Measuring Geodesics' Aperiodicity

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Abstract

Aperiodicity can arise both in the setting of sequences over a finite alphabet, and that of geodesics on a compact Riemannian surface. In both cases, aperiodicity itself provides no means to measure and compare different aperiodic objects one to another. For sequences the notion of ϕ-aperiodicity, by the function ϕ, provides a means for this. The aim of this thesis was to find an analogon in the setting of geodesics. This was done by defining f-aperiodicity of geodesics. The existence of f-aperiodic geodesics was proven for a very specific setting, namely that of a quotient of the hyperbolic surface of $\mathbb{H}$. This quotient was chosen in a specific way, such that a ϕ-aperiodic sequence could be chosen as the origin in the construction of the geodesic. Furthermore, this led to an easy way to define a flow-invariant subset of the unit tangent bundle of the compact Riemannian surface.

Zusammenfassung

Aperiodizität kann sowohl bei Folgen, als auch bei Geodäten auf kompakten Riemannschen Flächen eine Rolle spielen. In beiden Fällen liefert Aperiodizität an sich kein Mittel, verschiedene aperiodische Objekte zu messen und miteinander zu vergleichen. Für Folgen schafft der Begriff der ϕ-Aperiodizität mithilfe der Funktion ϕ Abhilfe. Ziel der vorliegenden Arbeit war es, ein Analogon für Geodäten zu finden. Dies geschah durch die Definition von f-aperiodischen Geodäten. Die Existenz von f-aperiodischen Geodäten wurde in einer sehr speziellen Situation gezeigt, nämlich die eines Quotienten der hyperbolischen Ebene $\mathbb{H}$. Der Quotient wurde spezifisch so gewählt, dass eine ϕ-aperiodische Folge als Ausgangspunkt für die Konstruktion der Geodäte dienen konnte. Diese Konstruktion brachte ausserdem eine einfache Art und Weise mit sich, eine flussinvariante Untermenge des Einheitstangentialvektorfeldes der kompakten Riemannschen Fläche zu konstruieren.

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1 Introduction

1.1 Overview

The notion of *aperiodicity* can arise, among others, in the context of sequences from a finite alphabet, or geodesics on compact manifolds. In both cases, aperiodicity is a generic property: a randomly generated sequence or geodesic will be aperiodic and not periodic. For example, almost any sequence consisting of two letters will at some point have a subword consisting of a repetition of only one of the letters of arbitrary length with probability 1, but this will not happen in any regulated way. Similarly, it would be exceptional for a generic geodesic on a compact surface to retrace its own image, although it will, also almost always with probability 1, come very close to retracing itself for an arbitrary length. Since aperiodicity is such a generic property, it would be interesting to categorize aperiodic elements, for example by measuring the aperiodicity. The fact that aperiodicity appears in such different contexts will be used in this thesis to draw conclusions in one setting (that of geodesics) by finding inspiration in the other setting (that of sequences).

The idea of this thesis is, first of all, to grasp the property of aperiocity from a quantitative point of view, i.e. to measure aperiodicity. In the setting of sequences this means that if the same subword appears twice in the same sequence, the subword in between those two appearances is long in comparison to the length of the reappearing subword: there is a non-trivial function ϕ which expresses the length of the word in between the reappearing subwords depending on the length of the reappearing subword. More precisely, for a function $\phi : \mathbb{N} \to \mathbb{N}^*$ with $\phi(\ell) \to \infty$ for $\ell \to \infty$, and a sequence $w \in \mathcal{W}$, the following implication holds $\forall i \in \mathbb{Z}$,

$\forall j \in \mathbb{N}^*$, $\forall \ell \in \mathbb{N}$:

$$[w(i) \cdots w(i+\ell)] = [w(i+j) \cdots w(i+j+\ell)] \quad \Rightarrow \quad j > \phi(\ell).$$

A sequence that has this kind of aperiodicity is called *ϕ-aperiodic*. The property of ϕ-aperiodicity is no longer generic: such a sequence with $\phi > 0$ is similarly exceptional to appear as a periodic one. In fact, almost any generic sequence over the alphabet $\mathcal{A} = \{0, 1\}$ will turn out to be ϕ-aperiodic for $\phi \equiv 0$. We are interested in the cases where $\phi > 0$ and ϕ is increasing. The growth properties of ϕ will give us a means to measure the aperiodicity of the sequence. In particular it is interesting to observe whether ϕ grows linearly or exponentially.

In this thesis, we will try to find a similar concept for measuring the aperiodicity in the setting of aperiodic geodesics on compact surfaces. Measuring the aperiodicity in the setting of geodesics means, in analogy to the setting of sequences, that the aperiodicity follows a rule, such as that if a geodesic comes close to itself, it travelled for a long time. The measuring will be done using a function f which will quantify the quality of the aperiodicity and such geodesics will be called *f-aperiodic*, in analogy to the ϕ-aperiodicity in the setting of sequences. The property of f-aperiodicity is no longer a generic property - again, in analogy to the setting of sequences. More precisely, if we consider a function $f : [1, \infty) \to (0, \infty)$, a geodesic γ is called *f-aperiodic* if for all $t \in \mathbb{R}$ and for all $s \geq 1$ we have $d(\dot\gamma(t + s), \dot\gamma(t)) \geq f(s)$, where $d(\cdot, \cdot)$ denotes the *Sasaki metric*, a metric on TM.

It is well known that on any compact Riemannian manifold there exist periodic geodesics. In this thesis, we study the question for which compact Riemannian manifolds there exist f-aperiodic geodesics. This thesis will provide an explicit construction, and hence an example, of a compact Riemannian manifold on which such a geodesic can be constructed. Hence in this thesis we will provide a constructive proof of the existence of such geodesics in specific settings; and therefore answer this qestion with "yes" under the given circumstances.

Now, consider the geodesic flow of a geodesic on a compact Riemannian manifold. One can define a set containing all the vectors of the image of the flow. Obviously, this will always be a flow-invariant subset of the unit tangent bundle. If the geodesic is periodic, this set will obviously be closed. If the geodesic is aperiodic, this is not necessarily the case. With the help of the function f that appears in the f-aperiodicity of a geodesic, we can always obtain a closed flow-invariant subset. Hence, on the other hand, this thesis will provide a way to find closed flow-invariant subsets of the unit tangent bundle of certain compact Riemannian manifolds. In other words, this means that aperiodicity is not a closed property, but f-aperiodicity is a closed property.

1.2 Results

In this section we shall elaborate on the results of this thesis somewhat more in detail as it was done in the overview of Section 1.1.

First, we define f-aperiodic geodesics, in analogy to the notion of ϕ-aperiodicity for sequences (see Section 3.1):

Definition 4.5. Consider a function $f : [1, \infty) \to (0, \infty)$. A geodesic γ is called *f-aperiodic* if for all $t \in \mathbb{R}$ and for all $s \geq 1$ we have $d(\dot\gamma(t+s), \dot\gamma(t)) \geq f(s)$.

We use the *Sasaki metric* $d(\cdot, \cdot)$ here (see Section 2.3), a metric on TM, since it seems adequate to not simply consider the points $\gamma(t)$ of the geodesic, but rather the tangent vectors $\dot\gamma(t)$, and a metric which also take into consideration the direction and the speed of the geodesic. We consider $s \geq 1$ and not simply $s > 0$ since we would like the geodesic γ to have travelled a little in between the two tangent vectors.

We then consider the following question:

Question. *For which positive non-trivial function f does there exist an f-aperiodic geodesic on a given compact Riemannian manifold?*

A first result will be Proposition 4.10 which states that f is monotonically decreasing and can be bounded by $f(s) \leq \frac{C}{s^{1/\dim SM}}$.

Proposition 4.10. *Let $f : [1, \infty) \to (0, \infty)$ be monotonically decreasing and continuous, M a compact Riemannian manifold. Define $C_f := \{v \in SM \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1\}$. Then, if $C_f \neq \emptyset$, we have $f(s) \leq \frac{C}{s^{1/\dim SM}}$, for some constant $C \in \mathbb{R}^+$.*

Then we will consider the following question:

Question. *Are there examples of compact Riemannian manifolds M for which we can find an explicit function f and an f-aperiodic geodesic on M?*

Theorem 6.8 will answer this question with yes in a very specific setting: we consider a certain quotient $\mathbb{D}$ of $\mathbb{H}$ which is a compact Riemannian manifold. In the construction which leads to $\mathbb{D}$ and the geodesic γ, and in the proof ot the theorem, the constants C and η will be defined. The function f is given implicitly:

Theorem 6.8. *Consider the geodesic $\gamma : \mathbb{R} \to \mathbb{D}$. There exist positive constants $\eta, C \in \mathbb{R}$, such that for all $t, s \in \mathbb{R}$, we have*

$$d(\dot{\gamma}(t), \dot{\gamma}(s)) \geq \min \left\{ |t-s|, \frac{C}{|t-s|^\eta} \right\}.$$

The proof of Theorem 6.8 will require a lot of constructional and technical results, all closely related to the very specific setting the theorem is proofed in: there will be a suitable tiling of the hyperbolic plane $\mathbb{H}$ equipped with a very specific coloring which yields a certain quotient $\mathbb{D}$ of $\mathbb{H}$; a quasi-geodesic, i.e. a polygonal chain, "following" the coloring of a sequence will be constructed on $\mathbb{H}$, which will have a geodesic "near" the polygonal chain and then eventually we will consider the projection of the geodesic on the quotient $\mathbb{D}$. This construction will lead to one very specific example for which Theorem 6.8 will be shown, and which can be also be extended easily to a certain family of further examples.

Another aspect of finding a suitable function f for f-aperiodic geodesics on $\mathbb{D}$ concerns subsets $C_f \subset S\mathbb{D}$ of the unit tangent bundle $S\mathbb{D}$, where

$$C_f := \{v \in S\mathbb{D} \mid d(\varphi_{t+s}(v), \varphi_t(v) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1)\}.$$

Such a set C_f is closed and flow-invariant. On a given compact Riemannian manifold M an f-aperiodic geodesic γ exists if and only if the set C_f is non-empty. This can be formulated as Theorem 6.8:

Theorem 6.10. *Consider γ from Section 6.1 on $\mathbb{D}$. Let*

$$f : [1, \infty) \to (0, \infty), f(s) := \frac{C}{s^\eta},$$

where $C, \eta \in \mathbb{R}^+$ are as in Theorem 6.8. Then,

$$C_f := \{v \in S\mathbb{D} \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1)\}$$

is non-empty and a closed flow-invariant subset of $S\mathbb{D}$.

1.3 Outline of the Thesis

First, after this introductory Chapter 1 with an overview over the thesis (Section 1.1), a short presentation of the main results (Section 1.2) and this outline of the thesis (Section 1.3), there will be a lot of necessary definitions in Chapter 2 and background information which will then help to explain the motivation of this thesis in detail. In Section 2.1 we will introduce some technical notations we will use later on, in Section 2.2 we will recall the definition of geodesics, in Section 2.3 we will define the Sasaki metric, in Section 2.4 we will recall the geodesic flow and in Section 2.5 we will recall Coxeter groups and how they can yield compact surfaces.

In Chapter 3 we will introduce aperiodic sequences. In Section 3.1 we will give the definition of aperiodic sequences, and in Section 3.2 we will present the famous example of the Morse-Thue sequence.

In Chapter 4 we will introduce aperiodic geodesics and closed flow-invariant subsets. In Section 4.1 we will define aperiodic geodesics and in Section 4.2 we will introduce closed flow-invariant subsets of the unit tangent bundle.

Chapter 5 and Chapter 6 contain the main arguments and results of this thesis which we are now ready to present. In Chapter 5 we will start with an aperiodic sequence and a suitable tiling of $\mathbb{H}$ with a convenient coloring (Section 5.1) and then construct a quasi-geodesic, i.e. a polygonal chain, which will "follow" the aperiodic sequence (Section 5.2). In Section 5.3 we will proof the existence of a geodesic which is "close" to the polygonal chain and in Section 5.4 we will give a systematic construction of such a geodesic. In Chapter 6 we will consider the projection of the geodesic in $\mathbb{H}$ onto a compact surface which is obtained as a quotient of $\mathbb{H}$. In Section 6.1 we will proof the first main result of this thesis, namely that the geodesic on the compact surface is aperiodic. In Section 6.2 we will present the second main result of this thesis, namely how this construction yields a closed flow-invariant subset of the unit tangent bundle, which will have Liouville-measure 0 and the geodesic flow will be ergodic.

Finally, in Chapter 7 we will first give a recapitulation of the results and their significance (Section 7.1), and then we will present some open questions related to the results of this thesis (Section 7.2).

2 Preliminaries

In this chapter we shall give some notions and definitions which will be used throughout this thesis.

2.1 Notation

Throughout this thesis we will use the following notation:

(i) By $\mathbb{N}$ we will denote the non-negative integers and by $\mathbb{N}^*$ the positive integers.

(ii) We will write $\mathbb{H}$ for the 2-dimensional hyperbolic space, i.e. $\mathbb{H} := \mathbb{H}^2$.

(iii) If we have $x, y \in \mathbb{H}$ two distinct points in the hyperbolic plane, we shall denote by xy the geodesic which connects these two points and by $\overline{xy}$ the geodesic segment from x to y.

(iv) By $\operatorname{dist}(\cdot, \cdot)$ we will denote the distance function. If we have $x, y, z \in \mathbb{H}$ points on the hyperbolic plane, we denote by $\operatorname{dist}(x, \overline{yz}) = \operatorname{dist}(x, yz)$ the distance from the point x to the geodesic connecting y and z.

(v) If we have $x, y, z \in \mathbb{H}$ points on the hyperbolic plane, we will denote by $\sphericalangle_x(y, z) = \sphericalangle(xy, xz)$ the angle, or to be more precise, the smaller of the two angles, between the geodesics xy and xz at their intersection point x.

2.2 Geodesics

This section is based on [2] and [9] which both should also be consulted for more details and background.

Denote by $M = (M, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the Euclidian metric, a *Riemannian manifold*, i.e. a *differentiable manifold* equipped with a *Riemannian metric.*

Let $\mathfrak{X}(M)$ denote the set of all C^∞ vector fields on M and $X, Y, Z \in \mathfrak{X}(M)$. An *(affine) connection* ∇ on M is a mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \to \nabla_X Y$ satisfying the following conditions for $f, g \in C^\infty(M, \mathbb{R})$:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

If for all $X, Y, Z \in \mathfrak{X}(M)$ the affine connection is *torsion-free*, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[\cdot,\cdot]$ denotes the *Lie-bracket*, and *Riemannian*, i.e. $Z\langle X, Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$, it is called the *Levi-Civita connection*. The Levi-Civita connection is unique for a given Riemannian manifold.

Given a Riemannian manifold M with Levi-Civita connection ∇, we have a unique correspondence between a vector field V along the differentiable curve $\gamma : I \to M$ and another vector field denoted by $\frac{D}{dt}V$, the *covariant derivative* of V along γ, such that the following conditions hold:

(i) $\frac{D}{dt}(V + W) = \frac{D}{dt}V + \frac{D}{dt}W$, where W is a vector field along γ,

(ii) $\frac{D}{dt}(fV) = \frac{d}{dt}fV + f\frac{D}{dt}V$, where f is a differentiable function and $f : I \to M$,

(iii) if V is induced by a vector field $Y \in \mathfrak{X}(M)$, i.e. we have $V(t) = Y(\gamma(t))$, then $\frac{D}{dt}V = \nabla_{\frac{d\gamma}{dt}} Y$.

Definition 2.1. A parametrized curve $\gamma : I \to M$ is called a *geodesic* if $\frac{D}{dt}\dot{\gamma}(t) = 0$ for all $t \in I$.

Remark 2.2. *A geodesic γ is self-parallel, i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, and straight.*

Any geodesic on a Riemannian manifold is parametrized proportional to arc length.

2.3 Sasaki Metric

This section is based on [1] and [9]. Consider an n-dimensional Riemannian manifold M and its *double tangent bundle* TTM, i.e. the tangent bundle of the tangent bundle TM. For $v \in TM$, $p = \pi(v)$ and $\pi : TM \to M$ the projection, we now consider a splitting of T_vTM into two n-dimensional complementary subspaces.

The first subspace is the *vertical space*

$$\mathcal{V}_v = \ker d_v\pi = T_vT_pM \subset T_vTM$$

where $d_v\pi : T_vTM \to T_pM$ for an arbitrary $v \in TM$. Since T_pM is a vector space, we may canonically identify T_vT_pM with T_pM. Recall that if we assign to every point p of a smooth manifold M an m-dimensional subspace D_p of T_pM, we say that an m-dimensional *distribution* $\mathcal{D}$ is given on M. Here, we assign to every $v \in TM$ the n-dimensional manifold T_pM. Therefore, we can denote by $\mathcal{V}$ the *vertical distribution* of TM, which consists of the fibres of π, i.e. the tangent spaces of M. The vertical distribution is smooth and integrable. The name derives from the convenient visualization of the tangent spaces of M as vertical to the manifold.

Now, we explain how we may naturally assign a complementary subspace $\mathcal{H}_v$ to $\mathcal{V}_v$ in T_vTM, using the given Levi-Civita connection and parallel translation. Let $c(t)$ be a curve which is tangent to $X \in T_pM$ at $t = 0$. The vector field $u(t)$ along c obtained by parallel translation of u along c may be considered as a curve in TM through u. We denote by $\widetilde{X}_u \in T_uTM$ the tangent vector to $u(t)$ at $t = 0$ which will be called the *horizontal lift* of X at $u \in TM$. The horizontal lift is well-defined, i.e. it does not depend on the choice of c. The space $\mathcal{H}_v$ of all horizontal lifts of tangent vectors in T_pM at u forms an n-dimensional subspace of T_vTM

and is called the *horizontal space* of T_vTM at u. We denote by $\mathcal{H}$ the *horizontal distribution*. Obviously, by their construction, we have $\mathcal{H}_v \cap \mathcal{V}_v = \{0\}$ and therefore $T_vTM = \mathcal{H}_v \oplus \mathcal{V}_v$.

Let $X = (X_h, X_v) \in T_vTM$, where $X_h \in \mathcal{H}_v$ and $X_v \in \mathcal{V}_v$. By $d_v\pi$ we may identify X_h with an element in T_pM. Therefore, we may write $T_vTM = \mathcal{H}_v \oplus \mathcal{V}_v = T_pM \oplus T_pM$ and $X = (X_h, X_v) \in T_pM \oplus T_pM$.

Now we can state the definition for the *Sasaki metric.*

Definition 2.3. The *Sasaki Metric* $d_S(\cdot,\cdot)$ or, if the context is clear, also simply $d(\cdot,\cdot)$, is a Riemannian metric on TM, based on a given Riemannian metric on M, denoted by $\langle\cdot,\cdot\rangle$.

For $X, Y \in T_vTM$, $\pi(v) = p$, we may use the splitting of T_vTM and write $X = (X_h, X_v) \in T_pM \oplus T_pM$ and $Y = (Y_h, Y_v) \in T_pM \oplus T_pM$. We now set

$$d_S(X,Y) = d_S\left((X_h, X_v), (Y_h, Y_v)\right) := \langle X_h, Y_h\rangle + \langle X_v, Y_v\rangle.$$

The set $S_pM := \{u \in T_pM \mid ||u|| = 1\}$ of unit tangent vectors at p forms the sphere of radius 1 in T_pM. Denote the *unit tangent bundle* by $SM = \bigcup_{p\in M} S_pM$. Since its codimension in TM is 1, SM is a $(2n-1)$-dimensional submanifold of TM. We now consider the Riemannian metric induced by the Sasaki metric on SM and denote it by $d(\cdot,\cdot)$. Define the vertical and horizontal distributions on SM analogously to before. Here, the vertical distribution is an $(n-1)$-dimensional vector bundle over SM and the horizontal distribution is the restriction of $\mathcal{H}$ to $\bigcup_{v\in SM} \mathcal{H}_v$. This leads to the following proposition.

Proposition 2.4. *Consider a shortest geodesic $\gamma : [0,t] \to M$ and a unit vector $u \in S_{\gamma(0)}M$. Then we get for the Sasaki metric $d_S(\cdot,\cdot)$, the metric $d(\cdot,\cdot)$ on M and the angle $\sphericalangle(\cdot,\cdot)$ between two vectors in SM:*

$$d_S(u, \dot\gamma(t)) \leq d(\gamma(0), \gamma(t)) + \sphericalangle(\dot\gamma(0), u).$$

For $u, w \in SM$ and a parallelly transported vector $pl_\gamma(u)$ of u along a shortest geodesic $\gamma : [0,t] \to M$ from $\pi(u)$ to $\pi(w)$ in general, we *get for the Sasaki metric:*

$$d_S(u,w) \leq d(\gamma(0),\gamma(t)) + d_S(pl_\gamma(u),w) =$$
$$= d(\pi(u),\pi(w)) + \sphericalangle(pl_\gamma(u),w).$$

Proof. (i) For a geodesic $\gamma : [0,t] \to M$, $\dot\gamma$ is a parallel vector field. Hence, $\dot\gamma$ is a geodesic in TM with only horizontal tangent vectors. Therefore, $d_S(\dot\gamma(0),\dot\gamma(t)) = d(\gamma(0),\gamma(t))$ and for any two vectors $u,v \in TM$ we have $d_S(u,v) \geq d(\pi(u),\pi(v))$. Analogously, this holds for SM.

(ii) Obviously, any curve in $S_{\pi(v)}M$ has only vertical tangent vectors. Therefore, it may be considered as a curve in S^{n-1} and we have $d_S(u,v) \leq \pi$ for $u,v \in SM$ with a common base point. In this case, $d_S(u,v)$ turns out to be exactly the angle between the geodesics γ_u and γ_v, where γ_v is the unique geodesic with $\gamma_v(0) = \pi(v)$ and $\dot\gamma_v(0) = v$; γ_u is defined analogously.

Overall, (i) and (ii) yield the desired relations. □

Remark 2.5. *In the case where a geodesic between any two points is unique, since then the parallel translation of a vector u at the starting point of the geodesic along the connecting geodesic between the two points is unique, we may define*

$$\tilde{d_S}(u,w) := d(\pi(u),\pi(w)) + \sphericalangle(pl_\gamma(u),w).$$

Because the triangle inequality is not always satisfied, $\tilde{d_S}(\cdot,\cdot)$ is not really a metric. Still, $\tilde{d_S}(\cdot,\cdot)$ can be used appropriately for estimating distances, since the sets $\tilde{\mathcal{O}}_\varepsilon(w) := \{u \in SM | \tilde{d_S}(u,w) < \varepsilon\}$, $\varepsilon > 0$, form a basis for the topology defined by $d_S(\cdot,\cdot)$:

$$\forall v \in SM \quad \forall \delta > 0 \quad \exists \varepsilon > 0 : \quad \mathcal{O}_\varepsilon(v) \subset \tilde{\mathcal{O}}_\delta(v) \subset \mathcal{O}_\delta(v),$$

where $\mathcal{O}_\mu(v), 0 < \mu \in \{\varepsilon,\delta\}$ are the open neighbourhoods of radius μ defined by $d_S(\cdot,\cdot)$. Therefore, whenever convenient, we will use $\tilde{d_S}(\cdot,\cdot)$ instead of the Sasaki metric. For the sake of notational simplicity, we shall write $d(\cdot,\cdot)$ or $d_S(\cdot,\cdot)$ respectively instead of $\tilde{d_S}(\cdot,\cdot)$.

2.4 Geodesic Flow

The two following definitions can be found for example in [9] or [4].

Definition 2.6. A *flow* on a manifold X is a continuous function $\varphi : \mathbb{R} \times X \to X$ satisfying $\varphi(0, x) = x$ and $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$ for all $t, s \in \mathbb{R}$ and $x \in X$. Given a flow φ, by φ_t we denote the diffeomorphism $\varphi_t : X \to X$, $\varphi_t(x) = \varphi(t, x)$.

Definition 2.7. A set $Y \subset X$ is called *flow-invariant*, if we have for all $t \in \mathbb{R}$ that $\varphi_t(Y) = Y$.

Example 2.8. The flow $\varphi : \mathbb{R} \times SM \to SM$ on the unit tangent bundle SM of a Riemannian manifold M defined by

$$\varphi(t, (p, v)) := (\gamma_{p,v}(t), \dot{\gamma}_{p,v}(t)),$$

where $\gamma_{p,v}$ is the unique geodesic with $\gamma_{p,v}(0) = p$ and $\dot{\gamma}_{p,v}(0) = v$, is called the *geodesic flow*.

As above we denote by φ_t the diffeomorphism $\varphi_t : SM \to SM$, $\varphi_t(p, v) = \varphi(t, (p, v))$.

As can be seen in more detail in [8], we may define the *one-form* α of TM, $\alpha_\theta(\xi) := d(\xi, g(\theta))$, where $d(\cdot, \cdot)$ denotes the Sasaki metric, $\xi, \theta \in SM$ and $g : TM \to TTM$ is the geodesic vector field, given by $g(\theta) := \frac{\partial}{\partial t}|_{t=0}\, \varphi_t(\theta)$, where φ_t is the geodesic flow. It can easily be seen that α annihilates the vertical space. If α is restricted to SM, we obtain a *contact form*, i.e. the $(2n-1)$-form $\alpha \wedge (d\alpha)^{n-1}$ never vanishes on SM. Furthermore, one can show that if M has finite volume, the volume form $\omega := \alpha \wedge (d\alpha)^{n-1}$ has finite integral over SM and hence gives rise to a probability measure μ, $\mu(X) := \frac{1}{\int_{SM} \omega} \int_X \omega$, where $X \subset SM$, defined on SM.

Definition 2.9. The probability measure μ on SM induced by the one-form α as described above is called the *Liouville measure*.

For the proof of the following proposition, see [8].

Proposition 2.10. *The Liouville measure is invariant under the geodesic flow on the unit tangent bundle.*

The following definition can for example be found in [4].

Definition 2.11. A flow φ_t on a compact manifold X is called *ergodic* with respect to a measure μ, if for any μ-measurable flow-invariant set $A \subset X$, we have either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

The following theorem can be found for example in [4]. It is proven using the so-called Hopf-argument.

Theorem 2.12. *The geodesic flow on a compact connected quotient of* $\mathbb{H}$ *with respect to the Liouville measure is ergodic.*

2.5 Coxeter Groups and Compact Surfaces

In this section we will first state the definition of Coxeter groups and then demonstrate how a compact surface can be obtained by considering a specific subgroup of a specific Coxeter group acting on $\mathbb{H}$. The following definition can be found for example in [3].

Definition 2.13. Let $S := \{s_1, s_2, \ldots, s_k\}$ be a finite set. A *Coxeter matrix* $M = (m_{ij})$ is a $k \times k$ symmetric matrix with $m_{ij} \in \mathbb{N}^* \cup \{\infty\}$, such that $m_{ij} = 1$ if $i = j$ and $m_{ij} > 1$ otherwise. Now we can define the *Coxeter group* as a group generated by the elements of S and with the relations of the form $(s_i s_j)^{m_{ij}} = 1$, where $(s_i s_j)^{\infty} = 1$ means that there is no relation between s_i and s_j. Furthermore, $m_{ii} = 1$ implies that all generators are of order 2 ant thus $s_i = s_i^{-1}$.

Remark 2.14. *Note that all reflections groups are descriptive examples of Coxeter groups.*

Consider the group Γ acting on $\mathbb{H}$ generated by reflections at walls which also generate a tiling of right-angled n-gons. This group Γ is a finitely generated Coxeter group. In [3] we have the following:

Lemma 2.15. *Any (finitely generated) Coxeter group is virtually torsion-free.*

Hence we have a torsion-free subgroup Δ of Γ of finite index.

Remark 2.16. *Since the coloring of the edges of the n-gons is invariant under the action of Γ, it is also invariant under the action of Δ.*

Proposition 2.17. *Consider the quotient space $\mathbb{D} := \mathbb{H}/\Delta$, where Δ is as above. Then, $\mathbb{D}$ is a compact surface.*

The following proof can be found in [4] or in more detail and with the background of Fuchsian groups, since this is what we have here, in [5].

Proof. We choose a Dirichlet domain D as a fundamental domain, where we set $D := D_p := \{x \in \mathbb{H} \mid d(x,p) \leq d(x, \gamma p), \forall \gamma \in \Delta\}$, for any point $p \in \mathbb{H}$. Obviously, for any $\gamma \in \Delta$, we have $D_{\gamma p} = \gamma(D_p)$. If $\gamma \neq \mathrm{id}$, the interiors of D_p and $D_{\gamma p}$ are disjoint. Since Δ is discrete, we only have finitely many $\gamma \in \Delta$ with $D_p \cap D_{\gamma p} \neq \emptyset$ and the intersection in these cases is a geodesic segment. Hence, D is a hyperbolic polygon which is bounded by finitely many edges. Hence, $\mathbb{H}/\Delta$ is compact if and only if D is compact. This is true by construction, since D tesselates $\mathbb{H}$. □

Remark 2.18. *A surface such as constructed in Proposition 2.17 cannot be embedded isometrically in $\mathbb{R}^3$.*

3 Aperiodic Sequences

In this chapter we will introduce aperiodic sequences and also present an example of a famous aperiodic sequence, the Morse-Thue sequence. Furthermore, we will introduce a way to "measure" the aperiodicity of a sequence and see how this applies to the Morse-Thue sequence.

3.1 Aperiodic Sequences

Consider a finite set $\mathcal{A}$, the *alphabet*, and denote the set of *words* of length m in $\mathcal{A}$ by $\mathcal{W}(m) = \{w : \{1, \dots, m\} \to \mathcal{A}\}$ and the *sequences* in $\mathcal{A}$ by $\mathcal{W} = \{w : \mathbb{Z} \to \mathcal{A}\}$. With $[w(i) \cdots w(i+\ell)]$ we denote the subword of $w \in \mathcal{W}$ starting at place $i \in \mathbb{Z}$ with length $(\ell + 1)$ for $\ell \in \mathbb{N}$.

Definition 3.1. A sequence is called *periodic* if there is a $j \in \mathbb{Z}$ such that $w(i) = w(i+j)$ for all $i \in \mathbb{Z}$. In other words, for subwords of the sequence we have for all $\ell \in \mathbb{N}$

$$[w(i) \cdots w(i+\ell)] = [w(i+j) \cdots w(i+j+\ell)].$$

Definition 3.2. A sequence is called *aperiodic* if it is not periodic, or, to be more precise, if for every $i \in \mathbb{Z}$, $j \in \mathbb{N}^*$ there is an $\ell \in \mathbb{N}^*$ such that

$$[w(i) \cdots w(i+\ell)] \neq [w(i+j) \cdots w(i+j+\ell)].$$

This definition of aperiodic sequences does not provide any information on the severeness of the aperiodicity, i.e. the relation of j to ℓ. In most cases, there will be no general relation between these two values. But in certain very rare cases of sequences, there is a relation that can be expressed by a function ϕ. This function will measure how far apart, i.e. how big j is, two identic subwords of a certain length ℓ at least are. This leads to the following definition which is taken from [10].

Definition 3.3. Let $\phi : \mathbb{N} \to \mathbb{N}^*$ be a function with $\phi(\ell) \to \infty$ if $\ell \to \infty$. A sequence $w \in \mathcal{W}$ is *ϕ-aperiodic*, if $\forall i \in \mathbb{Z}$, $\forall j \in \mathbb{N}^*$, and $\forall \ell \in \mathbb{N}$ the following implication holds:

$$[w(i) \cdots w(i+\ell)] = [w(i+j) \cdots w(i+j+\ell)] \quad \Rightarrow \quad j > \phi(\ell).$$

Remark 3.4. *Any ϕ-aperiodic sequence is obviously aperiodic.*

Remark 3.5. *Almost any given generic sequence with $\mathcal{A} = \{0, 1\}$ is ϕ-aperiodic with $\phi \equiv 0$: since for every $n \in \mathbb{N}$ there is an $i \in \mathbb{N}$ such that $w(i) = 0$, $w(i+1) = 0$, $\ldots$, $w(i+n+1) = 0$, we have $[w(i) \ldots w(i+n)] = [w(i+1) \ldots w(i+n+1)]$ and this implies $\phi(n) \leq 1$.*

Remark 3.6. *If we consider a sequence of sequences w_i, we understand convergence as pointwise convergence: we say, $w_i \to w$ if and only if for every j there is a $n_0(j)$, such that for all $n \geq n_0$ we have $w_n(j) = w(j)$, i.e. certain positions in the sequence w_i coincide with the according positions of w. Of course this can also be formulated in terms of open sets as a basis of a suitable topology, implementing the same idea.*

Consider a sequence of aperiodic sequences. Such a series can have a periodic sequence as an accumulation point.

On the other hand, if we consider sequence of ϕ-aperiodic sequences, their limit will also be a ϕ-aperiodic sequence. Hence we may call the property of being ϕ-aperiodic a closed *property.*

The following theorem from [10], where also the proof can be found, states some conditions for the existence of some examples of ϕ-aperiodic sequences. Later in this thesis we will use ϕ-aperiodic sequences that fulfill the requirements of this theorem and therefore exist.

Theorem 3.7. *If $|\mathcal{A}| \geq 5$, then there exists a sequence $w \in \mathcal{W}$ which is ϕ-aperiodic with $\phi(\ell) = 2^\ell$, for a strictly increasing function ϕ.*

Remark 3.8. *Note that any $w \in \mathcal{W}$ which is ϕ-aperiodic with $\phi(\ell) = 2^\ell$, does not contain any subwords $[w(i)w(i+1)]$, where $w(i) = w(i+1)$.*

3.2 The Morse-Thue Sequence

The *Morse-Thue sequence* was first introduced by Thue in [11] and then later rediscovered independently by Morse in [6] in a completely different context.

The Morse-Thue sequence is constructed the following way, using the alphabet $\mathcal{A} = \{0, 1\}$:

(i) start with the word [10],

(ii) replace the subword [1] by the subword [10] and replace the subword [0] by the subword [01],

(iii) iterate the replacing.

The first few iterations are:

$$
\begin{aligned}
&[10] \\
\rightsquigarrow\ &[1001] \\
\rightsquigarrow\ &[10010110] \\
\rightsquigarrow\ &[1001011001101001] \\
\rightsquigarrow\ &[10010110011010010110100110010110] \\
\rightsquigarrow\ &\ldots
\end{aligned}
$$

As can be seen immediately, the nature of the replacement rule doubles the length of the word with each iteration and puts the word of the last step as a subword in the first half of the word of the current step and attaches as the second half of the word a subword which appears as "inverse" to the first half, in the sense that wherever there was a 0 in the former step, there now is a 1, and vice versa. In fact, this observation can be used to formulate another rule for constructing the Morse-Thue sequence:

(i) start with the word $W_0 = [1]$

(ii) define recursively $W_{n+1} = W_n\overline{W_n}$, where $\overline{W_n}$ is obtained from W_n by replacing 1 by 0 and vice versa, and $W_n\overline{W_n}$ means that you attach the word $\overline{W_n}$ after the last letter of W_n.

Remark 3.9. *(i) The Morse-Thue sequence is aperiodic. A proof can be found for example in [6]. Furthermore, the Morse-Thue sequence does not contain a subword*

$$[w(i)\cdots w(i+m)w(i+m+1)\cdots w(i+2m+1)w(i+2m+2)],$$

where $[w(i)\cdots w(i+m)] = [w(i+m+1)\cdots w(i+2m+1)]$ *and* $[w(i) = w(i+2m+2)]$ *(see [7] for the proof). Using these properties, it can be shown that the Morse-Thue sequence is* ϕ*-aperiodic for* $\phi(\ell) = \ell$*.*

(ii) Obviously, the ϕ*-aperiodicity of the Morse-Thue sequence is linear. On the other hand, sequences such as the ones whose existence was proven in Theorem 3.7, have exponential aperiodicity since their associated function* ϕ *is exponential. Unfortunately, we cannot state such a sequence explicitely.*

Nonetheless, we will later in this thesis use sequences such as the ones in Theorem 3.7 and not the Morse-Thue sequence because of their exponential aperiodicity.

4 Aperiodic Geodesics and Flow-Invariant Closed Subsets

In this chapter we will introduce aperiodic geodesics. Similar as with aperiodic sequences, we will present a way of "measuring" their aperiodicity. This will be done by a function f. Then, one can also define a flow-invariant closed subset of the unit tangent bundle using f.

4.1 Aperiodic Geodesics

In the following $\gamma : \mathbb{R} \to M$ will denote a geodesic on a Riemannian manifold M.

Definition 4.1. A geodesic γ is called *periodic*, if there exists an $s \in \mathbb{R}$, $s \neq 0$, such that for all $t \in \mathbb{R}$ we have $\dot{\gamma}(t+s) = \dot{\gamma}(t)$.

Remark 4.2. *Note that a geodesic γ is periodic, if and only if there exists an $s \in \mathbb{R}$, $s \neq 0$, such that for all $t \in \mathbb{R}$ we have that $d(\dot{\gamma}(t+s), \dot{\gamma}(t)) = 0$, where $d(\cdot,\cdot)$ denotes the Sasaki metric.*

Definition 4.3. A geodesic γ is called *aperiodic*, if it is not periodic.

Remark 4.4. *Note that equivalently we can say that a geodesic γ is aperiodic, if and only if we have, for any $t \in \mathbb{R}$ and $s \in \mathbb{R}$, $s \neq 0$, $d(\dot{\gamma}(t+s), \dot{\gamma}(t)) > 0$, where $d(\cdot,\cdot)$ denotes the Sasaki metric.*

The formulation in Remark 4.4 allows us to stipulate the aperiodicity of a geodesic in a quantitative way:

Definition 4.5. Consider a function $f : [1, \infty) \to (0, \infty)$. A geodesic γ is called *f-aperiodic* if for all $t \in \mathbb{R}$ and for all $s \geq 1$ we have $d(\dot{\gamma}(t+s), \dot{\gamma}(t)) \geq f(s)$.

Remark 4.6. *(i) Since $f(s) > 0$, f-aperiodicity implies aperiodicity.*

(ii) The notion of f-aperiodicity is much stronger and much more precise than the notion of aperiodicity of a geodesic, since it grasps the aperiodicity in a quantitave way.

Remark 4.7. *Comparing Definition 3.2 of aperiodic sequences to the definition of aperiodic geodesics as formulated in Remark 4.4, there is a striking resemblance between the two concepts.*

Analogously, Definition 3.3 of ϕ-aperiodic sequences and Definition 4.5 for f-aperiodic geodesics, correspond.

This duality between the two concepts and the two settings will be used later in this thesis by using a ϕ-aperiodic sequence to construct an f-aperiodic geodesic on a compact Riemannian manifold.

4.2 Flow-Invariant Closed Subsets of SM

Now, we want to discuss subsets of SM which are invariant under the geodesic flow. We start by considering the orbit $\{\varphi_t(v) \mid t \in \mathbb{R}\}$ for a given vector $v \in SM$. If v is a periodic vector, this subset is a closed flow-invariant subset of SM. For a non-periodic vector, this is also flow-invariant, but not necessarily closed. However, we can obtain a closed flow-invariant subset from a non-periodic vector v by using a suitable function f which measures in a way the aperiodicity of the geodesic. Note that v is periodic if and only if $d(\varphi_{t+s}(v), \varphi_t(v)) = 0$ for all $t \in \mathbb{R}$ and for an $s \geq 1$, and v is aperiodic if and only if $d(\varphi_{t+s}(v), \varphi_t(v)) > 0$ for all $t \in \mathbb{R}$ and for all $s \geq 1$, where $d(\cdot, \cdot)$ denotes the Sasaki metric. Now, given a continuous function $f : [1, \infty) \to (0, \infty)$ consider the set

$$C_f := \{v \in SM \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1\}.$$

If $v \in C_f$, the geodesic γ_v is aperiodic and has the following property: if $d(\dot{\gamma}_v(t+s), \dot{\gamma}_v(t)) \leq \varepsilon$, for $\varepsilon > 0$, we have $\varepsilon \geq f(s) > 0$, i.e. if $s \in f^{-1}((0, \varepsilon])$, it takes at least the time $s_\varepsilon := \min f^{-1}((0, \varepsilon])$ until $\dot{\gamma}_v$ comes ε-close to itself. Thus, f describes how aperiodic the geodesic γ_v is, if γ_v is f-aperiodic. For a "more aperiodic" geodesic this will take, of course, longer than for a "less aperiodic" geodesic. Here, f shall be a function which is always positive, continuous and monotonically decreasing. For later reference we state the following lemma:

Lemma 4.8. *Let $f : [1, \infty) \to (0, \infty)$ be a monotonically decreasing and continuous function and M a compact Riemannian manifold. Define $C_f := \{v \in SM \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1\}$. Then, C_f is flow-invariant and closed.*

Remark 4.9. *It is not obvious whether there exist such functions f for which $C_f \neq \emptyset$.*

In this thesis, we will show that there are suitable Riemannian manifolds and suitable functions f which yield closed flow-invariant subsets C_f of suitable surfaces such that $C_f \neq \emptyset$. The intention will be to have a function f which is as slowly monotonically decreasing as possible, but still gives a non-empty closed flow-invariant subset C_f. Obviously, if we have for a function $g \geq f$, this implies $C_g \subset C_f$. Thus, in other words, we aim to find as small as possible a closed flow-invariant subset C_f which still fulfills $C_f \neq \emptyset$.

The following proposition on the other hand gives an upper bound for the function f on a given Riemannian manifold M.

Proposition 4.10. *Let $f : [1, \infty) \to (0, \infty)$ be monotonically decreasing and continuous, M a compact Riemannian manifold. Define $C_f := \{v \in SM \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1\}$. Then, if $C_f \neq \emptyset$, we have $f(s) \leq \frac{C}{s^{1/\dim SM}}$, for some constant $C \in \mathbb{R}^+$.*

Proof. Lemma 4.8 implies that C_f is a closed flow-invariant subset.

For M an n-dimensional manifold, we have $\dim(TM) = 2n$ for the manifold TM and $\dim(SM) = 2n - 1$, since $SM \subset TM$ with codimension 1 (see for example [9]).

We normalize $\mathrm{vol}(SM) = 1$ and $\mathrm{vol}(B_\varepsilon(x)) \geq \widetilde{C} \cdot \varepsilon^{2n-1}$, where $B_\varepsilon(x)$ is the open ball in SM of radius $\varepsilon > 0$ around $x \in SM$, and $\widetilde{C} \in \mathbb{R}^+$. Hence, if $x_1, \ldots, x_k \in SM$, $k \in \mathbb{N}$, are distinct points such that the $B_\varepsilon(x_i)$ are pairwise disjoint, then $k \leq \frac{1}{\widetilde{C} \cdot \varepsilon^{2n-1}}$.

Let γ be the geodesic on M which is associated with the geodesic flow φ_t. Now, consider $\dot{\gamma}(0), \dot{\gamma}(1), \dot{\gamma}(2), \ldots, \dot{\gamma}(T)$, $T \in \mathbb{N}$. Then we have that for $T > \frac{1}{\widetilde{C} \cdot \varepsilon^{2n-1}}$, there exist $i, j \in \{0, \ldots, T\}$ such that $d(\dot{\gamma}(i), \dot{\gamma}(j)) \leq 2\varepsilon$. Now, choose $T := \frac{1}{\widetilde{C} \cdot \varepsilon^{2n-1}} + \delta$, where $\delta > 0$ is arbitrary. Since we have $|i - j| \leq T$ and f is monotonically decreasing, this implies that $f(T) \leq \frac{C}{(T-\delta)^{1/\dim SM}}$, where $C := \frac{2}{\widetilde{C}^{2n-1}}$. With $\delta \to 0$, we obtain the result. □

Remark 4.11. *Consider a set C_f as defined in Proposition 4.10. There is, obviously, a geodesic γ on M which can be associated with C_f. This geodesic is f-aperiodic. Since C_f is closed and therefore contains the limit of any series of f-aperiodic geodesics, there exist many more geodesics that are f-aperiodic.*

Remark 4.12. *If we take for M a quotient of $\mathbb{H}$, we have according to Theorem 2.12 that the geodesic flow on M is ergodic. Hence we have that either $\mu(C_f) = 0$ or that $\mu(SM \setminus C_f) = 0$, where μ is the Liouville measure. If we had $\mu(SM \setminus C_f) = 0$, then C_f would be a dense subset. This cannot be the case, since the limit of a series in C_f is contained in C_f according to Remark 4.11. But not all geodesics are f-aperiodic and thus have a function f that allows to construct a set C_f. Hence we have that $\mu(SM \setminus C_f) = 0$.*

5 Construction of the Geodesic on $\mathbb{H}$

In this chapter we will start with a specific ϕ-aperiodic sequence as defined in Chapter 3. Using a suitable tiling of $\mathbb{H}$ with colored edges we will construct a polygonal chain that "follows" the coloring of the sequence. From the polygonal chain we will obtain a unique geodesic that also "follows" the coloring of the sequence.

5.1 Finding a Suitable Tiling of $\mathbb{H}$ and its Coloring

According to Theorem 3.7, we can assume a $\phi(\ell) = 2^{\ell}$-aperiodic sequence for $|\mathcal{A}| = 5$ to be given. Now, consider $\mathbb{H}$ and a tiling with regular right-angled n-gons, where $n := 5r$ and $r \in \mathbb{N}$. We will determine the minimal value of r later for Proposition 5.7 and Theorem 5.8, and Proposition 5.11 and Theorem 5.13, respectively.

Pick one of the n-gons and assign each of the elements of $\mathcal{A}$ to one of the edges of the n-gon in the following way (see also Figure 5.1.): first, pick an arbitrary edge of the n-gon, label it with the number 1 and then number the edges from 1 to $5r$. Now, assign a first element of $\mathcal{A}$ to edge number 1, a second element of $\mathcal{A}$ to edge numer $r+1$, a third element of $\mathcal{A}$ to edge number $2r+1$, and so on. We now have 5 labelled and $n-5$ unlabelled edges: between any two labelled edges there are $\frac{n-5}{5}$ unlabelled ones.

Now, we label all n-gons and the edges of the entire tiling, respectively, by applying the group action of the reflection group which is associated to the tiling (see also Figure 5.2).

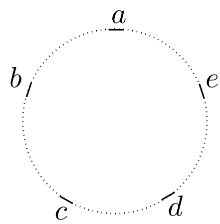

Figure 5.1: A schematic drawing of one of the n-gons with labelled edges, where $\mathcal{A} = \{a, b, c, d, e\}$ and the unlabelled edges are only outlined. The letter a is assigned to edge number 1, letter b is assigned to edge number $r + 1$, letter c is assigned to edge number $2r + 1$, letter d is assigned to edge number $3r + 1$ and letter e is assigned to egde number $4r + 1$.

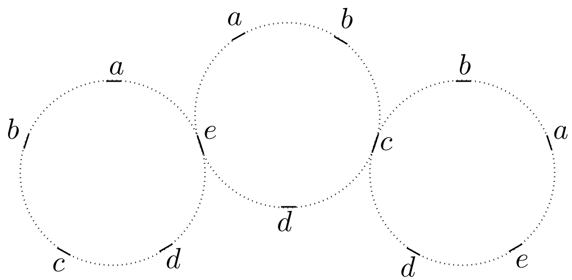

Figure 5.2: A schematic drawing of three labelled n-gons which illustrates how the labelling of one n-gon is distributed onto the whole tiling: the n-gon on the left is reflected at the edge, i.e. the wall, labelled e. The thus obtained n-gon in the middle is reflected at the wall labelled c. This is done with all edges, also the unlabelled ones.

Remark 5.1. *We will call a tiling with a coloring as described above a* 5-colored tiling*.*

5.2 Construction of the Polygonal Chain

In this section we first construct the polygonal chain and then state some preliminary technical results. Starting from this polygonal chain we will show in the next section that there exists a geodesic which follows the polygonal chain in the sense that it intersects the exact same sequence of edges of tiles (Theorem 5.8).

Definition 5.2. We construct the *polygonal chain* by connecting the middle points of adjacent tiles by following the elements of the sequence by intersecting the according colored edge of a tile to the adjacent one (see also Figure 5.3).

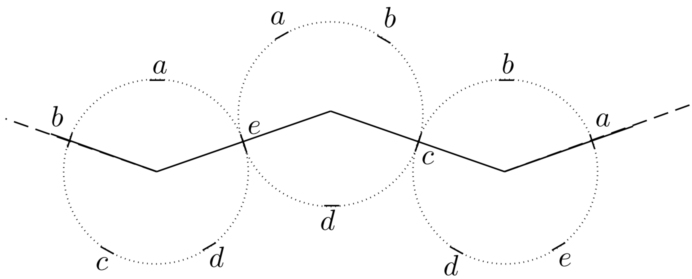

Figure 5.3: A schematic drawing to illustrate how the subword [*beca*] is represented by a polygonal chain.

We start with some preliminary technical results.

Lemma 5.3. *(i) The length a of one of the line segments of the polygonal chain is*

$$a = 2 \operatorname{arcosh} \frac{\sqrt{2}}{2 \sin \frac{\pi}{n}}.$$

(ii) The length b of one of the sides of the right-angled n-gon is

$$b = 2 \operatorname{arcosh} \frac{2 \cos \frac{\pi}{n}}{\sqrt{2}}.$$

Proof. See also Figure 5.4.

(i) Let b denote one of the edges of the n-gon. Consider the triangle with the sides $\frac{b}{2}$, $\frac{a}{2}$ and half a diagonal of the n-gon, denoted by c. Then, we have $\sphericalangle(\frac{b}{2}, c) = \frac{\pi}{4}$, $\sphericalangle(\frac{b}{2}, \frac{a}{2}) = \frac{\pi}{2}$ and $\sphericalangle(\frac{a}{2}, c) = \frac{\pi}{n}$. We have by hyperbolic trigonometry:

$$\cos\frac{\pi}{4} = \cosh\frac{a}{2}\sin\frac{\pi}{n},$$

and this is exactly what was claimed.

(ii) Consider again the triangle with the sides $\frac{b}{2}$, $\frac{a}{2}$ and half a diagonal of the n-gon, denoted by c. Hyperbolic trigonometry yields:

$$\cos\frac{\pi}{n} = \cosh\frac{b}{2}\sin\frac{\pi}{4},$$

and this is exactly what was claimed.

□

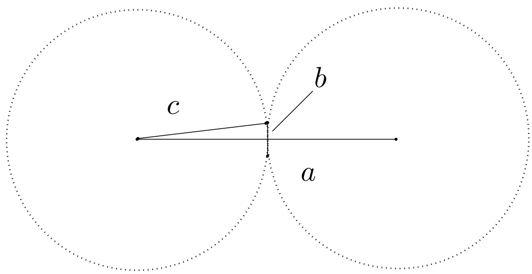

Figure 5.4: A schematic drawing of two n-gons and the sides a, b and c as used in the proof for Lemma 5.3.

Remark 5.4. *The polygonal chain described above never intersects the same n-gon more than once, since, as will be shown in Section 5.3, there exists a geodesic in a finite distance to the polygonial chain. This means that the polygonal chain connects two points of the boundary at infinity without ever intersecting itself, i.e. it can be described as always going farther away from itself.*

5.3 Existence of the Geodesic

For the construction of a geodesic with the desired properties and the proof of its existence, we need the following lemma:

Lemma 5.5. *Let* $\frac{\pi}{2} > \alpha_0 > 0$*. Let* $\ell_0 > 0$*, with* $\ell_0 \pi \leq \sin \alpha_0 \cosh \ell_0$*, and define* $h := \operatorname{arcosh} \frac{1}{\sin \frac{\alpha_0}{2}} \geq 0$*. Consider a triangle with vertices* x, y, z*, edges* $\ell_1 := xy$*,* $\ell_2 := xz$ *and* $m := yz$*, and angles* $\alpha := \sphericalangle(\ell_1, \ell_2)$*,* $\beta_1 := \sphericalangle(\ell_1, m)$ *and* $\beta_2 := \sphericalangle(\ell_2, m)$ *(see Figure 5.5). Assume* $\alpha \geq \alpha_0$ *and* $\ell_1, \ell_2 \geq \ell_0$*.*

Then the following hold:

(i) $d(x, m) \leq h$*,*

(ii) $m \geq \ell_1 + \ell_2 - 2h$*,*

(iii) $\beta_i \leq \frac{1}{\ell_i}, i \in \{1, 2\}$*.*

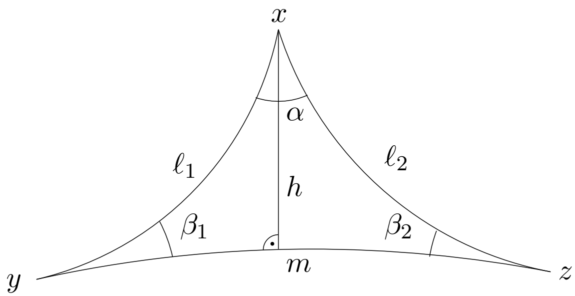

Figure 5.5: The triangle in Lemma 5.5.

Proof. (i) In order to determine h, consider a triangle where y and z are on the boundary at infinity and the angle at x is α_0. Denote by h_x the height starting at x. Hyperbolic trigonometry then yields:

$$h = h_x = \operatorname{arcosh} \frac{1}{\sin \frac{\alpha_0}{2}}.$$

Since this is monotonically decreasing for $\alpha_0 \in]0, \frac{\pi}{2}[$, and greater or equal than 0, we have $h \geq d(x, m)$.

For y and z not on the boundary, if α_0 is fixed, the height can only be smaller as for y and z on the boundary.

(ii) This follows immediately with the triangle inequality for the two triangles obtained by dividing the original triangle into two smaller ones along the height h_x starting at x.

(iii) We show $\beta_1 \leq \frac{1}{\ell_1}$, the proof for the inequality with β_2 is analogous.

Consider a triangle where z lies in the boundary at infinity and $\alpha = \alpha_0$. Since then $\beta_2 = 0$, we have

$$\cos \beta_2 = 1 = \sin \alpha_0 \sin \beta_1 \cosh \ell_1 - \cos \alpha_0 \cos \beta_1.$$

Since $|\cos \alpha_0 \cos \beta_1| \leq 1$, this yields

$$\sin \alpha_0 \sin \beta_1 \cosh \ell_1 \leq 2$$

and

$$\sin \beta_1 \cosh \ell_1 \leq \frac{2}{\sin \alpha_0}. \tag{5.1}$$

Consider now the height h_z starting at z (which still lies on the boundary at infinity). Since $\alpha_0 < \frac{\pi}{2}$, the point z' where h_z meets ℓ_1, is between x and y on ℓ_1. Considering the triangle yzz', yields $\beta_1 < \frac{\pi}{2}$, since $\beta_1 + \sphericalangle_{z'}(y, z) = \beta_1 + \frac{\pi}{2} < \pi$. With (5.1) and, since for $t \in [0, \frac{\pi}{2}]$ we have $\sin t \geq \frac{2}{\pi} t$, we obtain:

$$\frac{2}{\pi} \beta_1 \cosh \ell_1 \leq \frac{2}{\sin \alpha_0}$$

and therefore

$$\beta_1 \leq \frac{\pi}{\cosh \ell_1 \sin \alpha_0} \leq \frac{1}{\ell_1},$$

where the second inequality holds since we have

$$\ell_0 \pi \leq \sin \alpha_0 \cosh \ell_0$$

and therefore also

$$\ell_1 \pi \leq \sin \alpha_0 \cosh \ell_1$$

for $\ell_1 \geq \ell_0$, since $\forall x \in \mathbb{R} : x < \cosh x$.

If we consider a triangle where $\alpha > \alpha_0$ or z not on the boundary at infinity, i.e. a smaller angle for β_1, this still holds because of monotonicity.

□

Now, we will show that we can find a geodesic that is "near" the original polygonal chain, i.e. intersects the exact same sequence of 5-colored sides of the n-gons as the polygonal chain. We start with the original polygonal chain consisting of segments of length $\ell_0 := a$ as calculated in Lemma 5.3. Without loss of generality we choose one of the vertices of the polygonal chain and name it x_0. Again without loss of generality, we label all of the vertices along the polygonal chain as $\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots$ starting at x_0 and choosing a direction for the "positive" labelling.

First, we need to prove the following lemma:

Lemma 5.6. *Let $n = 110$. Consider a 5-colored regular right-angled tiling of $\mathbb{H}$ with n-gons and a polygonal chain as constructed above according to a given ϕ-aperiodic sequence. Consider vertices x_j, x_{j-1}, x_k, $k, j \in \mathbb{Z}$, of the polygonal chain, and $\overline{x_i x_{i+1}} = a := \ell_0$, $i \in \mathbb{Z}$, where a is as defined in Lemma 5.3.*

Then, for any angle of the type $\sphericalangle_{x_j}(x_{j-1}, x_k)$, we have

$$\sphericalangle_{x_j}(x_{j-1}, x_k) \leq \frac{1}{\ell_0}.$$

Proof. Let $\alpha_0 := \frac{2}{3} \cdot \frac{2\pi}{5} = \frac{4\pi}{15}$. We can easily verify by hyperbolic trigonometry that $\ell_0 \pi \leq \sin \alpha_0 \cosh \ell_0$ and $\ell_0 - 2h \geq 0$, where $h := \operatorname{arcosh}\left(\frac{1}{\sin(\frac{2\pi}{15})}\right)$.

Now we can proceed by induction on $m = |j - k|$.

For the base case $m = 1$, let $j := k + 1$. Since

$$\sphericalangle_{x_{k+1}}(x_k, x_{k+2}) \geq \frac{2\pi}{5} > \alpha_0$$

and

$$\overline{x_{k-2}x_{k-1}} = \overline{x_{k-1}x_k} \geq \ell_0,$$

this follows immediately with Lemma 5.5 for $n = 110$.

For the inductive step $m \curvearrowright m+1$ (see also Figure 5.6), first consider the angle $\sphericalangle_{x_j}(x_k, x_{j+1})$, where $m = j - k$. We have by hyperbolic trigonometry and the inductive hypothesis:

$$\sphericalangle_{x_j}(x_k, x_{j+1}) = \sphericalangle_{x_j}(x_{j-1}, x_{j+1}) - \sphericalangle_{x_j}(x_k, x_{j-1}) \geq \frac{2\pi}{5} - \frac{1}{\ell_0} \geq \alpha_0.$$

We have $\overline{x_j x_{j+1}} = \ell_0$ and by inductive hypothesis we have that $\overline{x_k x_{j-1}} \geq \ell_0$ and hence by Lemma 5.5 for the triangle with vertices x_k, x_{j-1}, x_j and $\sphericalangle_{x_{j-1}}(x_k, x_j) \geq \alpha_0$ we have $\overline{x_k x_j} \geq \ell_0$. Hence, we may apply Lemma 5.5 to the triangle with vertices x_k, x_j, x_{j+1} and $\sphericalangle_{x_j}(x_k, x_{j+1}) \geq \alpha_0$, and obtain $\sphericalangle_{x_{j+m+1}}(x_{j+m}, x_k) \leq \frac{1}{\ell_0}$. □

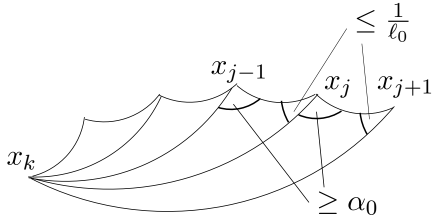

Figure 5.6: The angles used in the inductive step of the proof of Lemma 5.6

Now we may state and prove the following proposition:

Proposition 5.7. *Let $n = 110$. Consider a 5-colored regular right-angled tiling of $\mathbb{H}$ with n-gons and a polygonal chain as constructed above according to a given ϕ-aperiodic sequence. Then, in any triangle with vertices x_i, x_k and x_j of the polygonal chain, where $i, j, k \in \mathbb{Z}$ and $i < k < j$, we have*

$$\operatorname{dist}(x_k, \overline{x_i x_j}) \leq h := \operatorname{arcosh}\left(\frac{1}{\sin(\frac{2\pi}{15})}\right).$$

Proof. Let $\alpha_0 := \frac{2}{3} \cdot \frac{2\pi}{5} = \frac{4\pi}{15}$ and $\ell_0 := a$, where a is as defined in Lemma 5.3. We can easily verify by hyperbolic trigonometry that $\ell_0 \pi \leq \sin \alpha_0 \cosh \ell_0$ and $\ell_0 - 2h \geq 0$.

Consider the triangle with the vertices x_j, x_k, x_i and the angle $\sphericalangle_{x_k}(x_i, x_j)$ (see also Figure 5.7). In order to be able to apply Lemma 5.5, we need to verify that $\sphericalangle_{x_k}(x_i, x_j) \geq \alpha_0$, and $\overline{x_k x_i} \geq \ell_0$, $\overline{x_k x_j} \geq \ell_0$.

Obviously, we have

$$\sphericalangle_{x_k}(x_i, x_j) \geq \frac{2\pi}{5} - \sphericalangle_{x_k}(x_{k-1}, x_j) - \sphericalangle_{x_k}(x_i, x_{k+1}).$$

Using Lemma 5.6 and hyperbolic trigonometry, we obtain:

$$\sphericalangle_{x_k}(x_i, x_j) = \frac{2\pi}{5} - \sphericalangle_{x_k}(x_{k-1}, x_j) - \sphericalangle_{x_k}(x_i, x_{k+1}) \geq \frac{2\pi}{5} - 2\frac{1}{\ell_0} \geq \alpha_0.$$

Using the same argument as in the proof of Lemma 5.6, we obtain that $\overline{x_k x_i} \geq \ell_0$ and $\overline{x_k x_j} \geq \ell_0$.

Hence we may apply Lemma 5.5 to the triangle with vertices x_j, x_k, x_i and the angle $\sphericalangle_{x_k}(x_i, x_j) \geq \alpha_0$, and obtain

$$\mathrm{dist}(x_k, \overline{x_i x_j}) \leq h.$$

□

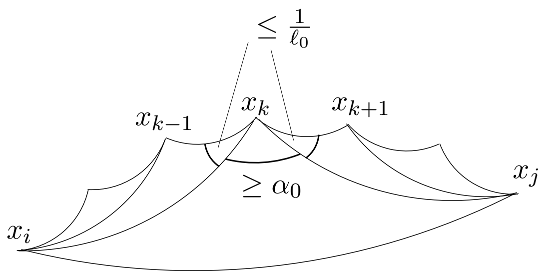

Figure 5.7: The angles used in the proof of Proposition 5.7

Theorem 5.8. *Given a $\phi(\ell) = 2^\ell$-aperiodic sequence for $|\mathcal{A}| = 5$ and a tiling of $\mathbb{H}$ with regular right-angled 110-gons which is 5-colored, there is a geodesic which intersects colored edges of the tiles in the exact same sequence as the given sequence.*

Proof. Consider labelled vertices and a polygonal chain such as in Lemma 5.6 and Proposition 5.7. With the proof of Lemma 5.6 we have that if $m = |j - k|$, we get a longer geodesic segment $\overline{x_j x_k}$ if m is chosen to be larger. Hence, if we choose $m := \infty$, we obtain a geodesic ray, starting at x_k. Without loss of generality we may choose $x_k = x_0$. We can therefore have a most simple polygonal chain consisting of two geodesic rays starting at x_0 in different directions. Applying, again, Proposition 5.7 to this situation, we obtain a geodesic within a maximal distance of h to all the vertices x_i, $i \in \mathbb{Z}$, which also intersects with the appropriate colored edges of the n-gons in the exact same order as the ϕ-aperiodic sequence, since $h \leq b$, where b is as defined in Lemma 5.3.

This geodesic is unique, since if there were two of them, they would have infinite distance somewhere, which is impossible by construction and Proposition 5.7.

□

Remark 5.9. *The above described systematic construction that leads eventually to Theorem 5.8 can also be done and verified by monotonicity for any tiling with $5r$-gons, where $r \geq 22$.*

Also, similar results can be obtained for $k \geq 5$, i.e. a greater number of colors in the original sequence, which can again be verified by monotonicity. For each k there will be a minimal r for the kr-gons such that the desired results may be obtained.

The example with 110-gons was chosen here, since it is the smallest possible number for which this result can be obtained such that the proof for Lemma 6.1 will work and hence we eventually may get Theorem 6.8.

Were we only to look for a geodesic in $\mathbb{H}$ intersecting the exact same sequence of walls as the polygonal chain, it would be sufficient to consider $r \geq 3$, since this is the smallest number for which Lemma 5.6 holds.

5.4 A Systematic Construction of the Geodesic

In this section we give an explicit and very systematic construction of a geodesic which intersects colored edges of a regular right-angled tiling of $\mathbb{H}$ with n-gons in the exact same sequence as does a given ϕ-aperiodic sequence with $|\mathcal{A}| = 5$. The construction will work for $n \geq 265$.

We will construct a geodesic by iteratively constructing a series of polygonal chains with longer edges and then taking the limit of this series. We start with a polygonal chain with labelled vertices $\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \dots$ as it was introduced in the last section. Now, starting at x_0 and connecting every other vertex in this polygonal chain with a new segment denoted by ℓ_1, we obtain a new polygonal chain, which has the vertices $\dots, x_{-6}, x_{-4}, x_{-2}, x_0, x_2, x_4, x_6, \dots$ and they are connected by segments in this order (see also Figure 5.8).

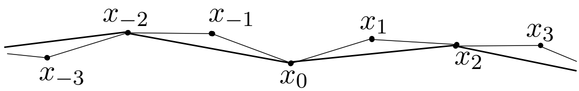

Figure 5.8: The original polygonal chain and the polygonal chain of the first step of the iteration.

In general, if we have the polygonal chain of step k of the iteration, where the segments connect the vertices

$$\dots, x_{-3\cdot 2^k}, x_{-2\cdot 2^k}, x_{-2^k}, x_0, x_{2^k}, x_{2\cdot 2^k}, x_{3\cdot 2^k}, \dots$$

in this order, we obtain the polygonal chain of step $k+1$ of the iteration by connecting every other vertice of the old polygonal chain. The new polygonal chain will connect the vertices

$$\dots, x_{-3\cdot 2^{k+1}}, x_{-2\cdot 2^{k+1}}, x_{-2^{k+1}}, x_0, x_{2^{k+1}}, x_{2\cdot 2^{k+1}}, x_{3\cdot 2^{k+1}}, \dots$$

in this order (see also Figure 5.9).

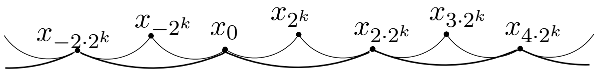

Figure 5.9: The polygonal chains of step k and step $k+1$ of the iteration.

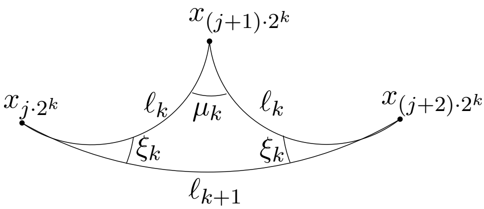

Figure 5.10: One of the considered triangles with sides that are segments of polygonal chains from different steps.

Throughout the construction, we will consider triangles consisting of sides of different polygonal chains. First, we consider a triangle, which will have two sides ℓ_k and one ℓ_{k+1}, as they occur throughout the iteration of constructing new polygonal chains. We denote the angle included by the two sides ℓ_k by μ_k and the two other angles by ξ_k (see also Figure 5.10).

Obviously, not all angles and sides of step k of the iteration are of the same size. But we may define the following:

Definition 5.10. Let $k \in \mathbb{N}$. We will use these notations in the following (see also Figure 5.10):

(i) $\mu_k := \inf_{j\in\mathbb{Z}} \sphericalangle_{x_{(j+1)\cdot 2^k}}(x_{j\cdot 2^k}, x_{(j+1)\cdot 2^k})$,

$\mu_0 := \inf_{j\in\mathbb{Z}} \sphericalangle_{x_{(j+1)}}(x_j, x_{(j+1)}) \geq \frac{2\pi}{5}$,

(ii) $\xi_k := \sup_{j\in\mathbb{Z}} \sphericalangle_{x_{j\cdot 2^k}}(x_{(j+1)\cdot 2^k}, x_{(j+2)\cdot 2^k})$,

$\xi_0 := \sup_{j\in\mathbb{Z}} \sphericalangle_{x_j}(x_{(j+1)}, x_{(j+2)})$,

(iii) $\ell_k := \inf_{j\in\mathbb{Z}} \operatorname{dist}(x_{j\cdot 2^k}, x_{(j+1)\cdot 2^k})$, $\ell_0 := a$, where a is as it was defined in Lemma 5.3,

(iv) $h_k := \sup_{j\in\mathbb{Z}} \operatorname{dist}(x_{(j+1)\cdot 2^k}, \overline{x_{j\cdot 2^k} x_{(j+2)\cdot 2^k}})$,
$h_0 := \sup_{j\in\mathbb{Z}} \operatorname{dist}(x_{j+1}, \overline{x_j, x_{j+2}})$.

Considering the above mentioned triangles, we can make sure that the polygonal chain obtained in step k of the iteration, is not too far away from the vertices which were in the polygonal chain of step $(k-1)$.

Now, we also have to make sure, that the polygonal chain of step k is close enough to any of the vertices, not just those from step $(k-1)$. This can be obtained by considering an edge connecting the vertices x_i and x_j, where $i, j \in \mathbb{Z}$ and $i < j$. This edge does not necessarily have to be one of the edges of a polygonal chain of the construction. Then, we consider the triangles $x_i x_k x_j$, where $i < k < j$ and for each of these triangles $d(x_k, x_i x_j)$ has to be small enough.

All of the above mentioned will be provided by the following proposition:

Proposition 5.11. *Let $k = 5$ the number of colors and $n = 265$ be the number of edges of the n-gons. Let $\alpha_0 := \frac{\pi}{5}$. Then, the following hold:*

(i) We have that $\ell_0 \pi \leq \sin\alpha_0 \cosh \ell_0$.

(ii) In the construction of the chains of polygons, the following hold for all $k \in \mathbb{N}$:

a) $\mu_k \geq \alpha_0$,

b) $\ell_k \geq (\frac{3}{2})^k \cdot \ell_0$,

c) $\xi_k \leq \frac{1}{\ell_{k-1}}$,

d) $h_k \leq h$, where h is as defined in Proposition 5.7.

(iii) *In any triangle with vertices x_i, x_k and x_j of the polygonal chain, where $i, j, k \in \mathbb{Z}$ and $i < k < j$, we have*

$$\operatorname{dist}(x_k, x_i x_j) \leq h.$$

Proof. (i) Let $\ell_0 := a$ from Lemma 5.3. Using Lemma 5.3, we can verify $\ell_0 \pi \leq \sin \alpha_0 \cosh \ell_0$.

(ii) We prove this by induction.

Base case:

a) For $k = 0$ we have $\mu_0 = \frac{2\pi}{5} \geq \alpha_0$.

b) For $k = 1$ we have by Lemma 5.5 and using the base case of (d) we have $h = \operatorname{arcosh} \frac{1}{\sin \frac{\varphi_0}{2}}$ and hence:

$$\ell_1 \geq 2\ell_0 - 2h_1 \geq 2\ell_0 - 2h \geq 2\ell_0 - 2h \geq \frac{3}{2}\ell_0.$$

c) For $k = 1$ this follows immediately with the base case of (a) and Lemma 5.5.

d) For $k = 0$ this follows immediately with the base case of (a) and Lemma 5.5.

Inductive step:

$k \curvearrowright k + 1$:

a) We have

$$\mu_{k+1} \geq \mu_0 - 2\sum_{j=1}^{k} \xi_j \underset{(*)}{\geq} 2\alpha_0 - 2\sum_{j=1}^{k} \frac{1}{\ell_{j-1}} =$$

$$= 2\alpha_0 - 2\sum_{j=1}^{k} \frac{1}{\left(\frac{3}{2}\right)^{j-1} \ell_0} = 2\alpha_0 - \frac{2}{\ell_0}\sum_{j=1}^{k} \left(\frac{2}{3}\right)^{j-1} =$$

$$= 2\alpha_0 - \frac{2}{\ell_0}\sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^{i} =$$

$$= 2\alpha_0 - \frac{2}{\ell_0}\left[\left(\sum_{i=0}^{k}\left(\frac{2}{3}\right)^i\right) - \left(\frac{2}{3}\right)^k\right] =$$

$$= 2\alpha_0 - \frac{2}{\ell_0}\left[\frac{1-\left(\frac{2}{3}\right)^{k+1}}{1-\frac{2}{3}} - \left(\frac{2}{3}\right)^k\right] =$$

$$= 2\alpha_0 - \frac{2}{\ell_0}\left[3 - \left(\frac{2}{3}\right)^k\right] \underset{(**)}{\geq} \alpha_0,$$

where $(*)$ holds because of the definitions of μ_0 and α_0, and because of the inductive hypothesis for (c), and $(**)$ holds by hyperbolic trigonometry for $n := 265 = 5 \cdot 53$. In fact, this is where, as was promised in the beginning of this section, we determine $r \geq 53$.

b) By Lemma 5.5 we have:

$$\ell_{k+1} \geq 2\ell_k - 2h_{k+1} \underset{(*)}{\geq} 2\ell_k - 2h \underset{(**)}{\geq} 2\left(\frac{3}{2}\right)^k \ell_0 - 2h =$$

$$= \left[\frac{3}{2}\left(\frac{3}{2}\right)^k \ell_0 + \frac{1}{2}\left(\frac{3}{2}\right)^k \ell_0\right] - 2h =$$

$$= \left(\frac{3}{2}\right)^{k+1} \ell_0 + \frac{1}{2}\left(\frac{3}{2}\right)^k \ell_0 - 2h \underset{(***)}{\geq} \left(\frac{3}{2}\right)^{k+1} \ell_0,$$

where $(*)$ holds because of (d), $(**)$ holds because of the inductive hypothesis of (b), and $(***)$ holds since

$$\frac{1}{2}\left(\frac{3}{2}\right)^k \ell_0 - 2h \geq 0$$

for $k \geq 1$, because we have $\ell_0 \geq 4h$.

c) This follows immediately with Lemma 5.5.

d) This follows immediately with the proof of Lemma 5.5.

(iii) This follows immediately with Proposition 5.7.

□

Remark 5.12. *The calculations in the inductive step for (ii) (a), make the choice of $r = 53$ and hence $n = 5 \cdot 53 = 265$ necessary: the inequalities there do not hold for smaller n.*

Now we can state and prove the main result of this chapter:

Theorem 5.13. *Given a $\phi(\ell) = 2^\ell$-aperiodic sequence for $|\mathcal{A}| = 5$ and a tiling of $\mathbb{H}$ with regular right-angled 265-gons which is 5-colored, there is a geodesic constructed systematically as described above which intersects colored edges of the tiles in the exact same sequence as the given sequence.*

Proof. With Proposition 5.11 we have that the segments of the polygonal chain grow with every step of the iteration. Therefore, for $n \to \infty$, we obtain two rays starting from x_0 in opposite directions. Furthermore, with Proposition 5.11 we have that any of the segments of any step of the construction is closer than h to any of the edges and any of the x_i, $i \in \mathbb{Z}$, of the original polygonal chains between its endpoints. Since x_0 was chosen arbitrarily, we may rename x_m to be x_0 and repeat the construction for this x_0. Since we may do this for any m and as many times as we want to, and each of these pairs of rays starting from the points x_0, x_m, x_{2m}, x_{3m}, $\ldots$ stays whithin the maximal distance of h of all x_i, $i \in \mathbb{Z}$, we obtain a series of such pairs. Taking the limit of $m \to \infty$ and sending thereby the x_0 to the boundary at infinity, we obtain a geodesic which stays within the maximal distance of h to all the x_i, $i \in \mathbb{Z}$ and also intersects with the appropriate colored edges of the n-gons in the exact same order as the ϕ-aperiodic sequence.

This geodesic is unique, since if there were two of them, they would have infinite distance somewhere, which is impossible, since they are both near the polygonal chains at all times. □

6 Behavior of the Geodesic on the Compact Surface

In this chapter we analyze the behavior of the geodesic $\widetilde{\gamma}$ constructed in Chapter 5, when projected by the projection pr from the universal covering $\mathbb{H}$ onto the quotient space $\mathbb{D} = \mathbb{H}/\Delta$ from Proposition 2.17. Hence we have $pr : \mathbb{H} \to \mathbb{D}$ and $Dpr : T\mathbb{H} \to T\mathbb{D}$. The geodesic $pr(\widetilde{\gamma})$ on $\mathbb{D}$ obtained this way shall be denoted by γ and will turn out to be f-aperiodic for a suitable f.

Let v_γ and u_γ be unit tangent vectors to the geodesic γ on $\mathbb{D}$, where $u_\gamma = \varphi_t(v_\gamma)$ with $\varphi_t(\cdot)$ the geodesic flow on $\mathbb{D}$, such that $d(u_\gamma, v_\gamma) < \varepsilon$, where $d(\cdot,\cdot)$ denotes the Sasaki metric and $\varepsilon > 0$.

Now consider $\tilde{\gamma}$, u and v, such that $pr(\tilde{\gamma}) = \gamma$, $Dpr(u) = u_\gamma$ and $Dpr(v) = v_\gamma$, where $u = \widetilde{\varphi}_t(v)$, u and v unit tangent vectors to $\tilde{\gamma}$, with $\widetilde{\varphi}_t(\cdot)$ the geodesic flow on $\mathbb{H}$, since pr is the projection of the universal covering. Let $\widetilde{\gamma}(t_v)$ be the starting point of v on $\widetilde{\gamma}$. Now, since $d(u_\gamma, v_\gamma) < \varepsilon$ on $\mathbb{H}$, and since the torsion-free subgroup Δ of the Coxeter group Γ (cf. Section 2.5) preserves the coloring of the reflection walls, there is a $\delta \in \Delta$ such that $d(\delta(u), v) < \varepsilon$ on $\mathbb{D}$. Let $\widetilde{\gamma}_u := \delta(\widetilde{\gamma})$. Then, $\widetilde{\gamma}$ and $\widetilde{\gamma}_u$ intersect a certain number of the same reflection walls. We will determine that number in the following, before stating the main results of this chapter, Theorem 6.8, which states how far a geodesic on $\mathbb{D}$ has to travel before coming ε-near to itself, i.e. will give an explicit definition of the function f for the f-aperiodicity, and Theorem 6.10 which states how the function f measuring the aperiodicity of a geodesic yields a closed flow-invariant subset of the unit tangent bundle.

6.1 Measuring the Aperiodicity of a Geodesic

In this section we will determine the f for the f-aperiodicity of the geodesic γ on $\mathbb{D}$. First, we need some technical results.

Lemma 6.1. *Let ω be a colored wall in $\mathbb{H}$, such that $\widetilde{\gamma}(t) \in \omega$ for a $t \in \mathbb{R}$. Then we have that $\frac{\pi}{4} \leq \sphericalangle(\widetilde{\gamma}, \omega) \leq \frac{3\pi}{4}$.*

Proof. Assume that $\alpha := \sphericalangle(\widetilde{\gamma}, \omega) < \frac{\pi}{4}$.

Denote by τ the geodesic connecting the points x_k and x_{k+1}, $k \in \mathbb{Z}$, of the geodesic chain (see also Figure 6.1). We have that $\tau(-\frac{a}{2}) = x_k$, $\tau(\frac{a}{2}) = x_{k+1}$ and $\tau(0) \in \omega$. Define for $t \in \mathbb{R}$ the function $f(t) := \operatorname{dist}(\widetilde{\gamma}(t), \tau)$. Obviously, f is a convex function and $f(t) \geq f'(0) \cdot t$. Hence, since we have that $f'(0) = \cos\alpha$, we obtain $f(t) \geq \cos\alpha \cdot t \geq \frac{\sqrt{2}}{2} t$.

Define $g(t) := \operatorname{dist}(\widetilde{\gamma}(t), x_{k+1})$, $t \in \mathbb{R}$. This implies

$$g(t) \geq f(t) = \operatorname{dist}(\widetilde{\gamma}(t), \tau) \geq \frac{\sqrt{2}}{2} t.$$

On the other hand we have $g(t) \geq \frac{a}{2} - t$. Hence we have

$$g(t) \geq \max\{\frac{\sqrt{2}}{2} t, \frac{a}{2} - t\} \geq h,$$

where the last inequality holds since $a \geq 5h$ for $n \geq 110$.

But this is a contradiction to the construction of $\widetilde{\gamma}$, hence the assumption was wrong.

□

Lemma 6.2. *There exists a positive constant $C \in \mathbb{R}$, with the following property:*

Let $\widetilde{\gamma}(t_0) \in \omega$, where ω is a (colored) wall (see also Figure 6.2), let $\dot{\widetilde{\gamma}}(t_v) = v$ and $\widetilde{\gamma}_u$ be a geodesic with $\dot{\widetilde{\gamma}}_u(t_u) = u$, where u is a unit vector with $d(u, \dot{\widetilde{\gamma}}(t_v)) := d(u, v) = \varepsilon$ and $t_v, t_u, t_0 \in \mathbb{R}$, where $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \in \mathbb{R}$. Assume that $\widetilde{\gamma}_u$ does not intersect ω.

Then, we have

$$|t_v - t_0| \geq \ln \frac{C}{\varepsilon}.$$

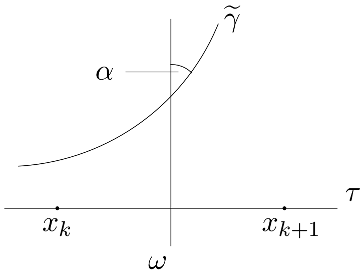

Figure 6.1: The situation in the proof for Lemma 6.1.

Proof. Assume that ω and $\widetilde{\gamma}_u$ do not intersect.

Let $\widetilde{\gamma}_\varepsilon$ be a perpendicular geodesic to $\widetilde{\gamma}$ which is going through $\widetilde{\gamma}_u(t_u)$. Let v' denote the parallel transport of v along $\widetilde{\gamma}$ to the intersection point of $\widetilde{\gamma}_\varepsilon$ and $\widetilde{\gamma}$, and let v'' denote the parallel transport of u along $\widetilde{\gamma}_\varepsilon$ to the intersection point of $\widetilde{\gamma}_\varepsilon$ and $\widetilde{\gamma}$ (also see Figure 6.2). Considering u, v, v'' and v', we obtain:

$$d(u,v) \leq \varepsilon \Longrightarrow \operatorname{dist}(\widetilde{\gamma}_u(t_u), \widetilde{\gamma}(t_v)) \leq \varepsilon$$

$$\Longrightarrow \operatorname{dist}(\widetilde{\gamma}_u(t_u), \widetilde{\gamma}(t_{v'})) \leq \varepsilon \Longrightarrow d(v,v') \leq \varepsilon$$

$$\Longrightarrow d(u,v') \leq 2\varepsilon.$$

If $\widetilde{\gamma}(t_{v'}) = \widetilde{\gamma}_u(t_u)$, we have $u = v''$, hence $d(u, v'') = 0$. Consider now $u \neq v''$.

Let $\widetilde{\gamma}_{v''}$ denote a geodesic starting at $\widetilde{\gamma}(t_v)$ which does not intersect ω, and ω' denote a geodesic starting at $\widetilde{\gamma}(t_0)$ which is asymptotic to $\widetilde{\gamma}_{v''}$ (see also Figure 6.3). If ω is asymptotic to $\widetilde{\gamma}_{v''}$, we have $\omega = \omega'$. We may now consider the triangle with edges $\widetilde{\gamma}$, ω' and $\widetilde{\gamma}_{v''}$. We have $\sphericalangle(\omega', \widetilde{\gamma}_{v''}) = 0$.

Consider $\widetilde{\gamma}_p$, the geodesic that intersects $\widetilde{\gamma}_\varepsilon$ at a right angle in the point which we shall denote by p and which is also asymptotic to $\widetilde{\gamma}_{v''}$ and ω'. For the triangle with edges $\widetilde{\gamma}_\varepsilon$, $\widetilde{\gamma}_p$ and $\widetilde{\gamma}_{v''}$, hyperbolic geometry yields $\cos(\sphericalangle(\widetilde{\gamma}_{v''}, \widetilde{\gamma}_\varepsilon)) = \tanh(|\widetilde{\gamma}(t'_v)p|)$, where $|\widetilde{\gamma}(t'_v)p|$ denotes the length of the section of $\widetilde{\gamma}$ from $\widetilde{\gamma}(t'_v)$ to p. Now, consider $\widetilde{\gamma}'_u$ a geodesic starting at $\widetilde{\gamma}_u(t_u)$ which is asymptotic to $\widetilde{\gamma}_p$. If $\widetilde{\gamma}_u$ is asymptotic to $\widetilde{\gamma}_p$, we have $\widetilde{\gamma}_u = \widetilde{\gamma}_{u'}$. Considering the triangle with edges $\widetilde{\gamma}_\varepsilon$, $\widetilde{\gamma}_p$ and $\widetilde{\gamma}_{u'}$, we obtain $\cos(\sphericalangle(\widetilde{\gamma}_{u'}, \widetilde{\gamma}_\varepsilon)) = \tanh(|\widetilde{\gamma}_u(t_u)p|)$, respectively. Since

$$|\left(\cos^{-1}(\tanh x)\right)'| = |-\sqrt{1-\tanh^2 x}| \leq 1$$

we have that $\cos^{-1}(\tanh x)$ is 1-Lipschitz and hence

$$|\sphericalangle(\widetilde{\gamma}_{u'}, \widetilde{\gamma}_\varepsilon) - \sphericalangle(\widetilde{\gamma}_{v''}, \widetilde{\gamma}_\varepsilon)| \leq \Big| |\widetilde{\gamma}(t_v)p| - |\widetilde{\gamma}_u(t_u)p| \Big| = \varepsilon.$$

Now, since $\frac{\pi}{2} = \sphericalangle(\widetilde{\gamma}_u, \widetilde{\gamma}_\varepsilon) = \sphericalangle(\widetilde{\gamma}_{v''}, \widetilde{\gamma}_\varepsilon) + \sphericalangle(\widetilde{\gamma}, \widetilde{\gamma}_{v''})$ we obtain $\sphericalangle(\widetilde{\gamma}, \widetilde{\gamma}_{v''}) < \varepsilon$. Hence we have $d(u, v'') \leq 2\varepsilon$. Since we already obtained $d(u, v'') = 0 \leq 2\varepsilon$ for $u = v''$, this holds in any case.

By Lemma 6.1 and the definition of ω', we have $\sphericalangle(\widetilde{\gamma}, \omega') < \frac{3\pi}{4}$, and hyperbolic trigonometry yields

$$\cosh(|t_v - t_0|) \geq \frac{1 + \cos\varepsilon \cdot \cos(\sphericalangle(\widetilde{\gamma}, \omega))}{\sin\varepsilon \cdot \sin(\sphericalangle(\widetilde{\gamma}, \omega))}.$$

Hence, we have

$$|t_v - t_0| \geq \operatorname{arcosh}\left(\frac{C}{\varepsilon}\right) = \ln\left(\frac{C}{\varepsilon} + \sqrt{\left(\frac{C}{\varepsilon}\right)^2 - 1}\right) \geq \ln\left(\frac{C}{\varepsilon}\right),$$

where $C \in \mathbb{R}^+$.

□

Remark 6.3. *For later reference we define* $\mathfrak{L}(\varepsilon) := \ln\frac{C}{\varepsilon}$.

Corollary 6.4. *Consider $\widetilde{\gamma}$ and $\widetilde{\gamma}_u$ as before. If a certain section of $\widetilde{\gamma}$ of length $t_1 > 1$, $t_1 \in \mathbb{R}$, intersects a certain sequence of walls, which are all also intersected by $\widetilde{\gamma}_u$, then there is a section of $\widetilde{\gamma}_u$ of length $t_2 > 1$, $t_2 \in \mathbb{R}$, in which it intersects exactly the same sequence of walls.*

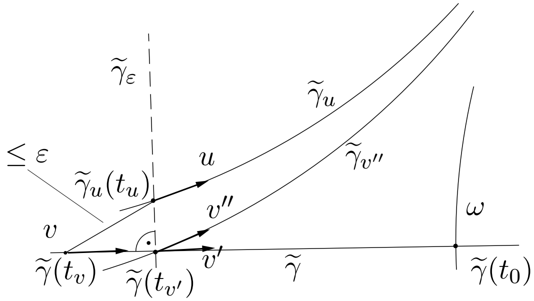

Figure 6.2: The situation in the proof for Lemma 6.2, focussing on v, v', v'' and u.

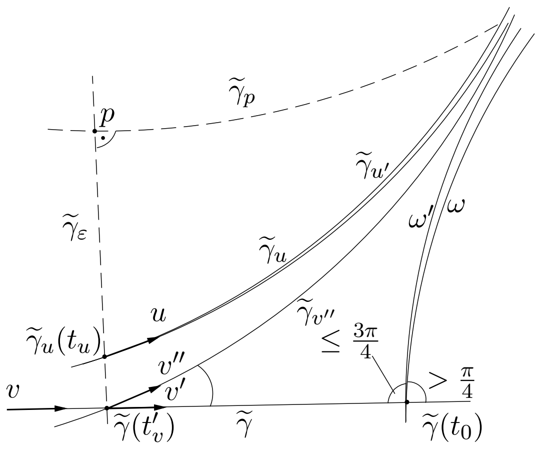

Figure 6.3: The situation in the proof for Lemma 6.2, focussing on the geodesics.

Proof. Consider a subsequence consisting of the walls ω_1 and ω_2 which is intersected by $\widetilde{\gamma}$ and also by $\widetilde{\gamma}_u$. Assume that there is a ω_3 which intersects $\widetilde{\gamma}_u$ such that $\widetilde{\gamma}_u$ intersects the subsequence $\omega_1\omega_3\omega_2$, but ω_3 does not intersect $\widetilde{\gamma}$. Since ω_3 and $\widetilde{\gamma}$ do not intersect, ω_3 intersects either ω_1 oder ω_2. But this is impossible, since walls never intersect by construction. Hence ω_3 cannot exist.

By induction we obtain the claim for sequences of any length $n \in \mathbb{N}$. $\square$

Lemma 6.5. *Let $\widetilde{\gamma}(t_1) \in \omega_1$ and $\widetilde{\gamma}(t_2) \in \omega_2$, $t_1, t_2 \in \mathbb{R}$, where ω_1 and ω_2 are two consecutive colored walls that are intersected by $\widetilde{\gamma}$. Then we have for n the number of edges of the right-angled regular polygons of the tiling:*

$$\frac{2(-\cos\left(\left(\frac{n}{5}-1\right)\frac{2\pi}{5}\right)+2\cos\left(\left(\frac{n}{5}-1\right)\frac{2\pi}{5}\right)\cdot\cos^2\frac{\pi}{n}-\cos^2\frac{\pi}{n})}{\cos\frac{2\pi}{n}-1} <$$

$$< |t_1 - t_2|$$

and

$$|t_1 - t_2| <$$

$$< \frac{2(-\cos\left(\left(2\frac{n}{5}+1\right)\frac{2\pi}{5}\right)+2\cos\left(\left(2\frac{n}{5}+1\right)\frac{2\pi}{5}\right)\cdot\cos^2\frac{\pi}{n}-\cos^2\frac{\pi}{n})}{\cos\frac{2\pi}{n}-1}.$$

Proof. The length $|t_1 - t_2|$ can be bounded by considering a triangle with two sides of length c which connect the middle point of one of the n-gon to one of its vertices, and the third side $\widetilde{x}$. Denote the angle between the two sides of length c by ν. Using Lemma 5.3 and hyperbolic trigonometry we have

$$c = \cosh\frac{a}{2}\cdot\cosh\frac{b}{2}.$$

The angle ν fulfills by construction (see also Figure 6.4)

$$\frac{104\pi}{265} = \left(\frac{n}{5}-1\right)\frac{2\pi}{5} = \nu_- < \nu < \nu_+ = \left(2\frac{n}{5}+1\right)\frac{2\pi}{5} = \frac{214\pi}{265}.$$

Again, using hyperbolic trigonometry, this yields what was claimed. $\square$

Remark 6.6. *For later reference we define $x := |t_1 - t_2|$.*

Corollary 6.7. *Let $t \in \mathbb{R}$. Then, $\widetilde{\gamma}|_{[t-\mathfrak{L}(\varepsilon),t+\mathfrak{L}(\varepsilon)]}$ intersects N consecutive colored walls, where $N \geq 2\cdot\frac{\mathfrak{L}(\varepsilon)}{x}$, $N \in \mathbb{N}$.*

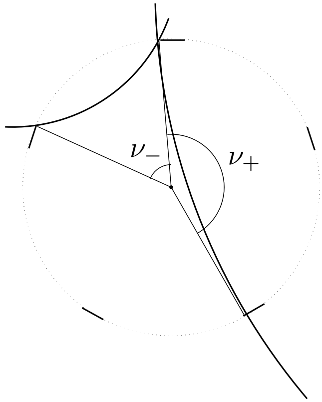

Figure 6.4: A schematic drawing of the minimal and maximal angle possible, ν_- and ν_+ respectively, which can appear for a triangle as used in the proof for Lemma 6.5.

Proof. This follows immediately using Lemma 6.2 and Lemma 6.5. □

Now we can state the main result of this section:

Theorem 6.8. *Consider the geodesic $\gamma : \mathbb{R} \to \mathbb{D}$. There exist positive constants $\eta, C \in \mathbb{R}$, such that for all $t, s \in \mathbb{R}$, we have*

$$d(\dot\gamma(t), \dot\gamma(s)) \geq \min\left\{ |t-s|, \frac{C}{|t-s|^\eta} \right\}.$$

Proof. For $|t-s| \leq 1$, we have $\min\{|t-s|, \frac{C}{|t-s|^\eta}\} = |t-s|$, since $\eta, C \geq 0$. By definition of the Sasaki metric, it follows immediately that $\varepsilon := d(\dot\gamma(t), \dot\gamma(s)) \geq |t-s|$.

For $|t-s| \geq 1$, we have $\min\{|t-s|, \frac{C}{|t-s|^\eta}\} = \frac{C}{|t-s|^\eta}$. Using Corollary 6.7, Theorem 3.7 and setting $\eta := \frac{2 \cdot \ln 2}{x}$, we obtain $\varepsilon \geq \frac{C}{|t-s|^\eta}$. □

Remark 6.9. *Theorem 6.8 tells us that γ is an f-aperiodic geodesic on $\mathbb{D}$ and even indicates an explicit definition of a function f, namely $f : [1, \infty) \to (0, \infty)$, $f(s) := \frac{C}{s^\eta}$, where $C, \eta \in \mathbb{R}^+$.*

6.2 A Closed Flow-Invariant Subset

In this section we will present a closed flow-invariant subset SM that emerges from the construction that eventually led to Theorem 6.8.

As it was done in Section 4.2, we consider a set C_f, in the situation described in Section 6.1: we consider the geodesic γ on $\mathbb{D}$ and the geodesic flow φ_t.

Now, we can state the main result of this section:

Theorem 6.10. *Consider γ from Section 6.1 on $\mathbb{D}$. Let*

$$f : [1, \infty) \to (0, \infty), f(s) := \frac{C}{s^\eta},$$

where $C, \eta \in \mathbb{R}^+$ are as in Theorem 6.8. Then,

$$C_f := \{v \in S\mathbb{D} \mid d(\varphi_{t+s}(v), \varphi_t(v)) \geq f(s), \forall t \in \mathbb{R}, \forall s \geq 1)\}$$

is non-empty and a closed flow-invariant subset of $S\mathbb{D}$.

Proof. This follows immediately from the existence of γ, Theorem 6.8 and Lemma 4.8. □

Remark 6.11. *The function f defined in Theorem 6.10 is in fact smaller than the function stated in Proposition 4.10, as it was predicted there.*

Remark 6.12. *With Remark 4.11 and Theorem 6.10 we have that there exist many other f-aperiodic geodesics on $\mathbb{D}$ besides γ.*

Corollary 6.13. *The set C_f has Liouville measure* 0 *and the geodesic flow φ_t is ergodic.*

Proof. This follows immediately from Theorem 6.10, using Remark 4.11 and Remark 4.12.

□

7 Conclusion

7.1 Recapitulation

In this thesis, we tried to adapt the concept of ϕ-aperiodicity for sequences which provides a means for measuring the aperiodicity of sequences to the setting of geodesics. Aperiodic geodesics on a compact Riemannian surface are as exceptional as aperiodic sequences. We introduced the notion of f-aperiodicity for geodesics in analogy to ϕ-aperiodicity of sequences to provide a means for measuring the aperiodicity of geodesics. So far, the existence of such a geodesic was only postulated and in no way proven.

To prove the existence of f-aperiodic geodesics, a very specific setting was chosen and constructed. The construction was aimed at obtaining a geodesic on a quotient of $\mathbb{H}$. Hence, the first step was to choose an adequate tiling of $\mathbb{H}$. The chosen tiling consisted of regular right-angled n-gons, where $n = 5r$ and $n, r \in \mathbb{N}^*$, was determined according to conditions that were given in different steps of the construction. Then, a ϕ-aperiodic sequence in $\mathcal{A}$, where $|\mathcal{A}| = 5$, was assumed to be given. The tiling was colored with 5 colors such that the colors were evenly distributed. Then, a polygonal chain was constructed on $\mathbb{H}$ in a way, that each segment would connect middle points of adjacent tiles and such that the segments intersected colored walls in the exact same sequence as the colors would appear in the ϕ-aperiodic sequence. Then, it was proven that there exists a geodesic which also intersect that exact same sequence of colored walls. In proving the existence of this geodesic γ, Lemma 5.5 turned out to be crucial: this lemma provided the means to control the growth of the height and the angles in triangles that appeared when making the segments of the polygonal chain longer. This was used in many steps of the calculations.

In the next step, we passed on to a compact quotient $\mathbb{D}$ of $\mathbb{H}$, by dividing by a suitable subgroup of the group associated with the tiling of $\mathbb{H}$. With the associated projection, the geodesic γ on $\mathbb{H}$ was projected to a geodesic $\widetilde{\gamma}$ on $\mathbb{D}$. It was then proven that $\widetilde{\gamma}$ is in fact f-aperiodic and the function f could even be stated explicitly. In this proof, it was crucial to use some properties of the ϕ-aperiodic sequence $\widetilde{\gamma}$ was based on and properties of γ on $\mathbb{H}$ which were used for determining properties of $\widetilde{\gamma}$ on $\mathbb{D}$ by applying the canonical projection from $\mathbb{H}$ to $\mathbb{D}$. Hence, the existence of some f-aperiodic geodesics under certain circumstances had been proven.

Then, practically as a corollary of the construction, by using $\widetilde{\gamma}$ and f on $\mathbb{D}$, we obtained a closed flow-invariant subset of SM. Flow invariant subsets are not hard to find, but it is not easy to find such a set which is closed. The existence of such a closed set implies the existence of more than one f-aperiodic geodesic on a specific $\mathbb{D}$ with a given f.

7.2 Open Questions

This thesis provides only a first glimpse on the task of measuring the aperiodicity of geodesics on compact Riemannian surfaces. Hence, there is a vast number of open questions related to the results of this thesis. Let us just mention two of them to give a general idea of where further related research might be going:

(i) Are there f-aperiodic geodesics with suitable non-trivial, positive and decreasing functions f on arbitrary compact Riemannian manifolds with negative curvature?

(ii) What is an "optimal" f such that $C_f \neq \emptyset$?

Since the constructions and proofs in this thesis have been chosen specifically to meet the requirements of the very specific situation considered here, it might most likely not be possible to generalize the results of this thesis sufficiently to answer all of the arising further questions using the same methods as those that were used here. Nevertheless, having some specific setting in which a result can be obtained might at least be motivation enough to keep trying to find more general results.

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