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Local stability analysis of a stochastic evolutionary financial market model with a risk-free asset*

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Abstract

An evolutionary model of a financial market with a risk-free asset is introduced and analyzed. The focus is on the study of local stability of the wealth dynamics through the application of recent results on the linearization and stability of random dynamical systems (Evstigneev, Pirogov and Schenk-Hoppé, Proceedings of the American Mathematical Society 139, 1061–1072, 2011). Conditions are derived for the linearization of the model at an equilibrium state which ensure local convergence of sample paths to this equilibrium. The paper also shows that concept of local stability is closely related to the notion of evolutionary stability. A locally evolutionarily stable investment strategy in the evolutionary model with a risk-free asset is derived, extending previous research. Our analysis can serve as an illustration of a mathematically correct approach to the study of local stability in stochastic systems. The method will be useful in the study of other economic and financial dynamic models involving randomness.

JEL classification: G11, G12.
Keywords: Evolutionary finance, risk-free asset, local stability, linearization, stochastic dynamics, random dynamical systems.

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1 Introduction

The contribution made in the paper is two-fold: (1) we extend the evolutionary stock market model presented in Evstigneev, Hens and Schenk-Hoppé (2006) by allowing investors to trade a risk-free asset; and (2) we apply to this model new results on the local stability of equilibria of random dynamical systems by Evstigneev, Pirogov and Schenk-Hoppé (2011).

Evolutionary finance provides an alternative way of thinking about financial markets. At its core it is a Darwinian view of markets, promoting the concepts of selection and survival over those of utility and consumption, see, e.g., the surveys Evstigneev, Hens and Schenk-Hoppé (2009,2011). The models developed in this line of research are behavioral and, due to their Darwinian nature, in stark contrast to the traditional schemes of utility maximization. This class of evolutionary model are also more amenable to numerical analysis and empirical investigation. Several characteristics are common to most of these models. A suitable classification is as follows; though the border lines are, admittedly, disputable and not each model or analysis necessarily has all of these features. Evolutionary finance is an approach that (a) builds only on observables; (b) does not assume rational expectations; (c) requires short-run equilibrium only; (d) focuses on the dynamics of prices and investors' wealth; (e) studies performance (such as average returns) or survival.

Permitting only observables as model ingredients, for instance, is diametrically opposed to classical general equilibrium models in which utility functions (an element that is not observable) are central to the theory. Though our modeling approach does not rule out that economic agents use utility functions to make decisions, they will have to base their actions on prices that have been revealed and on forecasts of future prices. Submitting a utility function to a Walrasian auctioneer is not possible (just as it cannot be done in reality).

Our approach also dispenses with the classical assumption on the perfect foresight of the economic agents. Though mainstream economics has a tendency of ignoring the issues surrounding this rational-expectation type assumption, it is instructive to recall Laffont (1989, page 85)'s comment "In the Radner equilibrium, price expectations are assumed to be exact for all agents. The agents do not necessarily agree on the probabilities of different states of nature, but they expect the same prices. The reader may be surprised by this assumption of perfect foresight. It should be viewed as a necessary methodological step. We must first understand how the economy performs with incomplete markets in the best case where expectations are correct."

In a model that does not assume expected utility maximization and perfect foresight, the notion of an equilibrium needs to be rethought. Evolutionary finance makes the least demanding assumption in this respect: it only assumes that the today's market for assets is cleared. (In the model presented in the present paper, all assets have non-negative payoffs and, naturally, should not be worthless.) This notion of equilibrium in the short-run captures the market-clearing feature of prices in limit order markets or auctions. Our equilibrium concept is only concerned with markets open today, prices in markets that will open at some future date will be found if and when that day arrives. Again this view is very different to general equilibrium where an equilibrium requires to determine today price systems for all future points in time and for every contingency.

In this paper, short-run equilibrium of prices is implemented as in the well-known Shapley-Shubik market game (Shapley and Shubik, 1977). Investors fix percentages, one number for each available asset, which determines the amount of money to be allocated to the purchase of an asset (percentage times the total available budget). Shapley and Shubik (1977) refer to these budget shares as 'fiscal rules.' In our work we refer to these as an investor's strategy or portfolio rule. Making investment decisions by choosing percentages is a common approach in asset allocation practice and theory. Many institutional (and also private) investors define their investment choice in this form. Pension funds typically would rethink their asset allocation on a regular basis (e.g. quarterly) and submit percentages to an investment team. This team is often internal for large pension funds. Professional financial advisors often ask private investors to choose percentages over different asset classes rather than individual assets. The implementation of such an investment strategy amounts to a rebalancing of the funds invested to maintain the specified percentages when asset prices change. In the presence of transaction costs, the optimal frequency to rebalance one's portfolio is a non-trival task, see, e.g., Kuhn and Luenberger (2010). A short-run equilibrium is defined as follows. In each period, all investors simultaneously announce their portfolio rules (budget shares) determining how much money should be invested in each of the available assets. Prices are then determined by the condition that instantaneous demand equals supply for each asset—ensuring market clearing.

The analysis of evolutionary finance models is (similarly to agent-based models) mainly concerned with the dynamics of asset prices and of the wealth of investors. Both are realized, observable quantities rather than theoretical constructs such as risk sharing or ex-post judgements on investor's asset allocation decisions. In line with this focus on dynamics, the ultimate success of an investor is measured by the performance such as average returns (in the short- and medium term) or survival (as a long term measure).

The maximization of utility from consumption is not an objective we believe to be useful for the understanding financial market dynamics. The main actors in the market, e.g., institutional investors, are concerned about goals such as growth or domination. These objectives are of an evolutionary nature and far removed from consumption maximization objectives of their clientele, if they have any. This reasoning is well supported by empirical facts. There is a growing literature, started by the seminal paper by Mankiw and Shapiro (1986), which finds that using asset returns (rather than consumption) as the argument in the stochastic discount factor gives much better fit when empirically fitting asset pricing models to actual asset prices.

The 'evolutionary finance philosophy' of modeling financial market dynamics is at odds with the stochastic general equilibrium approach. In that approach, agents maximize expected utility from consumption over an infinite time horizon, markets are competitive and agents have perfect foresight in the sense defined above. Blume and Easley (1992, 2006, 2009) study Milton Friedman's market selection hypothesis in this framework. Though the criterion of survival is at the core for Friedman's consideration, the economic agents are only concerned about utility from consumption, not about survival. Whether or not they 'wither away' over time is an issue of interest only to the modeler. The main finding is that, thanks to Pareto efficiency, one can establish a link between beliefs (the subjective probabilities for events) and consumption. When markets are complete, there is a clear-cut result on the accuracy of beliefs and asymptotically higher consumption. For incomplete markets, however, this link disappears and there appear to be no general results on survival.

Agent-based models in which investors are myopic mean-variance optimizers typically have one risky and one risk-free asset (Hommes and Wagener, 2009, and Chiarella, Dieci and He, 2009). The risky asset pays a random (i.i.d.) dividend and the risk-free asset has a constant yield. As a consequence of the agents' constant absolute risk aversion, savings in the money market grow over time while the investment in the risky asset is, on average, constant. This entails a dynamics with a vanishing contribution of the risky investment to the aggregate wealth. In our model this shortcoming is eliminated by letting the payoffs of all the risky assets grow (or decline) with the aggregate wealth in the economy. In models closely related to the above agent-based models, Anufriev and Dindo (2010) study the wealth dynamics when dividends grow at an exogenous rate. Chiarella and He (2001) and Anufriev and Bottazzi (2010) consider related models in which dividends grow at an endogenous rate, entailing constant dividend yields. These papers show that the assumption on the dividend process matters for price dynamics and market selection.

Introducing a money-market account (an asset which has no price risk) in the evolutionary finance model of Evstigneev, Hens and Schenk-Hoppé (2006) brings the model closer to practical finance and to classical asset pricing models. But it entails the difficulty of having to deal with an asset that is in unlimited supply. This is in contrast to the limited (exogenous) supply of the risky assets. The main issue arises from the fact that in this situation there is, in general, no simple equation describing the dynamics of aggregate wealth. The change in the aggregate wealth between two periods in time depends on the total investment in the risk-free asset and thus becomes a function of the investment strategies. Therefore one does not obtain a system of equations with dimension equal to the number of investors minus one when normalizing wealth by dividing with the aggregate wealth.

The present paper is also related to Hens and Schenk-Hoppé (2006). In that paper a model with only one risky asset, two investors and without consumption is studied. While our paper studies the local stability of the wealth dynamics in a more general model, Hens and Schenk-Hoppé (2006) obtain a global convergence result. Their main finding is that holding the risk-free bond does not ensure survival if the other investor only holds the risky asset. This assertion is proved under the assumption that either (a) there is no consumption or (b) that the gross return of the bond is dominated by the ratio of the dividend rate and the consumption rate in all states of the world. (The dividend rate is the total amount of dividends paid in a period divided by the total wealth of all the investors. The consumption rate is the ratio of the amount spent on consumption and the investor's wealth.) The interpretation of this finding relates to Tobin (1958) who argued that in the face of potential capital losses on bonds it is reasonable to hold cash as a means to transfer wealth over time. The result in Hens and Schenk-Hoppé (2006) shows that Tobin's argument is not correct from an evolutionary perspective.

Another innovation on this paper is that we also model changes in the net supply of risky assets over time and an asset-specific tax rate. Variations in the free-float of assets are quite common (e.g. through stock issues or buy-backs) and induce tracking errors in index-trackers. Taxes often vary substantially across asset classes and time, both of these features are captured in our model.

Finally it is worth mentioning the growing literature on empirical applications of evolutionary finance models, e.g., Hens, Schenk-Hoppé and Stalder (2002), Hens and Schenk-Hoppé (2004), and Hens, Lensberg, Schenk-Hoppé and Woehrmann (forth.), and the new approaches addressing the specification of the investors' strategies which is one of the main challenges in evolutionary finance models. One approach is to let strategies evolve as well.

A successful approach in this direction is applied in Lensberg and Schenk-Hoppé (2007) who use genetic programming to describe a process in which investors progressively improve their skills through imitation and repeated trial-and-error.

The paper is organized as follows. Section 2 presents the model, Section 3.1 derives the dynamics of investors' asset holdings and wealth, Section 3.2 proves existence of short-run equilibrium, and Section 3.3 derives a representation of the dynamics as a random dynamical system. Section 4 presents in detail the conditions for local stability and provides examples, and Section 5 concludes.

2 Model

We consider a market in which a risk-free and several risky assets are traded at discrete points in time $t = 0, 1, ...$ The $K \geq 1$ *risky assets* (*securities*) $k = 1, ..., K$ are risky in terms of their price and payoff. The total supply of asset k at date t is denoted by $V_{t,k}$ and the payoff (dividend paid as cash) of one unit of this asset is given by $D_{t+1,k}$. The market prices $p_{t,k}$ of these assets are determined *endogenously* through short-run equilibrium of supply and demand.

The risk-free asset has no price risk and is in unlimited supply. Its price is *exogenous*. We will use this price as a numeraire (cash) and express the market values of all the assets in the market in terms of this price. We will refer to holdings of the risk-free asset as balances in a bank account with the net interest rate β_t.

There are $N \geq 1$ *investors* (*traders*) acting in the market. Each investor $i = 1, ..., N$ has an initial endowment $w_0^i > 0$, which is cash. A *portfolio* of investor i at date $t = 0, 1, ...$ is specified by a vector $x_t^i = (x_{t,0}^i, x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^{K+1}$, where $x_{t,0}^i$ is the amount in the investor's bank account and $x_{t,k}^i$ ($k = 1, ..., K$) is the number of units of asset k held by the investor at time t. We do not allow short-selling which would entail a bankruptcy risk.

The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, ...,$ where s_t is a random element of a measurable space S_t. Asset prices $p_{t,k}$ and investors‘ portfolios x_t^i depend in general on the history

$$s^t := (s_1, ..., s_t)$$

of this process up to date t. Measurability (which is of vital importance in studying models with random components) of all the quantities defined in the description of the model will be discussed in Section 3.2.

An *investment (trading) strategy* of each investor i at date $t \geq 0$ is characterized by a vector of *investment proportions* $(\lambda^i_{t,0}, \lambda^i_{t,1}, ..., \lambda^i_{t,K})$, $\lambda^i_{t,k} = \lambda^i_{t,k}(s^t)$, according to which he/she plans to distribute the available budget between the assets and the bank account. Vectors $(\lambda^i_{t,0}, \lambda^i_{t,1}, ..., \lambda^i_{t,K})$ belong to the unit simplex

$$\Delta := \{(a_0, ..., a_K) \geq 0 : \; a_0 + ... + a_K = 1\}.$$

The condition rules out short positions.

Remark 1. We denote investment strategies by the vector of proportions $\lambda^i_{t,k}$, $k = 1, ..., K$, which are assigned to the assets with endogenous prices. Since strategies have to be in Δ one has that for a given $\lambda^i_t = (\lambda^i_{t,1}, ..., \lambda^i_{t,K})$, the relation $\lambda^i_{t,0} = 1 - \sum_{k=1}^K \lambda^i_{t,k}$ holds (with $\lambda^i_{t,0} \geq 0$).

At each date $t + 1 = 1, 2, ...$ the dividend $D_{t+1,k} \geq 0$ paid by one unit of asset k depends on the history of states of the world and the aggregate wealth of investors, $\bar{w}_t = w^1_t + ... + w^N_t$, where w^i_t is the wealth of investor i at time t. We model dividends as:

$$D_{t+1,k} = d_{t+1,k}\, \bar{w}_t \tag{1}$$

where the functions $d_{t+1,k} = d_{t+1,k}(s^{t+1})$ are assumed to satisfy

$$\sum_{k=1}^{K} d_{t+1,k} > 0. \tag{2}$$

This condition means that at each date and in each random situation at least one asset pays a strictly positive dividend. The total amount (the number of units) of asset k available in the market at date t is given by $V_{t,k} = V_{t,k}(s^t) > 0$.

Define

$$p_t = (1, p_{t,1}, ..., p_{t,K}),$$

where $p_{t,k}$ are the asset prices. The prices $p_{t,1}, ..., p_{t,K}$ will be determined endogenously. The price of the bond is set to $p_{t,0} = 1$ for all dates t.

The scalar product

$$\langle p_t, x^i_t \rangle = \sum_{k=0}^{K} p_{t,k} x^i_{t,k}$$

expresses the value of the investor i's portfolio x^i_t at date t in terms of the asset prices $p_{t,k}$.

At date $t = 0$ the investors have initial endowments $w^i_0 > 0$ ($i = 1, 2, ..., N$) in their bank accounts. These are their budgets at date 0. Investor i's total budget at date $t \geq 1$ is $B^i_t := \langle D_t + p_t, x^i_{t-1} \rangle$, where

$$D_t := (D_{t,0}, ..., D_{t,K}), \; D_{t,0} = \beta_t.$$

Here, the function $D_{t,0} = \beta(s^t) \geq 0$ represents the net interest rate paid on holdings in the bank accounts of all the investors at time t depending on the random situation s^t. The components $D_{t,k}$, $k = 1, ..., K$ are the (endogenously determined) dividends of the assets. The budget consists of two parts: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i (including interest) and the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i expressed in terms of the vector of today's prices p_t. The prices $p_{t,k}$, $k = 1, ..., K$ are defined below in terms of equilibrium between supply and demand.

Investment in each of the assets and holdings in the bank account are taxed at some rate $0 \leq \tau_{t,k}(s^t) < 1$, $k = 0, 1, ..., K$, the same for all the traders. The fraction of wealth actually invested in asset k is then $\alpha_{t,k} := 1 - \tau_{t,k}(s^t)$. In practice different asset classes are often taxed differently. One can also interpret the tax rate $\tau_{t,k}(s^t)$ as a consumption rate and, accordingly $\alpha_{t,k}$ as a saving rate. We assume that the functions $\alpha_{t,k}$ satisfy

$$\alpha_{t,k} < V_{t,k}/V_{t-1,k} \quad \text{for } k = 1, ..., K. \tag{3}$$

This condition holds, in particular, when the total number $V_{t,k}$ of each asset k does not decrease, i.e., when the right-hand side of (3) is not less than one. But assumption (3) also allows for a situation with decreasing $V_{t,k}$, as long as it does not decrease faster than $\alpha_{t,k}$. There is no condition on the growth rate of asset supply for the risk-free asset whose supply is infinitely elastic.

3 Equilibrium

3.1 Dynamic equilibrium

The random dynamics of prices and investors' portfolios and wealth is derived as a dynamic equilibrium in the model described above. Suppose the strategies $\lambda^1, ..., \lambda^N$ of all the investors, $\lambda^i = (\lambda_t^i(s^t))_{t,s^t}$, and their initial endowments $w_0^1, ..., w_0^N$ are given. The dynamic equilibrium is defined recursively by moving from time t to time $t+1$, starting at the initial time $t = 0$. The dynamics is random because it depends on the most current realization of the random component s_{t+1} which, in particular, determines the actual dividend payments.

At each date $t = 0, 1, ...$ each investor i possess wealth w_t^i and has selected some investment proportions $(\lambda_{t,0}^i, \lambda_{t,1}^i, ..., \lambda_{t,K}^i) \in \Delta$. The amount of cash invested in asset k by trader i is $\alpha_{t,k}\lambda_{t,k}^i w_t^i$ and the total amount invested in asset k is $\alpha_{t,k} \sum_{i=1}^N \lambda_{t,k}^i w_t^i$. The amount deposited with i's bank account is $\alpha_{t,0}\lambda_{t,0}^i w_t^i$ and the total amount kept by the investors in the bank is $\alpha_{t,0} \sum_{i=1}^N \lambda_{t,0}^i w_t^i$.

It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{t,k}$ of each asset $k = 1, ..., K$ from the equations

$$p_{t,k} V_{t,k} = \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i, \ k = 1, ..., K. \tag{4}$$

On the left-hand side of (4) we have the total value $p_{t,k}V_{t,k}$ of all the assets of the type k in the market (recall that the amount of each asset k at date t is $V_{t,k}$). The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (4). The price of the risk-free asset (bank account) is exogenous and set to $p_{t,0} = 1$; there is no market clearing condition for this asset.

The investment proportions $(\lambda_{t,0}^i, ..., \lambda_{t,K}^i)$ chosen by the traders $i = 1, ..., N$ at date t also determine their portfolios $(x_{t,0}^i, ..., x_{t,K}^i)$ at date t by the formula

$$x_{t,k}^i = \frac{\alpha_{t,k} \lambda_{t,k}^i w_t^i}{p_{t,k}}, \ k = 0, 1, ..., K. \tag{5}$$

Here, $x_{t,0}^i = \alpha_{t,0} \lambda_{t,0}^i w_t^i$ specifies the amount held in investor i's bank account. Formula (5) states that the current market value $p_{t,k}x_{t,k}^i$ of the kth position of the portfolio x_t^i of investor i is equal to the taxed fraction $\lambda_{t,k}^i$ of the i's investment budget w_t^i.

The wealth w_t^i of traders $i = 1, 2, ..., N$ are defined recursively. At the initial time $t = 0$, the wealth w_0^i of all the investors are given constants. But at each time $t = 1, 2, ...,$ the wealth is given by

$$w_t^i = \sum_{k=0}^{K} (D_{t,k} + p_{t,k}) x_{t-1,k}^i. \tag{6}$$

For the model to be well-defined, one needs to prove that the system of equations (4)–(6) possesses a unique, strictly positive price process $(p_{t,1}, ..., p_{t,K})$ (recall that $p_{t,0} \equiv 1$). The next section gives conditions ensuring its existence and uniqueness.

3.2 Existence of short-run equilibrium

Short-run equilibrium corresponds to the existence of a price system such that the market for each asset $k = 1, ..., K$ clears in each period in time. Portfolios as defined in (5) further require that $p_{t,k} > 0$, or equivalently, that the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Finally, measurability of all functions and prices needs to be considered and either assumed to proved. Measurability is vital when studying random systems as without it probability of events may not be well-defined.

(A) There is one investor, say i, with $w_0^i > 0$ and $\lambda_{t,k}^i > 0$ for $k = 1, ..., K$.

(B) The following functions of s^t are assumed to by measurable: investment strategies λ_t^i, asset supply $V_{t,k}$, tax rates $\tau_{t,k}$ $(t \geq 0)$, dividend rates $d_{t,k}$ and interest rate β_t $(t \geq 1)$.

Proposition 1 *Under assumption (A) there is a unique vector $p_t = (p_{t,1}, ..., p_{t,K})$ with $p_{t,k} > 0$ for all $k = 1, ..., K$ such that*

$$p_{t,k}V_{t,k} = \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i \sum_{m=0}^{K} (D_{t,m} + p_{t,m}) x_{t-1,m}^i, \; k = 1, ..., K \tag{7}$$

(recall that $p_{t,0} \equiv 1$). The solution is a measurable function if, in addition, assumption (B) holds.

Proof. We use a contraction argument to prove the assertion in Proposition 1. The price of the risk-free asset is set to $p_{t,0} = 1$ by definition. Therefore only prices $p_{t,k}$ with $k = 1, ..., K$ need to be considered.

Fix some s^t and consider the operator transforming a vector $p = (p_1, ..., p_K) \in R_+^K$ into the vector $q = (q_1, ..., q_K) \in R_+^K$ with coordinates

$$q_k = V_{t,k}^{-1} \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i \langle D_t + \tilde{p}, x_{t-1}^i \rangle, \; k = 1, ..., K,$$

where

$$\tilde{p} = (1, p_1, ..., p_K).$$

This operator is contracting in the norm $||p||_V := \sum_{k=1}^{K} |p_k| V_{t-1,k}$. Indeed, by virtue of assumption (A) we have

$$\alpha := \max_{k=1,...,K} \{\alpha_{t,k} V_{t-1,k} V_{t,k}^{-1}\} < 1,$$

and so

$$||q - q'||_V = \sum_{k=1}^{K} |q_k - q'_k| V_{t-1,k} \leq$$

$$\sum_{k=1}^{K} V_{t-1,k} V_{t,k}^{-1} \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \leq \alpha \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda_{t,k}^i |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \leq$$

$$\alpha \sum_{i=1}^{N} |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \leq \alpha \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p'_m| x_{t-1,m}^i =$$

$$\alpha \sum_{m=1}^{K} |p_m - p'_m| \sum_{i=1}^{N} x_{t-1,m}^i = \alpha \sum_{m=1}^{K} |p_m - p'_m| V_{t-1,m} = \alpha ||p - p'||_V,$$

where the last but one equality follows from (5). (Note that $\tilde{p}_0 - \tilde{p}'_0 = 0$.) By using the contraction principle, we obtain the existence and uniqueness of the solution to (7). Starting the iteration with $p = (1, 0, ..., 0)$, one finds that the solution must be non-negative.

Under assumption (A) the solution to (7) has all components strictly positive at each period in time t and for each history s^t. This can be seen as follows. Let p_0 be the non-negative solution to (7) with time $t = 0$. Then (A) implies that there is an index i such that $x_{0,m}^i > 0$ for $m = 1, ..., K$. Since $D_{1,m} > 0$ for at least one m, $\lambda_{1,k}^i$ for $k = 1, ..., K$ and $\alpha_{1,k} > 0$ for $k = 1, ..., K$, one has that the right-hand side of (7), and thus the left-hand side is strictly positive. Therefore $p_{1,k} > 0$ for all k. This, in turn, implies $x_{1,m}^i > 0$ for $m = 1, ..., K$, which allows to apply the same argument recursively.

Measurability of the (unique) solution to (7) follows from the fact that it can be expressed, under assumption (B), as the pointwise limit of measurable functions (e.g. when starting the iteration with any constant vector). □

3.3 Wealth dynamics

The random dynamics of the investors' wealth is obtained by combining the dynamic equilibrium relations (Section 3.1) and the existence and uniqueness results on short-run equilibrium (Section 3.2). This section presents the resulting dynamics and derives an explicit representation of the dynamics, which takes on the form of a random map on the space of investors' wealths.

From (4) and (5) we get

$$p_{t,k} = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i = \alpha_{t,k} \frac{\langle \lambda_{t,k}, w_t \rangle}{V_{t,k}}, \ k = 1, ..., K; \tag{8}$$

$$x_{t,k}^i = \frac{V_{t,k} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \ k = 1, ..., K; \tag{9}$$

where $t \geq 0$, $w_t := (w_t^1, ..., w_t^N)$ and $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$, $k = 0, ..., K$.

The relations $p_{t,0} = 1$ and $x_{t,0}^i = \alpha_{t,0} \lambda_{t,0}^i w_t^i$ (with $t \geq 1$) can be written in the same form by setting

$$V_{t,0} := \alpha_{t,0} \langle \lambda_{t,0}, w_t \rangle, \tag{10}$$

which gives

$$p_{t,0} = \alpha_{t,0} \frac{\langle \lambda_{t,0}, w_t \rangle}{V_{t,0}} = 1$$

and

$$x^i_{t,0} = \frac{V_{t,0} \lambda^i_{t,0} w^i_t}{\langle \lambda_{t,0}, w_t \rangle} = \alpha_{t,0} \lambda^i_{t,0} w^i_t.$$

So formulas (8) and (9) for the prices and portfolios are valid for all $k = 0, 1, 2, ...K$.

Consequently, we have

$$w^i_{t+1} = \sum_{k=0}^{K} (p_{t+1,k} + D_{t+1,k}) x^i_{t,k} =$$

$$\sum_{k=0}^{K} (\alpha_{t+1,k} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle}{V_{t+1,k}} + D_{t+1,k}) \frac{V_{t,k} \lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle} =$$

$$\sum_{k=0}^{K} (\alpha_{t+1,k} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k}) \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}.$$

By using the notation

$$\rho_{t+1,k} = \alpha_{t+1,k} V_{t,k} / V_{t+1,k},$$

we write

$$w^i_{t+1} = \sum_{k=0}^{K} \left[\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + (1 - \rho_{t+1,k}) \frac{D_{t+1,k} V_{t,k}}{1 - \rho_{t+1,k}} \right] \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}. \quad (11)$$

The equation expresses the investor's wealth as an aggregate of his position in each asset multiplied by the sum of asset re-sale price (adjusted for re-investment and dilution due to the changes in the number of outstanding shares) and the effective dividend payment of the asset.

The system of equations (11) can be written in more compact vector notation as:

$$[\mathrm{Id} - X_t \Delta \rho_{t+1} \Lambda_{t+1}] \, w_{t+1} = X_t \Delta V_t D_{t+1} + (1 + \beta_{t+1}) \Delta \lambda_{t,0} \, w_t \quad (12)$$

where $X_t \in R^{N \times K}$ is the matrix of all the investors' period-t portfolio holdings in all assets with endogenous prices (the ith row is given by $(x^i_{t,1}, ..., x^i_{t,K})$) with

$$(X_t)_{ik} = \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}, i = 1, ..., N, \; k = 1, ..., K.$$

Moreover, $\Lambda_{t+1} \in R^{K\times N}$ the matrix of the period-$t+1$ investment strategies where column i is given by investor i's investment proportions for the assets with endogenous prices, i.e. the transposed of the vector $(\lambda^i_{t+1,1}, ..., \lambda^i_{t+1,K})$:

$$(\Lambda_{t+1})_{ki} = \lambda^i_{t+1,k}.$$

For every vector $y \in R^K$, we denote by Δy the matrix in $R^{K\times K}$ with entries y_k $(k = 1, ..., K)$ along the diagonal and zero otherwise.

The system of equations (12) is equivalent to

$$[\mathrm{Id} - X_t \Delta\rho_{t+1}\Lambda_{t+1}]\, w_{t+1} = [X_t \Delta V_t D_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0}\, w_t]\,. \qquad (13)$$

The matrix $\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}$ is invertible if $\max_k \rho_{t+1,k} < 1$ because this condition ensures that the diagonal of this matrix is column-dominant. One therefore obtains an equivalent representation in explicit form:

$$w_{t+1} = [\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}]^{-1} [X_t\Delta V_t D_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0}\, w_t]\,. \qquad (14)$$

The interpretation of (14) is straightforward. The wealth of all the investors in period $t+1$ is determined by their individual dividend and interest income, multiplied by a matrix representing the price changes in the assets with endogenous prices.

The system (14) also covers the case where the resale price of all assets (except the bank account) is zero. Assets purchased at time t are only claims to a (random) payoff at time $t+1$ but they have to resale value (like a lottery ticket after the draw). Setting $\rho_{t+1,k} = 0$, $k = 1, ..., K$, the matrix on the right-hand side of (14) becomes the identity matrix. The interpretation is that assets are re-issued in each period in time, cf. Evstigneev, Hens and Schenk-Hoppé (2008).

The dynamics (14) can be represented by the iteration of a random map:

$$w_{t+1} = h_{t+1}(s^{t+1}, w_t).$$

It might be important to point out for clarity that, in general, this dynamics is not given by the iteration of i.i.d. maps because the process s^t is arbitrary.

In the following, we will assume for simplicity of presentation that:

(C) $V_{t,k} = 1$ for $k = 1, ..., K$ and $\alpha_{t+1,k} = \alpha$ for $k = 0, 1, ..., K$, with $0 < \alpha < 1$.

The first condition says that the firm neither issues new shares nor carries out buy-backs. The second assumption says that all asset classes are taxed at the same rate. Under assumption (C) (14) can be written as

$$w_{t+1} = [\mathrm{Id} - \alpha X_t\Lambda_{t+1}]^{-1} [X_t D_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0}\, w_t]\,. \qquad (15)$$

It might be of interest to briefly discuss the case with only one investor. The dynamics (15) with $N = 1$ is equivalent to

$$w^1_{t+1} = \frac{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda^1_{t,0}}{1 - \alpha(1 - \lambda^1_{t+1,0})}\, w^1_t \tag{16}$$

where $\bar{d}_{t+1} = \sum_{k=1}^{K} d_{t+1,k}$ (with $D_{t+1,k} = d_{t+1,k} w^1_t$). The endogenous prices of assets $k = 1, ..., K$ are given by $p_{t,k} = \lambda^1_{t,k} w^1_t$ which implies that the random dividend yield is $D_{t+1,k}/p_{t,k} = d_{t+1,k}/\lambda^1_{t,k}$ and the market valuation of asset k relative to asset j is given by $p_{t,k}/p_{t,j} = \lambda^1_{t,k}/\lambda^1_{t,j}$.

Whether the wealth in the market on average grows, declines or does not exhibit a trend depends on the growth rate of the random coefficient of w^1_t on the right-hand side of (16).

4 Local stability and evolutionarily stable strategies

The local stability of steady states in which one investor owns the entire wealth is studied by applying recent results by Evstigneev, Pirogov and Schenk-Hoppé (2011). Their paper presents conditions on the linearized system at an equilibrium of random dynamical systems which ensure local stability. The method will be useful in the study of other economic and financial dynamic models involving randomness. The analysis presented here can serve as an illustration of a mathematically correct approach to the study of local stability in stochastic systems.

In the study of the local stability of the wealth dynamics, we focus on the case of two investors. This simplifies the presentation without sacrificing generality (see Remark 3 below). Local stability in the two-investor case is concerned with the state of the dynamics in which one investor (say, investor 1) has no wealth, and the other investor (investor 2) has strictly positive wealth. It is obvious from (15) that this situation is invariant under the dynamics. Local stability means that the dynamics starting from a state in which investor 1 is provided with a (sufficiently) small amount of wealth will asymptotically revert to the state where investor 1 possesses no wealth. The comparison of the performance of two investment strategies ($N = 2$) will be carried out by analyzing the ratio of their wealths which is described by a one-dimensional random dynamics. The case of an arbitrary number of investors is briefly discussed in a remark. It turns our that there are no essential differences between the two and the N investor case. The local stability conditions in the latter case are characterized by $N-1$ independent conditions that each take the form of a condition in the two investor case.

On an intuitive level the concept of local stability applied here is closely related to the notion of stability in evolutionary game theory, see e.g., Weibull (1995). In that setting, one is interested in the population dynamics where individual players (typically a continuum) can follow two different strategies. Stability is defined under given a replicator dynamics which describes the frequency of the two types in the population. Given a situation in which most players follow one strategy (the incumbent strategy) and a small number of players follow the other "mutant" strategy, the latter group asymptotically becomes extinct. While evolutionary game theory considers population sizes, we study evolution *in pecunia*.

4.1 Dynamics of wealth ratio in the case of two investors

The dynamics of the ratio of the two investors' wealth can be derived from the system (15). This ratio compares the wealth of one investor relative to that of the other. Let $N = 2$, and define the ratio of the investors' wealth as

$$z_t := w_t^1/w_t^2.$$

The ratio is well-defined if investor 2 is fully diversified, i.e., $\min_k \lambda_{t,k}^2(s^t) > 0$ for all t, s^t, and he has strictly positive initial wealth, $w_0^2 > 0$. This is condition (A), assuming that $i = 2$ (if needed after relabeling of the investors). Therefore the process $w_t = (w_t^1, w_t^2)$ is well-defined with $w_t^1 \geq 0$ and $w_t^2 > 0$.

We now demonstrate that the random dynamic of the process z_t is one-dimensional. The map describing this dynamic is derived from (15) as follows. Denote the strategy λ_t^i, $i = 1, 2$, by the transposed of $(\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ which defines the proportions of wealth invested in the assets with endogenous prices, the money market account holdings are determined by $1 - \sum_{k=1}^K \lambda_{t,K}^i$ and w_t^i. The portfolio holdings in the assets with endogenous prices are denoted by $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$.

One has

$$[\mathrm{Id} - \alpha X_t \Lambda_{t+1}]^{-1} = \frac{1}{\det(\mathrm{Id} - \alpha X_t \Lambda_{t+1})} \begin{bmatrix} 1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle & \alpha\langle x_t^1, \lambda_{t+1}^2\rangle \\ \alpha\langle x_t^2, \lambda_{t+1}^1\rangle & 1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle \end{bmatrix}$$

with

$$\det(\mathrm{Id} - \alpha X_t \Lambda_{t+1}) = (1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle)(1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle) - \alpha^2\langle x_t^1, \lambda_{t+1}^2\rangle\langle x_t^2, \lambda_{t+1}^1\rangle.$$

The portfolios of the two investors (in the assets with endogenous prices) can be written as functions of z_t:

$$x_k^1(z_t, \lambda_t) = \frac{\lambda_{t,k}^1 z_t}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}$$

and

$$x_k^2(z_t, \lambda_t) = \frac{\lambda_{t,k}^2}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}.$$

We denote $x_t^i = (x_1^i(z_t, \lambda_t), ..., x_K^i(z_t, \lambda_t))$, $i = 1, 2$.

One further has

$$D_{t+1,k} = d_{t+1,k}(s^{t+1})(w_t^1 + w_t^2) = d_{t+1,k}(s^{t+1})(z_t + 1)w_t^2$$

for $k = 1, ..., K$, and

$$w_t = \begin{pmatrix} w_t^1 \\ w_t^2 \end{pmatrix} = w_t^2 \begin{pmatrix} z_t \\ 1 \end{pmatrix}.$$

This yields

$$X_t D_{t+1} + (1+\beta_{t+1})\Delta\lambda_{t,0}\, w_t = \left[(z_t + 1)X_t d_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0} \begin{pmatrix} z_t \\ 1 \end{pmatrix}\right] w_t^2.$$

These considerations show that the ratio $z_{t+1} = w_{t+1}^1/w_{t+1}^2$, where (w_{t+1}^1, w_{t+1}^2) is uniquely defined by (15) for a given $z_t \geq 0$ and a $w_t^2 > 0$, is independent of w_t^2.

Combining the above, one obtains (after some lengthy but elementary mathematical operations) the dynamic of the process z_t:

$$z_{t+1} = \frac{\begin{array}{c}[1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle][(z_t + 1)\langle x_t^1, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^1 z_t] \\ +\alpha\langle x_t^1, \lambda_{t+1}^2\rangle[(z_t + 1)\langle x_t^2, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^2]\end{array}}{\begin{array}{c}\alpha\langle x_t^2, \lambda_{t+1}^1\rangle[(z_t + 1)\langle x_t^1, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^1 z_t] \\ +[1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle][(z_t + 1)\langle x_t^2, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^2]\end{array}}. \quad (17)$$

Equation (17) shows that the ratio of the two investor's wealth can indeed be described as a one dimensional stochastic system. The right-hand side can be interpreted as follows. The ratio of the investor 1's wealth relative to that of the other investor 2 is driven by a comparison of the investor's dividend and interest income, adjusted for the impact of the change in the endogenous asset prices. The representation makes use of the fact that the units of an asset not owned by investor 2 must be in the possession of investor 1.

The stochastic equation (17) defines a random dynamical system in discrete time on the space R_+ of non-negative numbers. That is, for a given state $z_t \in R_+$, the right-hand side of (17) defines the state of the system, $z_{t+1} \in R_+$, at the next point in time $t+1$. This new state depends on the realization of the exogenous shock s_{t+1} and time. In this way, (17) generates a random path from a given initial state $z_0 \geq 0$. For the general theory of random dynamical system the mathematically-minded reader is referred to Arnold (1998). A survey of this theory within the economic context is provided, e.g., in Schenk-Hoppé (2001).

We make the following assumption:

(D) Strategies, idiosyncratic risk of asset payoffs and interest rate only depend on the process s^t, i.e., $\lambda_t(s^t) = \lambda(s^t)$, $d_{t,k}(s^t) = d_k(s^t)$ and $\beta_t(s^t) = \beta(s^t)$, and the process s^t is stationary and ergodic (with invariant probability measure P).

Under assumptions (A)–(D), we can express the equation (17) as

$$z_{t+1} = f(s^{t+1}, z_t) \tag{18}$$

where the right-hand side is a function of z_t and the process s^{t+1} only.

Observe that the situation in which investor 2 owns all the wealth (and investor 1 owns nothing) is a steady state of (18). If $w_0^1 = 0$ and $w_0^2 > 0$, then $z_0 = w_0^1/w_0^2 = 0$. Indeed this is a steady state because

$$f(s^{t+1}, 0) = 0$$

by the definition of the function f in (17).

Considering the ratio of the investors' wealth $z_t = w_t^1/w_t^2$ needs a little more explanation because the following asymptotic property $z_t \to 0$ does not necessarily imply $w_t^1 \to 0$. The convergence $z_t \to 0$ implies that investor 1's wealth asymptotically becomes small relative to that of investor 2, i.e., the wealth of investor 1 diminishes relative to that of investor 2. This definition does not rule out that the wealth of both investors can grow over time but it says that the wealth of investor 2 grows faster than that of investor 1. Indeed, given a positive net interest rate, wealth can grow without bound. For instance if the investor places an amount of his wealth in the bank account and reinvests all interest income.

4.2 Sufficient conditions ensuring local stability

We now present the conditions ensuring local stability of the state $z = 0$ (in which investor 2 owns all the wealth). The sufficiency of these conditions follows from the recent results presented in Evstigneev, Pirogov and Schenk-Hoppé (2011). The findings in that paper relevant to the current case are Theorem 2 and Remark 3.

In the following we assume that

$$E \ln \min_{k=0,\ldots,K} \lambda^2_{t,k} > -\infty; \tag{19}$$

$$E \ln^+ \beta_{t+1} < \infty; \text{ and } E \ln^+ d_{t+1,k} < \infty \text{ for } k = 1, ..., K. \tag{20}$$

Recall the dividend payment per unit of asset k is given by $D_{t+1,k} = d_{t+1,k}\bar{w}_t$ with the aggregate wealth defined as $\bar{w}_t = \sum_{i=1}^N w_t^i$.

These two integrability assumptions can be interpreted as conditions ensuring that everything that happens in the model at exponential speed is caused by the wealth dynamics rather than by changes in the 'ingredients' (investment strategies and dividend and interest payments). Note that for instance, condition (19) is stronger than the assumption $\lambda^2_{t,k} > 0$ for all t, s^t which was needed to ensure that the model is well-defined. Log-integrability conditions of the above type are common in stochastic dynamic models.

Proposition 2 *The steady state $z = 0$ of (18) is locally stable, if*

$$E \ln f'(s^{t+1}, 0) < 0 \tag{21}$$

with $f'(s^{t+1}, 0)$ denoting the derivative of the right-hand side of (18) evaluated at $z = 0$.

Local stability of a stochastic dynamical system is defined here as follows. There exists a neighborhood $U(\omega)$ (a random set), $\omega = (s^t)_{t=0}^\infty \in S^\infty$, of the steady state $z = 0$ such that for almost all ω: for each initial value $z_0 \in U(\omega)$, the sample path $z_t \to 0$. Under condition (21), the convergence is exponentially fast with constant (i.e., non-random) rate given by $E \ln f'(s^{t+1}, 0)$. The condition (21) is analogue to those in deterministic dynamical systems where local stability can be verified by studying the derivative at the steady state. In stochastic dynamic models, the condition can be interpreted as ensuring that the dynamics is locally contracting *on average*, see Evstigneev, Pirogov and Schenk-Hoppé (2011).

The economic interpretation of local stability in the present model is that if $z = 0$ is locally stable, then the 'incumbent' investment strategy λ^2 is locally stable against the 'mutant' strategy λ^1.

The derivative of $f(s^{t+1}, z)$ at $z = 0$ (which defines the linearization of the system (18) at the steady state $z = 0$) can be found after some elementary but lengthy calculations as:

$$f'(s^{t+1}, 0) = \frac{[1 - \alpha(1 - \lambda^2_{t+1,0})][\langle \eta_t, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda^1_{t,0}]}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda^2_{t,0}} + \alpha\langle \eta_t, \lambda^2_{t+1}\rangle \tag{22}$$

where

$$\eta_{t,k} = \frac{\lambda^1_{t,k}}{\lambda^2_{t,k}},\ k = 1, ..., K, \tag{23}$$

and $d_{t+1} = (d_{t+1,1}, ..., d_{t+1,K})$. Note that the derivative $f'(s^{t+1}, 0) > 0$.

Proof of Proposition 2. The proof is an the application Theorem 1 in Evstigneev, Pirogov and Schenk-Hoppé (2011) which requires to verify the conditions (B1) and (B2) defined in Section 2 of their paper.

First note that the dynamics (18) is also well-defined for a larger set values of z_t than $[0, \infty)$ because the right-hand side of this equation actually makes sense for negative values of z_t as well—provided they are not too small. This property is useful because it ensures that the derivative of the dynamics at $z = 0$ can be understood in the usual sense rather than as a directional derivative: by extending the dynamics (18) to the space $X = (-\infty, \infty)$, as we do in the next paragraph, the point $z = 0$ becomes an interior point.

The extension can be done as follows: Note that the right-hand side of (18) is well-defined even for negative $z_t \geq -\varepsilon(s^t)$, as long as $\lambda^2_{t,k} - \varepsilon(s^t)\lambda^1_{t,k} > 0$ (which ensures that the portfolios are well-defined) and the denominator is strictly positive. The first condition is satisfied for each $\varepsilon(s^t) < \min_k(\lambda^2_{t,k}/\lambda^1_{t,k})$. Existence of an $\varepsilon(s^t) > 0$ such that the second condition holds follows from the fact that the denominator is continuous in z_t and that it is larger than $(1-\alpha)(1+\beta_{t+1})\lambda^2_{t,0} > 0$ for $z_t = 0$. Let us assume a suitable $\varepsilon(s^t) > 0$ is chosen and fixed. Define the dynamics by (18) for all $z_t > -\varepsilon(s^t)$ and by -1 (or any other constant) for $z_t \leq -\varepsilon(s^t)$. The stochastic dynamics is then well-defined on the set $X = (-\infty, \infty)$, and the (random but trivial) set $X(\omega) = [0, \infty) \subset X$ is invariant under the dynamics.

Condition (B1) in Evstigneev, Pirogov and Schenk-Hoppé (2011) requires that there exist random variables $L(\omega)$ and $\delta(\omega)$ with $E|\ln L| < \infty$ and $E|\ln \delta| < \infty$ such that

$$|f(s^t, z) - f(s^t, 0)| \leq L(\omega)|z - 0|$$

for all $z \in X(\omega)$ with $0 \leq z \leq \delta(\omega)$, where $\omega = (s^t)^\infty_{t=0}$. This condition can be interpreted as local Lipschitz continuity with a log-integrable Lipschitz 'constant.'

Fix any constant $\delta \in (0, 1)$ and let $\delta(\omega) \equiv \delta$. Since $f(s^t, 0) = 0$ and $f(s^t, z) \geq 0$ for $z \geq 0$, the above inequality is equivalent to

$$\frac{f(s^t, z)}{z} \leq L(s^t)$$

for all $z > 0$. Indeed, for every $z_t > 0$, one has

$$\frac{f(s^t, z_t)}{z_t} \leq \frac{(z_t + 1)\langle \zeta^1_t, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda^1_{t,0}}{(1-\alpha)(1+\beta_{t+1})\lambda^2_{t,0}} + \frac{\alpha\langle \zeta^1_t, \lambda^1_{t+1}\rangle}{1 - \alpha\langle x^1_t, \lambda^1_{t+1}\rangle}$$

where

$$\zeta_{t,k}^1 = \frac{x_{t,k}^1}{z_t} = \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}.$$

Since $\zeta_{t,k}^1 \le M(s^t) := \max_k(\lambda_{t,k}^1/\lambda_{t,k}^2)$ and $\beta_{t+1} \ge 0$ one has

$$\frac{f(s^t, z_t)}{z_t} \le 2\frac{M(s^t)\bar{d}_{t+1} + 1 + \beta_{t+1}}{(1-\alpha)\lambda_{t,0}^2} + \frac{\alpha M(s^t)}{1-\alpha}$$

where it is used that $(z_t + 1)\langle \zeta_t^1, d_{t+1}\rangle \le 2M(s^t)\bar{d}_{t+1}$ for $0 \le z_t \le \delta < 1$ and that $\langle \zeta_t^1, \lambda_{t+1}^1\rangle \le M(s^t)$. The right-hand side of the last inequality is not less than one and it therefore suffices to verify that

$$E \ln^+ \left[2\frac{M(s^t)\bar{d}_{t+1} + 1 + \beta_{t+1}}{(1-\alpha)\lambda_{t,0}^2} + \frac{\alpha M(s^t)}{1-\alpha}\right] < \infty.$$

But this property follows from the above integrability assumptions (19) and (20).

Condition (B2) in Evstigneev, Pirogov and Schenk-Hoppé (2011) is identical to (21) because $f'(s^{t+1}, 0) > 0$ (see (22)). Therefore their Theorem 1 implies the assertion in the Proposition. □

Remark 2. Condition (21) is in particular satisfied if

$$E_{s^t} \ln f'(s^{t+1}, 0) \le 0 \quad \text{for } P \text{ almost all } s^t \tag{24}$$

and the inequality is strict on a set of s^t which has strictly positive probability. Here $E_{s^t} = E[\cdot \mid s^t]$ denotes the conditional expectation under the invariant measure P.

Remark 3. We briefly comment on the N investor case. Consider a market with N investment strategies, $\lambda^1, ..., \lambda^N$. In contrast to the above one has $N-1$ ratios $z_t^i = w_t^i/w_t^N$. One can work out the analogous steps to the above and derive sufficient conditions for the local stability of $z = (0, ..., 0)$ using the results in Evstigneev, Pirogov and Schenk-Hoppé (2011). It turns out that these $N-1$ conditions correspond to pairwise comparisons of λ^N and λ^i, $i = 1, ..., N-1$, each of which takes the form (21).

4.3 Evolutionarily stable strategy

In this section we derive an investment strategy $\lambda^*(s^t)$ such that the dynamics (17) whose local properties we studied in the previous section has the following property: if investor 2 follows strategy $\lambda^*(s^t)$ and investor 1 uses a different strategy $\lambda^1(s^t)$, then the state $z = 0$ is locally stable. 'Different' means $P(\lambda^*(s^t) = \lambda^1(s^t)) < 1$ (ergodicity of P implies that this probability is independent of t). We call a strategy with this property a (locally) *evolutionarily stable strategy*.

The economic interpretation of this property is that an incumbent λ^*-investor is *unbeatable*. If his 'mutant' competitor plays any different investment strategy, then the wealth ratio reverts to zero (locally). If the mutant also plays λ^*, then the wealth ratio remains equal to its initial value. A discussion of unbeatable investment strategies in a full game-theoretic evolutionary finance model is provided in Amir, Evstigneev and Schenk-Hoppé (2010).

The method to derive conditions characterizing (and hopefully being able to fully identify) evolutionarily stable strategies is as follows. One shows that the derivative (24) is a (strictly) concave function of $\lambda^1(s^t)$ for any given investment strategy $\lambda^2(s^t)$. Therefore there is a 'best response' strategy where best response refers to the growth rate obtained by choosing $\lambda^1(s^t)$ for given $\lambda^2(s^t)$. Any evolutionarily stable strategy is characterized by the fact that this best response to the investment strategy $\lambda^2 = \lambda^*$ is λ^* itself (for all s^t). We now carry out this program and derive conditions characterizing evolutionarily stable strategies.

For convenience we denote by $\lambda^i(s^t)$ the vectors of all investment proportions, $\lambda^i(s^t) = (\lambda^i_0(s^t), \lambda^i_1(s^t), ..., \lambda^i_K(s^t))$.

First, note that for a given process $\lambda^2(s^t)$, the map

$$\lambda^1(s^t) \mapsto E_{s^t} \ln f'(s^{t+1}, 0), \quad \Delta \to [-\infty, \infty]$$

is strictly concave for each s^t if the right-hand side of (22) is not constant in s_{t+1} on a set of strictly positive P-measure. Second, observe that for $\lambda^1(s^t) = \lambda^2(s^t)$, $f'(s^{t+1}, 0) = 1$ for all s^{t+1} and thus $E_{s^t} \ln f'(s^{t+1}, 0) = 0$.

One has the conditions

$$\frac{\partial E_{s^t} \ln f'(s^{t+1}, 0)}{\partial \lambda^1_k(s^t)}\Big|_{\lambda^1(s^t)=\lambda^*(s^t)} = c \quad k = 0, 1, ..., K. \tag{25}$$

with c a constant. This constant is determined by the condition $\sum_{k=0}^K \lambda^*_{t,k} = 1$ as follows: Using (22), one finds

$$E_{s^t}\left([1 - \alpha(1 - \lambda^*_{t+1,0})] \frac{1 + \beta_{t+1}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda^*_{t,0}} \right) = c \tag{26}$$

and, for $k = 1, .., K$,

$$E_{s^t}\left([1 - \alpha(1 - \lambda^*_{t+1,0})] \frac{d_{t+1,k}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda^*_{t,0}} + \alpha\lambda^*_{t+1,k} \right) = c\lambda^*_{t,k}. \tag{27}$$

Adding (27) over $k = 1, ..., K$ and (26) (after multiplying with $\lambda^*_{t,0}$) one obtains

$$c(\sum_{k=0}^{K} \lambda^*_{t,k}) = E_{s^t}\left([1 - \alpha(1 - \lambda^*_{t+1,0})] \frac{\bar{d}_{t+1} + (1+\beta_{t+1})\lambda^*_{t,0}}{\bar{d}_{t+1} + (1+\beta_{t+1})\lambda^*_{t,0}} + \alpha \sum_{k=1}^{K} \lambda^*_{t+1,k} \right). \tag{28}$$

Therefore the constant $c = 1$.

The investment strategy λ^*_t is obtain by first determining $(\lambda^*_{t,0})$ by solving

$$E_{s^t}\left([1 - \alpha(1 - \lambda^*_{t+1,0})] \frac{1 + \beta_{t+1}}{\bar{d}_{t+1} + (1+\beta_{t+1})\lambda^*_{t,0}} \right) = 1 \tag{29}$$

(subject to the constraint that $0 \le \lambda^*_{t,0} \le 1$ for all t) and, using this solution, to solve for $k = 1, .., K$

$$E_{s^t}\left([1 - \alpha(1 - \lambda^*_{t+1,0})] \frac{d_{t+1,k}}{\bar{d}_{t+1} + (1+\beta_{t+1})\lambda^*_{t,0}} + \alpha \lambda^*_{t+1,k} \right) = \lambda^*_{t,k}. \tag{30}$$

This procedure determines an investment strategy through the solution $\lambda^*(s^t) \in \Delta$ obtained for each s^t. By construction the strategy $\lambda^*(s^t)$ is (locally) evolutionary stable in the sense that the wealth of an investor following this strategy is locally stable in a market in which the other investor uses any strategy different to $\lambda^*(s^t)$.

It follows from (30) that $\lambda^*_{t,k} > 0$ for all $k = 1, ..., K$ (and from (29) that $\lambda^*_{t,0} < 1$). However, in general the holdings in the money market can be zero, i.e., it can happen that $\lambda^*_{t,0} = 0$.

4.4 Example with explicit solution

An explicit solution to the problem of finding a locally stable investment strategy can be given under certain conditions. Let us assume that the process s^t is Markovian, and that the aggregate dividend factor $\bar{d}(s^{t+1}) = \sum_{k=1}^{K} d_k(s^{t+1})$ and the interest rate $\beta(s^{t+1})$ are both constants denoted by $\bar{d}$ and β, respectively. Under these conditions, (29) is equivalent to

$$(1+\beta)(1 - \alpha + \alpha E_{s_t} \lambda^*_{t+1,0}) = \bar{d} + (1+\beta)\lambda^*_{t,0}$$

which has the constant solution

$$\lambda^*_0 = 1 - \frac{\bar{d}}{(1+\beta)(1-\alpha)}.$$

This solution is interior (i.e., $\lambda_0^* \in (0,1)$) if $\bar{d} < (1+\beta)(1-\alpha)$. Otherwise one obtains a 'corner solution' in which the investor does not place money in the money market. The first case has been studied in Hens and Schenk-Hoppé (2006) where a global convergence result is obtained in the case of only one risky asset.

Inserting the last term in (30) gives

$$\lambda_{t,k}^* - \alpha E_{s_t} \lambda_{t+1,k}^* = E_{s_t} \left((1 - \alpha\lambda_0^*) \frac{d_{t+1,k}}{\bar{d} + (1+\beta)\lambda_0^*} \right).$$

The right-hand side of this equation is equal to

$$\frac{1 - \alpha(1 - \lambda_0^*)}{\bar{d} + (1+\beta)\lambda_0^*} E_{s_t} d_{t+1,k} = \frac{1}{1+\beta} E_{s_t} d_{t+1,k}.$$

Thus, one needs to solve

$$\lambda_{t,k}^* - \alpha E_{s_t} \lambda_{t+1,k}^* = \frac{1}{1+\beta} E_{s_t} d_{t+1,k}$$

which has the solution

$$\lambda_{t,k}^* = \frac{1}{1+\beta} \sum_{m=1}^{\infty} \alpha^{m-1} E_{s_t} d_{t+m,k}. \tag{31}$$

This investment strategy allocates wealth across all of the available assets in proportions corresponding to the discounted expected (relative) dividend payoffs. The asset valuation implied by this strategy is a net present value in relative terms.

5 Conclusion

In many applications local stability of stochastic dynamical systems is often verified numerically or by using incorrect methods. The main aim of this paper is to illustrate, in a financial market model, a mathematically correct technique to prove local stability of steady states in random dynamical systems. The approach applies recent results in Evstigneev, Pirogov and Schenk-Hoppé (2011). The model analyzed generalizes the evolutionary finance model introduced in Evstigneev, Hens and Schenk-Hoppé (2006) by including a money market account. The local stability conditions derived here are used to describe an evolutionary stable investment strategy. An explicit solution is derived in the Markov case. We also provide a short introduction to evolutionary finance theory.

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