\left(\|x\|^{\frac{1-d}{2}} \right) \qquad \text{as } \|x\| \to \infty. \tag{2.5b}$$

Here, $\frac{\partial}{\partial r}$ denotes the derivative in the radial direction $x/\|x\|$. For numerical purposes, it is more convenient to reformulate this problem, which is posed on the unbounded domain Ω^c, as a problem posed posed on the bounded domain $\Omega_R^c := \Omega^c \cap B_R$. This is achieved with the aid of the Dirichlet-to-Neumann operator T_R. In order to introduce T_R, we let $B_R^c := \mathbb{R}^d \backslash \overline{B_R}$ and $\Gamma_R := \partial B_R$. It can be shown (see, e.g., [53]) that for given $h \in H^{1/2}(\Gamma_R)$ the problem

$$\text{Find } w \in H^1_{\mathrm{loc}}(B_R^c) \text{ s.t. } \begin{cases} \left(-\Delta - k^2\right) w = 0 & \text{in } B_R^c, \\ w = h & \text{on } \Gamma_R, \\ \left| \dfrac{\partial w}{\partial r} - \mathrm{i}\, k w \right| = o\left(\|x\|^{\frac{1-d}{2}} \right) & \|x\| \to \infty \end{cases}$$

has a unique weak solution. The *Dirichlet-to-Neumann map* $T_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is then defined as $h \mapsto (\frac{\partial}{\partial r} w)|_{\Gamma_R}$. With the aid of the operator T_R, we can rewrite (2.5) as the following problem on the bounded domain Ω_R^c:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega^c \cap B_R =: \Omega_R^c, \qquad u = g \quad \text{on } \Gamma, \tag{2.6a}$$
$$\partial_n u = T_R u \quad \text{on } \Gamma_R. \tag{2.6b}$$

The variational formulation of (2.6) is based on the spaces

$$V_R := \left\{ u|_{\Omega^c \cap B_R} : u \in H^1(\Omega^c) \right\} \quad \text{and} \quad V_{R,0} := \left\{ u|_{\Omega^c \cap B_R} : u \in H^1_0(\Omega^c) \right\} \tag{2.7}$$

and given by: Find $u \in V_R$ such that

$$u|_{\partial\Omega} = g \quad \text{and} \quad \int_{\Omega_R^c} \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} \right) - \int_{\Gamma_R} (T_R u)\, \overline{v} = \int_{\Omega_R^c} f \overline{v} \quad \forall v \in V_{R,0}. \tag{2.8}$$

The exterior Dirichlet problem is uniquely solvable, as discussed, for example, in [16], and as given next.

PROPOSITION 2.2. *Let Ω be a bounded Lipschitz domain. Then,* (2.8) *admits a unique solution $u \in V_R$ for all $g \in H^{1/2}(\Gamma)$ and $f \in V_{R,0}'$. The solution depends continuously on the data.*

2.2.1. Abstract Galerkin discretization. Given a finite-dimensional subspace $S \subset V_R$ and an approximation $g_S \in S$ to g, the conforming Galerkin of (2.8) reads: Find $u_S \in S$ such that

(2.9)

$$u_S|_{\partial\Omega} = g_S \quad \text{and} \quad \int_{\Omega_R^c} \nabla u_S \cdot \nabla \overline{v} - k^2 u_S \overline{v} - \int_{\Gamma_R} (T_R u_S)\, \overline{v} = \int_{\Omega_R^c} f v \quad \forall v \in S \cap V_{R,0}.$$

3. Abstract stability and convergence analysis. In this section, we identify in an abstract setting conditions on the approximation properties of ansatz spaces that ensure quasi optimality of a Galerkin discretization.

3.1. Variational formulations and adjoint problems. Many Helmholtz bvps can be cast in the following abstract form:

$$\text{(3.1)} \qquad \text{Find } u \in V \quad \text{s.t.} \qquad a(u,v) - b(u,v) = l(v) \qquad \forall v \in V.$$

Here, the space V is a suitable subspace of a Sobolev space $H^1(\widetilde{\Omega})$ that reflects the possible presence of essential Dirichlet boundary conditions. The sesquilinear form $a : H^1(\widetilde{\Omega}) \times H^1(\widetilde{\Omega}) \to \mathbb{C}$ has the form

$$\text{(3.2)} \qquad a\,(u,v) := \int_{\widetilde{\Omega}} \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v},$$

and the continuous sesquilinear form b encodes boundary conditions. Finally, l is a bounded antilinear functional on V. For example, the model problems of sections 2.1 and 2.2 (with additionally $g = 0$) have this form: In the setting of section 2.1, we may choose $\widetilde{\Omega} = \Omega$, $V = H^1(\Omega)$, and $b(u,v) = \mathrm{i}\, k \int_{\partial\Omega} u\overline{v}$; in the setting of section 2.2 with $g = 0$ we have $\widetilde{\Omega} = \Omega_R^c$, $V = V_{R,0}$, and $b(u,v) = \int_{\partial B_R} T_R u \overline{v}$.

The stability analysis will require adjoint problems associated with (3.1). That is, given an antilinear functional l on V, we consider

$$\text{(3.3)} \qquad \text{find } u \in V \text{ s.t.} \quad a(v,u) - b(v,u) = \overline{l(v)} \qquad \forall v \in V.$$

As a matter of convenience, we note that the adjoint problems for the Helmholtz problems of sections 2.1 and 2.2 are themselves Helmholtz problems, as shown next.

LEMMA 3.1.

(i) *Denote by* $S_k : (f,g) \mapsto u$ *the solution operator for the problem of section* 2.1. *Then the adjoint solution operator* $S_k^\star$ *for the problem: Find* $z \in H^1(\Omega)$ *such that*

$$\text{(3.4)} \qquad \int_\Omega \left(\nabla v \cdot \nabla \overline{z} - k^2 v \overline{z}\right) - \mathrm{i}\, k \int_{\partial\Omega} v\overline{z} = \int_\Omega v\overline{f} + \int_{\partial\Omega} v\overline{g} \qquad \forall v \in H^1(\Omega)$$

is given by $S_k^\star(f,g) = \overline{S_k(\overline{f},\overline{g})}$.

(ii) *Denote by* $S_k^c : (f,g) \mapsto u$ *the solution operator for the problem of section* 2.2. *For the special case* $g = 0$, *denote by* $S_k^{c,\star} : f \mapsto z$ *the solution operator for the adjoint problem*

(3.5)

$$\textit{Find } z \in V_{R,0} \textit{ s.t.} \int_{\Omega_R^c} \left(\nabla v \cdot \nabla \overline{z} - k^2 v \overline{z}\right) - \int_{\Gamma_R} T_R v \overline{z} = \int_{\Omega_R^c} v\overline{f} \qquad \forall v \in V_{R,0}.$$

Then, $S_k^{c,\star}(f) = \overline{S_k^c(\overline{f},0)}$.

Proof. We will only show (ii), since (i) is shown with similar ideas. By [49, Lem. 3.10] we have for the adjoint $T_R^\star$ (with respect to the $(\cdot,\cdot)_{L^2(\Gamma_R)}$ inner product) the representation $T_R^\star z = \overline{T_R \overline{z}}$. Hence, (3.5) is equivalent to finding $z \in V_{R,0}$ such that

$$\int_{\Omega_R^c} \nabla v \cdot \nabla \overline{z} - k^2 v \overline{z} - \int_{\Gamma_R} v T_R \overline{z} = \int_{\Omega_R^c} \overline{f} v \qquad \forall v \in V_{R,0}. \tag{3.6}$$

By replacing v with $\overline{v}$, we recognize that $\overline{z} = S_k^c(\overline{f}, 0)$, which then concludes the proof. ▯

3.2. Abstract stability and convergence analysis. It is well known that in the context of variational problems that satisfy a Gårding inequality, Galerkin methods are asymptotically quasi-optimal; i.e., quasi optimality is ensured if the ansatz space is sufficiently rich (see, e.g., [12, sect. 5.7], [58, Thm. 4.2.7], [60]). The following theorem restricts this general setting to one that is applicable to Helmholtz problems and formulates an abstract condition on the approximation properties of the ansatz space that guarantees quasi optimality. In particular, the model problems of sections 2.1 and 2.2 (with $g = 0$) are covered by the following theorem.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq \{2,3\}$, be a bounded Lipschitz domain. Let $V \subset H^1(\Omega)$ be a closed subspace, and let the sesquilinear form a be given by* (3.2). *Let the following additional hypotheses be true:*

(i) *$b : V \times V \to \mathbb{C}$ is a continuous sesquilinear form with*

$$|b(u,v)| \leq C_b \|u\|_{\mathcal{H},\Omega} \|v\|_{\mathcal{H},\Omega} \qquad \forall u, v \in V. \tag{3.7}$$

(ii) *There exist $\theta \geq 0$ and $\lambda > 0$ such that the following Gårding inequality holds:*

$$\operatorname{Re}\left(a(u,u) - b(u,u)\right) + \theta k^2 \|u\|^2_{L^2(\Omega)} \geq \lambda \|u\|^2_{\mathcal{H},\Omega} \qquad \forall u \in V. \tag{3.8}$$

(iii) *The adjoint problem*

$$\text{Find } z \in V \text{ s.t.} \qquad a(v,z) - b(v,z) = (v,f)_{L^2(\Omega)} \qquad \forall v \in V \tag{3.9}$$

is uniquely solvable for every $f \in L^2(\Omega)$. Let $\widetilde{S}_k^\star : f \mapsto z$ denote this solution operator with (possibly k-dependent norm)

$$C_{adj} := \sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{\|\widetilde{S}_k^\star f\|_{\mathcal{H},\Omega}}{\|f\|_{L^2(\Omega)}}. \tag{3.10}$$

Let $S \subset V$ be a closed subspace and define the adjoint approximability

$$\eta(S) := \sup_{f \in L^2(\Omega)\setminus\{0\}} \inf_{v \in S} \frac{\|\widetilde{S}_k^\star f - v\|_{\mathcal{H},\Omega}}{\|f\|_{L^2(\Omega)}}. \tag{3.11}$$

Then, the condition

$$\theta k \eta(S) \leq \frac{\lambda}{2(1 + C_b)} \tag{3.12}$$

implies the following statements:

1. *The discrete inf-sup condition is satisfied:*

$$\inf_{u\in S\setminus\{0\}} \sup_{v\in S\setminus\{0\}} \frac{|a(u,v)-b(u,v)|}{\|u\|_{\mathcal{H},\Omega}\|v\|_{\mathcal{H},\Omega}} \geq \frac{\lambda}{2+\lambda/(1+C_b)+2\theta k C_{adj}} > 0. \tag{3.13}$$

2. *The Galerkin method based on S is quasi-optimal; i.e., for every $u \in \mathcal{H}$ there exists a unique $u_S \in S$ with $a(u-u_S,v) - b(u-u_S,v) = 0$ for all $v \in S$, and there holds*

$$\|u-u_S\|_{\mathcal{H},\Omega} \leq \frac{2}{\lambda}(1+C_b) \inf_{v\in S} \|u-v\|_{\mathcal{H},\Omega}, \tag{3.14}$$

$$\|u-u_S\|_{L^2(\Omega)} \leq (1+C_b)\eta(S)\|u-u_S\|_{\mathcal{H},\Omega}. \tag{3.15}$$

Proof. The proof follows very closely the proofs of [49, Thms. 4.2, 4.3]. Details can be found in [48, Appendix B]. ◻

Theorem 3.2 is applicable to the model problems of sections 2.1 and 2.2 with $\theta = 2$ and $\lambda = 1$ as we now show.

COROLLARY 3.3. *Let $k \geq k_0 > 0$.*

(i) *For the model problem of section 2.1 the assumptions of Theorem 3.2 are satisfied for the choices $V = H^1(\Omega)$, $\theta = 2$, $\lambda = 1$, and a constant $C_b > 0$ that depends solely on Ω.*

(ii) *For the model problem of section 2.2 with $g = 0$ the assumptions of Theorem 3.2 are satisfied for the choices $V = V_{R,0}$ (see (2.7)), $\theta = 2$, $\lambda = 1$, and a constant $C_b > 0$ that depends solely on k_0 and R.*

In both cases, $C_{adj} < \infty$ (but possibly k-dependent) for any $k \geq k_0$.

Proof. To see (3.3) we note that $b(u,v) = \mathrm{i}k(u,v)_{L^2(\partial\Omega)}$. By [49, Cor. 3.2] the constant C_b is bounded uniformly in k. By Lemma 3.1 and Proposition 2.1 the solvability of the adjoint problem is ensured. From $\operatorname{Re} b(u,u) = 0$, it follows that the Gårding inequality (3.8) is satisfied with $\theta = 2$ and $\lambda = 1$.

To see (3.3) we observe $b(u,v) = \int_{\partial B_R} T_R u\overline{v}$. Next, [49, Lem. 3.3] gives a bound for C_b that is uniform in k; additionally, [49, Lem. 3.3] provides $\operatorname{Re} b(u,u) \leq -CR^{-1}\|u\|^2_{L^2(\partial B_R)} \leq 0$, so that again $\theta = 2$ and $\lambda = 1$ are valid choices. The unique solvability of the adjoint problem follows again by Lemma 3.1 and Proposition 2.2. ◻

The usefulness of Theorem 3.2 rests on the ability to quantify the adjoint approximability $\eta(S)$ in terms of the wavenumber k and properties of the approximation space S. Since $\eta(S)$ depends on the solution operator $S_k^\star$ of some adjoint Helmholtz problems, we need a regularity for these operators in which the influence of k is made explicit. This is the purpose of the following section 4. There, we construct for the model problems of section 2 for every $f \in L^2(\Omega)$ a splitting

$$\widetilde{S}_k^\star f = C_{k,\mathcal{A}}(f) u_{\mathcal{A},f} + C_{H^2}(f) u_{H^2,f} \quad \text{with } u_{H^2,f} \in \mathcal{H}_{\text{ell}},\ u_{\mathcal{A},f} \in \mathcal{H}_{\text{osc}}(\gamma,k) \tag{3.16}$$

for some fixed $\gamma = O(1)$ which depends on Ω but not on k and f (recall from (1.7) the sets $\mathcal{H}_{\text{ell}}$, $\mathcal{H}_{\text{osc}}$). We will also show that the quantities $C_{k,\mathcal{A}}(f)$ and $C_{H^2}(f)$ are bounded uniformly for all $f \in L^2(\Omega)$:

$$C_{H^2} := \sup_{f\in L^2(\Omega)} |C_{H^2}(f)| < \infty, \qquad C_{k,\mathcal{A}} := \sup_{f\in L^2(\Omega)} |C_{k,\mathcal{A}}(f)| < \infty. \tag{3.17}$$

Accepting this decomposition result for the moment, we can formulate the following result.

LEMMA 3.4. *Let $\gamma = O(1)$ be such that the operator $\widetilde{S}_k^\star$ admits a splitting of the form* (3.16) *with C_{H^2}, $C_{k,\mathcal{A}}$ given by* (3.17). *The adjoint approximability $\eta(S)$ defined in* (3.11) *is then bounded by*

$$\eta(S) \leq C_{k,\mathcal{A}} \eta_{\mathcal{A}}(S) + C_{H^2} \eta_{H^2}(S), \tag{3.18}$$

where

$$\eta_{\mathcal{A}}(S) := \sup_{v \in \mathcal{H}_{\mathrm{osc}}(\gamma,k)\setminus\{0\}} \inf_{w \in S} \|v - w\|_{\mathcal{H}} \quad \textit{and} \quad \eta_{H^2}(S) := \sup_{\substack{v \in H^2(\Omega) \\ \|v\|_{H^2(\Omega)} = 1}} \inf_{w \in S} \|v - w\|_{\mathcal{H}}.$$

Proof. The proof follows by the triangle inequality; see also [49, Lem. 5.10]. □

The important conclusion of Lemma 3.4 is that the stability and convergence estimates for Helmholtz problems follow from two types of approximation properties: $\eta_{\mathcal{A}}(S)$ measures the approximation property of the Galerkin space S for analytic highly oscillating functions, and $\eta_{H^2}(S)$ measures the standard approximation property of S for H^2-functions. We mention at this point that our analysis in section 4 will show that the constant C_{H^2} in (3.18) can be bounded uniformly in k and that $C_{k,\mathcal{A}}$ in (3.18) will have—due to our assumptions (cf. Assumptions 4.8, 4.18 ahead)—a polynomial growth in k. We emphasize that estimates for $\eta_{\mathcal{A}}(S)$, $\eta_{H^2}(S)$ involve neither any stability nor any regularity issues for Helmholtz problems. Finally, Lemma 3.1 informs us that the adjoint problems of those presented in sections 2.1 and 2.2 are structurally very similar to the "original" ones; in section 4 we will therefore focus on the regularity theory for the model problems of sections 2.1 and 2.2 and discuss the regularity of the adjoint problems only briefly.

4. Stable decompositions of the Helmholtz solutions.

4.1. Preliminaries.

In this section, we will develop the theoretical tools that will be used for the regularity estimates of the Helmholtz problems.

4.1.1. Frequency splitting.

The key ingredients of our refined regularity results are a frequency splitting of the right-hand side and some estimates of the solution operators applied to the high- and low frequency parts of the right-hand side. We start with introducing the frequency splitting. For functions on $\mathbb{R}^d$, the splitting is defined via the Fourier transform, and for functions on closed surfaces of finite domains, it is defined via the composition of a lifting operator of the boundary data with the frequency splitting for functions in $\mathbb{R}^d$. We recall the definition of the Fourier transform for functions with compact support,

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle} u(x)\, dx \qquad \forall \xi \in \mathbb{R}^d,$$

and the inversion formula

$$u(x) = \mathcal{F}^{-1}(\hat{u})(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi \qquad \forall x \in \mathbb{R}^d.$$

- For functions $f \in L^2(\mathbb{R}^d)$ the *high frequency filter* $H_{\mathbb{R}^d}$ and the *low frequency filter* $L_{\mathbb{R}^d}$ are defined by

$$\mathcal{F}(L_{\mathbb{R}^d} f) = \chi_{\eta k} \mathcal{F}(f), \qquad \mathcal{F}(H_{\mathbb{R}^d} f) = (1 - \chi_{\eta k}) \mathcal{F}(f), \tag{4.1a}$$

  where $\chi_{\eta k}$ is the characteristic function of the ball $B_{\eta k}(0)$.

- Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $E_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^d)$ be the extension operator of Stein [62, Chap. VI]. Then for $f \in L^2(\Omega)$ we set

$$L_\Omega f := \left(L_{\mathbb{R}^d}(E_\Omega f)\right)|_\Omega \quad \text{and} \quad H_\Omega f := \left(H_{\mathbb{R}^d}(E_\Omega f)\right)|_\Omega . \tag{4.1b}$$

- Let $\partial\Omega$ be smooth or (in two dimensions) polygonal. We remind the reader of the space $H^{1/2}_{pw}(\partial\Omega)$ introduced in (1.3) and define operators $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$ as follows. For smooth boundaries, there exists a lifting operator G^N with the mapping property $G^N : H^s(\partial\Omega) \to H^{3/2+s}(\Omega)$ for every $s > 0$ and $\partial_n G^N g = g$. For polygonal domains, we have the existence of a simplified lifting operator $G^N : H^{1/2}_{pw}(\partial\Omega) \to H^2(\Omega)$ with $\partial_n G^N g = g$ (see, e.g., Lemma A.1 for details). We then define $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$ by

$$H^N_{\partial\Omega}(g) := \partial_n H_\Omega(G^N(g)), \qquad L^N_{\partial\Omega}(g) := \partial_n L_\Omega(G^N(g)). \tag{4.1c}$$

  In particular, for both smooth domains and polygons, we have $H^N_{\partial\Omega} : H^{1/2}_{pw}(\partial\Omega) \to H^{1/2}_{pw}(\partial\Omega)$ and $L^N_{\partial\Omega} : H^{1/2}_{pw}(\partial\Omega) \to H^{1/2}_{pw}(\partial\Omega)$.

Remark 4.1. One has significant freedom in the choice of the lifting operator G^N. Here, we selected G^N independent of k. For the Dirichlet problem in section 4.3 we will select the corresponding lifting operator G^D in a k-dependent manner. This could likewise be done here and would alter the k-dependence for the "analytic" part in the decomposition result, Theorem 4.10.

LEMMA 4.2. *Let $\eta > 1$ be the parameter appearing in the definition of $H_{\mathbb{R}^d}$ in* (4.1a). *Then, the frequency splitting via* (4.1a) *satisfies for all $0 \le s' \le s$ the estimates*

$$\|H_{\mathbb{R}^d} f\|_{H^{s'}(\mathbb{R}^d)} \le C_{s',s} (\eta k)^{s'-s} \|f\|_{H^s(\mathbb{R}^d)} \qquad \forall f \in H^s(\mathbb{R}^d), \tag{4.2}$$

$$\|H_\Omega f\|_{H^{s'}(\Omega)} \le C_{s',s} (\eta k)^{s'-s} \|f\|_{H^s(\Omega)} \qquad \forall f \in H^s(\Omega). \tag{4.3}$$

If $\partial\Omega$ is smooth, then the operator $H^N_{\partial\Omega}$ satisfies for $0 \le s' \le s$

$$\|H^N_{\partial\Omega} g\|_{H^{s'}(\partial\Omega)} \le C_{s',s} (\eta k)^{s'-s} \|g\|_{H^s(\partial\Omega)}. \tag{4.4}$$

For smooth or polygonal $\partial\Omega$, we have for $s' \in \{0, 1/2\}$ and $s = 1/2$

$$\|H^N_{\partial\Omega} g\|_{H^{s'}_{pw}(\partial\Omega)} \le C_{s',s} (\eta k)^{s'-s} \|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \tag{4.5}$$

In particular, in (4.2)–(4.5) *one can select, for any $s' < s$ and any $q \in (0,1)$, a parameter $\eta > 1$ such that $C_{s',s}\eta^{-(s-s')} \le q < 1$.*

Proof. Standard properties of the Fourier transformation give for $s \ge s'$ and $f \in H^s(\mathbb{R}^d)$

$$\begin{aligned}
\|H_{\mathbb{R}^d} f\|^2_{H^{s'}(\mathbb{R}^d)} &\le C_{s'} \int_{\mathbb{R}^d \setminus B_{\eta k}(0)} \left(1 + \|\xi\|^{2s'}\right) |\mathcal{F}(f)|^2 \\
&\le C_{s'} \sup_{r \ge \eta k} \frac{1 + r^{2s'}}{1 + r^{2s}} \int_{\mathbb{R}^d \setminus B_{\eta k}(0)} \left(1 + \|\xi\|^{2s}\right) |\mathcal{F}(f)|^2 \\
&\le C_{s',s} (\eta k)^{2(s'-s)} \|f\|^2_{H^s(\mathbb{R}^d)},
\end{aligned}$$

where in the last estimate we used our standing assumption (2.1). The corresponding estimate for H_Ω follows from the properties of $H_{\mathbb{R}^d}$ and the continuity properties of the Stein extension operator E_Ω. We mention in passing that this argument also works for Lipschitz domains Ω.

The estimate (4.4) for the case of smooth $\partial\Omega$ and $0 < s' \le s$ follows from the continuity properties of the trace operator. The limiting case $s' = 0$ is shown by a multiplicative trace inequality by observing that for $\zeta > 1/2$ we have $\|u\|_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)}^{1-1/(2\zeta)} \|u\|_{H^\zeta(\Omega)}^{1/(2\zeta)}$ (see, e.g., [45, Thm. A.2] for a short proof). Using this with $\zeta := s + 1/2$ and recalling the definition of $H^N_{\partial\Omega}$ as in (4.1c), we get

$$\begin{aligned}\|H^N_{\partial\Omega} g\|_{L^2(\partial\Omega)} &\lesssim \|\nabla H_\Omega G^N g\|_{L^2(\Omega)}^{1-1/(2s+1)} \|\nabla H_\Omega G^N g\|_{H^{s+1/2}(\Omega)}^{1/(2s+1)} \\ &\lesssim (\eta k)^{-(s+1/2)(1-1/(2s+1))} \|G^N g\|_{H^{s+3/2}(\Omega)} \lesssim (\eta k)^{-s} \|G^N g\|_{H^{s+3/2}(\Omega)} \\ &\lesssim (\eta k)^{-s} \|g\|_{H^s(\Omega)}.\end{aligned}$$

Finally, we consider the case of polygonal domains $\Omega \subset \mathbb{R}^2$. The result follows by the same arguments as above if one observes that the mapping $v \mapsto \partial_n v$ maps $H^2(\Omega)$ into $H^{1/2}_{pw}(\partial\Omega)$. ∎

The low frequency part represents an analytic function, as can be seen from the following lemma.

LEMMA 4.3. *The low frequency parts of the splittings* (4.1a), (4.1b) *satisfy*

$$\|\nabla^p L_{\mathbb{R}^d} f\|_{L^2(\mathbb{R}^d)} \le (\eta k)^p \|f\|_{L^2(\mathbb{R}^d)} \quad \forall p \in \mathbb{N}_0, \quad \forall f \in L^2(\mathbb{R}^d), \tag{4.6}$$

$$\|\nabla^p L_\Omega f\|_{L^2(\Omega)} \le C (\eta k)^p \|f\|_{L^2(\Omega)} \quad \forall p \in \mathbb{N}_0, \quad \forall f \in L^2(\Omega). \tag{4.7}$$

The constant C in (4.7) *is independent of p, η, and k. If $f \in H^s(\Omega)$ for some $s \ge 0$, then the following stronger estimates are valid:*

$$\|\nabla^p L_\Omega f\|_{L^2(\Omega)} \le C (\eta k)^{p-s} \|f\|_{H^s(\Omega)} \qquad \forall f \in H^s(\Omega), \ \forall p \in \mathbb{N}_0, \ p \ge s. \tag{4.8}$$

Again, the constant $C > 0$ is independent of p, η, and k.

For $s > 0$ the operator $L^N_{\partial\Omega}$ is obtained as the normal trace on $\partial\Omega$ of an entire function, viz., $L^N_{\partial\Omega} g = n\cdot\nabla L_\Omega(G^N g)|_{\partial\Omega} = n\cdot\nabla L_{\mathbb{R}^d} E_\Omega G^N g|_{\partial\Omega}$, where $L_\Omega G^N g$ satisfies the following:

- *If $\partial\Omega$ is smooth and $g \in H^s(\partial\Omega)$ for some $s > 0$, then*

$$\begin{aligned}\|L_\Omega G^N g\|_{H^{3/2+s}(\Omega)} &\lesssim \|g\|_{H^s(\partial\Omega)}, \\ \|\nabla^p L_\Omega G^N g\|_{L^2(\Omega)} &\lesssim (\eta k)^{p-3/2-s} \|g\|_{H^s(\partial\Omega)} \qquad \forall p \in \mathbb{N}_0, \ p \ge s + 3/2.\end{aligned}$$

- *If Ω is a polygon, then*

$$\|L_\Omega G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$

$$\|\nabla^{p+2} L_\Omega G^N g\|_{L^2(\Omega)} \lesssim (\eta k)^p \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \qquad \forall p \in \mathbb{N}_0. \tag{4.9}$$

In particular, for analytic boundaries $\partial\Omega$ we have that $L^N_{\partial\Omega} g$ is an analytic function, and for polygonal Ω the function $L^N_{\partial\Omega} g$ is piecewise analytic on $\partial\Omega$.

Proof. We recall the multinomial formula $\sum_{|\alpha|=n} \frac{n!}{\alpha!} \prod_{i=1}^d \xi_i^{2\alpha_i} = (\sum_{i=1}^d \xi_i^2)^n$. Then, by Parseval's relation we have, for all $p \in \mathbb{N}_0$,

$$\|\nabla^p L_\Omega f\|_{L^2(\Omega)} \le \|\nabla^p L_{\mathbb{R}^d} E_\Omega f\|_{L^2(\mathbb{R}^d)} = \sqrt{\int_{B_{\eta k}(0)} \|\xi\|^{2p} |\mathcal{F}(E_\Omega f)|^2}$$

$$\le \sqrt{\int_{B_{\eta k}(0)} \|\xi\|^{2(p-s)} |\|\xi\|^s \mathcal{F}(E_\Omega f)|^2} \le C (\eta k)^{p-s} \|f\|_{H^s(\Omega)}. \tag{4.10}$$

Hence, (4.8) is shown; (4.6), (4.7) are seen by similar arguments. The estimates for $L_\Omega G^N g$ follow from (4.7), (4.8), and stability properties of the lifting G^N. ∎

Note that the statements of Lemma 4.2 and 4.3 imply that the splittings $f = L_\Omega f + H_\Omega f$ and $g = L_{\partial\Omega}^N g + H_{\partial\Omega}^N$ are *stable* in appropriate scales of Sobolev norms.

Remark 4.4. As remarked in the statement of Lemma 4.3, $L_\Omega f$ and $L_\Omega G^N$ are restrictions to Ω of the entire functions $L_{\mathbb{R}^d} E_\Omega f$ and $L_{\mathbb{R}^d} E_\Omega G^N$. Inspection of the proof of Lemma 4.3 then reveals that, in all bounds for these functions, the domain of integration Ω may in fact be enlarged to consist of all of $\mathbb{R}^d$.

4.1.2. The Newton potential N_k. With the aid of the Green's function

$$G_k(z) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k\|z\|), & d=2, \\ \frac{\mathrm{e}^{\mathrm{i}\,k\|z\|}}{4\pi\|z\|}, & d=3, \end{cases}$$

we define the Newton potential operator N_k by

$$(N_k f)(x) := \int_{\mathbb{R}^d} G_k(x-y) f(y)\, dy \qquad \forall x \in \mathbb{R}^d. \tag{4.11}$$

For functions $f \in L^2(\mathbb{R}^d)$ with *compact* support, the function $N_k(f)$ is the unique (weak) solution of the inhomogeneous Helmholtz equation in $\mathbb{R}^d$ with (outgoing) Sommerfeld radiation conditions [39]:

$$\left(-\Delta - k^2\right) u = f \quad \text{in } \mathbb{R}^d, \qquad \left|\frac{\partial u}{\partial r} - \mathrm{i}\,ku\right| = o\left(\|x\|^{\frac{1-d}{2}}\right) \quad \text{as } \|x\| \to \infty; \tag{4.12}$$

here, $\frac{\partial}{\partial r}$ denotes the derivative in the radial direction $x/\|x\|$.

The following lemma is a direct consequence of [49, Lem. 3.4].

LEMMA 4.5 (properties of N_k). *For $f \in L^2(\mathbb{R}^d)$ with* $\operatorname{supp} f \subset B_R$*, the function $u := N_k(f)$ satisfies $-\Delta u - k^2 u = f$ on B_R. Additionally, for every $q \in (0,1)$ one can select $\eta > 1$ (appearing in the definition of the operator $H_{\mathbb{R}^d}$ in* (4.1a)*) such that*

$$\|N_k(H_{\mathbb{R}^d} f)\|_{\mathcal{H},B_R} \le k^{-1} q \|f\|_{L^2(\mathbb{R}^d)}, \tag{4.13a}$$

$$\|N_k(H_{\mathbb{R}^d} f)\|_{H^2(B_R)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{4.13b}$$

4.1.3. The operator S_k^Δ. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, let S_k^Δ be the solution operator for the modified Helmholtz equation with Robin boundary conditions; i.e., $u = S_k^\Delta(g)$ solves

$$-\Delta u + k^2 u = 0 \quad \text{in } \Omega, \qquad \partial_n u - \mathrm{i} k u = g \quad \text{on } \partial\Omega. \tag{4.14}$$

The operator S_k^Δ has the following regularity properties.

LEMMA 4.6 (properties of S_k^Δ). *Let Ω be a bounded Lipschitz domain. Then for $g \in L^2(\partial\Omega)$ the function $u = S_k^\Delta g$ satisfies*

$$\|u\|_{\mathcal{H},\Omega} \lesssim \|g\|_{H^{-1/2}(\partial\Omega)}, \tag{4.15}$$

$$\|u\|_{\mathcal{H},\Omega} \lesssim k^{-1/2} \|g\|_{L^2(\partial\Omega)}, \tag{4.16}$$

$$\|u\|_{L^2(\partial\Omega)} \lesssim k^{-1} \|g\|_{L^2(\partial\Omega)}. \tag{4.17}$$

If $\partial\Omega$ is sufficiently smooth or if Ω is a convex polygon (in two dimensions), then the following shift theorem is true: If $g \in H_{pw}^{1/2}(\partial\Omega)$, then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \lesssim \|g\|_{H_{pw}^{1/2}(\partial\Omega)} + k^{1/2} \|g\|_{L^2(\partial\Omega)}. \tag{4.18}$$

Proof. The function u satisfies

$$\int_\Omega \nabla u \cdot \nabla \overline{v} + k^2 \int_\Omega u\overline{v} - \mathrm{i}k \int_{\partial\Omega} u\overline{v} = \int_{\partial\Omega} g\overline{v} \qquad \forall v \in H^1(\Omega).$$

Taking $v = u$ and considering the real and imaginary parts separately yields immediately the bounds (4.15), (4.17), (4.16).

Since u satisfies

$$-\Delta u + k^2 u = 0, \qquad \partial_n u = g + \mathrm{i}ku,$$

the standard shift theorem (which is applicable for smooth $\partial\Omega$ and convex polygons with piecewise $H^{1/2}$-Neumann data [25, Cor. 4.4.3.8]) gives

$$\|u\|_{H^2(\Omega)} \lesssim k^2 \|u\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)} + k\|u\|_{H^{1/2}(\partial\Omega)}.$$

Using (4.16), we get (4.18). ∎

LEMMA 4.7 (properties of $S_k^\Delta \circ H^N_{\partial\Omega}$). *Let Ω have a smooth boundary, or let Ω be a convex polygon. Let $q \in (0,1)$. Then there exists $\eta > 1$ defining the high frequency filter $H^N_{\partial\Omega}$ such that for every $g \in H^{1/2}_{pw}(\partial\Omega)$ there holds*

$$\|S_k^\Delta(H^N_{\partial\Omega} g)\|_{\mathcal{H},\Omega} \le qk^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$
$$\|S_k^\Delta(H^N_{\partial\Omega} g)\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)}.$$

Proof. The combination of (4.16) and Lemma 4.2 gives the first estimate. The second estimate follows from (4.18) and, again, Lemma 4.2. ∎

4.2. The case of a bounded domain with Robin boundary conditions. We consider the following problem:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d, \qquad \partial_n u - \mathrm{i}ku = g \quad \text{on } \partial\Omega. \tag{4.19}$$

Our analysis will be based on the following assumption.

Assumption 4.8. The solution operator $(f,g) \mapsto u := S_k(f,g)$ for (4.19) satisfies

$$\|u\|_{\mathcal{H},\Omega} \le C_S k^\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right) \tag{4.20}$$

for some C_S and $\alpha \ge 0$ independent of k.

Remark 4.9. Assumption 1.1 is in fact fulfilled with $\alpha = 5/2$ for Lipschitz domains [20]. For smooth domains that are star-shaped with respect to a ball or convex polygons/polyhedra, Assumption 1.1 is valid with $\alpha = 0$ [43, Prop. 8.1.4], [18, 31].

The goal of this section is the proof of the following result. (Concerning the weight functions $\Phi_{p,\vec{\beta},k}$ appearing in its statement, we remind the reader of our convention introduced in section 1.1, namely, $\Phi_{p,\vec{\beta},k} \equiv 1$ if Ω has an analytic boundary.)

THEOREM 4.10 (decomposition for bounded domain). *Let Assumptions* 1.1 *and* 4.8 *be valid. Then there exist constants C, $\gamma > 0$, $\vec{\beta} \in [0,1)^J$ independent of $k \ge k_0$ such that for every $f \in L^2(\Omega)$ and $g \in H^{1/2}_{pw}(\partial\Omega)$ the solution $u = S_k(f,g)$ of* (4.19) *can be written as $u = u_{\mathcal{A}} + u_{H^2}$, where, for all $p \in \mathbb{N}_0$,*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \le Ck^\alpha \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right), \tag{4.21a}$$

$$\|\Phi_{p,\vec{\beta},k} \nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \le C\gamma^p k^{\alpha-1} \max\{p,k\}^{p+2} \cdot \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right), \tag{4.21b}$$

$$\|u_{H^2}\|_{H^2(\Omega)} + k\|u_{H^2}\|_{\mathcal{H},\Omega} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}\right). \tag{4.21c}$$

Proof. The proof is based on Lemmata 4.15 and 4.16 below. By linearity of the operator S_k it suffices to consider the decomposition of $u = S_k(f,0)$ and $u = S_k(0,g)$ separately. Writing $f^{(0)} := f$, we get for $S_k(f^{(0)},0)$ from Lemma 4.15 that

$$u = u_{\mathcal{A}}^{(0)} + u_{H^2}^{(0)} + S_k(f^{(1)},0) \quad \text{for some } f^{(1)} \in L^2(\Omega),$$

where $u_{\mathcal{A}}^{(0)}$, $u_{H^2}^{(0)}$ satisfy the desired bounds and $\|f^{(1)}\|_{L^2(\Omega)} \leq q\|f^{(0)}\|_{L^2(\Omega)}$ for some $q \in (0,1)$. Hence, we may iterate the argument and can write u as a sum of series (one of analytic functions and one of H^2-functions) that can be bounded (in appropriate norms) by geometric series. For the decomposition of $S_k(0,g)$ we proceed completely analogously. ∎

Remark 4.11. For the case of polygonal Ω Theorem 4.10 merely asserts the *existence* of *some* $\overrightarrow{\beta} \in [0,1)^J$ with the stated properties. The proof of Lemmata 4.15 and 4.16 relies on [44]. A closer inspection of the proofs there reveals that, for convex Ω, any $\overrightarrow{\beta} \in (0,1)^J$ may be chosen.

In view of Lemma 3.1, the following corollary is evident.

COROLLARY 4.12. *Under the hypotheses of Theorem* 4.10*, the statement of Theorem* 4.10 *holds verbatim for the adjoint solution operator* $(f,g) \mapsto S_k^\star(f,g)$ *(see* (3.4)*).*

LEMMA 4.13 (analyticity of $S_k(L_\Omega f, L_{\partial\Omega}^N g)$). *Let Assumption* 1.1 *be valid, and let* $\eta > 1$ *appearing in the definition of* L_Ω *and* $L_{\partial\Omega}^N$ *be fixed. Then there exist constants* C*,* $K > 0$*,* $\overrightarrow{\beta} \in [0,1)^J$ *independent of* k *such that, for every* $g \in H_{pw}^{1/2}(\partial\Omega)$ *and* $f \in L^2(\Omega)$*, the function* $u_{\mathcal{A}} = S_k(L_\Omega f, L_{\partial\Omega}^N g)$ *is analytic on* Ω *and satisfies for all* $p \in \mathbb{N}_0$ *the estimates*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq Ck^{\alpha}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H_{pw}^{1/2}(\partial\Omega)}\right), \tag{4.22}$$

$$\|\Phi_{p,\overrightarrow{\beta},k}\nabla^{p+2}u_{\mathcal{A}}\|_{L^2(\Omega)} \leq CK^p \max\{k,p+2\}^{p+2}k^{\alpha-1} \cdot \left(\|f\|_{L^2(\Omega)} + \|g\|_{H_{pw}^{1/2}(\partial\Omega)}\right). \tag{4.23}$$

Proof. We first restrict our attention here to the case of polygonal Ω with edges Γ_j, $j = 1,\ldots,N_\Gamma$, and remark on the case of analytic $\partial\Omega$ at the end of the proof.

Let $u := S_k(L_\Omega f, L_{\partial\Omega}^N g)$. Set $\widetilde{f} := L_\Omega f$ and $\widetilde{g} := L_{\partial\Omega}^N g = \partial_n L_\Omega G^N g$. From Lemma 4.3 we have that $\widetilde{f}$ is an entire function. Note that for any Γ_j there exists an open neighborhood T_j of $\overline{\Gamma_j}$ such that the normal $n_j : \Gamma_j \to \mathbb{S}_1$ can be extended to an analytic function $n_j^\star : T_j \to \mathbb{R}^2$. (In the present case of a polygon, this is trivial since n_j is a constant vector.) We set $G_j := \langle n_j^\star, \nabla L_\Omega G^N g\rangle$ and assume that the open neighborhood T_j of $\overline{\Gamma_j}$ is such that G_j is analytic on T_j. (In view of Lemma 4.3, which asserts that G_j is an entire function, this is again trivial.) We note $G_j|_{\Gamma_j} = \widetilde{g}$. Furthermore, from Lemma 4.3, we have the following estimates:

$$\|\nabla^p \tilde{f}\|_{L^2(\Omega)} \lesssim (\eta k)^p \|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0, \tag{4.24}$$

$$\|G_j\|_{L^2(\Omega\cap T_j)} \leq \|\nabla L_\Omega G^N g\|_{L^2(\Omega)} \leq \|L_\Omega G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{H_{pw}^{1/2}(\partial\Omega)}, \tag{4.25}$$

$$\|\nabla^{p+1} G_j\|_{L^2(\Omega\cap T_j)} \lesssim \|\nabla^{p+2} L_\Omega G^N g\|_{L^2(\Omega)} \overset{(4.9)}{\lesssim} (\eta k)^p \|g\|_{H_{pw}^{1/2}(\partial\Omega)} \quad \forall p \in \mathbb{N}_0. \tag{4.26}$$

The bounds (4.25), (4.26) for $p = 0$ together with the multiplicative trace inequality give $\|\tilde{g}\|_{L^2(\partial\Omega)} \lesssim \|g\|_{H_{pw}^{1/2}(\partial\Omega)}$. This bound together with (4.24) and Assumption 4.8 implies (4.22).

The regularity estimate (4.23) will be derived by applying [44, Prop. 5.4.5, Rem. 5.4.6] and estimating the constants therein. To that end, we set $\varepsilon := 1/k$ and note that $u_{\mathcal{A}}$ solves

$$-\varepsilon^2 \Delta u - u = \varepsilon^2 \tilde{f} \quad \text{on } \Omega, \qquad \varepsilon^2 \partial_n u = \varepsilon(\varepsilon \tilde{g} + \mathrm{i}\, u) \quad \text{on } \partial\Omega.$$

Then [44, Prop. 5.4.5] is applicable with

$$\begin{array}{lllll} C_f = \varepsilon^2 \|f\|_{L^2(\Omega)} O(1), & C_{G_1} = \varepsilon \|g\|_{H^{1/2}(\partial\Omega)} O(1), & C_{G_2} = \varepsilon, & C_b = 0, & C_c = 1, \\ \gamma_f = O(1), & \gamma_{G_1} = O(1), & \gamma_{G_2} = O(1), & \gamma_b = 0, & \gamma_c = 0, \end{array}$$

resulting in the existence of constants C, $K > 0$ and $\overrightarrow{\beta} \in [0,1)^J$ with

$$\begin{aligned} &\|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} S_k u\|_{L^2(\Omega)} \\ &\quad \lesssim K^{p+2} \max\{p+2, k\}^{p+2} \left(k^{-2} \|f\|_{L^2(\Omega)} + k^{-1} \|u\|_{\mathcal{H},\Omega} + k^{-1} \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \right) \end{aligned}$$

for all $p \in \mathbb{N}_0$. Inserting (4.22) and using $\alpha \geq 0$, we arrive at (4.23).

For the case of analytic $\partial\Omega$, we proceed analogously. The main difference is that is suffices to consider a single tubular neighborhood T of $\partial\Omega$ and that the analytic extension $n^\star$ of the normal vector is no longer constant on T. Therefore, the estimate (4.26) (we write G instead of G_j) is replaced with

$$\|\nabla^{p+1} G\|_{L^2(\Omega \cap T)} \lesssim \gamma^p \max\{p, \eta k\}^p \|g\|_{H^{1/2}_{pw}(\partial\Omega)} \quad \forall p \in \mathbb{N}_0$$

for a constant γ that reflects the size of the domain of analyticity of $n^\star$. The remainder of the proof follows the above arguments but appeals to [44, Remark 5.4.6]. ∎

Remark 4.14. The k-dependence in the estimates of Lemma 4.13 is likely to be suboptimal. One reason for this is that we treated the contributions stemming from the boundary data g rather generously in order to cover the cases of domains with analytic boundaries and polygons in a unified way. However, sharper estimates are available for the lifting G^N for the case of smooth domains than for the polygonal case, and therefore sharper estimates are possible for the case of analytic boundaries. Additionally, the possibility of using k-dependent liftings G^N (as is done in the case of the exterior Dirichlet problem in section 4.3) has not been explored here. Nevertheless, the present estimates are already sufficient for our purposes in section 5: in the context of high order methods such as hp-FEM, analytic functions satisfying bounds of the form (4.22), (4.23) can be approximated at an exponential rate if the scale resolution condition (1.1) is satisfied. Sharper estimates concerning the k-dependence will only lead to improved estimates for the constant C in (1.1).

LEMMA 4.15 (properties of $S_k(f,0)$). *Let Assumptions* 1.1 *and* 4.8 *be valid. Let* $q \in (0,1)$. *Then there exist constants* C, $K > 0$, $\overrightarrow{\beta} \in [0,1)^J$ *independent of* k *such that for every* $f \in L^2(\Omega)$ *the function* $u = S_k(f,0)$ *can be written as* $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, *where*

$$\begin{aligned} \|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} &\leq C k^{\alpha} \|f\|_{L^2(\Omega)}, \\ \|\Phi_{p,\overrightarrow{\beta},k} \nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} &\leq C k^{\alpha-1} K^p \max\{p+2, k\}^{p+2} \|f\|_{L^2(\Omega)} \qquad \forall p \in \mathbb{N}_0, \\ \|u_{H^2}\|_{\mathcal{H},\Omega} &\leq q k^{-1} \|f\|_{L^2(\Omega)}, \\ \|u_{H^2}\|_{H^2(\Omega)} &\leq C \|f\|_{L^2(\Omega)}, \end{aligned}$$

and the remainder $\widetilde{u} = S_k(\widetilde{f}, 0)$ *satisfies*

$$-\Delta \widetilde{u} - k^2 \widetilde{u} = \widetilde{f}, \qquad \partial_n \widetilde{u} - \mathrm{i} k \widetilde{u} = 0,$$

where

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq q \|f\|_{L^2(\Omega)}.$$

Proof. Define

$$u_{\mathcal{A}}^{\mathrm{I}} := S_k(L_\Omega f, 0), \qquad u_{H^2}^{\mathrm{I}} := N_k(H_\Omega f).$$

Here, the parameter η defining the filter operators L_Ω and H_Ω is still at our disposal and will be selected at the end of the proof. Then, $u_{\mathcal{A}}^{\mathrm{I}}$ satisfies the desired bounds by Lemma 4.13. Lemma 4.5 gives

$$\|u_{H^2}^{\mathrm{I}}\|_{\mathcal{H},\Omega} \leq q' k^{-1} \|f\|_{L^2(\Omega)} \qquad \text{and} \qquad \|u_{H^2}^{\mathrm{I}}\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

Here, the parameter $q' \in (0,1)$ depends on η and is still at our disposal. In fact, in view of the statement of Lemma 4.5, it can be made arbitrarily small by taking η sufficiently large.

The function $u^{\mathrm{I}} := u - (u_{\mathcal{A}}^{\mathrm{I}} + u_{H^2}^{\mathrm{I}})$ solves

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = 0, \qquad \partial_n u^{\mathrm{I}} - \mathrm{i} k u^{\mathrm{I}} = \mathrm{i} k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{I}. \tag{4.27}$$

We note with the multiplicative trace inequality that

$$\begin{aligned}
\|\mathrm{i} k u_{H^2}^{\mathrm{I}}\|_{L^2(\partial\Omega)} &\lesssim k \|u_{H^2}^{\mathrm{I}}\|_{L^2(\Omega)}^{1/2} \|u_{H^2}^{\mathrm{I}}\|_{H^1(\Omega)}^{1/2} \lesssim k^{1/2} \|u_{H^2}^{\mathrm{I}}\|_{\mathcal{H},\Omega} \lesssim q' k^{-1/2} \|f\|_{L^2(\Omega)}, \\
\|\mathrm{i} k u_{H^2}^{\mathrm{I}}\|_{H^{1/2}(\partial\Omega)} &\lesssim k \|u_{H^2}^{\mathrm{I}}\|_{H^1(\Omega)} \lesssim q' \|f\|_{L^2(\Omega)}, \\
\left\|\partial_n u_{H^2}^{\mathrm{I}}\right\|_{L^2(\partial\Omega)} &\lesssim \left\|\nabla u_{H^2}^{\mathrm{I}}\right\|_{L^2(\Omega)}^{1/2} \left\|u_{H^2}^{\mathrm{I}}\right\|_{H^2(\Omega)}^{1/2} \lesssim \sqrt{\frac{q'}{k}} \, \|f\|_{L^2(\Omega)}, \\
\left\|\partial_n u_{H^2}^{\mathrm{I}}\right\|_{H_{pw}^{1/2}(\partial\Omega)} &\lesssim \left\|u_{H^2}^{\mathrm{I}}\right\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.
\end{aligned}$$

This implies in particular

$$\|\, \mathrm{i}\, k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{\mathrm{I}} \|_{H_{pw}^{1/2}(\partial\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{4.28}$$

Next, we define the functions $u_{\mathcal{A}}^{\mathrm{II}}$ and $u_{H^2}^{\mathrm{II}}$ by

$$u_{\mathcal{A}}^{\mathrm{II}} := S_k(0, L_{\partial\Omega}^N(\mathrm{i} k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{\mathrm{I}})), \qquad u_{H^2}^{\mathrm{II}} := S_k^{\Delta}(H_{\partial\Omega}^N(\mathrm{i} k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{\mathrm{I}})).$$

Then, the analytic part $u_{\mathcal{A}}^{\mathrm{II}}$ again satisfies the desired analyticity bounds by Lemma 4.13. For the function $u_{H^2}^{\mathrm{II}}$ we obtain from Lemma 4.7 the estimates

$$\begin{aligned}
\|u_{H^2}^{\mathrm{II}}\|_{\mathcal{H},\Omega} &\leq q' k^{-1} \|\mathrm{i} k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{\mathrm{I}}\|_{H_{pw}^{1/2}(\partial\Omega)} \lesssim q' k^{-1} \|f\|_{L^2(\Omega)}, \\
\|u_{H^2}^{\mathrm{II}}\|_{H^2(\Omega)} &\lesssim \|\mathrm{i} k u_{H^2}^{\mathrm{I}} - \partial_n u_{H^2}^{\mathrm{I}}\|_{H_{pw}^{1/2}(\partial\Omega)} \lesssim \|f\|_{L^2(\Omega)}.
\end{aligned}$$

We now set $u_{\mathcal{A}} := u_{\mathcal{A}}^{\mathrm{I}} + u_{\mathcal{A}}^{\mathrm{II}}$ and $u_{H^2} := u_{H^2}^{\mathrm{I}} + u_{H^2}^{\mathrm{II}}$ and conclude for the function $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ that it satisfies

$$-\Delta \widetilde{u} - k^2 \widetilde{u} = \widetilde{f} := 2k^2 u_{H^2}^{\mathrm{II}}, \qquad \partial_n \widetilde{u} - \mathrm{i} k \widetilde{u} = 0.$$

For $\widetilde{f}$ we compute

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq 2k\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|f\|_{L^2(\Omega)}.$$

Hence, by taking η sufficiently large so that q' is sufficiently small, we arrive at the desired bound. ∎

LEMMA 4.16 (properties of $S_k(0,g)$). *Let Assumptions* 1.1, 4.8 *be valid. Let* $q \in (0,1)$. *Then there exist constants* C, $\gamma > 0$, $\overrightarrow{\beta} \in [0,1)^J$ *independent of* k *such that for every* $g \in H^{1/2}_{pw}(\partial\Omega)$ *the function* $u = S_k(0,g)$ *can be written as* $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, *where*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},\Omega} \leq Ck^{\alpha}\|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$
$$\|\Phi_{p,\overrightarrow{\beta},k}\nabla^{p+2}u_{\mathcal{A}}\|_{L^2(\Omega)} \leq Ck^{\alpha-1}\gamma^p \max\{p+2,k\}^{p+2}\|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$
$$\|u_{H^2}\|_{\mathcal{H},\Omega} \leq qk^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \qquad \|u_{H^2}\|_{H^2(\Omega)} \leq C\|g\|_{H^{1/2}_{pw}(\partial\Omega)}.$$

The remainder $\widetilde{u} = S_k(0,\widetilde{g})$ *satisfies for a* $\widetilde{g}$ *with* $\|\widetilde{g}\|_{H^{1/2}_{pw}(\partial\Omega)} \leq q\|g\|_{H^{1/2}_{pw}(\partial\Omega)}$

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0, \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = \widetilde{g}.$$

Proof. The proof is very similar to that of Lemma 4.15. Define

$$u^{\mathrm{I}}_{\mathcal{A}} := S_k(0, L^N_{\partial\Omega}g) \qquad \text{and} \qquad u^{\mathrm{I}}_{H^2} := S^{\Delta}_k(H^N_{\partial\Omega}g),$$

where S^{Δ}_k is the solution operator for (4.14). Then $u^{\mathrm{I}}_{\mathcal{A}}$ is analytic and satisfies the desired analyticity estimates by Lemma 4.13. For $u^{\mathrm{I}}_{H^2}$ we have by Lemma 4.7

$$\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \leq q'k^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \tag{4.29}$$

$$\|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega)} \lesssim \|g\|_{H^{1/2}_{pw}(\partial\Omega)}, \tag{4.30}$$

where the parameter $q' < 1$ is at our disposal and depends on η defining $H^N_{\partial\Omega}$ and $L^N_{\partial\Omega}$. Hence, the function $u^{\mathrm{I}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2}$ satisfies

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = -2k^2 u^{\mathrm{I}}_{H^2}, \qquad \partial_n u^{\mathrm{I}} - \mathrm{i}ku^{\mathrm{I}} = g$$

together with

$$\|2k^2 u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim k\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}. \tag{4.31}$$

Next, we define $u^{\mathrm{II}}_{\mathcal{A}}$ and $u^{\mathrm{II}}_{H^2}$ by

$$u^{\mathrm{II}}_{\mathcal{A}} := S_k(L_\Omega(2k^2u^{\mathrm{I}}_{H^2}), 0) \qquad \text{and} \qquad u^{\mathrm{II}}_{H^2} := N_k(H_\Omega(2k^2u^{\mathrm{I}}_{H^2})).$$

Here, in order to apply the operator N_k, we extend $H_\Omega\left(2k^2u^{\mathrm{I}}_{H^2}\right)$ by zero outside of Ω. By Lemma 4.13 and (4.31), we see that $u^{\mathrm{II}}_{\mathcal{A}}$ satisfies the desired analyticity estimates. For the function $u^{\mathrm{II}}_{H^2}$, we obtain from Lemma 4.5

$$\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},\Omega} \leq q'k^{-1}\|2k^2u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim q'\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim {q'}^2k^{-1}\|g\|_{H^{1/2}_{pw}(\partial\Omega)},$$
$$\|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega)} \lesssim \|2k^2u^{\mathrm{I}}_{H^2}\|_{L^2(\Omega)} \lesssim k\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},\Omega} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}.$$

We set $u_{\mathcal{A}} := u_{\mathcal{A}}^{\rm I} + u_{\mathcal{A}}^{\rm II}$ and $u_{H^2} := u_{H^2}^{\rm I} + u_{H^2}^{\rm II}$. Then $u_{\mathcal{A}}$ and u_{H^2} satisfy the desired estimates, and $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0, \qquad \partial_n\widetilde{u} - \mathrm{i}k\widetilde{u} = \widetilde{g} := \mathrm{i}ku_{H^2}^{\rm II} - \partial_n u_{H^2}^{\rm II}$$

with

$$\begin{aligned}\|\widetilde{g}\|_{H^{1/2}_{pw}(\partial\Omega)} &\le k\|u_{H^2}^{\rm II}\|_{H^{1/2}(\partial\Omega)} + \left\|\partial_n u_{H^2}^{\rm II}\right\|_{H^{1/2}_{pw}(\Omega)} \\ &\lesssim k\|u_{H^2}^{\rm II}\|_{\mathcal{H},\Omega} + \|u_{H^2}^{\rm II}\|_{H^2(\Omega)} \lesssim q'\|g\|_{H^{1/2}_{pw}(\partial\Omega)}.\end{aligned}$$

The result follows by selecting η sufficiently large so that q' is sufficiently small. ∎

4.3. The exterior Dirichlet problem.

4.3.1. Main result. In the present section, we study the problem (2.5) or, equivalently, (2.6) of section 2.2, which we recall for convenience:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega^c, \qquad u|_{\partial\Omega} = g, \tag{4.32a}$$

$$\left|\frac{\partial u}{\partial r} - \mathrm{i}\,ku\right| = o\left(\|x\|^{\frac{1-d}{2}}\right) \qquad \text{as } \|x\| \to \infty. \tag{4.32b}$$

Throughout this section, we will make the following assumption concerning the bounded Lipschitz domain Ω and the right-hand side f.

Assumption 4.17.
1. $\partial\Omega$ is analytic.
2. $\operatorname{supp} f \subset B_R$ for fixed R.

We recall that the solution operator S_k^c for problem (4.32) and the adjoint solution operator $S_k^{c,\star}$ have been introduced in Lemma 3.1. Concerning the mapping properties of S_k^c, we will make a polynomial growth assumption, as follows.

Assumption 4.18. The solution operator S_k^c for the Helmholtz problem (2.6) satisfies

$$\|u\|_{\mathcal{H},R} \le C_S k^\alpha \left(\|f\|_{L^2(\Omega_R^c)} + k\|g\|_{1/2,\mathcal{H},\partial\Omega}\right) \tag{4.33}$$

for some C_S, $\alpha \ge 0$ independent of k, where

$$\|v\|^2_{\mathcal{H},R} := k^2\|v\|^2_{L^2(\Omega_R^c)} + |v|^2_{H^1(\Omega_R^c)}.$$

Remark 4.19. Assumption 4.18 is true with $\alpha = 0$ for star-shaped Ω. This is shown for the case $g = 0$ in [16, Lem. 3.5]. The case $g \neq 0$ can be reduced to the case $g = 0$ via a lifting argument in the standard way: Given $g \in H^{1/2}(\partial\Omega)$, Lemma 4.22 provides a lifting $u_g = G^D g$ with $u_g|_{\partial\Omega} = g$ and $-\Delta u_g - k^2 u_g = -2k^2 u_g$ in $\Omega^c \cap B_{2R}$. Using a suitable cut-off function χ, the function $\tilde{u} := \chi u_g$ satisfies $\tilde{u}|_{\partial\Omega} = g$, $\tilde{u} \equiv 0$ outside a ball of radius R, $\|\tilde{u}\|_{\mathcal{H},R} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}$, and $\|-\Delta\tilde{u} - k^2\tilde{u}\|_{L^2(\Omega_R^c)} \lesssim k\|u_g\|_{\mathcal{H},R} \lesssim k\|g\|_{1/2,\mathcal{H},\partial\Omega}$.

Sharper bounds than those stipulated in Assumption 4.18 are available for special geometries, e.g., circles and spheres in [53, Thm. 2.6.2].

The main result of this section is a decomposition result for the solution of (2.6). We show in Theorem 4.20 that the solution can be decomposed into a part with finite Sobolev regularity featuring k-independent regularity constants and an analytic part with k-explicit bounds on all derivatives.

THEOREM 4.20 (decomposition for exterior Dirichlet problem). *Let Assumptions* 4.17 *and* 4.18 *be satisfied. For $f \in L^2(\Omega^c_R)$ and $g \in H^{3/2}(\partial\Omega)$ the solution $u = S^c_k(f,g)$ of* (2.5a) *admits a decomposition $u = u_{\mathcal{A}} + u_{H^2}$, where for all $p \geq 2$*

$$\begin{aligned} \|u_{\mathcal{A}}\|_{\mathcal{H},\Omega^c_R} &\lesssim k^\alpha \left(\|f\|_{L^2(\Omega^c_R)} + k\|g\|_{1/2,\mathcal{H},\partial\Omega}\right), \\ \|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega^c_R)} &\lesssim \gamma^p \max\{p,k\}^p \left(k^{\alpha-1}\|f\|_{L^2(\Omega^c_R)} + (k+k^\alpha)\|g\|_{1/2,\mathcal{H},\partial\Omega}\right), \\ \|u_{H^2}\|_{\mathcal{H},R} &\lesssim k^{-1}\|f\|_{L^2(\Omega^c_R)} + \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ \|u_{H^2}\|_{H^2(\Omega^c_R)} &\lesssim \|f\|_{L^2(\Omega^c_R)} + k\|g\|_{3/2,\mathcal{H},\partial\Omega}. \end{aligned}$$

Proof. The proof is a consequence of the lemmata of section 4.3.3 by reasoning as in the proof of Theorem 4.10. ∎

COROLLARY 4.21. *Theorem* 4.20 *holds verbatim (with $g = 0$) for the adjoint solution operator $S^{c,\star}_k$ in view of Lemma* 3.1.

The following two subsections provide details of the proof of Theorem 4.20.

4.3.2. *k*-dependent lifting operators for Dirichlet problems.

LEMMA 4.22 (lifting operator G^D from $\partial\Omega$ to $\Omega^c \cap B_{2R}$). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with smooth $\partial\Omega$. Then there exists a trace lifting operator $G^D : H^{1/2}(\partial\Omega) \to H^1(\Omega^c \cap B_{2R})$ such that for a constant $C > 0$ independent of k*

$$\|G^D g\|_{\mathcal{H},2R} \leq C\|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.34}$$

$$\|G^D g\|_{H^2(\Omega^c_R)} \leq Ck\|g\|_{3/2,\mathcal{H},\partial\Omega} \qquad \textit{if additionally } g \in H^{3/2}(\partial\Omega). \tag{4.35}$$

Moreover, the lifting $G^D g$ is given explicitly by the solution u_g of

$$-\Delta u_g + k^2 u_g = 0 \quad \textit{in } \Omega^c \cap B_{2R} =: \Omega^c_{2R}, \qquad u_g|_{\partial\Omega} = g, \qquad u_g|_{\partial B_{2R}} = 0. \tag{4.36}$$

Proof. Step 1. We start with an estimate on a tubular neighborhood of $\partial\Omega$. For a univariate function $v \in H^1(0,1)$, we get for $\delta \in (0,1)$ from $v(x) = u(0) + \int_0^x v'(t)\,dt$ the bound $\|v\|_{L^2(0,\delta)} \leq C(\sqrt{\delta}|v(0)| + \delta\|v'\|_{L^2(0,1)})$, for a constant $C > 0$ that is independent of δ. Upon introducing the tubular neighborhood $S_\delta := \{x \in \Omega^c \,|\, \operatorname{dist}(x,\partial\Omega) < \delta\}$ of $\partial\Omega$, this univariate estimate implies

$$\|v\|_{L^2(S_\delta)} \leq C\left(\sqrt{\delta}\|v\|_{L^2(\partial\Omega)} + \delta\|v\|_{H^1(\Omega^c\cap B_{2R})}\right) \qquad \forall v \in H^1(\Omega^c \cap B_{2R}), \tag{4.37}$$

where, for sufficiently small δ, the constant $C > 0$ is independent of δ. Next, we select $\delta = 1/k$ and a cut-off function $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\chi \equiv 1$ on $\partial\Omega$, $\|\nabla^j\chi\|_{L^\infty(\mathbb{R}^d)} \leq Ck^j$, $j \in \{0,1\}$, $\operatorname{supp}\chi \cap \Omega^c \subset \overline{S_{1/k}}$. Then, we arrive at

$$\|\chi v\|_{\mathcal{H},R} \lesssim \left(\sqrt{k}\|v\|_{L^2(\partial\Omega)} + \|v\|_{H^1(\Omega^c\cap B_{2R})}\right) \qquad \forall v \in H^1(\Omega^c \cap B_{2R}). \tag{4.38}$$

Step 2. Let $\widetilde{u}_g$ solve

$$-\Delta\widetilde{u_g} = 0 \quad \text{on } \Omega^c \cap B_{2R}, \qquad \widetilde{u}_g|_{\partial\Omega} = g, \qquad \widetilde{u_g}|_{\partial B_{2R}} = 0.$$

Then $\|\widetilde{u}_g\|_{H^1(\Omega^c\cap B_{2R})} \lesssim \|g\|_{H^{1/2}(\partial\Omega)}$. The solution u_g of (4.36) is the minimizer in the $\|\cdot\|_{\mathcal{H},2R}$-norm over all functions that satisfy the boundary conditions. Therefore, we get, in view of (4.38),

$$\|u_g\|_{\mathcal{H},2R} \leq \|\chi\widetilde{u}_g\|_{\mathcal{H},2R} \lesssim \sqrt{k}\|g\|_{L^2(\partial\Omega)} + \|\widetilde{u}_g\|_{H^1(\Omega^c\cap B_{2R})} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega},$$

which is (4.34).

Step 3. To get the H^2 estimate, we use elliptic regularity to conclude

$$\|u_g\|_{H^2(\Omega^c\cap B_{2R})} \lesssim k^2\|u_g\|_{L^2(\Omega^c\cap B_{2R})} + \|g\|_{H^{3/2}(\partial\Omega)} \lesssim k\|u_g\|_{\mathcal{H},2R} + \|g\|_{H^{3/2}(\partial\Omega)} \\ \lesssim k\|g\|_{3/2,\mathcal{H},\partial\Omega},$$

which finishes the proof. ◻

With the aid of the lifting operator G^D of Lemma 4.22 we define the frequency splitting of the Dirichlet traces in terms of the operators $L^D_{\Omega^c_R}$ and $H^D_{\Omega^c_R}$ as follows:

$$\begin{aligned} L^D_{\Omega^c_R} g &:= \left(L_{\Omega^c\cap B_{2R}} G^D g\right)\big|_{\Omega^c_R}, & H^D_{\Omega^c_R} g &:= \left(H_{\Omega^c\cap B_{2R}} G^D g\right)\big|_{\Omega^c_R}, \\ L^D_{\partial\Omega} g &:= \left(L^D_{\Omega^c_R} g\right)\Big|_{\partial\Omega}, & H^D_{\partial\Omega} g &:= \left(H^D_{\Omega^c_R} g\right)\Big|_{\partial\Omega}. \end{aligned} \tag{4.39}$$

In view of the stability properties of the operators $L_{\Omega^c\cap B_{2R}}$, $H_{\Omega^c\cap B_{2R}}$, given by Lemmata 4.2, 4.3 we get (with $\eta > 1$ defining these operators)

$$\|H^D_{\Omega^c_R} g\|_{\mathcal{H},R} + \|L^D_{\Omega^c_R} g\|_{\mathcal{H},R} \overset{(4.3),\,(4.8)}{\lesssim} \|G^D g\|_{\mathcal{H},R} \overset{(4.34)}{\lesssim} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.40a}$$

$$\|\nabla^p L^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} \overset{(4.8),(4.34)}{\lesssim} (\eta k)^p \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \in \mathbb{N}_0, \tag{4.40b}$$

$$\|H^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} \overset{(4.3)}{\lesssim} (\eta k)^{-1}\|G^D g\|_{H^1(\Omega^c\cap B_{2R})} \overset{(4.34)}{\lesssim} (\eta k)^{-1}\|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.40c}$$

$$\|H^D_{\Omega^c_R} g\|_{\mathcal{H},R} \overset{(4.3)}{\lesssim} (\eta k)^{-1}\|G^D g\|_{H^2(\Omega^c\cap B_{2R})} \overset{(4.35)}{\lesssim} \eta^{-1}\|g\|_{3/2,\mathcal{H},\partial\Omega}, \tag{4.40d}$$

$$\|H^D_{\Omega^c_R} g\|_{H^2(\Omega^c_R)} \overset{(4.3)}{\lesssim} \|G^D g\|_{H^2(\Omega^c\cap B_{2R})} \overset{(4.35)}{\lesssim} k\|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.40e}$$

Remark 4.23. The trace theorem in the multiplicative form yields

$$\begin{aligned} \|u\|_{1/2,\mathcal{H},\partial\Omega} &\lesssim \|u\|_{\mathcal{H},R} & &\forall u \in H^1\left(\Omega^c_R\right), \\ \|u\|_{3/2,\mathcal{H},\partial\Omega} &\lesssim k^{-1}\|u\|_{H^2\left(\Omega^c_R\right)} + \|u\|_{\mathcal{H},R} & &\forall u \in H^2\left(\Omega^c_R\right). \end{aligned}$$

Hence, from (4.40) it follows that

$$\left\|H^D_{\partial\Omega} g\right\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.40a)}{\lesssim} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \tag{4.41a}$$

$$\left\|H^D_{\partial\Omega} g\right\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.40d)}{\lesssim} \eta^{-1}\|g\|_{3/2,\mathcal{H},\partial\Omega}, \tag{4.41b}$$

$$\left\|H^D_{\partial\Omega} g\right\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k^{-1}\left\|H^D_{\Omega^c_R} g\right\|_{H^2\left(\Omega^c_R\right)} + \left\|H^D_{\Omega^c_R} g\right\|_{\mathcal{H},R} \overset{(4.40e),(4.40d)}{\lesssim} \|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.41c}$$

4.3.3. Proof of Theorem 4.20.

LEMMA 4.24 (analysis of $S^c(L_{\Omega^c} f, 0)$). *Let Assumptions* 4.17 *and* 4.18 *be satisfied. Let* $\eta > 1$ *defining* L_{Ω^c} *be fixed. Then the function* $u = S^c_k(L_{\Omega^c} f, 0)$ *is analytic in an open neighborhood of* $\overline{\Omega^c_R}$ *and satisfies*

$$\|\nabla^p S^c_k(L_{\Omega^c} f, 0)\|_{L^2(\Omega^c_R)} \lesssim k^{\alpha-1}\gamma^p \max\{p,k\}^p \|f\|_{L^2(\Omega^c)}.$$

Proof. Note that by replacing R in (2.6) by $2R$ and denoting the corresponding solution by u_{2R} we have $u = u_{2R}|_{\Omega^c_R}$. As a consequence it suffices to apply from [44, sect. 5.5] the interior estimates and the local estimates at the boundary $\Gamma = \partial\Omega$. In other words, [44, Thm. 5.3.10] directly applies to this situation. Rewriting the equation satisfied by u as

$$-\varepsilon^2 \Delta u - u = \varepsilon^2 L_{\Omega^c} f, \qquad \varepsilon := 1/k,$$

and noting that $\varepsilon^2 L_{\Omega^c} f$ satisfies

$$\|\nabla^p (L_{\Omega^c} f)\|_{L^2(B_{2R})} \lesssim (\eta k)^p \|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0,$$

we may apply [44, Thm. 5.3.10] with $\mathcal{E} = \varepsilon = 1/k$, $C_c = 1$, $\gamma_f = O(1)$, $C_f = O(\varepsilon^2 \|f\|_{L^2(\Omega^c_R)})$, and $k\|u\|_{L^2(\Omega^c \cap B_{2R})} + \|\nabla u\|_{L^2(\Omega^c \cap B_{2R})} \lesssim k^\alpha \|f\|_{L^2(\Omega^c_R)}$ to get

$$\begin{aligned} \|\nabla^{p+2} u\|_{L^2(\Omega^c_R)} &\lesssim K^p \max\{p+2, k\}^{p+2} \left(k^{-2} \|f\|_{L^2(\Omega^c_R)} + k^{-1} \|u\|_{\mathcal{H},R}\right) \\ &\lesssim K^p \max\{p+2, k\}^{p+2} k^{\alpha-1} \|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0, \end{aligned}$$

where we have exploited the assumption $\alpha \geq 0$. ◻

LEMMA 4.25 (analysis of $S^c(0, L^D_{\partial\Omega} g)$). *Let Assumptions* 4.17 *and* 4.18 *be satisfied. Let $\eta > 1$ defining $L^D_{\partial\Omega}$ be fixed. Then the function $u = S^c_k(0, L^D_{\partial\Omega} g)$ is analytic on an open neighborhood of $\overline{\Omega^c_R}$ and satisfies*

$$\begin{aligned} \|S^c_k(0, L^D_{\partial\Omega} g)\|_{\mathcal{H},R} &\lesssim k^{\alpha+1} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ \|\nabla^p S^c_k(0, L^D_{\partial\Omega} g)\|_{L^2(\Omega^c_R)} &\lesssim (k + k^\alpha) \gamma^p \max\{p, k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \geq 2. \end{aligned}$$

Proof. Assumption 4.18 gives us $\|u\|_{\mathcal{H},R} \lesssim k^{\alpha+1} \|g\|_{1/2,\mathcal{H},\partial\Omega}$. Next, interior regularity as derived in [44, Prop. 5.5.1] gives

$$\|\nabla^{p+2} u\|_{L^2(\Omega^c_R \setminus S)} \lesssim K^{p+2} \max\{p, k\}^{p+2} k^{-1} \|u\|_{\mathcal{H},R} \qquad \forall p \in \mathbb{N}_0,$$

where S is a tubular neighborhood of $\partial\Omega$ of width $O(1)$. These are the desired bounds away from $\partial\Omega$. For the behavior of u near $\partial\Omega$, we write $u = \tilde{u} + L^D_{\Omega^c_R} g$ and set $\tilde{f} := -\Delta L^D_{\Omega^c_R} g + k^2 L^D_{\Omega^c_R} g$. Then, (4.40) gives us

$$\begin{aligned} \|\nabla^p L^D_{\Omega^c_R} g\|_{L^2(\Omega^c_R)} &\lesssim (\eta k)^{p-1} \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \in \mathbb{N}_0, \\ \|\nabla^p \tilde{f}\|_{L^2(\Omega^c_R)} &\lesssim (\eta k)^{p+1} \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \in \mathbb{N}_0. \end{aligned}$$

Near $\partial\Omega$, the function $\tilde{u}$ satisfies $-\Delta\tilde{u} - k^2 \tilde{u} = -\tilde{f}$ together with $\tilde{u}|_{\partial\Omega} = 0$. Hence, [44, Thm. 5.3.10] gives us

$$\|\nabla^{p+2} \tilde{u}\|_{L^2(S)} \lesssim \max\{p+2, k\}^{p+2} \left(k \|g\|_{1/2,\mathcal{H},\partial\Omega} + k^{-1} \|\tilde{u}\|_{\mathcal{H},R}\right) \qquad \forall p \in \mathbb{N}_0.$$

This concludes the argument. ◻

LEMMA 4.26 (decomposition of $S^c(f, 0)$). *Let Assumptions* 4.17 *and* 4.18 *be satisfied. Let $q \in (0, 1)$ be given. Then for $f \in L^2(\Omega^c_R)$ the solution $u = S^c_k(f, 0)$ can be written as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where $u_{\mathcal{A}}$ is analytic on $\overline{\Omega^c_R}$, $u_{H^2} \in H^2(\Omega^c_R)$, and*

$$\begin{aligned} \|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega^c_R)} &\lesssim k^{\alpha-1} \gamma^p \max\{p, k\}^p \|f\|_{L^2(\Omega^c_R)} \qquad \forall p \in \mathbb{N}_0, \\ \|u_{H^2}\|_{\mathcal{H},R} \leq q k^{-1} \|f\|_{L^2(\Omega^c_R)}, &\qquad \|u_{H^2}\|_{H^2(\Omega^c_R)} \lesssim \|f\|_{L^2(\Omega^c_R)}. \end{aligned}$$

Additionally, $\widetilde{u} = S^c_k(\widetilde{f}, 0)$ for a function $\widetilde{f} \in L^2(\Omega^c_R)$ with $\|\widetilde{f}\|_{L^2(\Omega^c)} \leq q \|f\|_{L^2(\Omega^c)}$.

Proof. Extend f by zero to $\mathbb{R}^d$ (and denote again by f the extended function). Define for an $\eta > 1$ to be chosen below

$$u^{\mathrm{I}}_{\mathcal{A}} := S^c_k(L_{\mathbb{R}^d} f, 0), \qquad u^{\mathrm{I}}_{H^2} := N_k(H_{\mathbb{R}^d} f).$$

By Lemma 4.24 we know that $u^{\mathrm{I}}_{\mathcal{A}}$ is analytic and satisfies the desired bounds. Lemma 4.5 implies that $u^{\mathrm{I}}_{H^2}$ satisfies (by choosing η suitably)

$$\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \le q k^{-1} \|f\|_{L^2(\Omega^c_R)}, \qquad \|u^{\mathrm{I}}_{H^2}\|_{H^2(B_R)} \lesssim \|f\|_{L^2(\Omega^c_R)}.$$

The trace inequality gives us $\|u^{\mathrm{I}}_{H^2}\|_{H^{1/2}(\partial\Omega)} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$, and the multiplicative trace inequality (cf. (4.28)) provides $\sqrt{k}\|u^{\mathrm{I}}_{H^2}\|_{L^2(\partial\Omega)} \le \|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$. That is, we have $\|u^{\mathrm{I}}_{H^2}\|_{1/2,\mathcal{H},\partial\Omega} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}$. Furthermore, we have from the trace estimate $\|u^{\mathrm{I}}_{H^2}\|_{H^{3/2}(\partial\Omega)} \lesssim \|u^{\mathrm{I}}_{H^2}\|_{H^2(B_R)}$ that $\|u^{\mathrm{I}}_{H^2}\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k^{-1}\|f\|_{L^2(\Omega^c_R)}$. The function $u^{\mathrm{I}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2})$ satisfies

$$-\Delta u^{\mathrm{I}} - k^2 u^{\mathrm{I}} = 0 \quad \text{in } \Omega^c, \qquad u^{\mathrm{I}}|_{\partial\Omega} = -u^{\mathrm{I}}_{H^2}|_{\partial\Omega}, \qquad u^{\mathrm{I}} \text{ satisfies (4.32b)}.$$

Let $u^{\mathrm{II}} := G^D\, u^{\mathrm{I}}_{H^2}\big|_{\partial\Omega}$, where G^D is the lifting operator of Lemma 4.22. Then

$$\|u^{\mathrm{II}}\|_{\mathcal{H},R} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}, \qquad \|u^{\mathrm{II}}\|_{H^2(\Omega^c_R)} \lesssim \|f\|_{L^2(\Omega^c_R)}. \tag{4.42}$$

For a smooth cut-off function $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\operatorname{supp}\chi \subset B_R$ and $\chi \equiv 1$ near $\partial\Omega$, we define $\widetilde{f} := \Delta(\chi u^{\mathrm{II}}) + k^2\chi u^{\mathrm{II}}$. In view of the definition of G^D, we have $-\Delta u^{\mathrm{II}} + k^2 u^{\mathrm{II}} = 0$, which allows us to obtain with the aid of (4.42)

$$\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}.$$

Next, we observe that the function χu^{II} satisfies the following equations and estimates:

$$\chi u^{\mathrm{II}}|_{\partial\Omega} = u^{\mathrm{I}}_{H^2}|_{\partial\Omega}, \qquad -\Delta\chi u^{\mathrm{II}} - k^2\chi u^{\mathrm{II}} = -\widetilde{f} \quad \text{in } \Omega^c, \qquad \chi u^{\mathrm{II}} \text{ satisfies (4.32b)},$$

$$\|\chi u^{\mathrm{II}}\|_{\mathcal{H},R} \lesssim qk^{-1}\|f\|_{L^2(\Omega^c_R)}, \qquad \|\chi u^{\mathrm{II}}\|_{H^2(\Omega^c_R)} \lesssim \|f\|_{L^2(\Omega^c_R)}.$$

We now set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}}$, $u_{H^2} := u^{\mathrm{I}}_{H^2} + \chi u^{\mathrm{II}}$, which satisfy the desired estimates. The function $\widetilde{u} := u - u_{\mathcal{A}} - u_{H^2} = u - u^{\mathrm{I}}_{\mathcal{A}} - u^{\mathrm{I}}_{H^2} - \chi u^{\mathrm{II}}$ satisfies

$$\widetilde{u}|_{\partial\Omega} = 0, \qquad -\Delta\widetilde{u} - k^2\widetilde{u} = \widetilde{f} \text{ in } \Omega^c, \qquad \widetilde{u} \text{ satisfies (4.32b)}.$$

Hence, $\widetilde{u} = S^c(\widetilde{f}, 0)$. Readjusting the constant q by enlarging η suitably concludes the argument. ∎

LEMMA 4.27 (decomposition of $S^c_k(0,g)$). *Let Assumptions* 4.17 *and* 4.18 *be satisfied. Let $q \in (0,1)$ be given. Then for $g \in H^{3/2}(\partial\Omega)$ the solution $u = S^c_k(0,g)$ can be decomposed as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$ with*

$$\|u_{\mathcal{A}}\|_{\mathcal{H},R} \lesssim k^{\alpha+1}\|g\|_{1/2,\mathcal{H},\partial\Omega},$$
$$\|\nabla^p u_{\mathcal{A}}\|_{L^2(\Omega^c_R)} \lesssim (k+k^\alpha)\gamma^p \max\{p,k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega} \qquad \forall p \ge 2,$$
$$\|u_{H^2}\|_{\mathcal{H},R} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}, \quad \|u_{H^2}\|_{\mathcal{H},R} \le q\|g\|_{3/2,\mathcal{H},\partial\Omega}, \quad \|u_{H^2}\|_{H^2(\Omega^c_R)} \lesssim k\|g\|_{3/2,\mathcal{H},\partial\Omega}.$$

The remainder $\widetilde{u}$ is given by $\widetilde{u} = S^c_k(0,\widetilde{g})$, where the function $\widetilde{g}$ satisfies the bounds $\|\widetilde{g}\|_{3/2,\mathcal{H},\partial\Omega} \le q\|g\|_{3/2,\mathcal{H},\partial\Omega}$ and $\|\widetilde{g}\|_{1/2,\mathcal{H},\partial\Omega} \le q\|g\|_{1/2,\mathcal{H},\partial\Omega}$.

Proof. For $\eta > 1$ chosen below, we split $g = L^D_{\partial\Omega} g + H^D_{\partial\Omega} g$ and define

$$u^{\mathrm{I}}_{\mathcal{A}} := S^c_k(0, L^D_{\partial\Omega} g), \qquad u^{\mathrm{I}}_{H^2} := G^D(H^D_{\partial\Omega} g),$$

where G^D is the trace lifting operator of Lemma 4.22. The function $u^{\mathrm{I}}_{\mathcal{A}}$ satisfies the desired analytic regularity estimates (cf. Lemma 4.25). From (4.41), we get the three estimates

$$\|H^D_{\partial\Omega} g\|_{1/2,\mathcal{H},\partial\Omega} \lesssim \|g\|_{1/2,\mathcal{H},\partial\Omega}, \qquad \|H^D_{\partial\Omega} g\|_{1/2,\mathcal{H},\partial\Omega} \lesssim q\|g\|_{3/2,\mathcal{H},\partial\Omega},$$
$$\|H^D_{\partial\Omega} g\|_{3/2,\mathcal{H},\partial\Omega} \lesssim \|g\|_{3/2,\mathcal{H},\partial\Omega},$$

where the parameter $q \in (0,1)$ depends on the choice of η and is still at our disposal. Lemma 4.22 gives

$$\|u^{\mathrm{I}}_{H^2}\|_{\mathcal{H},R} \lesssim \|H^D_{\partial\Omega} g\|_{1,2,\mathcal{H},\partial\Omega} \lesssim \begin{cases} \|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ q\|g\|_{3/2,\mathcal{H},\partial\Omega}, \end{cases} \tag{4.43a}$$

$$\|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega^c_R)} \lesssim k\|H^D_{\partial\Omega} g\|_{3/2,\mathcal{H},\partial\Omega} \lesssim k\|g\|_{3/2,\mathcal{H},\partial\Omega}. \tag{4.43b}$$

Next, we let $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\operatorname{supp}\chi \subset B_R$ and $\chi \equiv 1$ near $\partial\Omega$ and define $\widetilde{f} := \Delta(\chi u^{\mathrm{I}}_{H^2}) + k^2\chi u^{\mathrm{I}}_{H^2}$. By the definition of G^D we have $-\Delta u^{\mathrm{I}}_{H^2} + u^{\mathrm{I}}_{H^2} = 0$; a calculation using (4.43a) then shows

$$\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim \begin{cases} k\|g\|_{1/2,\mathcal{H},\partial\Omega}, \\ qk\|g\|_{3/2,\mathcal{H},\partial\Omega}. \end{cases} \tag{4.44}$$

Next, the function $u^{\mathrm{II}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + \chi u^{\mathrm{I}}_{H^2})$ satisfies

$$-\Delta u^{\mathrm{II}} - k^2 u^{\mathrm{II}} = \widetilde{f} \quad \text{in } \Omega^c, \qquad u^{\mathrm{II}}|_{\partial\Omega} = 0, \qquad u^{\mathrm{II}} \text{ satisfies (4.32b).}$$

We introduce the function $u^{\mathrm{II}}_{\mathcal{A}} := S^c_k(L_{\Omega^c}\widetilde{f}, 0)$ and $u^{\mathrm{II}}_{H^2} := N_k(H_{\Omega^c}\widetilde{f})$, and get from Lemma 4.24 that $u^{\mathrm{II}}_{\mathcal{A}}$ is analytic with the following bounds for all $p \in \mathbb{N}_0$:

$$\begin{aligned}\|\nabla^p u^{\mathrm{II}}_{\mathcal{A}}\|_{L^2(\Omega^c_R)} &\lesssim k^{\alpha-1}\gamma^p \max\{p,k\}^p \left\| L_{\Omega^c}\widetilde{f}\right\|_{L^2(\Omega^c_R)} \\ &\overset{(4.44)}{\lesssim} k^{\alpha}\gamma^p \max\{p,k\}^p \|g\|_{1/2,\mathcal{H},\partial\Omega}.\end{aligned}$$

The function $u^{\mathrm{II}}_{H^2}$ satisfies, by Lemma 4.5 and (4.44),

$$\|u^{\mathrm{II}}_{H^2}\|_{\mathcal{H},R} \lesssim qk^{-1}\|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim q\|g\|_{1/2,\mathcal{H},\partial\Omega},$$
$$\|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega^c_R)} \lesssim \|\widetilde{f}\|_{L^2(\Omega^c_R)} \lesssim qk\|g\|_{3/2,\mathcal{H},\partial\Omega}.$$

Set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{II}}_{\mathcal{A}}$ and $u_{H^2} := \chi u^{\mathrm{I}}_{H^2} + u^{\mathrm{II}}_{H^2}$. Then $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta\widetilde{u} - k^2\widetilde{u} = 0, \qquad \widetilde{u}|_{\partial\Omega} = -u^{\mathrm{II}}_{H^2}, \qquad \widetilde{u} \text{ satisfies (4.32b).}$$

Therefore, $\widetilde{u} = S^c(0, -u^{\mathrm{II}}_{H^2})$ and, for $s \in \{1/2, 3/2\}$,

$$\|\widetilde{u}\|_{s,\mathcal{H},\partial\Omega} = \|u^{\mathrm{II}}_{H^2}\|_{s,\mathcal{H},\partial\Omega} \lesssim q\|g\|_{s,\mathcal{H},\partial\Omega}.$$

Choosing η sufficiently large such that q is suitably small now concludes the argument. ∎

5. Application to *hp*-finite elements. The present section shows how the regularity theory developed in section 4 is applicable in the context of high order finite element spaces. We proceed in two steps: section 5.1 quantifies $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$ (see Lemma 3.4), and section 5.2 applies these results to the specific examples of section 2.1, 2.2.

5.1. *hp*-FEM approximation results for $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$. This section is devoted to the estimates of the adjoint approximation properties $\eta_{\mathcal{A}}(S)$ and $\eta_{H^2}(S)$ in the case where S is chosen as an *hp*-finite element space.

We have performed the regularity theory in section 4 for domains with analytic boundaries and polygons. These two cases require different types of meshes that we now introduce.

5.1.1. Domains with analytic boundary. We adopt the setting of [17]. The triangulation $\mathcal{T}_h$ consists of elements which are the image of the reference triangle (in two dimensions) or the reference tetrahedron (in three dimensions). We do not allow hanging nodes and assume—as is standard—that the element maps of elements sharing an edge or a face induce the same parametrization on that edge or face. The maximal mesh width is denoted by $h := \max_{K\in\mathcal{T}_h} \operatorname{diam} K$. Additionally, we make the following assumption on the element maps $F_K : \widehat{K} \to K$.

Assumption 5.1 (quasi-uniform regular triangulation). Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and R_K is an analytic map. Furthermore, with the notation $\widetilde{K} := A_K(K)$, the maps R_K and A_K satisfy for constants C_{affine}, C_{metric}, $\gamma > 0$ independent of h

$$\|A_K'\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h, \qquad \|(A_K')^{-1}\|_{L^\infty(\widehat{K})} \le C_{\text{affine}} h^{-1},$$
$$\|(R_K')^{-1}\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}}, \qquad \|\nabla^n R_K\|_{L^\infty(\widetilde{K})} \le C_{\text{metric}} \gamma^n n! \qquad \forall n \in \mathbb{N}_0.$$

Remark 5.2. Triangulations satisfying Assumption 5.1 can be obtained by patchwise construction of the mesh: Let $\mathcal{T}^{macro}$ be a *fixed* triangulation (with curved elements) with analytic element maps that resolves the geometry. If the triangulation $\mathcal{T}_h$ is obtained by quasi-uniform refinements of the reference element $\widehat{K}$ and the final mesh is obtained by mapping the subdivisions of the reference element with the macro element maps, then the resulting element maps satisfy the assumptions of Assumption 5.1.

For meshes $\mathcal{T}_h$ satisfying Assumption 5.1 with element maps F_K we denote the usual space of piecewise (mapped) polynomials by

$$S^{p,1}(\mathcal{T}_h) := \{u \in H^1(\Omega) \,|\, \forall K \in \mathcal{T}_h : u|_K \circ F_K \in \mathcal{P}_p\}, \tag{5.1}$$

where $\mathcal{P}_p$ denotes the space of polynomials of degree p.

PROPOSITION 5.3. *Let $\partial\Omega$ be analytic. Let Assumption* 5.1 *be satisfied. Let the parameter $\gamma > 0$ appearing in the definition of $\mathcal{H}_{\text{osc}}(\gamma, k)$ in* (1.7) *be fixed. Then for $\eta_{\mathcal{A}}$, η_{H^2} introduced in Lemma* 3.4 *there holds*

$$\eta_{H^2}(S) \le C\frac{h}{p}\left(1 + \frac{kh}{p}\right), \qquad \eta_{\mathcal{A}}(S) \le C\left(\left(\frac{h}{h+\sigma}\right)^p + k\left(\frac{kh}{\sigma p}\right)^p\right)\left(1 + \frac{kh}{p}\right),$$

where C, $\sigma > 0$ are independent of k, h, p.

Proof. The proofs of both estimates are simple consequences of the procedure in [49, Thm. 5.5]. ∎

5.1.2. Polygonal domains. For simplicity, we restrict our attention here to a special situation, namely, affine, shape-regular triangulations of the polygon Ω that consist of (a) quasi-uniform triangulations (with mesh size h) away from the vertices and (b) geometric meshes in an $O(h)$-neighborhood of the vertices. We mention already now that $h = O(p/k)$ will be a choice of particular interest. We denote by A_j, $j = 1, \ldots, J$, the vertices of the polygon Ω. The ball with radius ch about A_j is denoted by $B_{ch}(A_j)$.

Assumption 5.4. For $h > 0$, $L \in \mathbb{N}$, $\sigma \in (0, 1)$ the mesh $\mathcal{T}_h(L)$ is an affine, shape-regular triangulation of Ω such that the following hold:

1. The restriction of $\mathcal{T}_h(L)$ to $\Omega\backslash(\cup_{j=1}^{J} B_{ch}(A_j))$ is a quasi-uniform triangulation of that set with mesh size h. Like the shape-regularity constants, the constant c is independent of h, L.
2. For each vertex A_j, the set restriction of $\mathcal{T}_h(L)$ to $B_h(A_j) \cap \Omega$ is a geometric mesh with grading factor $\sigma \in (0, 1)$ and L layers (see, e.g., [61] for the precise definition).

We mention that the smallest elements in the triangulation are those abutting the vertices, and they are of size $h_{min} = O(h\sigma^L)$. Furthermore, the number of elements in $\mathcal{T}_h(L)$ is given by $|\mathcal{T}_h(L)| = O(h^{-2} + L)$.

Remark 5.5. The meshes of Assumption 5.4 are based on a geometric refinement in an $O(h)$-neighborhood of the vertices. The corresponding hp-finite element spaces with suitable choices of p, L, and h (see Theorem 5.8) can be regarded as spaces of (quasi-) minimal dimension which guarantee unique solvability of the arising Galerkin discretizations and quasi-optimal error estimates.

Further enrichments of these finite element spaces merely need to focus on the approximability of the solution u. Good choices of the mesh $\mathcal{T}$ and the polynomial degree p of the enriched space depend on regularity properties of the solution and can be selected either in an a priori or an a posteriori way.

On the geometric meshes of Assumption 5.4, we consider the $S^{p,1}(\mathcal{T}_h(L))$ as defined in (5.1). We have the following approximation results.

Proposition 5.6. *Let $\mathcal{T}_h(L)$ be a triangulation of the polygon Ω that satisfies Assumption* 5.4. *Let the parameters $\gamma > 0$ and $\overrightarrow{\beta} \in [0, 1)^J$ appearing in the definition of $\mathcal{H}_{\mathrm{osc}}(\gamma, k)$ in* (1.7) *be fixed. Assume*

$$\frac{kh}{p} < \widetilde{C}, \tag{5.2a}$$

$$L \geq C'p, \tag{5.2b}$$

for some $\widetilde{C}, C' > 0$. Then for some c, b, $\sigma_0 > 0$ independent of h, k, p there holds with $\beta_{\max} = \max_{j=1,\ldots,J} \beta_j$

$$\eta_{H^2}(S) \leq C\frac{h}{p}, \qquad \eta_{\mathcal{A}}(S) \leq Ck\left((hk)^{1-\beta_{\max}} e^{ckh-bp} + \left(\frac{kh}{\sigma_0 p}\right)^p\right).$$

Proof. Since the meshes $\mathcal{T}_h(L)$ are finer than quasi-uniform meshes with mesh size h, the bound for $\eta_{H^2}(S)$ follows by standard arguments.

To see the bound for $\eta_{\mathcal{A}}(S)$, we apply the approximation theory of [44, Chap. 3]. Let $u \in \mathcal{H}_{\mathrm{osc}}(\gamma, k)$, and define the approximation $v \in S^{p,1}(\mathcal{T}_h(L))$ elementwise with the aid of the operator Π_p^∞ of [44, Thm. 3.2.20]: $v|_K \circ F_K := \Pi_p^\infty(u \circ F_K)$, where F_K is the element map for K. We note that the elements of $\mathcal{T}_h(L)$ can be divided into two categories, namely, those belonging to a geometric mesh near the vertices, $\mathcal{T}_j^{geo}$, $j = 1, \ldots, J$, and those in a quasi-uniform mesh $\mathcal{T}_h^{unif}$ of mesh size h.

Let us first consider the error $u-v$ near the vertices. Let S be a fixed sector with apex A_j, where A_j is a vertex of the polygon Ω. In the notation of [44, Chap. 3], the assumption $u \in \mathcal{H}_{\rm osc}(\gamma,k)$ means $u \in \mathcal{B}^2_{\beta,1/k}(S,C_u,\gamma)$, where $C_u = O(1)$. Then, [44, Lemma 3.4.7] gives (inspection of the proof of [44, Lem. 3.4.7] shows that it is applicable with $H = O(h)$)

$$\sum_{K\in\mathcal{T}_j^{geo}} \|u-v\|^2_{\mathcal{H},K} \leq \left(1+\frac{k^2h^2}{p^4}\right) \sum_{K\in\mathcal{T}_j^{geo}} p^4 \|u-v\|^2_{L^\infty(K)} + \|\nabla(u-v)\|^2_{L^2(K)}$$

$$\lesssim \left(1+\frac{k^2h^2}{p^4}\right) k^2 \left\{(hk)^{2-2\beta_j}\, \mathrm{e}^{chk-bp} + p^7 \left(hk\sigma^L\right)^{2-2\beta_j}\right\}, \tag{5.3}$$

where we applied Hölder's inequality for the first estimate. The constant $b>0$ is independent of h, k, and p. The factor $(1+k^2h^2/p^4)$ can be bounded in view of the assumption (5.2a) and $p \geq 1$. Next, in view of the assumption on L in (5.2b), we arrive at

$$\sum_{K\in\mathcal{T}_j^{geo}} \|u-v\|^2_{\mathcal{H},K} \lesssim k^2 (kh)^{2(1-\beta_j)} e^{ckh-bp}, \tag{5.4}$$

where we suitably adjusted the constant $b>0$. This is the desired estimate for the elements in $\mathcal{T}_j^{geo}$. For the remaining elements in $\mathcal{T}_h^{unif}$, we proceed by standard approximation arguments as follows. For each $K \in \mathcal{T}_h^{unif}$ set

$$C_K^2 := \sum_{n\geq 0} \left(\frac{1}{2\gamma \max\{k,n\}}\right)^{2(n+2)} \|\Phi_{n,\vec{\beta},k} \nabla^{n+2} u\|^2_{L^2(K)}.$$

Then, $\sum_{K\in\mathcal{T}_h(L)} C_K^2 \leq 2$. Consider an element K with $\mathfrak{d} := \operatorname{dist}(K,A_j) \geq ch$ for all vertices A_j. Abbreviate $\beta := \beta_{\max}$. Then, for all $n \in \mathbb{N}_0$ (cf. [44, Lem. 4.2.2])

$$\|\nabla^{n+2}u\|_{L^2(K)} \leq C_K (2\gamma)^{n+2} \max\{n,k\}^{n+2} \left(\max\left\{1, \frac{\min\{1,\frac{n+1}{k+1}\}}{\mathfrak{d}}\right\}\right)^{n+\beta} =: RHS. \tag{5.5}$$

By distinguishing the three cases (a) $n \geq k$, (b) $n \leq k$ together with $n+1 \leq (k+1)\mathfrak{d}$, and (c) $n \leq k$ together with $n+1 > (k+1)\mathfrak{d}$, we arrive at

$$RHS \lesssim C_K \min\{1,k\mathfrak{d}\}^{2-\beta} (2\gamma)^{n+2} \max\{k, n/\mathfrak{d}\}^{n+2} \quad \forall n \in \mathbb{N}_0.$$

Combining now [49, Lem. C.2] with [44, Thm. 3.2.20] gives the existence of some C, $\sigma_0 > 0$ such that for $q \in \{0,1\}$

$$h^q \|u-v\|_{H^q(K)} \leq C C_K \min\{1,k\mathfrak{d}\}^{2-\beta} \left(\left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p+1} + \left(\frac{kh}{\sigma_0 p}\right)^{p+1}\right). \tag{5.6}$$

We distinguish the cases $\mathfrak{d} \geq 1/k$ and $\mathfrak{d} < 1/k$. For $\mathfrak{d} \geq 1/k$ we have in view of $\mathfrak{d} \geq ch$

$$\left(k+h^{-1}\right) \min\{1,k\mathfrak{d}\}^{2-\beta} \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/d}\right)^{p+1} = \left(k+h^{-1}\right) \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p+1}$$

$$\lesssim k \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p} \lesssim k \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{1-\beta} \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p-1+\beta}$$

$$\lesssim k \min\{1, h/\mathfrak{d}\}^{1-\beta} \left(\frac{1}{c\sigma_0 + 1}\right)^{p} \lesssim k \min\{1, hk\}^{1-\beta} \left(\frac{1}{c\sigma_0+1}\right)^{p}, \tag{5.7}$$

where we additionally exploited the monotonicity properties of $x \mapsto (x/(\sigma_0+x))^{p-1+\beta}$ and $p \geq 1$ together with $\beta \geq 0$. For the case $h \lesssim \mathfrak{d} < 1/k$ we have

$$(k+h^{-1}) \min\{1, k\mathfrak{d}\}^{2-\beta} \lesssim h^{-1}(k\mathfrak{d})^{2-\beta} = k(kh)^{1-\beta} \left(\frac{\mathfrak{d}}{h}\right)^{2-\beta}$$
$$\lesssim k \min\{1, kh\}^{1-\beta} \left(\frac{\mathfrak{d}}{h}\right)^{2-\beta}.$$

Exploiting again the monotonicity properties of $x \mapsto (x/(1+x))^{p-1+\beta}$ together with $p \geq 1$, $\beta \geq 0$, and $\mathfrak{d} \geq ch$, we conclude also for the case $\mathfrak{d} < 1/k$

$$(5.8) \qquad \left(k+h^{-1}\right) \min\{1, k\mathfrak{d}\}^{2-\beta} \left(\frac{h/\mathfrak{d}}{\sigma_0 + h/\mathfrak{d}}\right)^{p+1} \lesssim k \min\{1, kh\}^{1-\beta} \left(\frac{1}{c\sigma_0+1}\right)^p.$$

Inserting the estimates (5.7), (5.8) into (5.6), we get

$$k\|u-v\|_{L^2(K)} + |u-v|_{H^1(K)} \lesssim C_K k \left(\min\{1, kh\}^{1-\beta} \left(\frac{1}{c\sigma_0+1}\right)^p + \left(\frac{kh}{\sigma_0 p}\right)^p\right).$$

By summing over all elements K that are in the quasi-uniform mesh $\mathcal{T}_h^{unif}$ and recalling that $\sum_{K\in\mathcal{T}} C_K^2 \leq 2$, we obtain

$$(5.9) \qquad \sqrt{\sum_{K\in\mathcal{T}_h^{unif}} \|u-v\|^2_{\mathcal{H},K}} \lesssim k \left(\min\{1, kh\}^{1-\beta} \left(\frac{1}{c\sigma_0+1}\right)^p + \left(\frac{kh}{\sigma_0 p}\right)^p\right).$$

Combining (5.4) with (5.9) and appropriately adjusting constants proves the claim of the proposition. ∎

5.2. Stability and convergence analysis of *hp*-FEM for the model problems of section 2. In view of the oscillatory nature of solutions of Helmholtz problems, it is reasonable to expect that a minimal condition for stability is that the dimension N of the ansatz space has to satisfy $N = O(k^d)$. The next theorem shows that, indeed, the polynomial degree p and the mesh size h can be selected such that the resulting approximation space has dimension $N = O(k^d)$ and at the same time ensures quasi-optimality of the Galerkin FEM.

Since we will refer to the same hypotheses several times in this section, we formulate them as an assumption, as follows.

Assumption 5.7.

(1) If the model problem of section 2.1 (cf. (2.3)) is considered, then Assumptions 1.1 and 4.8 are valid and the data satisfy $f \in L^2(\Omega)$ and $g \in H^{1/2}_{pw}(\partial\Omega)$. The discrete formulation is (2.4). If Ω has an analytic boundary, then the approximation space S described in section 5.1.1 is used; if Ω is a polygon, then the *hp*-FEM space S described in section 5.1.2 is employed with the additional assumption $L = O(p)$.

(2) If the exterior Dirichlet problem (2.8) is considered, Assumptions 4.17 and 4.18 are valid. The data satisfy $f \in L^2(\Omega_R^c)$ and $g = 0$ on $\partial\Omega$.[1] The Galerkin

[1]The assumption $g = 0$ is made here to avoid further consistency estimates. Note that, for the analysis, we assumed $g \in H^{3/2}(\partial\Omega)$, which can be transformed in the standard way to the case of homogeneous boundary conditions by a trace lifting of g to some function $u_g \in H^2(\Omega_R^c)$ and then modifying the right-hand side f.

method (2.9) with $g_S = 0$ is based on the $V_{R,0}$-conforming subspace of the hp-FEM spaces described in section 5.1.1. The DtN-operator T_R is assumed to be realized exactly.

THEOREM 5.8 (discrete stability of hp-FEM). *Assume the set-up of Assumption 5.7 and $k > k_0 > 1$. Then there exist constants δ, $\widetilde{C} > 0$ that are independent of h, p, and k such that the conditions*

$$\frac{kh}{p} \leq \delta \quad \text{and} \quad p \geq 1 + \tilde{C} \log k \tag{5.10}$$

imply the following for constants C_b and C that are independent of h, p, k, and f, g:

(i) *The discrete problem* (2.4) *has a unique solution and*

$$\|u - u_S\|_{\mathcal{H},\Omega} \leq 2(1 + C_b) \inf_{v\in S} \|u - v\|_{\mathcal{H},\Omega},$$

$$\|u - u_S\|_{L^2(\Omega)} \leq C \frac{h}{p} \inf_{v\in S} \|u - v\|_{\mathcal{H},\Omega}.$$

(ii) *The discrete problem* (2.9) *has a unique solution and*

$$\|u - u_S\|_{\mathcal{H},\Omega_R^c} \leq 2(1 + C_b) \inf_{v\in S} \|u - v\|_{\mathcal{H},\Omega_R^c},$$

$$\|u - u_S\|_{L^2(\Omega_R^c)} \leq C \frac{h}{p} \inf_{v\in S} \|u - v\|_{\mathcal{H},\Omega_R^c}.$$

Proof. In the interest of brevity, we will not consider the case of geometric meshes but restrict our attention to the cases where quasi-uniform meshes that satisfy Assumption 5.1 are appropriate. From Lemma 3.4 and Proposition 5.3 we conclude

$$\eta(S) \leq C \left\{ C_{k,\mathcal{A}} \left(\left(\frac{h}{h + \sigma_0} \right)^p + k \left(\frac{kh}{\sigma_0 p} \right)^p \right) + C_{H^2} \frac{h}{p} \right\} \left(1 + \frac{kh}{p} \right). \tag{5.11}$$

Assumption 5.7 implies via Theorem 4.10 for the case of a bounded domain with Robin boundary conditions or via Theorem 4.20 for the case of an exterior Dirichlet problem that the constants $C_{k,\mathcal{A}}$, C_{H^2} may be assumed to have the form

$$C_{k,\mathcal{A}} = Ck^{\alpha-1} \quad \text{and} \quad C_{H^2} = C, \tag{5.12}$$

where C is independent of k, h, p. By Lemma 3.4, the stability condition (3.12) is therefore satisfied if

$$k^\alpha \left(\frac{h}{h + \sigma_0} \right)^p + k^{\alpha+1} \left(\frac{kh}{\sigma_0 p} \right)^p + \frac{hk}{p} \leq \rho \tag{5.13}$$

for some $\rho > 0$ that is independent of k, h, p. Without loss of generality, we require $\rho < 1$. By selecting δ sufficiently small, we can ensure $kh/p \leq \delta \leq \rho/3$ and $kh/(\sigma_0 p) \leq \delta/\sigma_0 \leq 1/2$. Finally, since the computational domain is bounded, we have $h/(\sigma_0 + h) \leq \theta < 1$. Therefore, the left-hand side of (5.13) can be bounded by $k^\alpha \theta^p + k^{\alpha+1} 2^{-p} + \rho/3$. This can be bounded by ρ if $p \geq \widetilde{C} \log k$ for sufficiently large $\widetilde{C}$. ∎

Remark 5.9. Let $k > k_0 > 1$. Selecting

$$p := 1 + \left\lceil \tilde{C} \log k \right\rceil, \qquad h := \frac{\delta p}{k}$$

(and $L = O(p)$ for the case of polygons) for the constants δ, $\widetilde{C} > 0$ of Theorem 5.8, we see that stability of the Galerkin method can be ensured with hp-FEM spaces of dimension $N := \dim S \sim (p/h)^d \sim k^d$. In other words, stability is given with a fixed number of degrees of freedom per wavelength.

Let us compare this with the lowest order FEM, i.e., the choice $p = 1$. In this case, the requirement (5.13) reads

$$k^\alpha h + k^{\alpha+1}(kh) + hk \leq \rho.$$

Even assuming $\alpha = 0$, this condition leads to the condition $k^2 h \lesssim 1$, so that the minimal number of unknowns of the $\mathcal{P}_1$-finite element space is $\dim S^{1,1} = O(k^{2d})$. This illustrates the substantial savings for the choice $p \approx c \log k$ (with c sufficiently large) over the lowest order case $p = 1$.

We close by stating convergence results for the hp-FEM applied to our model problems (I)—(III) listed in the introduction; we refer to sections 2.1 and 2.2 for their precise formulation. We restrict ourselves to the case $f \in L^2(\Omega)$ and $g \in H^{1/2}_{pw}(\partial\Omega)$.

COROLLARY 5.10. *Let Assumption 5.7 be satisfied. Abbreviate* $C_{f,g} := \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}_{pw}(\partial\Omega)}$. *Then there are constants* σ_0, $\tilde{C}$, $\delta > 0$ *such that the following hold:*

(I) *Let* Ω *be bounded with analytic boundary* $\partial\Omega$. *Let* u *solve* (2.3).
- *If* $p \geq 1 + \tilde{C}\log k$, *then the condition* $kh/p \leq \delta$ *implies the existence of the discrete solution* u_S *of* (2.4) *and the a priori estimate*

$$\|u - u_S\|_{\mathcal{H},\Omega} \lesssim C_{f,g}\frac{h}{p}. \tag{5.14a}$$

- *If* $p = O(1)$ *(i.e.,* p *is fixed independent of* h *and* p*), then the more restrictive condition* $k^{1+\omega}h \leq \delta$ *with* $\omega = \frac{\alpha+1}{p}$ *implies the existence of the discrete solution* u_S *and the a priori estimate*

$$\|u - u_S\|_{\mathcal{H},\Omega} \lesssim C_{f,g}\left(h + k^\alpha (kh)^p\right). \tag{5.14b}$$

(II) *Let* Ω *be a convex polygon. Let* u *solve* (2.3) *and assume hypothesis* (5.2). *Let* $\beta_{\max} \in (0,1)$ *be as in Proposition* 5.6, *which can—according to Remark* 4.11*—be chosen arbitrarily small.*
- *If* $p \geq 1 + \tilde{C}\log k$, *then the condition* $kh/p \leq \delta$ *implies the existence of the discrete solution* u_S *of* (2.4) *and the a priori estimate*

$$\|u - u_S\|_{\mathcal{H},\Omega} \lesssim C_{f,g}\left(\frac{h}{p}\right)^{1-\beta_{\max}}. \tag{5.14c}$$

- *If* $p = O(1)$ *(i.e.,* p *is fixed independent of* h *and* k*), then the condition* $k^{\alpha+1}(hk)^{1-\beta_{\max}} \leq \delta$ *implies the existence of the discrete solution* u_S *and*

$$\|u - u_S\|_{\mathcal{H},\Omega} \lesssim C_{f,g}(hk)^{1-\beta_{\max}} k^\alpha. \tag{5.14d}$$

(III) *For the exterior Dirichlet problem* (2.8), *the following hold:*
- *If* $p \geq 1 + \tilde{C}\log k$, *then the condition* $kh/p \leq \delta$ *implies the existence of the discrete solution* u_S *of* (2.9) *and the a priori estimate*

$$\|u - u_S\|_{\mathcal{H},\Omega_R^c} \lesssim C_{f,0}\frac{h}{p}. \tag{5.14e}$$

- *If $p = O(1)$ (i.e., p is fixed independent of h and k), then the condition $k^{1+\omega}h \leq \delta$, again with $\omega = \frac{\alpha+1}{p}$, implies the existence of the discrete solution u_S and the a priori estimate*

$$\|u - u_S\|_{\mathcal{H},\Omega_R^c} \lesssim C_{f,0}\left(h + k^{\alpha}(kh)^p\right). \tag{5.14f}$$

Appendix. Lifting.

LEMMA A.1. *Let $\Omega \subset \mathbb{R}^2$ be a polygon whose internal angles are different from 0, π, and 2π. Then there exists a linear operator $G : H^{1/2}_{pw}(\partial\Omega) \to H^2(\Omega)$ with $\partial_n G = g$ and $\|G\|_{H^2(\Omega)} \leq C\|g\|_{H^{1/2}_{pw}(\partial\Omega)}$.*

Proof. In the interest of brevity, we base the proof on the solvability theory in convex polygons.

Step 1. Let T be a (convex) triangle. Then one can infer from [25, Cor. 4.4.3.8] the existence of $C_T > 0$ such that for every $g \in H^{1/2}_{pw}(\partial T)$ with $\int_{\partial T} g = 0$ there holds for the solution $u \in H^2(T)$ of

$$-\Delta u = 0 \quad \text{in } T, \qquad \partial_n u = g \quad \text{on } \partial T, \qquad \int_T u = 0$$

the a priori bound $\|u\|_{H^2(T)} \leq C_T\|g\|_{H^{1/2}_{pw}(\partial T)}$.

Step 2. Let $S = \{(r\cos\varphi \,|\, r\sin\varphi) \,|\, 0 < r < 2\delta, 0 < \varphi < \omega\}$ with edges Γ_1, Γ_2 meeting at the origin. Set $\Gamma_{1,\delta} := \{(r,0) \,|\, 0 < r < \delta\}$, $\Gamma_{2,\delta} := \{(r\cos\omega, r\sin\omega) \,|\, 0 < r < \delta\}$.

For the case of a convex sector, i.e., $0 < \omega < \pi$, it is easy to construct with the aid of the first step a bounded linear operator $L : \prod_{i=1}^2\{u \in H^{1/2}(\Gamma_i) \,|\, \operatorname{supp} u \subset \overline{\Gamma_{i,\delta}}\} \to H^2(S)$ with $(\partial_n L(f_1, f_2))|_{\Gamma_i} = f_i$ $(i \in \{1,2\})$ $\|L(f_1,f_2)\|_{H^2(S)} \leq C\sum_{i=1}^2 \|f_i\|_{H^{1/2}(\Gamma_i)}$.

For the case of a nonconvex sector, i.e., $\pi < \omega < 2\pi$, let $S' := B_{2\delta} \setminus S$ and let $E : H^2(S') \to H^2(\mathbb{R}^2)$ be the extension operator of Stein [62, Chap. VI]. Then S' is a convex sector of the form considered above. Then it is easy to check that $(E(L(-f_1, -f_2)))|_{S'} \in H^2(S')$ has the desired lifting property for S.

Step 3. Localizing with the aid of partitions of unity, we can reduce the construction of the lifting to the question of liftings from an infinite line to a half space, and from two edges that meet at a common vertex V to the enclosed sector. The first case is well known (see, e.g., [25, Thm. 1.5.1.2]). The second case is covered by Step 2. ∎

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