Equations for the Ramification Loci of Outer Simple Linear Projections

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Abstract

Let $\widetilde{Z} \subseteq \mathbb{P}^n_K$ be a projective variety in the projective n-space over an algebraically closed field K of characteristic 0, and let $\widetilde{\pi} : \mathbb{P}^n_K \backslash \{\mathfrak{p}\} \to \mathbb{P}^{n-1}_K$ be a simple linear projection with centre $\mathfrak{p} \in \mathbb{P}^n_K \backslash \widetilde{Z}$. Denote $\pi : \widetilde{Z} \to Z := \widetilde{\pi}(\widetilde{Z}) \subseteq \mathbb{P}^{n-1}_K$ the restriction of $\widetilde{\pi}$ to $\widetilde{Z}$. Given a suitable linear embedding $\mathbb{P}^{n-1}_K \hookrightarrow \mathbb{P}^n_K$ avoiding $\mathfrak{p}$, the fibre $\pi^{-1}(\mathfrak{q})$ over a closed point $\mathfrak{q} \in Z$ can be considered as an effective divisor on the projective line spanned by $\mathfrak{q}$ and $\mathfrak{p}$, i.e., $\pi^{-1}(\mathfrak{q}) \cong \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$ for some integers $\lambda_1, \ldots, \lambda_e \in \mathbb{N}$ and distinct closed points $\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e \in \mathbb{P}^1_K$. The isomorphism class of the scheme $\pi^{-1}(\mathfrak{q})$ is determined by the integers $\lambda_1, \ldots, \lambda_e$. For a given partition $\lambda = (\lambda_1, \ldots, \lambda_e)$ of an integer $k \in \mathbb{N}$, the proper λ-ramification locus Z°_λ of π is the set of closed points of Z whose fibres are of isomorphism class determined by λ. In this thesis, we give equations describing the locally closed sets Z°_λ depending on equations defining $\widetilde{Z}$ and $\mathfrak{p}$.

To this end, we first show how to find equations defining the closed set Z_k of closed points of Z whose fibres are of length $\geqslant k$. These equations for Z_k are given by partial elimination ideals, which in turn can be computed using Gröbner bases. This yields an algorithm for computing Z_k as well as secant cones and secant loci. Also, we show that under certain conditions partial elimination ideals describe the length of the fibres of a multiple projection similarly as they do for simple projections.

We further study the coincident root locus X_λ, that is the projective variety of bilinear forms whose linear factors occur with multiplicities $\lambda_1, \ldots, \lambda_e$. We compute minimal sets of local generators of the fibre product of X_λ with its normalization, and we use these generators to find equations defining X_λ. Moreover, we use the explicitly computed local generators to overcome a gap in Chipalkatti's seminal paper 'On equations defining Coincident Root Loci' ([Ch1]). Then, we describe the singular locus of X_λ by comparing partitions and their coarsenings.

Putting together the previous results, we finally get equations describing Z°_λ in a way suited for explicit computations. We also obtain an open covering of $Z_k \backslash Z_{k+1}$ describing the behaviour of the fibres $\pi^{-1}(\mathfrak{q})$ via the elements of a Gröbner basis of the defining ideal of $\widetilde{Z}$.

We illustrate all our results with examples computed in [SINGULAR].

Zusammenfassung

Sei $\widetilde{Z} \subseteq \mathbb{P}^n_K$ eine projektive Varietät im n-dimensionalen projektiven Raum über einem algebraisch abgeschlossenen Körper K der Charakteristik 0, und sei $\widetilde{\pi} : \mathbb{P}^n_K \to \mathbb{P}^{n-1}_K$ eine einfache lineare Projektion von $\mathfrak{p} \in \mathbb{P}^n_K \backslash \widetilde{Z}$ aus. Es bezeichne $\pi : \widetilde{Z} \to Z := \widetilde{\pi}(\widetilde{Z}) \subseteq \mathbb{P}^{n-1}_K$ die Einschrängung von $\widetilde{\pi}$ auf $\widetilde{Z}$. Mit einer geeignete linearen Einbettung von $\mathbb{P}^{n-1}$ in $\mathbb{P}^n$ können wir die Faser $\pi^{-1}(\mathfrak{q})$ über einem abgeschlossenen Punkt $\mathfrak{q} \in Z$ als effetiven Divisor auf der projektiven Gerade, welche von $\mathfrak{q}$ und $\mathfrak{p}$ aufgespannt wird, auffassen, das heisst, $\pi^{-1}(\mathfrak{q}) \cong \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$ mit natürlichen Zahlen $\lambda_1, \ldots, \lambda_e \in \mathbb{N}$ und paarweise verschiedenen abgeschlossenen Punkten $\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e \in \mathbb{P}^1_K$. Die Isomorphieklasse von $\pi^{-1}(\mathfrak{q})$ wird durch $\lambda_1, \ldots, \lambda_e$ bestimmt. Ist $\lambda = (\lambda_1, \ldots, \lambda_e)$ eine gegebene Partition einer natürlichen Zahl $k \in \mathbb{N}$, so ist der eigentliche λ-Verzweigungsort Z°_λ von π die Menge der abgeschlossenen Punkte $\mathfrak{q}$ von Z, so dass die Isomrophieklassen der Fasern $\pi^{-1}(\mathfrak{q})$ durch λ bestimmt sind. Ziel dieser Dissertation ist es, abhängig von Gleichungen für $\widetilde{Z}$ und $\mathfrak{p}$ Gleichungen anzugeben, welche die lokal abgeschlossene Menge Z°_λ beschreiben.

Dazu erklären wir zuerst, wie man Gleichungen findet für die abgeschlossene Menge Z_k der abgeschlossenen Punkten von Z, deren Fasern mindestens Länge k besitzen. Diese Gleichungen für Z_k sind gegeben durch partielle Eliminationsideale, welche wiederum mit Hilfe von Gröbnerbasen berechnet werden können. So erhalten wir einen Algorithmus, mit dem Z_k ebenso berechnet werden kann wie Sekantenkegel und Sekantenörter. Wir zeigen ausserdem, wie unter gewissen Voraussetzung partielle Eliminationsideale die Fasern von mehrfachen Projektionen bestimmen, ähnlich wie bei einfachen Projektionen.

Weiter studieren wir den Ort der zusammenfallenden Nullstellen X_λ, das heisst die projektive Varietät von Binärformen, die als Produkt von Linearfaktoren mit Potenzen $\lambda_1, \ldots, \lambda_e$ geschrieben werden können. Wir berechnen minimale Mengen von lokalen Erzeugern des Faserprodukts des Orts der zusammenfallenden Nullstellen mit seiner Normalisierung, womit wir Gleichungen für X_λ bestimmen und eine Lücke in Chipalkattis Arbeit 'On equations defining Coincident Root Loci' ([Ch1]) schliessen können. Schliesslich beschreibenr wir den singulären Ort des Orts der zusammenfallenden Nullstellen durch Partitionen und deren Vergröberungen.

Bringen wir all diese Ergebnisse zusammen, finden wir die gesuchten Gleichungen, welche Z°_λ beschreiben, und zwar auf eine Art, welche für explizite Berechnungen geeignet ist. Ebenso erhalten wir eine Überdeckung von $Z_k \backslash Z_{k+1}$, welche mittels endlich vieler Elemente einer Gröbnerbasis von $\widetilde{Z}$ das Verhalten der Fasern über $Z_k \backslash Z_{k+1}$ beschreibt.

Wir erläutern alle Ergebnisse anhand von Beispielen, welche mit [Singular] berechnet wurden.

Contents

Chapter 0

Introduction

0.1 Background and motivation: Classification of projective varieties and the ramification of outer simple projections

A fundamental method in mathematics is to study an unknown and complicated subject by tracing it back to a well-known and (relatively) easy entity. In Algebraic Geometry, a classical instance of this method is the attempt to describe a non-degenerate projective variety Z in a projective $(n-1)$-space $\mathbb{P}_K^{n-1}$ over an algebraically closed field of characteristic zero as the image of a non-degenerate projective variety $\widetilde{Z} \subseteq \mathbb{P}_K^n$ under a simple outer projection $\widetilde{\pi} : \mathbb{P}_K^n \backslash \{\mathfrak{p}\} \to \mathbb{P}_K^{n-1}$ from a closed point $\mathfrak{p} \in \mathbb{P}_K^n \backslash \widetilde{Z}$.
Of course, this approach can only bear fruits if we know more about $\widetilde{Z}$ than about Z. This is usually no restrictive assumption. Indeed, the Δ-genus $\Delta(\bullet) := \deg(\bullet) - \operatorname{codim}(\bullet) - 1$ grows by a simple outer projection (for a treatment of the Δ-genus, see [Fu3]). For example, if the induced morphism

$$\pi : \widetilde{Z} \to Z = \widetilde{\pi}(\widetilde{Z}) \subseteq \mathbb{P}_K^{n-1}$$

is birational, it holds $\Delta(Z) = \Delta(\widetilde{Z}) + 1$. In general, this approach allows us to trace the projective variety Z back to a variety with smaller Δ-genus, operating on the principle that varieties of small Δ-genus are simple in some way. The results about the classification of non-degenerate varieties of small Δ-genus validate this principle: The smallest possible Δ-genus is 0, since for any non-degenerate projective variety $V \subseteq \mathbb{P}_K^n$ it holds $\deg(V) \geqslant \operatorname{codim}(V) + 1$. Non-degenerate projective varieties of Δ-genus 0, called varieties of minimal degree, have been completely classified by Bertini in 1907 (see [EHr], [Fu3], or [P]). They are either the projective space $\mathbb{P}_K^n$ itself, a quadric hypersurface, (a cone over) the Veronese surface in $\mathbb{P}_K^5$, or (a cone over) a smooth rational normal scroll.
The next case is that of Δ-genus 1, that is of varieties of almost minimal degree. Fujita shows in [Fu1] that a non-degenerate projective variety Z of Δ-genus 1 either is a normal del Pezzo variety, for which case he has a classification theory (compare [Fu2]), or it is the image of a variety $\widetilde{Z}$ of minimal degree under a simple outer projection $\widetilde{\pi} : \mathbb{P}_K^n \backslash \{\mathfrak{p}\} \to \mathbb{P}_K^{n-1}$. In the latter case, we can

understand Z by studying the projection $\pi : \widetilde{Z} \to Z$ and applying the known results about varieties of minimal degree (compare, for example, [BrS1] or [P]).
The next task, then, is the study of the finite morphism $\pi : \widetilde{Z} \to Z$. In general, morphisms of projective varieties are neither easily understood nor completely classified; but in the case of simple outer projections, we find ourselves in a more promising situation. Already the structure of the fibres $\pi^{-1}(\mathfrak{q})$ over the closed points $\mathfrak{q} \in Z$ tells us much about the behaviour of π and, consequently, about Z. For some recent work on fibres of (not necessarily simple) general linear projections, see for example [BeE1] and [BeE2]. In this work, we utilize the crucial property of the fibres of simple outer linear projections that they are subschemes of projective lines of finite length.
Said property allows an in-depth study of the ramification behaviour of π: We consider $\mathbb{P}^{n-1}_K$ as a linear subspace of $\mathbb{P}^n_K$ avoiding $\mathfrak{p}$; this embedding is not unique, but any linear embedding with $\mathfrak{p} \notin \mathbb{P}^{n-1}_K$ yields the same situation. For any closed point $\mathfrak{q} \in Z$, the fibre $\pi^{-1}(\mathfrak{q})$ is an effective divisor on the projective line $\langle \mathfrak{q}, \mathfrak{p} \rangle = \mathbb{P}^1_K$ spanned by $\mathfrak{q}$ and $\mathfrak{p}$ in $\mathbb{P}^n_K$. Hence, the fibre over $\mathfrak{q}$ is of the form

$$\pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$$

for some distinct closed points $\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e \in \mathbb{P}^1_K$ and positive integers $\lambda_1, \ldots, \lambda_e \in \mathbb{N}$. The partition $\lambda = (\lambda_1, \ldots, \lambda_e) \in \mathbb{N}^e_K$ of

$$k := \operatorname{length}_{\mathcal{O}_{\mathbb{P}^1_K}}(\mathcal{O}_{\pi^{-1}(\mathfrak{q})}) = \sum_{i=1}^{e} \lambda_e$$

now already determines the isomorphy class of the scheme $\pi^{-1}(\mathfrak{q})$ (but, of course, not the divisor $\pi^{-1}(\mathfrak{q}) \subseteq \mathbb{P}^1_K$ itself). Hence, the ramification behaviour of π is completely described by the strata

$$Z^\circ_\lambda := \left\{ \mathfrak{q} \in Z \text{ closed point} \;\middle|\; \begin{array}{l} \pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i \text{ for distinct} \\ \text{closed points } \widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e \in \langle \mathfrak{q}, \mathfrak{p} \rangle \end{array} \right\}$$

of closed points $\mathfrak{q} \in Z$ whose fibres yield the partition λ. Note that we indeed use a specific property of simple projections; it is not possible to describe the isomorphy class of the fibres of multiple projections by finitely many numerical invariants as in our case.
We now change our point of view: Starting with the projective variety $\widetilde{Z} \subseteq \mathbb{P}^n_K$, we study $Z = \widetilde{\pi}(\widetilde{Z})$ in relation to the projection centre $\mathfrak{p}$. This leads us to the classical question of how the projected image Z depends on the projection centre. In this work, we want to answer this question for the case of the ramification behaviour in an explicit way, that is we want to give a method to find a complete set of equations defining a certain closed set Z_λ containing Z°_λ as a locally closed subset as well as equations defining the complement $Z_\lambda \backslash Z^\circ_\lambda$. This set is

$$Z_\lambda = \bigcup_{\mu:\lambda \text{ refinement of } \mu} Z^\circ_\mu \cup \{\mathfrak{q} \in Z \text{ closed point} \mid \operatorname{length}(\pi^{-1}(\mathfrak{q}) > k\},$$

the union of all Z°_μ, where μ runs over all partitions of k such that λ is a refinement of μ – we call such a partition μ a *coarsening of* λ –, with the set of closed points of Z whose fibres are of length greater than k. We need this enlargement of Z°_λ to get a closed set. The subvariety Z_λ of Z is *the λ-ramification locus of π* occurring in the title of this work, while we call the locally closed set Z°_λ *the proper λ-ramification locus of π*. So, our goal is of a mostly algorithmic flavour even as it arises from a geometric question. The key to our algebraic considerations will be the mentioned fact that a fibre $\pi^{-1}(\mathfrak{q})$ is a subscheme of $\mathbb{P}^1_K$ of finite length, so that it can be described by a binary form $\mathfrak{f} \in K[x, y]$. A more geometric approach to the study of ramification loci can be found in [GrPe].

In order to find the equations defining the ramification loci, we first give equations defining the subvariety

$$Z_k := \{\mathfrak{q} \in Z \text{ closed point} \mid \text{length}(\pi^{-1}(\mathfrak{q}) \geqslant k\}$$

of closed points $\mathfrak{q}$ of Z such that $\text{length}(\pi^{-1}(\mathfrak{q})) \geqslant k$ for $k \in \mathbb{N}_0$. This will allow us to find finitely many equations determining the equations $\mathfrak{f}$ of all the fibres $\pi^{-1}(\mathfrak{q})$, and it also yields a method to determine equations defining secant cones and secant loci: The k-secant cone of $\widetilde{Z}$ with respect to $\mathfrak{p}$ is the union of all lines running through $\mathfrak{p}$ intersecting $\widetilde{Z}$ at least with length k, and the corresponding k-secant locus is the intersection of the k-secant cone with $\widetilde{Z}$. Those notions have made several appearances in recent works in Algebraic Geometry classifying projective varieties. For example, in [BrS1] Brodmann and Schenzel show that the cohomological and local properties of a non-del Pezzo variety $Z = \pi(\widetilde{Z})$ of almost minimal degree are governed by the 2-secant locus of the projected variety $\widetilde{Z}$ with respect to the projection centre, and Brodmann and Park give a complete stratification of all possible secant loci in this context in [BrP] (compare Example 3.5.3). Another occurrence of secant cones and secant loci (there called entry loci) can be found in recent works of Ionescu and Russo. In [R] and [IR1], they study irreducible projective varieties with quadratic secant loci (with respect to a general point in the secant variety). They further show the relation between varieties with quadratic secant loci and dual defective manifolds and the Hartshorne Conjecture in [IR2]. Hence, determining equations for secant cones and secant loci is a topic of interest in itself, and it was the subject of my recent article [Km]. As the equations for Z_k play an important rôle in determining those of Z_λ, we repeat [Km] here in Chapter 3 with only some small changes to better fit it into our present context.

The integers $\lambda_1, \ldots, \lambda_e$ arising from a fibre $\pi^{-1}(\mathfrak{q})$ equal the powers of the distinct linear factors of the binary form $\mathfrak{f}$ determining $\pi^{-1}(\mathfrak{q})$. Thus, in order to determine Z_λ, we also have to study the classic invariant theoretic question of what conditions a binary form must meet for its linear factors being distributed according to λ. Chapter 2 is devoted to this question and some related topics following Chipalkatti's work [Ch1].

The equations for Z_λ can be found in Chapter 4, where we combine the results of the previous two Chapters. Aside from this algebraic description of Z_λ, we also give a finite cover (U_g) of the locus Z°_k of closed points $\mathfrak{q}$ of Z with $\text{length}(\pi^{-1}(\mathfrak{q})) = k$ such that on any open set U_g, there is one equation g determining the behaviour of all fibres over closed points of U_g.

In the next Section, we give a more detailed (and therefore more technical) overview of our results, followed by an overview of the methods and the structure of this work, before we start the investigation of the equations of ramification loci in Chapter 1 by explaining several basic notions.

0.2 Results: Determining ramification loci

Consider the projective n-space $\mathbb{P}_K^n$ over an algebraically closed field K of characteristic 0. Let $\widetilde{Z} \subseteq \mathbb{P}_K^n$ be a projective variety, and denote by $\pi : \widetilde{Z} \to \mathbb{P}_K^{n-1}$ an outer simple linear projection of $\widetilde{Z}$ whose centre is a closed point $\mathfrak{p} \in \mathbb{P}_K^n \setminus \widetilde{Z}$. We want to study the fibres over the closed points of the variety $Z := \pi(\widetilde{Z}) \subseteq \mathbb{P}_K^{n-1}$ under this projection. For this, we always consider $\mathbb{P}_K^{n-1}$ as a linear subspace of $\mathbb{P}_K^n$ avoiding $\mathfrak{p}$; note that this embedding is not unique, but that this does not cause any trouble (compare Notation 3.1.1). The fibre over a closed point $\mathfrak{q} \in Z$ is $\pi^{-1}(\mathfrak{q}) = \langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n} \cap \widetilde{Z}$, the intersection of the line $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$ spanned by $\mathfrak{q}$ and $\mathfrak{p}$ with $\widetilde{Z}$. Since $\pi^{-1}(\mathfrak{q})$ is a closed subscheme of the line $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n} \cong \mathbb{P}_K^1$ not containing $\mathfrak{p}$, as a set it consists of finitely many closed points $\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e$; considered scheme-theoretically, each of the points $\widetilde{\mathfrak{q}}_i$ is furnished with its multiplicity in the fibre $\lambda_i = \operatorname{length}(\mathcal{O}_{\pi^{-1}(\mathfrak{q}), \widetilde{\mathfrak{q}}_i})$. We therefore write $\pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$. The length of the fibre $\pi^{-1}(\mathfrak{q})$ is the sum $k = \sum_{i=1}^{e} \lambda_i$ of this multiplicities.
For a fixed partition $\lambda = (\lambda_1, \ldots, \lambda_e) \in \mathbb{N}^e$ of an integer $k \in \mathbb{N}$, i.e., for a tuple of integers λ with $\sum_{i=1}^{e} \lambda_i = k$, the (proper) λ-ramification locus $Z_\lambda^\circ \subseteq Z$ is the locus of closed points of Z whose fibres are distributed according to λ, that is whose fibres each consist of e distinct points $\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_e$ with multiplicities $\lambda_1, \ldots, \lambda_e$, respectively. The question we now want to ponder is this:

Question 0.2.1. How can we describe $Z^\circ_{(\lambda_1,\ldots,\lambda_r)}$ by equations depending only on λ and the equations defining $\widetilde{Z}$ and $\mathfrak{p}$?

Let us take a closer look at the fibre $\pi^{-1}(\mathfrak{q})$ on the projective line $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}_K^n}$. This projective line has coordinate ring $K[x, y]$, and its closed points are in 1-to-1 relation with the linear forms $\mathfrak{l} \in K[x, y]_1$ (up to multiplication with the units K^*, which we will mostly ignore from now on). If we denote by $\mathfrak{l}_i$ the linear form corresponding to a closed point $\widetilde{\mathfrak{q}}_i \in \pi^{-1}(\mathfrak{q})$, then the fibre $\pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$ corresponds to the homogeneous polynomial $\mathfrak{f}_\mathfrak{q} := \prod_{i=1}^{e} \mathfrak{l}_i^{\lambda_i} \in K[x, y]_k$; we call such a polynomial a binary form of degree k. In particular, it holds $\pi^{-1}(\mathfrak{q}) = \operatorname{Proj}(K[x, y]/\mathfrak{f}_\mathfrak{q} K[x, y])$. Determining the number and multiplicities of points in $\pi^{-1}(\mathfrak{q})$ now becomes the same as determining the number and multiplicities of linear forms of $\mathfrak{f}_\mathfrak{q}$. Therefore, the above question is equivalent to the following one:

Question 0.2.2. How can we determine the distribution of the number of linear factors and their multiplicities of the binary forms $\mathfrak{f}_\mathfrak{q} \in K[x, y]$ corresponding to a closed point $\mathfrak{q} \in Z$ depending only on the equations defining $\widetilde{Z}$ and $\mathfrak{p}$?

The answer to this question is twofold: First, we need to determine the polynomials $\mathfrak{f}_\mathfrak{q}$, which shall be done in Chapter 3. Essentially, such a polynomial $\mathfrak{f}_\mathfrak{q}$ is determined in a natural and simple way by Theorem 3.1.9, which tells us that we can choose the restriction to $\langle \mathfrak{q}, \mathfrak{p} \rangle_{\mathbb{P}_K^n}$ of any equation of $\widetilde{Z}$ not vanishing in $\mathfrak{q}$ that is of minimal degree with respect to $\mathfrak{p}$ in the following sense: If x is an indeterminate in the homogeneous coordinate ring R of $\mathbb{P}_K^n$ not vanishing in $\mathfrak{p}$, then we can identify R with the polynomial ring in x over the homogeneous coordinate ring S of $\mathbb{P}_K^{n-1}$. Hence, any element f of R is a polynomial in x with coefficients in S, and as such it has a degree $\deg_x(f)$ and a leading coefficient $\mathrm{LC}_x(f)$ in S with respect to x. For $k \in \mathbb{N}_0$, we now define the k-th partial elimination ideal of the homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} \subseteq R$ of $\widetilde{Z}$ as

$$\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) := \{\mathrm{LC}_x(f) \mid f \in \mathfrak{I}_{\widetilde{Z}} \text{ and } \deg_x(f) \leqslant k\} \subseteq S.$$

Also setting $\mathfrak{K}_{-1}^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) := 0$, we get an ascending chain $\mathfrak{K}_{-1}^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) \subseteq \mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) \subseteq \mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})$ of ideals of S. Theorem 3.1.9 tells us that for any $\mathfrak{q} \in Z$, the degree of $\mathfrak{f}_{\mathfrak{q}}$ is the smallest number k such that $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})$ is not contained in the ideal of $\mathfrak{q}$. Denoting by

$$Z_k := \{\mathfrak{q} \in Z \text{ closed point } \mid \mathrm{length}(\pi^{-1}(\mathfrak{q})) \geqslant k\} \subseteq \mathbb{P}^{n-1}$$

the subvariety of Z of closed points whose fibre is of length at least k, we get as a first consequence an algebraic description of Z_k.

Proposition 0.2.3. *For all $k \in \mathbb{N}_0$, set-theoretically $Z_k = V_{\mathbb{P}^{n-1}}\left(\mathfrak{K}_{k-1}^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})\right)$.*

This result already describes the ramification behaviour of π in a coarse way; it has first been formulated by Marc Green in [G]. Chapter 3 contains this coarse description of the ramification of π. We also use Theorem 3.1.9 in Chapter 4 to get a finer description of the behaviour of the fibres: If the fibre over $\mathfrak{q} \in Z_k^{\circ} := Z_k \backslash Z_{k+1}$ has length k, there must be a polynomial $f \in \mathfrak{I}_{\widetilde{Z}}$ with $\deg_x(f) = k$ that does not vanish on the line $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$. The restriction of f to this line yields $\mathfrak{f}_{\mathfrak{q}}$. Using this argument, we can prove (see Theorem 4.1.3) the next result.

Theorem 0.2.4. *Let $k \in \mathbb{N}$. Then, the elements $f \in \mathfrak{I}_{\widetilde{Z}}$ with $\deg_x(f) = k$ determine an open covering (U_f) of Z_k°, where $U_f \subseteq Z_k^{\circ}$ is the locus of closed points $\mathfrak{q} \in Z_k^{\circ}$ such that $\mathrm{LC}_x(f)$ does not vanish in $\mathfrak{q}$. For any closed point $\mathfrak{q} \in U_f$, the restriction of f to $\langle \mathfrak{q}, \mathfrak{p} \rangle_{\mathbb{P}^n}$ is $\mathfrak{f}_{\mathfrak{q}}$ (up to multiplication with a unit).*

Now, if $x_0, \ldots, x_n$ are coordinates on $\mathbb{P}_K^n$ such that the ideal of $\mathfrak{p}$ is generated by $x_1, \ldots, x_n$ in R, we can set $x = x_0$ and $S = K[x_1, \ldots, x_n]$ and use Gröbner bases to easily read of partial elimination ideals as well as finitely many polynomials yielding an equivalent covering of Z_k°. More precisely, in Proposition 3.3.2 and Corollary 4.1.4, we prove the following statement:

Proposition 0.2.5. *Assume that $\mathfrak{p}$ is generated by the coordinates $x_1, \ldots, x_n$, and let G be a Gröbner basis of $\mathfrak{I}_{\widetilde{Z}}$ with respect to the lexicographic ordering on $R = K[x_0, \ldots, x_n]$. Then, set-theoretically Z_k is defined by the leading coefficients $\mathrm{LC}_{x_0}(g)$ of the (finitely many) elements $g \in G$ with $\deg_{x_0}(g) < k$. Moreover, the (finitely many) elements $g \in G$ with $\deg_{x_0}(g) = k$ determine an open covering (U_g) as above such that a closed point $\mathfrak{q} \in U_g$ belongs to Z_λ° if and only if the linear factors of the restriction of g to the line $\langle \mathfrak{q}, \mathfrak{p} \rangle_{\mathbb{P}^n}$ are distributed according to λ.*

As we now know how to find the polynomials $\mathfrak{f}_{\mathfrak{q}}$ which determine the fibres $\pi^{-1}(\mathfrak{q})$, we can continue with the second part of the answer to the question 0.2.2: the classical problem to determine equations in the coefficients of a binary form $\mathfrak{f} \in K[x, y]$ of degree $k \in \mathbb{N}$ that vanish if and only if the linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_e$ of $\mathfrak{f}$ appear with multiplicities $\lambda_1, \ldots, \lambda_e$ for a given partition $\lambda = (\lambda_1, \ldots, \lambda_e)$ of k, that is if and only if $\mathfrak{f} = \kappa \mathfrak{l}_1^{\lambda_1} \cdots \mathfrak{l}_e^{\lambda_e}$ for some $\kappa \in K^*$. In other words, we ask for the vanishing ideal of the coincident root locus with respect to λ

$$X_\lambda := \left\{ (a_0 : \cdots : a_k) \in \mathbb{P}_K^k \;\middle|\; \begin{array}{c} \sum_{r=0}^k a_r x^{k-r} y^r = \mathfrak{l}_1^{\lambda_1} \cdots \mathfrak{l}_e^{\lambda_e} \\ \text{for some linear factors } \mathfrak{l}_1, \ldots, \mathfrak{l}_e \in K[x,y]_1 \end{array} \right\}.$$

We study coincident root loci in Chapter 2. In particular, we show that the ideal of X_λ in $K[z_0, \ldots, z_0]$ is graded with respect to the grading induced by the weight $\omega(z_i) = i$ for $i \in \{0, \ldots, k\}$ (see Proposition 2.5.6). This yields generators of a forum useful for our study of ramification loci (see Corollary 2.5.7).

Corollary 0.2.6. *We can compute generators $F_1, \ldots, F_s$ of the ideal of X_λ in $K[z_0, \ldots, z_k]$ which are homogeneous with respect to the standard grading as well as homogeneous with respect to the grading induced by the weight $\omega(z_i) = i$ for $i \in \{0, \ldots, k\}$.*

Together with Proposition 0.2.5, we use this in Corollary 4.3.5 to finally state equations defining the subvariety

$$Z_\lambda := \left\{ \mathfrak{q} \in Z_k^\circ \text{ closed point} \mid \exists \tilde{\mathfrak{q}}_1, \ldots, \tilde{\mathfrak{q}}_e \in \widetilde{Z} : \pi^{-1}(\mathfrak{q}) = \sum_{i=1}^e \lambda_i \tilde{\mathfrak{q}}_i \right\} \cup Z_{k+1} \subseteq Z_k.$$

Proposition 0.2.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_e)$ be a partition of $k \in \mathbb{N}_0$. Assume in addition $\mathfrak{p} = \langle x_1, \ldots, x_n \rangle$, and let G be a Gröbner Basis of $\mathfrak{I}_{\widetilde{Z}}$ with respect to the lexicographic ordering. Let $F_1, \ldots, F_s$ be as in Corollary 0.2.6. For $g \in G$, we write $g = g_0 x_0^t + g_1 x_0^{t-1} + \cdots + g_t$ with $t := \deg_{x_0}(g)$. Then, set-theoretically, Z_λ is defined by the polynomials $F_i(g_0, \ldots, g_k)$ for $i \in \{1, \ldots, s\}$ and $g \in G$ with* $\deg_{x_0}(g) = k$ *together with the polynomials g_0 with $g \in G$ such that* $\deg_{x_0}(g) < k$.

Note that in the definition of Z_λ, the closed points $\tilde{\mathfrak{q}}_1, \ldots, \tilde{\mathfrak{q}}_e$ in the fibre over a closed point $\mathfrak{q}$ need not be distinct, and that Z_λ also can contain closed points whose fibres are of length greater than k. Both kind of closed points must be admitted in Z_λ to get a closed set; the proper λ-ramification locus Z_λ° then is a locally closed subset of Z_λ, and the complement of Z_λ° in Z_λ is the union of Z_{k+1} with all Z_μ°, were μ is a coarsening of λ with $\mu \neq \lambda$. From a geometric point of view, we can consider this union $\bigcup_\mu Z_\mu^\circ$ as the degeneracy locus of $Z_\lambda \backslash Z_{k+1}$. For example, $Z_{(2)}^\circ$ in $Z_{(1,1)} \subseteq Z_2$ consists of the closed points of Z_2° whose fibres consist of 1 double point, while the fibre over a closed point of $Z_{(1,1)}^\circ$ consists of two single points. We would expect the latter kind of fibre to be the generic case in $Z_{(1,1)}$. While we hence can not expect to be able to determine Z_λ° itself using equations, its complement in the closed set Z_λ is itself a closed set that can be determined in the same way as Z_λ, which allows an indirect description of Z_λ°. There is a similar difficulty in dealing with coincident root loci: For any coarsening μ of λ, it holds $X_\mu \subseteq X_\lambda$ since the linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_e$ in the definition of X_λ need not be distinct. For example, the discriminant for binary forms of degree 3 also vanishes in the coefficients of the third power $\mathfrak{l}^3$ of a linear form $\mathfrak{l}$. Indeed, the coincident root loci X_μ for certain, but not necessarily all coarsenings μ of λ make up the singular locus of X_λ. We will study this topic in-depth in Section 2.6.

Since we are able to explicitly compute $F_1, \ldots, F_s$, and since all our constructions commute with coordinate transformation, Proposition 0.2.7 yields an algorithm for computing the ideal of Z_λ for any $\mathfrak{p}$. This answers the two equivalent Questions 0.2.1 and 0.2.2.

0.3 Method and structure: Making explicit the locus of interest

Working toward Proposition 0.2.7 is our main task, but we also will follow some side paths. We divide this thesis as follows: Before we turn our attention to coincident root loci and the ramification of simple linear projections, we explain several basic notions and recall some facts and definitions in Chapter 1, whose flavour is one of Combinatorics and Commutative Algebra. The biggest part of this Chapter is devoted to the Hilbert multiplicity, notably system of multiplicity parameters (Section 1.4), which in my experience are well-known, but not described in this form in the standard literature.
In Chapter 2, we study the coincident root loci X_λ following Chipalkatti's work [Ch1]. Besides determining the generators $F_1, \ldots, F_s$ of the ideal of X_λ (Algorithm 2.5.3), our main goal is to fix a gap in the proof of Chipalkatti's crucial result [Ch1, Theorem 3.1], which states that $X_\lambda \times Y_\lambda$ is resolved by an Eagon-Northcott complex where Y_λ is the normalization of X_λ. The proof of this Theorem consists in showing that $\Gamma_\lambda := X_\lambda \times Y_\lambda$ equals the scheme T_φ which is resolved by the Eagon-Northcott complex associated to a specific morphism φ, and the missing part in [Ch1] is the proof that Γ_λ and T_φ are equal as schemes, not only as sets of closed points. We do this in Theorem 2.4.10 via explicitly giving minimal sets of local equations defining T_φ (and hence Γ_λ) and showing that the Jacobi matrix of T_φ contains the multiple of the identity matrix (of maximal rank) with a unit and hence that T_φ is smooth by the Jacobi-Criterion (Proposition 2.4.9). Since we know that T_φ equals Γ_λ set-theoretically and Γ_λ is smooth, this suffices to fix the gap. (Actually, it was this gap in a fascinating article which first led me to compute the generators of Γ_λ.) From the equations for Γ_λ, the ideal defining X_λ can be derived (Section 2.5). Also, we study in detail the relation between the singular locus of X_λ and the coarsenings of λ; again, this subject is already treated in [Ch1, Section 5] (compare Theorem 2.6.2). Here, we give a description of the singular locus of X_λ that I consider to be somewhat more intuitive (Proposition 2.6.5), and we illustrate it at several examples.
The definition and properties of partial elimination ideals (PEIs) make up the first part of Chapter 3, followed by the essential Theorem 3.1.9 which immediately leads to equations defining Z_k (Proposition 3.2.3). PEIs and their significance for Z_k first appeared in [G]. My study of them started because I was not entirely satisfied with the account given in [G] and because I did not understand the proof of [G, Proposition 6.2]. So, one of the goals of Chapter 3 is to give a more general description of PEIs, and we give an independent proof of [G, Proposition 6.2] in Proposition 3.2.3. We also use Proposition 3.2.3 to compute the ideals of the secant cones and secant loci of $\widetilde{Z}$ with respect to $\mathfrak{p}$ (Proposition 3.2.5) as well as to study the fibres of multiple linear projections with certain nice properties (Section 3.4). The last Section is devoted to several examples.

This Chapter contains a coarse description of the ramification of simple outer projections by giving the equations for Z_k. It has already been published in [Km]; we repeat this development here for the sake of completeness with slight modifications to better fit into our study of the fine structure of ramification loci.
We put the results of the previous Chapters together in Chapter 4, proving the above Theorem 0.2.4 (Theorem 4.1.3). We explain this result in detail by means of the example of the projection of a surface of degree 16 in $\mathbb{P}^4_{\mathbb{C}}$ in Section 4.2. Finally, we use Theorem 4.1.3, Proposition 3.2.3, and Corollary 2.5.7 to determine equations defining the ramification of simple linear projections in a way suited for explicit computations in Corollary 4.3.5, which is equivalent to Proposition 0.2.7.
Note that time and again, we use the same basic method in our study of ramification loci and coincident root loci: We describe how to explicitly determine equations describing them. Not only is the aim of our work to compute such equations for the ramification loci Z_λ, but we find them via the equations defining the closed sets Z_k as well as those defining the coincident root loci X_λ. The latter in turn are determined by equations for Γ_λ, which we also use to fix the gap in [Ch1, Theorem 3.1]. Hence, the deduction of equations defining the present loci under consideration might be considered to be the leitmotif of this thesis.

0.4 Notations and conventions

We fix several notations and conventions which we will use throughout. If they appear for the first time since several pages, we will refer to the relevant paragraph in this Section. The most basic and therefore never recited convention can be found in §0.

§0. We always work over an algebraically closed field K of characteristic 0; if we talk about algebras, vector spaces, tensor products etc. without further specifications, we always mean K-algebras, K-vector spaces, tensor products over K etc., respectively. Also, a K-algebra is never 0.

Combinatorics
§1. If $d \in \mathbb{N}_0$ is an integer, we write $[d] := \{1, \ldots, d\}$ for the set of integers running from 1 to d. Also, we denote by $[d]_0$ the ordinal number $d+1$, that is $[d]_0 = \{0, \ldots, d\}$.
§2. For elements $h_1, \ldots, h_m$ of a set H, we denote the tuple $(h_1, \ldots, h_m)$ by $\underline{h}$ if the size and order of this tuple is clear from the context. In the same way, a double underlining implies a tuple of tuples, e.g. $\underline{\underline{h}} = (\underline{h}_1, \ldots, \underline{h}_m) = ((h_{1,1}, \ldots, h_{1,n_1}), \ldots, (h_{m,1}, \ldots, h_{m,n_m}))$. We use the same notation for coordinates $\underline{a} = (a_0 : \ldots : a_m)$ of closed points of a projective m-space.
§3. We say that an assertion holds 'for all $t \gg 0$' if there is some $t_0 \in \mathbb{Z}$ such that the assertion holds for all $t > t_0$, and we use the phrase '$t \ll 0$' analogously.

Commutative Algebra
§4. By 'ring', we always mean a commutative ring with unit. Any polynomial ring over a ring is furnished with the standard grading.
§5. Ideals of any ring are denoted by Gothic letters, for example $\mathfrak{a}, \mathfrak{p}$, and $\mathfrak{I}$. Conversely, Gothic letters always indicate an object that can be considered as

an ideal of a ring.
§6. Let A be a ring, let M be an A-module, and let N be a submodule of M. For a subset $T \subseteq M$, we define

$$(N :_A T) := \{a \in A \mid \forall m \in T : am \in N\}.$$

If $T = \{t\}$ only contains one element, we write $(0 :_A t) = (0 :_A \{t\})$. For a subset $U \subseteq A$, we define

$$(N :_M U) := \{m \in M \mid \forall a \in U : um \in N\}.$$

This notation mostly occurs for $N = 0$, and the ideal $(0 :_A M) \subseteq A$ is the annihilator of M. We denote the set of associated prime ideals of M by $\mathrm{Ass}_A(M)$, that is

$$\mathrm{Ass}_A(M) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \exists m \in M : \mathfrak{p} = (0 :_A m)\}.$$

The notation $\mathrm{minAss}_A(M)$ means the set of minimal elements of $\mathrm{Ass}_A(M)$ with respect to the inclusion. (For this notations, compare [Ma, pp. 6, 38].)
§7. A positively graded ring $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ is homogeneous if $A_1 A = A_+ := \bigoplus_{m \in \mathbb{N}} A_m$. We denote the set of graded prime ideals of A by $^*\mathrm{Spec}(A)$. Also, we write $\mathrm{Proj}(A) = {}^*\mathrm{Spec}(A) \backslash \{A_+\}$. The saturation of an ideal $\mathfrak{a} \subseteq A$is denoted by $\mathfrak{a}^{\mathrm{sat}}$, that is

$$\mathfrak{a}^{\mathrm{sat}} = \bigcup_{t \in \mathbb{N}_0} (\mathfrak{a} :_A A_+^t).$$

§8. If N is a subset of a module M over a ring A, we denote by $\langle N \rangle_M$ the A-submodule of M generated by the elements of N, and if $m_1, \ldots, m_s$ are elements of a module M over a ring A, we denote by $\langle m_1, \ldots, m_s \rangle_M$ the submodule generated by those elements. If M is clear from the context, we may omit the index 'M'.
§9. Let A be a ring, M an A-module, and N an A-submodule of M. We denote by $\bar{\cdot} : M \twoheadrightarrow M/N$ the canonical epimorphism if there is no confusion to be expected. For example, if $\mathfrak{a}$ is an ideal of the ring A and $f \in A$, then $\overline{f} = f + \mathfrak{a} \in \overline{A} = A/\mathfrak{a}$.
§10. For a module M over a ring A and $m \in \mathbb{N}_0$, we denote the m-th symmetric power of M over A by $\mathrm{Sym}^d(V)$, that is

$$\mathrm{Sym}(M) = M^{\otimes_A m}/N = (\underbrace{M \otimes_A M \otimes_A \cdots \otimes_A M}_{m \times})/N,$$

where N is the A-submodule of $M^{\otimes_A m}$ generated by all elements of the form $c_1 \otimes \cdots \otimes c_m - c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(m)}$ for $c_1, \ldots, c_m \in M$ and a permutation σ of $[m]$.

Algebraic Geometry
§11. Let V be a K-vector space. We denote by $\mathbb{P}(V)$ its projectivization, that is the projective space of one-dimensional subspaces of V.
§12. By n we always denote a (fixed) integer. Let $R := K[x_0, \ldots, x_n]$ be the polynomial ring in $n + 1$ indeterminates $x_0, \ldots, x_n$ over K. As any polynomial ring, we always consider R to be furnished with the standard grading, and any use of 'graded' or 'homogeneous' is to be understood with respect to the standard grading.
§13. For any $m \in \mathbb{N}_0$, the projective m-space over K is denoted by $\mathbb{P}^m = \mathbb{P}^m_K$.

For example, $\mathbb{P}^n = \mathrm{Proj}(R)$. By $\mathrm{mProj}(\bullet)$ we denote the maximal elements of $\mathrm{Proj}(\bullet)$ with respect to the inclusion; we identify $\mathrm{mProj}(R)$ with the set of closed points of $\mathbb{P}^n$.

§14. The notation '$\dot{\in}$' means 'is a closed point of', that is $\mathfrak{p} \dot{\in} \mathbb{P}^n$ means that $\mathfrak{p} \in \mathrm{mProj}(R)$ is a closed point of $\mathbb{P}^n$. Since we identify closed points with homogeneous maximal prime ideals, closed points of $\mathbb{P}^m$, $m \in \mathbb{N}_0$, are denoted by Gothic letters. If we use homogeneous coordinates, we write them with the small Latin letters corresponding to the Gothic letter of the closed point. So, '$\mathfrak{p} = (p_0 : \cdots : p_n) \dot{\in} \mathbb{P}^n = \mathrm{Proj}(R)$' means that the $(n+1)$-tuple $\underline{p} = (p_0, \ldots, p_n) \in K^{n+1} \backslash \{\underline{0}\}$ is a coordinate representation of $\mathfrak{p}$ with respect to the coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$.

§15. Let $m \in \mathbb{N}_0$ and Z be a (closed) projective subscheme of the projective m-space $\mathbb{P}^m$ with homogeneous coordinate ring S. Then, $\mathfrak{I}_S(Z) \subseteq S$ denotes the homogeneous ideal of Z, that is the unique saturated ideal $\mathfrak{I} \subseteq S$ with $Z = \mathrm{Proj}(S/\mathfrak{I})$. We often omit the index 'S' and just write '$\mathfrak{I}(Z)$' if there is no confusion to be expected.

§16. Here, a projective variety in $\mathbb{P}^m$ is a reduced projective subscheme of $\mathbb{P}^m$; observe that a variety need not be irreducible. If $\mathfrak{I} \subseteq S$ is a graded ideal, we denote by $V_{\mathbb{P}^m}(\mathfrak{I})$ the projective variety defined by $\mathfrak{I}$, that is $V_{\mathbb{P}^m}(\mathfrak{I}) = \mathrm{Proj}(S/\sqrt{\mathfrak{I}})$. Again, we often omit the index '$\mathbb{P}^m$'.

§17. Let $m \in \mathbb{N}_0$, and let Ω_1, Ω_2 be linear subspaces of $\mathbb{P}^m$ with homogeneous ideals $\mathfrak{I}(\Omega_1)$ and $\mathfrak{I}(\Omega_2)$, respectively. Note that $\mathfrak{I}(\Omega_1)$ and $\mathfrak{I}(\Omega_2)$ are generated by linear forms in the homogeneous coordinate ring of $\mathbb{P}^m$. Then, by $\langle \Omega_1, \Omega_2 \rangle_{\mathbb{P}^m}$ we denote the subspace of $\mathbb{P}^m$ spanned by Ω_1 and Ω_2. It is defined by the ideal generated by $\mathfrak{I}(\Omega_1)_1 \cap \mathfrak{I}(\Omega_2)_1$, the common linear forms of $\mathfrak{I}(\Omega_1)$ and $\mathfrak{I}(\Omega_2)$. The dimension of $\langle \Omega_1, \Omega_2 \rangle_{\mathbb{P}^m}$ is $\dim(\Omega_1) + \dim(\Omega_2) + 1$ if $\Omega_1 \cap \Omega_2 = \varnothing$. For example, if $\mathfrak{p} \neq \mathfrak{q} \dot{\in} \mathbb{P}^n$, then $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$ is the projective line in $\mathbb{P}^n$ spanned by the closed points $\mathfrak{p}$ and $\mathfrak{q}$, and its ideal is generated by $\mathfrak{p}_1 \cap \mathfrak{q}_1$.

Chapter 1

Footings

This Chapter is devoted to make clear several notions that will be used later on; in particular, we explain our terminology concerning partitions, weights, and Hilbert polynomials, and we prove some basic facts about them as well as about saturation in $K[x, y]$.

1.1 Partitions

In this Section, we fix an integer $d \in \mathbb{N}_0$. For the notation $[\cdot]$, see 0.4 §1.

Definition 1.1.1. A *partition* λ *of* d is a tuple $(\lambda_1, \ldots, \lambda_e) \in \mathbb{N}^e$ for some $e \in \mathbb{N}_0$ such that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_e$ and $\lambda_1 + \cdots + \lambda_e = d$. We also write $\lambda = (1^{e_1}\, 2^{e_2} \ldots d^{e_d})$, where $e_r = \#\{i \in [e] \mid \lambda_i = r\}$ is the number of entries in λ that equal r. Denote $|\lambda| := e$ the number of entries in a partition $\lambda = (\lambda_1, \ldots, \lambda_e)$ of d.

In the above notation $\lambda = (1^{e_1}\, 2^{e_2} \ldots d^{e_d})$, we often omit entries of the form r^0, and it holds

$$e = \sum_{r=1}^{d} e_r \leqslant d \qquad \text{and} \qquad \sum_{r=1}^{d} r e_r = d.$$

Observe that there is no partition with $e > d$ and that $e = d$ if and only if $\lambda = (1^d) = (1, 1, \ldots, 1)$. Sometimes, it can be useful to set $e_r = 0$ for $r > d$ and write $\lambda = (1^{e_1} \ldots d^{e_d}\, (d+1)^0 \ldots m^0)$ for some $m \geqslant d$. We will use this notation from now on without further comment.

Remark and Definition 1.1.2. Let $\lambda = (1^{e_1} \ldots d^{e_d})$ be a partition of d. We call λ an *even partition of* d if $\lambda_1 = \cdots = \lambda_{|\lambda|}$; in this case $e_{\lambda_1} = |\lambda|$ and $\lambda_1 \cdot e_{\lambda_1} = d$.
Now, let $\delta = (1^{c_1} \ldots s^{c_s})$ be a second partition, but this time of of $s \in \mathbb{N}$. Then, we define a partition $\lambda \oplus \delta$ of $d + s$ by

$$\lambda \oplus \delta := (1^{e_1 + c_1} \ldots (d+s)^{e_{r+s} + c_{r+s}}).$$

Thus, $\lambda \oplus \delta \in \mathbb{N}_0^{|\lambda|+|\delta|}$ is the partition of $d+s$ with entries $\lambda_1, \ldots, \lambda_{|\lambda|}, \delta_1, \ldots, \delta_{|\delta|}$ in ascending order.
We call δ a *subpartition of* λ if $s \leqslant d$ and $c_r \leqslant e_r$ for all $r \in [s]$. In other words,

$\delta = (\delta_1, \ldots, \delta_{|\delta|})$ is a subpartition of $\lambda = (\lambda_1, \ldots, \lambda_{|\lambda|})$ if δ is composed of a subset of the entries $\lambda_1, \ldots, \lambda_{|\lambda|}$.
A partition $\mu = (1^{\varepsilon} \ldots d^{\varepsilon_d})$ of d is called *a coarsening of* λ if and only if λ is a refinement of μ, or equivalently, if there are subpartitions $\delta^{(1)}, \ldots, \delta^{(|\mu|)}$ of λ such that $\delta^{(1)} \oplus \cdots \oplus \delta^{(|\mu|)} = \lambda$ and $\delta^{(i)}$ is a partition of μ_i for all $i \in [|\mu|]$; in other words, we obtain μ by 'adding together some entries' of λ. We call $\underline{\delta} = (\delta^{(1)}, \ldots, \delta^{(|\mu|)})$ a *splitting of* μ *into* λ. Observe that the order of the partitions $\delta^{(1)}, \ldots, \delta^{(|\mu|)}$ is given by the order of $\mu_1, \ldots, \mu_{|\mu|}$, which we assume to be fixed. Any coarsening μ of λ has exactly one of the following three properties:

(0) μ *is of even unique splitting into* λ, that is there is exactly one splitting $\underline{\delta}$ of μ into λ, and for any $j \in [|\mu|]$, the partition $\delta^{(j)}$ of μ_j is even;

(1) μ *is of uneven unique splitting into* λ, that is there is exactly one splitting $\underline{\delta}$ of μ into λ, but there is an index $j \in [|\mu|]$ such that the partition $\delta^{(j)}$ of μ_j is not even;

(2) *the splitting of* μ *into* λ *is not unique,* that is there are different splittings $\underline{\delta}$ and $\underline{\gamma}$ of μ into λ.

Finally, we denote by Q_λ the set of all coarsenings of λ and by $Q_\lambda^{(i)}$ its subset of coarsenings fulfilling above condition (i) for $i \in \{0, 1, 2\}$. Observe that these subsets are pairwise disjoint, that their union is Q_λ, and that $\lambda \in Q_\lambda^{(0)}$.
A coarsening μ of λ is *strict* if $\mu \neq \lambda$. The subset of all strict coarsenings μ of λ is denoted by $Q_\lambda^\circ \subseteq Q_\lambda$.

Example 1.1.3. An example for an even unique splitting of μ in λ is the case $\lambda = (1, 1, 3)$ and $\mu = (2, 3)$; the only possible splitting of μ into λ are the even partitions $\delta^{(1)} = (1, 1)$ of $\mu_1 = 2$ and $\delta^{(2)} = (3)$ of $\mu_2 = 3$. The coarsening $(1, 3)$ of $(1, 1, 2)$ is of uneven unique splitting; the only possible splitting are the partitions $\delta^{(1)} = (1)$ of 1 and $\delta^{(2)} = (1, 2)$ of 3, of which the second is not even. Finally, the coarsening $\mu = (2, 2)$ of $(1, 1, 2)$ is not of unique splitting since we either can choose the partitions $\delta^{(1)} = (1, 1)$ of $\mu_1 = 2$ and $\delta^{(2)} = (2)$ of $\mu_1 = 2$ or the partitions $\gamma^{(1)} = (2)$ of μ_1 and $\gamma^{(2)} = (1, 1)$ of μ_2 as splittings of μ; those two splittings are not equal as we assume the order of the entries of μ to be fixed.

Lemma 1.1.4. *A partition* λ *of* d *is even if and only if* $Q_\lambda = Q_\lambda^{(0)}$.

Proof. First, assume that λ is even, that is $\lambda = (l^e)$ for some $l, e \in \mathbb{N}$ with $le = d$. Then, any coarsening $\mu \in Q_\lambda$ fulfils $\mu_j = le_j$ for all $j \in [|\mu|]$ for some partition $(e_1, \ldots, e_{|\mu|})$ of e. Hence, $\underline{\delta} = ((l^{e_1}), \ldots, (l^{e_{|\mu|}}))$ is the only splitting of μ into λ, and $\delta^{(j)}$ is even for all $j \in [|\mu|]$.
On the other hand, assume that λ is not even. Then, we find indices $i, j \in [|\lambda|]$ with $\lambda_i \neq \lambda_j$. Denote μ the coarsening of λ obtained by adding λ_i and λ_j and keeping the other entries of λ unchanged, that is $\mu = (\lambda_i + \lambda_j, \lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \widehat{\lambda_j}, \ldots, \lambda_{|\lambda|})$ (up to order) where $\widehat{\bullet}$ means that the entry $\bullet$ is omitted. Then, $((\lambda_i, \lambda_j), (\lambda_1), \ldots, \widehat{(\lambda_i)}, \ldots, \widehat{(\lambda_i)}, \ldots, (\lambda_{|\lambda|}))$ is a splitting of μ into λ containing the uneven partition (λ_i, λ_j). Hence, in this case, $Q_\lambda^{(1)} \cup Q_\lambda^{(2)} \neq \varnothing$. $\square$

1.2 Weighted graded polynomials

We now define weighted gradings auxiliary, and we prove two results about weighted graded polynomials that we will use later:

Remark and Definition 1.2.1. Let $d \in \mathbb{N}_0$, let x be an indeterminate, and let S be a ring. Any polynomial $f \in S[x]$ of degree $\deg(f) \leqslant d$ can be written

$$f \;=\; f_0 x^d + f_1 x^{d-1} + \cdots + f_d \;=\; \sum_{i=0}^{d} f_i x^{d-i}$$

with $f_0, \ldots, f_d \in S$. Now, consider the polynomial ring $A := K[z_0, \ldots, z_d]$. If S is a K-algebra, there is a natural inclusion $\iota : A \hookrightarrow B := S[z_0, \ldots, z_d]$, and for all $F \in A$ and $f = \sum_{i=0}^{d} f_i x^{d-i} \in S[x]$ of degree $\deg(f) \leqslant d$, we define

$$F(f) \;:=\; \iota(F)(f_0, \ldots, f_d) \;\in\; S.$$

Usually, we consider A and B to be furnished with the standard grading, i.e., all indeterminates z_r are of degree 1, and if we speak about 'homogeneous' without further specification, we always mean homogeneous with respect to the standard grading (compare 0.4 §4). But we also can furnish B with the weighted grading with weights $\omega = (0, 1, \ldots, d)$ for $(z_0, \ldots, z_d)$, i.e., the grading defined by $B = \bigoplus_{m \in \mathbb{N}_0} B_{\omega,m}$ with

$$B_{\omega,m} \;:=\; \sum_{\substack{\nu_0, \ldots, \nu_d \in \mathbb{N}_0 : \\ 0\nu_0 + \cdots + d\nu_d = m}} S z_0^{\nu_0} \cdots z_d^{\nu_d}$$

for $m \in \mathbb{N}_0$, so that $z_r \in B_{\omega,r}$ for $r \in \{0, \ldots, d\}$. In this situation, we use the expressions 'weighted graded' and 'weighted homogeneous' for graded and homogeneous with respect to the grading induced by the weight ω, respectively. For a weighted homogeneous element $a \in B_{\omega,m}$, the weight of a is its degree $\omega(a) = m$. Observe that $B_{\omega,0} = S[z_0]$.
Similarly, $A = \bigoplus_{m \in \mathbb{N}_0} A_{\omega,m}$ with $z_i \in A_{\omega,i}$ for $i \in \{0, \ldots d\}$ defines the weighted grading on A, and we use the same terminology. Note that $A \subseteq B$ is a weighted graded subring, and hence for a weighted graded ideal $\mathfrak{b} \subseteq B$, the restriction $\mathfrak{b} \cap A$ is also weighted graded.

Lemma 1.2.2. *Let $d \in \mathbb{N}_0$, and let S be a ring. Let $F \in B := S[z_0, \ldots, z_d]$ be homogeneous of degree m and weighted homogeneous of weight s. Let $f_0, \ldots, f_d \in S$, let $l \in \mathbb{N}_0$, and let $\alpha \in S$. Then*

$$F(\alpha^l f_0, \alpha^{l+1} f_1, \ldots, \alpha^{l+d} f_d) = \alpha^{tm+s} F(f_0, f_1, \ldots, f_d).$$

Proof. By linearity, it is enough to show the claim in the case that F is a monomial. So, let $M = \kappa \prod_{i=0}^{d} z_i^{\tau_i} \in B$ be a monomial $\kappa \in S \backslash \{0\}$ and $\tau_0, \ldots, \tau_d \in \mathbb{N}_0$. If M is homogeneous of degree m and weighted homogeneous of degree s, it holds

$$\sum_{i=0}^{d} \tau_i = m \quad \text{and} \quad \sum_{i=0}^{d} i\tau_i = s.$$

Thus,

$$\begin{array}{rcl} M(\alpha^t f_0, \alpha^{t+1} f_1, \ldots, \alpha^{t+d} f_d) & = & \kappa \prod_{i=0}^d (\alpha^{t+i} f_i)^{\tau_i} \\ & = & \kappa \prod_{i=0}^d \alpha^{\tau_i(t+i)} f_i^{\tau_i} \\ & = & \kappa \alpha^{\sum_{i=0}^d \tau_i(t+i)} \prod_{i=0}^d f_i^{\tau_i} \\ & = & \kappa \alpha^{t\sum_{i=0}^d \tau_i + \sum_{i=0}^d i\tau_i} \prod_{i=0}^d f_i^{\tau_i} \\ & = & \alpha^{tm+s} M(f_0, \ldots, f_d). \end{array}$$

□

For any ring S, an indeterminate x, and an element $f \in S[x]$, we denote by $\deg_x(f)$ the degree of f considered as a polynomial in x over S.

Lemma 1.2.3. *Let $t \in \mathbb{N}_0$, and let $S = K[y_1, \ldots, y_t]$ be the polynomial ring in t indeterminates over K. Let x be a further indeterminate. Let $d \in \mathbb{N}_0$, and let $f \in S[x] = K[x, y_1, \ldots, y_t]$ be homogeneous with respect to the standard grading on $K[x, y_1, \ldots, y_t]$ such that $\deg_x(f) = d$. Let $F \in A = K[z_0, \ldots, z_d]$ be homogeneous and weighted homogeneous. Then, $F(f) \in S$ is homogeneous.*

Proof. Write $f = \sum_{i=0}^d f_i x^{d-i}$ for $f_0, \ldots, f_d \in S$, and set $l := \deg(f_0)$. Then, for any $i \in [d]$, it holds $\deg(f_i) = l + i$. Let $m := \deg(F)$ and $s := \omega(F)$, that is $F \in A_m \cap A_{\omega,s}$. Let $M = z_0^{\tau_0} \cdots z_d^{\tau_d}$ be a monomial occurring in F. Then,

$$\sum_{i=0}^d \tau_i = m, \quad \text{and} \quad \sum_{i=0}^d i\tau_j = s.$$

Hence,

$$\deg(M(f)) = \deg(f_0^{\tau_0} \cdots f_d^{\tau_d}) = \sum_{i=0}^d \deg(f_i^{\tau_i}) = \sum_{i=0}^d \tau_i(l+i) = lm + s,$$

hence $M(f) \in A_{ml+s}$ for all monomials M occurring in F, and thus, $F(f) \in A_{lm+s}$ is homogeneous. □

1.3 Saturation and principal ideals

In this Section, we want prove a result (Lemma 1.3.4) about the relation between saturation and principal ideals in $K[x, y]$ which will be used in Chapter 3. We use the notations of 0.4, §§6-7.

Lemma 1.3.1. *Let $B = \bigoplus_{m \in \mathbb{N}_0} B_m$ be a Noetherian factorial positively graded domain such that $B_+ = \bigoplus_{m \in \mathbb{N}} B_m$ is not principal. Then, every principal ideal of B is saturated.*

Proof. Let $\mathfrak{a} \in B$ be a principal ideal with generator $a \in B$. As $B_0 \subseteq B$ is a domain and $B_0 \cong B/B_+$, the ideal B_+ is prime. As B is Noetherian and factorial, any prime ideal $\mathfrak{p} \in \operatorname{Spec}(B)$ of height 1 is principal. It follows that $\operatorname{height}(B_+) > 1$, thus $B_+ \nsubseteq \sqrt{\mathfrak{a}}$ since $\operatorname{height}(\sqrt{\mathfrak{a}}) = \operatorname{height}(\mathfrak{a}) = 1$. So, there is an irreducible element $b \in B_+$ with $b^t \notin \mathfrak{a} = aB$ for all $t \in \mathbb{N}$; in particular, $a \nmid b^t$. Now, let $c \in \mathfrak{a}^{\mathrm{sat}}$, so that there exists $t \in \mathbb{N}_0$ with $cB_+^t \subseteq \mathfrak{a}$. In particular, it holds $cb^t \in \mathfrak{a}$ and hence $a \mid cb^t$. As b is irreducible, this implies $a \mid c$ and hence $c \in \mathfrak{a}$. □

Corollary 1.3.2. *In a polynomial ring $K[y_1, \ldots, y_s]$ with $s > 1$, any graded principal ideal is saturated.*

Lemma 1.3.3. *Let B be a factorial Noetherian ring, and let $\mathfrak{a} \subseteq B$ be an ideal such that* $\operatorname{height}(\mathfrak{p}) = 1$ *for all* $\mathfrak{p} \in \operatorname{Ass}_B(B/\mathfrak{a})$. *Then, $\mathfrak{a}$ is a principal ideal.*

Proof. It holds $\operatorname{Ass}_B(B/\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ for an integer $l \in \mathbb{N}$ since B is Noetherian. As B is factorial and $\operatorname{height}(\mathfrak{p}_i) = 1$, the prime ideal $\mathfrak{p}_i$ is principal for all $i \in [l]$ (for $[\cdot]$, see 0.4 §1). Hence, there are prime elements $p_1, \ldots, p_l \in B$ with $\mathfrak{p} = p_i B$ for all $i \in [l]$. Now, for any $i \in [l]$, the local ring $B_{\mathfrak{p}_i}$ is a local ring of dimension 1 whose maximal ideal is principal; therefore, $B_{\mathfrak{p}_i}$ is a discrete valuation ring with uniformizing parameter $\frac{p_1}{1}$ (see [Ma, Theorem 11.2]), and we find an integer $t_i \in \mathbb{N}_0$ with $\mathfrak{a}_{\mathfrak{p}_i} = \frac{p_i^{t_i}}{1} B_{\mathfrak{p}_i}$. We aim to show that $\mathfrak{a} = (p_1^{t_1} \cdots p_l^{d_l})B$.
First, let $a \in \mathfrak{a}$. Then, for any $i \in [l]$, it holds in $B_{\mathfrak{p}_i}$ that $\frac{l}{1} = \frac{bp_i^{t_i}}{c}$ for some elements $b \in B, c \in B \backslash \mathfrak{p}$, hence $ca = bp_i^{t_i} \in \mathfrak{p}_i^{t_i}$. As $c \notin \mathfrak{p}_i$, it follows $a \in p_i^{t_i} B$. We get $a \in \bigcap_{i=1}^{l} p_i^{t_i} B = (p_1^{t_1} \cdots p_l^{t_l})B$, where the last equality holds as B is factorial.
Now, let $a \in (p_1^{t_1} \cdots p_l^{t_l})B$ be arbitrary. Then, for all $i \in [l]$, in $B_{\mathfrak{p}_i}$ it holds

$$\frac{a}{1} \in \frac{p_1^{t_1} \cdots p_l^{t_l}}{1} B_{\mathfrak{p}_i} = \frac{p_i^{t_i}}{1} B_{\mathfrak{p}_i} = \mathfrak{a} B_{\mathfrak{p}_i}.$$

So, there is an element $c_i \in B \backslash \mathfrak{p}_i$ with $c_i a \in \mathfrak{a}$. With $\mathfrak{b} := \langle c_1, \ldots, c_l \rangle_B$ it holds $a\mathfrak{b} \subseteq \mathfrak{a}$. As $\mathfrak{b} \nsubseteq \mathfrak{p}_i$ for all $i \in [l]$, there exists an element $b \in \mathfrak{b} \backslash \bigcup_{i=1}^{l} \mathfrak{p}_i$ by the prime avoidance principle ([E, Lemma 3.3]). It follows $ba \in \mathfrak{a}$ with $b \in B \backslash \bigcup \operatorname{Ass}_B(B/\mathfrak{a}) = \operatorname{NZD}_B(B/\mathfrak{a})$, hence $a \in \mathfrak{a}$. □

Lemma 1.3.4. *Any saturated graded ideal $\mathfrak{a} \subseteq B := K[x, y]$ of height 1 is a principal ideal.*

Proof. Let $\mathfrak{a} \subseteq B$ be a saturated ideal of height 1. Then, for $\mathfrak{p} \in \operatorname{Ass}_B(B/\mathfrak{a})$, it holds $\operatorname{height}(\mathfrak{p}) \geqslant 1$ by definition of $\operatorname{height}(\mathfrak{a})$. By the previous Lemma, it is enough to show $\operatorname{height}(\mathfrak{p}) \leqslant 1$. Assume the contrary! Then, since $\operatorname{Ass}_B(B/\mathfrak{a}) \subseteq {}^*\operatorname{Spec}(K[x, y])$, it follows $\mathfrak{p} = B_+$. So, there is an element $\overline{b} = b + \mathfrak{a} \in B/\mathfrak{a}$ with $B_+ = (0 :_B \overline{b})$, hence $bB_+ \subseteq \mathfrak{a}$. Since $\mathfrak{a}$ is saturated, it follows $b \in \mathfrak{a}$ and thus the contradiction $B = (\mathfrak{a} :_B b) = (0 :_B \overline{b}) = B_+$. □

1.4 Hilbert multiplicity

In this Section, we repeat some facts about Hilbert polynomials and the Hilbert multiplicities. These are obviously well known topics, treated in most text books in Commutative Algebra and Algebraic Geometry (e.g., see [B, VIII §10]). Nevertheless, I will explain the notion of multiplicity parameters and the associativity formula in the graded context and without use of the Hilbert-Samuel polynomial. Multiplicity parameters are the analogue in homogeneous rings to superficial elements in (semi-)local rings (see [B, VIII §10.5]). Most of this exposition is presented in the lecture 'Ausgewählte Kapitel der Kommutativen Algebra' by Markus Brodmann ([Br]), which unfortunately is unpublished.
Our standard assumptions in this Section will be that $A = \bigoplus_{m \in \mathbb{N}_0} A_m$ is a

homogeneous ring, that A_0 is an Artinian local ring, and that A is finitely generated over A_0. For the notation '$t \gg 0$', see 0.4 §3.

Definition 1.4.1. Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a finitely generated graded A-module. Note that M_t is finitely generated over A_0, and hence is of finite length for all $t \in \mathbb{Z}$. The *Hilbert function of M* is given by

$$h_M : \mathbb{Z} \to \mathbb{Z},\ t \mapsto \operatorname{length}_{A_0}(M_t).$$

Theorem 1.4.2 (Hilbert)**.** *Let $M = \bigoplus_{t \in \mathbb{Z}} M_m$ be a finitely generated graded A-module. There is a polynomial $p_M \in \mathbb{Q}[X]$ of degree $\dim(M) - 1$ such that $p_M(t) = h_M(t)$ for all $t \gg 0$.*

Proof. See [BsHe, Theorem 4.1.3]. ☐

Definition 1.4.3. Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a finitely generated graded A-module of dimension d. The uniquely determined polynomial p_M in the above Theorem is called the *Hilbert polynomial of M*. It can be written in the form

$$p_M(X) = \sum_{i=0}^{d-1} (-1)^i e_i(M) \binom{X+i}{i}$$

for some integers $e_0(M), \ldots, e_{d-1}(M) \in \mathbb{Z}$ (compare [BsHe, Lemma 4.1.4]). The *Hilbert multiplicity of M* is the coefficient $e_0(M)$ in the above formula if $d > 0$. If $d = 0$, the Hilbert multiplicity is defined as $e_0(M) := \operatorname{length}(M)$. Note that for $d > 0$, it holds

$$e_0(M) = \frac{\operatorname{LC}(p_M)}{(d-1)!} > 0$$

where $\operatorname{LC}(p_M)$ denotes the leading coefficient of the polynomial p_M.

Lemma 1.4.4. *Let $0 \to N \to M \to P \to 0$ be a sequence of finitely generated graded A-modules which is exact in high degrees, that is the induced sequence of A_0-modules $0 \to N_t \to M_t \to P_t \to 0$ is exact for all $t \gg 0$. Then, one of the following statements holds:*

i) $\dim(N) = \dim(M) = \dim(P)$ *and* $e_0(M) = e_0(N) + e_0(P)$*;*

ii) $\dim(N) = \dim(M) > \dim(P)$ *and* $e_0(M) = e_0(N)$*;*

iii) $\dim(N) < \dim(M) = \dim(P)$ *and* $e_0(M) = e_0(P)$*.*

Proof. For all $t \gg 0$, the sequence of A_0-modules $0 \to N_t \to M_t \to P_t \to 0$ is exact, hence $\operatorname{length}_{A_0}(M_t) = \operatorname{length}_{A_0}(N_t) + \operatorname{length}_{A_0}(P_t)$. It follows $h_M(t) = h_N(t) + h_P(t)$ for all $t \gg 0$, thus $p_M = p_N + p_P$. As $\dim(M) = \deg(p_M) + 1$, $\dim(N) = \deg(p_N) + 1$, and $\dim(P) = \deg(p_P) + 1$, comparing coefficients together with the relation between the Hilbert multiplicity and the leading coefficient of the Hilbert polynomial yields the claim. ☐

Proposition 1.4.5 (Associativity formula)**.** *Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a finitely generated graded A-module of dimension d with $M \neq 0$. Define $\mathcal{S} := \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \dim(A/\mathfrak{p}) = d\}$. Then,*

$$e_0(M) = \sum_{\mathfrak{p} \in \mathcal{S}} e_0(R/\mathfrak{p}) \cdot \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}).$$

Proof. As $\mathcal{S}$ is a subset of $\min(0 :_A M)$, the set of minimal prime ideals over $(0 :_A M)$, we first get $\#\mathcal{S} < \infty$. Let $\mathfrak{p} \in \mathcal{S}$. As $\mathfrak{p} \in \operatorname{Ass}_A(M)$, there are an integer $t \in \mathbb{Z}$ and an element $m \in M_t$ such that $\mathfrak{p} = (0 :_A m)$, hence $\mathfrak{p}(-t)$ is the kernel of the graded morphism $A(-t) \overset{\cdot m}{\to} M$ given by multiplication with m. So, we get an exact sequence of graded A-modules

$$\mathbb{S}: \quad 0 \to (A/\mathfrak{p})(-t) \to M \to M/Am \to 0.$$

Localizing this sequence, we get the exact sequence of $A_\mathfrak{p}$-modules

$$0 \to (A/\mathfrak{p})_\mathfrak{p} \to M_\mathfrak{p} \to (M/Am)_\mathfrak{p} \to 0.$$

Since $(A/\mathfrak{p})_\mathfrak{p} \cong A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is simple, we get

$$\forall \mathfrak{p} \in \mathcal{S}: \quad \operatorname{length}_{A_\mathfrak{p}}((M/Am)_\mathfrak{p}) = \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1.$$

In particular, for all prime ideals $\mathfrak{p} \in \mathcal{S}$, it holds $\mathfrak{p} \in \operatorname{Supp}(M/Am)$ if and only if $\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) > 1$ since $\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 1$ is equivalent to $M_\mathfrak{p} = 0$.
We prove our claim by induction on $\#\mathcal{S}$. First, consider the case $\#\mathcal{S} = 1$, i.e., $\mathcal{S} = \{\mathfrak{p}\}$, which we handle by induction on $\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p})$. If $\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 1$, then $\mathfrak{p} \notin \operatorname{Supp}(M/Am)$ for $m \in M_t$ homogeneous with $\mathfrak{p} = (0 :_A m)$, and thus $\operatorname{Supp}(M/Am) \subseteq \operatorname{Supp}(M) \backslash \mathcal{S}$. But by definition of $\mathcal{S}$, this means $\dim(M/Am) < \dim(M)$, and hence Lemma 1.4.4 applied to the above sequence $\mathbb{S}$ yields

$$\begin{aligned} e_0(M) &= e_0((R/\mathfrak{p})(-t)) &&= e_0(R/\mathfrak{p}) \\ &= e_0(R/\mathfrak{p}) \cdot \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) &&= \textstyle\sum_{\mathfrak{q} \in \mathcal{S}} e_0(R/\mathfrak{q}) \cdot \operatorname{length}_{A_\mathfrak{q}}(M_\mathfrak{q}). \end{aligned}$$

Next, assume $\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) > 1$ while still $\mathcal{S} = \{\mathfrak{p}\}$. Then $\mathfrak{p} \in \operatorname{Supp}(M/Am)$ and $\operatorname{length}_{A_\mathfrak{p}}((M/Am)_\mathfrak{p}) = \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1$. Because $\mathfrak{p} \in \operatorname{Supp}(M/Am) \subseteq \operatorname{Supp}(M)$, we get $\dim(M/Am) = \dim(M)$ by definition of $\mathcal{S}$. Hence, $\mathfrak{p}$ is the only element of $\operatorname{Supp}(M/Am)$ with $\dim(A/\mathfrak{p}) = \dim(M/Am)$, and by induction we get

$$e_0(M/Am) = e_0(A/\mathfrak{p}) \cdot \operatorname{length}_{A_\mathfrak{p}}((M/Am)_\mathfrak{p}) = e_0(A/\mathfrak{p}) \cdot \Big(\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1\Big).$$

Applying Lemma 1.4.4 to the sequence $\mathbb{S}$, we get

$$\begin{aligned} e_0(M) = e_0(A/\mathfrak{p}) + e_0(M/Am) &= e_0(A/\mathfrak{p}) + e_0(A/\mathfrak{p}) \cdot \Big(\operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1\Big) \\ = e_0(A/\mathfrak{p}) \cdot \operatorname{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) &= \textstyle\sum_{\mathfrak{q} \in \mathcal{S}} e_0(R/\mathfrak{q}) \cdot \operatorname{length}_{A_\mathfrak{q}}(M_\mathfrak{q}). \end{aligned}$$

This concludes the case $\#\mathcal{S} = \infty$. Next, we turn our attention to the case $\#\mathcal{S} > \infty$, and we fix $\mathfrak{p} = (0 :_A m) \in \mathcal{S}$ with $m \in M_t$. Let $\mathfrak{q} \in \mathcal{S} \backslash \{\mathfrak{p}\}$. Since $\mathfrak{p}, \mathfrak{q} \in \min(0 :_A M)$, it holds $\mathfrak{p} \nsubseteq \mathfrak{q}$, hence $(A/\mathfrak{p})_\mathfrak{q} = 0$. Localizing $\mathbb{S}$ at $\mathfrak{q}$, we get an isomorphism $M_\mathfrak{q} \cong (M/Am)_\mathfrak{q}$; in particular $\mathfrak{q} \in \operatorname{Supp}(M/Am) \subseteq \operatorname{Supp}(M)$ and therefore $\dim(M/Am) = \dim(A/\mathfrak{q}) = \dim(M)$. If we denote

$$\mathcal{S}_m := \{\mathfrak{q} \in \operatorname{Supp}(M/Am) \mid \dim(M/Am) = \dim(A/\mathfrak{q})\},$$

this means that $\mathcal{S} \backslash \{\mathfrak{p}\} = \mathcal{S}_m \backslash \{\mathfrak{p}\}$. We therefore get

$$\sum_{\mathfrak{q} \in \mathcal{S} \backslash \{\mathfrak{p}\}} e_0(A/\mathfrak{q}) \cdot \operatorname{length}_{A_\mathfrak{q}}(M_\mathfrak{q}) = \sum_{\mathfrak{q} \in \mathcal{S}_m \backslash \{\mathfrak{p}\}} e_0(A/\mathfrak{q}) \cdot \operatorname{length}_{A_\mathfrak{q}}((M/Am)_\mathfrak{q}).$$

Again, we continue by induction on $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p})$. Assume $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 1$. Then $\mathfrak{p} \notin \text{Supp}(M/Am)$, and therefore $\mathcal{S}_m = \mathcal{S}\backslash\{\mathfrak{p}\}$. By induction on $\#\mathcal{S}$, the right hand side of the above equation thus is equal to $e_0(M/Am)$. Again using $M_\mathfrak{q} \cong (M/Am)_\mathfrak{q}$ for $\mathfrak{q} \in \mathcal{S}_m\backslash\{\mathfrak{p}\}$ and applying Lemma 1.4.4 to $\mathbb{S}$, we get

$$\begin{aligned} e_0(M) &= e_0\left((R/\mathfrak{p})(-t)\right) + e_0\left(M/Am\right) \\ &= e_0(R/\mathfrak{p}) + \textstyle\sum_{\mathfrak{q}\in\mathcal{S}_m} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left((M/Am)_\mathfrak{q}\right) \\ &= e_0(R/\mathfrak{p}) + \textstyle\sum_{\mathfrak{q}\in\mathcal{S}\backslash\{\mathfrak{p}\}} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left(M_\mathfrak{q}\right) \\ &= \textstyle\sum_{\mathfrak{q}\in\mathcal{S}} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left(M_\mathfrak{q}\right). \end{aligned}$$

Finally, assume $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) > 1$. We already showed $\text{length}_{A_\mathfrak{p}}((M/Am)_\mathfrak{p}) = \text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1$, $\mathfrak{p} \in \mathcal{S}_m$, and $\mathcal{S}\backslash\{\mathfrak{p}\} = \mathcal{S}_m\backslash\{\mathfrak{p}\}$. Applying Lemma 1.4.4 once more to the sequence $\mathbb{S}$, we get by induction on $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p})$

$$\begin{aligned} e_0(M) &= e_0\left(A/\mathfrak{p})(-t)\right) + e_0\left(M/Am\right) \\ &= e_0\left(A/\mathfrak{p}\right) + \textstyle\sum_{\mathfrak{q}\in\mathcal{S}_m} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left((M/Am)_\mathfrak{q}\right) \\ &= e_0\left(A/\mathfrak{p}\right) + e_0\left(A/\mathfrak{p}\right)\left(\text{length}_{A_\mathfrak{p}}\left(M_\mathfrak{p}\right) - 1\right) \\ &\quad + \textstyle\sum_{\mathfrak{q}\in\mathcal{S}_m\backslash\{\mathfrak{p}\}} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left((M/Am)_\mathfrak{q}\right) \\ &= e_0\left(A/\mathfrak{p}\right)\text{length}_{A_\mathfrak{p}}\left(M_\mathfrak{p}\right) + \textstyle\sum_{\mathfrak{q}\in\mathcal{S}\backslash\{\mathfrak{p}\}} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left(M_\mathfrak{q}\right) \\ &= \textstyle\sum_{\mathfrak{q}\in\mathcal{S}} e_0\left(A/\mathfrak{q}\right)\cdot \text{length}_{A_\mathfrak{q}}\left(M_\mathfrak{q}\right). \end{aligned}$$

□

Complementing the above associativity formula, we also have the following useful Lemma for the study of projective schemes:

Lemma 1.4.6. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module, and let $\mathfrak{p} \in \text{Proj}(A)$. Then*

$$\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \text{length}_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}).$$

Proof. Let $t \in \mathbb{Z}$. Since $A_+ \nsubseteq \mathfrak{p}$, there is an element $a \in A_1\backslash\mathfrak{p}$. Then, $M(t)_{(\mathfrak{p})} \to M(t+1)_{(\mathfrak{p})}$, $\frac{m}{s} \mapsto \frac{am}{s}$ is an isomorphism of $A_{(\mathfrak{p})}$-modules, i.e., for all $t \in \mathbb{Z}$ it holds $M(t)_{(\mathfrak{p})} \cong M_{(\mathfrak{p})}$ and thus $\text{length}_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) = \text{length}_{A_{(\mathfrak{p})}}(M(t)_{(\mathfrak{p})})$. Moreover, if we ignore the gradings after localizing, $M_\mathfrak{p} = M(t)_\mathfrak{p}$ as $A_\mathfrak{p}$-modules, i.e., $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \text{length}_{A_\mathfrak{p}}(M(t)_\mathfrak{p})$ for all $t \in \mathbb{Z}$.
Denote $k := \text{length}_{A_\mathfrak{p}}(M_\mathfrak{p})$; we first assume that $k < \infty$ and proceed by induction on k. Assume $k = 0$, that is $M_\mathfrak{p} = 0$. Let $u = \frac{m}{s} \in M_{(\mathfrak{p})}$ with $m \in M_r$ and $s \in A_r\backslash\mathfrak{p}$ for some $r \in \mathbb{N}_0$. As $M_\mathfrak{p} = 0$, there is an element $a \in A\backslash\mathfrak{p}$ with $am = 0$. Clearly, there is some $i \in \mathbb{N}_0$ such that the i-th graded component a_i of a satisfies $a_i \in A\backslash\mathfrak{p}$. As m is homogeneous, it follows $a_i m = 0$, whence

$$u = \frac{m}{s} = \frac{a_i m}{a_i s} = \frac{0}{a_i s} = 0.$$

Therefore, $M_{(\mathfrak{p})} = 0$, and the case $k = 0$ is done.
Now assume $k \in \mathbb{N}$. As $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = k < \infty$, we have $\mathfrak{p}A_\mathfrak{p} \in \text{minAss}_{A_\mathfrak{p}}(M_\mathfrak{p})$. Indeed, assume that $\mathfrak{p}A_\mathfrak{p} \notin \text{minAss}_{A_\mathfrak{p}}(M_\mathfrak{p})$! As $M_\mathfrak{p} \neq 0$, the annihilator $(0 :_{A_\mathfrak{p}} M_\mathfrak{p})$ is contained in $\mathfrak{p}A_\mathfrak{p}$. As $\text{minAss}_{A_\mathfrak{p}}(M_\mathfrak{p})$ consists of the minimal prime ideals containing $(0 :_{A_\mathfrak{p}} M_\mathfrak{p})$, there is a prime ideal $\mathfrak{q} \in \text{minAss}_{A_\mathfrak{p}}(M_\mathfrak{p})$ with $\mathfrak{q} \subsetneqq \mathfrak{p}A_\mathfrak{p}$. With $m \in M_\mathfrak{p}$ such that $\mathfrak{q} = (0 :_{A_\mathfrak{p}} m)$, it follows that there is an

element $a \in \mathfrak{p}A_{\mathfrak{p}} \backslash \mathfrak{q}$ with $am \neq 0$. Since a is not an unit in $A_{\mathfrak{p}}$, we get a chain of submodules $\langle m \rangle \supsetneq \langle am \rangle \supsetneq \langle a^2 m \rangle \supsetneq \cdots \supsetneq \langle a^{k+1} m \rangle$ of $M_{\mathfrak{p}}$ of length $k+1$, a contradiction.
Whence $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{minAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ and thus $\mathfrak{p} \in \operatorname{Ass}_A(M)$. As $\mathfrak{p}$ is graded, there exist $t \in \mathbb{Z}$ and an injective morphism of graded A-modules $\iota : A/\mathfrak{p} \rightarrowtail M(t)$. Let $N := \operatorname{coker}(\iota)$, which is also a graded A-module. We get an exact sequence of graded A-modules

$$0 \to A/\mathfrak{p} \xrightarrow{\iota} M(t) \to N \to 0.$$

Localizing at $\mathfrak{p}$ and observing that $M(t)_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, we get an exact sequence of A-modules

$$0 \to (A/\mathfrak{p})_{\mathfrak{p}} \to M_{\mathfrak{p}} \to N_{\mathfrak{p}} \to 0.$$

As $\operatorname{length}_{A_{\mathfrak{p}}}((A/\mathfrak{p})_{\mathfrak{p}}) = \operatorname{length}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) = 1$, it follows

$$\operatorname{length}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 1 + \operatorname{length}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}).$$

Further, the module $(A/\mathfrak{p})_{(\mathfrak{p})} = A_{(\mathfrak{p})}/\mathfrak{p}_{(\mathfrak{p})}$ is the residue field of $A_{(\mathfrak{p})}$, hence it has length 1 over $A_{(\mathfrak{p})}$. Taking the 0-th graded component of the above sequence, we get the short exact sequence

$$0 \to (A/\mathfrak{p})_{(\mathfrak{p})} \to M_{(\mathfrak{p})} \to N_{(\mathfrak{p})} \to 0;$$

since the length is additive, we get $\operatorname{length}_{A_{(\mathfrak{p})}}(N_{(\mathfrak{p})}) = \operatorname{length}_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) - 1$. Now, we conclude by induction.
Assume now that $k = \infty$. Then $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Supp}_{A_{\mathfrak{p}}} \backslash \operatorname{minAss}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. So, there is a prime $\mathfrak{q} \in \operatorname{minAss}_A(M)$ with $\mathfrak{q} \subsetneq \mathfrak{p}$. In particular, we again get a monomorphism of graded A-modules $A/\mathfrak{q} \rightarrowtail M(s)$ for some $s \in \mathbb{Z}$. Consequently, we get an induced monomorphism of $A_{(\mathfrak{p})}$-modules $(A/\mathfrak{q})_{(\mathfrak{p})} \rightarrowtail (M(s))_{(\mathfrak{p})}$. As $0 \neq (\mathfrak{p}/\mathfrak{q})_{(\mathfrak{p})} \subsetneq (A/\mathfrak{q})_{(\mathfrak{p})}$, it follows that the local domain $(A/\mathfrak{q})_{(\mathfrak{p})}$ is of dimension greater than 0. Thus

$$\operatorname{length}_{A_{(\mathfrak{p})}}((A/\mathfrak{q})_{(\mathfrak{p})}) \geqslant \operatorname{length}_{(A/\mathfrak{q})_{(\mathfrak{p})}}((A/\mathfrak{q})_{(\mathfrak{p})}) = \infty.$$

Now, assume $s \in \mathbb{N}_0$. Since $\mathfrak{p} \neq A_+$, there is an element $w \in A_s \backslash \mathfrak{p}$, and multiplication with w induces an isomorphism $M_{(\mathfrak{p})} \cong M(s)_{(\mathfrak{p})}$. On the other hand, if $s < 0$, we can choose $w \in A_{-s} \backslash \mathfrak{p}$ and get by multiplication with w an isomorphism $(M(s))_{(\mathfrak{p})} \cong (M(s)(-s))_{(\mathfrak{p})} = M_{(\mathfrak{p})}$. Altogether, we get

$$\operatorname{length}_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) = \operatorname{length}_{A_{(\mathfrak{p})}}(M(s)_{(\mathfrak{p})}) \geqslant \operatorname{length}_{A_{(\mathfrak{p})}}((A/\mathfrak{q})_{(\mathfrak{p})}) = \infty,$$

which proves our claim. □

Remark and Definition 1.4.7. Let $M = \bigoplus_{t \in \mathbb{Z}} M_t$ be a finitely generated graded A-module of dimension d. By [BsHe, Remark 4.1.6], the *summatorial Hilbert function*

$$H_M : \mathbb{Z} \to \mathbb{N}_0, t \to \sum_{s \leqslant t} h_M(s)$$

is also polynomial, that is there is a polynomial $P_M \in \mathbb{Q}[X]$ of degree d such that $P_M(t) = H_M(t)$ for all $t \gg 0$. The polynomial P_M is called *the summatorial Hilbert polynomial of* M, and it holds

$$\operatorname{LT}(P_M) = \frac{e_0(M)}{d!} X^d$$

where $\operatorname{LT}(P_M)$ is the leading term of the polynomial P_M.

Proposition 1.4.8. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $t \in \mathbb{N}$ and let $a \in A_t$. Then:*

(a) $\dim(M/aM) \in \{d, d-1\}$*;*

(b) $\dim(M/aM) = d-1$ *if and only if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(M)$ with $= dim(R/\mathfrak{p}) = d$;*

(c) if $\dim(M/aM) = d-1$, *then $e_0(M/aM) \geqslant te_0(M)$, and equality holds if and only if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(M)$ with* $\dim(A/\mathfrak{p}) \geqslant d-1$.

Proof. (a) Since the module M is graded, the annihilator $(0 :_A M)$ is graded, too, and there is a maximal ideal $\mathfrak{m} \in \mathrm{Max}(A)$ with $(0 :_A M)+A_+ \subseteq \mathfrak{m}$ such that the maximal ideal $\mathfrak{m}/(0 :_A M) \subseteq A/(0 :_A M)$ satisfies $\mathrm{height}(\mathfrak{m}/(0 :_A M)) = d$. By the annihilator lemma (see [E, Proposition 10.8]), $\sqrt{(0 :_A (M/aM))} = \sqrt{aA + (0 :_A M)} \subseteq \mathfrak{m}$, and so

$$\begin{aligned}\dim(M/aM) &= \dim\left(A/(0 :_A M/aM)\right) &&\geqslant \mathrm{height}\left(\mathfrak{m}/(0 :_A (M/aM))\right)\\ &= \mathrm{height}\left(\mathfrak{m}/(aA + (0 :_A M)\right) &&\geqslant \mathrm{height}\left(\mathfrak{m}/(0 :_A M)\right) - 1\\ &= d-1.\end{aligned}$$

(b) Let $\mathcal{S} := \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \dim(A/\mathfrak{p}) = d\}$, and assume first that $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{S}$. Then, [E, Proposition 10.8] yields $(0 :_A (M/aM)) \subseteq \sqrt{aA + (0 :_A M)} \subseteq \mathfrak{p}$, hence $\mathfrak{p} \in \mathrm{Supp}_A(M/aM)$ and $\dim(M/aM) \geqslant \dim(A/\mathfrak{p}) = d$, therefore $\dim(M/aM) = d$. On the other hand, assume now $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{S}$, and let $\mathfrak{q} \in \mathrm{Supp}_A(M/aM)$, so that $a \in \mathfrak{q}$ and $(0 :_A M) \subseteq \mathfrak{q}$. In particular, there is a minimal prime $\mathfrak{p} \in \min(0 :_A M) \subseteq \mathrm{Ass}_A(M)$ with $\mathfrak{p} \subseteq \mathfrak{q}$. If $\mathfrak{p} \in \mathcal{S}$, we have $a \notin \mathfrak{p}$, hence $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\dim(A/\mathfrak{q}) < \dim(A/\mathfrak{p}) = d$. If $\mathfrak{p} \notin \mathcal{S}$, we have $\dim(A/\mathfrak{q}) \leqslant \dim(A/\mathfrak{p}) < d$. In either case, it follows $\dim(M/aM) < d$. Together with (a), this yields (b).
(c) For all $s \in \mathbb{Z}$, there is an exact sequence of A_0-modules

$$0 \to (0 :_M a)_{s-t} \to M_{s-t} \xrightarrow{a\cdot} M_s \to (M/aM)_t \to 0.$$

This sequence yields

$$\mathrm{length}_{A_0}\left((M/aM)_t\right) = \begin{pmatrix} \mathrm{length}_{A_0}(M_s) - \mathrm{length}_{A_0}(M_{s-t}) \\ +\,\mathrm{length}_{A_0}\left((0 :_M a)_{s-t}\right)\end{pmatrix}$$

for all $s \in \mathbb{Z}$. Adding this lengths, we get for the summatorial Hilbert polynomial

$$P_{M/aM}(X) = P_M(X) - P_M(X-t) + P_{(0:_M a)}(X-t).$$

Since $\mathrm{LT}(P_M(X)) = \frac{e_0(M)}{d!}X^d$, we have

$$\mathrm{LT}\left(P_M(X) - P_M(X-t)\right) = \frac{te_0(M)}{(d-1)!}X^{d-1}.$$

As $\deg(P_{M/aM}(X)) = \dim(M/aM) = d-1$, this implies $\deg(P_{(0:_M a)}(X-t)) \leqslant d-1$, and as $e_0(0 :_M a) \geqslant 0$, we get $e_0(M/aM) \geqslant te_0(M)$. Moreover, equality holds if and only if $\dim(0 :_M a) < d-1$. Therefore, we only need show that $\dim(0 :_M a) \geqslant d-1$ if and only if there exists $\mathfrak{p} \in \mathrm{Ass}_A(M)$ with $\dim(A/\mathfrak{p}) = d-1$ and $a \in \mathfrak{p}$.

Indeed, assume that there is such a $\mathfrak{p} \in \mathrm{Ass}_A(M)$ with $\dim(A/\mathfrak{p}) = d - 1$ and $a \in \mathfrak{p}$. As $\mathfrak{p} \in \mathrm{Ass}_A(M)$, there is some homogeneous element $m \in M$ with $\mathfrak{p} = (0 :_A m)$. As $a \in \mathfrak{p}$, we have $(0 :_M \mathfrak{p}) \subseteq (0 :_M a)$. If follows

$$(0 :_A (0 :_M a)) \subseteq (0 :_A (0 :_M \mathfrak{p})) \subseteq (0 :_A m) = \mathfrak{p},$$

so that $\dim(0 :_M a) \geqslant \dim(A/\mathfrak{p}) = d - 1$. On the other hand, if $\dim(0 :_M a) \geqslant d - 1$, then there is a minimal prime $\mathfrak{p} \in \min(0 :_A (0 :_M a))$ with $\dim(A/\mathfrak{p}) \geqslant d - 1$. But then, it also holds $\mathfrak{p} \in \mathrm{Ass}_A(0 :_M a) \subseteq \mathrm{Ass}_A(M)$ and $a \in (0 :_A (0 :_M a)) \subseteq \mathfrak{p}$. □

Definition 1.4.9. Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. let $t \in \mathbb{Z}$ and $a \in A_t$. We call a a *homogeneous parameter with respect to* M if and only if $\dim(M/aM) = d - 1$.
If a is a homogeneous parameter and also $e_0(M/aM) = te_0(M/aM)$, we call a a *multiplicity parameter with respect to* M. According to the above Proposition, a is a multiplicity parameter with respect to M if and only if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$ with $\dim(R/\mathfrak{p}) \geqslant d - 1$.

Corollary 1.4.10. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module. Let $a \in \mathrm{NZD}_A(M)$ be a homogeneous non zero divisor of M. Then a is a multiplicity parameter with respect to M.*

Proof. Since $a \in \mathrm{NZD}_A(M)$, it holds $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(M)$, and we are done by Proposition 1.4.8(c). □

Proposition 1.4.11. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $r \in [d]$, and let $a_1, \ldots, a_r \in A_+$ be homogeneous. Then $\dim(M/\sum_{i=1}^r a_iM) \geqslant d - r$. Also, the following conditions are equivalent:*

(i) a_i *is a homogeneous parameter with respect to* $M/\sum_{j=1}^{i-1} a_jM$ *for all* $i \in [r]$*;*

(ii) $\dim(M/\sum_{j=1}^{i} a_jM) = d - i$ *for all* $i \in [r]$*;*

(iii) $\dim(M/\sum_{j=1}^{r} a_jM) \leqslant d - r$*.*

Proof. For any $i \in [r]$, there is an isomorphism of graded A-modules

$$\left(M/\sum_{j=1}^{i-1} a_jM\right) \Big/ a_i \left(M/\sum_{j=1}^{i-1} a_jM\right) \cong M/\sum_{j=1}^{i} a_jM. \tag{1.1}$$

Using Proposition 1.4.8(a), we get the first claim by induction on r.
(i)⇒(ii): Condition (i) implies

$$\dim\left(\left(M/\sum_{j=1}^{i-1} a_jM\right) \Big/ a_i \left(M/\sum_{j=1}^{i-1} a_jM\right)\right) = \dim\left(M/\sum_{j=1}^{i} a_jM\right).$$

Using the above isomorphism, we get (ii) by induction on i.
(ii)⇒(iii): This is obvious.
(iii)⇒(i): By Proposition 1.4.8(a) and the isomorphism 1.1, (iii) immediately implies (i). □

Definition 1.4.12. Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $r \in [d]$ (for $[\cdot]$, see 0.4 §1), and let $a_1, \ldots, a_r \in A_+$ be homogeneous. If the equivalent conditions (i), (ii), and (iii) of the above Proposition hold, we call $a_1, \ldots, a_r$ a *system of homogeneous parameter with respect to M.*

Proposition 1.4.13. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $r \in [d]$, and let $t_1, \ldots, t_r \in \mathbb{Z}$. Let $a_1, \ldots, a_r \in A_+$ be a system of homogeneous parameters with respect to M such that $a_i \in A_{t_i}$ for all $i \in [r]$. Then*

$$e_0\left(M/\sum_{j=1}^{r} a_j M\right) \geqslant t_1 \cdots t_r e_0\left(M\right).$$

Moreover, the following conditions are equivalent:

(i) a_i is a multiplicity parameter with respect to $M/\sum_{j=1}^{i-1} a_j M$ for all $i \in [r]$;

(ii) $e_0(M/\sum_{j=1}^{i} a_j M) = t_1 \cdots t_i e_0\left(M\right)$ for all $i \in [r]$;

(iii) $e_0(M/\sum_{j=1}^{r} a_j M) \leqslant t_1 \cdots t_r e_0\left(M\right)$.

Proof. We get the first claim by induction on r using Proposition 1.4.8(c) and the isomorphism 1.1 of the previous proof.
(i)⇒(ii): Assume that (i) holds. Then

$$e_0\left(\left(M/\sum_{j=1}^{i-1} a_j M\right) \Big/ a_i \left(M/\sum_{j=1}^{i-1} a_j M\right)\right) = t_i e_0\left(M/\sum_{j=1}^{i} a_j M\right)$$

for all $i \in [r]$. Using the isomorphism 1.1, we get (ii) by induction on i.
(ii)⇒(iii): This is obvious.
(iii)⇒(i): By Proposition 1.4.8(c) and the isomorphism 1.1, (iii) immediately implies (i). □

Definition 1.4.14. Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $r \in [d]$, and let $t_1, \ldots, t_r \in \mathbb{Z}$. Let $a_1, \ldots, a_r \in A_+$ be a system of homogeneous parameters with respect to M such that $a_i \in A_{t_i}$ for all $i \in [r]$. If the equivalent conditions (i), (ii), and (iii) of Proposition 1.4.13 hold, we call $a_1, \ldots, a_r$ a *system of multiplicity parameters with respect to M.*

Corollary 1.4.15. *Let $M = \bigoplus_{t\in\mathbb{Z}} M_t$ be a finitely generated graded A-module with $d := \dim(M) > 0$. Let $r \in [d]$, and let $a_1, \ldots, a_r \in A$ be a homogeneous M-sequence. Then $a_1, \ldots, a_r$ is a system of multiplicity parameters with respect to M.*

Proof. This follow immediately by induction on r and Corollary 1.4.10. □

Chapter 2

Equations for coincident root loci

In this Chapter, we always denote an integer by $d \in \mathbb{N}_0$ and a partition of d with e entries by $\lambda = (\lambda_1, \dots, \lambda_e) \in \mathbb{N}^e$. If the integer r occurs e_r times in λ for $r \in \{1, \dots, d\}$, we again write $\lambda = (1^{e_1}\, 2^{e_2}\, \dots\, d^{e_d})$. Let V denote the two-dimensional K-vector space $K[x, y]_1$ of linear polynomials in indeterminates x and y over K. Throughout, we will make use of the conventions in Remark and Definition 2.0.2 and consider homogeneous polynomials in two variables to be equal if they are equal up to multiplication with units.
Let $\mathfrak{f}(x) = \sum_{j=0}^{d} f_j x^{d-j} \in K[x]$ be a polynomial of degree d. Then, as K is algebraically closed, $\mathfrak{f}(x)$ has d roots $\xi_1, \dots, \xi_d \in K$, so that $\mathfrak{f}(x) = f_0(x - \xi_1) \cdots (x - \xi_d)$. Now, several of the ξ_j might coincide; for example, they might be distributed according to λ, i.e., $\mathfrak{f}(x) = f_0(x - \xi_1)^{\lambda_1} \cdots (x - \xi_e)^{\lambda_e}$. In this Chapter, we want to determine algebraic conditions for this behaviour, that is equations for the coefficients $f_0, \dots, f_d$ of $\mathfrak{f}(x)$ whose vanishing is equivalent to the roots of $\mathfrak{f}(x)$ being distributed according to λ. The easiest case is well known: The discriminant of a polynomial vanishes if and only if the polynomial has at least one multiple root, i.e., if and only if its roots are distributed according to the partition $(1^{d-2}\, 2^1)$ of d. Observe that the discriminant also vanishes if the linear factors of $\mathfrak{f}(x)$ are distributed according to a coarsening of $(1^{d-2}\, 2^1)$, e.g., if $\mathfrak{f}(x) = (x - \varepsilon_1)^3(x - \varepsilon_2) \cdots (x - \varepsilon_{d-2})$. It is more natural to study this question for binary forms instead of polynomials in one variable, that is we ask for algebraic conditions such that the linear factors of a binary form of degree d are distributed according to λ. Note that there is a K-isomorphism between $K[x, y]_d$ and the vector space $K[x]_{\leqslant d}$ of polynomials in one indeterminate of degree less or equal than d given by $y \mapsto 1$ which respects the distribution of linear factors and roots, respectively.
The main object studied in this Chapter is the coincident root locus (abbreviated CRL) X_λ of binary forms $\mathfrak{f}$ of degree d over K which can be written in the form $\mathfrak{f} = \mathfrak{l}_1^{\lambda_1} \cdots \mathfrak{l}_e^{\lambda_e}$ with linear binary forms $\mathfrak{l}_1, \dots, \mathfrak{l}_e \in \mathbb{P}(\mathrm{Sym}^1(V))$. In our discussion, we closely follow the lead of Chipalkatti's work [Ch1], which mostly uses tools from modern algebraic geometry and representation theory. But the question for algebraic conditions for the linear factors of binary forms to be distributed according to λ is older than those branches of mathematics and harks back at

least to 19th century invariant theory; Chipalkatti cites Arthur Cayley's paper [C] as the first known treatment of this subject in this generality. Of course, the special case of the discriminant has been studied even earlier. Jerzy Weyman gives an in-depth study of the ideal of X_λ for $\lambda = (1^{d-p}p^1)$ with $p \in [d]$ in [W1],[W2], and [W3]; in particular, in [W1] he shows that for $p > \frac{d}{2}$, the ideal of $X_{(1^{d-p}p^1)}$ is generated in degree less than 5. For a beautiful invariant theoretic version of such algebraic conditions, see [Ch2]. A more geometric approach can be found in [Ka] (where the term 'discriminant variety' is used instead of 'coincident root locus').

As we will see, it is easier to study the closed subscheme $\Gamma_\lambda = X_\lambda \times Y_\lambda$ of $\mathbb{P}(\mathrm{Sym}^d(V)) \times Y_\lambda$ than X_λ, where Y_λ is the normalization of X_λ. Chipalkatti gives a resolution of Γ_λ in form of an Eagon-Northcott complex (see [EaNo]) and uses it to great effect. Here, we explicitly determine minimal sets of local equations defining Γ_λ and use them to fix a gap in the proof of [Ch1, Theorem 3.1] as well as to get the homogeneous ideal of X_λ in $K[z_0, \ldots, z_d]$. This also yields an algorithm for computing equations defining X_λ. Moreover, we study the singular locus of X_λ. But before the introduction of the CRL, we have to define binary forms:

Definition 2.0.1. Let S be a ring and let $d \in \mathbb{N}_0$. Let x and y be two indeterminates, and denote $S[x,y]_1 \cong xS \oplus yS$ the free S-module of rank two with basis (x,y). A *binary form* $\mathfrak{f} = \mathfrak{f}(x,y)$ *of degree d in x and y over S* is a closed point $\mathfrak{f} \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(S[x,y]_1))$ (for the notations compare 0.4 §§10, 12, 13).
A *linear (binary) form in x and y over S* is a binary form of degree 1 in x and y over S.

Note that we use Gothic letters to denote binary forms since we consider them as closed points of a projective space and therefore as ideals (see 0.4 §14). But at the same time, we also consider them as polynomials in two indeterminates as explained in

Remark and Definition 2.0.2. Keep the above notations, and let $S = K$. Our definition of binary forms coincides with the one of classical invariant theory in the following way: The group of units K^* acts multiplicatively on $\mathrm{Sym}^d(K[x,y]_1$, and we identify

$$\mathbb{P}(\mathrm{Sym}^d(K[x,y]_1)) = (K[x,y]_d \backslash \{0\})/K^*.$$

The monomials $x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d$ form a basis of $\mathrm{Sym}^d(K[x,y]_1)$. Moreover, $\mathfrak{f} = (f_0 : \cdots : f_d)$ is a coordinate representation of $\mathfrak{f} \in \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1)$ with respect to $x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d$ if and only if

$$\mathfrak{f} = (f_0x^d + f_1x^{d-1}y + \cdots + f_dy^d) \bmod K^* = \Big(\sum_{j=0}^{d} f_j x^{d-j} y^j\Big) \bmod K^*.$$

We call $\underline{f} = (f_0, \ldots, f_d)$ the *coefficients of* $\mathfrak{f}$. Coefficients $\underline{f} \in K^{d+1} \backslash \{\underline{0}\}$ determine a unique closed point $\mathfrak{f} \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$. On the other hand, the polynomial $\sum_{j=0}^d f_j x^{d-j} y^j \in K[x,y]_d$ is only determined by $\mathfrak{f} \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$ up to multiplication with elements of K^*. But as the the invariant theorists of the 19th century, we are only interested in properties of binary forms that do not change under multiplication with elements of K^*, notably in the number of roots and their multiplicities as explained above. We will omit the notation 'mod K^*', use the notation '$\simeq$' for 'equal modulo K^*', and write for $\mathfrak{f} = (f_0 : \cdots : f_d) \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$ and any $\kappa \in K^*$

$$\mathfrak{f} \simeq \sum_{j=0}^{d} f_j x^{d-j} y^j \simeq \sum_{j=0}^{d} \kappa f_j x^{d-j} y^j \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1)).$$

Now, let $d' \in \mathbb{N}_0$. For two binary forms $\mathfrak{f} \simeq \sum_{j=0}^{d} f_j x^{d-j} y^j \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$ and $\mathfrak{g} \simeq \sum_{j=0}^{d'} g_j x^{d'-j} y^j \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^{d'}(K[x,y]_1))$, we define $\mathfrak{f} \cdot \mathfrak{g}$ to be the binary form of degree $d + d'$ given by the product of polynomials, that is

$$\mathfrak{f} \cdot \mathfrak{g} \simeq \left(\sum_{j=0}^{d} f_j x^{d-j} y^j \right) \cdot \left(\sum_{j=0}^{d'} g_j x^{d'-j} y^j \right) \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^{d+d'}(K[x,y]_1)).$$

Since K is algebraically closed, for any binary form $\mathfrak{f}$ of degree d over K there are d linear binary forms $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$ over K with $\mathfrak{f} = \prod_{j=1}^{d} \mathfrak{l}_j$. Moreover, up to order this linear binary forms are uniquely determined. We call $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$ *the linear factors of* $\mathfrak{f}$.

Henceforth, we will make frequent use in this Chapter of the notations $[m] = \{1, \ldots, m\}$ and $[m]_0 = \{0, \ldots, m\}$ for $m \in \mathbb{N}_0$ (compare 0.4 §1).

2.1 Comparing coefficients

First, we want to make explicit the quite technical, but often used tool of comparing coefficients of two polynomials in our situation. In particular, we want to give a formula for the coefficients of a binary form of degree d whose linear factors are distributed according to λ. To this end, consider the projective spaces $\mathbb{P}(\mathrm{Sym}^{e_1}(V)), \ldots, \mathbb{P}(\mathrm{Sym}^{e_d}(V))$. Observe that for $e_r = 0$ the space $\mathbb{P}(\mathrm{Sym}^{e_r}(V)) = \mathbb{P}^0$ consists only of the closed point 1, which we will neglect throughout. For $r \in [d]$, we again write a binary form $\mathfrak{g}_r \mathrel{\dot\in} \mathbb{P}(\mathrm{Sym}^{e_r}(V))$ in polynomial form

$$\mathfrak{g}_r = (g_{r,0} : \cdots : g_{r,e_r}) \simeq \sum_{t=0}^{e_r} g_{r,t} x^{e_r - t} y^t \in (K[x,y]_{e_r} \backslash \{0\})/K^*.$$

Definition 2.1.1. Let $j \in \{0, \ldots, d\}$. We define a set

$$N_j = N_{\lambda,j} := \left\{ \begin{array}{l} \underline{\underline{\nu}} = (\underline{\nu}_1 = (\nu_{1,0}, \ldots, \nu_{1,e_1}), \ldots, \underline{\nu}_d) \in \mathbb{N}_0^{e_1+1} \times \cdots \times \mathbb{N}_0^{e_d+1} \mid \\ (\forall r \in [d] : \sum_{t=0}^{e_r} \nu_{r,t} = r) \wedge \sum_{r=1}^{d} (\sum_{t=0}^{e_r} t\nu_{r,t}) = j \end{array} \right\}.$$

For any $\underline{\underline{\nu}} \in N_j$, we define the integer

$$\beta\left(\underline{\underline{\nu}}\right) := \prod_{r=1}^{d} \frac{r!}{\nu_{r,0}! \nu_{r,1}! \cdots \nu_{r,e_r}!},$$

and for a tuple of tuples $\underline{\underline{h}} \in K^{e_1+1} \times \cdots \times K^{e_d+1}$ (compare 0.4 §2 for the notation) we write

$$\underline{\underline{h}}^{\underline{\underline{\nu}}} := \prod_{r=1}^{d} \prod_{t=0}^{e_r} h_{r,t}^{\nu_{r,t}}.$$

Lemma 2.1.2. *Let* $\mathfrak{g}_1 = (g_{1,0} : \cdots : g_{1,e_1}) \dot{\in} \mathbb{P}(\mathrm{Sym}^{e_1}(V)), \ldots, \mathfrak{g}_d = (g_{d,0} : \cdots : g_{d,e_d}) \dot{\in} \mathbb{P}(\mathrm{Sym}^{e_d}(V))$ *be binary forms of degree* $e_1, \ldots, e_d$*, respectively. Then*

$$\prod_{r=1}^{d} \mathfrak{g}_r^r = \sum_{j=0}^{d} \left(\sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) \underline{\underline{g}}^{\underline{\underline{\nu}}} \right) x^{d-j} y^j.$$

Proof. Denote

$$\Lambda := \{0, \ldots, e_1\} \times \{0, \ldots, e_2\}^2 \times \cdots \times \{0, \ldots, e_d\}^d.$$

Then, expanding the product

$$\prod_{r=1}^{d} \mathfrak{g}_r^r = \prod_{r=1}^{d} \left(g_{r,0} x^{e_r} + g_{r,1} x^{e_r - 1} y + \cdots + g_{r,e_r} y^{e_r} \right)^r,$$

we get a sum of terms of the form $g_{1,\tau_{1,1}} g_{2,\tau_{2,1}} g_{2,\tau_{2,2}} g_{3,\tau_{3,1}} \cdots g_{d,\tau_{d,d}} x^{d-j(\underline{\underline{\tau}})} y^{j(\underline{\underline{\tau}})}$ for $\underline{\underline{\tau}} \in \Lambda$ and some $j(\underline{\underline{\tau}}) \in [d]_0$. For any $r \in [d], t \in [r]_0$, the coefficient $g_{r,t}$ appears in the monomial $g_{r,t} x^{e_r - t} y^t$ of $g_{r,0} x^{e_r} + g_{r,1} x^{e_r - 1} y + \cdots + g_{r,e_r} y^{e_r}$, hence

$$j(\underline{\underline{\tau}}) = \tau_{1,1} + \tau_{2,1} + \tau_{2,2} + \tau_{3,1} + \cdots + \tau_{d,d} \leqslant \sum_{r=1}^{d} r e_r = d$$

for all $\underline{\underline{\tau}} \in \Lambda$. Thus, writing

$$\underline{\underline{g}}_{\underline{\underline{\tau}}} := g_{1,\tau_{1,1}} g_{2,\tau_{2,1}} g_{2,\tau_{2,2}} g_{3,\tau_{3,1}} \cdots g_{d,\tau_{d,d}} = \prod_{r=1}^{d} \prod_{j=1}^{r} g_{r,\tau_{r,j}}$$

for $\tau \in \Lambda$, we get

$$\begin{aligned} \textstyle\prod_{r=1}^{d} \mathfrak{g}_r^r &\simeq \textstyle\prod_{r=1}^{d} \left(g_{r,0} x^{e_r} + g_{r,1} x^{e_r - 1} y + \cdots + g_{r,e_r} y^{e_r} \right)^r \\ &= \textstyle\sum_{\underline{\underline{\tau}} \in \Lambda} g_{1,\tau_{1,1}} g_{2,\tau_{2,1}} g_{2,\tau_{2,2}} g_{3,\tau_{3,1}} \cdots g_{d,\tau_{d,d}} x^{d-j(\underline{\underline{\tau}})} y^{j(\underline{\underline{\tau}})} \\ &= \textstyle\sum_{\underline{\underline{\tau}} \in \Lambda} \underline{\underline{g}}_{\underline{\underline{\tau}}} x^{d-j(\underline{\underline{\tau}})} y^{j(\underline{\underline{\tau}})}. \end{aligned}$$

We now fix $j \in [d]_0$, and we want to find all $\underline{\underline{\tau}} \in \Lambda$ with $j(\underline{\underline{\tau}}) = j$; we denote the set of these $\underline{\underline{\tau}}$ by Λ_j, thus $\Lambda_j = \{\underline{\underline{\tau}} \in \Lambda \mid j(\underline{\underline{\tau}}) = j\}$. This is equivalent to finding all possible ways to distribute j 'balls' to the $1 + 2 + 3 + \cdots + d = \frac{d(d+1)}{2}$ 'urns' $\tau_{1,1}, \tau_{2,1}, \tau_{2,2}, \tau_{3,1}, \ldots, \tau_{d,d}$ such that $\tau_{r,k} \leqslant e_r$ for all $(r,k) \in [d] \times [r]$. Now, for $\underline{\underline{\tau}} \in \Lambda_j$ and any $r \in [d], t \in [e_r]_0$, we count the number of $\tau_{r,k}$ that equal t:

$$\nu_{r,t}(\underline{\underline{\tau}}) := \#\{k \in [r] \mid \tau_{r,k} = t\}.$$

Then, it holds

$$\left(\forall r \in [d] : \sum_{t=0}^{e_r} \nu_{r,t}(\underline{\underline{\tau}}) = r \right) \quad \text{and} \quad \sum_{r=1}^{d} \left(\sum_{t=0}^{e_r} t \nu_{r,t}(\underline{\underline{\tau}}) \right) = j.$$

Therefore,

$$\underline{\underline{\nu}}(\underline{\underline{\tau}}) := \left((\nu_{1,0}(\underline{\underline{\tau}}), \ldots, \nu_{1,e_1}(\underline{\underline{\tau}})), \ldots, (\nu_{d,0}(\underline{\underline{\tau}}), \ldots, \nu_{d,e_d}(\underline{\underline{\tau}})) \right) \in N_j$$

with

$$\underline{\underline{g}}_{\underline{\underline{\tau}}} = \prod_{r=1}^{d} g_{r,0}^{\nu_{r,0}(\underline{\underline{\tau}})} \cdots g_{r,e_r}^{\nu_{r,e_r}(\underline{\underline{\tau}})} = \underline{\underline{g}}^{\underline{\underline{\nu}}(\underline{\underline{\tau}})}.$$

Of course, switching $\tau_{r,k}$ and $\tau_{r,l}$ in $\underline{\underline{\tau}} \in \Lambda_j$ for $r \in [d]$ and $k, l \in [r]$ does neither change the product $g_{\underline{\underline{t}}}$ nor the power $\underline{\underline{\nu}}(\underline{\underline{\tau}})$.
We now claim that for any $\underline{\underline{\nu}} \in N_j$, there are exactly $\beta(\underline{\underline{\nu}})$ multi-indices $\underline{\underline{\tau}} \in \Lambda_j$ with $\underline{\underline{\nu}}(\underline{\underline{\tau}}) = \underline{\underline{\nu}}$. Indeed, for each $\underline{\underline{\nu}} = (\underline{\nu}_1, \ldots, \underline{\nu}_d) \in N_j$ and $r \in [d]$, there are

$$\begin{aligned}
\beta_r(\underline{\nu}_r) &:= \binom{r}{\nu_{r,0}}\binom{r-\nu_{r,0}}{\nu_{r,1}} \cdots \binom{r-\nu_{r,0}-\nu_{r,1}-\cdots-\nu_{r,e_r-1}}{\nu_{r,e_r}} \\
&= \frac{r!}{\nu_{r,0}!(r-\nu_{r,0})!} \frac{(r-\nu_{r,0})!}{\nu_{r,1}!(r-\nu_{r,0}-\nu_{r,1})!} \\
&\quad \cdots \frac{(r-\nu_{r,0}-\cdots-\nu_{r,e_r-2})!}{\nu_{r,e_r-1}!(r-\nu_{r,0}-\cdots-\nu_{r,e_r-1})!} \frac{(r-\nu_{r,0}-\cdots-\nu_{r,e_r-1})!}{\nu_{r,e_r}!(r-\nu_{r,0}-\cdots-\nu_{r,e_r})!} \\
&= \frac{r!}{\nu_{r,0}!\nu_{r,1}!\cdots\nu_{r,e_r}!}
\end{aligned}$$

ways to distribute first $\nu_{r,0}$ zeroes onto the r spots $\tau_{r,1}, \ldots, \tau_{r,r}$, then $\nu_{r,1}$ ones to the remaining $r - \nu_{r,0}$ spots, etc. Therefore, there are

$$\prod_{r=1}^{d} \beta_r(\underline{\nu}_r) = \beta(\underline{\underline{\nu}})$$

ways to distribute j onto $\tau_{1,1}, \tau_{2,1}, \ldots, \tau_{d,d}$ according to $\underline{\underline{\nu}}$.
Therefore, we get

$$\sum_{\underline{\underline{\tau}} \in \Lambda_j} \underline{\underline{g}}_{\underline{\underline{\tau}}} = \sum_{\underline{\underline{\nu}} \in N_j} \underline{\underline{g}}^{\underline{\underline{\nu}}}.$$

This yields our claim. □

2.2 The coincident root locus

The object of our interest is the coincident root locus (CRL) with multiplicities λ, which as a set is

$$X_\lambda = \{\mathfrak{f} \mathrel{\dot{\in}} \mathbb{P}(\mathrm{Sym}^d(V)) \mid \mathfrak{f} = \prod_{i=1}^{e} \mathfrak{l}_r^{\lambda_r} \text{ for linear binary forms } \mathfrak{l}_1, \ldots, \mathfrak{l}_e \in V\}.$$

Note that the linear forms $\mathfrak{l}_1, \ldots, \mathfrak{l}_e$ need not be distinct. The set X_λ is a projective variety in $\mathbb{P}(\mathrm{Sym}^d(V))$. For example, for the partition $\lambda = (2)$ of $d = 2$, the CRL $X_{(2)} \subseteq \mathbb{P}^2$ is given by the discriminant locus, that is it consists of closed points $(f_0 : f_1 : f_2)$ with $f_1^2 - 4f_0f_2 = 0$.

Definition 2.2.1. Let $\mathfrak{f} \mathrel{\dot{\in}} \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$ be a binary form of degree d. We say that *the linear factors of* $\mathfrak{f}$ *are distributed according to* λ if there are linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_e$ of $\mathfrak{f}$ with $\mathfrak{f} = \prod_{i=1}^{d} \mathfrak{l}_i^{\lambda_i}$.

The above description of X_λ is not sufficient for our purposes. Hence, we now give a definition of X_λ in the language of schemes following [Ch1, 2.4]. To this end, consider the map

$$\upsilon_r : \mathbb{P}(\mathrm{Sym}^{e_r}(V)) \to \mathbb{P}(\mathrm{Sym}^{re_r}(V)); \quad \mathfrak{g}_r \mapsto \mathfrak{g}_r^r = \left(\sum_{t=0}^{e_r} g_{r,t} x^{e_r-t} y^t\right)^r$$

and the multiplication map

$$\chi : \prod_{r=1}^{d} \mathbb{P}(\mathrm{Sym}^{re_r}(V)) \to \mathbb{P}(\mathrm{Sym}^d(V)); \quad (\mathfrak{h}_1, \ldots, \mathfrak{h}_d) \mapsto \prod_{r=1}^{d} \mathfrak{h}_r.$$

Definition 2.2.2. Keep the above notations. We define a scheme

$$Y_\lambda := \prod_{r=1}^{d} \mathbb{P}(\mathrm{Sym}^{e_r}(V))$$

and a morphism of schemes

$$\psi_\lambda := \chi \circ \prod_{r=1}^{d} v_r : Y_\lambda \to \mathbb{P}(\mathrm{Sym}^d(V));$$

on closed points, ψ_λ is given by

$$(\mathfrak{g}_1, \ldots, \mathfrak{g}_d) \mapsto \psi_\lambda\left((\mathfrak{g}_1, \ldots, \mathfrak{g}_d)\right) = \prod_{r=1}^{d} \mathfrak{g}_r^r.$$

Now, the *coincident root locus (CRL) X_λ with multiplicities* λ is the (scheme-theoretic) image of ψ_λ.

For the notion of scheme-theoretic image see [Ha, II, Ex. 3.11 (d)]. The CRL X_λ is a closed subscheme of $\mathbb{P}(\mathrm{Sym}^d(V))$, and since Y_λ is reduced, so is X_λ.

Remark 2.2.3. Note that Y_λ is the normalization of X_λ. Also, the dimension of X_λ is

$$\dim(X_\lambda) = \dim(Y_\lambda) = \sum_{r=1}^{d} e_r = e,$$

the number of entries in λ (compare [Ch1, Section 2.4]).

Unfortunately, the structure of X_λ is not simple, so we concentrate on the more accessible scheme Γ_λ:

Definition 2.2.4. Define

$$T_\lambda := \mathbb{P}(\mathrm{Sym}^d(V)) \times \prod_{r=1}^{d} \mathbb{P}(\mathrm{Sym}^{e_r}(V)) = \mathbb{P}(\mathrm{Sym}^d(V)) \times Y_\lambda.$$

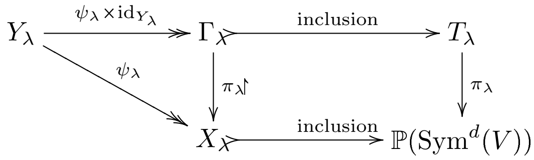
Then, there is a projective morphism $\psi_\lambda \times \mathrm{id}_{Y_\lambda} : Y_\lambda \to T_\lambda$, and we define a closed subscheme of T_λ by

$$\Gamma_\lambda := \mathrm{im}(\psi_\lambda \times \mathrm{id}_{Y_\lambda}).$$

Remark 2.2.5. Being the (scheme-theoretic) image of the irreducible scheme Y_λ under the projective morphism $\psi_\lambda \times \mathrm{id}_{Y_\lambda}$, the scheme Γ_λ is irreducible, too. Moreover, $\psi_\lambda \times \mathrm{id}_{Y_\lambda} : Y_\lambda \xrightarrow{\cong} \Gamma_\lambda$ is an isomorphism, so that Γ_λ is smooth. Let

$$\pi_\lambda : T_\lambda \to \mathbb{P}(\mathrm{Sym}^d(V))$$

be the projection to the first factor of T_λ. We then obtain the following commutative diagram:



A closed point of Γ_λ is of the form $(\mathfrak{f}, \mathfrak{g}_1, \ldots, \mathfrak{g}_d)$ with $\mathfrak{f} \dot{\in} \mathbb{P}(\mathrm{Sym}^d(V))$ and $\mathfrak{g}_r \dot{\in} \mathbb{P}(\mathrm{Sym}^{e_r}(V))$ for $r \in [d]$ such that $\mathfrak{f} = \mathfrak{g}_1 \cdot \mathfrak{g}_2^2 \cdots \mathfrak{g}_d^d$. For example, a closed point $((f_0 : f_1 : f_2), (g_0 : g_1)) \in \Gamma_{(2)} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ satisfies $f_0 = \kappa g_0^2, f_1 = 2\kappa g_0 g_1$, and $f_2 = \kappa g_1^2$ for some $\kappa \in K^*$.

2.3 About T_λ: notational considerations

In this and the next Section, we often omit λ from our notations, i.e. we write $Y = Y_\lambda, X = X_\lambda, \Gamma = \Gamma_\lambda, A^{(r)} = A^{(\lambda,r)}$, etc.
Let $Z_0, \ldots, Z_d, W_{1,0}, \ldots, W_{1,e_1}, W_{2,0}, \ldots, W_{d,e_d}$ be indeterminates, and write

$$\begin{array}{rl} & A^{(0)} = A^{(\lambda,0)} := K[\underline{Z}] = K[Z_0, \ldots, Z_d], \\ \forall r \in [d]: & A^{(r)} = A^{(\lambda,r)} := K[\underline{W}_r] = K[W_{r,0}, \ldots, W_{r,e_r}] \end{array}$$

for the coordinate rings of $\mathbb{P}(\mathrm{Sym}^d(V)), \mathbb{P}(\mathrm{Sym}^{e_1}(V)), \ldots, \mathbb{P}(\mathrm{Sym}^{e_d}(V))$. Then, the coordinate ring of T is the Segre product

$$A^{(\lambda)} := A^{(0)} \boxtimes A^{(1)} \boxtimes \cdots \boxtimes A^{(d)},$$

and a closed point $\mathfrak{t} \dot{\in} T_\lambda$ can be written

$$\begin{aligned} \mathfrak{t} &= (\mathfrak{f}, \mathfrak{g}_1, \ldots, \mathfrak{g}_d) \\ &= (\underline{f}, \underline{\underline{g}}) \\ &= ((f_0 : \cdots : f_d), (g_{1,0} : \cdots : g_{1,e_1}), \ldots, (g_{d,0} : \cdots : g_{d,e_d})) . \end{aligned}$$

Moreover, according to Remark 2.2.5, we have the equivalence

$$\begin{aligned} \mathfrak{f} &\simeq f_0 x^d + f_1 x^{d-1} y + \cdots + f_d y^d \\ &\simeq (g_{1,0} x^{e_1} + f_{1,1} x^{e_1 - 1} y + \cdots + g_{1,e_1} y^{e_1}) \\ &\quad \cdot (g_{2,0} x^{e_2} + g_{2,1} x^{e_2-1} y + \cdots + g_{2,e_2} y^{e_2})^2 \\ &\quad \cdots (g_{d,0} x^{e_d} + \cdots + g_{d,e_d} y^d)^d \\ &\simeq \prod_{r=1}^d \mathfrak{g}_r^r \end{aligned}$$

$$\Longleftrightarrow$$

$$\mathfrak{t} \dot{\in} \Gamma_\lambda.$$

To get an affine covering of $T = \mathbb{P}(\mathrm{Sym}^d(V)) \times \mathbb{P}(\mathrm{Sym}^{e_1}(V)) \times \cdots \times \mathbb{P}(\mathrm{Sym}^{e_d}(V))$, we can take the product of affine coverings of the factor spaces of T_λ. More precisely, let $r \in [d]$. Then, $(U_{r,i})_{i=0}^{e_r}$ is an affine covering of $\mathbb{P}(\mathrm{Sym}^{e_r}(V))$, where $U_{r,i}$ is the i-th standard affine chart of $\mathbb{P}(\mathrm{Sym}^{e_r}(V)) = \mathrm{Proj}(A^{(\lambda,r)})$. Writing

$$\begin{aligned} A^{(\lambda,r,i)} = A^{(r,i)} &:= K[\underline{w}_{r,i}] \\ &= K\left[w_{r,i,0} := \frac{W_{r,0}}{W_{r,i}}, \ldots, w_{r,i,e_r} := \frac{W_{r,e_r}}{W_{r,i}}\right] \end{aligned}$$

for $i \in [e_r]_0$, we have

$$U_{r,i} = \operatorname{Spec}\left(A^{(r,i)}\right) \cong \mathbb{A}_K^{e_r}.$$

On $\mathbb{P}(\operatorname{Sym}^d(V))$, we consider the affine covering $(U_i)_{i=0}^d$ where

$$U_i = \operatorname{Spec}\left(A^{(0,i)}\right) \cong \mathbb{A}_K^d$$

with

$$A^{(0,i)} = A^{(\lambda,0,i)} := K[\underline{z} = \underline{z}_i] = K\left[z_0 = z_{i,0} := \frac{Z_0}{Z_i}, \dots, z_d = z_{i,d} := \frac{Z_d}{Z_i}\right]$$

for $i \in [d]_0$. Now, let

$$\mathbb{I} = \mathbb{I}_\lambda := [d]_0 \times [e_1]_0 \times \cdots \times [e_d]_0 \subseteq \mathbb{N}_0^{d+1}.$$

Then,

$$\left(\underline{U_{\underline{i}}} := U_{i_0} \times U_{1,i_1} \times \cdots \times U_{d,i_d}\right)_{\underline{i}=(i_0,\dots,i_d)\in\mathbb{I}_\lambda}$$

is an open affine covering of T with

$$\underline{U_{\underline{i}}} = \operatorname{Spec}\left(A^{(\lambda,\underline{i})}\right) \cong \mathbb{A}_k^{d+e}$$

where

$$A^{(\underline{i})} = A^{(\lambda,\underline{i})} := A^{(0,i_0)} \otimes \bigotimes_{r=1}^{d} A^{(r,i_r)} = K[\underline{z} = \underline{z}_{i_0}, \underline{\underline{w}} = (\underline{w}_{1,i_1}, \dots, \underline{w}_{d,i_d})]$$

for all $\underline{i} \in \mathbb{I}$.
For any $\underline{i} \in \mathbb{I}$, we define the canonical morphism of K-algebras

$$\rho_{\underline{i}} = \rho_{\lambda,\underline{i}} : A^{(0)} \otimes \bigotimes_{r=1}^{d} A^{(r)} = K[\underline{Z}, \underline{W}_1, \dots, \underline{W}_d] \to A^{(\underline{i})}$$

given by

$$\begin{array}{rll} & Z_{i_0} \mapsto 1, & Z_j \mapsto z_{i_0,j} \text{ for } j \neq i_0 \\ \forall r \in [d]: & W_{r,i_r} \mapsto 1, & W_{r,t} \mapsto w_{r,i_r,t} \text{ for } t \neq i_r. \end{array}$$

Observe that

$$\forall \underline{i} \in \mathbb{I}: \quad \pi_\lambda(\underline{U_{\underline{i}}}) = U_{i_0}.$$

For $\underline{i} \in \mathbb{I}$, let

$$\Gamma_{\underline{i}} = \Gamma_{\lambda,\underline{i}} := \Gamma_\lambda \cap \underline{U_{\underline{i}}}.$$

For a closed point $\mathfrak{t} = (\underline{f}, \underline{\underline{g}}) \in \Gamma_{\underline{i}}$, the image $\pi_\lambda(\mathfrak{t}) \in U_{i_0} \subseteq \mathbb{P}(\operatorname{Sym}^d(V))$ is the binary form

$$f_0 x^d + \cdots + f_{i_0-1} x^{d-i_0+1} y^{i_0-1} + x^{d-i_0} y^{i_0} + f_{i_0+1} x^{d-i_0-1} y^{i_0+1} + \cdots + f_d y^d;$$

in particular, closed points of $\Gamma_{\underline{i}}$ with $i_0 = 0$ correspond to polynomials in x and y which are monic with respect to x.

2.4 Equations for Γ_λ

We now want to compute local equations for Γ_λ, that is equations of $\Gamma_{\underline{i}}$ for any $\underline{i} \in \mathbb{I}$. Recall that for a closed point $\mathfrak{t} = (\mathfrak{f}, \mathfrak{g}) = (\underline{f}, \underline{\underline{g}}) \,\dot{\in}\, \Gamma_\lambda$, by Lemma 2.1.2 it holds

$$\textstyle\sum_{j=0}^d f_j x^{d-j} y^j \quad \simeq \quad \mathfrak{f} \quad = \quad \prod_{r=1}^d \mathfrak{g}_r^r \quad \simeq \quad \sum_{j=0}^d \left(\sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) \underline{\underline{g}}^{\underline{\underline{\nu}}} \right) x^{d-j} y^j$$

(for the notations, see Definition 2.1.1).

Definition 2.4.1. For all $j \in \{0, \dots, d\}$, we set

$$\begin{array}{rcll} \Theta_j & = & \Theta_{\lambda,j} & := & \sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) 1 \otimes \underline{W}_1^{\nu_1} \otimes \dots \otimes \underline{W}_d^{\nu_d} \\ & & & = & \sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) 1 \otimes W_{1,0}^{\nu_{1,0}} \cdots W_{1,e_1}^{\nu_{1,e_1}} \otimes \dots \otimes W_{d,0}^{\nu_{d,0}} \cdots W_{d,e_d}^{\nu_{d,e_d}} \\ & & & \in & A^{(0)} \otimes \bigotimes_{r=1}^d A^{(r)}. \end{array}$$

For $\underline{i} \in \mathbb{I}$ and $j \in \{0, \dots, d\}$, we define

$$\theta_j = \theta_{\lambda,\underline{i},j} := \rho_{\lambda,\underline{i}}(\Theta_j) \in A^{(\underline{i})} = K[\underline{z}, \underline{\underline{w}}].$$

As above, $\rho_{\underline{i}} : A^{(0)} \otimes \bigotimes_{r=1}^d A^{(r)} \to A^{(\underline{i})}$ denotes the canonical morphism. Observe that $\theta_0, \dots, \theta_d \in K[\underline{\underline{w}}]$.

Now, for any matrix $(\Omega) = (\upsilon_{k,l})_{k,l}$ with entries $\upsilon_{k,l} \in A^{(0)} \otimes \bigotimes_{r=1}^d A^{(r)}$, and for any $\underline{i} \in \mathbb{I}$, we denote by $\left(\rho_{\underline{i}}(\Omega)\right) = \left(\rho_{\underline{i}}(\upsilon_{k,l})\right)_{k,l}$ the matrix with entries $\rho_{\underline{i}}(\upsilon_{k,l}) \in A^{(\underline{i})}$.

Definition 2.4.2. Define

$$(\Phi) = (\Phi_\lambda) := \begin{pmatrix} \Theta_{\lambda,0} & \cdots & \Theta_{\lambda,i_0} & \cdots & \Theta_{\lambda,d} \\ Z_0 \otimes 1 & \cdots & Z_{i_0} \otimes 1 & \cdots & Z_d \otimes 1 \end{pmatrix},$$

a $(2 \times (d+1))$-matrix over $A^{(0)} \otimes \bigotimes_{r=1}^d A^{(r)}$, and for any $\underline{i} \in \mathbb{I}^\lambda$, define

$$\left(\varphi_{\underline{i}}\right) = \left(\varphi_{\lambda,\underline{i}}\right) := \left(\rho_{\lambda,\underline{i}}(\Phi)\right) = \begin{pmatrix} \theta_0 & \cdots & \theta_{i_0-1} & \theta_{i_0} & \theta_{i_0+1} & \cdots & \theta_d \\ z_0 & \cdots & z_{i_0-1} & 1 & z_{i_0+1} & \cdots & z_d \end{pmatrix},$$

a $(2 \times (d+1))$-matrix over $A^{(\underline{i})}$. This matrix $\left(\varphi_{\underline{i}}\right)$ defines a morphism of free $\mathcal{O}_{\underline{U}_{\underline{i}}}$-modules

$$\varphi_{\underline{i}} = \varphi_{\lambda,\underline{i}} : \left(\mathcal{O}_{\underline{U}_{\underline{i}}}\right)^{\oplus d+1} \to \left(\mathcal{O}_{\underline{U}_{\underline{i}}}\right)^{\oplus 2}.$$

For any closed point $\mathfrak{t} \,\dot{\in}\, T$, we choose $\underline{i} \in \mathbb{I}$ with $\mathfrak{t} \in \underline{U}_{\underline{i}}$, write $\mathfrak{t} = (\underline{f}, \underline{\underline{g}})$ with $f_{i_0} = g_{1,i_1} = \dots = g_{d,i_d} = 1$, and set

$$\left(\varphi_{\underline{i}}(\mathfrak{t})\right) := \begin{pmatrix} \theta_0(\underline{\underline{g}}) & \cdots & \theta_{i_0-1}(\underline{\underline{g}}) & \theta_{i_0}(\underline{\underline{g}}) & \theta_{i_0+1}(\underline{\underline{g}}) & \cdots & \theta_d(\underline{\underline{g}}) \\ f_0 & \cdots & f_{i_0-1} & 1 & f_{i_0+1} & \cdots & f_d \end{pmatrix},$$

a $(2 \times (d+1))$-matrix over K. Then, for any $\underline{i}, \underline{i}' \in \mathbb{I}$ with $\mathfrak{t} = (\underline{f}, \underline{\underline{g}}) \in \underline{U}_{\underline{i}} \cap \underline{U}_{\underline{i}'}$, the matrices $\left(\varphi_{\underline{i}}(\mathfrak{t})\right)$ and $\left(\varphi_{\underline{i}'}(\mathfrak{t})\right)$ are equal up to multiplication with a unit:

$$\begin{pmatrix} \theta_{\lambda,\underline{i},0}(\underline{\underline{g}}) & \cdots & \theta_{\lambda,\underline{i},d}(\underline{\underline{g}}) \\ z_{i_0,0}(\underline{f}) & \cdots & z_{i_0,d}(\underline{f}) \end{pmatrix} = z_{i_0,i_0'}(\underline{f}) \begin{pmatrix} \theta_{\lambda,\underline{i}',0}(\underline{\underline{g}}) & \cdots & \theta_{\lambda,\underline{i}',d}(\underline{\underline{g}}) \\ z_{i_0',0}(\underline{f}) & \cdots & z_{i_0',d}(\underline{f}) \end{pmatrix},$$

meaning

$$\varphi_{\underline{i}}\upharpoonright_{\underline{U}_{\underline{i}}\cap\underline{U}_{\underline{i}'}} = \varphi_{\underline{i}'}\upharpoonright_{\underline{U}_{\underline{i}}\cap\underline{U}_{\underline{i}'}}.$$

Hence, the $\varphi_{\underline{i}}$ glue, and we obtain a morphism of $\mathcal{O}_{T_\lambda}$-bundles

$$\varphi = \varphi_\lambda : \mathcal{O}_{T_\lambda}^{\otimes d+1} \to \mathcal{O}_{T_\lambda}^{\otimes 2}.$$

Let

$$\mathcal{I} = \mathcal{I}_\lambda := \mathrm{Fitt}_0(\mathrm{coker}(\varphi_\lambda))$$

denote the 0-th Fitting ideal sheaf of $\mathrm{coker}(\varphi)$ (see [E, 20,2] for the definition of Fitting ideals). Then, by T_φ we denote the closed subscheme of T_λ determined by $\mathcal{I}_\lambda$.

Remark 2.4.3. The reader might wonder why we study the matrices $(\varphi_{\underline{i}})$ instead of (Φ). The reason is twofold: First of all, (Φ) is not a matrix over the coordinate ring $A^{(\lambda)}$ of T, that is the entries of (Φ) are not elements of the Segre product $A^{(0)} \boxtimes \cdots \boxtimes A^{(d)}$. But the minors of (Φ) are linear forms in $A^{(\lambda)}$. Moreover, for computing equations defining CRL, it is easier to work in the polynomial rings $A^{(\lambda,\underline{i})}$ than in the Segre product $A^{(\lambda)}$.

Example 2.4.4. (A) First, let us have a look at the easiest interesting example: the partition $\lambda = (2) = (1^0\,2^1)$ of 2. As usual, we ignore $e_1 = 0$, and we just write $W_0 = W_{2,0}, B_1 = W_{2,1}$. Obviously, $\Theta_0 = 1 \otimes W_0^2$ and $\Theta_2 = 1 \otimes W_1^2$. For Θ_1, we observe that there are exactly two possibilities to distribute 2 powers on the indices $(0,1)$ such that the sum of the products of the indices and their respective power equals 1: Either we choose 0 for the first power and 1 for the second, resulting in the summand $1 \otimes W_0W_1$, or we choose 1 for the first power and 0 for the second, resulting in $1 \otimes W_1W_0$. Of course, those two summands are equal, hence, $\Theta_1 = 1 \otimes 2W_0W_1$. This coincides with $\beta\left((0),(0,1)\right) = 1 \cdot \frac{2}{1!\cdot 1!} = 2$. Therefore,

$$[\varphi_{(2)}] = \begin{bmatrix} 1 \otimes W_0^2 & 1 \otimes 2W_0W_1 & 1 \otimes W_1^2 \\ Z_0 & Z_1 & Z_2 \end{bmatrix}.$$

If we look at the affine chart $U_{(0,0,0)}$, i.e., if we set $W_0 = Z_0 = 1$, and eliminate $w_1 = \frac{W_1}{W_0}$, then we get the expected equation $z_1^2 - 4z_2$ for a monic binary form of degree 2 with exactly 1 root with multiplicity 2.

(B) For a somewhat more involved example, let us consider the partition $\lambda = (2,2,3) = (2^2\,3^1)$ of 7; we ignore any occurrence of $e_1 = e_4 = e_5 = e_7 = 0$ as well as their indices $r \in \{1,4,5,6,7\}$. Again, clearly $\Theta_0 = 1 \otimes W_{2,0}^2 \otimes W_{3,0}^3$ and $\Theta_7 = 1 \otimes W_{2,2}^2 \otimes W_{3,1}^3$. For Θ_1, we can either shift one of the two powers of $W_{2,0}$ to $W_{2,1}$, or we can shift one of the three powers of $W_{3,0}$ to $W_{3,1}$. In the first case, we can choose one of two powers, hence $\beta((1,1,0),(3,0)) = 2$, and in the second case, we can choose one of three powers, hence $\beta((2,0,0),(2,1)) = 3$. Adding those possible summands, we get $\Theta_1 = 1 \otimes (2W_{2,0}W_{2,1} \otimes W_{3,0}^3 + 3W_{2,0}^2 \otimes W_{3,0}^2W_{3,1})$. For Θ_2, we get

$$N_2 = \left\{\left((1,0,1),(3,0)\right), \left((0,2,0),(3,0)\right), \left((1,1,0),(2,1)\right), \left((2,0,0),(1,2)\right)\right\}.$$

For the distribution of powers $((1,1,0),(2,1))$, we first have to choose one of two possible powers for $W_{2,1}$ and then one of three possible powers for $W_{3,1}$. Hence, the summand $1 \otimes W_{2,0}W_{2,1} \otimes W_{3,0}^2W_{3,1}$ occurs $\binom{2}{1} \cdot \binom{3}{1} = 6$ times. Alternatively, we just could compute $\beta(((1,1,0),(2,1))) = \frac{2!}{1!\cdot 1!\cdot 0!} \cdot \frac{3!}{2!\cdot 1!} = 6$. Computing the coefficients $\beta(\cdot)$ for the other elements of N_2, we find

$$\begin{aligned}\Theta_2 \quad = \quad 1 \quad \otimes \quad \Big(& \quad 2W_{2,0}W_{2,2} \otimes W_{3,0}^3 \quad + \quad W_{2,1}^2 \otimes W_{3,0}^3 \\ + & \quad 6W_{2,0}W_{2,1} \otimes W_{3,0}^2 W_{3,1} \quad + \quad 3W_{2,0}^2 \otimes W_{3,0}W_{3,1}^2 \quad \Big).\end{aligned}$$

Further carrying out this computations, we eventually find the matrix $(\varphi_{(2,2,3)})$ (see Figure (2.1) on the next page).

$$
\begin{pmatrix}
W_{2,0}^2 \otimes W_{3,0}^3 & \begin{matrix} 2W_{2,0}W_{2,1} \otimes W_{3,0}^3 \\ + \\ 3W_{2,0}^2 \otimes W_{3,0}^2 W_{3,1} \end{matrix} & \begin{matrix} 2W_{2,0}W_{2,2} \otimes W_{3,0}^3 \\ + \\ W_{2,1}^2 \otimes W_{3,0}^3 \\ + \\ 6W_{2,0}W_{2,1} \otimes W_{3,0}^2 W_{3,1} \\ + \\ 3W_{2,0}^2 \otimes W_{3,0} W_{3,1}^2 \end{matrix} & \begin{matrix} 2W_{2,1}W_{2,2} \otimes W_{3,0}^3 \\ + \\ 6W_{2,0}W_{2,2} \otimes W_{3,0}^2 W_{3,1} \\ + \\ 3W_{2,1}^2 \otimes W_{3,0}^2 W_{3,1} \\ + \\ 6W_{2,0}W_{2,1} \otimes W_{3,0} W_{3,1}^2 \\ + \\ W_{2,0}^2 \otimes W_{3,1}^3 \end{matrix} & \begin{matrix} W_{2,2}^2 \otimes W_{3,0}^3 \\ + \\ 6W_{2,1}W_{2,2} \otimes W_{3,0}^2 W_{3,1} \\ + \\ 6W_{2,0}W_{2,2} \otimes W_{3,0} W_{3,1}^2 \\ + \\ 3W_{2,1}^2 \otimes W_{3,0} W_{3,1}^2 \\ + \\ 2W_{2,0}W_{2,1} \otimes W_{3,1}^3 \end{matrix} & \begin{matrix} 3W_{2,2}^2 \otimes W_{3,0}^2 W_{3,1} \\ + \\ 6W_{2,1}W_{2,2} \otimes W_{3,0} W_{3,1}^2 \\ + \\ 2W_{2,0}W_{2,2} \otimes W_{3,1}^3 \\ + \\ W_{2,1}^2 \otimes W_{3,1}^3 \end{matrix} & \begin{matrix} 3W_{2,2}^2 \otimes W_{3,0} W_{3,1}^2 \\ + \\ 2W_{2,1}W_{2,2} \otimes W_{3,1}^3 \end{matrix} & W_{2,2}^2 \otimes W_{3,1}^3 \\
z_0 \otimes 1 & z_1 \otimes 1 & z_2 \otimes 1 & z_3 \otimes 1 & z_4 \otimes 1 & z_5 \otimes 1 & z_6 \otimes 1 & z_7 \otimes 1
\end{pmatrix}
$$

Figure 2.1: The (2×8)-matrix $[\varphi_{2,2,3}]$ of example 2.4.4 (B); due to of a lack of space, we omit $1 \otimes (\bullet)$ in the entries of the first row.

On the affine chart $\underline{U}_{\underline{i}}, \underline{i} \in \mathbb{I}$, by definition, the ideal

$$\mathfrak{I}^{(\underline{i})} = \mathfrak{I}^{(\lambda,\underline{i})} := H^0(\underline{U}_{\underline{i}}, \mathcal{I}_\lambda) \subseteq A^{(\lambda,\underline{i})}$$

of $T_\varphi \cap \underline{U}_{\underline{i}}$ is generated by the (2×2)-minors of the matrix $(\varphi_{\underline{i}})$. We can do better than this and generate $\mathfrak{I}^{(\underline{i})}$ by d elements, as we well prove soon. Before doing so, we formulate two Lemmas resulting in Proposition 2.4.7, which at first might seem mildly interesting, but which expresses the arguably most important property of our constructions. In order to formulate them, we introduce another notation: For $l \in \{1, 2\}$, $\underline{i} \in \mathbb{I}$ an $\mathsf{t} \,\dot{\in}\, \underline{U}_{\underline{i}}$, by $\left(\varphi_{\underline{i}}(\mathsf{t})\right)_{l,\bullet}$ we denote the l-th row of $\left(\varphi_{\underline{i}}(\mathsf{t})\right)$ as a $(d+1)$-tuple of elements of K.

Lemma 2.4.5. *For any $\underline{i} \in \mathbb{I}_\lambda$ and any $\mathsf{t} = (\underline{f}, \underline{\underline{g}}) \,\dot{\in}\, \underline{U}_{\underline{i}}$, it holds*

$$\left(\varphi_{\underline{i}}(\mathsf{t})\right)_{1,\bullet} \quad \neq \quad \underline{0} \quad \neq \quad \left(\varphi_{\underline{i}}(\mathsf{t})\right)_{2,\bullet}.$$

Proof. As $\underline{f}$ denotes a closed point in $\mathbb{P}(\mathrm{Sym}^d(V))$, the second inequality is clear. To show the first one, assume the contrary, i.e., assume $\theta_j(\underline{\underline{g}}) = 0$ for all $j \in [d]_0$. In particular,

$$0 = \theta_0(\underline{\underline{g}}) = \prod_{r=1}^{d} g_{r,0}^r$$

implies

$$M_0 := \{r \in [d]_0 \mid g_{r,0} = 0\} \neq \varnothing.$$

The summands of θ_1 are obtained (up to their multiplicity $\beta(\cdot)$) by shifting exactly one factor $w_{r,0}^r$ of θ_0 to $w_{r,0}^{r-1} w_{r,1}$; the summands of θ_2 are in turn (again, up to the multiplicities $\beta(\cdot)$) obtained by taking one of the summands of θ_1 and shifting one factor $w_{r,t}$ to $w_{r,t+1}$ and so on. Hence, after

$$t_1 := \sum_{r \in M_0} r$$

such steps, we find

$$\theta_{t_1}(\underline{\underline{g}}) = \left(\prod_{r \in M_0} g_{r,1}^r\right)\left(\prod_{r \in [d]_0 \setminus M_0} g_{r,0}^r\right) + C,$$

where

$$C = \sum_{\underline{\underline{\nu}} \in N_{t_1} \setminus \{\underline{\underline{\hat{\nu}}}\}} \prod_{r=1}^{e_r} g_{r,0}^{\nu_{r,0}} \cdots g_{r,e_r}^{\nu_{r,e_r}}$$

with

$$\underline{\underline{\hat{\nu}}} := ((0, 1, 0, \ldots, 0), (0, 2, 0, \ldots, 0), \ldots, (0, d, 0, \ldots, 0)).$$

In other words, C is a sum of products obtained by t_1 shifts $g_{r,t} \mapsto g_{r,t+1}$ of which at least one occurs for $r \notin M_0$. It follows that every summand of C contains a factor $g_{r,0}$ for some $r \in M_0$, thus,

$$C = 0.$$

Now, as by assumption $0 = \theta_{t_1}(\underline{\underline{b}})$ and by construction $\prod_{r\in[d]_0\setminus M_0} g_{r,0} \neq 0$, it follows
$$M_1 := \{r \in M_0 \mid g_{r,1} = 0\} \neq \varnothing.$$
We repeat this construction with $t_2 := t_1+\sum_{r\in M_1} r$, and with the same argument as above, we get $0 = \theta_{t_2}(\underline{\underline{g}}) = (\prod_{r\in M_1} g_{r,2}^r)\cdot c + 0$ with $c \neq 0$, hence
$$M_2 := \{r \in M_1 \mid g_{r,2} = 0\} \neq \varnothing.$$
Continuing inductively, we get a chain of sets $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq \cdots$ with $M_k \neq \varnothing$. But, after at most
$$k_0 \leqslant \max\{e_r \mid r \in M_0\}$$
such steps, we find an element $r \in M_{k_0}$ with $g_{r,0} = \cdots = g_{r,e_r} = 0$, a contradiction to $(g_{r,0} : \cdots : g_{r,e_r}) \dot{\in} \mathbb{P}(\mathrm{Sym}^{e_r}(V))$ a closed point. Hence, there is at least one $j \in [d]_0$ with $\theta_j(\underline{\underline{g}}) \neq 0$. □

Lemma 2.4.6. *Let $\underline{i} \in \mathbb{I}_\lambda$. Then, for any closed point $\mathfrak{t} \in \underline{U_{\underline{i}}}$, it holds*
$$\mathfrak{t} \in T_\varphi \Leftrightarrow \left(\theta_{i_0}(\mathfrak{t}) \in K^* \text{ and } \left(\varphi_{\underline{i}}(\mathfrak{t})\right)_{1,\bullet} = \theta_{i_0}(\mathfrak{t}) \cdot \left(\varphi_{\underline{i}}(\mathfrak{t})\right)_{2,\bullet}\right).$$

Proof. By [E, 20.2], the Fitting ideal $\mathfrak{I}^{(\underline{i})} = \mathrm{Fitt}_0(\mathrm{coker}(\varphi_{\underline{i}}))$ defines the closed subset of $\underline{U_{\underline{i}}}$ of closed points $\mathfrak{t}$ such that $\mathrm{rank}\left(\left(\varphi_{\underline{i}}(\mathfrak{t})\right)\right) < 2$. Now, by the above Lemma, $\mathrm{rank}\left(\left(\varphi_{\underline{i}}(\mathfrak{t})\right)\right) \leqslant 1$ is equivalent to the existence of a unit $\kappa_{\mathfrak{t}} \in K^*$ such that
$$\left(\varphi_{\underline{i}}(\mathfrak{t})\right)_{1,\bullet} = \kappa_{\mathfrak{t}} \cdot \left(\varphi_{\underline{i}}(\mathfrak{t})\right)_{2,\bullet}.$$
As $Z_{i_0}(\mathfrak{t}) = 1$, it also must hold $\kappa_{\mathfrak{t}} = \theta_{i_0}(\mathfrak{t})$. □

Proposition 2.4.7. *For any $\underline{i} \in \mathbb{I}_\lambda$, the element θ_{i_0} is a unit in $A^{(\underline{i})}/\mathfrak{I}^{(\underline{i})}$.*

Proof. $A^{(i)}/\mathfrak{I}^{(i)}$ is the coordinate ring of the affine variety $\Gamma_{\underline{i}}$, and θ_{i_0} and $\theta_{i_0}^{-1}$ are rational functions on $\Gamma_{\underline{i}}$ vanishing nowhere by Lemma 2.4.6. Hence, θ_{i_0} is a regular function on $\Gamma_{\underline{i}}$, and this proves our claim. □

Lemma 2.4.8. *Let $\underline{i} \in \mathbb{I}_\lambda$. The ideal $\mathfrak{I}^{(\lambda,\underline{i})} \subseteq A^{(\lambda,\underline{i})} = K[\underline{z},\underline{\underline{w}}]$ is generated by the $A^{(\lambda,\underline{i})}$-sequence of length d*
$$\theta_{i_0}z_0 - \theta_0, \ldots, \theta_{i_0}z_{i_0-1} - \theta_{i_0-1}, \theta_{i_0}z_{i_0+1} - \theta_{i_0+1}, \ldots, \theta_{i_0}z_d - \theta_d.$$

Proof. The elements $\theta_{i_0}z_j - \theta_j$, $j \in [d]_0\setminus\{i_0\}$ are (2×2)-minors of $\left(\varphi_{\underline{i}}\right)$, and they generate $\mathfrak{I}^{(\underline{i})}$ since for any $j,l \in [d]_0$, it holds
$$\theta_j z_l - \theta_l z_j = z_j(\theta_{i_0}z_l - \theta_l) - z_l(\theta_{i_0}z_j - \theta_j).$$
Moreover, as elements of the polynomial ring $A^{(\underline{i})} = \left(K[\underline{\underline{w}}]\right)[\underline{z}]$ over the integral domain $K[\underline{\underline{w}}]$ with θ_{i_0} a unit in $A^{(\underline{i})}/\mathfrak{I}^{(\underline{i})}$, they form an $A^{(\underline{i})}$-sequence. □

Hence, T_φ is a local complete intersection, and we can explicitly compute minimal sets of local equations. We can prove more than this:

Proposition 2.4.9. *T_φ is smooth.*

Proof. Let $\mathfrak{t} \in T_\varphi$ be a closed point. We want to show that the Jacobi matrix of T_φ at $\mathfrak{t}$ has maximal rank; this suffices to show the smoothness of T_φ. Let $\underline{i} \in \mathbb{I}$ with $\mathfrak{t} \in \underline{U_{\underline{i}}}$. The defining ideal of $T_\varphi \cap \underline{U_{\underline{i}}}$ is generated by the d elements $\theta_{i_0} z_0 - \theta_0, \ldots, \theta_{i_0} z_{i_0-1} - \theta_{i_0-1}, \theta_{i_0} z_{i_0+1} - \theta_{i_0+1}, \ldots, \theta_{i_0} z_d - w_d$, and let J be the Jacobi matrix of T_φ on $\underline{U_{\underline{i}}}$. We set

$$\hat{J} = \begin{pmatrix} \frac{\partial(\theta_{i_0} z_0 - \theta_0)}{\partial z_0} & \cdots & \frac{\partial(\theta_{i_0} z_0 - \theta_0)}{\partial z_d} & \frac{\partial(\theta_{i_0} z_0 - \theta_0)}{\partial u_{1,0}} & \cdots & \frac{\partial(\theta_{i_0} z_0 - \theta_0)}{\partial u_{d,e_d}} \\ \vdots & & & & & \vdots \\ \frac{\partial(\theta_{i_0} z_d - \theta_d)}{\partial z_0} & \cdots & \frac{\partial(\theta_{i_0} z_d - \theta_d)}{\partial z_d} & \frac{\partial(\theta_{i_0} z_d - \theta_d)}{\partial u_{1,0}} & \cdots & \frac{\partial(\theta_{i_0} z_d - \theta_d)}{\partial u_{d,e_d}} \end{pmatrix}.$$

We get J from $\hat{J}$ by deleting the row

$$\left(\frac{\partial\left(\theta_{i_0} z_{i_0} - \theta_{i_0}\right)}{\partial z_0}, \ldots, \frac{\partial\left(\theta_{i_0} z_{i_0} - \theta_{i_0}\right)}{\partial w_{d,e_d}}\right)$$

and the columns

$$\left(\frac{\partial\left(\theta_{i_0} z_0 - \theta_0\right)}{\partial z_{i_0}}, \ldots, \frac{\partial\left(\theta_{i_0} z_d - \theta_d\right)}{\partial z_{i_0}})\right)$$

and

$$\left(\frac{\partial\left(\theta_{i_0} z_0 - \theta_0\right)}{\partial w_{r,i_r}}, \ldots, \frac{\partial\left(\theta_{i_0} z_d - \theta_d\right)}{\partial w_{r,i_r}})\right), \ r \in [d].$$

Thus, J is a $(d \times (d+e))$-matrix. For any $j \in [d]_0$, it holds $\theta_j \in K[\underline{\underline{w}}]$, and therefore,

$$\frac{\partial\left(\theta_{i_0} z_j - \theta_j\right)}{\partial z_j} = \theta_{i_0} \quad \text{and} \quad \frac{\partial\left(\theta_{i_0} z_j - \theta_j\right)}{\partial z_k} = 0 \ \text{ for } k \neq j.$$

The first d columns of $J(\mathfrak{t})$ hence form a diagonal submatrix

$$\begin{bmatrix} \theta_{i_0}(\mathfrak{t}) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \theta_{i_0}(\mathfrak{t}) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \theta_{i_0}(\mathfrak{t}) \end{bmatrix}$$

As $\theta_{i_0}(\mathfrak{t}) \neq 0$, this means that $\operatorname{rank}(J(\mathfrak{t})) = d$. So, $J(\mathfrak{t})$ has maximal rank. □

We finally can give equations for $\Gamma_{\underline{i}}$ and prove [Ch1, Theorem 3.1]. Note that the proof that Γ and T_φ are equal is only executed in [Ch1] on the sets of closed points, that is the proof of their scheme-theoretical equality is missing.

Theorem 2.4.10. *As schemes,*

$$\Gamma_\lambda = T_\varphi.$$

In particular, the Eagon-Northcott complex of φ resolves $\mathcal{O}_\Gamma$. Moreover, for $i \in \mathbb{I}_\lambda$, the ideal of the affine part $\Gamma_{\underline{i}}$ of Γ_λ is generated by the d polynomials

$$\begin{aligned} \theta_{\lambda,\underline{i},i_0} z_{\underline{i},0} - \theta_{\lambda,\underline{i},0}, \quad \ldots, \quad \theta_{\lambda,\underline{i},i_0} z_{\underline{i},i_0-1} - \theta_{\lambda,\underline{i},i_0-1}, \\ \theta_{\lambda,\underline{i},i_0} z_{\underline{i},i_0+1} - \theta_{\lambda,\underline{i},i_0+1}, \quad \ldots, \quad \theta_{\lambda,\underline{i},i_0} z_{\underline{i},d} - \theta_{\lambda,\underline{i},d} \quad \in A^{(\lambda,\underline{i})}. \end{aligned}$$

Proof. By definition of the elements $\Theta_0, \ldots, \Theta_d \in \bigotimes_{r=0}^{d} A^{(r)}$ and the scheme Γ, Lemma 2.1.2, and Lemma 2.4.6, the schemes Γ and T_φ are set-theoretically equal, and as Γ as well as T_φ is smooth, they are scheme-theoretically equal. Hence, [Ch1, Theorem 3.1] indeed holds, and the Eagon-Northcott complex of φ resolves $\mathcal{O}_\Gamma$. The last claim is clear by Lemma 2.4.8. $\square$

2.5 Computing X_λ

We now want to explain how to compute the ideal of a CRL using the equations of Γ_λ we already found. Again, we will omit λ from our notations if no confusion is to be expected. Denote

$$\mathfrak{I} := \mathfrak{I}_\lambda = \mathfrak{I}_{A^{(0)}}(X_\lambda) \subseteq A^{(0)} = K[Z_0, \ldots, Z_d]$$

the homogeneous ideal of X_λ (compare 0.4 §15). We identify the indeterminates $Z_1, \ldots, Z_d$ with $z_1 = \frac{Z_1}{Z_0}, \ldots, z_d = \frac{Z_d}{Z_0}$, respectively, as well as Z_0 with a new indeterminate z_0; hence, we also write $A^{(0)} = K[z_0, \ldots, z_d]$. As before, we denote the standard affine charts of $\mathbb{P}(\mathrm{Sym}^d(V))$ by $U_0, \ldots, U_d$, and for any affine subscheme $V \subseteq U_j \subseteq \mathbb{P}(\mathrm{Sym}^d(V))$, where $j \in [d]_0$, we denote by $\overline{V}$ its projective closure in $\mathbb{P}(\mathrm{Sym}^d(V))$ with respect to U_j. First, we need the following observation:

Lemma 2.5.1.

$$\overline{X_\lambda \cap U_0} = X_\lambda.$$

Proof. Observe that $X_\lambda \cap U_0 \neq \varnothing$ since this intersection is the locus of binary forms which are monic with respect to x. In addition, the image X_λ of the irreducible scheme Y_λ under the projective morphism f_λ is irreducible and therefore connected. So, $X_\lambda \cap U_0$ is dense in X_λ. $\square$

Lemma 2.5.1 tells us that it suffices to compute the ideal

$$\begin{aligned} \mathfrak{I}^{(0)} = \mathfrak{I}^{(\lambda,0)} &:= \mathfrak{I}_{A^{(0,0)}}(X_\lambda \cap U_0) \\ &= \mathfrak{I}_\lambda \cap A^{(0,0)} \\ &= \mathfrak{I}_\lambda \cap K[z_1, \ldots, z_d] \end{aligned}$$

and homogenize it in z_0 to get $\mathfrak{I}_\lambda$, that is

$$\mathfrak{I}_\lambda = \overline{\mathfrak{I}^{(0)}}^{\hom_{z_0}}$$

where $\overline{\bullet}^{\hom_{z_0}}$ means homogenization in z_0. Computing $\mathfrak{I}^{(0)}$ is straightforward using the following result (for π_λ, compare Remark 2.2.5):

Lemma 2.5.2.

$$X_\lambda \cap U_0 = \pi_\lambda(\Gamma_{\lambda,\underline{0}}) = \pi_\lambda\left(\Gamma_\lambda \cap (U_0 \times U_{1,0} \times \cdots \times U_{d,i_d})\right).$$

Proof. With

$$\mathbb{I}_0 := \{1, \ldots, e_1\} \times \cdots \{1, \ldots, e_d\},$$

it holds

$$X_\lambda \cap U_0 = \pi(\Gamma \cap \pi^{-1}(U_0)) = \pi\left(\Gamma \cap \bigcup_{(i_1,\dots,i_d)\in\mathbb{I}_0} U_0 \times U_{1,i_1} \times \cdots \times U_{d,i_d}\right),$$

hence it suffices to show

$$\Gamma \cap \left(\bigcup_{(i_1,\dots,i_d)\in\mathbb{I}_0}\left(U_0 \times \prod_{r=1}^{d} U_{r,i_r}\right)\right) = \Gamma \cap \underbrace{\left(U_0 \times \prod_{r=1}^{d} U_{r,0}\right)}_{=U_{\underline{0}}}.$$

The inclusion '$\supseteq$' is clear. For '$\subseteq$', consider a closed point $\mathfrak{t} = (\underline{f}, \underline{\underline{g}}) \in \Gamma \cap \underline{U}_{(0,i_1,\dots,i_d)}$ for any $(i_1,\dots,i_d) \in \mathbb{I}_0$. Then, by Lemma 2.4.6 it holds

$$0 \neq \theta_0(\underline{\underline{g}}) = \prod_{r=1}^{d} g_{r,0}^{r}.$$

Hence, for all $r \in [d]$, we get $g_{r,0} \neq 0$ and $(g_{r,0} : \cdots : g_{r,e_r}) \in U_{r,0}$. But this just means $\mathfrak{t} \in \underline{U}_{\underline{0}}$, and we have proven our claim. □

Putting the above together proves the validity of the following algorithm for computing $\mathfrak{I}_\lambda$:

Algorithm 2.5.3. *Input:* A partition $\lambda = (1^{e_1} \dots d^{e_d})$.

1. For $j \in [d]$, determine the set N_j as defined in Definition 2.1.1. The computation of this set is straightforward, but we are not going to explicitly do it here.

2. For $j \in [d]$, compute
$$\theta_j = \sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) \underline{\underline{w}}^{\underline{\underline{\nu}}},$$
where $w_{1,0} = \cdots = w_{r,0} = 1$.

3. Define $\mathfrak{I}^{(\underline{0})}$ to be the ideal of $K[z_1,\dots,z_d,w_{1,1},\dots,w_{1,e_1},w_{2,1},\dots,w_{d,e_d}]$ generated by $z_1 - \theta_1, \dots, z_d - \theta_d$.

4. Compute $\mathfrak{I}^{(0)} := \mathfrak{I}^{(\underline{0})} \cap K[z_1,\dots,z_d]$, that is reduce $\mathfrak{I}^{(\underline{0})}$ by $\underline{\underline{w}}$. This can be done using a Gröbner basis with respect to an elimination ordering, but has high computational complexity in general.

5. Compute $\mathfrak{I}_\lambda = \overline{\mathfrak{I}^{(0)}}^{\hom_{z_0}}$, that is homogenize the generators in a Gröbner basis of $\mathfrak{I}^{(0)}$ at z_0.

Output: The homogeneous ideal $\mathfrak{I}_\lambda = \mathfrak{I}_\lambda \subseteq K[z_0,\dots,z_d]$ of the CRL X_λ via a set of homogeneous generators.

This algorithm indeed computes equations for X_λ by Theorem 3.1.9 and Lemmas 2.5.1 and 2.5.2.

Example 2.5.4. We again consider $\lambda = (2,2,3)$ as in Example 2.4.4. Having found $\big(\varphi_{(2,2,3)}\big)$, we immediately get generators of the ideal of $\Gamma_{\underline{0}}$ in the polynomial ring $A^{(\underline{0})} = K[z_1, \ldots, z_7, w_{2,1}, w_{2,2}, w_{3,1}]$ as follows:

$$\begin{array}{l}
z_1 - 2w_{2,1} - 3w_{3,1}, \\
z_2 - 2w_{2,2} - w_{2,1}^2 - 6w_{2,1}w_{3,1} - 3w_{3,1}^2, \\
z_3 - 2w_{2,1}w_{2,2} - 6w_{2,2}w_{3,1} - 3w_{2,1}^2w_{3,1} - 6w_{2,1}w_{3,1}^2 - w_{3,1}^3, \\
z_4 - w_{2,2}^2 - 6w_{2,1}w_{2,2}w_{3,1} - 6w_{2,2}w_{3,1}^2 - 3w_{2,1}^2w_{3,1}^2 - 2w_{2,1}w_{3,1}^3, \\
z_5 - 3w_{2,2}^2w_{3,1} - 6w_{2,1}w_{2,2}w_{3,1}^2 - 2w_{2,0}w_{2,2}w_{3,1}^3 - w_{2,1}^2w_{3,1}^3, \\
z_6 - 3w_{2,2}^2w_{3,1}^2 - 2w_{2,1}w_{2,2}w_{3,1}^3, \\
z_7 - w_{2,2}^2w_{3,1}^3.
\end{array}$$

To get generators for the ideal of $X_{(2,2,3)}$, we have to eliminate $w_{2,1}, w_{2,2}$, and $w_{3,1}$ from these polynomials.

Example 2.5.5. We want to compute the equations for the coincident root loci of partitions of 4. The CRL $X_{(1,1,2)}$ is determined by the discriminant, that is

$$\mathfrak{I}_{(1,1,2)} = \langle \Delta_4 \rangle \subseteq K[z_0, z_1, z_2, z_3, z_4]$$

with

$$\begin{array}{rcl}
\Delta_4 & = & 256z_0^3z_4^3 - 192z_0^2z_1z_3z_4^2 - 128z_0^2z_2^2z_4^2 + 144z_0^2z_2z_3^2z_4 \\
& & -27z_0^2z_3^4 + 144z_0z_1^2z_2z_4^2 - 6z_0z_1^2z_3^2z_4 - 8z_0z_1z_2^2z_3z_4 \\
& & +18z_0z_1z_2z_3^3 + 16z_0z_2^4z_4 - 4z_0z_2^3z_3^2 - 27z_1^4z_4^2 \\
& & +18z_1^3z_2z_3z_4 - 4z_1^3z_3^3 - 4z_1^2z_2^3z_4 + z_1^2z_2^2z_3^2.
\end{array}$$

To find the ideal of $X_{(1,3)}$,we have to look at the elements

$$\begin{array}{rcll}
z_{\underline{0},1} - \theta_{(1,3),\underline{0},1} & = & z_1 - (3w_{3,1} + w_{1,1}), & \\
z_{\underline{0},2} - \theta_{(1,3),\underline{0},2} & = & z_2 - (3w_{3,1}^2 + 3w_{1,1}w_{3,1}), & \\
z_{\underline{0},3} - \theta_{(1,3),\underline{0},3} & = & z_3 - (w_{3,1}^3 + 3w_{1,1}w_{3,1}^2), & \\
z_{\underline{0},4} - \theta_{(1,3),\underline{0},4} & = & z_4 - (w_{1,1}w_{3,1}^3) & \in \quad K[z_1, \ldots, z_4, w_{1,1}, w_{3,1}],
\end{array}$$

eliminate the indeterminates $w_{1,1}$ and $w_{3,1}$ from the ideal generated by them, and then homogenize at z_0. Doing this with help of the library [`elim.lib`] for [Singular], we get

$$\mathfrak{I}_{(1,3)} = \langle 12z_0z_4 - 3z_1z_3 + z_2^2, 27z_0z_3^2 + 27z_1^2z_4 - 27z_1z_2z_3 + 8z_2^3.\rangle$$

Repeating the same computations for the partitions $(2,2)$ and (4), we also get

$$\begin{array}{rcll}
\mathfrak{I}_{(2,2)} & = & \Big\langle & 8z_0^2z_3 - 4z_0z_1z_2 + z_1^3, \quad 16z_0^2z_4 + 2z_0z_1z_3 - 4z_0z_2^2 + z_1^2z_2, \\
& & & 8z_0z_1z_4 - 4z_0z_2z_3 + z_1^2z_3, \quad z_0z_3^2 - z_1^2z_4, \\
& & & 8z_0z_3z_4 - 4z_1z_2z_4 + z_1z_3^2, \quad 16z_0z_4^2 + 2z_1z_3z_4 - 4z_2^2z_4 + z_2z_3^2, \\
& & & 8z_1z_4^2 - 4z_2z_3z_4 + z_3^3 \quad \Big\rangle
\end{array}$$

and

$$\begin{array}{rcll}
\mathfrak{I}_{(4)} & = & \Big\langle & 8z_0z_2 - 3z_1^2, \quad 6z_0z_3 - z_1z_2, \quad 36z_0z_4 - z_2^2, \\
& & & 9z_1z_3 - 4z_2^2, \quad 6z_1z_4 - z_2z_3, \quad 8z_2z_4 - 3z_3^3 \quad \Big\rangle.
\end{array}$$

At the end of this Section, we prove that the ideal $\mathfrak{I}_\lambda$ is weighted homogeneous (see Remark and Definition 1.2.1).

Proposition 2.5.6. *Let λ be a partition of $d \in \mathbb{N}_0$. Then, the ideal $\mathfrak{I}_\lambda \subseteq A^{(0)} = K[z_0, \dots, z_d]$ is weighted graded.*

Proof. We furnish the ring

$$A^{(\underline{0})} = A^{(0)} \otimes K[\underline{\underline{w}}] = K[z_0, \dots, z_d, w_{1,1}, \dots, w_{1,e_1}, w_{2,1}, \dots, w_{d,e_d}]$$

with a weighted grading $A^{(\underline{0})} = \bigoplus_{m \in \mathbb{N}_0} A^{(\underline{0})}_{\omega,m}$ by giving the additional indeterminates weights $\omega(w_{r,t}) = t$ for all $(r,t) \in [d] \times [e_r]$. Then, $A^{(0)}$ is a weighted graded subring of $A^{(\underline{0})}$. Consider the ideal of $\Gamma_\lambda \cap \underline{U_0}$ in $A^{(\underline{0})}$; by Theorem 2.4.10 and Lemma 2.4.8, it is generated by the elements $z_1 - \theta_1, \dots, z_d - \theta_d \in A^{(\underline{0})}$. Fix $i \in [d]$, and consider the element θ_i. It is obtained by adding monomials of the form

$$\underline{\underline{w}}^{\underline{\underline{\nu}}} = w_{1,1}^{\nu_{1,1}} \cdots w_{1,e_q}^{\nu_{1,e_1}} w_{2,1}^{\nu_{2,1}} \cdots w_{d,e_d}^{\nu_{d,e_d}}$$

for $\underline{\underline{\nu}} \in N_{\lambda,i}$, where by definition

$$\sum_{r=1}^{d} \left(\sum_{t=1}^{e_r} t\nu_{r,t} \right) = i.$$

Hence,

$$\omega(\underline{\underline{w}}^{\underline{\underline{\nu}}}) = \sum_{r=1}^{d} \left(\sum_{t=1}^{e_r} \theta(w_{r,t})\nu_{r,t} \right) = \sum_{r=1}^{d} \left(\sum_{t=1}^{e_r} t\nu_{r,t} \right) = i$$

and therefore $\theta_i \in A^{(\underline{0})}_{\omega,j}$. As also $z_i \in B_{\omega,i}$, this means that $\mathfrak{I}(\Gamma_\lambda \cap \underline{U_0})$ is generated by weighted homogeneous elements and hence is a weighted graded ideal. It follows that $\mathfrak{I}(X_\lambda \cap U_0) = \mathfrak{I}(\Gamma_\lambda \cap \underline{U_0}) \cap A^{(0)}$ is a weighted graded ideal. As $\omega(z_0) = 0$, multiplying with z_0 does not change the weight of a monomial in $A^{(0)}$, hence, homogenizing a weighted homogeneous element at z_0 yields again a weighted homogeneous element. Therefore, the homogenization $\mathfrak{I}_\lambda$ of $\mathfrak{I}(X_{\lambda,\underline{0}})$ is weighted graded. ☐

Corollary 2.5.7. *There are generators $F_1, \dots, F_s$ of $\mathfrak{I}_Z \subseteq K[z_0, \dots, z_d]$ which are homogeneous with respect to the standard grading as well as weighted graded. Moreover, we can compute such generators.*

Proof. Let $H_1, \dots, H_t$ be generators of $\mathfrak{I}_\lambda$ which are homogeneous with respect to the standard grading; we can compute this generators, for example using Algorithm 2.5.3. Now, we take all the weighted homogeneous components of the elements $H_1, \dots, H_t$ and denote them by $F_1, \dots, F_s$. Since $\mathfrak{I}_Z$ is weighted graded, $F_1, \dots, F_s$ are contained in $\mathfrak{I}_Z$, and obviously we can write $H_1, \dots, H_t$ as linear combinations of them, that is $F_1, \dots, F_s$ indeed generate $\mathfrak{I}_\lambda$. These generators are homogeneous with respect to the standard grading since they are sums of monomials occurring in the homogeneous elements $H_1, \dots, H_t$. ☐

2.6 Some comments about the singular locus of X_λ

In [Ch1], we find a description of the singular locus of the CRL X_λ. First, we give the definition of the set S_λ.

Definition 2.6.1. Let $\lambda = (1^{e_1} 2^{e_2} \dots d^{e_d})$ be a partition of $d \in \mathbb{N}$. Then, a partition $\mu = (1^{\varepsilon_1} 2^{\varepsilon_2} \dots d^{\varepsilon_d})$ of d belongs to the set S_λ if and only if μ fulfils one of the following mutually exclusive conditions:

(a) There are distinct integers $r_1, r_2 \in [d]$ with

$$\varepsilon_{r_1} = e_{r_1} - 1, \quad \varepsilon_{r_2} = e_{r_2} - 1, \quad \varepsilon_{r_1+r_2} = e_{r_1+r_2} + 1,$$
$$\varepsilon_r = e_r \text{ for } r \in [d]\backslash\{r_1, r_2, r_1 + r_2\}.$$

(b) There are integers $r_1, r_2 \in [d], t \in \mathbb{N}$ with $t > 1$ and $r_1 = tr_2$ such that

$$e_{r_1} > 0 \quad \text{and} \quad \varepsilon_{r_1} = e_{r_1} + 1, \quad \varepsilon_{r_2} = e_{r_2} - t$$
$$\varepsilon_r = e_r \text{ for } r \in [d]\backslash\{r_1, r_2, \}.$$

(c) There exist pairwise distinct integers $r_1, r_2, r_3 \in [d]$ and integers $t_1, t_2 \in \mathbb{N}$ with $r_3 = t_1 r_1 = t_2 r_2$ such that

$$\varepsilon_{r_1} = e_{r_1} - t_1, \quad \varepsilon_{r_2} = e_{r_2} - t_2, \quad \varepsilon_{r_3} = e_{r_3} + 2,$$
$$\varepsilon_r = e_r \text{ for } r \in [d]\backslash\{r_1, r_2, r_3\}.$$

By $S_\lambda^{(\bullet)}$ we denote the subset of S_λ of partitions that fulfil the above condition $(\bullet)$ for $\bullet \in \{a, b, c\}$. Then, S_λ is the union of the pairwise disjoint sets $S_\lambda^{(a)}, S_\lambda^{(b)}$, and $S_\lambda^{(c)}$.

Observe that any partition $\mu \in S_\lambda$ is a coarsening of λ, hence, $X_\mu \subseteq X_\lambda$.

Theorem 2.6.2. *Let λ be a partition of $d \in \mathbb{N}$. Then, the singular locus of X_λ is $\bigcup_{\mu \in S_\lambda} X_\mu$. Moreover, for any singular closed point $\mathfrak{f}$ of X_λ, if $\psi_\lambda^{-1}(\mathfrak{f})$ is singleton, then $\mathfrak{f} \in \bigcup_{\mu \in S_\lambda^{(a)}} X_\mu$.*

Proof. See [Ch1, Theorem 5.4 and Proposition 5.5]. □

Let us give the singular locus of X_λ some more thoughts. First, we look at some examples:

Example 2.6.3. A) For $\lambda = (1, 1, 2)$, we have $S_\lambda^{(a)} = \{(1, 3)\}, S_\lambda^{(b)} = \{(2, 2)\}$, and $S_\lambda^{(c)} = \varnothing$, hence $S_{(1,1,2)} = \{(1, 3), (2, 2)\}$. Observe that the only coarsening of $(1, 1, 2)$ not contained in $S_{(1,1,2)}$ is (4), but that $X_{(4)} = X_{(1,3)} \cap X_{(2,2)}$. Therefore, the singular locus of $X_{(1,1,2)}$ consists of all closed points $\mathfrak{f} \mathrel{\dot\in} X_{(1,1,2)}$ such that there are more than exactly two linear factors of $\mathfrak{f}$ that happen to be equal to another linear factor.

B) For $\lambda = (1, 1, 3)$, it holds $S_\lambda^{(a)} = \{(1, 4)\}, S_\lambda^{(b)} = \varnothing = S_\lambda^{(c)}$. In particular, $(2, 3) \notin S_{(1,1,3)}$, hence a closed point $\mathfrak{f} \mathrel{\dot\in} X^\circ_{(2,3)} = X_{(2,3)}\backslash X_{(5)}$ is non-singular in

$X_{(1,1,3)}$. For such a closed point $\mathfrak{f} \dot\in X^\circ_{(2,3)}$, it holds $\mathfrak{f} = \mathfrak{l}_1 \cdot \mathfrak{l}_2 \cdot \mathfrak{l}_3 \cdot \mathfrak{l}_4 \cdot \mathfrak{l}_5$ with linear factors $\mathfrak{l}_1 = \mathfrak{l}_2 \neq \mathfrak{l}_3 = \mathfrak{l}_4 = \mathfrak{l}_5$; if we consider $\mathfrak{f}$ as a point of $X_{(1,1,3)}$ with $\mathfrak{f} = \mathfrak{g}_1 \cdot \mathfrak{g}_3^3$, then obviously $\mathfrak{g}_1 = \mathfrak{l}_1 \cdot \mathfrak{l}_2$ and $\mathfrak{g}_3 = \mathfrak{l}_3$. This means that the distribution of the linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_5$ to the factors $\mathfrak{g}_1, \mathfrak{g}_2$ is unique. On the other hand, if $\mathfrak{f} \dot\in X_{(1,4)} \subseteq X_{(1,1,3)}$, then $\mathfrak{f} = \mathfrak{l}_1 \cdot \mathfrak{l}_2 \cdots \mathfrak{l}_5$ with linear factors $\mathfrak{l}_1, \mathfrak{l}_2 = \cdots = \mathfrak{l}_5$, and we can choose three of the linear factors $\mathfrak{l}_2, \ldots, \mathfrak{l}_5$ to form $\mathfrak{g}_3$, while the last one belongs to $\mathfrak{g}_1$ – that is, each of the cases $\mathfrak{g}_1 = \mathfrak{l}_1 \cdot \mathfrak{l}_2, \mathfrak{g}_1 = \mathfrak{l}_1 \cdot \mathfrak{l}_3, \mathfrak{g}_1 = \mathfrak{l}_1 \cdot \mathfrak{l}_4$, or $\mathfrak{g}_1 = \mathfrak{l}_1 \cdot \mathfrak{l}_5$ can occur, and the distribution of the linear factors to the factors $\mathfrak{g}_1, \mathfrak{g}_3$ is not unique.

The last example gives raise to the next Proposition, which claims that the singular locus of X_λ for any λ consists of closed points $\mathfrak{f}$ such that the distribution of the linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$ to factors $\mathfrak{g}_1, \mathfrak{g}_2^2, \ldots, \mathfrak{g}_d^d$ of $\mathfrak{f}$ is not unique. In formulating and proving this result, we make use of the terminology introduced in Section 1.1.

Remark and Notation 2.6.4. Let $\mathfrak{f} \dot\in X_\lambda$ with $\mathfrak{g}_r \in K[x,y]_{e_r}$ for all $r \in [d]$ such that $\mathfrak{f} = \prod_{r=1}^d \mathfrak{g}_r^r$, and let $\mathfrak{l}_1, \ldots, \mathfrak{l}_d \dot\in \mathbb{P}(\mathrm{Sym}^1(K[x,y])$ be the linear factors of $\mathfrak{f}$. Taking only non-equal linear factors, we also find an integer $s \leqslant d$ and linear factors $\mathfrak{k}_1, \ldots, \mathfrak{k}_s \dot\in \mathbb{P}(\mathrm{Sym}^1(K[x,y]_1))$ with $\mathfrak{k}_i \neq \mathfrak{k}_j$ for $i \neq j$ such that $\mathfrak{f} = \prod_{j=1}^s \mathfrak{k}_j^{\mu_j}$ for a partition $\mu = (\mu_1, \ldots, \mu_s)$ of d. In this situation, μ is a coarsening of λ, and there is no strict coarsening υ of μ such that $\mathfrak{f} \in X_\upsilon$. By X°_μ we denote the subvariety of X_μ of points that are not contained in X_υ for any strict coarsening $\upsilon \in Q^\circ_\mu$. The subset $X^\circ_\mu \subseteq X_\lambda$ is locally closed since X_υ is closed for any $\upsilon \in Q_\lambda$ and thus X°_μ is open in its closure X_μ.
Now, let $\underline{\delta} = (\delta^{(1)}, \ldots, \delta^{(s)})$ be a splitting of μ into λ (see Remark and Definition 1.1.2). This splitting corresponds to the factorization

$$\mathfrak{f} = \prod_{j=1}^{s} \mathfrak{k}_j^{\mu_j} = \prod_{j=1}^{s} \prod_{i=1}^{|\delta^{(j)}|} \mathfrak{k}_j^{\delta_i^{(j)}}. \tag{2.1}$$

So, we can determine a factorization $\mathfrak{f} = \prod_{r=1}^d \mathfrak{g}_r^r$ by

$$\mathfrak{g}_r = \prod_{j=1}^{s} \prod_{\substack{i \in [|\delta^{(j)}|] \\ \delta_i^{(j)} = r}} \mathfrak{k}_j \tag{2.2}$$

for $r \in [d]$, that is by collecting all linear factors $\mathfrak{k}_j$ appearing in (2.1) with power r, counted with multiplicity. This factorization determines a closed point $(\mathfrak{g}_1, \ldots, \mathfrak{g}_d) \in \psi_\lambda^{-1}(\mathfrak{f}) \subseteq Y_\lambda$. On the other hand, given $(\mathfrak{g}_1, \ldots, \mathfrak{g}_r)$, we can determine a splitting $\underline{\delta}$ of μ into λ as follows: For any binary forms $\mathfrak{o}, \mathfrak{n}$ over K denote $\mathrm{ord}_{\mathfrak{o}}(\mathfrak{n})$ the highest power c with $\mathfrak{n}^c \mid \mathfrak{o}$. For $j \in [|\mu|]$, $i \in [\mu_j]$, let $c_i^{(j)} := \mathrm{ord}_{\mathfrak{g}_i}(\mathfrak{k}_i)$ be the highest power c with $\mathfrak{k}_j^c \mid \mathfrak{g}_i$. Then, $\delta^{(j)} = (1^{c_1^{(j)}} \ldots \mu_j^{c_{\mu_j}^{(j)}})$ is a factorization of $\mu_j = \mathrm{ord}_{\mathfrak{f}}(\mathfrak{k}_j)$, and $(\delta^{(1)}, \ldots, \delta^{(|\mu|)})$ is a splitting of μ into λ.

The three sets $Q_\lambda^{(0)}, Q_\lambda^{(1)}$, and $Q_\lambda^{(2)}$ (see 1.1.2) now correspond to the situation that

(1) the factors $\mathfrak{g}_1, \ldots, \mathfrak{g}_d$ and the distribution of the linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$ to them are unique,

(2) the factors $\mathfrak{g}_r$ are unique, but there is a linear factor $\mathfrak{l}_j$ dividing more than one such factor $\mathfrak{g}_r$, and

(3) the factors $\mathfrak{g}_r$ are not unique,

respectively. This leads to the following Proposition, in which $\#\psi_\lambda^{-1}(\mathfrak{f})$ denotes the number of (closed) points in the fibre of ψ_λ over a closed point $\mathfrak{f} \in X_\lambda$:

Proposition 2.6.5. *Let $\mathfrak{f} \dot{\in} X_\lambda$. Let $\mu \in Q_\lambda$ with $\mathfrak{f} \in X_\mu^\circ$. Then,*

a) $\mathfrak{f}$ is non-singular in X_λ if and only if $\mu \in Q_\lambda^{(0)}$, that is μ is of even unique splitting into λ;

b) $\mathfrak{f}$ is singular in X_λ with $\#\psi_\lambda^{-1}(\mathfrak{f}) = 1$ if and only if $\mu \in Q_\lambda^{(1)}$, that is μ is of uneven unique splitting into λ;

c) $\mathfrak{f}$ is singular in X_λ with $\#\psi_\lambda^{-1}(\mathfrak{f}) > 1$ if and only if $\mu \in Q_\lambda^{(2)}$, that is μ is not of unique splitting into λ.

Moreover, $\#\psi_\lambda^{-1}(\mathfrak{f})$ equals the number of different splittings of μ into λ.

Proof. Let $\underline{\delta}$ be a splitting of μ into λ, and let the factors $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ be obtained as in (2.2). First, assume that the splitting of μ into λ is not unique, so that we can find a splitting $\underline{\gamma} \neq \underline{\delta}$ of μ into λ. This is equivalent to

$$\mathfrak{f} = \prod_{j=1}^{|\mu|} \mathfrak{k}_j^{\mu_j} = \prod_{j=1}^{|\mu|} \prod_{i=1}^{|\gamma^{(j)}|} \mathfrak{k}_j^{\gamma_i^{(j)}}$$

for linear factors $\mathfrak{k}_1, \ldots, \mathfrak{k}_{|\mu|}$ of $\mathfrak{f}$. Writing

$$\mathfrak{h}_r := \prod_{j=1}^{|\mu|} \prod_{\substack{i \in [|\gamma^{(j)}|] \\ \gamma_i^{(j)} = r}} \mathfrak{k}_j,$$

we get a second factorization $\mathfrak{f} = \prod_{r=1}^d \mathfrak{h}_r^r$. As there is an index $j \in [|\mu|]$ with $m := \mu_j$ and $\delta^{(j)} = (1^{c_1} \ldots m^{c_m}) \neq \gamma^{(j)} = (1^{c'_1} \ldots m^{c'_m})$, there is an integer $i \in [m]$ with $c_i > c'_i$, and hence $\mathfrak{k}_j^{c_i} \mid \mathfrak{g}_i$ but $\mathfrak{k}_j^{c_i} \nmid \mathfrak{h}_i$. Thus, $(\mathfrak{h}_1, \ldots, \mathfrak{h}_d)$ is a second point in $\psi_\lambda^{-1}(q)$ not equal to $(\mathfrak{g}_1, \ldots, \mathfrak{g}_r)$. On the other hand, if there are two different points $(\mathfrak{g}_1, \ldots, \mathfrak{g}_r), (\mathfrak{h}_1, \ldots, \mathfrak{g}_r) \dot{\in} \psi_\lambda^{-1}(q)$, we can construct two splittings $\underline{\delta}, \underline{\gamma}$ of μ into λ as in Remark and Notation 2.6.4; since there is an index $r \in [d]$ with $\mathfrak{g}_r \neq \mathfrak{h}_r$, there is an index $j \in [|\mu|]$ with $\mathrm{ord}_{\mathfrak{g}_r}(\mathfrak{k}_j) \neq \mathrm{ord}_{\mathfrak{h}_r}(\mathfrak{k}_j)$. Hence, $\delta^{(j)} \neq \gamma^{(j)}$, and $\mu \in Q_\lambda^{(2)}$. This proves (c) and, by the same argument for more than two different partitions, the additional claim.
The case that $\mathfrak{f}$ is singular in X_λ but $\#\psi_\lambda^{-1}(\mathfrak{f}) = 1$ is equivalent to the existence of a linear factor $\mathfrak{l}$ dividing $\mathfrak{g}_r$ and $\mathfrak{g}_{r'}$ with $r, r' \in [d]$ and $r \neq r'$ by [Ch1, Corollary 5.8 and Proposition 5.1]; hence, $\mathrm{ord}_{\mathfrak{g}_r}(\mathfrak{l}) \neq 0 \neq \mathrm{ord}_{G_{r'}}(\mathfrak{l})$. This in turn is equivalent to the situation that $\delta^{(j)} = (1^{\mathrm{ord}_{\mathfrak{g}_1}(\mathfrak{l})} \ldots d^{\mathrm{ord}_{\mathfrak{g}_d}(\mathfrak{l})})$ is not even. This proves (b).
As $\mu \in Q_\lambda = Q_\lambda^{(0)} \cup Q_\lambda^{(1)} \cup Q_\lambda^{(2)}$ and this union is disjoint, (a) follows from (b) and (c). □

Corollary 2.6.6. *Let λ be a partition of $d \in \mathbb{N}$. Then, X_λ is smooth if and only if λ is even.*

Proof. This is clear by Proposition 2.6.5 and Lemma 1.1.4 □

Corollary 2.6.7. *For any partition λ of $d \in \mathbb{N}$, the CRL X_λ is either smooth, or the dimension of its singular locus* $\mathrm{Sing}(X_\lambda)$ *is* $\dim(X_\lambda) - 1$.

Proof. The locus X_λ is not smooth if and only if we find a coarsening $\mu = (1^{\varepsilon_1} \dots d^{\varepsilon_d})$ of $\lambda = (1^{e_1} \dots d^{e_d})$ by adding two different entries λ_i, λ_j and keeping the other entries of λ as in the proof of the previous Corollary. But this means $\varepsilon_{\lambda_i} = e_{\lambda_i} - 1$, $\varepsilon_{\lambda_j} = e_{\lambda_j} - 1$, and $\varepsilon_{\lambda_i+\lambda_j} = e_{\lambda_i+\lambda_j} + 1$, hence by Remark 2.2.3

$$\dim(X_\mu) = \sum_{r=1}^{d} \varepsilon_r = \sum_{r=1}^{d} e_r - 1 = \dim(X_\lambda) - 1.$$

By construction, $\mu \notin Q_\lambda^{(0)}$, and thus $X_\mu \subseteq \mathrm{Sing}(X_\lambda)$. □

Geometrically speaking, singularities of X_λ of type (b) in Proposition 2.6.5 are similar to cusps, while those of type (c) are similar to nodes and multiple points. Note that those two kinds of singularities can occur simultaneously, as we will see in the next example.

Example 2.6.8. (A) Let us consider the partitions of 5 and the respective CRL. All partitions of 5 are coarsenings of the trivial partition (1^5) with $X_{(1^5)} = \mathbb{P}(\mathrm{Sym}^5(V))$, and it holds

$$Q_{(1^5)} = \{(1,1,1,2),(1,1,3),(1,2,2),(1,4),(2,3),(5)\}.$$

First. we look at the singular locus of $X_{(1^3\,2^1)}$. We find $(1^3\,2^1)$ to be the only element of $Q^{(0)}_{(1^3\,2^1)}$, while

$$Q^{(1)}_{(1^3\,2^1)} = \{(1,1,3),(1,4),(5)\} \quad \text{and} \quad Q^{(2)}_{(1^3\,2^1)} = \{(1,2,2),(2,3)\}.$$

Closed points of $X^\circ_{(2,3)} \subseteq X_{(1,1,1,2)}$ are of the form $\mathfrak{l}_1^2\mathfrak{l}_2^3$ for some linear binary forms $\mathfrak{l}_1 \neq \mathfrak{l}_2$, and there are two points in the fibre $\psi_\lambda^{-1}(\mathfrak{l}_1^2\mathfrak{l}_2^3) \subseteq \mathbb{P}(\mathrm{Sym}^3(V)) \times \mathbb{P}(\mathrm{Sym}^1(V))$, namely $(\mathfrak{l}_1^2\mathfrak{l}_2, \mathfrak{l}_2)$ and $(\mathfrak{l}_2^3, \mathfrak{l}_1)$. The first one corresponds to the splitting $((1,1),(1,2))$ of $(2,3)$ into $(1,1,1,2)$, containing an uneven partition. As $\mathfrak{l}_2$ divides $\mathfrak{g}_1 = \mathfrak{l}_1^2\mathfrak{l}_2$ as well as $\mathfrak{g}_2 = \mathfrak{l}_2$, the induced morphism on tangent spaces

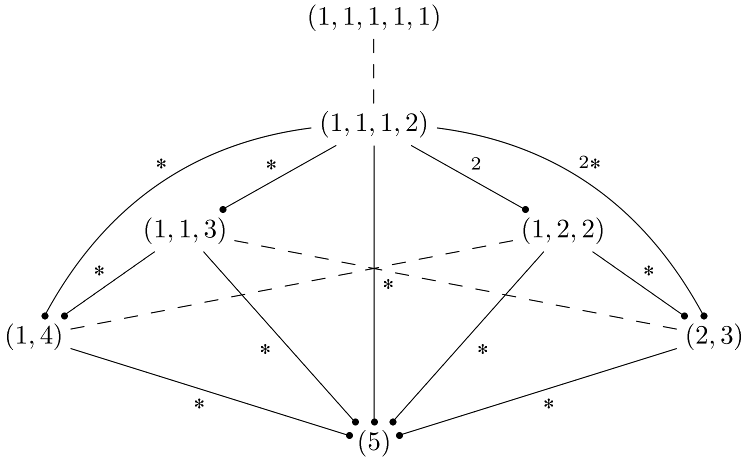
$$d\psi_{\lambda,(\mathfrak{l}_1^2\mathfrak{l}_2,\mathfrak{l}_2)} : T_{\mathbb{P}(\mathrm{Sym}^3(V))\times\mathbb{P}(\mathrm{Sym}^1(V)),(\mathfrak{l}_1^2\mathfrak{l}_2,\mathfrak{l}_2)} \to T_{\mathbb{P}(\mathrm{Sym}^5(V)),\mathfrak{l}_1^2\mathfrak{l}_2^3}$$

is not injective (see [Ch1, Corollary 5.8]). On the other hand, $(\mathfrak{l}_2^3, \mathfrak{l}_1) \in \psi_\lambda^{-1}(\mathfrak{l}_1^2\mathfrak{l}_2^2)$ corresponds to the splitting $((2),(1,1,1))$, containing only even partitions, that is

$$d\psi_{\lambda,(\mathfrak{l}_2^3,\mathfrak{l}_1)} : T_{\mathbb{P}(\mathrm{Sym}^3(V))\times\mathbb{P}(\mathrm{Sym}^1(V)),(\mathfrak{l}_1^2\mathfrak{l}_1,\mathfrak{l}_2)} \to T_{\mathbb{P}(\mathrm{Sym}^5(V)),\mathfrak{l}_1^2\mathfrak{l}_2^3}$$

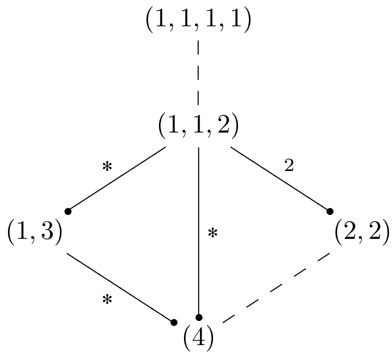
is injective. For the coarsening $(1,2,2)$ of $(1,1,1,2)$, both possible splittings $((1),(1,1),(2))$ and $((1),(2),(1,1))$ consist of even partitions, meaning that both

induced morphisms on tangent spaces for a closed point $\mathfrak{f} \in X^\circ_{(1,2,2)} \subseteq X_{(1,1,1,2)}$ are injective. Finally, the fibre $\psi^{-1}_{(1^3\,2^1)}(\mathfrak{f})$ of a closed point $\mathfrak{f} \in X^\circ_{(1,1,3)} \cup X^\circ_{(1,4)} \cup X^\circ_{(5)} = X_{(1,1,3)} \backslash X^\circ_{(2,3)}$ contains only one point, but the induced morphism on tangent spaces is not injective.
Looking at $(1,1,3)$, we see that the coarsening $(2,3)$ is of even unique splitting, hence a point in $X^\circ_{(2,3)}$ is non-singular in $X_{(1,1,3)}$. Continuing in this way, we can draw the following diagram:



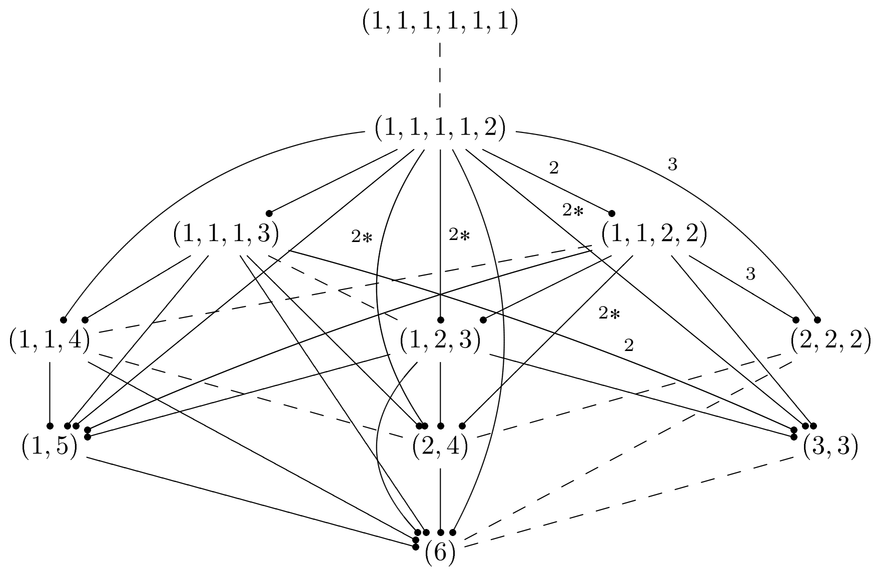
In this diagram, any line between two partitions λ above and μ below means that $X_\mu \subseteq X_\lambda$; we omitted most of the lines connected to the trivial partition (1^5). A dashed line means that closed points of X°_μ are non-singular in X_λ, while a line with a $\bullet$ means that X°_μ is contained in the singular locus of X_λ. The label of such a line indicates the type of singularity: A number denotes the number of points in the fibres of ψ_λ over points of X°_μ ('1' is always omitted), while an asterisk $*$ means that there is (at least) one induced morphism on tangent spaces which is not injective. Also, the dimension of X_λ equals the number of its rows counted from below, or equivalently, the length of a longest path to the coarsest splitting (5). E.g., $\dim(X_{(1,1,1,2)}) = 4$ as $(1,1,1,2)$ can be found in the fourth row from the bottom.

(B) For $d = 4$, we immediately get that $X_{(1^4)}, X_{(2^2)}$ and $X_{(4^1)}$ are smooth. The diagram of the CRL as above looks as follows:



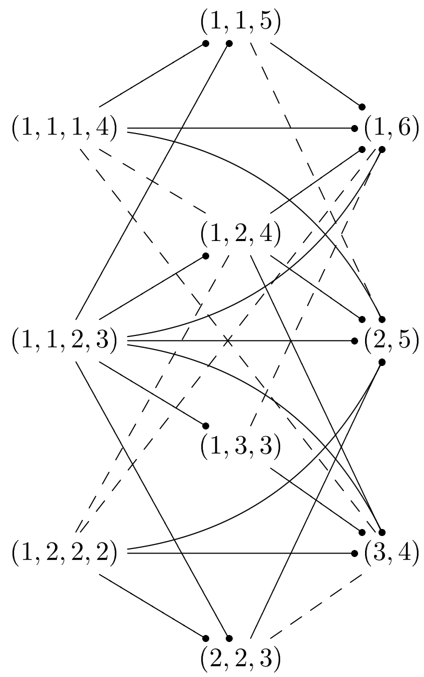
Using the Algorithm 2.5.3, we can compute equations for this CRLs: $X_{(1,1,2)}$ is defined by the discriminant for quartics, $X_{(1,3)}$ is defined by $12z_0z_4 - 3z_1z_3 + z_2^3$ and $27z_0z_3^2 + 27z_1^2z_4 - 27z_1z_2z_3 + 8z_2^3$, and $X_{(2,2)}$ is defined by 7 cubics (see Example 2.5.5). This means that for every partition λ of 4, the CRL X_λ is a locally complete intersection: $X_{(1,1,1,1)}, X_{(2,2)}$, and $X_{(4)}$ because they are smooth, and $X_{(1,1,2)}$ and $X_{(1,3)}$ because their homogeneous ideal is generated by a number of equations equal to their codimension.

(C) For $d = 6$, the diagram looks as follows if we omit any $*$ indicating singularities purely of type (b) in Proposition 2.6.5:



In this diagram, we note that the singular locus of X_λ looks more complicated the bigger $|\lambda| = \dim(X_\lambda)$ is, which is hardly remarkable. But we note also, that for fixed $|\lambda|$, that is on a fixed row of the diagram, the singular locus is more complicated the more different entries there are in λ. For example, any coarsening of $(1, 2, 3)$ yields singularities, only two of three coarsenings of $(1, 1, 4)$ yield singularities, while no coarsening of $(2, 2, 2)$ yields a singularity.

(D) To further illustrate the last remark, we draw the second to fourth row counted from below of the diagram for $d = 7$ (we omit the bottom row as (7) just yields singularities purely of type (b) for any partition $\lambda \notin \{(1^7), (7)\}$). Here, for clarity's sake, we swap rows and columns, and we omit any label indicating the type of singularity.



As above, the more different entries a partition λ exhibits, the more complete lines lead away from it and the more parts the singular locus of X_λ possesses. This observation is rooted in the fact that the more different entries a partition has, the more different and uneven subpartitions we can compose of them, leading to more splittings of coarsenings in $Q_\lambda^{(1)} \cup Q_\lambda^{(2)}$. Thus, the number of different entries in λ is a measure for the complexity of the singular locus in X_λ; Corollary 2.6.6 is just the simplest instance of this principle.

Chapter 3

On partial elimination ideals and equations of Z_k

In this Chapter, we return to the situation described in the Introduction: For a closed point $\mathfrak{p} \,\dot{\in}\, \mathbb{P}^n$ and a projective variety $\widetilde{Z} \subseteq \mathbb{P}^n$ with $\mathfrak{p} \notin \widetilde{Z}$, we consider the simple outer linear projection $\pi : \widetilde{Z} \to Z \subseteq \mathbb{P}^{n-1}$ with centre $\mathfrak{p}$. For all $k \in \mathbb{N}_0$, we want to determine the ideal of the closed set $Z_k \subseteq \mathbb{P}^{n-1}$ that consists of all closed points $\mathfrak{q} \,\dot{\in}\, Z$ such that the fibres $\pi^{-1}(\mathfrak{q})$ are of length greater or equal than k. We show that this ideals are the partial elimination ideals of $\mathfrak{I}(\widetilde{Z})$ with respect to $\mathfrak{p}$, and we give an algorithm for computing them. We also explain their connexion to secant cones and secant loci and shortly look at the use of partial elimination ideals in the study of multiple linear projections. Finally, we give some examples. The most important result of this Section for our aim of determining equations for Z_λ is Theorem 3.1.9, which explains how to get the degree of the polynomials defining the fibres $\pi^{-1}(\mathfrak{q})$.
Partial elimination ideals have been introduced by Marc Green in [G] in order to study generic initial ideals. In the same context, they have been used by Conca and Sidman in [CoSi]; these authors also showed the connexion of partial elimination ideals and Gröbner bases, which is the corner stone for our algorithmic considerations. In [HK], Han and Kwak also used partial elimination ideals in the study of inner projections. The results in this Chapter have been published in [Km].
Through the whole Chapter, we work with a fixed integer $n \in \mathbb{N}$ and $R = K[x_0, \ldots, x_n]$ (see 0.4 §§12 - 13).

3.1 Partial Elimination Ideals

Notation 3.1.1. Let $\mathfrak{p} \in \mathrm{mProj}(R)$ (see 0.4 §13), and let $K[\mathfrak{p}_1] \subseteq R$ be the graded K-subalgebra of R generated by the linear forms of $\mathfrak{p}$. Frequently, we just write $S := K[\mathfrak{p}_1]$ and consider S as the homogeneous ring of the projective space $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}_K(\mathfrak{p}) := \mathrm{Proj}(K[\mathfrak{p}_1])$. Now, let $x \in R_1 \backslash \mathfrak{p}_1$. We identify $S = R/xR$ and get an inclusion $\mathbb{P}^{n-1} = \mathrm{Proj}(R/xR) \hookrightarrow \mathbb{P}^n$. Hence, we always consider $\mathbb{P}^{n-1}$ as a linear subspace of $\mathbb{P}^n$ with the embedding determined by x.
Also, the K-vector space R_1 is generated by x and $\mathfrak{p}_1$, and therefore $R =$

$K[x,\mathfrak{p}_1] = S[x]$. Let $f \in R$. We may consider f as a polynomial in x over S and write $\deg_x(f)$ for the degree of $f \in S[x]$ in x. Furthermore, let $\mathrm{LC}_x(f)$ and $\mathrm{LT}_x(f)$ denote the leading coefficient and the leading term of f, respectively, as a polynomial in x over S. Note that, using these notations, we are not considering the standard grading on R but the one induced by $R = S[x]$.

Definition 3.1.2. Let $\mathfrak{a} \subseteq R$ be a graded ideal, let $\mathfrak{p} \in \mathrm{mProj}(R)$, and let $k \in \mathbb{N}_0$. We define the *k-th partial elimination ideal* (abbreviated PEI) *of* $\mathfrak{a}$ *with respect to* $\mathfrak{p}$ by

$$\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) := \{f \in K[\mathfrak{p}_1] \mid \forall x \in R_1 \backslash \mathfrak{p}_1 \,\exists g \in R : \deg_x(g) < k \wedge x^k f + g \in \mathfrak{a}\}.$$

$\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})$ is a graded ideal of S whose d-th graded component is given by

$$\begin{aligned}\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})_d &= \{f \in S_d \mid \forall x \in R_1 \backslash \mathfrak{p}_1 \,\exists g \in R_{d+k} : \deg_x(g) < k \wedge x^k f + g \in \mathfrak{a}_{d+k}\} \\ &= \{f \in S_d \mid \forall x \in R_1 \backslash \mathfrak{p}_1 \,\exists g \in (\mathfrak{p}^{d+1})_{d+k} : x^k f + g \in \mathfrak{a}_{d+k} \cap (\mathfrak{p}^d)_{d+k}\}.\end{aligned}$$

Finally, we define $\mathfrak{K}_{-1}^{\mathfrak{p}}(\mathfrak{a}) = 0$. In this way we get an ascending chain of graded ideals of S

$$0 = \mathfrak{K}_{-1}^{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathfrak{a} \cap S = \mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{a}) \subseteq \cdots \subseteq \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathfrak{K}_{k+1}^{\mathfrak{p}} \subseteq \cdots$$

For the remainder of this Section, we fix a graded ideal $\mathfrak{a} \subseteq R$ and a closed point $\mathfrak{p} \in \mathrm{mProj}(R)$.

Remark and Notation 3.1.3. For all $k \in \mathbb{Z}$, let

$$\widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a}) := \bigoplus_{d \in \mathbb{Z}} \left(\mathfrak{a}_d \cap (\mathfrak{p}^{d-k})_d\right),$$

where $\mathfrak{p}^d = R$ for $d \leqslant 0$. $\widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a})$ is a graded S-module, and it holds $\widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a}) \subseteq \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a})$ for all $k \in \mathbb{Z}$. Moreover, for any $k \in \mathbb{Z}$, we claim

$$\forall x \in R_1 \backslash \mathfrak{p}_1 : \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a}) = \{f \in \mathfrak{a} \mid \deg_x(f) \leqslant k\}.$$

This means that for each $x \in R_1 \backslash \mathfrak{p}_1$, we can write any element $f \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a})$ uniquely as $f = x^k f_0 + g$ with $f_0 \in S$ and $g \in R$ such that $\deg_x(g) < k$. Indeed, let $x \in R_1 \backslash \mathfrak{p}_1$, let $d \in \mathbb{N}_0$, and let $f \in \mathfrak{a}_d$ homogeneous. Then, $\deg_x(f) \leqslant k$ if and only if $f = x^k f_{d-k} + x^{k-1} f_{d-k+1} + \cdots + f_d$ with $f_j \in S_j$ for $j \in \{d-k, \ldots, d\}$, that is $f_j \in \mathfrak{p}^j$ for all $j \in \{d-k, \ldots, d\}$. This is equivalent to $f \in \mathfrak{p}^{d-k}$.

Lemma 3.1.4. *For all $k \in \mathbb{N}_0$ and all $x \in R_1 \backslash \mathfrak{p}_1$, there is an isomorphism of graded S-modules*

$$\zeta_k^x(\mathfrak{a}) : \left(\widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a}) / \widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a})\right)(-k) \xrightarrow{\cong} \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}), \overline{f} = x^k f_0 + g + \widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a}) \mapsto f_0.$$

Proof. Let $x \in R_1 \backslash \mathfrak{p}_1$, and let $k \in \mathbb{N}_0$. There is a morphism of graded S-modules

$$\widetilde{\zeta}_k^x(\mathfrak{a}) : \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(-k) \to S, f = x^k f_0 + g \mapsto f_0.$$

By definition, we find $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathrm{im}(\widetilde{\zeta}_k^x(\mathfrak{a}))$; we want to show that the reverse inclusion holds, too. So, let $y \in R_1 \backslash \mathfrak{p}_1$ be arbitrary, let $d \in \mathbb{N}_0$, and let $f_0 \in$

$\mathrm{im}(\widetilde{\zeta}_k^x(\mathfrak{a}))_d \subseteq S_d$ be homogeneous of degree d. Then $f_0 \in (\mathfrak{p}^d)_d \subseteq R_d$, and there is an element $g \in (\mathfrak{p}^{d+1})_{d+k}$ such that $x^k f_0 + g \in \mathfrak{a}_{d+k} \cap (\mathfrak{p}^d)_{d+k}$. As R_1 is generated by x and $\mathfrak{p}_1$ over K, we find elements $\kappa \in K\backslash\{0\}$ and $v \in \mathfrak{p}_1$ such that $x = \kappa y + v$, that is $x^k f_0 = \kappa^k y^k f_0 + u f_0$ for some $u \in \mathfrak{p}$. Therefore, $\frac{uf_0}{\kappa^k} + \frac{g}{\kappa^k} \in \mathfrak{p}^{d+1}$ and $y^k f_0 + \frac{uf_0}{\kappa^k} + \frac{g}{\kappa^k} \in \mathfrak{a}$, proving indeed $f_0 \in \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})$. On the other hand, it is easy to see that $\ker(\widetilde{\zeta}_k^x(\mathfrak{a})) = \widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a})(-k)$. This immediately gives the desired isomorphism. □

Corollary 3.1.5. *For all $x \in R_1\backslash\mathfrak{p}_1$ and all $k \in \mathbb{N}_0$ it holds*

$$\begin{array}{rcl}\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) & = & \bigoplus_{d\in\mathbb{Z}}\{f \in S_d \mid \exists g \in R_{d+k} : \deg_x(g) < k \wedge x^k f + g \in \mathfrak{a}_{d+k}\} \\ & = & \bigoplus_{d\in\mathbb{Z}}\{f \in S_d \mid \exists g \in (\mathfrak{p}^{d+1})_{d+k} : x^k f + g \in \mathfrak{a}_{d+k} \cap (\mathfrak{p}^d)_{d+k}\}.\end{array}$$

Proof. Let $x \in R_1\backslash\mathfrak{p}_1$, $k \in \mathbb{N}_0$, and write $\mathfrak{K}_k^x(\mathfrak{a}) := \bigoplus_{d\in\mathbb{Z}}\{f \in S_d \mid \exists g \in (\mathfrak{p}^{d+1})_{d+k} : x^k f + g \in \mathfrak{a}_{d+k} \cap \mathfrak{p}^d_{d+k}\}$. As in the above proof, there is an isomorphism of graded S-modules

$$\widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{a})/\widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a})(-k) \xrightarrow{\cong} \mathfrak{K}_k^x(\mathfrak{a}), x^k f_0 + g + \widetilde{\mathfrak{K}}_{k-1}^{\mathfrak{p}}(\mathfrak{a}) \mapsto f_0.$$

This gives $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \cong \mathfrak{K}_k^x(\mathfrak{a})$. As by definition $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \subseteq \mathfrak{K}_k^x(\mathfrak{a})$, we get our claim. □

Note that the formula of Corollary 3.1.5 is indeed the same as the one given in Section 0.2 as $S_d \subseteq \mathfrak{p}^d$.

Lemma 3.1.6. *Let $\gamma : R \xrightarrow{\cong} R$ be a graded ring automorphism, and let $k \in \mathbb{N}_0$. Then*

$$\gamma(\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})) = \mathfrak{K}_k^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{a})).$$

Proof. Let $x \in R_1\backslash\mathfrak{p}_1$. Then $\gamma(x) \in R_1\backslash\gamma(\mathfrak{p})_1$ and $\langle x, \mathfrak{p}_1\rangle_K = R_1 = \gamma(R_1) = \langle\gamma(x), \gamma(\mathfrak{p})\rangle_K$, where $\langle x, \mathfrak{p}_1\rangle_K$ denotes the K-vector space generated by x and $\mathfrak{p}_1$. Now let $d \in \mathbb{Z}$, and let $f \in S_d$. Then by Corollary 3.1.5 we see

$$\begin{array}{rcl} f \in \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})_d \subseteq K[\mathfrak{p}_1]_d & \Leftrightarrow & \exists g \in (\mathfrak{p}^{d+1})_{d+k} : x^k f + g \in \mathfrak{a}_{d+k} \cap (\mathfrak{p}^d)_{d+k} \\ & \Leftrightarrow & \exists g' \in (\gamma(\mathfrak{p})^{d+1})_{d+k} : \\ & & \gamma(x)^k\gamma(f) + g' \in \gamma(\mathfrak{a})_{d+k} \cap (\gamma(\mathfrak{p})^d)_{d+k} \\ & \Leftrightarrow & \gamma(f) \in \mathfrak{K}_k^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{a}))_d \subseteq K[\gamma(\mathfrak{p})_1]_d. \end{array}$$

□

Remark 3.1.7. Corollary 3.1.5 means that to compute PEIs it is enough to look at one element $x \in R_1\backslash\mathfrak{p}_1$, while Lemma 3.1.6 tells us that computing PEIs commutes with coordinate transformations. Definition 3.1.2 therefore indeed gives a generalization of the partial elimination ideals defined in [G, 6.1] which is independent of a choice of coordinates of R.

Remark and Notation 3.1.8. For the remainder of this Section, let $x \in R_1\backslash\mathfrak{p}_1$ be a linear form, so that R_1 is generated by x and $\mathfrak{p}_1$ over K. For a graded ideal $\mathfrak{q} \subseteq S = K[\mathfrak{p}_1]$ let

$$\overline{S} := S/((\mathfrak{a} \cap S) + \mathfrak{q})$$

(for the notation $\bar{\cdot}$, see 0.4 §9). Assume $\dim(\overline{S}) = 1$. According to the homogeneous Noether normalization (see [BsHe, Theorem 1.5.17]) , there is an element

$y \in S_1$ such that $K[y] \hookrightarrow \overline{S}$ is a finite integral extension (here and later we identify indeterminates and their residue classes if there is no danger of confusion). Furthermore, we can write

$$\overline{R} := R/((\mathfrak{a} \cap S)R + \mathfrak{q}R) = \overline{S}[x].$$

The ring extension $K[x, y] \hookrightarrow \overline{R}$ is finite and integral, too. Let

$$\overline{\mathfrak{p}} := \mathfrak{p}/((\mathfrak{a} \cap S)R + \mathfrak{q}R),$$

and let

$$\overline{\mathfrak{a}} := \mathfrak{a} + \mathfrak{q}R/((\mathfrak{a} \cap S)R + \mathfrak{q}R).$$

If $\mathfrak{a} \nsubseteq \mathfrak{p}$, it holds $\sqrt{\mathfrak{a} + \mathfrak{p}} = R_+$, and therefore, there exists an integer $t \in \mathbb{N}_0$ and an element $g \in \mathfrak{p}_t$ such that $x^t + g \in \mathfrak{a}$. Hence, $\deg_x(g) < t$ implies $1_S \in \mathfrak{K}_t^{\mathfrak{p}}(\mathfrak{a})$. So, if $(\mathfrak{a} \cap S) + \mathfrak{q} \neq S$, there exists an integer

$$k_{\mathfrak{q}} := \max\{k \in \mathbb{N}_0 \cup \{-1\} \mid \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \subseteq (\mathfrak{a} \cap S) + \mathfrak{q}\}.$$

Theorem 3.1.9. *Assume $\mathfrak{a} \nsubseteq \mathfrak{p}$. Let $\mathfrak{q} \subseteq S$ be a graded ideal such that $\dim(\overline{S}) = 1$, and such that $\overline{\mathfrak{a}}^{\text{sat}} \subseteq \overline{R}$ is a principal ideal. Then, any generator $\overline{h}$ of $\overline{\mathfrak{a}}^{\text{sat}}$ can be written as*

$$\overline{h} = h_0 x^{k_{\mathfrak{q}}+1} + \overline{g} \in \overline{R}_{k_{\mathfrak{q}}+1}$$

with $h_0 \in K \backslash \{0\}$ and $\overline{g} \in \overline{\mathfrak{p}}_{k_{\mathfrak{q}}+1}$.

Proof. Let $y \in \overline{S}_1$ such that $K[y] \subseteq \overline{S}$ is finite and integral. Let $\overline{h} \in \overline{R}$ be a homogeneous generator of $\overline{\mathfrak{a}}^{\text{sat}}$, and let $l := \deg(\overline{h})$. As $\overline{R} = K[x, \overline{\mathfrak{p}}_1]$, we can write

$$\overline{h} = h_0 x^l + \overline{g},$$

where $h_0 \in K$ and $\overline{g} \in \overline{\mathfrak{p}}$. As $\overline{\mathfrak{a}}^{\text{sat}} \nsubseteq \overline{\mathfrak{p}}$, it follows $h_0 \neq 0$.
So, we need only show that $l = k_{\mathfrak{q}} + 1$. Let $g \in \mathfrak{p}_l$ be a representative of $\overline{g}$. Then $h := h_0 x^l + g$ is a representative of $\overline{h}$. Since $\overline{h} \in \overline{\mathfrak{a}}^{\text{sat}}$, for $d \gg 0$ it holds $\overline{h}y^d \in \overline{\mathfrak{a}}_{d+l} \cap (\overline{\mathfrak{p}}^d)_{d+l}$ and therefore $hy^d \in (\mathfrak{a}_{d+l} \cap (\mathfrak{p}^d)_{d+l}) + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+l}$. Thus, there are elements $v \in \mathfrak{a}_{d+l} \cap (\mathfrak{p}^d)_{d+l}$ and $w \in ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+l}$ such that $hy^d = v + w$. In particular, $\mathrm{LT}_x(v + w) = h_0 x^l y^d$. As $v \in (\mathfrak{p}^d)_{d+l}$, it holds $\deg_x(v) \leqslant l$ and therefore $\deg_x(w) \leqslant l$. We write $w = x^l w_0 + \widetilde{w}$, where $w_0 \in S_d$ and $\widetilde{w} \in R_{d+l}$ with $\deg_x(\widetilde{w}) < l$. As $x^t \notin (\mathfrak{a} \cap S)R + \mathfrak{q}R$ for all $t \in \mathbb{N}_0$, it follows $w_0 \in (\mathfrak{a} \cap S) + \mathfrak{q}$, and as $y^t \notin (\mathfrak{a} \cap S) + \mathfrak{q}$ for all $t \in \mathbb{N}_0$, we finally get $w_0 \neq h_0 y^d$. This means that $\mathrm{LT}_x(v) = x^l(h_0 y^d - w_0)$ and hence $h_0 y^d - w_0 \in \mathfrak{K}_l^{\mathfrak{p}}(\mathfrak{a})$. If $l \leqslant k_{\mathfrak{q}}$, this would imply $y^d \in (\mathfrak{a} \cap S) + \mathfrak{q}$, a contradiction. It follows $l > k_{\mathfrak{q}}$.
On the other hand, let $k \in \mathbb{N}_0$ with $k < l$, let $d \in \mathbb{N}_0$, and let $f \in \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})_d$. We want to show that $f \in (\mathfrak{a} \cap S) + \mathfrak{q}$. There is an element $g \in (\mathfrak{p}^{d+1})_{d+k}$ such that $x^k f + g \in \mathfrak{a}_{d+k} \cap (\mathfrak{p}^d)_{d+k}$, that is

$$x^k f + g + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k} \in \overline{\mathfrak{a}}_{d+k}^{\text{sat}} \cap (\overline{\mathfrak{p}}^d)_{d+k} = 0,$$

where the last equality holds because of $\deg(\overline{h}) > k$ and $\overline{h} \notin \overline{\mathfrak{p}}$. Thus, $x^k f + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k} = -g + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k}$. If we assume $f \notin ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k}$, we therefore immediately get the contradiction

$$\begin{aligned} k &= \deg_x(x^k f + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k}) \\ &= \deg_x(-g + ((\mathfrak{a} \cap S)R + \mathfrak{q}R)_{d+k}) \quad \leqslant \quad \deg_x(g) \quad < \quad k. \end{aligned}$$

This proves $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) \subseteq (\mathfrak{a} \cap S) + \mathfrak{q}$ for all $k < l$ and therefore $l \leqslant k_{\mathfrak{q}} + 1$. □

This Theorem, as inconspicuous as it first may look, is the key to answer the questions posed in Section 0.2. It will provide us with the means to find the polynomials $\mathfrak{f}_\mathfrak{q}$ as well as equations describing the behaviour of the degree of said polynomials.

Proposition 3.1.10. *Assume $\mathfrak{a} \nsubseteq \mathfrak{p}$. Let $\mathfrak{q} \subseteq S$ be a graded ideal such that $\dim(\overline{S}) = 1$ and such that $\overline{\mathfrak{a}}^{\mathrm{sat}} \subseteq \overline{R}$ is a principal ideal. Then,*

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = (k_\mathfrak{q} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q})).$$

Proof. Let $y \in \overline{S}_1$ such that $K[y] \subseteq \overline{S}$ is finite and integral. Then, $x \in \mathrm{NZD}(\overline{R})$, hence x is a multiplicity parameter of degree 1 for $\overline{R}$ by Corollary 1.4.10. Therefore,

$$e_0(\overline{S}) = e_0(\overline{R}/x\overline{R}) = e_0(\overline{R}).$$

Now, let $\overline{h} \in \overline{R}$ be a homogeneous generator of $\overline{\mathfrak{a}}^{\mathrm{sat}}$. According to Lemma 3.1.9, we have $\overline{h} = h_0 x^{k_\mathfrak{q}+1} + \overline{g} \in R_{k_\mathfrak{q}+1}$ with $h_0 \in K\backslash\{0\}$ and $\overline{g} \in \overline{\mathfrak{p}}$, and since $x \in \mathrm{NZD}(\overline{R})$, it follows $\overline{h} \in \mathrm{NZD}(\overline{R})$, so $\overline{h}$ is a multiplicity parameter of degree $k_\mathfrak{q} + 1$ for $\overline{R}$. As a result, we finally get

$$\begin{array}{rclcl} e_0(R/(\mathfrak{a} + \mathfrak{q}R)) & = & e_0(\overline{R}/\overline{\mathfrak{a}}) & = & e_0(\overline{R}/\overline{\mathfrak{a}}^{\mathrm{sat}}) \\ & = & e_0(\overline{R}/\overline{h}R) & = & (k_\mathfrak{q} + 1) \cdot e_0(\overline{R}) \quad = \quad (k_\mathfrak{q} + 1) \cdot e_0(\overline{S}). \end{array}$$

The second equation holds as $\overline{\mathfrak{a}}_d = \overline{\mathfrak{a}}_d^{\mathrm{sat}}$ for all $d \gg 0$ and their Hilbert polynomials therefore are equal. □

As a corollary to Proposition 3.1.10, we get the main result about PEIs (see [CoSi, Theorem 3.5], [G, Proposition 6.2]):

Corollary 3.1.11. *Assume $\mathfrak{a} \nsubseteq \mathfrak{p}$, and let $\mathfrak{q} \in \mathrm{mProj}(S)$. Then*

$$\mathfrak{K}_k^\mathfrak{p}(\mathfrak{a}) \subseteq \mathfrak{q} \Leftrightarrow e_0(R/(\mathfrak{a} + \mathfrak{q}R)) > k.$$

Proof. We use the same notations as above. If $\mathfrak{K}_0^\mathfrak{p}(\mathfrak{a}) = \mathfrak{a} \cap S \nsubseteq \mathfrak{q}$, then $\sqrt{(\mathfrak{a} \cap S) + \mathfrak{q}} = S_+$ and $e_0(\overline{S}) = 0$. Now, assume $\mathfrak{a} \cap S \subseteq \mathfrak{q}$. Then $\overline{S} = S/\mathfrak{q} = K[y]$ for $y \in S_1\backslash\mathfrak{q}_1$, hence $\dim(\overline{S}) = 1 = e_0(\overline{S})$.
As $\dim(\overline{S}) = 1$, it holds $S_+ \nsubseteq \sqrt{\mathfrak{a} \cap S}$, therefore $R_+ \nsubseteq \sqrt{\mathfrak{a}}$ and $\overline{R}_+ \nsubseteq \sqrt{\overline{\mathfrak{a}}}$. Since $\overline{R}_+ = \langle x, y \rangle_{\overline{R}}$ is the only graded ideal of $\overline{R}$ of height 2, we get $\mathrm{height}(\overline{\mathfrak{a}}) < 2$. As $\mathfrak{a} \nsubseteq \mathfrak{p}$, it also holds $\sqrt{\overline{\mathfrak{a}}} \neq 0$, and therefore $\mathrm{height}(\overline{\mathfrak{a}}) = 1$, so $(\overline{\mathfrak{a}})^{\mathrm{sat}}$ is a principal ideal by Lemma 1.3.4, and we get our claim by Proposition 3.1.10. □

3.2 Equations for Z_k and secant cones

Throughout this Section as well as in the next Chapter, we will always use the following conventions and notations:

Remark and Notation 3.2.1. Let $\mathfrak{p} \dot{\in} \mathbb{P}^n$ be a closed point (compare 0.4 §14), and let $S := K[\mathfrak{p}_1]$ be the homogeneous K-subalgebra of R which is generated by the vector space $\mathfrak{p}_1$ of linear forms of $\mathfrak{p}$. The *(simple linear) projection with centre $\mathfrak{p}$ from $\mathbb{P}^n$* is the morphism

$$\tilde{\pi} : \mathbb{P}^n\backslash\{\mathfrak{q}\} \to \mathbb{P}^{n-1}$$

determined by the canonical inclusion $S \hookrightarrow R$. If $y_0, \dots, y_n \in R_1$ are coordinates of $\mathbb{P}^n$ with $y_0 \notin \mathfrak{p}_1$ and $\mathfrak{p} = \langle y_1, \dots, y_n \rangle_R$, then $y_1, \dots, y_n$ are coordinates on $\mathbb{P}^{n-1}$. We can write any closed point $\tilde{\mathfrak{q}} \mathrel{\dot\in} \mathbb{P}^n \backslash \{\mathfrak{p}\}$ with coordinates $\tilde{\mathfrak{q}} = (\tilde{q}_0 : \cdots : \tilde{q}_n)$, and it holds $\tilde{\pi}(\tilde{\mathfrak{q}}) = (\tilde{q}_1 : \cdots : \tilde{q}_n)$.
Also, let $x \in R_1 \backslash \mathfrak{p}_1$. We again consider $\mathbb{P}^{n-1}$ as a subspace of $\mathbb{P}^n$ by the identification $S = R/xR$, and we also identify $R = S[x]$.
Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a closed subscheme with homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} := \mathfrak{I}_S(\widetilde{Z})$ (compare 0.4 §15) such that $\mathfrak{p} \notin \widetilde{Z}$. The image $Z := \tilde{\pi}(\widetilde{Z})$ has homogeneous ideal $\mathfrak{I}_Z := \mathfrak{I}_{\widetilde{Z}} \cap S$, and the canonical inclusion $S/\mathfrak{I}_Z \hookrightarrow R/\mathfrak{I}_{\widetilde{Z}}$ determines a morphism of schemes

$$\pi := \tilde{\pi}\restriction_{\widetilde{Z}} : \widetilde{Z} \to Z.$$

We call π the *(simple linear) projection with centre* $\mathfrak{p}$ *of* $\widetilde{Z}$, or just the projection of $\widetilde{Z}$. Note that Z is a projective variety if $\widetilde{Z}$ is (compare 0.4 §16).
For a closed point $\mathfrak{q} \in Z$, the *fibre of* π *over* $\mathfrak{q}$ is the 0-dimensional scheme

$$\pi^{-1}(\mathfrak{q}) := \langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n} \cap \widetilde{Z}$$

(for the notation $\langle \cdot, \cdot \rangle_{\mathbb{P}^n}$, see 0.4 §17). The ideal of $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$ is $\mathfrak{q}R \subseteq R$, hence $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n} = \operatorname{Proj}(K[x,y])$ for $y \in S_1 \backslash \mathfrak{q}_1$. Also, $\mathfrak{I}_R(\pi^{-1}(\mathfrak{q})) = \mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R \subseteq R$, and we often consider

$$\pi^{-1}(\mathfrak{q}) = \operatorname{Proj}\left(R/(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R)\right) = \operatorname{Proj}\left(K[x,y]/(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R/\mathfrak{q}R)\right)$$

as a subscheme of $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$. The *length of the fibre over* $\mathfrak{q}$ is

$$\operatorname{length}\left(\pi^{-1}(\mathfrak{q})\right) := e_0\left(R/(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R)\right).$$

For a closed point $\tilde{\mathfrak{q}} \mathrel{\dot\in} \pi^{-1}(\mathfrak{q})$, the *multiplicity of* $\tilde{\mathfrak{q}}$ *in* $\pi^{-1}(\mathfrak{q})$ is

$$l_{\tilde{\mathfrak{q}}} := \operatorname{length}\left(\mathcal{O}_{\pi^{-1}(\mathfrak{q}), \tilde{\mathfrak{q}}}\right) = \operatorname{length}\left((R/\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R)_{(\tilde{\mathfrak{q}})}\right).$$

By Proposition 1.4.5 and Lemma 1.4.6, it holds

$$\operatorname{length}\left(\pi^{-1}(\mathfrak{q})\right) = \sum_{\tilde{\mathfrak{q}} \mathrel{\dot\in} \pi^{-1}(\mathfrak{q})} l_{\tilde{\mathfrak{q}}},$$

and we often write

$$\pi^{-1}(\mathfrak{q}) = \sum_{\tilde{\mathfrak{q}} \mathrel{\dot\in} \pi^{-1}(\mathfrak{q})} l_{\tilde{\mathfrak{q}}} \tilde{\mathfrak{q}}.$$

This last notation is due to the fact that we could consider $\pi^{-1}(\mathfrak{q})$ as an effective divisor on the projective line $\mathbb{P}^1$; as we are not interested in the embedding of $\pi^{-1}(\mathfrak{q})$ but in the distribution of $(l_{\tilde{\mathfrak{q}}})_{\tilde{\mathfrak{q}} \in \pi^{-1}(\mathfrak{q})}$, we will ignore this additional geometric approach.

Definition 3.2.2. For $k \in \mathbb{N}$, we define the *k-heterodyne locus of Z* as the set

$$Z_k := \{\mathfrak{q} \mathrel{\dot\in} Z \mid \operatorname{length}\left(\pi^{-1}(\mathfrak{q})\right) \geqslant k\} \subseteq \mathbb{P}^{n-1},$$

and the *proper k-heterodyne locus of Z* as the set

$$Z_k^\circ := Z_k \backslash Z_{k+1} = \{\mathfrak{q} \mathrel{\dot\in} Z \mid \operatorname{length}\left(\pi^{-1}(\mathfrak{q})\right) = k\} \subseteq \mathbb{P}^{n-1}.$$

Also, we set $Z_0 := \mathbb{P}^{n-1}$. In the next Proposition, we will see that the heterodyne loci are closed sets in $\mathbb{P}^{n-1}$, and if $\mathfrak{I}_{\widetilde{Z}}$ is reduced, we consider Z_k to be a subvariety of Z for all $k \in \mathbb{N}_0$. (The name 'heterodyne locus' has been chosen since it is the locus of closed points over which several points are combined to form a new one (at least in the generic case without multiple points in the fibre). 'Ramification locus' would be a more conventional name, but we use it already for something different in this work.)

Proposition 3.2.3. *For all $k \in \mathbb{N}_0$, set-theoretically*

$$Z_k = V_{\mathbb{P}^{n-1}}\left(\mathfrak{K}^{\mathfrak{p}}_{k-1}(\mathfrak{I}_{\widetilde{Z}})\right).$$

Proof. This follows immediately by Corollary 3.1.11. □

This result has first been formulated by M. Green in [G, Proposition 6.2]. Another proof can be found in [CoSi].

PEIs can not only be used to determine heterodyne loci, but also to compute secant cones and secant loci, as we now want to show:

Definition 3.2.4. Let $k \in \mathbb{N}_0$. A *k-secant line to $\widetilde{Z}$* is a line $\mathbb{L} \subseteq \mathbb{P}^n$ such that $\operatorname{length}(\widetilde{Z} \cap \mathbb{L}) \geqslant k$. We define the *$k$-secant cone* $\operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z})$ *of Z with vertex* $\mathfrak{p}$ as the closed subset of $\mathbb{P}^n$

$$\operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z}) := \{\mathfrak{p}\} \cup \bigcup \{\mathbb{L} \mid \mathbb{L} \text{ is a } k\text{-secant line to } \widetilde{Z} \text{ with } \mathfrak{p} \in \mathbb{L}\}$$

furnished with its structure of reduced closed subscheme of $\mathbb{P}^n$. Next, we define the *k-secant locus of $\widetilde{Z}$ with respect to* $\mathfrak{p}$ as the closed subscheme of $\mathbb{P}^n$

$$\Sigma^k_{\mathfrak{p}}(\widetilde{Z}) := \widetilde{Z} \cap \operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z}).$$

Some authors also use the the term 'entry locus' instead of 'secant locus'. Observe that $\operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z}) = \operatorname{Join}(\mathfrak{p}, \Sigma^k_{\mathfrak{p}}(\widetilde{Z}))$ is the (embedded) join of $\mathfrak{p}$ and $\Sigma^k_{\mathfrak{p}}(\widetilde{Z})$; this is the reason why we demand $\operatorname{Sec}^k_{\mathfrak{p}}(Z)$ to be reduced (see [FOV]). Example 3.5.2 shows the importance of defining the secant cone to be reduced.

Proposition 3.2.5. *Let $k \in \mathbb{N}_0$. Then*

$$\operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z}) = \operatorname{Proj}\left(R/\sqrt{\mathfrak{K}^{\mathfrak{p}}_{k-1}(\mathfrak{I}_{\widetilde{Z}})R}\right)$$

and

$$\Sigma^k_{\mathfrak{p}}(\widetilde{Z}) = \operatorname{Proj}\left(R/\left(\mathfrak{I}_{\widetilde{Z}} + \sqrt{\mathfrak{K}^{\mathfrak{p}}_{k-1}(\mathfrak{I}_{\widetilde{Z}})R}\right)\right).$$

Proof. Let $\mathfrak{q} \in \mathbb{P}^{n-1} = \operatorname{mProj}(S)$; the homogeneous ideal of the projective line $\langle \mathfrak{q}, \mathfrak{p} \rangle_{\mathbb{P}^n} \subseteq \mathbb{P}^n$ is $\mathfrak{q}R \in \operatorname{Proj}(R)$. According to Corollary 3.1.11, it therefore holds

$$\begin{aligned} V_{\mathbb{P}^{n-1}}\left(\mathfrak{K}^{\mathfrak{p}}_{k-1}(\mathfrak{I}_{\widetilde{Z}})\right) &= \{\mathfrak{q} \in \pi(\widetilde{Z}) \mid e_0\left(R/(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}R)\right) \geqslant k\} \\ &= \{\mathfrak{q} \in \pi(\widetilde{Z}) \mid \operatorname{length}(\widetilde{Z} \cap \langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}) \geqslant k\} = \pi\left(\Sigma^k_{\mathfrak{p}}(\widetilde{Z})\right). \end{aligned}$$

But as the closure of $\pi^{-1}(\pi(\Sigma^k_{\mathfrak{p}}(\widetilde{Z}))) \subseteq \mathbb{P}^n$ is just $\operatorname{Sec}^k_{\mathfrak{p}}(\widetilde{Z})$, we get the first equation. The second equations follows by definition. □

3.3 Computational Aspects

We keep the notations of 3.2.1, and we introduce a new one:

Notation 3.3.1. Let $G \subseteq R$ be a set of polynomials in $K[x_0, \dots, x_n]$, and let $k \in \mathbb{N}_0$. Then, we denote

$$\mathrm{In}_k(G) := \{\mathrm{LC}_{x_0}(g) \in S \mid g \in G \wedge \deg_{x_0}(g) \leqslant k\}$$

and

$$\mathrm{In}_k^\circ(G) := \{\mathrm{LC}_{x_0}(g) \in S \mid g \in G \wedge \deg_{x_0}(g) = k\} = \mathrm{In}_k(G) \setminus \mathrm{In}_{k-1}(G).$$

Proposition 3.3.2. *Let $\mathfrak{a} \subseteq R$ be a graded ideal. Assume $\mathfrak{p} := \langle x_1, \dots, x_n \rangle_R \in$ $\mathrm{mProj}(R)$ (for notations, compare 0.4 §§8 and 13). Denote* Lex *the lexicographical ordering on R, and let G be a Gröbner basis of $\mathfrak{a}$ with respect to the lexicographical term ordering* Lex. *For all $k \in \mathbb{N}_0$, the set $\mathrm{In}_k(G)$ is a Gröbner basis of $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})$ with respect to the lexicographical term ordering* $\mathrm{Lex}\restriction$ *on $S = K[\mathfrak{p}_1] = K[x_1, \dots, x_n]$.*

Proof. As before, we write $\mathrm{LC}_{x_0}(\cdot)$ and $\deg_{x_0}(\cdot)$ for leading coefficient and degree as polynomial in $R = S[x_0]$. For any term ordering σ, we denote by LC_σ and $\deg_\sigma$ the leading coefficient and degree with respect to σ, respectively. In the same way, we use the notations $\mathrm{LT}_\bullet$ and $\mathrm{In}_\bullet$ for leading terms and initial ideals, respectively, with $\bullet$ either a term ordering or x_0.
Note that $\mathrm{Lex}\restriction$ on $S = K[x_1, \dots, x_n]$ is induced by Lex on R. Let $G = \{g_1, \dots, g_t\}$ with $\deg_{x_0}(g_1) \leqslant \deg_{x_0}(g_2) \leqslant \dots \leqslant \deg_{x_0}(g_t)$. We define for $k \in \mathbb{N}_0$ an integer

$$s_k := \begin{cases} 0 & \text{if } \deg_{x_0}(g_1) > k, \\ \max\{i \in [n] \mid \deg_{x_0}(g_i) \leqslant k\} & \text{otherwise.} \end{cases}$$

Now, fix $k \in \mathbb{N}_0$, and let $i \in \{1, \dots, s_k\}$. Then,

$$g_i x_0^{k-\deg_{x_0}(g_i)} \in \mathfrak{a} \quad \text{and} \quad \deg_{x_0}\left(g_i x_0^{k-\deg_{x_0}(g_i)}\right) = k,$$

therefore $\mathrm{LC}_{x_0}(g_i) \in \mathfrak{K}_k(\mathfrak{a})$ by Corollary 3.1.5. It follows

$$\mathrm{LT}_{\mathrm{Lex}\restriction}(\mathrm{LC}_{x_0}(g_i)) \in \mathrm{In}_{\mathrm{Lex}\restriction}(\mathfrak{K}_k(\mathfrak{a})),$$

and it suffices to show

$$\mathrm{In}_{\mathrm{Lex}\restriction}(\mathfrak{K}_k(\mathfrak{a})) \subseteq \langle \mathrm{LT}_{\mathrm{Lex}\restriction}(\mathrm{LC}_{x_0}(g_1)), \dots, \mathrm{LT}_{\mathrm{Lex}\restriction}(\mathrm{LC}_{\mathrm{Lex}\restriction}(g_{s_k})) \rangle_S.$$

Indeed, let $f \in \mathfrak{K}_k(\mathfrak{a})$. There exists $g \in R$ such that $\deg_{x_0} < k$ and $\widetilde{f} := x_0^k f + g \in \widetilde{\mathfrak{K}}_k(\mathfrak{a}) \subseteq \mathfrak{a}$. Since $\mathrm{In}_{\mathrm{Lex}}(\mathfrak{a})$ is generated by the leading terms of G and $\deg_{x_0}(g_i) > k = \deg_{x_0}(\widetilde{f})$ for all $i \in \{s_k + 1, \dots, t\}$, we get

$$x_0^k \, \mathrm{LT}_{\mathrm{Lex}\restriction}(f) = \mathrm{LT}_{\mathrm{Lex}}(\widetilde{f}) \in \langle \mathrm{LT}_{\mathrm{Lex}}(g_1), \dots, \mathrm{LT}_{\mathrm{Lex}}(g_{s_k}) \rangle_S,$$

and thus

$$\begin{aligned} \mathrm{LT}_{\mathrm{Lex}\restriction}(f) &\in \langle \mathrm{LC}_{x_0}(\mathrm{LT}_{\mathrm{Lex}}(g_1)), \dots, \mathrm{LC}_{x_0}(\mathrm{LT}_{\mathrm{Lex}}(g_{s_k})) \rangle_S \\ &= \langle \mathrm{LT}_{\mathrm{Lex}\restriction}(\mathrm{LC}_{x_0}(g_1)), \dots, \mathrm{LT}_{\mathrm{Lex}\restriction}(\mathrm{LC}_{x_0}(g_{s_k})) \rangle_S. \end{aligned}$$

□

The above Proposition has first been shown in [CoSi, Proposition 3.4].

Remark 3.3.3. We keep the notations of Proposition 3.3.2, but we assume further that $\mathfrak{a} \nsubseteq \mathfrak{p}$, that is $R_+ \subseteq \sqrt{\mathfrak{a}+\mathfrak{p}}$. So, there is an integer $t \in \mathbb{N}_0$ such that $x_0^t \in \mathfrak{a}+\mathfrak{p}$, hence x_0^t is contained in the initial ideal $\mathrm{In}_{\mathrm{Lex}}(\mathfrak{a}+\mathfrak{p})$. But as $\mathfrak{p} = (x_1, \ldots, x_n)$, this means $x_0^t \in \mathrm{In}_{\mathrm{Lex}}(\mathfrak{a})$. Therefore, there must be an element $g_0 \in G$ such that $\mathrm{LT}_{x_0}(g_0) = x_0^s$ for some $s \leqslant t$ and $\mathfrak{K}_s^{\mathfrak{p}}(\mathfrak{a}) = K[\mathfrak{p}_1]$.

Algorithm 3.3.4. (A) Using Proposition 3.3.2, we obtain the following method for computing equations defining heterodyne loci:

Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a closed subscheme with homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} \subseteq R$, and let $\mathfrak{p} \dot{\in} \mathbb{P}^n$ such that $\mathfrak{p} \notin Z$.

1. Choose a linear coordinate transformation $\gamma : R \overset{\cong}{\to} R$ such that $\gamma(\mathfrak{p}) = (x_1, \ldots, x_n)$.
2. Compute a Gröbner basis G of $\gamma(\mathfrak{I}_{\widetilde{Z}})$ with respect to Lex.
3. Choose $k_{\mathfrak{p}} \in \mathbb{N}_0$ such that $\mathfrak{K}_k^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{I}_{\widetilde{Z}})) = K[\mathfrak{p}_1]$ for all $k \geqslant k_{\mathfrak{p}}$. An integer $k_{\mathfrak{p}}$ with this property exists by Remark 3.3.3.
4. Compute the partial elimination ideals $\mathfrak{K}_0^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{I}_{\widetilde{Z}})), \ldots, \mathfrak{K}_{k_{\mathfrak{p}}-1}^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{I}_{\widetilde{Z}}))$. This can easily be done using Proposition 3.3.2.
5. Set $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) := \gamma^{-1}(\mathfrak{K}_k^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{I}_{\widetilde{Z}})))$ for $k \in \{0, \ldots, k_{\mathfrak{p}}-1\}$. Lemma 3.1.6 guarantees that we indeed get the partial elimination ideals of $\mathfrak{I}_{\widetilde{Z}}$ with respect to $\mathfrak{p}$.
6. Compute $\sqrt{\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R}$ for $k \in \{0, \ldots, k_{\mathfrak{p}}-1\}$.
7. Compute $\sqrt{\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R + \mathfrak{I}_{\widetilde{Z}}}$ for $k \in \{0, \ldots, k_{\mathfrak{p}}-1\}$.

According to 3.2.3, the ideals $\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}), \ldots, \mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})$ computed in step 5 define set-theoretically the heterodyne loci $Z_1 = Z, \ldots, Z_{k_{\mathfrak{p}}}$, while $Z_{k_{\mathfrak{p}}+1} = \varnothing$. Moreover, by Proposition 3.2.5, for any $k \in \{0, \ldots, k_{\mathfrak{p}}-1\}$, the $(k+1)$-secant cone of $\widetilde{Z}$ with respect to $\mathfrak{p}$ is defined by the homogeneous ideal $\sqrt{\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R}$, while the $(k+1)$-secant loci of $\widetilde{Z}$ with respect $\mathfrak{p}$ is defined by the homogeneous ideal $\sqrt{\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R + \mathfrak{I}_{\widetilde{Z}}}$. As $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R = R$ for all $k \geqslant k_{\mathfrak{p}}$, the higher secant loci $\Sigma_{\mathfrak{p}}^k(\widetilde{Z})$ are empty. Note that the ideals computed in steps 6 and 7 indeed define secant cones and secant loci scheme-theoretically.

(B) The above method contains some choices. We can replace these choices with explicit terms and get the following algorithm:

Input: The homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} \subseteq R$ of a closed subscheme $\widetilde{Z} \subseteq \mathbb{P}^n$, and a minimal system of generators $y_1, \ldots, y_n \in R_1$ of the closed point $\mathfrak{p} \in \mathrm{Proj}(R)$. Consider R to be furnished with the lexicographical term order.

1.1. Compute $l := \min\{i \in \{0, \ldots, n\} \mid x_i \notin \mathfrak{p}\}$.

1.2. Define the coordinate transformation $\gamma : R \xrightarrow{\tilde{=}} R$ to be the inverse of $x_0 \mapsto x_l$, $x_1 \mapsto y_1, \ldots, x_n \mapsto y_n$. Then indeed $\gamma(\mathfrak{p}) = (x_1, \ldots, x_n)$. Calculate $\gamma(\mathfrak{I}_{\tilde{Z}})$.

2. Compute a Gröbner basis G of $\gamma(\mathfrak{I}_{\tilde{Z}})$, for example using the Buchberger algorithm.

3. Set $k_{\mathfrak{p}} := \max\{\deg_{x_0}(g) \mid g \in G\}$. Then $\mathfrak{K}_k^{\gamma(\mathfrak{p})}(\gamma(\mathfrak{I}_{\tilde{Z}})) = K[\mathfrak{p}_1]$ for all $k \geqslant k_{\mathfrak{p}}$ according to Remark 3.3.3.

4. For all $k \in \{0, \ldots, k_{\mathfrak{p}} - 1\}$, set $\mathrm{In}_k(G) := \{\mathrm{LC}_{x_0}(g) \mid g \in G \wedge \deg_{x_0}(g) \leqslant k\}$.

5. Set $\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}}) := \gamma^{-1}(\mathrm{In}_k(G))K[\mathfrak{p}_1]$ for $k \in \{0, \ldots, k_{\mathfrak{p}} - 1\}$.

6. Compute $\sqrt{\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}})R}$ for $k \in \{0, \ldots, k_{\mathfrak{p}} - 1\}$, for example using the algorithm of Krick and Logar (see [KrL]).

7. Compute $\sqrt{\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}})R} + \mathfrak{I}_{\tilde{Z}}$ for $k \in \{0, \ldots, k_{\mathfrak{p}} - 1\}$.

Output: Ideals $\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}), \ldots, \mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})$ (via a Gröbner basis) of $Z_1, \ldots, Z_{k_{\mathfrak{p}}}$ (set-theoretically); ideals $\sqrt{\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})R}, \ldots, \sqrt{\mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_Z)R}$ (via a finite set of generators) of $\mathrm{Sec}_1^{\mathfrak{p}}(\tilde{Z}), \ldots, \mathrm{Sec}_{k_{\mathfrak{p}}}^{\mathfrak{p}}(\tilde{Z})$ (scheme-theoretically); ideals $\sqrt{\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})R} + \mathfrak{I}_{\tilde{Z}}, \ldots, \sqrt{\mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_Z)R} + \mathfrak{I}_{\tilde{Z}}$ (via a finite set of generators) of $\Sigma_1^{\mathfrak{p}}(\tilde{Z}), \ldots, \Sigma_{k_{\mathfrak{p}}}^{\mathfrak{p}}(\tilde{Z})$ (scheme-theoretically).

3.4 Multiple Projections

In this Section, we take a short look at the relation of PEIs with multiple projections, that is we show two ways to use PEIs to compute the heterodyne loci of multiple projections fulfilling certain assumptions.

Let $\hat{S} \subseteq R$ be a homogeneous graded K-subalgebra, that is there exist an integer $t \in \{0, \ldots, n\}$ and linearly independent elements $y_0, \ldots, y_t \in R_1$ such that $R = \hat{S}[y_0, \ldots, y_t]$. For a graded ideal $\mathfrak{a} \subseteq R$, let $\hat{\mathfrak{a}} := \mathfrak{a} \cap \hat{S}$. Note that there is a natural inclusion map $\hat{S}/\hat{\mathfrak{a}} \hookrightarrow R/\mathfrak{a}$; we therefore consider $\hat{S}/\hat{\mathfrak{a}}$ as a graded K-subalgebra of $R/\mathfrak{a}$.

Proposition 3.4.1. *Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $(R/\mathfrak{a})_m = (\hat{S}/\hat{\mathfrak{a}})_m$ for all $m \gg 0$. Let $\hat{\mathfrak{p}} \in \mathrm{mProj}(\hat{S})$ such that $\hat{\mathfrak{a}} \nsubseteq \hat{\mathfrak{p}}$, let $S := K[\hat{\mathfrak{p}}_1] \subseteq \hat{S}$, and let $\mathfrak{q} \subseteq S$ be a graded ideal such that $\dim(S/((\hat{\mathfrak{a}} \cap S) + \mathfrak{q})) = 1$, and such that $\hat{a} + \mathfrak{q}\hat{S}/((\hat{a} \cap S)\hat{S} + \mathfrak{q}\hat{S})^{\mathrm{sat}}$ is a principal ideal. Denote $l_{\mathfrak{q}} := \max\{l \in \mathbb{N}_0 \cup \{-1\} \mid \mathfrak{K}_l^{\hat{\mathfrak{p}}}(\hat{\mathfrak{a}}) \subseteq \hat{\mathfrak{a}} \cap S + \mathfrak{q}\}$. Then*

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = (l_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q})).$$

Proof. Let $m \gg 0$. As $(R/\mathfrak{a})_m = (\hat{S}/\hat{\mathfrak{a}})_m$ and $R_m = S[y_0, \ldots, y_t]_m$, it must hold $\sum_{i=0}^t y_i R_{m-1} \subseteq \mathfrak{a}_m$. It follows $(\langle y_0, \ldots, y_t \rangle_R)_m \subseteq \mathfrak{a}_m$ and in particular $(\langle y_0, \ldots, y_t \rangle \cdot \mathfrak{q})_m \subseteq \mathfrak{a}_m$. Moreover, $\mathfrak{q}R = \mathfrak{q}\hat{S} + (y_0, \ldots, y_t)\mathfrak{q}$. It follows

$$\begin{aligned}(R/(\mathfrak{a} + \mathfrak{q}R))_m &= (R/(\mathfrak{a} + \mathfrak{q}\hat{S}))_m \\ &= ((\hat{S}/\hat{\mathfrak{a}})/(\hat{\mathfrak{a}} + \mathfrak{q}\hat{S}/\hat{\mathfrak{a}}))_m = (\hat{S}/(\hat{\mathfrak{a}} + \mathfrak{q}\hat{S}))_m.\end{aligned}$$

Thus, by Proposition 3.1.10

$$\begin{aligned} e_0(R/(\mathfrak{a}+\mathfrak{q}R)) &= e_0(\hat{S}/(\hat{\mathfrak{a}}+\mathfrak{q}\hat{S})) \\ &= (l_{\mathfrak{q}}+1)\cdot e_0(S/((\hat{\mathfrak{a}}\cap S)+\mathfrak{q})) \\ &= (l_{\mathfrak{q}}+1)\cdot e_0(S/((\mathfrak{a}\cap S)+\mathfrak{q})). \end{aligned}$$

□

Now, let $t \in \mathbb{N}_0$, and let $\Omega = \mathbb{P}^t \subseteq \mathbb{P}^n$ be a linear subspace of dimension t with homogeneous ideal $\mathfrak{I}_\Omega := \mathfrak{I}_R(\Omega) \subseteq R$, and let

$$\pi := \pi_\Omega : \mathbb{P}^n \backslash \Omega \to \mathbb{P}^{n-t-1}$$

be the multiple linear projection with centre Ω; this projection is given by $S = S^{(\Omega)} := K[(\mathfrak{I}_\Omega)_1] \hookrightarrow R$. Then, we can choose a decomposition

$$\pi = \pi_t \circ \pi_{t-1} \circ \cdots \circ \pi_0,$$

where $\pi_i : \mathbb{P}^{n-i}\backslash\{\mathfrak{p}^{(i)}\} \to \mathbb{P}^{n-i-1}$ is a simple linear projection with centre $\mathfrak{p}^{(i)}$ for $i \in \{0,\ldots,t\}$. If we denote the homogeneous rings of the projective spaces $\mathbb{P}^{n-1},\ldots,\mathbb{P}^{n-t}$ by $\hat{S}^{(0)},\ldots,\hat{S}^{(t-1)}$, this decomposition is given by

$$S \hookrightarrow \hat{S}^{(t-1)} \hookrightarrow \cdots \hat{S}^{(0)} \hookrightarrow R.$$

Note that $\mathfrak{I}_\Omega \cap S^{(t-1)} = \mathfrak{p}^{(t)}$ is a closed point in $\mathbb{P}^{n-t}(\mathfrak{p}^{(t-1)})$.

Corollary 3.4.2. *Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a closed subscheme with homogeneous ideal $\mathfrak{I}_{\widetilde{Z}}$, and assume that there is a decomposition $\pi = \pi_t \circ \cdots \circ \pi_0$ such that $(\pi_{t-1} \circ \cdots \circ \pi_0)\restriction_{\widetilde{Z}} : \widetilde{Z} \to \pi_t^{-1}(\pi(\widetilde{Z}))$ is an isomorphism. Let $\mathfrak{q} \in \mathbb{P}^{n-t-1}$ be a closed point. Then, for all $k \in \mathbb{N}_0$*

$$\text{length}\left(\widetilde{Z} \cap \langle \mathfrak{q}, \Omega\rangle_{\mathbb{P}^n}\right) > k \Leftrightarrow \mathfrak{K}_k^{\mathfrak{I}_\Omega \cap \hat{S}^{(t-1)}}(\mathfrak{I}_{\widetilde{Z}} \cap \hat{S}^{(t-1)}) \subseteq \mathfrak{q}.$$

Proof. As by assumption $(\pi_{t-1}\circ\cdots\circ\pi_0)\restriction_{\widetilde{Z}}$ is an isoprojection, we get $(R/\mathfrak{I}_{\widetilde{Z}})_m = (\hat{S}^{(t-1)}/(\mathfrak{I}_{\widetilde{Z}} \cap \hat{S}^{(t-1)}))_m$ for all $m \gg 0$. Hence, our claim follows by Proposition 3.4.1. □

Notation 3.4.3. For the remainder of this Section, let $\mathfrak{L} \in \text{Proj}(R)$ be a linearly generated ideal of height $n-1$, so that $\mathbb{L} := \text{Proj}(R/\mathfrak{L}) = \mathbb{P}^1 \subseteq \mathbb{P}^n$ is a projective line. Let $S = S^{(\mathbb{L})} := K[\mathfrak{L}_1] \subseteq R$; the twofold projection $\pi_\mathbb{L} : \mathbb{P}^n\backslash\mathbb{L} \twoheadrightarrow \mathbb{P}^{n-2}$ is given by $S \hookrightarrow R$. Let $\mathfrak{p}, \mathfrak{p}' \dot{\in} \mathbb{L}, \mathfrak{p} \neq \mathfrak{p}'$, and let $\hat{S} := K[\mathfrak{p}_1], \hat{S}' := K[\mathfrak{p}'_1] \subseteq R$. Consider the projections $\pi : \mathbb{P}^n\backslash\{\mathfrak{p}\} \to \mathbb{P}^{n-1}$ and $\pi' : \mathbb{P}^n\backslash\{\mathfrak{p}'\} \to \mathbb{P}^{n-1}$ given by $\hat{S} \hookrightarrow R$ and $\hat{S}' \hookrightarrow R$, respectively, as well as $\hat{\pi} : \mathbb{P}^{n-1}\backslash\{\hat{\mathfrak{p}}\} \to \mathbb{P}^{n-2}$ and $\hat{\pi}' : \mathbb{P}^{n-1}\backslash\{\hat{\mathfrak{p}}'\} \to \mathbb{P}^{n-2}$ given by $S \hookrightarrow \hat{S}$ and $S \hookrightarrow \hat{S}'$, respectively, where $\hat{\mathfrak{p}} := \pi'(\mathfrak{p}) = \pi'(\mathbb{L}\backslash\{\mathfrak{p}'\})$ and $\hat{\mathfrak{p}}' := \pi(\mathfrak{p}') = \pi(\mathbb{L}\backslash\{\mathfrak{p}\})$. Then,

$$\pi_\mathbb{L} = \hat{\pi}' \circ \pi = \hat{\pi} \circ \pi' : \mathbb{P}^n\backslash\mathbb{L} \to \mathbb{P}^{n-2}$$

are two decompositions of π.

Definition 3.4.4. Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $R_+ \subseteq \sqrt{\mathfrak{a}+\mathfrak{L}}$. We call $\mathfrak{p}, \mathfrak{p}' \in \text{mProj}(R) \cap \text{Var}(\mathfrak{L})$ a *clever decomposition of $\mathfrak{L}$ with respect to $\mathfrak{a}$* if

$$\mathfrak{a}^{\text{sat}} = \left((\mathfrak{a} \cap K[\mathfrak{p}_1])R + (\mathfrak{a} \cap K[\mathfrak{p}'_1])R\right)^{\text{sat}}.$$

Remark 3.4.5. Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a closed subscheme with homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} \subseteq R$ such that $\widetilde{Z} \cap \mathbb{L} = \varnothing$. Geometrically, Definition 3.4.4 means that $\mathfrak{p}, \mathfrak{p}' \mathrel{\dot{\in}} \mathbb{L}$ are a *clever decomposition of* $\mathbb{L}$ *with respect to* $\widetilde{Z}$ if

$$\widetilde{Z} = \operatorname{Join}(\pi(\widetilde{Z}), \mathfrak{p}) \cap \operatorname{Join}(\pi'(\widetilde{Z}), \mathfrak{p}') = \operatorname{Join}(\widetilde{Z}, \mathfrak{p}) \cap \operatorname{Join}(\widetilde{Z}, \mathfrak{p}').$$

Proposition 3.4.6. *Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $R_+ \subseteq \sqrt{\mathfrak{a} + \mathfrak{L}}$, and let $\mathfrak{p}, \mathfrak{p}' \in \operatorname{mProj}(R) \cap \operatorname{Var}(\mathfrak{L})$ be a clever decomposition of $\mathfrak{L}$ with respect to $\mathfrak{a}$. Let $\mathfrak{q} \in \operatorname{mProj}(S)$, and let $k_\mathfrak{q} := \max\{k \in \mathbb{N}_0 \cup \{-1\} \mid \mathfrak{K}_k^{\mathfrak{L} \cap \hat{S}}(\mathfrak{a} \cap \hat{S}) \subseteq \mathfrak{q}\}$, $k'_\mathfrak{q} := \max\{k \in \mathbb{N}_0 \cup \{-1\} \mid \mathfrak{K}_k^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{a} \cap \hat{S}') \subseteq \mathfrak{q}\}$. Then,*

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = (k_\mathfrak{q} + 1) \cdot (k'_\mathfrak{q} + 1).$$

Proof. We write $\overline{R} := R/\mathfrak{q}R$. Let $x, x' \in R_1$ such that $\mathfrak{p} = \mathfrak{L} + xR$ and $\mathfrak{p}' = \mathfrak{L} + x'R$, and let $y \in S_1 \subseteq R_1$ such that $\mathfrak{q}R + yR = \mathfrak{L}$, that is $\hat{S}/\mathfrak{q}\hat{S} = K[y]$ and $\overline{R} = K[x, x', y]$. By Lemma 3.1.9, there is a homogeneous element

$$\overline{h} = h_0 x^{k_\mathfrak{q}+1} + h_1 x^{k_\mathfrak{q}} y + \cdots + h_{k_\mathfrak{q}+1} y^{k_\mathfrak{q}+1} \in K[x, y]_{k_\mathfrak{q}+1} \subseteq \overline{R}_{k_\mathfrak{q}+1},$$

where $h_0, \ldots, h_{k_\mathfrak{q}+1} \in K$ such that $h_0 \neq 0$ and

$$(\mathfrak{a} \cap \hat{S}) + \mathfrak{q}\hat{S}/\mathfrak{q}\hat{S} = \overline{hS}.$$

Analogously, $\mathfrak{a} \cap \hat{S} + \mathfrak{q}\hat{S}/\mathfrak{q}\hat{S}$ is generated by

$$\overline{h}' = h'_0 x'^{k'_\mathfrak{q}+1} + h'_1 x'^{k'_\mathfrak{q}} y + \cdots + h'_{k'_\mathfrak{q}+1} y^{k'_\mathfrak{q}+1} \in K[x', y]_{k'_\mathfrak{q}+1} \subseteq \overline{R}_{k'_\mathfrak{q}+1},$$

where $h'_0, \ldots, h'_{k,_\mathfrak{0}+1} \in K$ such that $h'_0 \neq 0$. As $h_0, h'_0 \neq 0$, it follows that $\overline{h}, \overline{h}' \subseteq \overline{R} = K[x, x', y]$ is an $\overline{R}$-regular sequence. So, by Corollary 1.4.15, $\overline{h}, \overline{h}'$ is a system of multiplicity parameters of degrees $k_\mathfrak{q} + 1, k'_\mathfrak{q} + 1$ for $\overline{R}$. Further, for all $m \gg 0$

$$\begin{aligned}
(\mathfrak{a} + \mathfrak{q}R/\mathfrak{q}R)_m &= \mathfrak{a}_m^{\text{sat}} + (\mathfrak{q}R)_m/(\mathfrak{q}R)_m \\
&= \Big((\mathfrak{a} \cap \hat{S})R + (\mathfrak{a} \cap \hat{S}')R\Big)_m + (\mathfrak{q}R)_m/(\mathfrak{q}R)_m \\
&= \Big((\mathfrak{a} \cap \hat{S} + \mathfrak{q}\hat{S}/\mathfrak{q}\hat{S})\overline{R}\Big)_m + \Big((\mathfrak{a} \cap \hat{S}' + \mathfrak{q}\hat{S}'/\mathfrak{q}\hat{S}')\overline{R}\Big)_m \\
&= \Big((\overline{h}, \overline{h}')\overline{R}\Big)_m .
\end{aligned}$$

Therefore, we get

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = e_0(\overline{R}/(\overline{h}, \overline{h}')\overline{R}) = (k_\mathfrak{q} + 1) \cdot (k'_\mathfrak{q} + 1).$$

□

Corollary 3.4.7. *Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a closed subscheme such that $\widetilde{Z} \cap \mathbb{L} = \varnothing$, and let $\mathfrak{p}, \mathfrak{p}' \mathrel{\dot{\in}} \mathbb{L}$ be a clever decomposition of $\mathbb{L}$ with respect to $\widetilde{Z}$. Then, for all closed points $\mathfrak{q} \mathrel{\dot{\in}} \mathbb{P}^{n-2}$*

$$\operatorname{length}\left(\widetilde{Z} \cap \langle \mathfrak{q}, \mathbb{L} \rangle\right) = \operatorname{length}\left(\pi(\widetilde{Z}) \cap \langle \mathfrak{q}, \pi(\mathfrak{p}') \rangle\right) \cdot \operatorname{length}\left(\pi'(\widetilde{Z}) \cap \langle \mathfrak{q}, \pi'(\mathfrak{p}) \rangle\right).$$

Proof. Clear by Proposition 3.4.6 and Proposition 3.1.10. □

Remark 3.4.8. The next obvious question here would be what conditions on $\mathfrak{L}$ and $\mathfrak{a}$ imply the existence of a clever decomposition. We are not going to answer this open question here; for now, we are not interested in clever decompositions themselves but in their usefulness for studying examples (see Example 3.5.5).

3.5 Examples

We now consider some examples to illustrate the results in this Chapter. We use the notations of the previous Sections as well as of 0.4, in particular §§8 and 14. All computations were done in [SINGULAR].

Example 3.5.1. First, we take a look at the condition in Proposition 3.1.10 that $\overline{\mathfrak{a}}$ must be a principal ideal:
(A) Let $R := K[x_0, \dots, x_4]$, let $\mathfrak{p} := \langle x_1, \dots, x_4 \rangle_R$, let

$$\mathfrak{a} := \langle x_0^4 + x_1^2 x_2^2, x_0^2 x_1 - x_3^3, x_2^2 - x_3^2, x_0 x_2 + x_4^2 \rangle \subseteq R,$$

and let $\mathfrak{q} := \langle x_3, x_4 \rangle \subseteq S = K[x_1, \dots, x_4]$. Then, $(\mathfrak{a} \cap S) + \mathfrak{q} = \langle x_2^2, x_3, x_4 \rangle_S$ and $x_2 \in \mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{a})$, so $k_{\mathfrak{q}} = 0$ and $e_0(\overline{S}) = e_0(K[x_1, \dots, x_4]/(x_2^2, x_3, x_4)) = 2$. Furthermore,

$$\mathfrak{a} + \mathfrak{q}R = \langle x_0^4, x_0^2 x_1, x_0 x_2, x_2^2, x_3, x_4 \rangle_R,$$

and therefore

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = 3 > (k_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q})).$$

(B) Keep R, $\mathfrak{p}$, S and $\mathfrak{q}$ of part (A), and let

$$\mathfrak{a} := \langle x_0^4 + x_0 x_1^3, x_0^3 x_1 + x_1^4 + x_3^4, x_0^2 x_2 + x_4^3, x_2^2 \rangle \subseteq R.$$

Then,

$$\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{a}) = (\mathfrak{a} \cap S) + \langle x_3^4, x_1 x_4^3 \rangle_S \subseteq (\mathfrak{a} \cap S) + \mathfrak{q} = \langle x_2^2, x_3, x_4 \rangle_S,$$

but $x_2 \in \mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{a})$, so $k_{\mathfrak{q}} = 1$ and $e_0(\overline{S}) = 2$. On the other hand,

$$\mathfrak{a} + \mathfrak{q}R = \langle x_0^4 + x_0 x_1^3, x_0^3 x_1 + x_1^4, x_0^2 x_2, x_0 x_1^3 x_2, x_1^4 x_2, x_2^2, x_3, x_4 \rangle_R$$

and hence

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = 3 < (k_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + q)).$$

(C) Again, keep R, $\mathfrak{p}$ and S as in part (A), but now let

$$\mathfrak{a} := \langle x_0^5 + x_0^2 x_1^3, x_0^4 x_1 + x_0 x_1^4 + x_3^5, x_0^3 x_1^2 + x_1^5 + x_4^5, x_0^3 x_2, x_2^3 \rangle_R,$$

and let $\mathfrak{q} := (x_1 x_2, x_3, x_4) \subseteq S$. Then,

$$\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{a}) = (\mathfrak{a} \cap S) + \langle x_1^3 x_2, x_4^5, x_3^5 \rangle_R \subseteq (\mathfrak{a} \cap S) + \mathfrak{q} = \langle x_1 x_2, x_2^3, x_2, x_4 \rangle_S,$$

but $x_2 \in \mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{a})$, so $k_{\mathfrak{q}} = 2$ and $e_0(\overline{S}) = 1$. On the other hand,

$$\overline{\mathfrak{a}}^{\text{sat}} = \langle x_0^3 + x_1^3, \overline{x}_2 \rangle \subseteq \overline{R} = K[x_0, x_1, \overline{x}_2],$$

and therefore

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = 3 = (k_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q})).$$

(D) Now, let $R = K[x_0, \dots, x_n]$, $\mathfrak{p} \in \text{mProj}(R)$, let $\mathfrak{a} \subseteq R$ be a graded ideal such that $\mathfrak{a} \nsubseteq \mathfrak{p}$, and let $\mathfrak{q} \subseteq S := K[\mathfrak{p}_1]$ be a graded ideal such that $\dim(\overline{S}) = 1$.

Then, $\overline{\mathfrak{a}}_0 := (\overline{\mathfrak{a}} \cap K[y_0, y_1])^{\mathrm{sat}}$ is a principal ideal; the same argument as used in the proof of Theorem 3.1.9 shows that a homogeneous generator $\overline{h}$ of $\overline{\mathfrak{a}}_0$ is of degree $\geqslant k_{\mathfrak{q}} + 1$. As $\overline{\mathfrak{a}}_0 \overline{R} \subseteq \overline{\mathfrak{a}}^{\mathrm{sat}}$, it follows

$$e_0(R/(\mathfrak{a} + \mathfrak{q}R)) \leqslant e_0(\overline{R}/\overline{\mathfrak{a}}_0\overline{R}) = e_0(\overline{R}/\overline{h}\overline{R}) \geqslant (k_{\mathfrak{q}} + 1)e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q})).$$

Now, (A) and (B) above prove that both inequalities between $e_0(R/(\mathfrak{a} + \mathfrak{q}R))$ and $(k_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q}))$ can occur if $\overline{\mathfrak{a}}^{\mathrm{sat}}$ is not a principal ideal. But the condition that $\overline{\mathfrak{a}}^{\mathrm{sat}}$ is a principal ideal is not necessary for $e_0(R/(\mathfrak{a} + \mathfrak{q}R)) = (k_{\mathfrak{q}} + 1) \cdot e_0(S/((\mathfrak{a} \cap S) + \mathfrak{q}))$, as (C) shows. Indeed, we conjecture that this equality always holds if $\overline{S}_+$ can be generated by two elements and the degree of a generator of $\overline{\mathfrak{a}}_0$ is $k_{\mathfrak{q}} + 1$.

Example 3.5.2. In this example, we illustrate why we demanded the secant cone to be reduced. Let $\widetilde{Z} \subseteq \mathbb{P}^3$ be the subscheme defined by the homogeneous ideal

$$\mathfrak{I}_{\widetilde{Z}} := \langle x_0^4, x_0^3 x_1, x_0^2 x_1^2 + x_0 x_3^3, x_0 x_1 x_2^2 + x_1^4, x_1 x_3^3 \rangle \subseteq R := K[x_0, x_1, x_2, x_3].$$

As $\sqrt{\mathfrak{I}_{\widetilde{Z}}} = \langle x_0, x_1 \rangle$ by [`primdec.lib`], the underlying set of $\widetilde{Z}$ is just a line. While $\widetilde{Z}$ is not reduced, computing the saturation of $\mathfrak{I}_{\widetilde{Z}}$ with the procedure 'sat' from [`elim.lib`], we see that $\mathfrak{I}_{\widetilde{Z}}$ is saturated. Computing the PEIs of $\mathfrak{I}_{\widetilde{Z}}$ with respect to the point $\mathfrak{p} = (1 : 0 : 0 : 0) \in \mathbb{P}^3$, we get

$$\begin{array}{ll}
\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) = \langle x_1 x_3^3, x_1^9 \rangle & \subseteq S := K[x_1, x_2, x_3], \\
\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) = \langle x_1 x_3^3, x_1^5 - x_2^2 x_3^3, x_1 x_2^2, x_3^6 \rangle, & \\
\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) = \langle x_1^2, x_1 x_2^2, x_3^3 \rangle, & \\
\mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) = \langle x_1, x_3^3 \rangle, & \\
\mathfrak{K}_4^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}) = S. &
\end{array}$$

Using [`elim.lib`] again, we compute

$$\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})^{\mathrm{sat}} = \mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})^{\mathrm{sat}} = \mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}}).$$

Hence, the first and second PEI of $\mathfrak{I}_{\widetilde{Z}}$ with respect to $\mathfrak{p}$ are not saturated; their saturation indices are 8 and 4, respectively. Furthermore, we see that $\Sigma_{\mathfrak{p}}^2(\widetilde{Z}) = \Sigma_{\mathfrak{p}}^3(\widetilde{Z}) = \Sigma_{\mathfrak{p}}^4(\widetilde{Z})$ are equal as sets and consist just of the point $\widetilde{\mathfrak{q}} = (0 : 0 : 1 : 0)$. The line $\langle \widetilde{\mathfrak{q}}, \mathfrak{p} \rangle_{\mathbb{P}^n}$ is a 4-secant to $\widetilde{Z}$.
On the other hand, the extension ideals $\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R$, $\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R$ and $\mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R$ are saturated, meaning that the schemes $\mathrm{Proj}(R/\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)$, $\mathrm{Proj}(R/\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)$ and $\mathrm{Proj}(R/\mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)$ are different; they are non-reduced and therefore not equal to the k-secant cones of $\widetilde{Z}$ for $k \in \{2, 3, 4\}$. Another consultation of [Singular] tells us that

$$(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)^{\mathrm{sat}} = (\mathfrak{I}_{\widetilde{Z}} + \mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)^{\mathrm{sat}} = (\mathfrak{I}_{\widetilde{Z}} + \mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R)^{\mathrm{sat}} = \langle x_0^4, x_1, x_3^3 \rangle,$$

while

$$\begin{aligned}
\left(\mathfrak{I}_{\widetilde{Z}} + \sqrt{\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R}\right)^{\mathrm{sat}} &= \left(\mathfrak{I}_{\widetilde{Z}} + \sqrt{\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R}\right)^{\mathrm{sat}} \\
&= \left(\mathfrak{I}_{\widetilde{Z}} + \sqrt{\mathfrak{K}_3^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})R}\right)^{\mathrm{sat}} = \langle x_0^4, x_1, x_3 \rangle_R.
\end{aligned}$$

$\mathfrak{A}_\mathfrak{p}$	$\mathfrak{K}_1^\mathfrak{p}(\mathfrak{I}_{\widetilde{Z}})$	$\mathrm{Sec}^2_\mathfrak{p}(\widetilde{Z})$	$\mathfrak{K}_1^\mathfrak{p}(\mathfrak{I}_{\widetilde{Z}})R$ $+\mathfrak{I}_{\widetilde{Z}}/\mathfrak{K}_1^\mathfrak{p}(\mathfrak{I}_{\widetilde{Z}})R$	$\Sigma^2_\mathfrak{p}(\widetilde{Z})$
$\{7,10\}$	$\left\langle \begin{array}{c} x_0,\ldots,x_6, \\ x_8,x_9 \end{array} \right\rangle$	$\mathbb{P}^1$	$\langle x_7x_{10}\rangle$ $\subseteq K[x_7,x_{10}]$	Two points
$\{9\}$	$\langle x_0,\ldots,x_8\rangle$	$\mathbb{P}^1$	$\langle x_9^2\rangle$ $\subseteq K[x_9,x_{10}]$	Double point
$\{0,10\}$	$\langle x_2,\ldots,x_9\rangle$	$\mathbb{P}^2$	$\langle x_0x_{10}\rangle$ $\subseteq K[x_0,x_1,x_{10}]$	Two lines
$\{5\}$	$\left\langle \begin{array}{c} x_0,\ldots,x_3, \\ x_7,\ldots,x_{10} \end{array} \right\rangle$	$\mathbb{P}^2$	$\langle x_4x_6-x_5^2\rangle$ $\subseteq K[x_4,x_5,x_6]$	Smooth conic
$\{0,3\}$	$\left\langle \begin{array}{c} x_4,x_5,x_6, \\ x_8,x_9,x_{10} \end{array} \right\rangle$	$\mathbb{P}^3$	$\langle x_0x_3-x_1x_2\rangle$ $\subseteq K[x_0,\ldots,x_3]$	Quadric surface
$\{4,9\}$	$\left\langle \begin{array}{c} x_0,\ldots,x_3, \\ x_5,\ldots,x_8, \\ x_{10},x_4-x_9 \end{array} \right\rangle$	p	$\langle x_9^2\rangle \subseteq K[x_9]$	$\varnothing$

Table 3.1: Examples of all possible secant loci of $\widetilde{Z} = S(1,1,2,3)$

Hence, we indeed have to demand that $\mathrm{Sec}^k_\mathfrak{p}(\widetilde{Z})$ is reduced; if we omitted this condition, we would get $\mathrm{length}(\widetilde{Z} \cap \mathrm{Sec}^4_\mathfrak{p}(\widetilde{Z})) = 12$, where $\mathrm{Sec}^4_\mathfrak{p}(\widetilde{Z}) = \langle \mathfrak{p}, \widetilde{\mathfrak{q}}\rangle$ is just a line. But $\langle \mathfrak{p}, \widetilde{\mathfrak{q}}\rangle$ certainly is no 12-secant line to $\widetilde{Z}$.

Example 3.5.3. Let $\widetilde{Z} \subseteq \mathbb{P}^n$ be a smooth rational normal scroll and of codimension at least 2. Let $\mathfrak{p} \dot{\in} \mathbb{P}^n\backslash\widetilde{Z}$ be a closed point. Then, according to [BrP, Theorem 3.2], either

(a) $\mathrm{Sec}^2_\mathfrak{p}(\widetilde{Z}) = \mathbb{P}^1$ and $\Sigma^2_\mathfrak{p}(\widetilde{Z}) \subseteq \mathbb{P}^1$ is either a double point or the union of two simple points.

(b) $\mathrm{Sec}^2_\mathfrak{p}(\widetilde{Z}) = \mathbb{P}^2$ and $\Sigma^2_\mathfrak{p}(\widetilde{Z}) \subseteq \mathbb{P}^2$ is either a smooth conic or the union of two lines $\mathbb{L}, \mathbb{L}' \subseteq \widetilde{Z}$.

(c) $\mathrm{Sec}^2_\mathfrak{p}(\widetilde{Z}) = \mathbb{P}^3$ and $\Sigma^2_\mathfrak{p}(\widetilde{Z}) \subseteq \mathbb{P}^3$ is a smooth quadric surface.

(d) $\mathrm{Sec}^2_\mathfrak{p}(\widetilde{Z}) = \varnothing$, i.e., $\mathfrak{p} \notin \mathrm{Sec}(\widetilde{Z})$.

Now, let us consider the scroll $\widetilde{Z} = S(1,1,2,3) \subseteq \mathbb{P}^{10}$. Using Algorithm 3.3.4, it is easy to find 6 points of $\mathbb{P}^{10}\backslash S(1,1,2,3)$ such that every one of the possible six secant loci occurs (see Table 3.1; there $\mathfrak{A}_\mathfrak{p}$ of a closed point $\mathfrak{p} = (p_0 : \cdots : p_{10}) \dot{\in} \mathbb{P}^{10}\backslash\widetilde{Z}$ is the set of indices $i \in \{0,\ldots,10\}$ such that $p_i = 1$ for $i \in \mathfrak{A}_p$ and $p_i = 0$ otherwise).

Example 3.5.4 (Example 7.4(E) in [BrS2])**.** Let $n = 10$. Consider the rational normal scroll $\widetilde{W} := S(1,8) \subseteq \mathbb{P}^{10}_K$ with homogeneous ideal $\mathfrak{I}_{\widetilde{W}} \subseteq R$, and let $\mathbb{L} \subseteq \mathbb{P}^{10}_K$ be the line given by $\mathfrak{L} = \langle x_0, x_1, x_2, x_5, \ldots, x_{10}\rangle \subseteq R$. Let $\pi_\mathbb{L} : \mathbb{P}^{10}_K\backslash\mathbb{L} \to \mathbb{P}^8_K$ be the double linear projection given by $S := K[x_0, x_1, x_2, x_5, \ldots, x_{10}] \hookrightarrow R$, and let $W := \pi_\mathbb{L}(\widetilde{W}) \subseteq \mathbb{P}^8_K$. The homogeneous ideal $\mathfrak{I}_W \subseteq S$ of W is given by 18 quadrics and one quartic $Q = x_1^3x_2 - x_0^3x_5$.
Now, let us consider the secant loci of $\widetilde{W}$ with respect to the points of $\mathbb{L}$.

According to [C-J], the secant variety of $\widetilde{W}$ is given by the ideal $\mathfrak{M}$ generated by the (3×3)-minors of the matrix

$$M := \begin{pmatrix} x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{pmatrix},$$

so that $\mathfrak{M} + \mathfrak{L} = \langle x_0, x_1, x_2, x_4^3, x_5, \dots, x_{10} \rangle \subseteq R$. Therefore, $\mathrm{Sec}(\widetilde{W}) \cap \mathbb{L}$ contains just one point $\mathfrak{p} = (0 : 0 : 0 : 1 : 0 : \cdots : 0)$ with homogeneous ideal $\langle x_0, x_1, x_2, x_4, \dots, x_{10} \rangle \in \mathrm{mProj}(R)$. The partial elimination ideals of $\mathfrak{I}_{\widetilde{W}}$ with respect to $\mathfrak{p}$ are

$$\begin{aligned} \mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{W}}) &= \mathfrak{I}_{\widetilde{W}} \cap K[\mathfrak{p}_1], \\ \mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{W}}) &= \langle x_0, x_1, x_4, \dots, x_{10} \rangle, \\ \mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{W}}) &= K[\mathfrak{p}_1], \end{aligned}$$

so that

$$\mathrm{Sec}^2_{\mathfrak{p}}(\widetilde{W}) = V_{\mathbb{P}^{10}_K}(\mathfrak{K}_1^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{W}})R) = \mathbb{P}^1 \subseteq \mathbb{P}^{10}_K$$

(for the notation $V_{\mathbb{P}^m}(\cdot)$, see 0.4 §16) and

$$\Sigma^2_{\mathfrak{p}}(\widetilde{W}) = \widetilde{W} \cap \mathrm{Sec}^2_{\mathfrak{p}}(\widetilde{W}) = \{\tilde{\mathfrak{w}} := (0 : 0 : 1 : 0 : \cdots : 0)\}.$$

Moreover, $\mathrm{length}(\widetilde{W} \cap \mathrm{Sec}^2_{\mathfrak{p}}(\widetilde{W})) = 2$. So, $\mathrm{Sec}^2_{\mathfrak{p}}(\widetilde{W})$ is a line which intersects $\widetilde{W}$ in one point $\tilde{\mathfrak{w}}$ with multiplicity 2. It holds

$$\langle \tilde{\mathfrak{w}}, \mathbb{L} \rangle_{\mathbb{P}^n} \cap \widetilde{W} = \{\tilde{\mathfrak{w}}\} \text{ and } \mathrm{length}\left(\langle \tilde{\mathfrak{w}}, \mathbb{L} \rangle_{\mathbb{P}^n} \cap \widetilde{W}\right) = 3,$$

meaning that $\tilde{\mathfrak{w}}$ lies with multiplicity 3 over its image $\pi(\tilde{\mathfrak{w}}) \in W$ (for the notation $\langle \cdot, \cdot \rangle_{\mathbb{P}^n}$, see 0.4 §17).
Let us consider the PEIs corresponding to the decomposition $\pi_{\mathbb{L}} = \hat{\pi} \circ \pi'$, where

$$\pi' : \mathbb{P}^{10} \backslash \{\mathfrak{p}' := (0 : 0 : 0 : 0 : 1 : 0 : \cdots : 0)\} \to \mathbb{P}^9_K$$

is given by $\hat{S}' := K[x_0, x_1, x_2, x_3, x_5, \dots, x_{10}] \hookrightarrow R$, and

$$\hat{\pi} : \mathbb{P}^9_K \backslash \{\hat{\mathfrak{p}} := \pi'(\mathfrak{p}) = (0 : 0 : 0 : 1 : 0 : \cdots : 0)\} \to \mathbb{P}^8_K$$

is given by $S \hookrightarrow \hat{S}'$. Then, the PEIs of $\mathfrak{I}_{\widetilde{W}}$ with respect to $\mathfrak{p}'$ are

$$\mathfrak{K}_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}}) = \mathfrak{I}_{\widetilde{W}} \cap \hat{S}', \mathfrak{K}_1^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}}) = (\hat{S}')_+, \mathfrak{K}_2^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}}) = \hat{S}',$$

where $\mathfrak{I}_{\widetilde{W}} \cap \hat{S}'$ is generated by 18 quadrics in S and 9 quadrics and 1 cubic in $\hat{S}' \backslash S$. The equality $\mathfrak{K}_1^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}}) = (\hat{S}')_+$ means that π' is an isomorphism in accordance with $\mathfrak{p}' \notin \mathrm{Sec}(\widetilde{W})$. For the PEIs of $\mathfrak{K}_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}})$ with respect to $\hat{\mathfrak{p}}$ we get

$$\begin{aligned} \mathfrak{K}_0^{\hat{\mathfrak{p}}}(\mathfrak{K}_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}})) &= \mathfrak{I}_W, \\ \mathfrak{K}_1^{\hat{\mathfrak{p}}}(\mathfrak{K}_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}})) &= \langle x_0, x_1^3, x_5, \dots, x_{10} \rangle, \\ \mathfrak{K}_2^{\hat{\mathfrak{p}}}(K_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}})) &= \langle x_0, x_1, x_5, \dots, x_{10} \rangle, \\ \mathfrak{K}_3^{\hat{\mathfrak{p}}}(\mathfrak{K}_0^{\mathfrak{p}'}(\mathfrak{I}_{\widetilde{W}})) &= S. \end{aligned}$$

Looking at these PEIs, Corollary 3.4.2 tells us that $\pi_{\mathbb{L}}(\tilde{\mathfrak{w}}) = (0 : 0 : 1 : 0 : \cdots : 0)$ is indeed the only point $\mathfrak{q}$ of W such that the length of the fibre $(\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathfrak{q})$ is 3;

for every other point $\mathfrak{q} \in W$ the length of the fibre is length$(\widetilde{W} \cap \langle \mathfrak{q}, \mathbb{L} \rangle_{\mathbb{P}^{10}}) = 1$. Finally, for example (on use of the command 'reduce' for [Singular])

$$x_2x_5 - x_3x_4 \in \mathfrak{I}_{\widetilde{W}} \backslash ((\mathfrak{I}_{\widetilde{W}} \cap K[\mathfrak{p}_1])R + (\mathfrak{I}_{\widetilde{W}} \cap K[\mathfrak{p}'_1])R)^{\mathrm{sat}},$$

so $\mathfrak{p}, \mathfrak{p}'$ is not a clever decomposition of $\mathbb{L}$.
The ideal $\mathfrak{E} := ((\mathfrak{I}_W)_2 S :_S Q) = \langle x_5, \ldots, x_{10} \rangle \subseteq S$ defines a projective plane $\mathbb{E} = \mathbb{P}^2 \subseteq \mathbb{P}^8$. The intersection $W \cap \mathbb{E}$ is the quartic defined in $\mathbb{E}$ by $\overline{Q} = x_1^3 x_2$. Let $\mathbb{L}_1$ be the projective line in $\mathbb{E}$ defined by x_1, in $\mathbb{P}^8$ by $\mathfrak{L}_1 = \langle x_1, x_5, \ldots x_{10} \rangle_S$. In $\mathbb{P}^{10}$, $\mathfrak{L}_1 R$ defines the projective three-space $\langle \mathbb{L}, \mathbb{L}_1 \rangle$, and $\mathfrak{I}_{\widetilde{W}} + \mathfrak{L}_1 R = \langle x_0x_4, x_2x_4 - x_3^2, x_3x_4, x_4^2, x_1, x_5, \ldots, x_{10} \rangle \subseteq R$, hence as a set $(\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathbb{L}_1) = \langle \mathbb{L}, \mathbb{L}_1 \rangle_{\mathbb{P}^{10}} = V_{\mathbb{P}^{10}}(x_1, x_3, x_4, \ldots, x_{10})$ is the ruling line $\widetilde{\mathbb{L}}_1$ on $\widetilde{W}$ which contains $\widetilde{\mathfrak{w}}$. Moreover, $e_0(R/(\mathfrak{I}_{\widetilde{W}} + \mathfrak{L}_1 R)) = 3$, that is '$\widetilde{\mathbb{L}}_1$ lies with length 3 over $\mathbb{L}_1$'.

Example 3.5.5. Let $\widetilde{W} = S(1,8) \subseteq \mathbb{P}^{10}$ be as in Example 3.5.4, but now consider the line $\mathbb{L} = \mathbb{P}^1 \subseteq \mathbb{P}^{10}$ given by the ideal $\mathfrak{L} = \langle x_0, x_1, x_2, x_4, \ldots, x_8, x_{10} \rangle$. Let $\mathfrak{p}, \mathfrak{p}' \dot{\in} \mathbb{L}$ be the closed points given by the ideals $\mathfrak{p} := \langle x_0, x_1, x_2, x_4, \ldots, x_{10} \rangle$, $\mathfrak{p}' := \langle x_0, \ldots, x_8, , x_{10} \rangle \in \mathrm{mProj}(R) \cap \mathrm{Var}(\mathfrak{L})$, and let $\hat{S} := K[\mathfrak{p}_1], \hat{S}' := K[\mathfrak{p}'_1]$. We can use [BrPS] to easily compute the dimensions of the secant loci $\Sigma^2_{\mathfrak{p}}(\widetilde{W})$ and $\Sigma^2_{\mathfrak{p}'}(\widetilde{W})$: Consider the (3×3)-matrix M as in Example 3.5.4, and denote by $M(\mathfrak{p})$ the matrix given by the coordinates $\mathfrak{p} = (0:0:0:1:0:\cdots:0)$. Then

$$\dim(\Sigma^2_{\mathfrak{p}}(\widetilde{W})) = 2 - \mathrm{rank}(M(\mathfrak{p})) = 0.$$

Similarly, we compute $\dim(\Sigma^2_{\mathfrak{p}'}(\widetilde{W})) = 0$. Thus, both secant loci $\Sigma^2_{\mathfrak{p}}(\widetilde{W})$ and $\Sigma^2_{\mathfrak{p}'}(\widetilde{W})$ consist of finitely many closed points of $\widetilde{W}$.
Next, a short computation using the library [`elim.lib`] for [Singular] shows

$$\Big((\mathfrak{I}_{\widetilde{W}} \cap \hat{S})R + (\mathfrak{I}_{\widetilde{W}} \cap \hat{S}')R\Big)^{\mathrm{sat}} = \mathfrak{I}_{\widetilde{W}}^{\mathrm{sat}} = \mathfrak{I}_{\widetilde{W}},$$

so $\mathfrak{p}, \mathfrak{p}'$ is a clever decomposition of $\mathbb{L}$ with respect to $\widetilde{W}$. Another consultation of Singular gives

$$\begin{array}{ll} \mathfrak{K}_0^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) = \mathfrak{I}_{\widetilde{W}} \cap S = \mathfrak{K}_0^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}'), & \\ \mathfrak{K}_1^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) = (x_0, x_1, x_2, x_4, \ldots, x_8), & \mathfrak{K}_2^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) = S, \\ \mathfrak{K}_1^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') = (x_0, x_1, x_4, \ldots, x_8, x_{10}), & \mathfrak{K}_2^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') = S. \end{array}$$

Thus, we can compute Table 3.2, giving us the ideals $\mathfrak{K}_l^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) + \mathfrak{K}_k^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') \subseteq S$.
So, we see that for $\mathfrak{q} = (0:0:1:0:\cdots:0), \mathfrak{q}' = (0:\cdots:0:1) \in W := \pi_{\mathbb{L}}(\widetilde{W})$, it holds

$$\mathfrak{K}_0^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) + \mathfrak{K}_1^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') = \mathfrak{q}$$

and

$$\mathfrak{K}_1^{\mathfrak{L} \cap \hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) + \mathfrak{K}_0^{\mathfrak{L} \cap \hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') = \mathfrak{q}',$$

i.e., length$((\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathfrak{q}))$ = length$((\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathfrak{q}')) = 2$ by Proposition 3.4.6. Indeed, we can compute that

$$\Sigma^2_{\mathfrak{p}}(\widetilde{W}) = \{(\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathfrak{q})\} = 2\widetilde{\mathfrak{w}} \quad \text{and} \quad \Sigma^2_{p'}(\widetilde{W}) = \{(\pi_{\mathbb{L}} \restriction_{\widetilde{W}})^{-1}(\mathfrak{q}')\} = 2\widetilde{\mathfrak{w}}'$$

l \ k	0	1	2
0	$\mathfrak{I}_{\widetilde{W}} \cap S$	$\langle x_0, x_1, x_4, \ldots, x_8, x_{10}\rangle$	S
1	$\langle x_0, x_1, x_2, x_4, \ldots, x_8\rangle$	S	S
2	S	S	S

Table 3.2: $\mathfrak{K}_l^{\mathfrak{L}\cap\hat{S}}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}) + \mathfrak{K}_k^{\mathfrak{L}\cap\hat{S}'}(\mathfrak{I}_{\widetilde{W}} \cap \hat{S}') \subseteq S$

both consist of a double point, where $\widetilde{\mathfrak{w}} := (0 : 0 : 1 : 0 : \cdots : 0$ and $\widetilde{\mathfrak{w}}' := (0 : \cdots : 0 : 1)$.

Chapter 4

Equations for Z_λ

In this last Chapter, we finally explain how to obtain equations for the ramification loci Z_λ of a simple linear projection $\pi : \widetilde{Z} \to Z$. For this purpose, we first give an open covering of the proper heterodyne loci Z_k° which already determines the behaviour of the polynomials $\mathfrak{f}_\mathfrak{q}$ for $\mathfrak{q} \in Z_k^\circ$ with respect to their linear factors (see Section 0.2); moreover, we show how to obtain a finite open covering of this kind using Gröbner bases. We study in detail the example of a surface of degree 16 in $\mathbb{P}^4_\mathbb{C}$, illustrating the structure and several noteworthy facts on ramification loci. Finally, we use the results of Chapter 2 to explicitly determine the ideals of the ramification loci of π in a way suited for computations.

Through this Chapter, we use the notations introduced in 3.1.1, 3.1.3, and 3.2.1 as well as 0.4, and we always consider $R = K[x_0, \dots, x_n]$ to be furnished with the lexicographical term ordering. Also, we finally give a formal definition of the ramification locus of a simple linear projection:

Definition 4.0.1. Consider the outer simple linear projection $\pi : \widetilde{Z} \to Z$ with centre $\mathfrak{p}$ as before. Let $k \in \mathbb{N}$, and let $\lambda = (\lambda_1, \dots, \lambda_e)$ be a partition of k. The *λ-ramification locus of* π is the set

$$Z_\lambda := \left\{ \mathfrak{q} \mathrel{\dot\in} Z_k \mid \exists \widetilde{\mathfrak{q}}_1, \dots, \widetilde{\mathfrak{q}}_e \mathrel{\dot\in} \widetilde{Z} : \pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i \right\} \cup Z_{k+1}.$$

Note that for each closed point $\mathfrak{q} \mathrel{\dot\in} Z$, the fibre $\pi^{-1}(\mathfrak{q})$ is a divisor on $\mathbb{P}^1 = \operatorname{Proj}(R/\mathfrak{q}R)$, thus the notation $\sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i$ (compare Remark and Notation 3.2.1). Also note that the closed points $\widetilde{\mathfrak{q}}_1, \dots, \widetilde{\mathfrak{q}}_e$ in the sum need not be pairwise different; if some of them happen to be equal for $\mathfrak{q} \in Z_\lambda$, then also $\mathfrak{q} \in Z_\mu$ for a strict coarsening μ of λ (for partitions and refinements, see Remark and Definition 1.1.2). We will show that Z_λ is closed; if $\widetilde{Z}$ is reduced, we consider Z_λ to be a subvariety of the projective variety Z.

The *proper λ-ramification locus of* π is the set

$$Z_\lambda^\circ := \left\{ \mathfrak{q} \in Z_k \mid \exists \widetilde{\mathfrak{q}}_1, \dots, \widetilde{\mathfrak{q}}_e \mathrel{\dot\in} \widetilde{Z} : \pi^{-1}(\mathfrak{q}) = \sum_{i=1}^{e} \lambda_i \widetilde{\mathfrak{q}}_i \text{ and } \widetilde{\mathfrak{q}}_i \neq \widetilde{\mathfrak{q}}_j \text{ for } i \neq j \right\}.$$

Note that

$$Z^\circ_\lambda \;=\; Z_\lambda \backslash \left(Z_{k+1} \cup \bigcup_{\mu \in Q^\circ_\lambda} Z_\mu \right),$$

hence we will see that Z°_λ is relatively open in Z.

4.1 Covering Z°_k

For two non-zero polynomials f, g over K in the same indeterminates, we again denote by $\operatorname{ord}_f(g)$ the maximal integer $t \in \mathbb{N}_0$ with $g^t \mid f$.

Lemma 4.1.1. *Let $f \in K[x,y]\backslash\{0\}$ be a homogeneous polynomial in two indeterminates, and let $\mathfrak{p} \in \operatorname{mProj}(K[x,y])$ be generated by a linear form $l \in K[x,y]_1$. Then*

$$\operatorname{length}_{K[x,y]_\mathfrak{p}}\left((K[x,y]/fK[x,y])_\mathfrak{p}\right) = \operatorname{ord}_f(l).$$

Proof. Denote $t := \operatorname{ord}_f(l)$. If $t = 0$, then $l \nmid f$ and $\mathfrak{p} \notin \operatorname{Var}(fK[x,y])$, hence $(K[x,y]/fK[x,y])_\mathfrak{p} = 0$. Assume $t > 0$. Then, $K[x,y]_\mathfrak{p}$ is a regular local ring of dimension 1, that is it is a discrete valuation ring (see [Ma, Theorem 11.2]), and every ideal of $K[x,y]_\mathfrak{p}$ is generated by a power of $\frac{l}{1}$. Hence, $(fK[x,y])_\mathfrak{p} = \frac{f}{1}K[x,y]_\mathfrak{p}$ is generated by $\frac{l^t}{1}$. and

$$\frac{f}{1}K[x,y]_\mathfrak{p} \subseteq \frac{l^{t-1}}{1}K[x,y]_\mathfrak{p} \subseteq \cdots \subseteq \frac{l}{1}K[x,y]_\mathfrak{p} \subseteq K[x,y]_\mathfrak{p}$$

is the only maximal ascending chain of submodules of $K[x,y]_\mathfrak{p}$ starting with $\frac{f}{1}K[x,y]_\mathfrak{p}$. The length of this chain is t, and since $(K[x,y]/fK[x,y])_\mathfrak{p} \cong K[x,y]_\mathfrak{p}/\frac{f}{1}K[x,y]_\mathfrak{p}$, this yields the claim. □

Remark and Notation 4.1.2. Let $x \in R_1\backslash\mathfrak{p}_1$. For $\mathfrak{q} \,\dot{\in}\, \mathbb{P}^{n-1}$, let $y \in S_1\backslash\mathfrak{q}_1$. We again use the identifications $R/\mathfrak{q}R = K[x,y]$ and $S = K[\mathfrak{q}_1][y]$ (compare Remark and Notation 3.1.8). Let $f \in R$ be a homogeneous element of degree $t \in \mathbb{N}_0$, and denote $d := \deg_x(f)$, so that we can write $f = \sum_{i=0}^d f_i x^{d-i}$ with $f_i \in S_{t-d+i}$ for $i \in \{0,\dots,d\}$. Assume $f \notin \mathfrak{q}R$, whence $0 \neq \overline{f} = f + \mathfrak{q}R \in K[x,y]_t$. For all $i \in \{0,\dots,d\}$, we have $y^{t-d} \mid \overline{f}_i := f_i + \mathfrak{q}R$, and

$$\mathfrak{f}_\mathfrak{q} := \frac{1}{y^{t-d}}\,(f + \mathfrak{q}R) \mod K^* \in \mathbb{P}\left(\operatorname{Sym}^d(K[x,y]_1)\right)$$

is a binary form of degree d in x and y over K (see Definition 2.0.1).
There is a unique partition $\lambda_\mathfrak{q}(f)$ of d such that

$$\mathfrak{f}_\mathfrak{q} \;\dot{\in}\; X^\circ_{\lambda_\mathfrak{q}(f)} \;=\; X_{\lambda_\mathfrak{q}(f)} \backslash \bigcup_{\mu \in Q^\circ_{\lambda_\mathfrak{q}(f)}} X_\mu$$

(see Remark and Notation 2.6.4). We call $\lambda_\mathfrak{q}(f)$ the *root multiplicity of f over $\mathfrak{q}$*.
Also, note that $\lambda_\mathfrak{q}(f)$ is independent of our choice of coordinates, and in particular of the indeterminates x and y. Indeed, let $\gamma : R \overset{\cong}{\to} R$ be a graded

isomorphism, and let $\mathfrak{f} \simeq \sum_{i=0}^{d} f_i x^{d-i} y^i \in \mathbb{P}(\mathrm{Sym}^d(K[x,y]_1))$ be a binary form of degree d with linear factors $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$. Using the natural inclusion $K[x,y] \hookrightarrow R$, we set $\gamma(\mathfrak{f}) \simeq \sum_{i=0}^{d} \gamma(f_i)\gamma(x)^{d-i}\gamma(y)^i$, etc., and immediately get $\gamma(\mathfrak{f}) = \gamma(\mathfrak{l}_1)\cdots\gamma(\mathfrak{l}_d)$. So, the distributions of the linear factors of $\mathfrak{f}$ and $\gamma(\mathfrak{f})$ are equal.

Theorem 4.1.3. *Let $k \in \mathbb{N}$, let λ be a partition of k, and let $\mathfrak{q} \in Z_k^\circ$. There is a homogeneous element $f \in \mathfrak{I}_{\tilde{Z}}$ such that*

$$f \notin \mathfrak{q}R \text{ and } \deg_x(f) = k.$$

Moreover, for any such f, it holds

$$\mathfrak{q} \in Z_\lambda^\circ \Leftrightarrow \lambda_{\mathfrak{q}}(f) = \lambda.$$

Proof. According to Proposition 3.2.3, it holds $\mathfrak{K}_{k-1}^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}) \subseteq \mathfrak{q}$ and $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}) \nsubseteq \mathfrak{q}$; hence there is a homogeneous element $f \in \mathfrak{I}_{\tilde{Z}}$ with $\deg_x(f) = k$ and $\mathrm{LC}_x(f) \notin \mathfrak{q}$. Therefore $f \notin \mathfrak{q}R$.
Now let us consider the projective line $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$ with coordinate ring $\overline{R} := R/\mathfrak{q}R = K[x,y]$ with $x \in R_1 \backslash \mathfrak{p}_1$ and $y \in S_1 \backslash \mathfrak{q}_1$ (for notations, see 0.4 §§17 and 9). According to Lemma 1.3.4 and Theorem 3.1.9, the intersection $\langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n} \cap \tilde{Z} = \pi^{-1}(\mathfrak{q}) \subseteq \langle \mathfrak{p}, \mathfrak{q} \rangle_{\mathbb{P}^n}$ has ideal $\mathfrak{I}_{\tilde{Z}} + \mathfrak{q}R/\mathfrak{q}R = \overline{hR}$ for some homogeneous element $\overline{h} \in \overline{R}_k$. According to Lemma 1.4.6 and Lemma 4.1.1, the number of closed points in $\pi^{-1}(\mathfrak{q})$ and their multiplicities equal the number of linear factors of $\overline{h}$ and their multiplicities, respectively. □

In the next Corollary and henceforth, we use the notations introduced in 3.3.1.

Corollary 4.1.4. *Assume that $\mathfrak{p} = \langle x_1, \ldots, x_n \rangle_R$ is generated by the indeterminates $x_1, \ldots, x_n$. Let G be a Gröbner Basis of $\mathfrak{I}_{\tilde{Z}}$ with respect to the lexicographic ordering. Then, for any $k \in \mathbb{N}$, the elements h of $\mathrm{In}_k^\circ G$ define an open covering $(U_h)_{h \in \mathrm{In}_k^\circ(G)}$ of Z_k° by*

$$U_h = \{\mathfrak{q} \,\dot{\in}\, Z_k^\circ \mid h \notin \mathfrak{q}\}$$

for $h \in \mathrm{In}_k^\circ(G)$. Moreover, it holds

$$\forall h \in \mathrm{In}_k^\circ(G)\, \forall \mathfrak{q} \in U_h : \mathfrak{q} \in Z_{\lambda_{\mathfrak{q}}(h)}.$$

Proof. Let $k \in \mathbb{N}$, and let $\mathfrak{q} \in Z_k^\circ$. As $Z_k^\circ = V_{\mathbb{P}^{n-1}}(\mathfrak{K}_{k-1}^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})) \backslash V_{\mathbb{P}^{n-1}}(\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}))$ by Proposition 3.2.3 (for the notation $V_{\mathbb{P}^m}(\cdot)$, see 0.4 §16) and $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})$ is generated by $\mathrm{In}_k(G)$ by Proposition 3.3.2, there exists $h \in \mathrm{In}_k^\circ(G)$ with $h \notin \mathfrak{q}$. Therefore, $(U_h)_{h \in \mathrm{In}_k^\circ(G)}$ indeed is an open covering of Z_k°. The second claim now follows by Theorem 4.1.3. Also observe that according to this Theorem $\lambda_{\mathfrak{q}}(h) = \lambda_{\mathfrak{q}}(h')$ for any $h, h' \in \mathrm{In}_k^\circ(G)$ and any $\mathfrak{q} \,\dot{\in}\, U_h \cap U_{h'}$. □

Example 4.1.5. Let us consider a simple example: Assume that $n = 2$, $\mathfrak{p} = \langle x_1, x_2 \rangle \subseteq R = K[x_0, x_1, x_2]$, and $\tilde{Z} \subseteq \mathbb{P}^2$ is the curve defined by some quadric $f = x_0^2 + bx_0 + c \in R$ with $b \in S_1, c \in S_2$, where $S = K[x_1, x_2]$. Then $Z = \pi(\tilde{Z}) = \mathbb{P}^1$, and as $\mathfrak{K}_1^{\mathfrak{p}}(fR) = \mathfrak{K}_0^{\mathfrak{p}}(fR) = 0$ and $\mathfrak{K}_2^{\mathfrak{p}}(fR) = S$, we see

$$Z_2 = Z = \mathbb{P}^1 \text{ and } Z_3 = \varnothing.$$

Now, if and only if $\mathfrak{q} = (q_1 : q_2) \in Z_{(2)}$ with $y \in S_1 \backslash \mathfrak{q}$, then the binary form

$$\mathfrak{f}_{\mathfrak{q}} \simeq x_0^2 + b(q_1, q_2)x_0 y + c(q_1, q_2)y^2$$

has exactly one solution. Thus,

$$\mathfrak{q} \in S_{(2)} \Leftrightarrow b(q_1, q_2)^2 - 4c(q_1, q_2) = 0 \Leftrightarrow b^2 - 4c \in \mathfrak{q}.$$

Note that the condition $b(q_1, q_2)^2 - 4c(q_1, q_2) = 0$ is independent of the chosen coordinates of $\mathfrak{q}$. This means that $Z_{(2)} = V_{\mathbb{P}^1}(b^2 - 4c)$, so that the closed set $Z_{(2)} \subseteq Z_2^\circ = Z$ is determined by the discriminant of f.

The above example motivates the next Lemma, which states that, for $\mathfrak{p} = \langle x_1, \dots, x_n \rangle_R$, there always is a Gröbner basis of $\mathfrak{I}_{\tilde{Z}}$ of a shape making it straightforward to read of the covering of Corollary 4.1.4 and study equations of ramification loci.

Remark 4.1.6. Keep $\mathfrak{p} = \langle x_1, \dots, x_n \rangle_R$, and let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \nsubseteq \mathfrak{p}$. Then, according to Remark and Notation 3.1.8, for $k \ll 0$ it holds $1 \in \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a})$; therefore we may define $l(\mathfrak{a}) := \min\{k \in \mathbb{N}_0 \mid \mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{a}) = S\}$.

Lemma 4.1.7. *Let $\mathfrak{a} \subseteq R$ be a graded ideal with $\mathfrak{a} \notin \mathfrak{p}$. Then, there is a Gröbner basis $G = \{f_0, \dots, f_s\}$ of $\mathfrak{a}$ with respect to the lexicographic ordering such that $f_0 = x_0^{l(\mathfrak{a})} + g$ with $g \in R_{l(\mathfrak{a})}$ and $\deg_{x_0}(g) < l(\mathfrak{a})$, and moreover, it holds $\deg_{x_0}(f_i) < l(\mathfrak{a})$ for any $i \in \{1, \dots, s\}$.*

Proof. Denote $l := l(\mathfrak{a})$. Let $\hat{G} = \{\hat{f}_1, \dots, \hat{f}_s\}$ be a homogeneous Gröbner basis of $\mathfrak{a}$; we may assume that $\deg_{x_0}(f_1) \geqslant \deg_{x_0}(f_2) \geqslant \cdots \geqslant \deg_{x_0}(f_s)$. As $\mathfrak{K}_{l(\mathfrak{a})}^{\mathfrak{p}}(\mathfrak{a}) = S$, there is an element $g \in R_l$ with $\deg_{x_0}(g) < l$ such that $f_0 := x_0^l + g \in \mathfrak{a}$. Joining f_0 to $\hat{G}$ again yields a Gröbner basis $\hat{G}_0$ of $\mathfrak{a}$ with respect to Lex.
We denote $c := \max\{i \in \{1, \dots, s\} \mid \deg_{x_0}(\hat{f}_i) \geqslant l\}$; the elements $f_{c+1} := \hat{f}_{c+1}, \dots, f_s := \hat{f}_s$ then already fulfil the condition $\deg_{x_0}(f_i) < l$. So, let $i \in \{1, \dots, c\}$. There are elements $\hat{t}_i, \hat{s}_i \in R$ with $\hat{t}_i \neq 0, \deg_{x_0}(\hat{s}_i) < l$ such that

$$\hat{f}_i = \hat{t}_i x_0^l + \hat{s}_i.$$

For

$$\hat{f}_i^{(1)} := \hat{t}_i f_0 - \hat{f}_i \in \mathfrak{a},$$

it holds

$$\hat{f}_i = \hat{t}_i f_0 - \hat{f}_j^{(1)} \in \langle f_0, \hat{f}_i^{(1)} \rangle_R.$$

As moreover $\mathrm{LT}_{\mathrm{Lex}}(\hat{f}_i) = \mathrm{LT}_{\mathrm{Lex}}(\hat{t}_i)x_0^l \in x_0^k R = \mathrm{LT}_{\mathrm{Lex}}(f_0)R$, we can replace $\hat{f}_i$ in $\hat{G}_0$ with $\hat{f}_i^{(1)}$; in doing so, $\hat{G}_0$ stays a Gröbner basis of $\mathfrak{a}$ with respect to Lex. Further, as $\deg_{x_0}(g) < l$, it holds $\deg_{x_0}(\hat{f}_i^{(1)}) < \deg_{x_0}(\hat{f}_i)$. Hence we can repeat this construction to get a sequence $\hat{f}_i^{(1)}, \hat{f}_i^{(2)}, \dots, \hat{f}_i^{(m_i)}$ of elements of $S[x_0]$ with strictly decreasing x_0-degree, where m_i is the first integer m with $\deg_{x_0}(\hat{f}_{\mathfrak{J}}^{(m)}) < l$.
With $f_1 := \hat{f}_1^{(m_1)}, \dots, f_c := \hat{f}_c^{(m_c)}$, we get a Gröbner basis $G = \{f_0, \dots, f_s\}$ of the desired form. $\square$

Remark 4.1.8. Given a Gröbner basis G of $\mathfrak{I}_{\widetilde{Z}}$ as in Lemma 4.1.7 and assuming $\mathfrak{p} = \langle x_1, \ldots, x_n \rangle$, the coverings $(U_h)_{h \in \mathrm{In}_k^\circ(G)}$ for $k \in \mathbb{N}$ can easily be determined. Observe that the above proof contains an algorithm to compute the Gröbner basis G from $\hat{G}$, which latter can be computed from any set of homogeneous generators of the ideal $\mathfrak{I}_{\widetilde{Z}}$ (see Algorithm A.3.1).
For a general $\mathfrak{p} \in \mathrm{mProj}(R)$ with $\mathfrak{I}_{\widetilde{Z}} \nsubseteq \mathfrak{p}$, we first can determine a coordinate transformation $\gamma : R \xrightarrow{\cong} R$ with $\gamma(\mathfrak{p}) = \langle x_1, \ldots, x_n \rangle$ (see Algorithm 3.3.4), compute a Gröbner basis G of $\gamma(\mathfrak{I}_{\widetilde{Z}})$ as in Lemma 4.1.7, and read of $\mathrm{In}_k^\circ(G)$ for $k \in \mathbb{N}_0$. Transforming the coordinates back, Lemma 3.1.6 shows that $\gamma^{-1}(\mathrm{In}_k(G))$ generates $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\widetilde{Z}})$, hence that $(U_h)_{h \in \gamma^{-1}(\mathrm{In}_k^\circ(G))}$ yields a covering as in Corollary 4.1.4.

4.2 Example: Projecting a surface of degree 16 in $\mathbb{P}^4$

To illustrate the results of the last Section, we now consider in detail an example computed with [Singular]: Let $K = \mathbb{C}$ and $n = 4$. Let $\widetilde{Z}$ be the surface in $\mathbb{P}^4_{\mathbb{C}}$ defined by

$$\mathfrak{I}_{\widetilde{Z}} := \langle x_0^4 - x_0^3 x_1 - x_0 x_3^3 - x_4^4, x_0^2 x_2^2 + x_1 x_3^3 \rangle \subseteq R := \mathbb{C}[x_0, x_1, x_2, x_3, x_4];$$

using [`primdec.lib`], we first compute a primary decomposition of $\mathfrak{I}_{\widetilde{Z}}$ and find that $\mathfrak{I}_{\widetilde{Z}}$ is prime. Also, we compute $\mathrm{height}(\mathfrak{I}_{\widetilde{Z}}) = 2$. Next, using [`sing.lib`], we compute the singular locus of $\widetilde{Z}$, which is determined by the perfect ideal

$$\mathfrak{I}(\mathrm{Sing}(\widetilde{Z})) = \langle x_0^4 - x_0^3 x_1 - x_4^4, x_0 x_2, x_2 x_4, x_3 \rangle.$$

A primary decomposition of $\mathfrak{I}(\mathrm{Sing}(\widetilde{Z}))$ is given by

$$\langle x_0^4 - x_0^3 x_1 - x_4^4, x_0 x_2, x_2 x_4, x_3 \rangle = \langle x_0, x_3, x_4 \rangle \cap \langle x_0^4 - x_0^3 x_1 - x_4^4, x_2, x_3 \rangle,$$

which means that the singular locus $\mathrm{Sing}(\widetilde{Z})$ of $\widetilde{Z}$ consists of the line $\widetilde{L} = \mathrm{Proj}(R/\langle x_0, x_3, x_4 \rangle)$ and the plane quartic $\widetilde{Q} = \mathrm{Proj}(R/\langle x_0^4 - x_0^3 x_1 - x_4^4, x_2, x_3 \rangle)$. Consulting [`sing.lib`] once more, we also find that $\widetilde{Q} \subseteq \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_4]) = \mathbb{P}^2 \subseteq \mathbb{P}^4$ has exactly one singular closed point $(0 : 1 : 0 : 0 : 0)$.
Let $\widetilde{\pi} : \mathbb{P}^4 \backslash \{p\} \to \mathbb{P}^3$ denote the simple linear projection with centre $\mathfrak{p} := \langle x_1, x_2, x_3, x_4 \rangle \in \mathrm{mProj}(R)$; written in coordinates, this means $\mathfrak{p} = (1 : 0 : 0 : 0;)) \dot{\in} \mathbb{P}^4$, and the projection $\widetilde{\pi}$ corresponds to the inclusion of rings $S := \mathbb{C}[x_1, x_2, x_3, x_4] \hookrightarrow R$. Denote

$$\pi := \widetilde{\pi}\restriction_{\widetilde{Z}} : \widetilde{Z} \to Z := \widetilde{\pi}(\widetilde{Z}).$$

We want to study the ramification loci of π in Z, which are part of the singular locus of Z, and determine their ideals in S.
Using again [Singular], we compute a Gröbner basis G of $\mathfrak{I}_{\widetilde{Z}}$ with respect to

the lexicographic order:

$$
\begin{array}{lll}
g_1 & := & x_0^4 - x_0^3x_1 - x_0x_3^3 - x_4^4, \\
g_2 & := & x_0^2x_1x_3^3 - x_0x_1^2x_3^3 + x_0x_2^2x_3^3 + x_2^2x_4^4, \\
g_3 & := & x_0^2x_2^2 + x_1x_3^3, \\
g_4 & := & x_0x_1^2x_2^2x_3^3 - x_0x_2^4x_3^3 + x_1^2x_3^6 - x_2^4x_4^4, \\
g_5 & := & x_0x_1^2x_3^6 - x_0x_2^4x_4^4 - x_1^3x_3^6 + x_1x_2^2x_3^6, \\
g_6 & := & x_0x_2^6x_4^4 - x_0x_2^4x_3^6 + x_1^3x_2^2x_3^6 + x_1^2x_3^9 - x_1x_2^4x_3^6 - x_2^4x_3^3x_4^4, \\
g_7 & := & x_1^5x_2^2x_3^9 + x_1^4x_3^{12} - 2x_1^3x_2^4x_3^9 - 2x_1^2x_2^4x_3^6x_4^4 + x_1x_2^6x_3^9 + x_2^8x_4^8.
\end{array}
$$

This gives us the ideal

$$\mathfrak{I}_Z = \mathfrak{I}_{\widetilde{Z}} \cap S = \langle g_7 \rangle$$

of Z as well as the partial elimination ideals

$$
\begin{array}{lll}
\mathfrak{K}_{\mathfrak{p}}^0(\mathfrak{I}_{\widetilde{Z}}) & = & \mathfrak{I}_{\widetilde{Z}}, \\
\mathfrak{K}_{\mathfrak{p}}^1(\mathfrak{I}_{\widetilde{Z}}) & = & \langle g_7, x_2^6x_4^4 - x_2^4x_3^6, x_1^2x_3^6 - x_2^4x_4^4, x_1^2x_2^2x_3^3 - x_2^4x_3^3 \rangle \\
 & = & \langle x_1^2x_2^2x_3^3 - x_2^4x_3^3, x_1^2x_3^6 - x_2^4x_4^4 \rangle, \\
\mathfrak{K}_{\mathfrak{p}}^2(\mathfrak{I}_{\widetilde{Z}}) & = & \mathfrak{K}_{\mathfrak{p}}^1(\mathfrak{I}_{\widetilde{Z}}) + \langle x_2^2, x_1x_3^3 \rangle \\
 & = & \langle x_2^2, x_1x_3^3 \rangle, \\
\mathfrak{K}_{\mathfrak{p}}^3(\mathfrak{I}_{\widetilde{Z}}) & = & \mathfrak{K}_{\mathfrak{p}}^2(\mathfrak{I}_{\widetilde{Z}}), \\
\mathfrak{K}_{\mathfrak{p}}^4(\mathfrak{I}_{\widetilde{Z}}) & = & S.
\end{array}
$$

Note that not all partial elimination ideals of $\mathfrak{I}_{\widetilde{Z}}$ are reduced even though $\widetilde{Z}$ is reduced. Next, we use [`primdec.lib`] to compute

$$
\begin{array}{lllll}
\mathfrak{I}(Z_1) & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^0(\mathfrak{I}_{\widetilde{Z}})} & = & \mathfrak{I}_{\widetilde{Z}}, \\
\mathfrak{I}(Z_2) & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^1(\mathfrak{I}_{\widetilde{Z}})} & = & \langle x_1^2x_2x_4 - x_2^3x_4, x_1^2x_3 - x_2^2x_3, \\
 & & & & x_1x_2^2x_4^4 - x_1x_3^6, x_2^3x_4^4 - x_2x_3^6 \rangle, \\
\mathfrak{I}(Z_3) & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^2(\mathfrak{I}_{\widetilde{Z}})} & = & \langle x_2, x_1x_3 \rangle, \\
\mathfrak{I}(Z_4) & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^3(\mathfrak{I}_{\widetilde{Z}})} & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^2(\mathfrak{I}_{\widetilde{Z}})}, \\
\mathfrak{I}(Z_5) & = & \sqrt{\mathfrak{K}_{\mathfrak{p}}^4(\mathfrak{I}_{\widetilde{Z}})} & = & S.
\end{array}
$$

The first things we notice are $Z_5 = \varnothing \neq Z_4$ and $Z_4 = Z_3$; hence, we only have to study Z_2 and Z_4 to get a complete picture of the ramification of π.

First, we compute a primary decomposition of $\mathfrak{I}(Z_2)$; by [`primdec.lib`], we get the seven prime ideals

$$
\begin{array}{lll}
\mathfrak{J}_1 & := & \langle x_3, x_4 \rangle, \\
\mathfrak{J}_2 & := & \langle x_1 - x_2, x_2x_4^2 - x_3^3 \rangle, \\
\mathfrak{J}_3 & := & \langle x_1 - x_2, x_2x_4^2 + x_3^3 \rangle, \\
\mathfrak{J}_4 & := & \langle x_1 + x_2, x_2x_4^2 - x_3^3 \rangle, \\
\mathfrak{J}_5 & := & \langle x_1 + x_2, x_2x_4^2 + x_3^3 \rangle, \\
\mathfrak{J}_6 & := & \langle x_1, x_2 \rangle, \\
\mathfrak{J}_7 & := & \langle x_2, x_3 \rangle.
\end{array}
$$

(Note that [`primdec.lib`] only works over base field $\mathbb{Q}$; but looking at above prime ideals, we see that this is no problem in our situation.) This ideals define three lines

$$\begin{array}{lllll} L_1 & := & \mathrm{Proj}(S/\mathfrak{J}_1) & = & \mathrm{Proj}(\mathbb{C}[x_1,x_2]), \\ L_6 & := & \mathrm{Proj}(S/\mathfrak{J}_6) & = & \mathrm{Proj}(\mathbb{C}[x_3,x_4]), \\ L_7 & := & \mathrm{Proj}(S/\mathfrak{J}_7) & = & \mathrm{Proj}(\mathbb{C}[x_1,x_4]) \quad \subseteq \quad \mathrm{Proj}(\mathbb{C}[x_1,x_2,x_3,x_4]), \end{array}$$

and four cubics, each contained in a plane:

$$\begin{array}{lllll} C_2 & := & \mathrm{Proj}(S/\mathfrak{J}_2) & \subseteq & \mathrm{Proj}(S/\langle x_1 - x_2\rangle), \\ C_3 & := & \mathrm{Proj}(S/\mathfrak{J}_3) & \subseteq & \mathrm{Proj}(S/\langle x_1 - x_2\rangle), \\ C_4 & := & \mathrm{Proj}(S/\mathfrak{J}_4) & \subseteq & \mathrm{Proj}(S/\langle x_1 + x_2\rangle), \\ C_5 & := & \mathrm{Proj}(S/\mathfrak{J}_5) & \subseteq & \mathrm{Proj}(S/\langle x_1 + x_2\rangle). \end{array}$$

Also, observe that each of this cubics is a semi-cubical parabola; more precisely, if, for example, we look at C_2 in the plane $\mathbb{E}_2 := \mathrm{Proj}(R/\langle x_1 - x_2\rangle)$ on the affine chart determined by $x_1 = x_2 = 1$ of $\mathbb{E}_2$, we get the semi-cubical parabola $x_4^2 - x_3^3 = 0$ with cusp in $(1:1:0:0) \in \mathbb{E}_2 \subseteq \mathbb{P}^3$. Then, we can consider $C_3 \subseteq \mathbb{E}_2$ as the semi-cubical parabola in $\mathbb{E}_2$ with cusp in $(1:1:0:0)$ that we get by 'reflecting C_2 at the x_3-axis'. Similarly, the curve C_4 is a semi-cubical parabola in $\mathbb{E}_4 := \mathrm{Proj}(R/\langle x_1 + x_2\rangle)$ with cusp in $(1:-1:0:0)$, and $C_5 \subseteq \mathbb{E}_4$ is its reflection at the x_3-axis.
Now $\mathfrak{J}_6 \cap \mathfrak{J}_7 = \mathfrak{I}(Z_3)$, hence,

$$Z_2^\circ = (L_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)\backslash(L_6 \cup L_7)$$

consists of one line and four cubics, each contained in a plane, minus their intersection with the two lines L_6 and L_7. For any closed point $\mathfrak{q} \mathrel{\dot\in} Z_2^\circ$, the fibre $\pi^{-1}(q)$ either consists of two single points or one double point. We now want to determine those two sets $Z_{(1,1)}^\circ$ and $Z_{(2)}^\circ$.
First, consider the line L_1. Any closed point $\mathfrak{q} \mathrel{\dot\in} L_1$ is of the form $\mathfrak{q} = (q_1 : q_2 : 0 : 0)$ with $(q_1, q_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, and as $(1:0:0:0) \in L_7 \nsubseteq Z_2^\circ$, we may assume $q_2 = 1$. But this means $x_2^2 \notin \mathfrak{q}$ while $x_3 \in \mathfrak{q}$, and the behaviour of the fibre $\pi^{-1}(\mathfrak{q})$ is determined by $\mathfrak{g}_\mathfrak{q} \simeq x_0^2$ (see Theorem 4.1.3). As x_0^2 has one double root, i.e., $x_0^2 \in X_{(2)}$, we get $\mathfrak{q} \in Z_{(2)}^\circ$ and hence $L_1\backslash\{(1:0:0:0)\} \subseteq Z_{(2)}^\circ$. We also can compute the preimage $\pi^{-1}(L_1)$: It is determined by

$$\mathfrak{J}_1 R + \mathfrak{I}_{\widetilde{Z}} = \langle x_3, x_4, x_0^4 - x_0^3 x_1 - x_4^4, x_0^2 x_2^2\rangle.$$

The radical of this ideal in turn has a primary decomposition

$$\sqrt{\mathfrak{J}_1 R + \mathfrak{I}_{\widetilde{Z}}} = \langle x_0, x_3, x_4\rangle \cap \langle x_0 - x_1, x_2, x_3, x_4\rangle,$$

meaning that $\pi^{-1}(L_1) = \widetilde{L} \cup \{(1:1:0:0:0)\}$. As $\pi((1:1:0:0:0)) = (1:0:0:0) \in L_1 \cap L_7$, this means that $\pi^{-1}(L_1 \cap Z_{(2)}^\circ) = \widetilde{L} \subseteq \mathrm{Sing}(\widetilde{Z})$.

A closed point $\mathfrak{q} \mathrel{\dot\in} C_2$ can be written as $\mathfrak{q} = (q_1 : q_1 : \sqrt[3]{q_1 q_4^2} : q_4)$ with $(q_1, q_4) \in \mathbb{C}^2\backslash\{(0,0)\}$, and as $(0:0:0:1) \in L_6 \nsubseteq Z_2^\circ$, we may assume $q_1 = 1$. Again, for another indeterminate $y \in S_1\backslash\mathfrak{q}_1$, this means that $\mathfrak{g}_\mathfrak{q} = g_3 + \mathfrak{q} \simeq x_0^2 + q_4^2 y$ determines the fibre $\pi^{-1}(\mathfrak{q})$. But $x_0^2 + q_4^2 y \in X_{(2)}$ if and only if $q_4 = 0$, thus

$$C_2 \cap Z_{(2)}^\circ = \{(1:1:0:0)\} = C_2 \cap L_1$$

and

$$C_2\backslash\{(1:1:0:0),(0:0:0:1)\} \subseteq Z^\circ_{(1,1)}.$$

Hence, the only closed point $\mathfrak{q}$ of the semi-cubical parabola C_2 with one double point in the fibre $\pi^{-1}(q)$ is the cusp of C_2, which also lies on the image of the line $\widetilde{L} \subseteq \mathrm{Sing}(\widetilde{Z})$. The same holds for the other three semi-cubical parabolas: Closed points of C_3 can be written as $(q_1 : q_1 : \sqrt[3]{-q_1q_4^2} : q_4)$ with $(q_1, q_4) \in \mathbb{C}^2\backslash\{(0,0)\}$, and we get as above

$$C_3 \cap Z^\circ_{(2)} = \{(1:1:0:0)\} = C_3 \cap L_1$$

and

$$C_3\backslash\{(1:1:0:0),(0:0:0:1)\} \subseteq Z^\circ_{(1,1)}.$$

Closed points of C_4 can be written as $(q_1 : -q_1 : \sqrt[3]{-q_1q_4^2} : q_4)$ with $(q_1, q_4) \in \mathbb{C}^2\backslash\{(0,0)\}$, and closed points of C_5 are of the form $(q_1 : -q_1 : \sqrt[3]{q_1q_4^2} : q_4)$ with $(q_1, q_4) \in \mathbb{C}^2\backslash\{(0,0)\}$. Again, the same argument as above yields

$$C_4 \cap Z^\circ_{(2)} = C_5 \cap Z^\circ_{(2)} = \{(1:-1:0:0)\} = C_4 \cap L_1 = C_5 \cap L_1$$

and

$$(C_4 \cup C_5)\backslash\{(1:-1:0:0),(0:0:0:1)\} \subseteq Z^\circ_{(1,1)}.$$

Altogether, we get the following picture of Z°_2:

$$Z_{(2)^\circ} = L_1\backslash\{(1:0:0:0)\} = \mathrm{Spec}\left(\mathbb{C}\left[\frac{x_2}{x_1}\right]\right)$$

is an affine line embedded in $\mathbb{P}^3$, and

$$\begin{aligned} Z^\circ_{(1,1)} &= (C_2 \cup C_3 \cup C_4 \cup C_5)\backslash\{(1:1:0:0),(1:-1:0:0),(0:0:0:1)\} \\ &= (C_2 \cup C_3 \cup C_4 \cup C_5)\backslash(L_1 \cup L_6 \cup L_7) \end{aligned}$$

consists of the closed points lying on a cubic C_2, C_3, C_4, or C_5, but not on any one of the lines L_1, L_6, and L_7.

As we have seen, Z_4 consists of the two lines $L_6 = \mathrm{Proj}(K[x_3, x_4])$ and $L_7 = \mathrm{Proj}(K[x_1, x_4])$. Since g_1 is the only element of G of x_0-degree 4, for any closed point $\mathfrak{q} = (q_1 : 0 : q_3 : q_4) \in Z_4$, the number of points and their multiplicities in the fibre $\pi^{-1}(\mathfrak{q})$ equals the number of linear factors and their multiplicities of $\mathfrak{g}_\mathfrak{q} = g_1 + \mathfrak{q} \simeq x_0^4 - q_1x_0^3y - q_3^3x_0y^3 - q_4^4y^4 \in \mathbb{C}[x_0, y]$ (see again 4.1.3). Hence, we need the equations for the coincident root loci of partitions of 4, which we computed in Example 2.5.5.

First, we consider $L_6 = \mathrm{Proj}(\mathbb{C}[x_3, x_4]) \subseteq \mathbb{P}^3$. A closed point $\mathfrak{q} \,\dot\in\, L_6$ can be written $\mathfrak{q} = (0:0:q_3:q_4)$ for some $(q_3, q_4) \in \mathbb{C}^2\backslash\{(0,0\}$, thus, the fibre $\pi^{-1}(\mathfrak{q})$ is determined by $x_0^4 - q_3^3x_0y^3 - q_4^4y^4$. Looking at the ideal $\mathfrak{I}_{(4)}$, we see that a closed point $\mathfrak{q} \,\dot\in\, Z_{(4)} \cap L_6$ must fulfil

$$6q_3^3 = 0 \;\text{ and }\; 36q_4^4 = 0, \qquad \text{hence,} \qquad q_3 = q_4 = 0,$$

a contradiction. Therefore $L_6 \cap Z_{(4)} = \varnothing$.

Similarly, $\mathfrak{q} \in Z_{(1,3)} \cap L_6$ would imply the contradiction $12q_4^4 = 0$ and $27q_3^6 = 0$,

and $\mathfrak{q} \in Z_{(2,2)} \cap L_6$ would imply $8q_3^3 = 0$ and $8q_4^4 = 0$. Thus, $Z_{(1,3)} \cap L_6 = \varnothing = Z_{(2,2)} \cap L_6$. Of course, either one of these last two contradictions also would yield $L_6 \cap Z_{(4)} = \varnothing$ since $X_{(4)} = X_{(1,3)} \cap X_{(2,2)}$.
On the other hand, a closed point $\mathfrak{q} \dot{\in} Z_{(1,1,2)} \cap L_6$ only has to fulfil $256q_4^{12} - 27q_3^{12} = 0$, meaning

$$q_4 = \sqrt[12]{\frac{3^3}{4^4}q_3^{12}} = \frac{\sqrt[4]{3}}{\sqrt[3]{4}}\xi q_3,$$

where ξ is a 12-th root of unity. Thus, the set

$$Z_{(1,1,2)} \cap L_6 = \left\{ \left(0:0:1:\frac{\sqrt[4]{3}}{\sqrt[3]{4}}\xi\right) \in \mathbb{P}^3 \;\middle|\; \xi \in \mathbb{C} \wedge \xi^{12} = 1 \right\}$$

contains 12 closed points.
If we now look at the preimage $\pi^{-1}(L_6) = \langle \mathfrak{p}, L_6 \rangle_{\mathbb{P}^4} \cap \widetilde{Z}$, we find its ideal to be

$$\mathfrak{J}_6 R + \mathfrak{J}_{\widetilde{Z}} = \langle x_1, x_2, x_0^4 - x_0^3 x_3 - x_4^4 \rangle,$$

that is, $\pi^{-1}(L_6)$ is a quartic in the plane $\operatorname{Proj}(R/\langle x_1, x_2 \rangle$. Its intersection with $\operatorname{Sing}(\widetilde{Z})$ is

$$\begin{aligned} \langle \mathfrak{p}, L_6 \rangle \cap \operatorname{Sing}(\widetilde{Z}) &= \operatorname{Proj}(R/\langle x_1, x_2, x_3, x_0^4 - x_4^4 \rangle \\ &= \left\{ \begin{array}{l} (1:0:0:0:1), (1:0:0:0:-1), \\ (1:0:0:0:i), (1:0:0:0:-i) \end{array} \right\} \\ &= \langle \mathfrak{p}, L_6 \rangle \cap \widetilde{Q} \end{aligned}$$

and thus consists of four closed points whose images under π are simple. Hence, for any $\mathfrak{q} \dot{\in} Z_{(1,1,2)} \cap L_6$, the double point in the fibre $\pi^{-1}(\mathfrak{q})$ must lay on a tangent to $\widetilde{Z}$ through $\mathfrak{p}$.

Next, we turn our attention to the line $L_7 = \operatorname{Proj}(\mathbb{C}[x_1, x_4]) \subseteq \mathbb{P}^3$ whose closed points $\mathfrak{q}$ can be written $\mathfrak{q} = (q_1 : 0 : 0 : q_4)$ with $(q_1, q_4) \in \mathbb{C}^2 \backslash \{(0,0\}$. The fibre $\pi^{-1}(\mathfrak{q})$ over a closed point $\mathfrak{q} \dot{\in} L_7$ thus is determined by the binary form $x_0^4 - q_1 x_0^3 y - q_4^4 y^4$ which belongs to the coincident root locus $X_{(4)}$ if and only if $3q_1^2 = 0$ and $36q_4^4 = 0$, hence, if $q_1 = q_4 = 0$ which is not possible. Therefore, $Z_{(4)} \cap L_7 = \varnothing$.
For $\mathfrak{q} \in Z_{(1,3)} \cap L_7$, it must hold $12q_4^4 = 0$ and $27q_1^2 q_4^4 = 0$, meaning $q_4 = 0$ and $q_1 \in \mathbb{C} \backslash \{0\}$, and we get

$$Z_{\{1,3\}} \cap L_7 = \{(1:0:0:0)\}.$$

Looking at the ideal $\mathfrak{J}_{(2,2)}$ of $X_{(2,2)}$ in Example 2.5.5, we see that a closed point $\mathfrak{q} = (q_1 : 0 : 0 : q_4) \dot{\in} Z_{\{2,2\}} \cap L_7$ would have to fulfil $q_1^3 = 0$ and $16q_4^4 = 0$, a contradiction to $(q_1, q_4) \neq (0,0)$. Hence, $Z_{(2,2)} \cap L_7 = \varnothing$.
Finally, $\mathfrak{q} = (q_1 : 0 : 0 : q_4) \in Z_{(1,1,2)} \cap L_7$ if and only if

$$\Delta_4(x_0^4 - q_1 x_0^3 y - q_4^4 y^4) = 256q_4^{12} - 27q_1^4 q_4^8 = q_4^8(256q_4^4 - 27q_1^4) = 0,$$

that is either if $q_4 = 0$, resulting in the closed point $(1:0:0:0)$ that we already found to belong to $Z_{(1,3)}$, or else $q_4 \neq 0 \neq q_1$ and $q_4 = \sqrt[4]{\frac{256}{27}q_1^4}$, hence,

$$Z^{\circ}_{(1,1,2)} \cap L_7 = \left\{ \begin{array}{ll} \left(1:0:0:\frac{4}{\sqrt[4]{3^3}}\right), & \left(1:0:0:-\frac{4}{\sqrt[4]{3^3}}\right), \\ \left(1:0:0:\frac{4}{\sqrt[4]{3^3}}i\right), & \left(1:0:0:-\frac{4}{\sqrt[4]{3^3}}i\right) \end{array} \right\}.$$

Now, we look at $\pi^{-1}(L_7) = \langle \mathfrak{p}, L_7 \rangle \cap \widetilde{Z}$; its ideal is

$$\mathfrak{I}_7 R + \mathfrak{I}_{\widetilde{Z}} = \langle x_2, x_3, x_0^4 - x_0^3 x_1 - x_4^4 \rangle,$$

that is

$$\pi^{-1}(L_7) = \widetilde{Q} \subseteq \mathrm{Sing}(\widetilde{Z}).$$

The only singular closed point $(0:1:0:0:0)$ of $\widetilde{Q}$ lies in the fibre over $(1:0:0:0)$, the only point of $Z_{(1,3)}$. Also, for any $\mathfrak{q} \dot{\in} Z^\circ_{(1,1,2)} \cap L_7$, the line $\langle \mathfrak{p}, \mathfrak{q} \rangle$ is tangent to $\widetilde{Z}$.

Putting everything together, we get the following stratification on $Z_4 \backslash Z^\circ_{(1,1,1,1)}$:

$$\begin{array}{rcl} Z_4 &=& L_6 \cup L_7; \\ Z^\circ_{(4)} &=& \varnothing; \\ Z^\circ_{(1,3)} &=& \{(1:0:0:0)\}; \\ Z^\circ_{(2,2)} &=& \varnothing; \\ Z^\circ_{(1,1,2)} &=& \left(\begin{array}{c} \left\{ \left(0:0:1:\frac{\sqrt[4]{3}}{\sqrt[3]{4}}\xi\right) \in \mathbb{P}^3 \;\middle|\; \xi \in \mathbb{C} \wedge \xi^{12} = 1 \right\} \\ \cup \\ \left\{ \left(1:0:0:\frac{4}{\sqrt[4]{3^3}}\xi\right) \in \mathbb{P}^3 \;\middle|\; \xi \in \mathbb{C} \wedge \xi^4 = 1 \right\} \end{array} \right). \end{array}$$

Associated to this stratification, which only consists of closed sets, are the ideals

$$\begin{array}{rcl} \mathfrak{I}(Z_4) &=& \langle x_2, x_1 x_3 \rangle; \\ \mathfrak{I}(Z_{(4)}) &=& S; \\ \mathfrak{I}(Z_{(1,3)}) &=& \langle x_2, x_3, x_4 \rangle; \\ \mathfrak{I}(Z_{(2,2)}) &=& S; \\ \mathfrak{I}(Z_{(1,1,2)}) &=& \langle x_1, x_2, x_3^{12} - \frac{27}{256} x_4^{12} \rangle \cap \langle x_2, x_3, x_1^4 - \frac{256}{27} x_4^4 \rangle \end{array}$$

According to [Singular], a minimal set of generators for the last ideal is

$$\mathfrak{I}(Z_{(1,1,2)}) = \langle x_2, x_1 x_3, 27 x_1^5 - 256 x_1 x_4^4, 3^6 x_1^4 x_4^8 + 2^{16} x_3^{12} - 2^8 \cdot 3^3 x_4^{12} \rangle.$$

4.3 The ideal of Z_λ

After the example above culminating in equations for the respective ramification loci, we now give a description of the homogeneous ideal of Z_λ in general. We keep the standard notations (see 3.1.3 and 3.2.1); in particular, let $x \in R_1 \backslash \mathfrak{p}_1$.

Lemma 4.3.1. *Let $k \in \mathbb{N}_0$, and let $\mathfrak{q} \in \mathrm{mProj}(S)$ with $\mathfrak{K}^{\mathfrak{p}}_k(\mathfrak{I}_{\widetilde{Z}}) \subseteq \mathfrak{q}$. Then, for all $f \in \mathfrak{I}_{\widetilde{Z}}$ with $\deg_x(f) \leqslant k$, it holds $f \in \mathfrak{q}R$.*

Proof. Let $f \in \mathfrak{I}_{\widetilde{Z}}$ with $\deg_x(f) \leqslant k$. Then $\overline{f} := f + \mathfrak{q}R \in (\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}/\mathfrak{q}R)^{\mathrm{sat}} \subseteq R/\mathfrak{q}R$ with $\deg_x(\overline{f}) \leqslant \deg_x(f) \leqslant k$. According to Theorem 3.1.9, $(\mathfrak{I}_{\widetilde{Z}} + \mathfrak{q}/\mathfrak{q}R)^{\mathrm{sat}}$ is generated by an element of x-degree bigger than k, hence $\overline{f} = 0$ and therefore $g \in \mathfrak{q}R$. □

Notation 4.3.2. For $k \in \mathbb{N}_0$, we define

$$\widetilde{\mathfrak{K}}^{\mathfrak{p}}_k(\mathfrak{I}_{\widetilde{Z}})^\circ := \{ g \in \mathfrak{I}_{\widetilde{Z}} \mid \deg_{y_0}(g) = k \}.$$

We also use the notations of Section 2.5, that is $\mathfrak{I}_\lambda \subseteq A^{(0)} = K[z_0, \ldots, z_k]$ is the homogeneous vanishing ideal of the CRL $X_\lambda \subseteq \mathbb{P}(\mathrm{Sym}^k(K[x,y]_1))$ for a partition λ of k. Any polynomial $f \in R = S[x]$ with $\deg_x(f) = k$ still can be written $f = \sum_{i=0}^k f_i x^{k-i}$ with $f_0, \ldots f_k \in S$. Using the canonical inclusion $K[z_0, \ldots, z_k] \hookrightarrow S[z_0, \ldots, z_k]$, for $F \in A^{(0)}$, we again denote (compare Remark and Definition 1.2.1)

$$F(f) = F(f_0, \ldots, f_k) \in S.$$

We finally are ready to formulate and prove our main theorem about the ideals of the ramification loci Z_λ:

Theorem 4.3.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_e)$ be a partition of $k \in \mathbb{N}_0$. Then, for any closed point $\mathfrak{q} \,\dot{\in}\, Z_k$, it holds*

$$\mathfrak{q} \in Z_\lambda \Leftrightarrow \forall F \in \mathfrak{I}_\lambda \, \forall f \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ : \, F(f) \in \mathfrak{q}.$$

Proof. First, assume $\mathfrak{q} \in Z_\lambda$. If $\mathfrak{q} \in Z_{k+1}$, then $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}) \subseteq \mathfrak{q}$, and hence $\widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ \subseteq \mathfrak{q}R$ according to Lemma 4.3.1. Thus, as $x \notin \mathfrak{q}R$ and $\mathfrak{q}R$ is a prime ideal, for all $f = \sum_{i=0}^k f_i x^{k-i} \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ$, it holds $f_i \in \mathfrak{q}$ for all $i \in \{0, \ldots, k\}$, and therefore $F(f) = F(f_0, \ldots, f_i) \in \mathfrak{q}$ for all $F \in A^{(0)}$.
If, on the other hand, $\mathfrak{q} \in Z_k^\circ$, then for $f = \sum_{i=0}^k f_i x^{k-i} \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ$ either $f \in \mathfrak{q}R$ and hence $F(f) \in \mathfrak{q}$ for all $F \in A^{(0)}$ as above, or $f \notin \mathfrak{q}R$, that is $0 \neq f + \mathfrak{q}R \in R/\mathfrak{q}R$. In the latter case, by Proposition 4.1.3 $\mathfrak{f}_\mathfrak{q} \in X_\lambda$ for $y \in S_1 \backslash \mathfrak{q}_1$. Denote $t := \deg(f)$. By definition,

$$(\overline{f}_0, \ldots, \overline{f}_k) := \left(\frac{f_0 + \mathfrak{q}}{y^{t-k}}, \frac{f_1 + \mathfrak{q}}{y^{t-k+1}}, \ldots, \frac{f_k + \mathfrak{q}}{y^t} \right) \in K^{k+1}$$

are coefficients of the binary form $\mathfrak{f}_\mathfrak{q}$. For a homogeneous and weighted homogeneous polynomial $F \in \mathfrak{I}_\lambda$ of (standard) degree m and weighted degree s, Lemma 1.2.2 therefore yields

$$\begin{aligned} F(f_0 + \mathfrak{q}, \ldots, f_k + \mathfrak{q}) &= F(y^{t-k}\overline{f}_0, \ldots, y^t \overline{f}_k) \\ &= y^{(t-k)m+s} F(\overline{f}_0, \ldots, \overline{f}_k) \quad = \quad 0, \end{aligned}$$

hence $F(f) \in \mathfrak{q}$. Since $\mathfrak{I}_\lambda$ is homogeneous and weighted homogeneous by Proposition 2.5.6, this shows the implication '$\Rightarrow$'.
Now, assume $F(f) \in \mathfrak{q}$ for all $F \in \mathfrak{I}_\lambda$ and all $f \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ$. If $f \in \mathfrak{q}R$ for all $f \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ$, then $\mathfrak{K}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}) \subseteq \mathfrak{q}$, and hence $\mathfrak{q} \in Z_{k+1} \subseteq Z_\lambda$. On the other hand, if there is an element $f = \sum_{i=0}^k f_i x^{k-i} \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ$ with $f \notin \mathfrak{q}R$, then $F(f) \in \mathfrak{q}$ implies $F(f_0 + \mathfrak{q}, \ldots, f_k + \mathfrak{q}) = 0 \in S/\mathfrak{q}$. If F is homogeneous and weighted homogeneous, with the same notations and arguments as before we get $F(\overline{f}_0, \ldots, \overline{f}_k) = 0$, hence $\mathfrak{f}_\mathfrak{q} \in X_\lambda$. Another application of Proposition 4.1.3 finishes the proof. □

Corollary 4.3.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_e)$ be a partition of $k \in \mathbb{N}_0$. Then, set-theoretically,*

$$Z_\lambda = V_{\mathbb{P}^n} \left(\mathfrak{K}_{k-1}^{\mathfrak{p}} + \langle F(f) \mid F \in \mathfrak{I}_\lambda, f \in \widetilde{\mathfrak{K}}_k^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})^\circ \rangle \right).$$

Recall that according to Corollary 2.5.7, we can compute generators $F_1, \ldots, F_s$ of $\mathfrak{I}_\lambda$ which are homogeneous and weighted homogeneous.

Corollary 4.3.5. *Let $\lambda = (\lambda_1, \dots, \lambda_e)$ be a partition of $k \in \mathbb{N}_0$. Additionally, assume $\mathfrak{p} = \langle x_1, \dots, x_n \rangle$ and $x = x_0$, and let G be a Gröbner Basis of $\mathfrak{I}_{\tilde{Z}}$ with respect to the lexicographic ordering. Let $F_1, \dots, F_s$ be homogeneous and weighted homogeneous generators of $\mathfrak{I}_\lambda \in K[z_0, \dots, z_k]$. Then, set-theoretically,*

$$\mathfrak{I}(Z_\lambda) = \langle \mathrm{In}_{k-1}(G) \cup \{F_i(f) \mid i \in [s], f \in \widetilde{\mathfrak{K}}^{\mathfrak{p}}_k(\mathfrak{I}_{\tilde{Z}})^\circ\} \rangle.$$

Proof. $\mathfrak{K}^{\mathfrak{p}}_{k-1}(\mathfrak{I}_{\tilde{Z}})$ is generated by $\mathrm{In}_{k-1}(G)$, and according to Corollary 4.1.4, a closed point $\mathfrak{q} \in Z_k^\circ$ belongs to Z_λ if and only if there exists $f = \sum_{i=0}^k f_i x^{k-i} \in \widetilde{\mathfrak{K}}^{\mathfrak{p}}_k(\mathfrak{I}_{\tilde{Z}})^\circ$ with $f_\mathfrak{q} \in X_\lambda$; this is equivalent to $F_i(f) \in \mathfrak{q}$ for all $i \in [s]$ by Theorem 4.3.3. Also, we note that for all $i \in [s]$ and $f \in \widetilde{\mathfrak{K}}^{\mathfrak{p}}_k(\mathfrak{I}_{\tilde{Z}})^\circ$, the element $F_i(f) \in S$ is homogeneous by Lemma 1.2.3. □

Remark 4.3.6. By Corollary 2.5.7, we indeed can compute homogeneous and weighted homogeneous generators of $\mathfrak{I}_\lambda$. Therefore, the last Corollary gives an algorithm for computing the ideal of Z_λ if $\mathfrak{p} = \langle x_1, \dots, x_n \rangle$. For a general $\mathfrak{p} \in \mathbb{P}^n$, we again can use a coordinate transformation γ with $\gamma(\mathfrak{p}) = \langle x_1, \dots, x_n \rangle$ as before to compute $\mathfrak{I}(Z_\lambda)$ (see Algorithm A.3.2). By Remark and Notation 4.1.2, coordinate transformation has no effect on the number of linear factors of a binary form and their multiplicities and therefore on the number of closed points in the fibres $\pi^{-1}(\mathfrak{q})$ and their multiplicities.
While the above Corollary only explains how to compute equations for Z_λ, we can derive a description for the proper λ-ramification locus from it: It holds

$$Z_\lambda^\circ = Z_\lambda \setminus \left(Z_{k+1} \cup \bigcup_{\mu \in Q_\lambda^\circ} Z_\mu \right)$$

(for Q_λ°, see Remark and Definition 1.1.2). We can compute equations defining the closed sets Z_μ for $\mu \in Q_\lambda$ using Corollary 4.3.5 and equations defining the closed set Z_{k+1} using Algorithm 3.3.4. The locally closed set Z_λ° is the set of closed points for which the equations for Z_λ are satisfied, but for which at least one of the equations for Z_{k+1} and Z_μ for $\mu \in Q_\lambda^\circ$ is not satisfied.

Appendix A

Algorithms

Here, we collect all algorithms which were developed in this thesis for the convenience of a potential user and implementor. We still work over a fixed algebraically closed base field K of characteristic 0.

A.1 Computing coincident root loci

We repeat Algorithm 2.5.3 for computing the ideal of the coincident root locus (CRL) X_λ with multiplicities λ. For CRLs, see Section 2.2. For partitions, see Definition 1.1.1.

Algorithm A.1.1. *Input:* A partition λ of $d \in \mathbb{N}$ given by the numbers $e_1, \ldots, e_d \in \mathbb{N}_0$ with $\lambda = (1^{e_1} \ldots d^{e_d})$.

1. For $j \in \{1, \ldots, d\}$, set

$$N_j := \left\{ \begin{array}{l} \underline{\underline{\nu}} = (\underline{\nu}_1 = (\nu_{1,0}, \ldots, \nu_{1,e_1}), \ldots, \underline{\nu}_d) \in \mathbb{N}_0^{e_1+1} \times \cdots \times \mathbb{N}_0^{e_d+1} \mid \\ (\forall r \in \{1, \ldots, d\} : \sum_{t=0}^{e_r} \nu_{r,t} = r) \wedge \sum_{r=1}^{d} \left(\sum_{t=0}^{e_r} t\nu_{r,t}\right) = j \end{array} \right\}.$$

   We can compute N_j recursively using Algorithm A.1.2 below.

2. For $j \in \{1, \ldots, d\}$ and $\underline{\underline{\nu}} \in N_j$, compute

$$\beta\left(\underline{\underline{\nu}}\right) := \prod_{r=1}^{d} \frac{r!}{\nu_{r,0}!\nu_{r,1}! \cdots \nu_{r,e_r}!}.$$

3. Set $A^{(\underline{0})} := K[z_1, \ldots, z_d, w_{1,1}, \ldots, w_{1,e_1}, w_{2,1}, \ldots, w_{d,e_d}]$ furnished with the lexicographical ordering where $z_i, w_{r,t}$ are indeterminates.

4. For $j \in \{1, \ldots, d\}$, compute

$$\theta_j = \sum_{\underline{\underline{\nu}} \in N_j} \beta(\underline{\underline{\nu}}) \underline{\underline{w}}^{\underline{\underline{\nu}}},$$

   where $w_{1,0} = \cdots = w_{r,0} = 1$.

5. Define $\mathfrak{I}^{(\underline{0})}$ to be the ideal of $A^{(\underline{0})}$ generated by $z_1 - \theta_1, \ldots, z_d - \theta_d$.

6. Compute $\mathfrak{I}^{(0)} := \mathfrak{I}^{(\underline{0})} \cap K[z_1, \ldots, z_d]$, that is reduce $\mathfrak{I}^{(\underline{0})}$ by $\underline{\underline{w}}$.

7. Compute $\mathfrak{I}_\lambda = \overline{\mathfrak{I}^{(0)}}^{\hom_{z_0}}$, that is homogenize the generators in a Gröbner basis of $\mathfrak{I}^{(0)}$ at z_0.

Output: The homogeneous ideal $\mathfrak{I}_\lambda = \mathfrak{I}_\lambda \subseteq K[z_0, \ldots, z_d]$ of the CRL X_λ via a set of homogeneous generators.

This algorithm indeed computes equations for X_λ by Theorem 3.1.9 and Lemmas 2.5.1 and 2.5.2.

Algorithm A.1.2. *Input:* A partition λ of $d \in \mathbb{N}$ given by the numbers $e_1, \ldots, e_d \in \mathbb{N}_0$ with $\lambda = (1^{e_1} \ldots d^{e_d})$.

1. Set $\Lambda := \mathbb{N}_0^{e_1+1} \times \cdots \times \mathbb{N}_0^{e_d+1}$.

2. Set $N_0 := \{((e_1+1, 0, \ldots, 0), \ldots, (e_d+1, 0, \ldots, 0))\} \subseteq \Lambda$.

3. Set $N_1 = \cdots = N_d = \varnothing \subseteq \Lambda$.

4. For j running from 1 to d, consider all $\underline{\underline{\nu}} \in N_{j-1}$. For all $r \in \{1, \ldots, d\}$ and all $t \in \{0, \ldots, e_r - 1\}$, if $\nu_{r,t} > 0$, then set
$$\begin{aligned}\underline{\underline{\nu}}' \quad := \quad & \big(\underline{\nu}_1, \ldots, \underline{\nu}_{r-1},\\ & (\nu_{r,0}, \ldots, \nu_{r,t-1}, \nu_{r,t} - 1, \nu_{r,t+1} + 1, \nu_{r,t+2}, \ldots, \nu_{r,e_r}),\\ & \underline{\nu}_{r+1}, \ldots, \underline{\nu}_d\big)\,,\end{aligned}$$
and adjoin $\underline{\underline{\nu}}'$ to N_j, i.e., redefine $N_j := N_j \cup \{\underline{\underline{\nu}}'\}$.

Output: The sets $N_0, \ldots, N_d$.

Looking at the definition of the sets N_j in Algorithm A.1.1, it is easy to see that Algorithm A.1.2 indeed yields these sets (also compare the proof of Lemma 2.4.5).

A.2 Computing the heterodyne loci Z_k, secant cones, and secant loci

For the definitions of heterodyne loci, secant cones, and secant loci, see Definitions 3.2.2 and 3.2.4.

Algorithm A.2.1. *Input:* The homogeneous ideal $\mathfrak{I}_{\widetilde{Z}} \subseteq R := K[x_0, \ldots, x_n]$ of a closed subscheme $\widetilde{Z} \subseteq \mathbb{P}^n = \mathrm{Proj}(R)$, and a minimal system of generators $y_1, \ldots, y_n \in R_1$ of the homogeneous ideal of the closed point $\mathfrak{p} \in \mathbb{P}^n \backslash \widetilde{Z}$. Consider R to be furnished with the lexicographical term order.

1. Compute $l := \min\{i \in \{0, \ldots, n\} \mid x_i \notin \mathfrak{p}\}$.

2. Define the coordinate transformation $\gamma : R \xrightarrow{\cong} R$ to be the inverse of $x_0 \mapsto x_l$, $x_1 \mapsto y_1, \ldots, x_n \mapsto y_n$. Compute $\gamma(\mathfrak{I}_{\widetilde{Z}})$.

3. Compute a Gröbner basis G of $\gamma(\mathfrak{I}_{\widetilde{Z}})$, for example using the Buchberger algorithm.

4. Set $k_{\mathfrak{p}} := \max\{\deg_{x_0}(g) \mid g \in G\}$.
5. For all $k \in \{0, \dots, k_{\mathfrak{p}} - 1\}$, set $\mathrm{In}_k(G) := \{\mathrm{LC}_{x_0}(g) \mid g \in G \wedge \deg_{x_0}(g) \leqslant k\}$.
6. Set $\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}}) := \gamma^{-1}(\mathrm{In}_k(G))K[\mathfrak{p}_1]$ for $k \in \{0, \dots, k_{\mathfrak{p}} - 1\}$.
7. Compute $\sqrt{\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}})R}$ for $k \in \{0, \dots, k_{\mathfrak{p}} - 1\}$, for example using the algorithm of Krick and Logar (see [KrL]).
8. Compute $\sqrt{\mathfrak{K}_k^p(\mathfrak{I}_{\tilde{Z}})R} + \mathfrak{I}_{\tilde{Z}}$ for $k \in \{0, \dots, k_{\mathfrak{p}} - 1\}$.

Output: The ideals $\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}}), \dots, \mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})$ (via a Gröbner basis) of $Z_1, \dots, Z_{k_{\mathfrak{p}}}$ (set-theoretically); the ideals $\sqrt{\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})R}, \dots, \sqrt{\mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_Z)R}$ (via a finite set of generators) of $\mathrm{Sec}_1^{\mathfrak{p}}(\tilde{Z}), \dots, \mathrm{Sec}_{k_{\mathfrak{p}}}^{\mathfrak{p}}(\tilde{Z})$; the ideals $\sqrt{\mathfrak{K}_0^{\mathfrak{p}}(\mathfrak{I}_{\tilde{Z}})R} + \mathfrak{I}_{\tilde{Z}}, \dots, \sqrt{\mathfrak{K}_{k_{\mathfrak{p}}-1}^{\mathfrak{p}}(\mathfrak{I}_Z)R} + \mathfrak{I}_{\tilde{Z}}$ (via a finite set of generators) of $\Sigma_1^{\mathfrak{p}}(\tilde{Z}), \dots, \Sigma_{k_{\mathfrak{p}}}^{\mathfrak{p}}(\tilde{Z})$. For all $k > k_{\mathfrak{p}}$, it holds $Z_k = \mathrm{Sec}_k^{\mathfrak{p}}(\tilde{Z}) = \Sigma_k^{\mathfrak{p}}(\tilde{Z}) = \varnothing$.

For a discussion of the above Algorithm, see Algorithm 3.3.4.

A.3 Computing equations describing the ramification loci

For the definitions of ramification loci and proper ramification loci, see Definition 4.0.1. We first give an algorithm for computing a Gröbner basis describing the ramification behaviour on a proper heterodyne locus as in Theorem 4.1.3 and Corollary 4.1.4.

Algorithm A.3.1. *Input:* The homogeneous ideal $\mathfrak{I}_{\tilde{Z}} \subseteq R := K[x_0, \dots, x_n]$ of a closed subscheme $\tilde{Z} \subseteq \mathbb{P}^n = \mathrm{Proj}(R)$, and a minimal system of generators $y_1, \dots, y_n \in R_1$ of the homogeneous ideal of the closed point $\mathfrak{p} \in \mathbb{P}^n \backslash \tilde{Z}$. Consider R to be furnished with the lexicographical term order.

1. Compute $l := \min\{i \in \{0, \dots, n\} \mid x_i \notin \mathfrak{p}\}$.
2. Define the coordinate transformation $\gamma : R \xrightarrow{\cong} R$ to be the inverse of $x_0 \mapsto x_l$, $x_1 \mapsto y_1, \dots, x_n \mapsto y_n$. Compute $\gamma(\mathfrak{I}_{\tilde{Z}})$.
3. Compute a Gröbner basis $\hat{G}$ of $\gamma(\mathfrak{I}_{\tilde{Z}})$, for example using the Buchberger algorithm.
4. Find the element $f_0 = \kappa x_0^{k_{\mathfrak{p}}} + g \in \hat{G}$ for an unit $\kappa \in K^*$, an integer $k_{\mathfrak{p}} \in \mathbb{N}_0$, and an element $g \in R$ with $\deg_{x_0}(g) < k_{\mathfrak{p}}$. If there is more than one element of $\hat{G}$ of this form, choose one with the smallest degree in x_0.
5. Set $G := \{f_0\}$.
6. For all elements $f \in \hat{G} \backslash \{f_0\}$, do the following procedure: If $\deg_{x_0}(f) < k_{\mathfrak{p}}$, then join f to G, i.e., redefine $G = G \cup \{f\}$.
If $\deg_{x_0}(f) \geqslant k_{\mathfrak{p}}$, then take all the monomials of f whose degree is greater

or equal than $k_\mathfrak{p}$ and add them to the polynomial r (so that $f = r + s$ with $\deg_{x_0}(s) < k_\mathfrak{p}$). Set

$$t := \frac{r}{x_0^{k_\mathfrak{p}}}.$$

Replace f with $tf_0 - f$. Repeat until $\deg_{x_0}(f) < k_\mathfrak{p}$. Redefine $G = G \cup \{f\}$.

7. Compute $\gamma^{-1}(G)$.

Output: The set $\gamma^{-1}(G)$. For any $k \in \mathbb{N}_0$, a finite covering of the proper k-ramification locus Z_k° determining the ramification behaviour of the linear projection with centre $\mathfrak{p}$ as in Theorem 4.1.3 is given by the elements $f \in \gamma^{-1}(G)$ with $\deg_{x_0}(\gamma(f)) = k$.

Step 4 in the above algorithm works since there is an element $x_0^{k_\mathfrak{p}} + g \in \mathfrak{I}_{\tilde{Z}}$ with $k_\mathfrak{p} \in \mathbb{N}_0$ and $\deg_{x_0}(g) < k_\mathfrak{p}$ (see Remark 3.3.3). As the leading terms of $\hat{G}$ generate the initial ideal of $\mathfrak{I}_{\tilde{Z}}$, there must be an element $f_0 \in \hat{G}$ of the form claimed in step 4. The algorithm as a whole terminates and computes the set $\gamma^{-1}(G)$ as claimed according to the proof of Lemma 4.1.7 and Remark 4.1.8.

The next algorithm uses Algorithms A.1.1 and A.2.1 to compute the ideals of the ramification loci.

Algorithm A.3.2. *Input:* A partition λ of $k \in \mathbb{N}$ given by the numbers $e_1, \ldots, e_d \in \mathbb{N}_0$ with $\lambda = (1^{e_1} \ldots d^{e_d})$ (compare Definition 1.1.1), the homogeneous ideal $\mathfrak{I}_{\tilde{Z}} \subseteq R := K[x_0, \ldots, x_n]$ of a closed subscheme $\tilde{Z} \subseteq \mathbb{P}^n = \mathrm{Proj}(R)$, and a minimal system of generators $y_1, \ldots, y_n \in R_1$ of the homogeneous ideal of the closed point $\mathfrak{p} \in \mathbb{P}^n \backslash \tilde{Z}$. Consider R to be furnished with the lexicographical term order.

1. Compute homogeneous generators $F_1, \ldots, F_s$ of the ideal $\mathfrak{I}_\lambda$ of X_λ in $K[z_0, \ldots, z_d]$ using Algorithm A.1.1.

2.1 Compute a Gröbner basis G of $\mathfrak{K}_{k-1}^\mathfrak{p}(\mathfrak{I}_{\tilde{Z}})$ using Algorithm A.2.1.

2.2 Determine all elements $f_1, \ldots, f_t \in G$ with $\deg_{x_0}(\gamma(f_j)) = k$ for $j \in \{1, \ldots, t\}$ where γ is as in Algorithm A.2.1.

3. For $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$, compute $F_i(f_j)$ (for the notation, see Remark and Definition 1.2.1).

Output: The ideal $\mathfrak{I}(Z_\lambda)$ of the λ-ramification locus of the linear projection of $\tilde{Z}$ with centre $\mathfrak{p}$ via its generators $F_i(f_j)$ for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$.

For a discussion of the above Algorithm, see Corollary 4.3.5 and Remark 4.3.6.

Appendix B

Open Questions

Here, we collect some open questions that arise from the results in this thesis. I gave all of them some few thoughts, but did not study them in-depth.

Problem 1. Are coincident root loci locally Cohen-Macaulay? In general, they are nor arithmetically Cohen-Macaulay according to Chipalkatti (see [Ch1, Section 6]). But the close relationship between the smooth scheme Γ_λ, whose local Jacobi matrices contain a multiple of the identity matrix of maximal rank with a unit (see Section 2.4), and X_λ for a partition λ of $d \in \mathbb{N}_0$ suggests the conjecture that X_λ still might be locally Cohen-Macaulay. Also, for $d = 4$ this certainly holds (compare Example 2.5.5). It might be possible to answer this question by studying the singular points of X_λ using the description of the singular locus of X_λ in Section 2.6.

Problem 2. Given a projective subscheme $\widetilde{Z} \subseteq \mathbb{P}^n$ and a projective line $\mathbb{L} \subseteq \mathbb{P}^n$, when is there a clever decomposition $\mathfrak{p}, \mathfrak{p}'$ of $\mathbb{L}$ with respect to $\widetilde{Z}$ (see Definition 3.4.4)? Such a decomposition certainly does not always exist; e.g., if $\widetilde{Z}$ is a curve, then $\mathrm{Join}(\widetilde{Z}, \mathfrak{p})$ and $\mathrm{Join}(\widetilde{Z}, \mathfrak{p}')$ are surfaces, and if $\mathfrak{p}, \mathfrak{p}'$ are a clever decomposition, then $\widetilde{Z} = \mathrm{Join}(\widetilde{Z}, \mathfrak{p}) \cap \mathrm{Join}(\widetilde{Z}, \mathfrak{p}')$ is the scheme-theoretic intersection of two surfaces (compare Remark 3.4.5). Analogously for any $\widetilde{Z}$, if $\mathfrak{q}, \mathfrak{q}'$ are a clever decomposition, then $\widetilde{Z}$ is the intersection of two varieties of dimension $\dim(\widetilde{Z}) + 1$. Of course, this does not hold for all subschemes of $\mathbb{P}^n$; a classical example is the twisted cubic in $\mathbb{P}^3$ (see [Ha, I, Ex. 2.17]). So, it is not always possible to find clever decompositions. A condition whose fulfilment guarantees the existence of a clever decomposition would be useful in the study of examples of double projections.

Problem 3. Given a projective subscheme $\widetilde{Z} \subseteq \mathbb{P}^n$, what are the equations describing the secant stratification of $\widetilde{Z}$? A secant strata of $\widetilde{Z}$ is a maximal subset W of $\mathbb{P}^n \backslash \widetilde{Z}$ such that for any closed points $\mathfrak{q}, \mathfrak{q}' \in W$, the 2-secant loci $\Sigma^2_{\mathfrak{p}}(\widetilde{Z})$ and $\Sigma^2_{\mathfrak{p}'}(\widetilde{Z})$ are isomorphic. In [BrP], there is a geometric description of the secant stratification in the case that $\widetilde{Z}$ is a scroll (compare Example 3.5.3). But even knowing this geometric description, it is not obvious how to describe the secant stratification by equations. It would be useful to find a way to determine such equations for scrolls or even arbitrary varieties.
Starting from the results in this work, it might be possible to find equations de-

scribing the secant stratification of a projective variety $\widetilde{Z}$. The direct approach would be to look for algebraic conditions at two closed points $\mathfrak{p}, \mathfrak{p}' \in \mathbb{P}^n \backslash \widetilde{Z}$ such that there is a graded automorphism $\gamma : R \xrightarrow{\cong} R$ with

$$\gamma\left(\sqrt{\mathfrak{K}_2^{\mathfrak{p}}(\mathfrak{I}_Z)R} + \mathfrak{I}_{\widetilde{Z}}\right) = \sqrt{\mathfrak{K}_2^{\mathfrak{p}'}(\mathfrak{I}_Z)R} + \mathfrak{I}_{\widetilde{Z}}$$

(compare Proposition 3.2.5). But since the above equation contains radicals and sums of ideals, this approach gets very difficult soon – I tried it in the case of scrolls, but was not able to bring it to a useful conclusion.

I think a more promising approach arises from the description of the ramification loci in Chapter 4. Let $\pi : \widetilde{Z} \to Z$ and $\pi' : \widetilde{Z} \to Z'$ be the linear projections with the centres $\mathfrak{p}$ and $\mathfrak{p}'$, respectively. If the stratifications of the proper heterodyne loci Z_k° and $(Z')_k^\circ$ into proper ramification loci Z_λ° and $(Z')_\lambda^\circ$, respectively, are isomorphic for all $k \geqslant 2$, then we may expect that $\Sigma_{\mathfrak{p}}^2(\widetilde{Z}) \cong \Sigma_{\mathfrak{p}'}^2(\widetilde{Z})$. As said stratification of the proper heterodyne loci is determined by an open affine covering which in turn is determined by the elements of a Gröbner basis (see 4.1.4), I make the following conjecture: Let $x \in R\backslash\mathfrak{p}_1$, and consider R to be the polynomial ring in x and n linear forms generating $\mathfrak{p}$. Furnish $R = K[x, \mathfrak{p}_1]$ with the lexicographic ordering where x is the biggest indeterminate, and denote this ordering by Lex_x. Define an ordering $\mathrm{Lex}_{x'}$ on R for $x' \in R\backslash\mathfrak{p}'$ analogously. Now, the conjecture is that $\mathfrak{p}$ and $\mathfrak{p}'$ belong to the same secant stratum of $\widetilde{Z}$ if and only if there is an automorphism $\gamma : R \xrightarrow{\cong} R$ which maps a Gröbner basis of $\widetilde{Z}$ in $K[x, \mathfrak{p}_1]$ with respect to Lex_x to a Gröbner basis of $\widetilde{Z}$ in $K[x', \mathfrak{p}'_1]$ with respect to $\mathrm{Lex}_{x'}$.

Hence, finding equations describing the secant stratification of a variety $\widetilde{Z}$ could consist in first proving the above conjecture. Then, we would have to find all occurring isomorphy classes of 2-secant cones with respect to $\widetilde{Z}$ as it was done for scrolls in [BrP]. Finally, we would have to study the behaviour of Gröbner bases under coordinate transformations. In general, the image of a Gröbner basis under a graded ring automorphism need not be a Gröbner basis with respect to the matching ordering. We must ask for conditions on the ordering on R (which in our situation is induced by $\mathfrak{p}$ and x), on an ideal $\mathfrak{a}$ (in our situation $\mathfrak{I}_{\widetilde{Z}}$), and on a graded automorphism γ on R, such that Gröbner bases of $\mathfrak{a}$ are mapped to Gröbner bases of $\gamma(\mathfrak{a})$. Finding such conditions might be of some interest on its own. An obvious first idea is to look at generic initial ideals (see [E]). But since our main goal is the secant stratification, we are in particular interested in certain closed sets, i.e., in non-generic cases. Hence, generic initial ideals perhaps do not cover the situation we are interested in.

A weaker form of this problem is to determine equations for the stratum of closed points $\mathfrak{p} \in \mathbb{P}^n\backslash\widetilde{Z}$ such that $\Sigma_{\mathfrak{p}}^2(\widetilde{Z})$ has a given dimension. A solution to this problem for scrolls is given in [BrPS].

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