Finiteness of leading monomial ideals and critical cones of characteristic varieties

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Abstract: Diese Dissertation besteht aus zwei Teilen. Im ersten Teil geben wir einen in sich abgeschlossenen und einheitlichen Ansatz zu einigen Endlichkeitsergebnissen über Leitmonomideale von Idealen im Polynomring bezüglich verschiedener Typen von totalen Monomordnungen. Die Ergegnisse in diesem Teil sind weitgehend nicht neu und können in den Arbeiten anderer Autoren gefunden werden, entweder basierend auf verschiedenen Ansätzen oder angewandt auf verschiedene Kontexte. Wir verallgemeinern einen Teil dieser Resultate auf Vektorräume, die zum Polynomring isomorph sind, einen Teil auf die grosse Klasse der zulässigen Algebren, welche zumindest die Klasse der Algebren von auflösbarem Typ umfasst. In der Literatur werden Leitmonomideale meistens nur bezüglich Monoidordnungen von Nt0 mit t N studiert, weil diese Ordnungen eine ergebnisreiche Divisionstheorie induzieren. In diesem Rahmen stellt der Macaulay'sche Basissatz den Schlüssel zu den Endlichkeitsresultaten für Leitmonomideale dar. Wir betrachten Leitmonomideale bezüglich Totalordnungen, Gradordnungen, Halbgruppenordnungen, Monoidordnungen, und gradverträglicher Monoidordnungen. Es stellt sich heraus, dass ein Ideal im Polynomring höchstens endlich viele bezüglich der Inklusion minimale Leitmonomideale besitzt, die aus Totalordnungen stammen. Weiter besitzt ein Ideal höchstens endlich viele minimale Leitmonomideale bezüglich Gradordnungen. Durch Monoidordnungen induzierte Leitmonomideale sind wegen einer hier bewiesenen leicht verallgemeinerten Version des Macaulay'schen Basissatzes minimal, und es folgt so, dass es nur endlich viele Leitmonomideale bezüglich Monoidordnungen zu einem gegebenen Ideal gibt. Anfangs hatten wir geplant, die Existenz von universellen Gröbnerbasen in zulässigen Algebren mithilfe der erwähnten Endlichkeitsresultate durch Nachahmung des klassischen Beweises im Polynomring zu zeigen. Das war unsere ursprüngliche Motivation, diese Endlichkeitseigenschaften zu untersuchen. In der Tat folgt aber die Existenz universeller Gröbnerbasen schon aus der Tatsache, dass die Totalordnungen auf einer gegebenen Menge einen kompakten topologischen Raum bilden und die zulässigen Algebren noethersch sind. Mit diesem Thema beenden wir den ersten Teil der vorliegenden Arbeit. Der zweite und innovative Teil dieser Dissertation stellt den Inhalt unseres Artikels [13] dar, welcher im Dezember 2010 zur Publikation in den Transactions of the American Mathematical Society angenommen worden ist. Hier widmen wir uns den charakteristischen Varietäten von Moduln über Weylalgebren. Diese affinen Variet¨aten werden mit gewichteten Gradfiltrierungen eines endlich erzeugten Moduls über einer Weilalgebra konstruiert. Zun¨achst erinnern wir also einige Tatsachen über filtrierte Moduln und deren assoziierte graduierte Moduln. Für filtrierte Moduln über filtrierten kommutativen Ringen zeigen wir, dass der Annullator des assoziierten graduierten Moduls radikalgleich ist zum assoziierten graduierten Ideal des filtrierten Annullators. Ein klassischer Satz von Bernstein besagt, dass die einem gegebenen Modul zugehörigen charakteristischen Varietäten nach dem Grad und nach der Ordnung die gleiche Krulldimension haben. In der Tat haben alle charakteristischen Varietäten eines Moduls die gleiche Krulldimension. Dies wird üblicherweise durch homologische Methoden gezeigt. Wir betten den erwähnten Dimensionssatz in den grösseren Zusammenhang einer Deformationstheorie von gewichteten Gradfiltrierungen und Monomordnungen ein. Unser deformationstheoretischer Ansatz wendet universelle Gröbnerbasen an, und die erwähnte Dimensionsgleichheit folgt als Korollar aus einem tieferen Resultat. Charakteristische Varietäten zeigen nämlich ein bemerkenswertes Verhalten, wenn man ihre definierenden Filtrierungen durch gewisse Adjustierungen der Gewichtung deformiert. Genauer wird eine charakteristische Varietät durch solche

Deformationen in ihren eigenen kritischen Kegel übergeführt. Dies erlaubt, eine nichtendliche Filtrierung so zu deformieren, dass die entstehende Filtrierung endlich wird und die zu ihr assoziierte charakteristische Variet¨at gerade der kritische Kegel der ursprünglichen Varietät ist. Daraus folgt die Dimensionsgleichheit. Ein Grund hierfür ist, dass eine affine Variet¨at die gleiche Krulldimension wie ihr kritischer Kegel hat. Ein weiterer Grund ist, dass die Krull- und die GK-Dimension eines endlich erzeugten Moduls über einer endlich erzeugten kommutativen K-Algebra übereinstimmen. Ein dritter Grund ist, dass die GK-Dimension eines endlich filtrierten Moduls beim Übergang zum assoziierten graduierten Modul erhalten bleibt. Unser Resultat stellt auch einen ersten Schritt zur Klassifikation der charakteristischen Varietäten dar. Wir waren aber nicht in der Lage, eine solche Klassifikation in voller Allgemeinheit durchzuführen. Wir haben uns deshalb auf charakteristische Varietäten von zyklischen Moduln über der ersten Weylalgebra beschränkt und eine approximierte Klassifikation durch ein Computerexperiment berechnet. Das Experiment zeigt, dass der Gewichtsraum N20 r(0, 0) der Gradfiltrierungen in halbkegelförmige Gebiete unterteilt werden kann, welche jeweils zur selben charakteristische Varietät führen. Auf Grund dieses Experiments können wir auch eine obere Schranke für die Anzahl dieser charakteristischen Varietäten in Termen von Totalgraden der Elemente einer universellen Gröbnerbasis vermuten. Im Hinblick auf eine Arbeit von Aschenbrenner und Leykin [2] kann diese obere Schranke auch in Termen von Totalgraden von Erzeugern des Ideals angegeben werden, das den gegebenen zyklichen Modul definiert. Wir beenden den zweiten Teil mit einem Resultat von Skoda über die Lokalisierung von filtrierten Moduln. Mithilfe eines leichten Lemmas können wir Skodas Ergebnis eine geometrische Interpretation in unserem Kontext geben. Im ersten Anhang geben wir einen direkteren Beweis der Existenz von universellen Gröbnerbasen in Weylalgebren basierend auf den Divisionseigenschaften dieser Algebren und auf der Kompaktheit des topologischen Raums der Monoidordnungen. Im zweiten Anhang listen wir das Computerprogramm auf, das wir für das erw¨ahnte Experiment geschrieben haben.

Finiteness of Leading Monomial Ideals and Critical Cones of Characteristic Varieties

Dissertation
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A Mireille

«Penso a volte che noi siamo come il vento che trascorre impalpabile.»

Cesare Pavese

Zusammenfassung

Diese Dissertation besteht aus zwei Teilen. Im ersten Teil geben wir einen in sich abgeschlossenen und einheitlichen Ansatz zu einigen Endlichkeitsergebnissen über Leitmonomideale von Idealen im Polynomring bezüglich verschiedener Typen von totalen Monomordnungen. Die Ergegnisse in diesem Teil sind weitgehend nicht neu und können in den Arbeiten anderer Autoren gefunden werden, entweder basierend auf verschiedenen Ansätzen oder angewandt auf verschiedene Kontexte. Wir verallgemeinern einen Teil dieser Resultate auf Vektorräume, die zum Polynomring isomorph sind, einen Teil auf die grosse Klasse der zulässigen Algebren, welche zumindest die Klasse der Algebren von auflösbarem Typ umfasst.

In der Literatur werden Leitmonomideale meistens nur bezüglich Monoidordnungen von $\mathbb{N}_0^t$ mit $t \in \mathbb{N}$ studiert, weil diese Ordnungen eine ergebnisreiche Divisionstheorie induzieren. In diesem Rahmen stellt der Macaulay'sche Basissatz den Schlüssel zu den Endlichkeitsresultaten für Leitmonomideale dar.

Wir betrachten Leitmonomideale bezüglich Totalordnungen, Gradordnungen, Halbgruppenordnungen, Monoidordnungen, und gradverträglicher Monoidordnungen. Es stellt sich heraus, dass ein Ideal im Polynomring höchstens endlich viele bezüglich der Inklusion minimale Leitmonomideale besitzt, die aus Totalordnungen stammen. Weiter besitzt ein Ideal höchstens endlich viele minimale Leitmonomideale bezüglich Gradordnungen. Durch Monoidordnungen induzierte Leitmonomideale sind wegen einer hier bewiesenen leicht verallgemeinerten Version des Macaulay'schen Basissatzes minimal, und es folgt so, dass es nur endlich viele Leitmonomideale bezüglich Monoidordnungen zu einem gegebenen Ideal gibt.

Anfangs hatten wir geplant, die Existenz von universellen Gröbnerbasen in zulässigen Algebren mithilfe der erwähnten Endlichkeitsresultate durch Nachahmung des klassischen Beweises im Polynomring zu zeigen. Das war unsere ursprüngliche Motivation, diese

Endlichkeitseigenschaften zu untersuchen. In der Tat folgt aber die Existenz universeller Gröbnerbasen schon aus der Tatsache, dass die Totalordnungen auf einer gegebenen Menge einen kompakten topologischen Raum bilden und die zulässigen Algebren noethersch sind. Mit diesem Thema beenden wir den ersten Teil der vorliegenden Arbeit.

Der zweite und innovative Teil dieser Dissertation stellt den Inhalt unseres Artikels [13] dar, welcher im Dezember 2010 zur Publikation in den *Transactions of the American Mathematical Society* angenommen worden ist. Hier widmen wir uns den charakteristischen Varietäten von Moduln über Weylalgebren.

Diese affinen Varietäten werden mit gewichteten Gradfiltrierungen eines endlich erzeugten Moduls über einer Weilalgebra konstruiert. Zunächst erinnern wir also einige Tatsachen über filtrierte Moduln und deren assoziierte graduierte Moduln. Für filtrierte Moduln über filtrierten kommutativen Ringen zeigen wir, dass der Annullator des assoziierten graduierten Moduls radikalgleich ist zum assoziierten graduierten Ideal des filtrierten Annullators.

Ein klassischer Satz von Bernstein besagt, dass die einem gegebenen Modul zugehörigen charakteristischen Varietäten nach dem Grad und nach der Ordnung die gleiche Krulldimension haben. In der Tat haben alle charakteristischen Varietäten eines Moduls die gleiche Krulldimension. Dies wird üblicherweise durch homologische Methoden gezeigt.

Wir betten den erwähnten Dimensionssatz in den grösseren Zusammenhang einer Deformationstheorie von gewichteten Gradfiltrierungen und Monomordnungen ein. Unser deformationstheoretischer Ansatz wendet universelle Gröbnerbasen an, und die erwähnte Dimensionsgleichheit folgt als Korollar aus einem tieferen Resultat. Charakteristische Varietäten zeigen nämlich ein bemerkenswertes Verhalten, wenn man ihre definierenden Filtrierungen durch gewisse Adjustierungen der Gewichtung deformiert. Genauer wird eine charakteristische Varietät durch solche Deformationen in ihren eigenen kritischen Kegel übergeführt.

Dies erlaubt, eine nichtendliche Filtrierung so zu deformieren, dass die entstehende Filtrierung endlich wird und die zu ihr assoziierte charakteristische Varietät gerade der kritische Kegel der ursprünglichen Varietät ist. Daraus folgt die Dimensionsgleichheit. Ein Grund hierfür ist, dass eine affine Varietät die gleiche Krulldimension wie ihr kritischer Kegel hat. Ein weiterer Grund ist, dass die Krull- und die GK-Dimension eines endlich erzeugten Moduls über einer endlich erzeugten kommutativen K-Algebra übereinstimmen. Ein dritter Grund ist, dass die GK-Dimension eines endlich filtrierten Moduls beim Übergang zum assoziierten graduierten Modul erhalten bleibt.

Unser Resultat stellt auch einen ersten Schritt zur Klassifikation der charakteristischen Varietäten dar. Wir waren aber nicht in der Lage, eine solche Klassifikation in voller Allgemeinheit durchzuführen. Wir haben uns deshalb auf charakteristische Varietäten von zyklischen Moduln über der ersten Weylalgebra beschränkt und eine approximierte Klassifikation durch ein Computerexperiment berechnet. Das Experiment zeigt, dass der Gewichtsraum $\mathbb{N}_0^2 \setminus \{(0,0)\}$ der Gradfiltrierungen in halbkegelförmige Gebiete unterteilt werden kann, welche jeweils zur selben charakteristische Varietät führen. Auf Grund dieses Experiments können wir auch eine obere Schranke für die Anzahl dieser charakteristischen Varietäten in Termen von Totalgraden der Elemente einer universellen Gröbnerbasis vermuten. Im Hinblick auf eine Arbeit von Aschenbrenner und Leykin [2] kann diese obere Schranke auch in Termen von Totalgraden von Erzeugern des Ideals angegeben werden, das den gegebenen zyklichen Modul definiert.

Wir beenden den zweiten Teil mit einem Resultat von Škoda über die Lokalisierung von filtrierten Moduln. Mithilfe eines leichten Lemmas können wir Škodas Ergebnis eine geometrische Interpretation in unserem Kontext geben.

Im ersten Anhang geben wir einen direkteren Beweis der Existenz von universellen Gröbnerbasen in Weylalgebren basierend auf den Divisionseigenschaften dieser Algebren und auf der Kompaktheit des topologischen Raums der Monoidordnungen.

Im zweiten Anhang listen wir das Computerprogramm auf, das wir für das erwähnte Experiment geschrieben haben.

Abstract

This dissertation consists of two parts. In the first part we give a self-contained and unified approach to some finiteness results on leading monomial ideals of a polynomial ring with respect to various types of total orderings of the monomials. To a large extent the results of this part are not new and may be found in the work of other authors either relying on different approaches or applied in different contexts. We generalize a part of these results to vector spaces isomorphic to a polynomial ring, a part to the large class of admissible algebras, which comprehends at least the class of algebras of solvable type.

In the literature leading monomial ideals with respect to monoid orderings of $\mathbb{N}_0^t$ with $t \in \mathbb{N}$ are the main object of study because these orderings induce a fruitful division theory. In this context Macaulay's Basis Theorem is the key to finiteness results on leading monomial ideals.

We consider leading monomial ideals with respect to total orderings, degree orderings, semigroup orderings, monoid orderings, and degree-compatible monoid orderings. It turns out that an ideal of a polynomial ring admits at most finitely many minimal leading monomial ideals arising from total orderings, of course minimal with respect to inclusion. Furthermore an ideal possesses at most finitely many minimal leading monomial ideals with respect to degree orderings. Due to a slightly generalized version of Macaulay's Basis Theorem shown here, leading monomial ideals induced by monoid orderings are minimal, thus an ideal has only finitely many leading monomial ideals with respect to monoid orderings.

Initially, inspired by the classical proof for polynomial rings, we planned to show the existence of universal Gröbner bases in admissible algebras by the finiteness results mentioned above. This was our original motivation for investigating these finiteness properties. But, indeed, the existence of universal Gröbner bases already follows from the fact that the set of all total orderings on any given set builds a compact topological space and that

admissible algebras are noetherian. With this topic we conclude the first part of our work.

The second and innovative part of this dissertation is a slightly more detailed version of our article [13], which in December 2010 was accepted for publication on the *Transactions of the American Mathematical Society*. Here we dedicate ourselves to the characteristic varieties of modules over Weyl algebras.

These affine varieties are constructed by providing a finitely generated module over a Weyl algebra with weighted filtrations and forming their associated graded modules. Therefore we first recall some facts over filtered modules and their associated graded modules. For any filtered module over a filtered commutative ring we show that the annihilator of the associated graded module is equal (up to taking radicals) to the associated graded ideal of the filtered annihilator.

A classical theorem of Bernstein states that the characteristic varieties by degree and by order of a given module have the same Krull dimension. Indeed all characteristic varieties of a module have the same dimension. This is usually proved by homological methods.

We embed the mentioned dimension theorem in the wider context of a deformation theory of weighted degree filtrations and monomial orderings. Our deformation-theoretic approach applies universal Gröbner bases, and the mentioned equality of dimensions follows as a corollary of a deeper result. Namely, characteristic varieties denote a remarkable behaviour when one deforms their defining filtrations by certain adjustments of the weights. More precisely, by such adjustments a characteristic variety is stepwise deformed into its own critical cone.

This permits to deform a nonfinite filtration in such a manner that the resulting filtration becomes finite and the characteristic variety associated to it is the critical cone of the original variety. From this follows the wanted dimension equality. A reason is that a variety has the same Krull dimension as its own critical cone. A further reason is that the Krull and Gelfand–Kirillov dimension of a finitely generated module over a finitely generated K-algebra agree. A third reason is that the Gelfand–Kirillov dimension of a finitely filtered module is preserved when passing to the associated graded module.

Our result represents also a first step in trying to classify characteristic varieties. We were not able to perform such a classification in full generality. Therefore we have focused on characteristic varieties of cyclic modules over the first Weyl algebra and have calculated an approximated classification by a computer experiment. The experiment shows that the weight space $\mathbb{N}_0^2 \setminus \{(0,0)\}$ of the filtrations can be subdivided in semicone-shaped regions such that each region corresponds to the same characteristic variety. On the basis of this experiment we can also conjecture a higher bound for the number of these characteristic varieties in terms of total degree of elements of a universal Gröbner basis. In view of a work of Aschenbrenner and Leykin [2], this higher bound can be given also in terms of total degrees of generators of the ideal that defines the considered cyclic module.

We end the second part with a result of Škoda on localizations of filtered modules. By means of an easy lemma we give a geometric interpretation to Škoda's results in our context.

In the first appendix we furnish a more direct proof of the existence of universal Gröbner bases in Weyl algebras using the division properties of these algebras together with the compactness of the topological space of monoid orderings.

In the second appendix we list the computer program that we wrote to perform the mentioned computer experiment.

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Lebenslauf

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Contents

Part 1

Spaces of total orderings and universal Gröbner bases

Introduction

The aim of this part is to give a unified and self-contained approach to some fundamental results on leading monomial ideals and universal Gröbner bases. Our approach relies solely on a topological method inspired by previous work of Sikora. The main results are not new and may be found in the work of Aschenbrenner, Aschenbrenner and Pong, Becker, and Maclagan, where they were proved either by different methods or in different contexts. Precise bibliographical reference will be given below.

Similarly as Schwartz did in [36], Sikora introduced in [37], for semigroups S, a natural topology $\mathcal{U}(S)$ on the set $\mathrm{TO}(S)$ of the total orderings on S. Then he proved that $\mathrm{TO}(S)$ is compact with respect to $\mathcal{U}(S)$. This can be done actually for any set S.

We consider a polynomial ring $K[X] = K[X_1, \ldots, X_t]$ over a field K, where $t \in \mathbb{N}$, and several sorts of total orderings on the set $M = \{X^\nu \mid \nu \in \mathbb{N}_0^t\}$ of the monomials of $K[X]$, namely, the following subsets of $\mathrm{TO}(M)$:

(1) the set $\mathrm{WO}(M)$ of the total well-orderings on M;

(2) the set $\mathrm{FO}_1(M) = \{\leq \in \mathrm{TO}(M) \mid m \in M \Rightarrow 1 \leq m\}$ of the 1-founded orderings on M;

(3) the set $\mathrm{CO}(M) = \{\leq \in \mathrm{TO}(M) \mid X^\upsilon \leq X^\nu \Rightarrow X^{\upsilon+\gamma} \leq X^{\nu+\gamma}\}$ of the compatible orderings, or semigroup orderings, on M;

(4) the set $\mathrm{DO}(M) = \{\leq \in \mathrm{TO}(M) \mid p \in K[X] \Rightarrow \deg(p) = \deg(\mathrm{LM}_{\leq}(p))\}$ of the degree orderings on M;

(5) the set $\mathrm{AO}(M) = \mathrm{FO}_1(M) \cap \mathrm{CO}(M)$ of the admissible orderings, or monoid orderings, on M;

(6) the set $\mathrm{DCO}(M) = \mathrm{DO}(M) \cap \mathrm{CO}(M)$ of the degree-compatible orderings on M.

Then we have the following results:

(1) $\mathrm{FO}_1(M)$ is closed in $\mathrm{TO}(M)$;

(2) $\mathrm{CO}(M)$ is closed in $\mathrm{TO}(M)$;

(3) $\mathrm{DO}(M)$ is closed in $\mathrm{TO}(M)$ and $\mathrm{DO}(M) \subseteq \mathrm{WO}(M) \cap \mathrm{FO}_1(M)$;

(4) $\mathrm{AO}(M)$ is closed in $\mathrm{TO}(M)$ and $\mathrm{AO}(M) = \mathrm{WO}(M) \cap \mathrm{CO}(M)$;

(5) $\mathrm{DCO}(M)$ is closed in $\mathrm{TO}(M)$;

(6) $\mathrm{DCO}(M)$ is nowhere dense in $\mathrm{DO}(M)$ if $t > 1$, otherwise $\mathrm{DCO}(M) = \mathrm{DO}(M)$.

The Venn diagram in Figure 1.1 sketches the situation.

After these preliminaries, given any $\mathfrak{S} \subseteq \mathrm{TO}(M)$ and any $E \subseteq K[X]$, first we consider the set $\mathscr{lm}_{\mathfrak{S}}(E) = \{\mathrm{LM}_{\leq}(E) \mid \leq \in \mathfrak{S}\}$ of the leading monomial ideals $\mathrm{LM}_{\leq}(E)$ of E with respect to the total orderings $\leq \in \mathfrak{S}$ and the set $\mathscr{min}_{\mathfrak{S}}(E)$ of the minimal elements of $\mathscr{lm}_{\mathfrak{S}}(E)$ with respect to the inclusion relation $\subseteq$, and show that $\mathscr{min}_{\mathfrak{S}}(E)$ is finite if $\mathfrak{S}$ is closed in $\mathrm{TO}(M)$.

The proof goes as follows. The set $\min_E(\mathfrak{S})$ of the elements $\leq \in \mathfrak{S}$ such that $\mathrm{LM}_{\leq}(E)$ is $\subseteq$-minimal in $\mathscr{lm}_{\mathfrak{S}}(E)$ is closed in $\mathfrak{S}$, and hence $\min_E(\mathfrak{S})$ is compact under our hypothesis on $\mathfrak{S}$. Thus the quotient space $\min_E(\mathfrak{S})/\sim_E$ of $\min_E(\mathfrak{S})$, where $\leq \sim_E \leq'$ if and only if $\mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\leq'}(E)$, is compact. Since $\min_E(\mathfrak{S})/\sim_E$ is also discrete, it follows that $\min_E(\mathfrak{S})/\sim_E$ is finite. Of course, there exists a canonical bijection between $\min_E(\mathfrak{S})/\sim_E$ and $\mathscr{min}_{\mathfrak{S}}(E)$.

As a consequence, because $\mathrm{DO}(M)$ is closed in $\mathrm{TO}(M)$, one has that $\mathscr{min}_{\mathrm{DO}(M)}(E)$ is finite.

When considering closed subsets $\mathfrak{S}$ of $\mathrm{AO}(M)$, we obtain a similar and well-known finiteness result. Indeed, in this case, if I is an ideal of $K[X]$, the Macaulay Basis Theorem

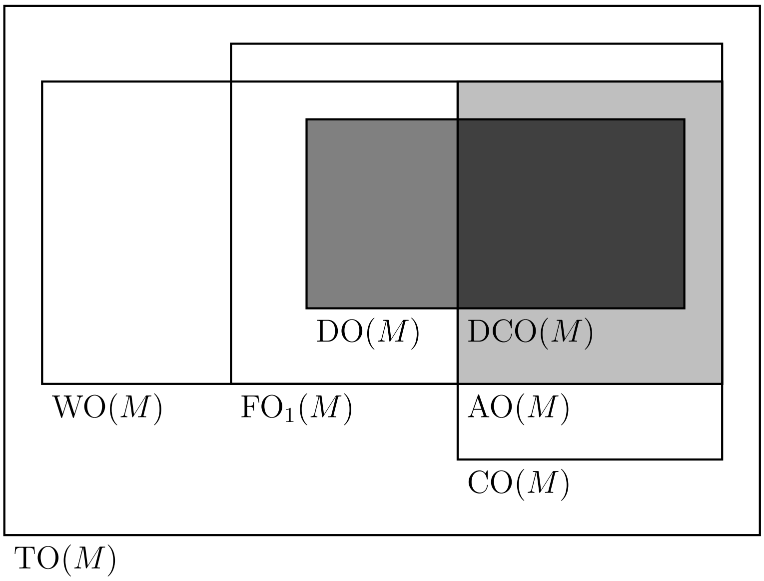


Figure 1.1: Subspaces of total orderings of monomials

holds and comes to our aid as it implies that $\mathscr{lm}_{\mathfrak{S}}(I) = \mathscr{min}_{\mathfrak{S}}(I)$, which we already know to be finite.

Actually these finiteness results, deduced from the mentioned compactness property, can be infered in greater generality by another approach from a theorem of Maclagan [30], that is,

Theorem. *For every infinite sequence $I_0, I_1, \ldots$ of monomial ideals of $K[X]$ there are $i < j$ such that $I_i \supseteq I_j$.*

Therefore one may even drop the hypothesis that $\mathfrak{S}$ be closed: if the set $\mathscr{min}_{\mathfrak{S}}(E)$ was infinite, then there were orderings $\leq$ and $\leq'$ in $\mathscr{min}_{\mathfrak{S}}(E)$ such that $\mathrm{LM}_{\leq}(E) \supsetneq \mathrm{LM}_{\leq'}(E)$, contradicting that $\leq$ is a minimalizer of E in $\mathfrak{S}$.

Maclagan's theorem in turn was later deduced from a still more general statement proved by Aschenbrenner and Pong in [3], namely,

Theorem. *If the set S ordered by $\leq$ is noetherian, then the set $S^{(\geq)}$ of all infinite decreasing sequences $(s_0, s_1, \ldots)$ of elements $s_0 \geq s_1 \geq \ldots$ of S, ordered componentwise, is noetherian, too.*

Thus they immediately obtain Maclagan's theorem restated as

Corollary. *The set of monomial ideals in $K[X]$, ordered by reverse inclusion, is noetherian.*

Next let Φ be a K-module isomorphism of V in $K[X]$ and consider the K-basis $N = \Phi^{-1}(M)$ of V. Then Φ induces a homeomorphism ϕ of $\mathrm{TO}(N)$ in $\mathrm{TO}(M)$. Now, given a total ordering $\preceq$ on N, we may speak of the $\preceq$-leading component $\mathrm{lm}_{\preceq}(v) \in N$ in the unique representation $v = \sum_{n \in N} c_n n$ with $c_n \in K \setminus \{0\}$ of any element $v \in V$ as a K-linear combination over N. Further, given $H \subseteq V$, we consider the ideal

$$\mathrm{LM}_{\preceq}(H) = \langle \Phi(\mathrm{lm}_{\preceq}(h)) \mid h \in H \rangle = \langle \mathrm{LM}_{\phi(\preceq)}(\Phi(h)) \mid h \in H \rangle$$

of $K[X]$. For all $H \subseteq V$, $E \subseteq K[X]$, $\preceq \in \mathrm{TO}(N)$, $\leq \in \mathrm{TO}(M)$, $\mathfrak{T} \subseteq \mathrm{TO}(N)$, $\mathfrak{S} \subseteq \mathrm{TO}(M)$ we have:

(1) $\mathrm{LM}_{\preceq}(H) = \mathrm{LM}_{\phi(\preceq)}(\Phi(H))$ and $\mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\phi^{-1}(\leq)}(\Phi^{-1}(E))$;

(2) $\mathcal{lm}_{\mathfrak{T}}(H) = \mathcal{lm}_{\phi(\mathfrak{T})}(\Phi(H))$ and $\mathcal{lm}_{\mathfrak{S}}(E) = \mathcal{lm}_{\phi^{-1}(\mathfrak{S})}(\Phi^{-1}(E))$;

(3) $\mathcal{min}_{\mathfrak{T}}(H) = \mathcal{min}_{\phi(\mathfrak{T})}(\Phi(H))$ and $\mathcal{min}_{\mathfrak{S}}(E) = \mathcal{min}_{\phi^{-1}(\mathfrak{S})}(\Phi^{-1}(E))$.

Thus what we have said above about $K[X]$ easily translates to V. With one exception: assuming that $\mathfrak{T}$ is closed in $\mathrm{AO}(N)$, the equality $\mathcal{lm}_{\mathfrak{T}}(H) = \mathcal{min}_{\mathfrak{T}}(H)$ holds so far only under the hypothesis that $H = \Phi^{-1}(I)$ for some ideal I of $K[X]$.

Therefore, when considering the set $\mathrm{AO}(N) = \phi^{-1}(\mathrm{AO}(M))$ of the admissible orderings on N, we replace the K-module V by an associative but not necessarily commutative K-algebra A that is a domain and is isomorphic to $K[X]$ as a K-module. Assuming similar multiplicativity properties of A on taking leading monomials as in the case of $K[X]$, we prove a generalized version of the Macaulay Basis Theorem, which then implies the equality $\mathcal{lm}_{\mathfrak{T}}(J) = \mathcal{min}_{\mathfrak{T}}(J)$ for each closed $\mathfrak{T} \subseteq \mathrm{AO}(N)$ and each (left, right, two-sided) ideal $J \subseteq A$.

Finally, similarly as Becker did in [6], [7], [8] for universal standard bases in power series rings over a field and similarly as Aschenbrenner did in [1] for universal standard bases in power series rings over an arbitrary commutative ring, we follow this topological approach and apply the results obtained so far to show that every (left, right, two-sided) ideal of a

K-algebra A as above admits a $\mathfrak{T}$-universal Gröbner basis, where $\mathfrak{T}$ is any closed subset of $\mathrm{DO}(N)$. To prove a similar result for closed subsets $\mathfrak{T}$ of $\mathrm{AO}(N)$, we have to require that A is a domain and is multiplicative on taking leading monomials over $\mathfrak{T}$.

These proofs of theorems about universal Gröbner bases do not rely on the finiteness of the total number of leading monomial ideals of a given ideal. Indeed, the statements about universal Gröbner bases as well as the finiteness results both descend directly from some of the topological properties of total orderings and, partly, from the generalized Macaulay Basis Theorem.

General remark

In this part all the statements involving ideals of noncommutative rings are proved only for left ideals. These statements translate word by word to right and two-sided ideals, too.

1.1 Topological spaces of total orderings on sets

In this section, let S be a set.

Definition 1.1.1. A *total ordering* on S is a binary relation $\preceq$ on S such that it holds antisymmetry: $a \preceq b \wedge b \preceq a \Rightarrow a = b$, transitivity: $a \preceq b \wedge b \preceq c \Rightarrow a \preceq c$, totality: $a \preceq b \vee b \preceq a$, for all $a, b, c \in S$. Totality implies reflexivity: $a \preceq a$ for all $a \in S$. The nonempty set of all total orderings on S is denoted $\mathrm{TO}(S)$.

Given any ordered pair $(a, b) \in S \times S$, let $\mathfrak{U}_{(a,b)}$ be the set of all total orderings $\preceq$ on S for which $a \preceq b$. Let $\mathcal{U}(S)$ be the coarsest topology of S for which all the sets $\mathfrak{U}_{(a,b)}$ are open. This is the topology for which $\{\mathfrak{U}_{(a,b)} \mid (a, b) \in S \times S\}$ is a subbasis, that is, the open sets in $\mathcal{U}(S)$ are precisely the unions of finite intersections of sets of the form $\mathfrak{U}_{(a,b)}$. Observe that $\mathfrak{U}_{(a,a)} = \mathrm{TO}(S)$ and that $\mathfrak{U}_{(a,b)} = \mathrm{TO}(S) \setminus \mathfrak{U}_{(b,a)}$ if $a \neq b$, so that the sets $\mathfrak{U}_{(a,b)}$ are also closed.

Let $\mathbf{S}$ be any *filtration* of S, that is, $\mathbf{S} = (S_i)_{i \in \mathbb{N}_0}$ is a family of subsets S_i of S such that (a) $S_0 = \varnothing$, (b) $S_i \subseteq S_{i+1}$ for all $i \in \mathbb{N}_0$, (c) $S = \bigcup_{i \in \mathbb{N}_0} S_i$. We define the function $d_{\mathbf{S}} : \mathrm{TO}(S) \times \mathrm{TO}(S) \to \mathbb{R}$ by $d_{\mathbf{S}}(\preceq', \preceq'') = 2^{-r}$ with $r = \sup\{i \in \mathbb{N}_0 \mid \preceq'\restriction_{S_i} = \preceq''\restriction_{S_i}\}$. Here $\restriction$ denotes restriction. Firstly, it holds $\{0\} \subseteq \mathrm{Im}(d_{\mathbf{S}}) \subseteq [0, 1]$. As $\mathbf{S}$ is exhaustive by (c), we have that $d_{\mathbf{S}}(\preceq', \preceq'') = 0$ if and only if $\preceq' = \preceq''$. Secondly, $d_{\mathbf{S}}(\preceq', \preceq'') = d_{\mathbf{S}}(\preceq'', \preceq')$. Finally,

$d_{\mathbf{S}}(\preceq', \preceq''') \leq d_{\mathbf{S}}(\preceq', \preceq'') + d_{\mathbf{S}}(\preceq'', \preceq''')$, since $d_{\mathbf{S}}(\preceq', \preceq''') \leq \max\{d_{\mathbf{S}}(\preceq', \preceq''), d_{\mathbf{S}}(\preceq'', \preceq''')\}$. Thus $d_{\mathbf{S}}$ is a metric on $\mathrm{TO}(S)$, dependent on the choice of the filtration $\mathbf{S}$ of S.

Theorem 1.1.2. *Assume that there exists a filtration $\mathbf{S} = (S_i)_{i\in\mathbb{N}_0}$ of S such that each of the sets S_i is finite. Let $\mathcal{N}(S)$ be the topology of S induced by the metric $d_{\mathbf{S}}$, that is more precisely, $\mathfrak{N} \in \mathcal{N}(S)$ if and only if $\mathfrak{N}$ is a union of finite intersections of sets of the form $\mathfrak{N}_r(\preceq) = \{\preceq' \in \mathrm{TO}(S) \mid d_{\mathbf{S}}(\preceq, \preceq') < 2^{-r}\}$ where $r \in \mathbb{N}_0$ and $\preceq \in \mathrm{TO}(S)$. Then it holds $\mathcal{N}(S) = \mathcal{U}(S)$, in particular the topology $\mathcal{N}(S)$ is independent of the choice of $\mathbf{S}$, and the topology $\mathcal{U}(S)$ is Hausdorff.*

Proof. Let $r \in \mathbb{N}_0$ and $\preceq \in \mathrm{TO}(S)$. We claim that $\mathfrak{N}_r(\preceq) \in \mathcal{U}(S)$. Let $\mathfrak{U} = \bigcap_{(a,b)} \mathfrak{U}_{(a,b)}$, where the intersection is taken over all ordered pairs $(a, b) \in S_{r+1} \times S_{r+1}$ with $a \preceq b$. Then $\preceq \in \mathfrak{U} \in \mathcal{U}(S)$. Hence $\preceq' \in \mathfrak{N}_r(\preceq)$ if and only if $\preceq'\restriction_{S_{r+1}} = \preceq\restriction_{S_{r+1}}$, and this is the case if and only if it holds $a \preceq' b \Leftrightarrow a \preceq b$ for all $(a, b) \in S_{r+1} \times S_{r+1}$, which is true if and only if $\preceq' \in \mathfrak{U}$. Thus $\mathfrak{N}_r(\preceq) = \mathfrak{U}$, and this shows that $\mathcal{N}(S) \subseteq \mathcal{U}(S)$.

On the other hand, let $(a, b) \in S \times S$ be any ordered pair. We claim that the set $\mathfrak{U}_{(a,b)}$ is open with respect to the metric $d_{\mathbf{S}}$. Let $\preceq \in \mathfrak{U}_{(a,b)}$, so that $a \preceq b$. We find $r \in \mathbb{N}_0$ such that $(a, b) \in S_{r+1} \times S_{r+1}$. If $\preceq' \in \mathfrak{N}_r(\preceq)$, then $\preceq'\restriction_{S_{r+1}} = \preceq\restriction_{S_{r+1}}$, in particular $a \preceq' b$, so that $\preceq' \in \mathfrak{U}_{(a,b)}$, thus $\mathfrak{N}_r(\preceq) \subseteq \mathfrak{U}_{(a,b)}$. Hence $\mathfrak{U}_{(a,b)}$ is open with respect to $\mathcal{N}(S)$, and we conclude that $\mathcal{U}(S) \subseteq \mathcal{N}(S)$. □

Convention 1.1.3. Henceforth, unless otherwise stated, whenever we refer to topological properties of $\mathrm{TO}(S)$, we always intend that $\mathrm{TO}(S)$ is provided with the topology $\mathcal{U}(S)$. Subsets of $\mathrm{TO}(S)$ are tacitly furnished with their relative topology with respect to $\mathcal{U}(S)$. Quotient sets of $\mathrm{TO}(S)$ by equivalence relations are equipped with their quotient topology with respect to $\mathcal{U}(S)$.

Theorem 1.1.4. *If the set S is countable, then $\mathrm{TO}(S)$ is compact.*

Proof. As S is countable, we find a filtration $\mathbf{S} = (S_i)_{i\in\mathbb{N}_0}$ of S consisting of finite subsets S_i of S. Since $\mathrm{TO}(S)$ is a metric space with respect to the metric $d_{\mathbf{S}}$ introduced above, see 1.1.2, it is sufficient to prove that each sequence of elements of $\mathrm{TO}(S)$ admits a subsequence convergent in $\mathrm{TO}(S)$. Let $(\preceq_j)_{j\in\mathbb{N}_0}$ be any sequence of total orderings on S. Without loss of generality we can assume that the elements $\preceq_j$ are pairwise distinct.

As there exist only finitely many distinct total orderings on S_0, we find an infinite subsequence $(\preceq_{j_k^0})_{k\in\mathbb{N}_0}$ of $(\preceq_j)_{j\in\mathbb{N}_0}$ whose members all agree on S_0. As there exist only finitely many distinct total orderings on S_1, we find an infinite subsequence $(\preceq_{j_k^1})_{k\in\mathbb{N}_0}$ of $(\preceq_{j_k^0})_{k\in\mathbb{N}_0}$ whose members all agree on S_1. Going on in this manner, we construct a family $((\preceq_{j_k^i})_{k\in\mathbb{N}_0})_{i\in\mathbb{N}_0}$ of infinite sequences $(\preceq_{j_k^i})_{k\in\mathbb{N}_0}$ of total orderings on S such that for each $i \in \mathbb{N}_0$ the members of $(\preceq_{j_k^i})_{k\in\mathbb{N}_0}$ all agree on S_i and $(\preceq_{j_k^{i+1}})_{k\in\mathbb{N}_0}$ is a subsequence of $(\preceq_{j_k^i})_{k\in\mathbb{N}_0}$. Putting $\preceq^i = \preceq_{j_i^i}$, we thus obtain a subsequence $(\preceq^i)_{i\in\mathbb{N}_0}$ of $(\preceq_j)_{j\in\mathbb{N}_0}$.

Now let $\preceq^\infty$ be the binary relation on S defined by $a \preceq^\infty b \Leftrightarrow a \preceq^i b$ for almost all i. One easily verifies that $\preceq^\infty$ is antisymmetric and transitive. Let $a, b \in S$. We find $r \in \mathbb{N}_0$ with $a, b \in S_r$. It holds $a \preceq^r b$ or $b \preceq^r a$, say $a \preceq^r b$. As $\preceq^{r+1}$ is a member of the subsequence $(\preceq_{j_k^{r+1}})_{k\in\mathbb{N}_0}$ of the sequence $(\preceq_{j_k^r})_{k\in\mathbb{N}_0}$ which contains $\preceq^r$ and whose members all agree on S_r, it follows $a \preceq^{r+1} b$, and inductively $a \preceq^i b$ for all $i \geq r$, thus $a \preceq^\infty b$. Hence $\preceq^\infty$ is a total ordering on S.

For all $r \in \mathbb{N}_0$ and all $i \geq r+1$ it holds $\preceq^i \in \mathfrak{N}_r(\preceq^\infty)$. Indeed, let $r \in \mathbb{N}_0$ and let $i \geq r+1$. It is enough to show that $\preceq^i$ and $\preceq^\infty$ agree on S_{r+1}. Let $a, b \in S_{r+1}$. As before we have the implications $a \preceq^i b \Rightarrow a \preceq^{i+1} b \Rightarrow a \preceq^{i+2} b \Rightarrow \ldots$, whence the implication $a \preceq^i b \Rightarrow a \preceq^\infty b$. On the other hand, assume that $a \preceq^\infty b$, say. If not $a \preceq^i b$, then $b \preceq^i a$ by totality, and as above it follows $b \preceq^\infty a$, whence $a = b$ by antisymmetry, and so $a \preceq^i b$ by reflexivity, a contradiction. Thus $\preceq^i$ and $\preceq^\infty$ agree on S_{r+1}. It follows $\preceq^i \to \preceq^\infty$ for $i \to \infty$. □

Definition 1.1.5. A *filter* over a set X is a subset $\mathcal{F}$ of the power set $\mathcal{P}(X)$ of X that enjoys the properties (a) $X \in \mathcal{F}$, (b) $\varnothing \notin \mathcal{F}$, (c) $A \subseteq B \subseteq X \wedge A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$, (d) $A \in \mathcal{F} \wedge B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

An *ultrafilter* over X is a filter $\mathcal{L}$ over X that fulfills (e) $A \subseteq X \Rightarrow A \in \mathcal{L} \vee X \setminus A \in \mathcal{L}$. The disjunction in (e) is exclusive by (d) and (b). Equivalently, an ultrafilter over X is a maximal filter over X with respect to inclusion.

Remark 1.1.6. Prof. Dr. Matthias Aschenbrenner of the University of California, Los Angeles, made us kindly aware that in Theorem 1.1.4 one may drop the hypothesis that S be countable.

Indeed, let S be any set and suppose by contradiction that $\mathrm{TO}(S)$ is not compact. Then we find an infinite index set I and families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ of elements $a_i, b_i \in S$

such that $(\mathfrak{U}_{(a_i,b_i)})_{i\in I}$ is a covering of $\mathrm{TO}(S)$ which admits no finite subcovering. Thus for each finite subset $s \subseteq I$ there exists $\preceq_s \in \mathrm{TO}(S)$ such that $\preceq_s \notin \bigcup_{i\in s} \mathfrak{U}_{(a_i,b_i)}$, that is, for all $i \in s$ it holds $a_i \succ_s b_i$.

Let I^* be the set of all nonempty finite subsets of I. For each $s \in I^*$ let us define $s^* = \{t \in I^* \mid s \subseteq t\}$. As $s \in s^*$ for all $s \in I^*$ and $s_1^* \cap s_2^* = (s_1 \cup s_2)^*$ for all $s_1, s_2 \in I^*$, the set $\mathcal{S} = \{s^* \mid s \in I^*\}$ has the finite intersection property, that is, any finite intersection of elements of $\mathcal{S}$ is nonempty. Thus $\mathcal{F} = \{Y \in \mathcal{P}(I^*) \mid \exists\, n \in \mathbb{N}\, \exists\, Z_1, \ldots, Z_n \in \mathcal{S} : Z_1 \cap \ldots \cap Z_1 \subseteq Y\}$ is a filter over I^* that extends $\mathcal{S}$. Hence, by the Ultrafilter Lemma, which descends from Zorn's Lemma, there exists an ultrafilter $\mathcal{L}$ over I^* that extends $\mathcal{F}$, so that $s^* \in \mathcal{L}$ for all $s \in I^*$.

We fix a family $(\preceq_s)_{s\in I^*}$ of total ordering $\preceq_s$ on S as above and define a binary relation $\preceq$ on S by $x \preceq y \Leftrightarrow \{s \in I^* \mid x \preceq_s y\} \in \mathcal{L}$. By axioms (d) and (b) of 1.1.5, $\preceq$ is antisymmetric. By axioms (d) and (c) of 1.1.5, $\preceq$ is transitive. By axioms (e) and (c) of 1.1.5, $\preceq$ is total. So $\preceq \in \mathrm{TO}(S)$. On the other hand, by our choice of the orderings $\preceq_s$, it holds $a_i \succ b_i$ for all $i \in I$, thus $\preceq \notin \bigcup_{i\in I} \mathfrak{U}_{(a_i,b_i)} = \mathrm{TO}(S)$, a contradiction.

Definition 1.1.7. For each $a \in S$ let $\mathrm{FO}_a(S) = \{\preceq \in \mathrm{TO}(S) \mid \forall\, b \in S : a \preceq b\}$, the set of all *a-founded orderings* on S.

Corollary 1.1.8. *For each $a \in S$ the set $\mathrm{FO}_a(S)$ is closed in $\mathrm{TO}(S)$. Hence, if the set S is countable, then the subspace $\mathrm{FO}_a(S)$ of $\mathrm{TO}(S)$ is compact.*

Proof. It holds $\mathrm{FO}_a(S) = \bigcap_{b\in S} \mathfrak{U}_{(a,b)}$, thus $\mathrm{FO}_a(S)$ is closed in $\mathrm{TO}(S)$ as each $\mathfrak{U}_{(a,b)}$ is closed in $\mathrm{TO}(S)$ as observed in 1.1.1. If S is countable, then $\mathrm{TO}(S)$ is compact by 1.1.4, and hence, as a closed subset of a compact set, $\mathrm{FO}_a(S)$ equipped with its relative topology is compact. □

Remark 1.1.9. Also in 1.1.8 we may drop the countability hypothesis on S by 1.1.6.

1.2 Leading monomial ideals from total orderings

Let $t \in \mathbb{N}$, let K be a field, and let $K[X]$ denote the commutative polynomial ring $K[X_1, \ldots, X_t]$.

Reminder & Definition 1.2.1. The countable set $M = \{X^\nu \mid \nu \in \mathbb{N}_0^t\}$ of the *monomials of* $K[X]$ is a basis of the K-module $K[X]$, often referred to as the *canonical* K*-basis of* $K[X]$. We fix once for all this K-basis M of $K[X]$.

Thus each $p \in K[X]$ can be written in *canonical form* as $\sum_{\nu \in \mathrm{supp}(p)} \alpha_\nu X^\nu$ for a uniquely determined finite subset $\mathrm{supp}(p)$ of $\mathbb{N}_0^t$ such that $\alpha_\nu \in K \smallsetminus \{0\}$ for all $\nu \in \mathrm{supp}(p)$. Notice that $\mathrm{supp}(p) = \varnothing$ if and only if $p = 0$.

For each $p \in K[X]$ let us define the subset $\mathrm{Supp}(p) = \{X^\nu \mid \nu \in \mathrm{supp}(p)\}$ of M, which we call the *support of* p. Clearly, $\mathrm{Supp}(p) = \varnothing$ if and only if $p = 0$. We also put $\mathrm{Supp}(E) = \bigcup_{e \in E} \mathrm{Supp}(e)$ for each subset E of $K[X]$.

For each $p \in K[X] \smallsetminus \{0\}$ and each $\leq \in \mathrm{TO}(M)$ we denote by $\mathrm{LM}_\leq(p)$ the uniquely determined maximal element of $\mathrm{Supp}(p)$ with respect to $\leq$ and call $\mathrm{LM}_\leq(p)$ the *leading monomial of* p *with respect to* $\leq$. In this situation, there exists a unique $\alpha \in K \smallsetminus \{0\}$ such that either $p - \alpha\,\mathrm{LM}_\leq(p) = 0$ or $\mathrm{LM}_\leq(p - \alpha\,\mathrm{LM}_\leq(p)) < \mathrm{LM}_\leq(p)$. Such element α is denoted $\mathrm{LC}_\leq(p)$ and called the *leading coefficient of* p *with respect to* $\leq$.

For each $E \subseteq K[X]$ and each $\leq \in \mathrm{TO}(M)$ we denote the ideal $\langle \mathrm{LM}_\leq(e) \mid e \in E \smallsetminus \{0\}\rangle$ of $K[X]$ by $\mathrm{LM}_\leq(E)$ and call $\mathrm{LM}_\leq(E)$ the *leading monomial ideal of* E *with respect to* $\leq$.

Finally, let $\ell m_{\mathfrak{S}}(E) = \{\mathrm{LM}_\leq(E) \mid \leq \in \mathfrak{S}\}$, for $E \subseteq K[X]$ and $\mathfrak{S} \subseteq \mathrm{TO}(M)$, be the set of all *leading monomial ideals of* E *from* $\mathfrak{S}$.

Remark 1.2.2. We shall, almost always tacitly, make use of the following well-known results, see [20, II.4.2 & II.4.4].

Let $N \subseteq \mathbb{N}_0^t$. Then a monomial X^υ of $K[X]$ lies in the ideal $\langle X^\nu \mid \nu \in N\rangle$ of $K[X]$ if and only if there exists $\gamma \in N$ such that X^γ divides X^υ.

From this it follows that two monomials ideals are equal if and only if they contain the same monomials.

Remark 1.2.3. If $p \in K[X]$ and $\leq, \leq' \in \mathrm{TO}(M)$ are such that $\leq$ and $\leq'$ agree on $\mathrm{Supp}(p)$, then clearly $\mathrm{LM}_\leq(p) = \mathrm{LM}_{\leq'}(p)$.

Hence, if $\leq, \leq' \in \mathrm{TO}(M)$ and $F \subseteq K[X]$ are such that $\leq$ and $\leq'$ agree on $\mathrm{Supp}(F)$, then $\mathrm{LM}_\leq(F) = \langle \mathrm{LM}_\leq(f) \mid f \in F\rangle = \langle \mathrm{LM}_{\leq'}(f) \mid f \in F\rangle = \mathrm{LM}_{\leq'}(F)$.

In this situation, if in addition we have $F \subseteq E \subseteq K[X]$ and $\mathrm{LM}_\leq(F) = \mathrm{LM}_\leq(E)$, then clearly $\mathrm{LM}_\leq(E) \subseteq \mathrm{LM}_{\leq'}(E)$.

Definition 1.2.4. Let $E \subseteq K[X]$ and let $\mathfrak{S} \subseteq \mathrm{TO}(M)$. We say that $\leq' \in \mathfrak{S}$ is a *minimalizer of E in $\mathfrak{S}$* if the implication $\mathrm{LM}_{\leq}(E) \subseteq \mathrm{LM}_{\leq'}(E) \Rightarrow \mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\leq'}(E)$ is true for all $\leq \in \mathfrak{S}$, that is, if $\mathrm{LM}_{\leq'}(E)$ is a minimal element of $\mathscr{lm}_{\mathfrak{S}}(E)$ with respect to $\subseteq$.

We denote the set of all minimalizers of E in $\mathfrak{S}$ by $\min_E(\mathfrak{S})$. We write $\mathscr{min}_{\mathfrak{S}}(E)$ for the set $\mathscr{lm}_{\min_E(\mathfrak{S})}(E) = \{\mathrm{LM}_{\leq}(E) \mid \leq \in \min_E(\mathfrak{S})\}$ of all *minimal leading monomial ideals of E from* $\mathfrak{S}$.

Lemma 1.2.5. *Let $E \subseteq K[X]$ and $\mathfrak{S} \subseteq \mathrm{TO}(M)$. Then $\min_E(\mathfrak{S})$ is a closed subset of $\mathfrak{S}$. Hence, if $\mathfrak{S}$ is closed in $\mathrm{TO}(M)$, then $\min_E(\mathfrak{S})$ is compact.*

Proof. We may choose a filtration $(S_i)_{i\in\mathbb{N}_0}$ of M consisting of finite subsets S_i of S. Let $\leq \in \mathfrak{S}$ be any accumulation point of $\min_E(\mathfrak{S})$. Thus for each $r \in \mathbb{N}_0$ there exists $\leq_r \in \min_E(\mathfrak{S}) \cap \mathfrak{N}_r(\leq) \setminus \{\leq\}$. Since $K[X]$ is noetherian, there exists a finite set $F \subseteq E$ such that $\mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\leq}(F)$. We can find $r \in \mathbb{N}_0$ such that $\mathrm{Supp}(F) \subseteq S_{r+1}$. We fix then $\leq_r \in \min_E(\mathfrak{S}) \cap \mathfrak{N}_r(\leq) \setminus \{\leq\}$. Thus $\leq$ and $\leq_r$ agree on S_{r+1} and in particular on $\mathrm{Supp}(F)$. From 1.2.3 it follows $\mathrm{LM}_{\leq}(E) \subseteq \mathrm{LM}_{\leq_r}(E)$. As $\leq \in \mathfrak{S}$ and $\leq_r \in \min_E(\mathfrak{S})$, it follows $\mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\leq_r}(E)$. Hence $\mathrm{LM}_{\leq}(E)$ is a minimal element of $\mathscr{lm}_{\mathfrak{S}}(E)$ with respect to $\subseteq$, that is, $\leq \in \min_E(\mathfrak{S})$. Therefore $\min_E(\mathfrak{S})$ contains all its accumulation points in $\mathfrak{S}$, and hence $\min_E(\mathfrak{S})$ is closed in $\mathfrak{S}$. The statement about compactness follows now from 1.1.4. □

Definition 1.2.6. Let $E \subseteq K[X]$ and $\mathfrak{S} \subseteq \mathrm{TO}(M)$. We define an equivalence relation $\sim_E$ on $\min_E(\mathfrak{S})$ by $\leq \sim_E \leq' \Leftrightarrow \mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\leq'}(E)$. We also provide the set $\min_E(\mathfrak{S})/\sim_E$ of the equivalence classes of $\min_E(\mathfrak{S})$ with respect to $\sim_E$ with its quotient topology.

Remark 1.2.7. Let $E \subseteq K[X]$ and $\mathfrak{S} \subseteq \mathrm{TO}(M)$. By 1.2.5, $\min_E(\mathfrak{S})/\sim_E$ is compact whenever $\mathfrak{S}$ is closed in $\mathrm{TO}(M)$.

Theorem 1.2.8. *Let $E \subseteq K[X]$ and $\mathfrak{S} \subseteq \mathrm{TO}(M)$. Then $\min_E(\mathfrak{S})/\sim_E$ is discrete. Hence, if $\mathfrak{S}$ is closed in $\mathrm{TO}(M)$, then $\min_E(\mathfrak{S})/\sim_E$ is finite.*

Proof. Let $\pi_E : \min_E(\mathfrak{S}) \to \min_E(\mathfrak{S})/\sim_E$ be the natural projection that maps each $\leq$ to its equivalence class $[\leq]$ with respect to $\sim_E$. Let $\leq \in \min_E(\mathfrak{S})$. It is enough to show that $\{[\leq]\}$ is open in $\min_E(\mathfrak{S})/\sim_E$. Put $\mathfrak{U} = \pi_E^{-1}([\leq])$. By definition, $\{[\leq]\}$ is open in $\min_E(\mathfrak{S})/\sim_E$ if and only if $\mathfrak{U}$ is open in $\min_E(\mathfrak{S})$.

We may assume that $\mathfrak{U} \neq \varnothing$. Let $\leq' \in \mathfrak{U}$. We aim to find an open subset $\mathfrak{V}$ of $\min_E(\mathfrak{S})$ such that $\leq' \in \mathfrak{V} \subseteq \mathfrak{U}$. As $K[X]$ is noetherian, there exists a finite subset F of E with $\mathrm{LM}_{\leq'}(F) = \mathrm{LM}_{\leq'}(E)$. Let $(S_i)_{i \in \mathbb{N}_0}$ be a filtration of M by finite sets S_i. As the set $\mathrm{Supp}(F)$ is finite, we find $r \in \mathbb{N}_0$ such that $\mathrm{Supp}(F) \subseteq S_{r+1}$. Put $\mathfrak{V} = \mathfrak{N}_r(\leq') \cap \min_E(\mathfrak{S})$. Of course, $\mathfrak{V}$ is open in $\min_E(\mathfrak{S})$ and $\leq' \in \mathfrak{V}$.

We claim that $\mathfrak{V} \subseteq \mathfrak{U}$. Let $\leq'' \in \mathfrak{V}$. Then $\leq'$ and $\leq''$ agree on S_{r+1} and hence on $\mathrm{Supp}(F)$. It follows $\mathrm{LM}_{\leq'}(E) \subseteq \mathrm{LM}_{\leq''}(E)$, as we have already observed in 1.2.3. Because $\leq'' \in \min_E(\mathfrak{S})$ and $\leq' \in \mathfrak{S}$, we obtain $\mathrm{LM}_{\leq'}(E) = \mathrm{LM}_{\leq''}(E)$. Thus $[\leq''] = [\leq'] = [\leq]$, that is, $\leq'' \in \mathfrak{U}$.

Hence $\mathfrak{V} \subseteq \mathfrak{U}$, so $\mathfrak{U}$ is open in $\min_E(\mathfrak{S})$. We have proved that $\min_E(\mathfrak{S})/\sim_E$ is discrete. If $\mathfrak{S}$ is closed in $\mathrm{TO}(M)$, then $\min_E(\mathfrak{S})/\sim_E$ is also compact by 1.2.7, and hence finite. □

Corollary 1.2.9. *For each $E \subseteq K[X]$ and each closed $\mathfrak{S} \subseteq \mathrm{TO}(M)$ the set $\mathcal{min}_{\mathfrak{S}}(E)$ is finite, that is, there exist at most finitely many distinct minimal leading monomial ideals of E from $\mathfrak{S}$.*

Proof. The statement follows from 1.2.8 as clearly there exists a bijection between the sets $\mathcal{min}_{\mathfrak{S}}(E)$ and $\min_E(\mathfrak{S})/\sim_E$ given by $\mathrm{LM}_{\leq}(E) \mapsto [\leq]$ for all $\leq \in \min_E(\mathfrak{S})$. □

1.3 Leading monomial ideals from degree orderings

We keep the notation of the previous section.

Definition 1.3.1. For all $\leq \in \mathrm{TO}(M)$ and $p \in K[X] \setminus \{0\}$ one has $\deg(\mathrm{LM}_{\leq}(p)) \leq \deg(p)$, where $\deg(-)$ denotes the total degree function on $K[X]$. A *degree ordering on M* or *of $K[X]$* is a total ordering $\leq$ on M such that $\deg(\mathrm{LM}_{\leq}(p)) = \deg(p)$ for all $p \in K[X] \setminus \{0\}$. The set of all degree orderings on M is denoted $\mathrm{DO}(M)$.

Example 1.3.2. For each $\leq \in \mathrm{TO}(M)$ the binary relation $\leq_{\deg}$ on M defined by

$$m \leq_{\deg} m' \Leftrightarrow \deg(m) < \deg(m') \vee (\deg(m) = \deg(m') \wedge m \leq m')$$

is a degree ordering of $K[X]$.

Proposition 1.3.3. *It holds* $\mathrm{DO}(M) \subseteq \mathrm{FO}_1(M)$.

Proof. Let $\leq\ \in \mathrm{DO}(M)$. Suppose $\leq\ \notin \mathrm{FO}_1(M)$. Then there exists $m \in M$ such that $1 \not\leq m$. So $m < 1$ by totality. It follows $\mathrm{LM}_{\leq}(m+1) = 1$, thus $\deg(\mathrm{LM}_{\leq}(m+1)) = 0$. But m is a monomial different than 1, hence $\deg(m+1) > 0$, a contradiction. ☐

Reminder 1.3.4. Let S be a set. We recall that a *partial ordering on* S is a reflexive, transitive, and antisymmetric binary relation on S, and that a partial ordering $\preceq$ on S is said a *well-ordering on* S if each nonempty subset T of S admits a minimal element with respect to $\preceq$, that is, for each $T \subseteq S$ with $T \neq \varnothing$ there exists $t' \in T$ such that for each $t \in T$ it holds the implication $t \preceq t' \Rightarrow t = t'$.

If $\preceq$ is a total ordering of S, then $\preceq$ is a well-ordering on S precisely when each nonempty subset T of S admits a minimum, that is, for each $T \subseteq S$ with $T \neq \varnothing$ there exists $t' \in T$ such that for each $t \in T$ it holds $t' \preceq t$.

Notation 1.3.5. For each set S we denote by $\mathrm{WO}(S)$ the set of all total orderings on S that are also well-orderings on S.

Proposition 1.3.6. *It holds* $\mathrm{DO}(M) \subseteq \mathrm{WO}(M)$.

Proof. Let $\leq\ \in \mathrm{DO}(M)$. Let $\varnothing \neq T \subseteq M$. Suppose that there exists no minimum in T with respect to $\leq$. Let $t_0 \in T$. We find $t_1 \in T$ such that $t_1 < t_0$, and then find $t_2 \in T$ such that $t_2 < t_1$, and then... Thus there exists in T an infinite strictly descending chain $\ldots < t_2 < t_1 < t_0$.

For each $k \in \mathbb{N}_0$ it holds $\deg(t_k) \geq \deg(t_{k+1})$. Indeed, let $k \in \mathbb{N}_0$ and consider the polynomial $t_k + t_{k+1}$. We have $\mathrm{LM}_{\leq}(t_k + t_{k+1}) = t_k$ as $t_k > t_{k+1}$. Since $\leq\ \in \mathrm{DO}(M)$, it follows $\deg(t_k + t_{k+1}) = \deg(t_k)$. Hence $\deg(t_k) \geq \deg(t_{k+1})$.

Therefore we can write $\ldots \leq \deg(t_2) \leq \deg(t_1) \leq \deg(t_0)$. Now, for each $d \in \mathbb{N}_0$ there exist only finitely many distinct monomials of degree d. Hence we can find a sequence $(k_i)_{i\in\mathbb{N}_0}$ of integers k_i with $k_0 = 0$ and $k_i < k_{i+1}$ with the property that the strict descending chain $\ldots < \deg(t_{k_2}) < \deg(t_{k_1}) < \deg(t_{k_0})$ in $\mathbb{N}_0$ is infinite, and this is absurd. ☐

Lemma 1.3.7. $\mathrm{DO}(M)$ *is a closed subset of* $\mathrm{TO}(M)$ *and hence compact.*

Proof. Let $(S_i)_{i\in\mathbb{N}_0}$ be a filtration of M consisting of finite sets S_i. Let $\leq\ \in \mathrm{TO}(M)$ be an accumulation point of $\mathrm{DO}(M)$. For each $r \in \mathbb{N}_0$ there exists $\leq_r\ \in \mathrm{DO}(M) \cap \mathfrak{N}_r(\leq) \setminus \{\leq\}$, so that $\leq$ and $\leq_r$ agree on S_{r+1}. Let $p \in K[X] \setminus \{0\}$. We can find $r \in \mathbb{N}_0$ such that

$\operatorname{Supp}(p) \subseteq S_{r+1}$. We choose $\leq_r$ as above, and so $\mathrm{LM}_{\leq}(p) = \mathrm{LM}_{\leq_r}(p)$, thus $\deg(\mathrm{LM}_{\leq}(p)) = \deg(\mathrm{LM}_{\leq_r}(p)) = \deg(p)$ as $\leq_r$ is a degree ordering. Hence $\leq \in \mathrm{DO}(M)$. Therefore $\mathrm{DO}(M)$ contains all its accumulation points in $\mathrm{TO}(M)$ and so is closed in $\mathrm{TO}(M)$. Since $\mathrm{TO}(M)$ is compact by 1.1.4, it follows that $\mathrm{DO}(M)$ is compact. □

Corollary 1.3.8. *For each $E \subseteq K[X]$ and each closed $\mathfrak{S} \subseteq \mathrm{DO}(M)$ the set $min_{\mathfrak{S}}(E)$ is finite, that is, there exist at most finitely many distinct minimal leading monomial ideals of E from $\mathfrak{S}$.*

Proof. Clear by 1.2.9 and 1.3.7. □

1.4 Action of K-module isomorphisms

We keep the notation of the previous section. Further, let V be a K-module such that there exists a K-module isomorphism Φ of V in $K[X]$, and put $N = \Phi^{-1}(M)$, so that N is a countable K-basis of V. Sometimes we denote the inverse of Φ by Ψ.

Remark 1.4.1. We have a map $\phi : \mathrm{TO}(N) \to \mathrm{TO}(M)$ such that for any given $\preceq \in \mathrm{TO}(N)$ it holds $\Phi(n)\,\phi(\preceq)\,\Phi(n') \Leftrightarrow n \preceq n'$ for all $n, n' \in N$.

Indeed, fixed any $\preceq \in \mathrm{TO}(N)$, simply define $m\,\phi(\preceq)\,m' \Leftrightarrow \Phi^{-1}(m) \preceq \Phi^{-1}(m')$ for all $m, m' \in M$. Then $\phi(\preceq)$ is uniquely determined by $\preceq$ in virtue of the surjectivity of Φ^{-1}, and $\phi(\preceq)$ is total and hence reflexive and is transitive as $\preceq$ is. The antisymmetry of $\phi(\preceq)$ follows immediately from the injectivity of Φ^{-1}.

In a similar way, there exists a map $\psi : \mathrm{TO}(M) \to \mathrm{TO}(N)$ such that for any given $\leq \in \mathrm{TO}(M)$ it holds $\Psi(m)\,\psi(\leq)\,\Psi(m') \Leftrightarrow m \leq m'$ for all $m, m' \in M$.

The maps ϕ and ψ are inverse of each other, thus they are isomorphisms of sets. Indeed, they are more, as the following theorem asserts.

Theorem 1.4.2. *The bijection ϕ of 1.4.1 is a homeomorphism of* $\mathrm{TO}(N)$ *in* $\mathrm{TO}(M)$.

Proof. We only have to show that ϕ is continuous and open. Since ϕ is bijective, it is enough to check this for one choice of subbases of $\mathrm{TO}(N)$ and $\mathrm{TO}(M)$.

For each $(n, n') \in N \times N$ one has $\phi(\mathfrak{U}_{(n,n')}) = \mathfrak{U}_{(\Phi(n),\Phi(n'))}$, thus ϕ is open. For each $(m, m') \in M \times M$ it holds $\phi^{-1}(\mathfrak{U}_{(m,m')}) = \mathfrak{U}_{(\Phi^{-1}(m),\Phi^{-1}(m'))}$, hence ϕ is continuous. □

Definition & Remark 1.4.3. Each $v \in V$ can be written in *canonical form* as a sum $\sum_{n \in \mathrm{Supp}(v)} \alpha_n n$ for a uniquely determined finite subset $\mathrm{Supp}(v)$ of N such that $\alpha_n \in K \setminus \{0\}$ for all $n \in \mathrm{Supp}(v)$. We call $\mathrm{Supp}(v)$ the *support of* v. For each subset H of V let $\mathrm{Supp}(H) = \bigcup_{h \in H} \mathrm{Supp}(h)$.

In the notation of 1.2.1, one has $\mathrm{Supp}(\Phi(v)) = \Phi(\mathrm{Supp}(v))$ for all $v \in V$, and hence $\mathrm{Supp}(\Phi(H)) = \Phi(\mathrm{Supp}(H))$ for all $H \subseteq V$. Conversely, $\mathrm{Supp}(\Psi(p)) = \Psi(\mathrm{Supp}(p))$ for all $p \in K[X]$, and hence $\mathrm{Supp}(\Psi(E)) = \Psi(\mathrm{Supp}(E))$ for all $E \subseteq K[X]$.

Given any $\preceq \in \mathrm{TO}(N)$, for each $v \in V \setminus \{0\}$ we denote by $\mathrm{lm}_{\preceq}(v)$ the uniquely determined maximal element of $\mathrm{Supp}(v)$ with respect to $\preceq$.

In the notation of 1.4.1, observe that $\mathrm{LM}_{\phi(\preceq)}(\Phi(v)) = \Phi(\mathrm{lm}_{\preceq}(v))$ for all $v \in V \setminus \{0\}$. For each $v \in V \setminus \{0\}$ we write $\mathrm{LM}_{\preceq}(v)$ for $\mathrm{LM}_{\phi(\preceq)}(\Phi(v))$, and with abuse of language we call $\mathrm{LM}_{\preceq}(v)$ the *leading monomial of* v *with respect to* $\preceq$. In this situation, we denote $\mathrm{LC}_{\phi(\preceq)}(\Phi(v))$ by $\mathrm{LC}_{\preceq}(v)$ or $\mathrm{lc}_{\preceq}(v)$, and with abuse of language we call $\mathrm{LC}_{\preceq}(v)$ alias $\mathrm{lc}_{\preceq}(v)$ the *leading coefficient of* v *with respect to* $\preceq$. Observe that either $v - \mathrm{lc}_{\preceq}(v)\,\mathrm{lm}_{\preceq}(v) = 0$ or $\mathrm{lm}_{\preceq}(v - \mathrm{lc}_{\preceq}(v)\,\mathrm{lm}_{\preceq}(v)) \prec \mathrm{lm}_{\preceq}(v)$.

For each $\preceq \in \mathrm{TO}(N)$ and each $H \subseteq V$ we denote the ideal $\langle \mathrm{LM}_{\preceq}(h) \mid h \in H \setminus \{0\} \rangle$ of $K[X]$ by $\mathrm{LM}_{\preceq}(H)$, and again with abuse of language we call $\mathrm{LM}_{\preceq}(H)$ the *leading monomial ideal of* H *with respect to* $\preceq$.

Further, for each $H \subseteq V$ and each $\mathfrak{T} \subseteq \mathrm{TO}(N)$ let $\mathscr{lm}_{\mathfrak{T}}(H) = \{\mathrm{LM}_{\preceq}(H) \mid \preceq \in \mathfrak{T}\}$ be the set of all *leading monomial ideals of* H *from* $\mathfrak{T}$.

Similarly as in 1.2.4, given $H \subseteq V$ and $\mathfrak{T} \subseteq \mathrm{TO}(N)$, we say that $\preceq \in \mathrm{TO}(N)$ is a *minimalizer of* H *in* $\mathfrak{T}$ if $\mathrm{LM}_{\preceq}(H)$ is a minimal element of $\mathscr{lm}_{\mathfrak{T}}(H)$ with respect to $\subseteq$.

We denote the set of all minimalizers of H in $\mathfrak{T}$ by $\min_H(\mathfrak{T})$. We write $\mathscr{min}_{\mathfrak{T}}(H)$ for the set $\mathscr{lm}_{\min_H(\mathfrak{T})}(H) = \{\mathrm{LM}_{\preceq}(H) \mid \preceq \in \min_H(\mathfrak{T})\}$ of all *minimal leading monomial ideals of* H *from* $\mathfrak{T}$.

Remark 1.4.4. Let $\mathfrak{T} \subseteq \mathrm{TO}(N)$ and $H \subseteq V$. The homeomorphism $\phi\restriction_{\mathfrak{T}} : \mathfrak{T} \to \phi(\mathfrak{T})$ induces a homeomorphism $\overline{\phi\restriction_{\mathfrak{T}}} : \mathfrak{T}/{\sim_H} \to \phi(\mathfrak{T})/{\sim_{\Phi(H)}}$ such that $\pi_{\Phi(H)} \circ \phi\restriction_{\mathfrak{T}} = \overline{\phi\restriction_{\mathfrak{T}}} \circ \pi_H$, where $\sim_H$ is the equivalence relation on $\mathfrak{T}$ given by $\preceq \sim_H \preceq'$ if and only if $\mathrm{LM}_{\preceq}(H) = \mathrm{LM}_{\preceq'}(H)$, and $\sim_{\Phi(H)}$ is the equivalence relation on $\phi(\mathfrak{T})$ defined as in 1.2.6, and π_H and $\pi_{\Phi(H)}$ are the respective natural projections.

Remark 1.4.5. Given $H \subseteq V$ and $\mathfrak{T} \subseteq \mathrm{TO}(N)$, it follows immediately from the definitions that $\mathrm{LM}_{\preceq}(H) = \mathrm{LM}_{\phi(\preceq)}(\Phi(H))$ for all $\preceq \in \mathfrak{T}$. Conversely, given $E \subseteq K[X]$

and $\mathfrak{S} \subseteq \mathrm{TO}(M)$, one has $\mathrm{LM}_{\leq}(E) = \mathrm{LM}_{\psi(\leq)}(\Psi(E))$ for all $\leq \in \mathfrak{S}$. It immediately follows that $\mathcal{lm}_{\mathfrak{T}}(H) = \mathcal{lm}_{\phi(\mathfrak{T})}(\Phi(H))$ and $\mathcal{lm}_{\mathfrak{S}}(E) = \mathcal{lm}_{\psi(\mathfrak{S})}(\Psi(E))$, and even that $\mathcal{min}_{\mathfrak{T}}(H) = \mathcal{min}_{\phi(\mathfrak{T})}(\Phi(H))$ and $\mathcal{min}_{\mathfrak{S}}(E) = \mathcal{min}_{\psi(\mathfrak{S})}(\Psi(E))$.

Theorem 1.4.6. *Let $H \subseteq V$ and let $\mathfrak{T} \subseteq \mathrm{TO}(N)$ be closed. Then $\mathcal{min}_{\mathfrak{T}}(H)$ is finite, that is, there exist at most finitely many distinct minimal leading monomial ideals of H from $\mathfrak{T}$.*

Proof. Clear by 1.4.5, 1.4.2, and 1.2.9. □

Definition 1.4.7. We put $\mathrm{DO}(N) = \phi^{-1}(\mathrm{DO}(M))$, and call $\mathrm{DO}(N)$ the set of all *degree orderings on N.*

Remark 1.4.8. Clearly, $\mathrm{FO}_{\Phi^{-1}(1)}(N) = \phi^{-1}(\mathrm{FO}_1(M))$ and $\mathrm{WO}(N) = \phi^{-1}(\mathrm{WO}(M))$. Hence $\mathrm{DO}(N) \subseteq \mathrm{FO}_{\Phi^{-1}(1)}(N) \cap \mathrm{WO}(N)$ by 1.3.3 and 1.3.6. Moreover, by 1.4.2 and 1.3.7, $\mathrm{DO}(N)$ is closed in $\mathrm{TO}(N)$ and compact.

Theorem 1.4.9. *For each $H \subseteq V$ and each closed $\mathfrak{T} \subseteq \mathrm{DO}(N)$ the set $\mathcal{min}_{\mathfrak{T}}(H)$ is finite, that is, there exist at most finitely many distinct minimal leading monomial ideals of H from $\mathfrak{T}$.*

Proof. Clear by 1.4.5 and 1.3.8. □

1.5 $\mathfrak{T}$-multiplicative algebras of countable type

We keep the notation of the previous section.

Definition 1.5.1. An *algebra of countable type* is given by a quadruple $A_K^{t,\Phi} = (A, K, t, \Phi)$ consisting of an associative, not necessarily commutative algebra A over a field K, a non-negative integer t, and a K-module isomorphism Φ of A in $K[X_1, \ldots, X_t]$.

If $A_K^{t,\Phi}$ is an algebra of countable type and if M is the canonical K-basis $\{X^\nu \mid \nu \in \mathbb{N}_0^t\}$ of $K[X_1, \ldots, X_t]$, then $N = \Phi^{-1}(M)$ is a countable K-basis of A, which we call the *canonical basis of $A_K^{t,\Phi}$.*

Given any subset $\mathfrak{T}$ of the set $\mathrm{TO}(N)$ of all total orderings on N, we say that $A_K^{t,\Phi}$ or simply A is *multiplicative on $\mathfrak{T}$* or *$\mathfrak{T}$-multiplicative* if A is a domain and in the notation of 1.4.3 it holds $\mathrm{LM}_{\preceq}(ab) = \mathrm{LM}_{\preceq}(a)\,\mathrm{LM}_{\preceq}(b)$ for all $a, b \in A \setminus \{0\}$ and all $\preceq \in \mathfrak{T}$.

Henceforth in this section, let $A_K^{t,\Phi}$ be an algebra of countable type. We write $K[X]$ for $K[X_1,\ldots,X_t]$ and fix the canonical K-bases M and N of $K[X]$ and $A_K^{t,\Phi}$, respectively. Now we may make use of the notation introduced in 1.4.3. And yet another… Macaulay Basis Theorem, that is, a slight generalization of a classical result.

Theorem 1.5.2. *Let $\preceq\,\in \mathrm{WO}(N)$, assume that $A_K^{t,\Phi}$ is multiplicative on $\{\preceq\}$, let L be a left ideal of A, put $B = M \setminus \mathrm{LM}_\preceq(L)$, and let $\overline{} : K[X] \to K[X]/\Phi(L)$ be the residue class epimorphism of K-modules. Then the image $\overline{B}$ of B under $\overline{}$ is a K-basis of $K[X]/\Phi(L)$.*

Proof. We first show that $\overline{B}$ generates $K[X]/\Phi(L)$ over K. Suppose it is not the case. Let $\overline{W} = \sum_{b\in B} K\overline{b}$. Then the set $P = \{p \in K[X] \setminus \{0\} \mid \overline{p} \notin \overline{W}\}$ is nonempty. Thus, with $\leq\, = \phi(\preceq)$, the subset $Q = \{\mathrm{LM}_\leq(p) \mid p \in P\}$ of M is nonempty. As $\phi(\preceq) \in \mathrm{WO}(M)$, see 1.4.8, we may choose $p \in P$ such that $\mathrm{LM}_\leq(p)$ is minimal in Q with respect to $\leq$. It holds $\overline{\mathrm{Supp}(p) \setminus \{\mathrm{LM}_\leq(p)\}} \subseteq \overline{W}$. Indeed, if there existed $m \in \mathrm{Supp}(p) \setminus \{\mathrm{LM}_\leq(p)\}$ such that $\overline{m} \notin \overline{W}$, then we would have $m \in P$ and hence $m = \mathrm{LM}_\leq(m) \in Q$, and this would contradict the minimality of $\mathrm{LM}_\leq(p)$ as clearly $m < \mathrm{LM}_\leq(p)$. It follows $\overline{\mathrm{LM}_\leq(p)} \notin \overline{W}$ as otherwise we would have $\overline{\mathrm{Supp}(p)} \subseteq \overline{W}$ and hence the contradiction $\overline{p} \in \overline{W}$. Therefore $\mathrm{LM}_\leq(p) \in \mathrm{LM}_\preceq(L)$ as otherwise we would have $\mathrm{LM}_\leq(p) \in B$ and hence the contradiction $\overline{\mathrm{LM}_\leq(p)} \in \overline{B} \subseteq \overline{W}$. Thus we find $x \in L \setminus \{0\}$ such that $\mathrm{LM}_\preceq(x) \mid \mathrm{LM}_\leq(p)$, see 1.2.2. So we find $n \in N$ with $\mathrm{LM}_\leq(p) = \Phi(n)\,\mathrm{LM}_\preceq(x) = \mathrm{LM}_\preceq(n)\,\mathrm{LM}_\preceq(x) = \mathrm{LM}_\preceq(nx)$, where this last equality holds by multiplicativity of $A_K^{t,\Phi}$ on $\{\preceq\}$. With $q = \mathrm{LC}_\leq(p)\,\mathrm{LC}_\leq(\Phi(nx))^{-1}\,\Phi(nx)$ we obtain $q \in \Phi(L)$ as L is a left ideal and $\Phi(L)$ is a K-module, and of course we have $\mathrm{LM}_\leq(p) = \mathrm{LM}_\leq(q)$ and $\mathrm{LC}_\leq(p) = \mathrm{LC}_\leq(q)$. Now we consider $r = p - q$. It holds $\overline{r} = \overline{p}$. Thus $\overline{r} \notin \overline{W}$. But then in particular $r \neq 0$, and hence clearly $\mathrm{LM}_\leq(r) < \mathrm{LM}_\leq(p)$, thus $r \notin P$ by the minimality of $\mathrm{LM}_\leq(p)$, so that $\overline{r} \in \overline{W}$, a contradiction.

Next we show that $\overline{B}$ is linearly independent over K. Suppose to the contrary that there exist $r \in \mathbb{N}$ and $\alpha_1,\ldots,\alpha_r \in K \setminus \{0\}$ and pairwise distinct $\overline{b}_1,\ldots,\overline{b}_r \in \overline{B}$ such that $\alpha_1\overline{b}_1 + \ldots + \alpha_r\overline{b}_r = \overline{0}$. Then any respective representatives $b_1,\ldots,b_r \in B$ of $\overline{b}_1,\ldots,\overline{b}_r$ are pairwise distinct and $\alpha_1 b_1 + \ldots + \alpha_r b_r = \Phi(y)$ for some $y \in L$. Of course, $y \neq 0$ as the monomials $b_1,\ldots,b_r$ are linearly independent over K. It follows $\mathrm{LM}_\leq(\Phi(y)) = b_i \in B$ for some $i \in \{1,\ldots,r\}$. Therefore $\mathrm{LM}_\leq(\Phi(y)) \in B \cap \mathrm{LM}_\leq(\Phi(L))$, that is, $\mathrm{LM}_\preceq(y) \in B \cap \mathrm{LM}_\preceq(L)$ by 1.4.5. But, by definition, $B \cap \mathrm{LM}_\preceq(L) = \varnothing$, a contradiction. □

Corollary 1.5.3. *Let* $\preceq, \preceq' \in \mathrm{WO}(N)$*, assume that* $A_K^{t,\Phi}$ *is multiplicative on* $\{\preceq, \preceq'\}$*, and let* L *be a left ideal of* A *such that* $\mathrm{LM}_{\preceq}(L) \subseteq \mathrm{LM}_{\preceq'}(L)$*. Then* $\mathrm{LM}_{\preceq}(L) = \mathrm{LM}_{\preceq'}(L)$*.*

Proof. Put $B = M \setminus \mathrm{LM}_{\preceq}(L)$ and $B' = M \setminus \mathrm{LM}_{\preceq'}(L)$. Let $\overline{} : K[X] \to K[X]/\Phi(L)$ be the residue class homomorphism (of K-modules). Suppose that $\mathrm{LM}_{\preceq}(L) \subsetneq \mathrm{LM}_{\preceq'}(L)$. Then $B \supsetneq B'$, hence $\overline{B} \supseteq \overline{B'}$.

If it held $\overline{B} = \overline{B'}$, then we would find $b \in B \setminus B'$ and $b' \in B'$ such that $\overline{b} = \overline{b'}$, hence $b - b' \in \Phi(L)$, thus $\mathrm{LM}_{\phi(\preceq)}(b - b') \in \mathrm{LM}_{\phi(\preceq)}(\Phi(L)) = \mathrm{LM}_{\preceq}(L)$; on the other hand, either $\mathrm{LM}_{\phi(\preceq)}(b - b') = b$ or $\mathrm{LM}_{\phi(\preceq)}(b - b') = b'$, in any case $\mathrm{LM}_{\phi(\preceq)}(b - b') \in B$, a contradiction.

Therefore $\overline{B} \supsetneq \overline{B'}$. But, by 1.5.2, $\overline{B}$ and $\overline{B'}$ are K-bases of $K[X]/\Phi(L)$, hence the one cannot strictly contain the other, a contradiction. □

Corollary 1.5.4. *Let* $\mathfrak{T} \subseteq \mathrm{WO}(N)$*, assume that* $A_K^{t,\Phi}$ *is multiplicative on* $\mathfrak{T}$*, and let* L *be a left ideal of* A*. Then* $\mathscr{lm}_{\mathfrak{T}}(L) = \mathscr{min}_{\mathfrak{T}}(L)$*. In particular, if* $\mathfrak{T}$ *is closed in* $\mathrm{TO}(N)$*, then* $\mathscr{lm}_{\mathfrak{T}}(L)$ *is finite, that is,* L *admits at most finitely many distinct leading monomial ideals from* $\mathfrak{T}$*.*

Proof. By 1.5.3, $\mathfrak{T} = \min_L(\mathfrak{T})$, thus $\mathscr{lm}_{\mathfrak{T}}(L) = \mathscr{min}_{\mathfrak{T}}(L)$, which is finite by 1.4.6 if $\mathfrak{T}$ is closed in $\mathrm{TO}(N)$. □

1.6 Admissible orderings

We keep the notation of the previous section.

Definition 1.6.1. A *compatible ordering on* M or *of* $K[X]$ is a total ordering $\leq$ on M such that for all $\upsilon, \nu, \gamma \in \mathbb{N}_0^t$ it holds compatibility: $X^\upsilon \leq X^\nu \Rightarrow X^{\upsilon+\gamma} \leq X^{\nu+\gamma}$.

Compatible orderings are also known as *semigroup orderings*. The set of all compatible orderings of $K[X]$ is denoted by $\mathrm{CO}(M)$.

We also consider the set of *compatible orderings on* N or *of* $A_K^{t,\Phi}$ or simply *of* A defined as $\mathrm{CO}(N) = \phi^{-1}(\mathrm{CO}(M))$.

Proposition 1.6.2. $\mathrm{CO}(M)$ *and* $\mathrm{CO}(N)$ *are closed in* $\mathrm{TO}(M)$ *and* $\mathrm{TO}(N)$*, respectively, and hence compact.*

Proof. Let $(S_i)_{i\in\mathbb{N}_0}$ be a filtration of M consisting of finite sets S_i. Let $\leq\ \in \mathrm{TO}(M)$ be an accumulation point of $\mathrm{CO}(M)$. Thus, by definition, for each $r \in \mathbb{N}_0$ there exists

$\leq_r \in \mathrm{CO}(M) \cap \mathfrak{N}_r(\leq) \setminus \{\leq\}$, so that $\leq_r$ and $\leq$ agree on S_{r+1}. Choose any $\upsilon, \nu \in \mathbb{N}_0^t$ and assume that $X^\upsilon \leq X^\nu$, say. Let $\gamma \in \mathbb{N}_0^t$. Then we find $r \in \mathbb{N}_0$ such that S_{r+1} contains the monomials $X^\upsilon, X^\nu, X^{\upsilon+\gamma}, X^{\nu+\gamma}$. There exists $\leq_r$ as above that agrees with $\leq$ on S_{r+1}, so that $X^\upsilon \leq_r X^\nu$. Since $\leq_r$ is a compatible ordering of $K[X]$, it follows $X^{\upsilon+\gamma} \leq_r X^{\nu+\gamma}$. Therefore $X^{\upsilon+\gamma} \leq X^{\nu+\gamma}$. Hence $\leq \in \mathrm{CO}(M)$. Thus $\mathrm{CO}(M)$ contains all its accumulation points in $\mathrm{TO}(M)$ and hence $\mathrm{CO}(M)$ is closed in $\mathrm{TO}(M)$. Since $\mathrm{TO}(M)$ is compact by 1.1.4, $\mathrm{CO}(M)$ is compact. Since ϕ is a homeomorphism by 1.4.2, also $\mathrm{CO}(N)$ is closed in $\mathrm{TO}(N)$ and compact. □

Definition 1.6.3. $\mathrm{AO}(M) = \mathrm{FO}_1(M) \cap \mathrm{CO}(M)$ is the set of all *admissible orderings on* M or *of* $K[X]$, and $\mathrm{AO}(N) = \mathrm{FO}_{\Phi^{-1}(1)}(N) \cap \mathrm{CO}(N)$ is the set of all *admissible orderings on* N or *of* $A_K^{t,\Phi}$ or simply *of* A. Observe that $\phi^{-1}(\mathrm{AO}(M)) = \mathrm{AO}(N)$. Admissible orderings are also known as *monoid orderings*.

Remark 1.6.4. One sees that this definition of admissible ordering on M and on N is equivalent to the one given in [23], and it is equivalent to the notion of term orderings given in [35] in the case of Weyl algebras under the assumption that $\Phi(1) = 1$.

Remark 1.6.5. An admissible ordering of $K[X]$ is thus a total ordering $\leq$ on M enjoying the properties of well-foundedness: $1 \leq X^\nu$, and compatibility: $X^\upsilon \leq X^\nu \Rightarrow X^{\upsilon+\gamma} \leq X^{\nu+\gamma}$. Since M is a K-basis of $K[X]$, these axioms are equivalent to: $1 < X^\nu$ whenever $\nu \neq 0$, and $X^\upsilon < X^\nu \Rightarrow X^{\upsilon+\gamma} < X^{\nu+\gamma}$.

Example 1.6.6. The *lexicographic ordering* $\leq_{\mathrm{lex}}$ on M defined by

$$X^\upsilon <_{\mathrm{lex}} X^\nu :\Leftrightarrow (\exists\, i \in \{1, \ldots, t\} : (1 \leq j < i \Rightarrow \upsilon_j = \nu_j) \wedge \upsilon_i < \nu_i)$$

for all $\upsilon, \nu \in \mathbb{N}_0^t$ is an admissible ordering of $K[X]$, see [21, Example 1.29(1)].

Example 1.6.7. Fixed $\leq \in \mathrm{AO}(M)$ and $\omega \in \mathbb{N}_0^t$, the ω*-weighted* $\leq$*-ordering* $\leq_\omega$ defined by

$$X^\upsilon <_\omega X^\nu :\Leftrightarrow (\omega \cdot \upsilon < \omega \cdot \nu) \vee (\omega \cdot \upsilon = \omega \cdot \nu \wedge X^\nu < Y^\upsilon)$$

for all $\upsilon, \nu \in \mathbb{N}_0^t$ is an admissible ordering of $K[X]$, see Exercise 12 in [20, II.4].

Proposition 1.6.8. $\mathrm{AO}(M)$ *and* $\mathrm{AO}(N)$ *are closed in* $\mathrm{TO}(M)$ *and* $\mathrm{TO}(N)$*, respectively, and hence compact.*

Proof. Clear by 1.6.2, 1.1.8, and 1.1.4. ☐

Proposition 1.6.9. $\mathrm{AO}(M) = \mathrm{WO}(M) \cap \mathrm{CO}(M)$ *and* $\mathrm{AO}(N) = \mathrm{WO}(N) \cap \mathrm{CO}(N)$.

Proof. By [20, II.4.6], $\mathrm{FO}_1(M) \cap \mathrm{CO}(M) = \mathrm{WO}(M) \cap \mathrm{CO}(M)$. As ϕ^{-1} is injective and as $\phi^{-1}(\mathrm{CO}(M)) = \mathrm{CO}(N)$ and $\phi^{-1}(\mathrm{FO}_1(M)) = \mathrm{FO}_{\Phi^{-1}(1)}(N)$ and $\phi^{-1}(\mathrm{WO}(M)) = \mathrm{WO}(N)$, the second claim follows. ☐

1.7 Degree-compatible orderings

We keep the notation of the previous section.

Example 1.7.1. It holds $\mathrm{DO}(M) \nsubseteq \mathrm{CO}(M)$ and hence $\mathrm{DO}(N) \nsubseteq \mathrm{CO}(N)$. Indeed, any degree ordering $\leq$ of $K[Y, Z]$ such that $1 < Y < Z < YZ < Y^2 < Z^2 < \dots$ is not compatible because compatibility would force $Y^2 < YZ$ from $Y < Z$.

Also it holds $\mathrm{CO}(M) \nsubseteq \mathrm{DO}(M)$ and hence $\mathrm{CO}(N) \nsubseteq \mathrm{DO}(N)$. Indeed, the lexicographic ordering $\leq_{\mathrm{lex}}$ of $K[Y, Z]$ induced for instance by $Y <_{\mathrm{lex}} Z$ is a compatible ordering but not a degree ordering since $\deg(\mathrm{LM}_{\leq}(Y + Z^2)) = \deg(Y) = 1 \neq 2 = \deg(Y + Z^2)$ for instance.

Remark & Definition 1.7.2. It is not to expect that there exist interesting K-algebras of countable type that are multiplicative on $\mathrm{DO}(M)$ since even $K[X]$ is not. For a degree ordering $\leq$ of $K[Y, Z]$ such that $1 < Y < Z < Y^2 < Z^2 < YZ < \dots$ for instance, it holds $\mathrm{LM}_{\leq}((Y + Z)^2) = YZ \neq Z^2 = \mathrm{LM}_{\leq}(Y + Z)\,\mathrm{LM}_{\leq}(Y + Z)$.

Therefore we shall consider the set $\mathrm{DCO}(M) = \mathrm{DO}(M) \cap \mathrm{CO}(M)$ of the *degree-compatible orderings on* M or *of* $K[X]$ and the set $\mathrm{DCO}(N) = \mathrm{DO}(N) \cap \mathrm{CO}(N)$ of the *degree-compatible orderings on* N or *of* $A_K^{t,\Phi}$ or simply *of* A.

Of course, it holds $\mathrm{DCO}(N) = \phi^{-1}(\mathrm{DCO}(M))$. Moreover, $\mathrm{DCO}(M) \subseteq \mathrm{AO}(M)$ by 1.3.3, and hence $\mathrm{DCO}(N) \subseteq \mathrm{AO}(N)$. Finally, by 1.3.7 and 1.4.8 and by 1.6.2, $\mathrm{DCO}(M)$ and $\mathrm{DCO}(N)$ are closed in $\mathrm{TO}(M)$ and $\mathrm{TO}(N)$, respectively, and compact.

Proposition 1.7.3. *If* $t > 1$*, where* t *is the number of indeterminates, then* $\mathrm{DCO}(M)$ *is nowhere dense in* $\mathrm{DO}(M)$*, and so is* $\mathrm{DCO}(N)$ *in* $\mathrm{DO}(N)$.

Proof. Consider the filtration $(S_i)_{i \in \mathbb{N}_0}$ of M given by $S_i = \{m \in M \mid \deg(m) < i\}$. Suppose that some ordering $\leq$ lies in the interior $\mathrm{DCO}(M)^\circ$ of the closed subset $\mathrm{DCO}(M)$ of $\mathrm{DO}(M)$. Then there exists a neighbourhood of $\leq$ open in $\mathrm{DO}(M)$ that is contained in

$\mathrm{DCO}(M)^\circ$, that is, we find $r \in \mathbb{N}_0$ such that $\mathfrak{N}_r(\leq) \cap \mathrm{DO}(M) \subseteq \mathrm{DCO}(M)$. Since $S_1 = \{1\}$, we have $\mathfrak{N}_0(\leq) = \mathrm{TO}(M)$. As $\mathrm{DCO}(M) \subsetneq \mathrm{DO}(M)$, it follows $r \geq 1$. Assume that $X_1 < X_2$, say. Then $X_1^{r+2} < X_1^{r+1}X_2$ by compatibility. Let $\leq'$ be the total ordering on M given by $X_1^{r+1}X_2 <' X_1^{r+2}$ and $m \leq' m' \Leftrightarrow m \leq m'$ whenever $(m, m') \in M \times M \setminus \{(X_1^{r+1}X_2, X_1^{r+2})\}$. Then $\leq' \in \mathfrak{N}_r(\leq) \cap \mathrm{DO}(M)$, so that $\leq' \in \mathrm{DCO}(M)$. As $r \geq 1$, we have that $\leq$ and $\leq'$ agree on S_2, thus $X_1 <' X_2$. By compatibility it follows $X_1^{r+2} <' X_1^{r+1}X_2$, a contradiction. Now we conclude by 1.4.2. □

Remark 1.7.4. If $t = 1$, then $|\mathrm{DO}(M)| = |\mathrm{DCO}(M)| = 1 = |\mathrm{DCO}(N)| = |\mathrm{DO}(N)|$, thus $\mathrm{DCO}(M) = \mathrm{DO}(M)$ and $\mathrm{DCO}(N) = \mathrm{DO}(N)$.

Example 1.7.5. For each $\leq \in \mathrm{AO}(M)$ the binary relation $\leq_{\deg}$ on M defined by

$$m \leq_{\deg} m' \Leftrightarrow \deg(m) < \deg(m') \vee (\deg(m) = \deg(m') \wedge m \leq m').$$

is a degree-compatible ordering of $K[X]$. More generally, the admissible orderings of Example 1.6.7 are degree-compatible orderings whenever $\omega \neq 0$ or $\leq \in \mathrm{DCO}(M)$.

Remark 1.7.6. By 1.5.4, for each $H \subseteq A$ and each $\mathfrak{T} \subseteq \mathrm{DCO}(N)$ the set $\mathscr{lm}_{\mathfrak{T}}(H)$ is finite. In particular, by 1.6.6, 1.6.7, and 1.7.5, the set $\mathscr{lm}_{\mathrm{DCO}(N)}(H)$ is nonempty and finite.

1.8 $\mathfrak{T}$-admissible algebras

We keep the notation of the previous section.

Definition 1.8.1. Let $\mathfrak{T} \subseteq \mathrm{AO}(N)$. We say that $A_K^{t,\Phi}$ or simply A is *$\mathfrak{T}$-admissible* if $A_K^{t,\Phi}$ is multiplicative on $\mathfrak{T}$. We say that $A_K^{t,\Phi}$ or simply A is *admissible* if $A_K^{t,\Phi}$ is $\mathrm{AO}(N)$-admissible. We say that $A_K^{t,\Phi}$ or simply A is *degree-compatible* if $A_K^{t,\Phi}$ is $\mathrm{DCO}(N)$-admissible.

Example 1.8.2. In the terminology of [23], every K-algebra that is of solvable type with respect to all admissible orderings is admissible. This follows indeed from [23, 1.5].

For instance, if K has characteristic 0, then every Weyl algebra W over K is isomorphic as a K-module to a commutative polynomial ring over K, see [19, I.2.1], and W clearly fulfills the axioms [23, 1.2] of an algebra of solvable type for all admissible orderings on its canonical K-basis, so that W is multiplicative on these orderings by [23, 1.5].

Example 1.8.3. If K has characteristic 0, then the universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ of finite length over K is degree-compatible. Indeed, let $X = \{x_1, \ldots, x_r\}$ be a finite K-basis of $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt Theorem, see [31, II] for instance, especially 2.13, 2.14, and 2.22, there exist then a canonical K-module monomorphism $h : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ and a countable K-basis $Y = \{y_1^{\nu_1} \cdots y_r^{\nu_r} \mid (\nu_1, \ldots, \nu_r) \in \mathbb{N}_0^r\}$ of $U(\mathfrak{g})$ with $y_i = h(x_i)$ such that $[y_j, y_k] = \sum_{1 \le i \le r} c_{ijk} y_i$ for some $c_{ijk} \in K$. Thus, $U(\mathfrak{g})$ is isomorphic as a K-module to the commutative polynomial ring $K[X_1, \ldots X_r]$ by an isomorphism that maps y_i to X_i, and the relations $y_k y_j = y_j y_k - \sum_{1 \le i \le r} c_{ijk} y_i$ imply by [23, 1.2 & 1.5] that $U(\mathfrak{g})$ is multiplicative on $\mathrm{DCO}(Y)$.

Theorem 1.8.4. *Let* $\mathfrak{T} \subseteq \mathrm{AO}(N)$ *be a closed subset. Assume that* $A_K^{t,\Phi}$ *is* $\mathfrak{T}$*-admissible. Let* L *be a left ideal of* A*. Then* $\ell m_{\mathfrak{T}}(L)$ *is finite, that is,* L *admits only finitely many distinct leading monomial ideals from* $\mathfrak{T}$*. In particular, if* $A_K^{t,\Phi}$ *is admissible, then the nonempty set* $\ell m_{\mathrm{AO}(N)}(L)$ *is finite.*

Proof. It is all clear by 1.5.4, 1.6.8, 1.6.9, and by 1.4.2, 1.6.6, 1.6.7, 1.8.2. □

Remark 1.8.5. Notice that by 1.7.6 we already know this result for subspaces $\mathfrak{T}$ of $\mathrm{DCO}(N)$ without having to assume that A be multiplicative on $\mathfrak{T}$ nor that L be a left ideal.

1.9 Gröbner bases

We keep the notation of the previous section.

Definition 1.9.1. Let $A_K^{t,\Phi}$ be an algebra of countable type, L be a left ideal of A, N denote the canonical K-basis of $A_K^{t,\Phi}$, and $\preceq$ be a total ordering on N. A *Gröbner basis of L with respect to* $\preceq$ is a finite subset G of L such that $L = \sum_{g \in G} Ag$ and $\mathrm{LM}_{\preceq}(L) = \mathrm{LM}_{\preceq}(G)$.

Remark 1.9.2. The definition of Gröbner basis given here is equivalent to the one given in [23] if one restricts to admissible orderings and algebras of solvable type, see [23, 3.8].

This definition is also equivalent to the one given in [35] when further restricting to Weyl algebras.

By [27, II.4.2] it is less general than the one given in [27, II.3.2(ii)], but it is equivalent to the definition given in [27, III.1.1] when restricting to admissible orderings and free K-algebras $K\langle X_\lambda \mid \lambda \in \Lambda \rangle$, Λ any index set.

Definition 1.9.3. Let $A_K^{t,\Phi}$ be an algebra of countable type, let L be a left ideal of A, and let N denote the canonical K-basis of $A_K^{t,\Phi}$.

Given any $\mathfrak{T} \subseteq \mathrm{TO}(N)$, we say that a finite subset U of L is a *$\mathfrak{T}$-universal Gröbner basis of L* if U is a Gröbner basis of L with respect to all elements of $\mathfrak{T}$.

In the following we call the $\mathfrak{T}$-universal Gröbner bases in $\mathfrak{T}$-admissible algebras simply *universal Gröbner bases.*

We fix here an algebra $A_K^{t,\Phi}$ of countable type and as usually denote its canonical K-basis by N.

Theorem 1.9.4. *Assume that A is left noetherian, let L be a left ideal of A, and let $\preceq$ be a total ordering on N. Then L admits a Gröbner basis with respect to $\preceq$.*

Proof. Suppose that L admits no Gröbner basis with respect to $\preceq$. Since A is left noetherian, there exists a finite subset F_0 of L such that $L = AF_0$. It holds $\mathrm{LM}_\preceq(F_0) \subsetneq \mathrm{LM}_\preceq(L)$ as F_0 is not a Gröbner basis. Thus there exists $x_1 \in L \smallsetminus \{0\}$ with $\mathrm{LM}_\preceq(x_1) \notin \mathrm{LM}_\preceq(F_0)$. Put $F_1 = F_0 \cup \{x_1\}$. Again $\mathrm{LM}_\preceq(F_1) \subsetneq \mathrm{LM}_\preceq(L)$ as F_1 is not a Gröbner basis. Thus there exists $x_2 \in L \smallsetminus \{0\}$ with $\mathrm{LM}_\preceq(x_2) \notin \mathrm{LM}_\preceq(F_1)$. Put $F_2 = F_1 \cup \{x_2\}$. Again $\mathrm{LM}_\preceq(F_2) \subsetneq \mathrm{LM}_\preceq(L)$ as F_2 is not a Gröbner basis... We find in this manner an infinite chain $\mathrm{LM}_\preceq(F_0) \subsetneq \mathrm{LM}_\preceq(F_1) \subsetneq \mathrm{LM}_\preceq(F_2) \subsetneq \dots$ of ideals of $K[X]$, in contradiction to the noetherianity of $K[X]$. □

Theorem 1.9.5. *Assume that there exists $\preceq \in \mathrm{WO}(N)$ with the property that $A_K^{t,\Phi}$ is multiplicative on $\{\preceq\}$. Let L be a left ideal of A and F be a finite subset of L such that* $\mathrm{LM}_\preceq(L) = \mathrm{LM}_\preceq(F)$. *Then $L = \sum_{f\in F} Af$.*

Proof. Trivially, we have $\sum_{f\in F} Af \subseteq L$. Suppose that $\sum_{f\in F} Af \subsetneq L$. Then the set $U = \{\mathrm{LM}_\preceq(l) \mid l \in L \smallsetminus \sum_{f\in F} Af\}$ is nonempty. We have $\leq\, = \phi(\preceq) \in \mathrm{WO}(M)$, and hence there exists $l \in L \smallsetminus \sum_{f\in F} Af$ such that $u = \mathrm{LM}_\preceq(l)$ is minimal in U with respect to $\leq$. Since $u \in \mathrm{LM}_\preceq(L) = \mathrm{LM}_\preceq(F)$, we can write $u = \sum_{f\in F\smallsetminus\{0\}} p_f \,\mathrm{LM}_\preceq(f)$ for some family $(p_f)_{f\in F\smallsetminus\{0\}}$ of polynomials. Because $u \in M$ and M is a K-basis of $K[X]$, we find $m \in \bigcup_{f\in F\smallsetminus\{0\}} \mathrm{Supp}(p_f) \subseteq M$ and $g \in F \smallsetminus \{0\}$ such that $u = m\,\mathrm{LM}_\preceq(g)$. Put $n = \Phi^{-1}(m)$. As $n \in N$, clearly $n \neq 0$. Since A is a domain, it follows $ng \neq 0$. Now put $h = l - \mathrm{lc}_\preceq(l)\,\mathrm{lc}_\preceq(ng)^{-1} ng$. Then $h \in L \smallsetminus \sum_{f\in F} Af$, thus $\mathrm{LM}_\preceq(h) \in U$. On the other hand, $\mathrm{LM}_\preceq(ng) = \mathrm{LM}_\preceq(n)\,\mathrm{LM}_\preceq(g) = m\,\mathrm{LM}_\preceq(g) = u = \mathrm{LM}_\preceq(l)$, so that $\mathrm{LM}_\preceq(h) < \mathrm{LM}_\preceq(l)$, a contradiction. □

Corollary 1.9.6. *Assume that there exists* $\preceq \in \mathrm{WO}(N)$ *such that* $A_K^{t,\Phi}$ *is multiplicative on* $\{\preceq\}$*. Then* A *is left noetherian.*

Proof. Let L be a left ideal of A. As $K[X]$ is noetherian, we find a finite subset F of L such that $\mathrm{LM}_{\preceq}(F) = \mathrm{LM}_{\preceq}(L)$. By 1.9.5, F is a generating set of L. Thus every left ideal of A is finitely generated. □

Corollary 1.9.7. *Assume that there exists* $\preceq \in \mathrm{WO}(N)$ *such that* $A_K^{t,\Phi}$ *is multiplicative on* $\{\preceq\}$*. Then for each left ideal* L *of* A *and each total ordering* $\preceq'$ *on* N *there exists a Gröbner basis of* L *with respect to* $\preceq'$*.*

Proof. Clear by 1.9.4 and 1.9.6. □

1.10 Universal Gröbner bases in admissible algebras

We keep the notation of the previous section.

Lemma 1.10.1. *Let* $\preceq, \preceq' \in \mathrm{WO}(N)$ *such that* $A_K^{t,\Phi}$ *is multiplicative on* $\{\preceq, \preceq'\}$*. Let* L *be a left ideal of* A *and* G *be a Gröbner basis of* L *with respect to* $\preceq$*. If* $\preceq$ *and* $\preceq'$ *agree on* $\mathrm{Supp}(G)$*, then* $\mathrm{LM}_{\preceq}(L) = \mathrm{LM}_{\preceq'}(L)$ *and* G *is a Gröbner basis of* L *with respect to* $\preceq'$*.*

Proof. Because $\preceq$ and $\preceq'$ agree on $\mathrm{Supp}(G)$, it follows that $\phi(\preceq)$ and $\phi(\preceq')$ agree on $\Phi(\mathrm{Supp}(G)) = \mathrm{Supp}(\Phi(G))$. Hence $\mathrm{LM}_{\phi(\preceq)}(\Phi(G)) = \mathrm{LM}_{\phi(\preceq')}(\Phi(G))$ by 1.2.3. From 1.4.5 it follows $\mathrm{LM}_{\preceq}(L) = \mathrm{LM}_{\preceq}(G) = \mathrm{LM}_{\preceq'}(G) \subseteq \mathrm{LM}_{\preceq'}(L)$. As $\mathrm{TO}(N)$ is a Hausdorff space, see 1.1.2, points are closed, so $\{\preceq, \preceq'\}$ is closed in $\mathrm{TO}(N)$. Thus $\mathcal{lm}_{\{\preceq,\preceq'\}}(L) = \mathcal{min}_{\{\preceq,\preceq'\}}(L)$ by 1.5.4, and hence $\mathrm{LM}_{\preceq}(L) = \mathrm{LM}_{\preceq'}(L)$, and therefore $\mathrm{LM}_{\preceq'}(G) = \mathrm{LM}_{\preceq'}(L)$. □

Lemma 1.10.2. *Let* $\mathfrak{T} \subseteq \mathrm{WO}(N)$ *such that* $A_K^{t,\Phi}$ *is multiplicative on* $\mathfrak{T}$*. Let* L *be a left ideal of* A *and let* F *be a finite subset of* L*. Then the set* $\mathfrak{U}_L(F)$ *of all* $\preceq \in \mathfrak{T}$ *such that* F *is a Gröbner basis of* L *with respect to* $\preceq$ *is open in* $\mathfrak{T}$*.*

Proof. Let $(S_i)_{i\in\mathbb{N}_0}$ be a filtration of N consisting of finite sets S_i. There exists $r \in \mathbb{N}_0$ such that the finite subset $\mathrm{Supp}(F)$ of N lies in S_{r+1}. We may assume that $\mathfrak{U}_L(F) \neq \varnothing$, so that $\mathfrak{T} \neq \varnothing$. Let $\preceq \in \mathfrak{U}_L(F)$. Thus F is a Gröbner basis of L with respect to $\preceq$. Consider the open neighbourhood $\mathfrak{N}_r(\preceq) \cap \mathfrak{T}$ of $\preceq$ in $\mathfrak{T}$ and let $\preceq' \in \mathfrak{N}_r(\preceq) \cap \mathfrak{T}$. Then $\preceq$ and $\preceq'$ agree on S_{r+1} and in particular on $\mathrm{Supp}(F)$. By 1.10.1, F is a Gröbner basis of L with respect to $\preceq'$, that is, $\preceq' \in \mathfrak{U}_L(F)$. Hence $\preceq \in \mathfrak{N}_r(\preceq) \cap \mathfrak{T} \subseteq \mathfrak{U}_L(F)$, and $\mathfrak{U}_L(F)$ is open in $\mathfrak{T}$. □

Remark 1.10.3. Let $\varnothing \neq \mathfrak{T} \subseteq \mathrm{WO}(N)$ such that $A_K^{t,\Phi}$ is multiplicative on $\mathfrak{T}$. Let L be a left ideal of A. Then, by 1.9.7, for each $\preceq\, \in \mathfrak{T}$ there exists a Gröbner basis $G_\preceq$ of L with respect to $\preceq$. Thus, in the notation of 1.10.2, clearly $\preceq\, \in \mathfrak{U}_L(G_\preceq)$ for each $\preceq\, \in \mathfrak{T}$. Hence, by 1.10.2, $\bigcup_{\preceq \in \mathfrak{T}} \mathfrak{U}_L(G_\preceq)$ is an open covering of $\mathfrak{T}$.

Theorem 1.10.4. *Let $\varnothing \neq \mathfrak{T} \subseteq \mathrm{WO}(N)$ such that $\mathfrak{T}$ is closed in $\mathrm{TO}(N)$ and $A_K^{t,\Phi}$ is multiplicative on $\mathfrak{T}$. Let L be a left ideal of A. Then L admits a $\mathfrak{T}$-universal Gröbner basis.*

Proof. In the notation of 1.10.3, $\bigcup_{\preceq \in \mathfrak{T}} \mathfrak{U}_L(G_\preceq)$ is an open covering of $\mathfrak{T}$, where each $G_\preceq$ is a Gröbner basis of L with respect to $\preceq$. As $\mathrm{TO}(N)$ is compact and $\mathfrak{T}$ is closed in $\mathrm{TO}(N)$, $\mathfrak{T}$ is compact. Hence we can find $s \in \mathbb{N}$ and $\preceq_1, \ldots, \preceq_s \in \mathfrak{T}$ such that $\bigcup_{1 \le j \le s} \mathfrak{U}_L(G_{\preceq_j})$ is a finite open covering of $\mathfrak{T}$. We claim that $U = \bigcup_{1 \le j \le s} G_{\preceq_j}$ is a $\mathfrak{T}$-universal Gröbner basis of L. Indeed, let $\preceq_0 \in \mathfrak{T}$. Then there exists $j \in \{1, \ldots, s\}$ such that $\preceq_0 \in \mathfrak{U}_L(G_{\preceq_j})$. Thus $G_{\preceq_j}$ is a Gröbner basis of L with respect to $\preceq_0$. As $G_{\preceq_j} \subseteq U$, of course also U is a Gröbner basis of L with respect to $\preceq_0$. Since the choice of $\preceq_0$ in $\mathfrak{T}$ was arbitrary, we conclude that U is a $\mathfrak{T}$-universal Gröbner basis of L. □

Corollary 1.10.5. *Let $\mathfrak{T}$ be a nonempty closed subset of $\mathrm{AO}(N)$ such that $A_K^{t,\Phi}$ is $\mathfrak{T}$-admissible. Then for each left ideal L of A there exists a $\mathfrak{T}$-universal Gröbner basis of L. In particular, every left ideal of an admissible or degree-compatible algebra has a universal Gröbner basis.* □

Remark 1.10.6. To effectively compute a $\mathfrak{T}$-universal Gröbner basis, one should start walking among the orderings in $\mathfrak{T}$ and pick some ones that allow to cover $\mathfrak{T}$ as in 1.10.3. But how to pluck the right flowers in that vast meadow? The following Lemma 1.10.7 *might* be of help. Once one thinks to have located a suitable kind of orderings, that is, an appropriate subset $\mathfrak{D}$ of $\mathfrak{T}$, if one is able to show that $\mathfrak{D}$ is dense in $\mathfrak{T}$, then one can indeed restrict the own search to $\mathfrak{D}$. This fact might be the first step toward the construction of a "topological algorithm" that computes a $\mathfrak{T}$-universal Gröbner basis.

Lemma 1.10.7. *In the hypotheses of 1.10.4, let $\mathfrak{D}$ be a dense subset of $\mathfrak{T}$. Then we can find finitely many $\preceq_1, \ldots, \preceq_s$ in $\mathfrak{D}$ and respective Gröbner bases $G_1, \ldots, G_s$ of L such that $\bigcup_{1 \le j \le s} G_j$ is a $\mathfrak{T}$-universal Gröbner basis of L.*

Proof. As $\mathfrak{T}$ is compact, we find finitely many $\preceq'_1, \ldots, \preceq'_s \in \mathfrak{T}$ such that $\mathfrak{T} = \bigcup_{1 \le j \le s} \mathfrak{U}_L(G_j)$, where each G_j is a Gröbner basis of L with respect to $\preceq'_j$. Then $\bigcup_{1 \le j \le s} G_j$ is a $\mathfrak{T}$-universal Gröbner basis of L.

Because $\mathfrak{D}$ is dense in $\mathfrak{T}$ and each $\mathfrak{U}_L(G_j)$ is an open neighbourhood of $\preceq'_j$ in $\mathfrak{T}$, for $1 \leq j \leq s$ we find $\preceq_j \in \mathfrak{D} \cap \mathfrak{U}_L(G_j)$. Thus each G_j is a Gröbner basis of L with respect to $\preceq_j$. □

Example 1.10.8. The orderings $\preceq$ given by

$$\Phi^{-1}(X^{\upsilon}) \preceq \Phi^{-1}(X^{\nu}) \Leftrightarrow X^{\Gamma\upsilon} \leq_{\text{lex}} X^{\Gamma\nu}$$

with Γ a $t \times t$-matrix with entries in $\mathbb{N}_0$ constitute a dense subset of $\text{AO}(N)$. This follows easily from [2, p. 6].

Definition 1.10.9. Let (X, d) be a metric space and let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. We say that $Y \subseteq X$ is *ε-dense in X* if for all $x \in X$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$.

Lemma 1.10.10. *In the hypotheses of 1.10.4, assume that there exists $r \in \mathbb{N}_0$ such that for all $\preceq \in \mathfrak{T}$ and all Gröbner bases $G_{\preceq}$ of L with respect to $\preceq$ and all $g \in G_{\preceq}$ it holds $\deg(\Phi(g)) \leq r$. Let $\mathbb{S} = (S_i)_{i \in \mathbb{N}_0}$ be the filtration of N given by $S_i = \Phi^{-1}(M_{\leq i-1})$. Let $\mathfrak{D}$ be a $\frac{1}{r}$-dense subset of $\mathfrak{T}$ with respect to the metric $d_{\mathbb{S}}\restriction_{\mathfrak{T}}$ induced by $\mathbb{S}$. Then we can find finitely many $\preceq_1, \ldots, \preceq_s$ in $\mathfrak{D}$ and respective Gröbner bases $G_1, \ldots, G_s$ of L such that $\bigcup_{1 \leq j \leq s} G_j$ is a $\mathfrak{T}$-universal Gröbner basis of L.*

Proof. We find $s \in \mathbb{N}$ and $\preceq'_1, \ldots, \preceq'_s \in \mathfrak{T}$ and $G_1, \ldots, G_s \subseteq L$ such that each G_j is a $\preceq'_j$-Gröbner basis of L and $U = \bigcup_{1 \leq j \leq s} G_j$ is a $\mathfrak{T}$-universal Gröbner basis of L.

It holds $\text{Supp}(U) \subseteq S_{r+1}$. Because $\mathfrak{D}$ is $\frac{1}{r}$-dense in $\mathfrak{T}$, for $1 \leq j \leq s$ there exists $\preceq_j \in \mathfrak{D} \cap \mathfrak{N}_r(\preceq'_j)$. Since $\preceq'_j$ and $\preceq_j$ agree on $\text{Supp}(U)$ and hence on $\text{Supp}(G_j)$, by 1.10.1 G_j is a Gröbner basis of L with respect to $\preceq_j$. □

Remark 1.10.11. Assume that $A_K^{t,\Phi}$ is a quadric algebra of solvable type, this means, $\Phi^{-1}(X_i)\Phi^{-1}(X_j) = \Phi^{-1}(X_j)\Phi^{-1}(X_i) + \Phi^{-1}(p_{ij})$ for polynomials $p_{ij} \in K[X]$ at most of degree 2. Assume further that L can be generated by finitely many elements $x_1, \ldots, x_q$ such that $\deg(\Phi(x_h)) \leq d$ for $1 \leq h \leq q$. As proved in [2], for each $\preceq \in \text{AO}(N)$ there exists a Gröbner basis $G_{\preceq}$ of L with respect to $\preceq$ such that $\deg(\Phi(g)) \leq 2(\frac{d^2+2d}{2})^{2^{t-1}}$ for all $g \in G_{\preceq}$. Therefore there exists a $\mathfrak{T}$-universal Gröbner basis U of L such that $\deg(\Phi(u)) \leq 2(\frac{d^2+2d}{2})^{2^{t-1}}$ for all $u \in U$, for one can construct U as a union of (finitely many) such Gröbner bases $G_{\preceq}$.

Remark 1.10.12. An alternative, "classical" proof of 1.10.5 involves a division and a reduction algorithm:

(i) Assume that $A_K^{t,\Phi}$ is multiplicative on $\{\preceq\}$ for some $\preceq \in \mathrm{WO}(N)$. Let $a \in A$, let $F \subseteq L$ be finite, and put $\leq = \phi(\preceq)$. Then there are $r \in A$ and $(q_f)_{f\in F} \in A^{\oplus F}$ such that:

  (a) $a = \sum_{f\in F} q_f + r$,

  (b) $\forall\, f \in F : (f \neq 0 \Rightarrow \forall\, s \in \mathrm{Supp}(r) : \mathrm{LM}_{\preceq}(f) \nmid \Phi(s))$,

  (c) $a \neq 0 \Rightarrow (\forall\, f \in F : (q_f f \neq 0 \Rightarrow \mathrm{LM}_{\preceq}(q_f f) \leq \mathrm{LM}_{\preceq}(a)))$.

(ii) Let $\preceq \in \mathrm{AO}(N)$ such that $A_K^{t,\Phi}$ is multiplicative on $\{\preceq\}$. Let L be a left ideal of A. Let G be a Gröbner basis of L with respect to $\preceq$. One can then transform G by applying repeatedly the following procedures:

  (a) If there exists $g \in G \setminus \{0\}$ such that $\mathrm{LM}_{\preceq}(g) \in \mathrm{LM}_{\preceq}(G \setminus \{g\})$, then replace G by $G \setminus \{g\}$.

  (b) If there exist $g \in G \setminus \{0\}$ and $n \in \mathrm{Supp}(g) \setminus \{\mathrm{LM}_{\preceq}(g)\}$ with $n \in \mathrm{LM}_{\preceq}(G \setminus \{g\})$, then divide g by $G \setminus \{g\}$ as in (i), so that $g = \sum_{f\in G\setminus\{g\}} q_f f + r$, and replace G by $(\{r\} \cup G) \setminus \{g\}$, which is equal to $\{r\} \cup (G \setminus \{g\})$ in this case.

After finitely many steps both conditions become false, and the process terminates with a *reduced* Gröbner basis G of L with respect to $\preceq$, that is, for each $g \in G$ and each $n \in \mathrm{Supp}(g)$ it holds $n \notin \mathrm{LM}_{\preceq}(G \setminus \{g\})$.

(iii) Let $\mathfrak{T}$ be a closed subset of $\mathrm{AO}(N)$ such that $A_K^{t,\Phi}$ is $\mathfrak{T}$-admissible. Let L be a left ideal of A. Then there exist at most finitely many leading monomial ideals of L from $\mathfrak{T}$, therefore we find a finite subset $\mathfrak{U}$ of $\mathfrak{T}$ such that $\ell m_{\mathfrak{U}}(L) = \ell m_{\mathfrak{T}}(L)$. For each $\preceq \in \mathfrak{U}$ we may choose a reduced Gröbner basis $G_{\preceq}$ of L with respect to $\preceq$. Then $\bigcup_{\preceq\in\mathfrak{U}} G_{\preceq}$ is a $\mathfrak{T}$-universal Gröbner basis of L.

Part 2

Characteristic varieties over Weyl algebras

Introduction

Let $n \in \mathbb{N}$, let W be the n^{th} Weyl algebra over a field K of characteristic 0, and let $\Omega = \{\omega \in \mathbb{N}_0^{2n} \mid \omega_i + \omega_{i+n} > 0 \text{ for } 1 \leq i \leq n\}$. For each $\omega \in \Omega$ consider the ω-degree filtration $\mathrm{F}^\omega W = (\mathrm{F}_i^\omega W)_{i\in\mathbb{Z}}$ of W and any good $\mathrm{F}^\omega W$-filtration $\mathrm{F}^\omega M = (\mathrm{F}_i^\omega M)_{i\in\mathbb{Z}}$ of a left W-module M. We construct the associated graded ring $\mathrm{G}^\omega W = \bigoplus_{i\in\mathbb{Z}} \mathrm{F}_i^\omega W/\mathrm{F}_{i-1}^\omega W$ and the associated graded module $\mathrm{G}^\omega M = \bigoplus_{i\in\mathbb{Z}} \mathrm{F}_i^\omega M/\mathrm{F}_{i-1}^\omega M$. Then, indeed, $\mathrm{G}^\omega W$ is a ring canonically isomorphic to the commutative polynomial ring $K[X, Y]$ in the indeterminates $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, and $\mathrm{G}^\omega M$ is a finitely generated $K[X, Y]$-module. For a fixed $\omega \in \Omega$, the radical ideal $\sqrt{(0 : \mathrm{G}^\omega M)}$ of $K[X, Y]$ is independent of the choice of a good $\mathrm{F}^\omega W$-filtration $\mathrm{F}^\omega M$ of M. So we may define the ω-characteristic variety of M as the closed subset $\mathcal{V}^\omega(M) = \mathrm{Var}(0 : \mathrm{G}^\omega M)$ of $\mathrm{Spec}(K[X, Y])$.

We mention here a couple of good reasons to study such characteristic varieties. First, the Gelfand–Kirillov dimension of M equals the Krull dimension of $\mathcal{V}^\omega(M)$ for all $\omega \in \Omega$. We give a proof of this fact in the present work.

Second, the support of M as module over the commutative polynomial subring $K[X]$ of W is the precisely the projection to $\mathrm{Spec}(K[X])$ of the characteristic variety of M by order, see [24, 2.8], or see [12, 11.28] for more details.

Similarly as above, we consider the ν-degree filtrations $\mathrm{F}^\nu K[X, Y]$ of $K[X, Y]$, $\nu \in \mathbb{N}_0^{2n}$, good $\mathrm{F}^\nu K[X, Y]$-filtrations $\mathrm{F}^\nu N$ of $K[X, Y]$-modules N, the rings $\mathrm{G}^\nu K[X, Y]$, canonically isomorphic to $K[X, Y]$, and the finitely generated $K[X, Y]$-modules $\mathrm{G}^\nu N$. Again, for a fixed $\nu \in \mathbb{N}_0^{2n}$, the radical ideal $\sqrt{(0 : \mathrm{G}^\nu N)}$ of $K[X,Y]$ does not depend on the choice of a good $\mathrm{F}^\nu K[X,Y]$-filtration $\mathrm{F}^\nu N$ of N.

The main result of this part is that for each $\nu \in \mathbb{N}_0^{2n}$ there exists $s_0 \in \mathbb{N}_0$ such that for all $s \in \mathbb{N}$ with $s > s_0$ and all $\omega \in \Omega$ in $K[X,Y]$ it holds

$$\sqrt{(0 : \mathrm{G}^\nu \mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} M)}. \tag{1}$$

Observe that s_0 does not depend on ω. We can choose the lowest such s_0 in $\mathbb{N}_0$, denoted $\kappa_\nu(M)$. If L is a left ideal of W, we give an upper bound for $\kappa_\nu(W/L)$ in terms of total degrees of elements of universal Gröbner bases of L, more precisely,

$$\kappa_\nu(W/L) \leq \gamma_\nu(L), \tag{2}$$

where

$$\gamma_\nu(L) = \inf_U \sup_{u\in U} \deg^\nu(u),$$

the infimum being taken over all universal Gröbner bases U of L.

A case with evident geometrical meaning is when $\nu = (1) = (1,\ldots,1) \in \mathbb{N}_0^{2n}$. The equality (1) says that the "affine deformations" $\mathcal{V}^{(1)+s\omega}(M)$ of $\mathcal{V}^\omega(M)$ stabilize for large s to the critical cone $\mathcal{C}^\omega(M) = \mathrm{Var}(0 : \mathrm{G}^{(1)}\mathrm{G}^\omega M)$ of $\mathcal{V}^\omega(M)$. Thus the minimal limit beyond which this occurs, namely, $\kappa(M) = \kappa_{(1)}(M)$, is —surprisingly— an invariant of M. Upper bounds for the greatest total degree of Gröbner bases and of reduced Gröbner bases of a left ideal L of W are given in [2] in terms of greatest total degrees of systems of generators of L, and hence, combining both results, we obtain an upper bound for $\kappa(W/L)$ also in such terms.

The critical cone C of an affine variety $V \subseteq \mathbb{A}^r$ over an algebraically closed field F is the cone with vertex at the origin $O \in \mathbb{A}^r$ tangent to V at infinity. In other words, C consists of all lines through O along whose directions V goes to infinity. To construct C, we choose an injection $\iota : \mathbb{A}^r \rightarrowtail \mathbb{P}^r$ of $\mathbb{A}^r$ into the projective space $\mathbb{P}^r$ over F and put

$$C = \iota^{-1}\left(\bigcup\nolimits_{P\in\overline{\iota(V)}\smallsetminus\iota(V)} \ell_P\right),$$

where $\overline{\iota(V)}$ is the projective closure of $\iota(V)$ in $\mathbb{P}^r$ and ℓ_P is the projective line through the points $\iota(O)$ and P. One has that C does not depend on the choice of ι. Algebraically, if I is any ideal of $F[Z_1,\ldots,Z_r]$ that defines V, then C is defined by the ideal J generated by the homogeneous components of greatest total degree of the polynomials in I, that is, J is the leading form ideal of I by total degree. Again, C does not depend on the choice of I.

As a further consequence of the equality (1), we are able to give an easy proof that $\operatorname{Kdim}_{K[X,Y]} \mathrm{G}^{\omega} M = \operatorname{GKdim}_W M$ for all $\omega \in \Omega$. Thus, without having to appeal to sophisticated homological methods as in classical proofs, we have shown in particular that the characteristic varieties $\mathcal{V}^{\omega}(M)$, $\omega \in \Omega$, all have the same Krull dimension. The key point is that (1) allows in some sense to pass from nonfinite to finite filtrations, and Gelfand–Kirillov dimension behaves well with finite discrete filtrations: $\operatorname{GKdim}_{\mathrm{G}^{\omega} W} \mathrm{G}^{\omega} M = \operatorname{GKdim}_W M$ whenever $\mathrm{F}^{\omega} M$ is finite and discrete. The second point is that Gelfand–Kirillov dimension and Krull dimension agree in the category of finitely generated modules over any fixed finitely generated algebra over a field.

Fixed a left ideal L of W, we give an upper bound for the number $\chi(L)$ of distinct ideals $\mathrm{G}^{\omega} L$, $\omega \in \Omega$, and hence of distinct ω-characteristic varieties of W/L, namely,

$$\chi(L) \leq \inf_U \prod_{u \in U} \sum_{0 \leq k \leq \# \operatorname{supp}(u)} \binom{\# \operatorname{supp}(u)}{k}, \tag{3}$$

the infimum being taken over all universal Gröbner bases of L. The equality (1) let us conjecture a second upper bound in the case when W is the 1^{st} Weyl algebra, namely,

$$\chi(L) \leq 2^{1+\gamma(L)} + 1, \tag{4}$$

where $\gamma(L) = \gamma_{(1)}(L)$. As mentioned afore, by [2] it follows an upper bound for $\gamma(L)$ in terms of total degrees of generators of L.

In Section 2.1 we recall some known facts about filtered rings and modules as well as their associated graded rings and modules. To keep our treatment self-contained, and because these results are scattered over a vast literature, see for instance [28] and [33], we give a proof of the few statements we need. Some simple results, namely, 2.1.29, 2.1.31, and 2.1.32, about the behaviour of filtered modules and their associated graded modules with respect to taking radicals, we did not find in the literature.

In Section 2.2 we introduce Weyl algebras and state some of their basic properties, which are a generalization of results that can be found for instance in [19].

Section 2.3 is about Gröbner bases in Weyl algebras. Here, too, we recall known facts, important in the next section, in particular the existence of universal Gröbner bases for left ideals, and a very tight relation between the Gröbner bases of ω-filtered left ideals and the Gröbner bases of their associated graded ideals.

In Section 2.4 we define ω-characteristic varieties of a left W-module M as some particular affine spectra, and not as algebraic zero sets, as it is usual, for there is no reason here to work only over algebraically closed fields. Then we prove our main result (1) about the defining annihilators of such varieties.

In Section 2.5 we apply (1) to provide an easier proof of the known result that the ω-characteristic varieties of M all have the same Krull dimension as ω varies in Ω. Indeed, their dimension is equal to the Gelfand–Kirillov dimension of M.

Finally in Section 2.6 we perform a computer experiment in order to try to classify the ω-characteristic varieties of a cyclic left W-module W/L, where L is a left ideal L of W. This experiment let us conjecture an upper bound for their number, namely (4).

2.1 Filtrations and gradings

In this section we give a short review on filtered rings and modules and the graded rings and modules associated to them. Most of this material can be found or inferred from the books of Constantin Năstăsescu, Freddy van Oystaeyen, and Huishi Li, among which we particularly appreciate [28]. We provide in particular a proof of 2.1.29, 2.1.31, and 2.1.32, which we did not find in the literature.

Definition 2.1.1. A *filtration* $\mathcal{R}$ of a ring R is a family $(\mathrm{F}_i\mathcal{R})_{i\in\mathbb{Z}}$ of additive subgroups $\mathrm{F}_i\mathcal{R}$ of R with: (a) $R=\bigcup_{i\in\mathbb{Z}}\mathrm{F}_i\mathcal{R}$, (b) $\mathrm{F}_{i-1}\mathcal{R}\subseteq\mathrm{F}_i\mathcal{R}$, (c) $r\in\mathrm{F}_i\mathcal{R}\wedge s\in\mathrm{F}_j\mathcal{R}\Rightarrow rs\in\mathrm{F}_{i+j}\mathcal{R}$, (d) $i<0\Rightarrow\mathrm{F}_i\mathcal{R}=0$, (e) $1\in\mathrm{F}_0\mathcal{R}$, so that $\mathrm{F}_0\mathcal{R}$ is a subring of R and each $\mathrm{F}_i\mathcal{R}$ is a left $\mathrm{F}_0\mathcal{R}$-submodule of R.

If the ring R is provided with a filtration $\mathcal{R}$, we say that the ordered pair $(R,\mathcal{R})$ is a *filtered ring*.

Let $(R,\mathcal{R})$ and $(S,\mathcal{S})$ be filtered rings. A *homomorphism of* $(R,\mathcal{R})$ *in* $(S,\mathcal{S})$ is a ring homomorphism ϕ of R in S such that $\phi(\mathrm{F}_i\mathcal{R})\subseteq\mathrm{F}_i\mathcal{S}$.

Definition 2.1.2. Let $(R,\mathcal{R})$ be a filtered ring. An $\mathcal{R}$*-filtration* $\mathcal{M}$ of a left R-module M is a family $(\mathrm{F}_i\mathcal{M})_{i\in\mathbb{Z}}$ of additive subgroups $\mathrm{F}_i\mathcal{M}$ of M such that: (a) $M=\bigcup_{i\in\mathbb{Z}}\mathrm{F}_i\mathcal{M}$, (b) $\mathrm{F}_{i-1}\mathcal{M}\subseteq\mathrm{F}_i\mathcal{M}$, (c) $r\in\mathrm{F}_i\mathcal{R}\wedge m\in\mathrm{F}_j\mathcal{M}\Rightarrow rm\in\mathrm{F}_{i+j}\mathcal{M}$, so that each $\mathrm{F}_i\mathcal{M}$ is a left $\mathrm{F}_0\mathcal{R}$-submodule of M.

If the left R-module M is provided with an $\mathcal{R}$-filtration $\mathcal{M}$, we say that the ordered pair $(M,\mathcal{M})$ is an $\mathcal{R}$*-filtered left* R*-module* or simply a *left* $(R,\mathcal{R})$*-module*. Observe that a filtered ring is also a filtered left module over itself.

Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be left $(R, \mathcal{R})$-modules. An *$(R, \mathcal{R})$-homomorphism of $(M, \mathcal{M})$ in $(N, \mathcal{N})$* is a left R-module homomorphism ϕ of M in N such that $\phi(\mathrm{F}_i\mathcal{M}) \subseteq \mathrm{F}_i\mathcal{N}$.

Definition 2.1.3. Let $(R, \mathcal{R})$ be a filtered ring and $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. For any left R-submodule N of M we can canonically construct the *induced $\mathcal{R}$-filtrations* $\mathcal{N} = (\mathrm{F}_i\mathcal{M} \cap N)_{i\in\mathbb{Z}}$ of N and $\mathcal{M}/\mathcal{N} = (\mathrm{F}_i\mathcal{M} + N/N)_{i\in\mathbb{Z}}$ of M/N. In this situation we call $(N, \mathcal{N})$ a *submodule of* $(M, \mathcal{M})$ and $(M/N, \mathcal{M}/\mathcal{N})$ a *quotient module of* $(M, \mathcal{M})$. Similarly, if I is a left ideal of R and $\mathcal{I}$ is the induced $\mathcal{R}$-filtration of I, we say that $(I, \mathcal{I})$ is a *left ideal of* $(R, \mathcal{R})$.

Definition 2.1.4. Let $(R, \mathcal{R})$ be a filtered ring. The *associated graded ring* $\mathrm{G}\mathcal{R}$ *of R with respect to* $\mathcal{R}$ is the commutative group $\bigoplus_{i\in\mathbb{Z}} \mathrm{F}_i\mathcal{R}/\mathrm{F}_{i-1}\mathcal{R}$ provided with a multiplication given by $(r_i + \mathrm{F}_{i-1}\mathcal{R})_{i\in\mathbb{Z}}\ (s_j + \mathrm{F}_{j-1}\mathcal{R})_{j\in\mathbb{Z}} = (\sum_{i+j=k} r_i s_j + \mathrm{F}_{k-1}\mathcal{R})_{k\in\mathbb{Z}}$, which indeed turns $\mathrm{G}\mathcal{R}$ into a ring.

Let $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. The *associated graded left* $\mathrm{G}\mathcal{R}$*-module* $\mathrm{G}\mathcal{M}$ *of M with respect to* $\mathcal{M}$ is the commutative group $\bigoplus_{i\in\mathbb{Z}} \mathrm{F}_i\mathcal{M}/\mathrm{F}_{i-1}\mathcal{M}$ with a $\mathrm{G}\mathcal{R}$-action defined by $(r_i + \mathrm{F}_{i-1}\mathcal{R})_{i\in\mathbb{Z}}\ (m_j + \mathrm{F}_{j-1}\mathcal{M})_{j\in\mathbb{Z}} = (\sum_{i+j=k} r_i m_j + \mathrm{F}_{k-1}\mathcal{M})_{k\in\mathbb{Z}}$, which indeed turns $\mathrm{G}\mathcal{M}$ into a left $\mathrm{G}\mathcal{R}$-module.

$\mathrm{G}\mathcal{R}$ is precisely the associated graded left $\mathrm{G}\mathcal{R}$-module of R with respect to $\mathcal{R}$. We denote the i^{th} homogeneous component $\mathrm{F}_i\mathcal{M}/\mathrm{F}_{i-1}\mathcal{M}$ of $\mathrm{G}\mathcal{M}$ by $\mathrm{G}_i\mathcal{M}$. Then $\mathrm{G}_0\mathcal{R}$ is a subring of $\mathrm{G}\mathcal{R}$ and each $\mathrm{G}_i\mathcal{M}$ is a left $\mathrm{G}_0\mathcal{R}$-submodule of $\mathrm{G}\mathcal{M}$.

Remark 2.1.5. Let $(R, \mathcal{R})$ be a filtered ring, $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be left $(R, \mathcal{R})$-modules, and ϕ be a homomorphism of $(X, \mathcal{X})$ in $(Y, \mathcal{Y})$. We have canonical $\mathrm{F}_0\mathcal{R}$-module homomorphisms $\mathrm{F}_i\mathcal{X}/\mathrm{F}_{i-1}\mathcal{X} \to \mathrm{F}_i\mathcal{Y}/\mathrm{F}_{i-1}\mathcal{Y}$ whose direct sum is a graded left $\mathrm{G}\mathcal{R}$-module homomorphism $\mathrm{G}\mathcal{X} \to \mathrm{G}\mathcal{Y}$.

If $(N, \mathcal{N}) \rightarrowtail (M, \mathcal{M}) \twoheadrightarrow (P, \mathcal{P})$ is a *strict exact sequence of $(R, \mathcal{R})$-modules*, that is, $N \overset{\nu}{\rightarrowtail} M \overset{\pi}{\twoheadrightarrow} P$ is an exact sequence of R-modules with $\nu(\mathrm{F}_i\mathcal{N}) = \mathrm{F}_i\mathcal{M} \cap \mathrm{Im}(\nu)$ and $\pi(\mathrm{F}_i\mathcal{M}) = \mathrm{F}_i\mathcal{P} \cap \mathrm{Im}(\pi)$, then there is an exact sequence $\mathrm{G}\mathcal{N} \rightarrowtail \mathrm{G}\mathcal{M} \twoheadrightarrow \mathrm{G}\mathcal{P}$ of graded left $\mathrm{G}\mathcal{R}$-modules.

In particular, if $(N, \mathcal{N})$ is a submodule of $(M, \mathcal{M})$ and $(M/N, \mathcal{M}/\mathcal{N})$ is a quotient module of $(M, \mathcal{M})$, then we obtain an exact sequence $\mathrm{G}\mathcal{N} \rightarrowtail \mathrm{G}\mathcal{M} \twoheadrightarrow \mathrm{G}\mathcal{M}/\mathcal{N}$, so that $\mathrm{G}\mathcal{M}/\mathcal{N} \cong \mathrm{G}\mathcal{M}/\mathrm{G}\mathcal{N}$ as graded left $\mathrm{G}\mathcal{R}$-modules.

Remark 2.1.6. Let $(R, \mathcal{R})$ be a filtered ring, $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module, and $(N, \mathcal{N})$ be a submodule of $(M, \mathcal{M})$. By 2.1.5 we may write $\mathrm{G}\mathcal{N} \subseteq \mathrm{G}\mathcal{M}$.

Assume that $N \subsetneq M$. Then the set $I = \{i \in \mathbb{Z} \mid \mathrm{F}_i\mathcal{M} \nsubseteq N\}$ is nonempty. Assume further that the $\mathcal{R}$-filtration $\mathcal{M}$ is *discrete*, that is, $\mathrm{F}_i\mathcal{M} = 0$ for $i \ll 0$. Then I admits a unique least element i_0. If it held $\mathrm{G}\mathcal{N} = \mathrm{G}\mathcal{M}$, then $\mathrm{G}\mathcal{M}/\mathcal{N} \cong \mathrm{G}\mathcal{M}/\mathrm{G}\mathcal{N} = 0$, thus $(\mathrm{F}_i\mathcal{M}+N)/(\mathrm{F}_{i-1}\mathcal{M}+N) \cong \mathrm{G}_i\mathcal{M}/\mathcal{N} = 0$ for all $i \in \mathbb{Z}$, so $\mathrm{F}_i\mathcal{M} \subseteq \mathrm{F}_i\mathcal{M} + N = \mathrm{F}_{i-1}\mathcal{M} + N$ for all $i \in \mathbb{Z}$, in particular $\mathrm{F}_{i_0}\mathcal{M} \subseteq \mathrm{F}_{i_0-1}\mathcal{M} + N \subseteq N + N = N$ as $i_0 - 1 \notin I$, therefore $i_0 \notin I$, a contradiction.

So, assuming that $\mathcal{M}$ is discrete, we have the implication $N \subsetneq M \Rightarrow \mathrm{G}\mathcal{N} \subsetneq \mathrm{G}\mathcal{M}$, the property of *strict monotony* of G for discrete filtrations.

Without discreteness, strict monotony of G is no longer valid in general. As a simple example, assume that $M \neq 0$ and that M is endowed with its trivial filtration $\mathcal{M}$ given by $\mathrm{F}_i\mathcal{M} = M$ for all $i \in \mathbb{Z}$. Then it always holds $\mathrm{G}\mathcal{N} = \mathrm{G}\mathcal{M} = 0$.

Remark 2.1.7. Let $(R, \mathcal{R})$ be a filtered ring. Assume that $\mathcal{R}$ is *commutative*, that is, $r \in \mathrm{F}_i\mathcal{R} \wedge s \in \mathrm{F}_jR \Rightarrow rs - sr \in \mathrm{F}_{i+j-1}\mathcal{R}$. Then the ring $\mathrm{G}\mathcal{R}$ is commutative. In this situation let $(I, \mathcal{I})$ be a left ideal of $(R, \mathcal{R})$ and consider the quotient module $(R/I, \mathcal{R}/\mathcal{I})$ of $(R, \mathcal{R})$. Then $\mathrm{G}\mathcal{I} = (0 : \mathrm{G}\mathcal{R}/\mathcal{I})$ as ideals of $\mathrm{G}\mathcal{R}$ by 2.1.5.

Definition 2.1.8. Let $(R, \mathcal{R})$ be a filtered ring and let $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. We define the function $\deg^{\mathcal{M}} : M \to \mathbb{Z} \cup \{-\infty\}$ by $\deg^{\mathcal{M}}(m) = \inf\{i \in \mathbb{Z} \mid m \in \mathrm{F}_i\mathcal{M}\}$ for all $m \in M$ and call $\deg^{\mathcal{M}}$ the *$\mathcal{M}$-degree function of M*. In particular, $\deg^{\mathcal{M}}(0) = -\infty$. If $(N, \mathcal{N})$ is a left submodule of $(M, \mathcal{M})$, then $\deg^{\mathcal{N}}(n) = \deg^{\mathcal{M}}(n)$ for all $n \in N$. Further it holds $\deg^{\mathcal{M}}(m+n) \leq \max\{\deg^{\mathcal{M}}(m), \deg^{\mathcal{M}}(n)\}$ and $\deg^{\mathcal{M}}(rm) \leq \deg^{\mathcal{R}}(r) + \deg^{\mathcal{M}}(m)$ for all $r \in R$ and all $m, n \in M$.

We convene that $\mathrm{F}_{-\infty}\mathcal{M} = 0$ and $\mathrm{G}_{-\infty}\mathcal{M} = 0$. For each $i \in \mathbb{Z} \cup \{-\infty\}$ let us consider the left $\mathrm{F}_0\mathcal{R}$-module epimorphism $\sigma_i^{\mathcal{M}} : \mathrm{F}_i\mathcal{M} \to \mathrm{G}_i\mathcal{M}$ given by $m \mapsto m + \mathrm{F}_{i-1}\mathcal{M}$. Now we define the *$\mathcal{M}$-symbol map $\sigma^{\mathcal{M}} : M \to \mathrm{G}\mathcal{M}$ of M* by $m \mapsto \sigma_d^{\mathcal{M}}(m)$ where $d = \deg^{\mathcal{M}}(m)$. We call $\sigma^{\mathcal{M}}(m)$ the *$\mathcal{M}$-symbol of m*. If $(N, \mathcal{N})$ is a left submodule of $(M, \mathcal{M})$, then the image of $\sigma^{\mathcal{N}}(n)$ in $\mathrm{G}\mathcal{M}$ is precisely $\sigma^{\mathcal{M}}(n)$. Moreover, in general, $\sigma^{\mathcal{M}}$ is not additive, and $\sigma^{\mathcal{M}}$ is multiplicative precisely when $\deg^{\mathcal{M}}(rm) = \deg^{\mathcal{R}}(r) + \deg^{\mathcal{M}}(m)$ for all $r \in R$ and all $m \in M$.

Remark 2.1.9. Let $(R, \mathcal{R})$ be a filtered ring, $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module, and $(N, \mathcal{N})$ be a submodule of $(M, \mathcal{M})$. The image $\sigma^{\mathcal{N}}(N)$ consists precisely of all homogeneous elements of the graded left $\mathrm{G}\mathcal{R}$-module $\mathrm{G}\mathcal{N}$, whereas $\sigma^{\mathcal{M}}(N)$ consists of the homogeneous elements of the graded left $\mathrm{G}\mathcal{R}$-submodule $\mathrm{G}\mathcal{N}$ of $\mathrm{G}\mathcal{M}$.

In particular $\mathrm{G}\mathcal{N}$ is generated by $\sigma^{\mathcal{N}}(N)$ as a left $\mathrm{G}\mathcal{R}$-module, and $\mathrm{G}\mathcal{N}$ is generated by $\sigma^{\mathcal{M}}(N)$ as a left $\mathrm{G}\mathcal{R}$-submodule of $\mathrm{G}\mathcal{M}$, and for any subset U of N we have that $\sigma^{\mathcal{N}}(U)$ generates $\mathrm{G}\mathcal{N}$ as a left $\mathrm{G}\mathcal{R}$-module if and only if $\sigma^{\mathcal{M}}(U)$ generates $\mathrm{G}\mathcal{N}$ as a left $\mathrm{G}\mathcal{R}$-submodule of $\mathrm{G}\mathcal{M}$.

Proposition 2.1.10. *Let $(R, \mathcal{R})$ be a commutatively filtered ring. Let I be a left ideal of R and $\mathcal{I}$ and $\mathcal{R}/\mathcal{I}$ be the induced $\mathcal{R}$-filtrations of I and R/I, respectively. Then it holds $(0 : \mathrm{G}\mathcal{R}/\mathcal{I}) = \mathrm{G}\mathcal{I} = \sum_{x \in I} \mathrm{G}\mathcal{R}\, \sigma^{\mathcal{R}}(x)$ as ideals of $\mathrm{G}\mathcal{R}$.*

Proof. Clear by 2.1.7 and 2.1.9. □

Remark 2.1.11. Let $(R, \mathcal{R})$ be a filtered ring and $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. If U is a system of generators of M other than M, then $\mathrm{G}\mathcal{M}$ is not generated by $\sigma^{\mathcal{M}}(U)$, in general.

For instance consider the commutative polynomial ring $R = \mathbb{C}[X]$ provided with the filtration $\mathcal{R}$ given by $\mathrm{F}_i\mathcal{R} = \{r \in R \mid \deg(r) \leq i\}$. Put $(M, \mathcal{M}) = (R, \mathcal{R})$. Obviously $\{X, X+1\}$ is a system of generators of M. Further $\mathrm{G}\mathcal{R} \cong \mathbb{C}[X]$ as rings and $\mathrm{G}\mathcal{M} \cong \mathbb{C}[X]$ as $\mathbb{C}[X]$-modules. By these isomorphisms we can write $\sigma^{\mathcal{M}}(X+1) = X = \sigma^{\mathcal{M}}(X)$. Hence $\mathrm{G}\mathcal{R}\, \sigma^{\mathcal{M}}(\{X, X+1\}) = \mathbb{C}[X]\, X \subsetneq \mathbb{C}[X]$.

Remark 2.1.12. The converse of 2.1.11 is partially true. If $\mathcal{M}$ is discrete and $U \subseteq M$ is such that $\sigma^{\mathcal{M}}(U)$ generates $\mathrm{G}\mathcal{M}$ over $\mathrm{G}\mathcal{R}$, then U generates M over R.

Indeed, let N be the left R-submodule of M generated by U. Because $M = \bigcup_{i \in \mathbb{Z}} \mathrm{F}_i\mathcal{M}$, it is sufficient to prove that $\mathrm{F}_i\mathcal{M} \subseteq N$ for all $i \in \mathbb{Z}$. Since $\mathcal{M}$ is discrete, there exists $i_0 \in \mathbb{Z}$ such that $\mathrm{F}_i\mathcal{M} \subseteq N$ for all $i \leq i_0$. Let $i > i_0$. We inductively assume that $\mathrm{F}_{i-1}\mathcal{M} \subseteq N$ and show that $\mathrm{F}_i\mathcal{M} \subseteq N$. We can assume that $0 \notin U$ and $\mathrm{F}_{i-1}\mathcal{M} \subsetneq \mathrm{F}_i\mathcal{M}$. We choose then $m \in \mathrm{F}_i\mathcal{M} \setminus \mathrm{F}_{i-1}\mathcal{M}$. Then $\sigma^{\mathcal{M}}(m) = \sum_{u \in U} \overline{r}_u \sigma^{\mathcal{M}}(u)$ for some elements $\overline{r}_u \in \mathrm{G}\mathcal{R}$. As the element $\sigma^{\mathcal{M}}(m)$ is homogeneous of degree $i \in \mathbb{Z}$ in $\mathrm{G}\mathcal{M}$, we can assume that every $\overline{r}_u$ is either zero or homogeneous of degree $i - d_u$, where $d_u = \deg^{\mathcal{M}}(u)$ and without restriction $d_u \in \mathbb{Z}$ as $u \neq 0$ and $\mathcal{M}$ is discrete. Thus for each $u \in U$ we find $r_u \in \mathrm{F}_{i-d_u}\mathcal{R}$ with $\overline{r}_u =$

$r_u + \mathrm{F}_{i-d_u-1}\mathcal{R}$. It follows $m + \mathrm{F}_{i-1}\mathcal{M} = \sigma^{\mathcal{M}}(m) = \sum_{u\in U}(r_u + \mathrm{F}_{i-d_u-1}\mathcal{R})(u + \mathrm{F}_{d_u-1}\mathcal{M}) = \sum_{u\in U} r_u u + \mathrm{F}_{i-1}\mathcal{M}$, so $m - \sum_{u\in U} r_u u \in \mathrm{F}_{i-1}\mathcal{M}$. Using the inductive hypothesis, we have $m - \sum_{u\in U} r_u u \in N$, and by the definition of N it follows $m \in N$, and we are done.

Remark 2.1.13. Let $(R, \mathcal{R})$ be a filtered ring. We can provide the graded ring $\mathrm{G}\mathcal{R}$ with its *filtration* $\mathcal{G}\mathcal{R}$ *induced by the grading* given by $\mathrm{F}_i\mathcal{G}\mathcal{R} = \bigoplus_{j\le i} \mathrm{G}_j\mathcal{R}$. Then we construct the graded ring $\mathrm{G}\mathcal{G}\mathcal{R}$ associated to the filtered ring $(\mathrm{G}\mathcal{R}, \mathcal{G}\mathcal{R})$. Since for each i one has a left module isomorphism $\mathrm{F}_i\mathcal{R} \cong \mathrm{F}_i\mathcal{G}\mathcal{R}$ over the isomorphic rings $\mathrm{F}_0\mathcal{R} \cong \mathrm{F}_0\mathcal{G}\mathcal{R}$, there exists a graded ring isomorphism $\mathrm{G}\mathcal{R} \cong \mathrm{G}\mathcal{G}\mathcal{R}$.

Analogously, if $(M, \mathcal{M})$ is a left $(R, \mathcal{R})$-module, we have an isomorphism $\mathrm{G}\mathcal{M} \cong \mathrm{G}\mathcal{G}\mathcal{M}$ of graded left modules over the isomorphic graded rings $\mathrm{G}\mathcal{R} \cong \mathrm{G}\mathcal{G}\mathcal{R}$, where $\mathcal{G}\mathcal{M}$ is the filtration of $\mathrm{G}\mathcal{M}$ given by $\mathrm{F}_i\mathcal{G}\mathcal{M} = \bigoplus_{j\le i} \mathrm{G}_j\mathcal{M}$.

We have already encountered the notion of discrete and of commutative filtrations. Now we introduce some other classes of filtrations, among which the class of good filtrations is particularly remarkable.

Definition 2.1.14. Let $(R, \mathcal{R})$ be a filtered ring and M be a left R-module. An $\mathcal{R}$-filtration $\mathcal{M}$ of M is *good* if there exist $s \in \mathbb{N}_0$ and $m_1, \ldots, m_s \in M$ and $p_1, \ldots, p_s \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ it holds $\mathrm{F}_i\mathcal{M} = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\, m_j$. Since $1 \in \mathrm{F}_0\mathcal{R}$, we have then $m_j \in \mathrm{F}_{p_j}\mathcal{M}$.

Remark 2.1.15. In the notation of 2.1.14, any good $\mathcal{R}$-filtration $\mathcal{M}$ of M is discrete as $\mathcal{R}$ is discrete by definition.

Example 2.1.16. Let $(R, \mathcal{R})$ be a filtered ring and M be a finitely generated left R-module. For each finite system of generators $m_1, \ldots, m_s \in M$ of M and each $p_1, \ldots, p_s \in \mathbb{Z}$ there exists a *standard* good $\mathcal{R}$-filtration $\mathcal{M}$ of M given by $\mathrm{F}_i\mathcal{M} = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\, m_j$.

Proposition 2.1.17. *Let $(R, \mathcal{R})$ be a filtered ring and $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. If the $\mathcal{R}$-filtration $\mathcal{M}$ is good, then the left $\mathrm{G}\mathcal{R}$-module $\mathrm{G}\mathcal{M}$ is finitely generated.*

Proof. There exist $s \in \mathbb{N}_0$ and $m_1, \ldots, m_s \in M \setminus 0$ and $p_1, \ldots, p_s \in \mathbb{Z}$ with $m_j \in \mathrm{F}_{p_j}\mathcal{M}$ such that for all $i \in \mathbb{Z}$ it holds $\mathrm{F}_i\mathcal{M} = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\, m_j$. Let $\overline{u} \in \mathrm{G}\mathcal{M}$ be a homogeneous element of degree $i \in \mathbb{Z}$. We can write $\overline{u} = u + \mathrm{F}_{i-1}\mathcal{M}$ for some $u \in \mathrm{F}_i\mathcal{M}$, and therefore $u = \sum_{j=1}^{s} r_j m_j$ for elements $r_j \in \mathrm{F}_{i-p_j}\mathcal{R}$. Hence $\overline{u} = \sum_{j=1}^{s} r_j m_j + \mathrm{F}_{i-1}\mathcal{M} = \sum_{j=1}^{s}(r_j + \mathrm{F}_{i-p_j-1}\mathcal{R})(m_j + \mathrm{F}_{p_j-1}\mathcal{M})$, and so we see that $\mathrm{G}\mathcal{M}$ is generated by the symbols $\sigma^{\mathcal{M}}(m_1), \ldots, \sigma^{\mathcal{M}}(m_s)$. See also [28, Lemma I.5.4(2)]. □

Definition 2.1.18. Let $(R, \mathcal{R})$ be a filtered ring, $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module, and $(m_k)_{k\in\mathbb{N}}$ be a sequence of elements m_k of M.

Then $(m_k)_{k\in\mathbb{N}}$ is said to be an $\mathcal{M}$-*Cauchy sequence* if for each $j \in \mathbb{Z}$ there exists $n_j \in \mathbb{N}$ such that for all $k, l \geq n_j$ it holds $m_k - m_l \in \mathrm{F}_j\mathcal{M}$.

And $(m_k)_{k\in\mathbb{N}}$ is said to be $\mathcal{M}$-*convergent* to $m \in M$ if for each $j \in \mathbb{Z}$ there exists $n_j \in \mathbb{N}$ such that for all $k \geq n_j$ it holds $m_k - m \in \mathrm{F}_j\mathcal{M}$.

If every $\mathcal{M}$-Cauchy sequence of elements of M is $\mathcal{M}$-convergent, then $\mathcal{M}$ is said to be *complete*.

If $\bigcap_{j\in\mathbb{Z}} \mathrm{F}_j\mathcal{M} = \{0\}$, then $\mathcal{M}$ is called *separated* or *Hausdorff*.

Remark 2.1.19. Discrete filtrations are complete and, trivially, separated. So are, in particular, our ring filtrations and any good module filtrations.

Proposition 2.1.20. *Let $(R, \mathcal{R})$ be a filtered ring and $(M, \mathcal{M})$ be a left $(R, \mathcal{R})$-module. If the $\mathcal{R}$-filtration $\mathcal{M}$ is separated and the left $\mathrm{G}\mathcal{R}$-module $\mathrm{G}\mathcal{M}$ is finitely generated, then $\mathcal{M}$ is good.*

Proof. If $\mathrm{G}\mathcal{M} = 0$, then $\mathrm{F}_{i-1}\mathcal{M} = \mathrm{F}_i\mathcal{M}$ for all $i \in \mathbb{Z}$, hence $\mathrm{F}_i\mathcal{M} = 0$ for all $i \in \mathbb{Z}$ because $\mathcal{M}$ is discrete, thus $M = 0$, and $\mathcal{M}$ is good.

So we can assume that $\mathrm{G}\mathcal{M} \neq 0$. There exist $s \in \mathbb{N}$ and $m_1, \ldots, m_s \in M \setminus 0$ with $\mathrm{G}\mathcal{M} = \sum_{j=1}^{s} \mathrm{G}\mathcal{R}\, \sigma^{\mathcal{M}}(m_j)$. Let $p_j = \deg^{\mathcal{M}}(m_j)$ for $1 \leq j \leq s$. Then each $p_j \in \mathbb{Z}$ because $\mathcal{M}$ is discrete. Let $p = \min\{p_j \mid 1 \leq j \leq s\}$. Since the ring $\mathrm{G}\mathcal{R}$ is positively graded, $\mathrm{G}_i\mathcal{M} = 0$ for all $i < p$, that is, $\mathrm{F}_{i-1}\mathcal{M} = \mathrm{F}_i\mathcal{M}$ for all $i < p$, and hence $\mathrm{F}_i\mathcal{M} = 0$ for all $i < p$ because $\mathcal{M}$ is discrete.

We claim that $\mathrm{F}_i\mathcal{M} = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\ m_j$ for all $i \in \mathbb{Z}$. We only have to show that $\mathrm{F}_i\mathcal{M} \subseteq \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\ m_j$, since the converse inclusion is trivial. By what we have said above, this claim is obvious for all $i < p$. So, let $i \geq p$. Let $m \in \mathrm{F}_i\mathcal{M}$. If $m \in \mathrm{F}_{i-1}\mathcal{M}$, then $m \in \sum_{j=1}^{s} \mathrm{F}_{i-1-p_j}\mathcal{R}\ m_j$ by induction hypothesis, and hence $m \in \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\, m_j$ for $\mathrm{F}_{i-1-p_j}\mathcal{R} \subseteq \mathrm{F}_{i-p_j}\mathcal{R}$. So we can assume that $m \notin \mathrm{F}_{i-1}\mathcal{M}$. It follows $\deg^{\mathcal{M}}(m) = i$, that is, $\sigma^{\mathcal{M}}(m)$ is a homogeneous element of degree i in $\mathrm{G}\mathcal{M}$. We can write $\sigma^{\mathcal{M}}(m) = \sum_{j=1}^{s} \sigma^{\mathcal{R}}(x_j)\sigma^{\mathcal{M}}(m_j)$ for elements $x_j \in R$ with either $x_j = 0$ or $x_j \in \mathrm{F}_{i-p_j}\mathcal{R} \setminus \mathrm{F}_{i-1-p_j}\mathcal{R}$. It follows $m + \mathrm{F}_{i-1}\mathcal{M} = \sum_{j=1}^{s} x_j m_j + \mathrm{F}_{i-1}\mathcal{M}$. So, by induction, $m - \sum_{j=1}^{s} x_j m_j = \sum_{j=1}^{s} y_j m_j$ for elements $y_j \in \mathrm{F}_{i-1-p_j}\mathcal{R} \subseteq \mathrm{F}_{i-p_j}\mathcal{R}$. Thus $m = \sum_{j=1}^{s}(x_j + y_j)m_j \in \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\ m_j$, and we are done. ∎

Remark 2.1.21. Alternatively, to prove 2.1.20, as $\mathcal{R}$ is discrete and hence complete, we can appeal to [28, Theorem I.5.7].

Corollary 2.1.22. *Let $(R,\mathcal{R})$ be a filtered ring, $(M,\mathcal{M})$ be a left $(R,\mathcal{R})$-module, and $(N,\mathcal{N})$ be a submodule of $(M,\mathcal{M})$, so that by definition $\mathcal{N}$ is the $\mathcal{R}$-filtration of N induced by $\mathcal{M}$. If the ring $\mathrm{G}\mathcal{R}$ is left noetherian and the $\mathcal{R}$-filtration $\mathcal{M}$ is good, then also $\mathcal{N}$ is good.*

Proof. By 2.1.17, $\mathrm{G}\mathcal{M}$ is left noetherian, and so is $\mathrm{G}\mathcal{N}$. By 2.1.15, $\mathcal{M}$ is discrete, and so is $\mathcal{N}$. We conclude by 2.1.19 and 2.1.20. ☐

Remark 2.1.23. Let $(R,\mathcal{R})$ be a filtered ring and let $(M,\mathcal{M})$ be a left $(R,\mathcal{M})$-module. Given any left R-submodule N of M, if the $\mathcal{R}$-filtration $\mathcal{M}$ is good, then the induced $\mathcal{R}$-filtration $\mathcal{M}/\mathcal{N}$ of M/N is good. Indeed, in the notation of 2.1.14, one immediately sees that $\mathrm{F}_i\mathcal{M}/\mathcal{N} = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\;(m_j + N)$.

Remark 2.1.24. Summarizing, if $(R,\mathcal{R})$ is a filtered ring such that $\mathrm{G}\mathcal{R}$ is left noetherian and if M is a finitely generated left R-module, then there exists a good $\mathcal{R}$-filtration $\mathcal{M}$ of M (2.1.16) such that induced submodule and quotient module filtrations are good (2.1.22, 2.1.23) and $\mathrm{G}\mathcal{M}$ is left noetherian (2.1.17).

Let us introduce the important notion of equivalent filtrations. We shall see that certain ideals having a geometric meaning do not depend on the choice of equivalent filtrations and hence on the choice of good filtrations.

Definition 2.1.25. Let $(R,\mathcal{R})$ be a filtered ring and M be a left R-module. Two $\mathcal{R}$-filtrations $\mathcal{M}'$ and $\mathcal{M}''$ of M are *equivalent* or *of bounded difference* if there exists $r \in \mathbb{N}$, or equivalently $r \in \mathbb{Z}$, such that $\mathrm{F}_{i-r}\mathcal{M}'' \subseteq \mathrm{F}_i\mathcal{M}' \subseteq \mathrm{F}_{i+r}\mathcal{M}''$ for all $i \in \mathbb{Z}$. This defines indeed an equivalence relation among the $\mathcal{R}$-filtrations of M.

Proposition 2.1.26. *Let $(R,\mathcal{R})$ be a filtered ring and $(M,\mathcal{M}')$ and $(M,\mathcal{M}'')$ be left $(R,\mathcal{R})$-modules. If the $\mathcal{R}$-filtrations $\mathcal{M}'$ and $\mathcal{M}''$ are good, then they are equivalent.*

Proof. This is shown in [28, Lemma I.5.3]. For the sake of completeness we provide a proof of this easy but important statement.

For all $i \in \mathbb{Z}$ we can write $\mathrm{F}_i\mathcal{M}' = \sum_{j=1}^{s} \mathrm{F}_{i-p_j}\mathcal{R}\; u_j$ for some $s \in \mathbb{N}$ and $p_j \in \mathbb{Z}$ and $u_j \in \mathrm{F}_{p_j}\mathcal{M}'$, and we can write $\mathrm{F}_i\mathcal{M}'' = \sum_{k=1}^{t} \mathrm{F}_{i-q_k}\mathcal{R}\; v_k$ for some $t \in \mathbb{N}$ and $q_k \in \mathbb{Z}$ and

$v_k \in \mathrm{F}_{q_k}\mathcal{M}''$. We have that each $u_j \in \mathrm{F}_{h_j}\mathcal{M}''$ for some $h_j \in \mathbb{Z}$, thus $u_j = \sum_{k=1}^{t} b_{jk}v_k$ with $b_{jk} \in \mathrm{F}_{h_j-q_k}\mathcal{R}$. Notice that all u_j, p_j, h_j, v_k, q_k are independent of i.

Now let $i \in \mathbb{Z}$ and $m \in \mathrm{F}_i\mathcal{M}'$. We can write $m = \sum_{j=1}^{s} a_j u_j$ where $a_j \in \mathrm{F}_{i-p_j}\mathcal{R}$. Therefore $m = \sum_{k=1}^{t} c_k v_k$ where $c_k = \sum_{j=1}^{s} a_j b_{jk}$. Setting $h = \max\{h_j \mid 1 \le j \le s\}$ and $p = \min\{p_j \mid 1 \le j \le s\}$ and $r'' = h - p$, we immediately see that $c_k \in \mathrm{F}_{i+r''-q_k}\mathcal{R}$. Thus $m \in \sum_{k=1}^{t} \mathrm{F}_{i+r''-q_k}\mathcal{R}\ v_k = \mathrm{F}_{i+r''}\mathcal{M}''$, and so $\mathrm{F}_i\mathcal{M}' \subseteq \mathrm{F}_{i+r''}\mathcal{M}''$. By construction r'' is independent of i.

Analogously, we find $r' \in \mathbb{Z}$, independent of i, such that $\mathrm{F}_i\mathcal{M}'' \subseteq \mathrm{F}_{i+r'}\mathcal{M}'$. Putting $r = \max\{r', r''\}$, we finally get $\mathrm{F}_{i-r}\mathcal{M}' \subseteq \mathrm{F}_i\mathcal{M}'' \subseteq \mathrm{F}_{i+r}\mathcal{M}'$ for all $i \in \mathbb{Z}$. □

Theorem 2.1.27. *Let $(R, \mathcal{R})$ be a filtered ring such that the ring filtration $\mathcal{R}$ is commutative. Let $(M, \mathcal{M}')$ and $(M, \mathcal{M}'')$ be left $(R, \mathcal{R})$-modules such that the $\mathcal{R}$-filtrations $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent. Then $\sqrt{(0 : \mathrm{G}\mathcal{M}')} = \sqrt{(0 : \mathrm{G}\mathcal{M}'')}$.*

Proof. In [28, Lemma III.4.1.9] the claim is stated for good filtrations, but the authors actually prove it for the more general case of equivalent filtrations. For completeness we report here a proof from [16, Satz 8.2].

It is enough to show that $(0 : \mathrm{G}\mathcal{M}') \subseteq \sqrt{(0 : \mathrm{G}\mathcal{M}'')}$. Let $\overline{a} \in (0 : \mathrm{G}\mathcal{M}')$ be any homogeneous element of degree $i \in \mathbb{Z}$. Thus $\overline{a} = a + \mathrm{F}_{i-1}\mathcal{R}$ for some $a \in \mathrm{F}_i\mathcal{R} \smallsetminus \mathrm{F}_{i-1}\mathcal{R}$. Let $m'' \in \mathrm{G}\mathcal{M}''$ be any homogeneous element of degree $j \in \mathbb{Z}$. So $m'' = m + \mathrm{F}_{j-1}\mathcal{M}''$ for some $m \in \mathrm{F}_j\mathcal{M}'' \smallsetminus \mathrm{F}_{j-1}\mathcal{M}''$. As $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent, we find $r \in \mathbb{N}$ such that for all $h \in \mathbb{Z}$ it holds $\mathrm{F}_{h-r}\mathcal{M}' \subseteq \mathrm{F}_h\mathcal{M}'' \subseteq \mathrm{F}_{h+r}\mathcal{M}'$. Hence $m \in \mathrm{F}_{j+r}\mathcal{M}'$. Because $a + \mathrm{F}_{i-1}\mathcal{R}$ annihilates $m + \mathrm{F}_{j+r-1}\mathcal{M}'$ in $\mathrm{G}\mathcal{M}'$, we have that $am \in \mathrm{F}_{i+j+r-1}\mathcal{M}'$. Correspondingly, since $a + \mathrm{F}_{i-1}\mathcal{R}$ annihilates $am + \mathrm{F}_{i+j+r-2}\mathcal{M}'$ in $\mathrm{G}\mathcal{M}'$, it follows $a^2 m \in \mathrm{F}_{2i+j+r-2}\mathcal{M}'$, and so forth. Thus $a^{2r+1}m \in \mathrm{F}_{(2r+1)i+j+r-(2r+1)}\mathcal{M}' = \mathrm{F}_{(2r+1)i+j-r-1}\mathcal{M}' \subseteq \mathrm{F}_{(2r+1)i+j-1}\mathcal{M}''$. On the other hand, $\overline{a}^{2r+1}m'' = a^{2r+1}m + \mathrm{F}_{(2r+1)i+j-1}\mathcal{M}''$. It follows $\overline{a}^{2r+1}m'' = 0$ in $\mathrm{G}\mathcal{M}''$. Since r is independent of m'', we obtain $\overline{a}^{2r+1}\mathrm{G}\mathcal{M}'' = 0$, thus $\overline{a} \in \sqrt{(0 : \mathrm{G}\mathcal{M}'')}$. □

Corollary 2.1.28. *Let $(R, \mathcal{R})$ be a filtered ring such that the ring filtration $\mathcal{R}$ is commutative. Let $(M, \mathcal{M}')$ and $(M, \mathcal{M}'')$ be left $(R, \mathcal{R})$-modules such that the $\mathcal{R}$-filtrations $\mathcal{M}'$ and $\mathcal{M}''$ are good. Then $\sqrt{(0 : \mathrm{G}\mathcal{M}')} = \sqrt{(0 : \mathrm{G}\mathcal{M}'')}$.*

Proof. Clear by 2.1.26 and 2.1.27. □

Finally let us investigate how graded modules associated to filtrations behave with respect to building annihilators and to taking radicals.

Proposition 2.1.29. *Let* $(R, \mathcal{R})$ *be a filtered commutative ring,* M *be an* R*-module and* N *be an* R*-submodule of* M*. Providing the annihilators* $(0:M)$*,* $(0:N)$*,* $(0:M/N)$ *in* R *with the respective induced* $\mathcal{R}$*-filtrations, denoted* $(0:\mathcal{M})$*,* $(0:\mathcal{N})$*,* $(0:\mathcal{M}/\mathcal{N})$*, it holds* $\sqrt{\mathrm{G}(0:\mathcal{M})} = \sqrt{\mathrm{G}(0:\mathcal{N})} \cap \sqrt{\mathrm{G}(0:\mathcal{M}/\mathcal{N})}$ *in* $\mathrm{G}\mathcal{R}$.

Proof. Let $\overline{x} \in \mathrm{G}(0:\mathcal{N}) \cap \mathrm{G}(0:\mathcal{M}/\mathcal{N})$ be a homogeneous element of degree $i \in \mathbb{Z}$. We find $u \in \mathrm{F}_i(0:\mathcal{N}) = \mathrm{F}_i\mathcal{R} \cap (0:N)$ and $v \in \mathrm{F}_i(0:\mathcal{M}/\mathcal{N}) = \mathrm{F}_i\mathcal{R} \cap (0:M/N)$ such that $u + \mathrm{F}_{i-1}\mathcal{R} = \overline{x} = v + \mathrm{F}_{i-1}\mathcal{R}$. Because $v \in (0:M/N)$, it holds $vM \subseteq N$. Since $u \in (0:N)$, it follows $uvM = 0$. Hence $uv \in (0:M)$. Since $u \in \mathrm{F}_i\mathcal{R}$ and $v \in \mathrm{F}_i\mathcal{R}$, it follows $uv \in \mathrm{F}_{2i}\mathcal{R} \cap (0:M) = \mathrm{F}_{2i}(0:\mathcal{M})$. So $\overline{x}^2 = uv + \mathrm{F}_{2i-1}\mathcal{R} \in \mathrm{G}(0:\mathcal{M})$, thus $\overline{x} \in \sqrt{\mathrm{G}(0:\mathcal{M})}$. We have obtained that $\mathrm{G}(0:\mathcal{N}) \cap \mathrm{G}(0:\mathcal{M}/\mathcal{N}) \subseteq \sqrt{\mathrm{G}(0:\mathcal{M})}$, whereas, on the other hand, since $(0:M) \subseteq (0:N) \cap (0:M/N)$, it follows from 2.1.6 that $\mathrm{G}(0:\mathcal{M}) \subseteq \mathrm{G}(0:\mathcal{N}) \cap \mathrm{G}(0:\mathcal{M}/\mathcal{N})$. Now we pass to the radicals. □

Remark 2.1.30. Let $(R, \mathcal{R})$ be a filtered ring and $\phi : M \to N$ be a an isomorphism of left R-modules. If $\mathcal{M}$ is an $\mathcal{R}$-filtration of M, then there exists an $\mathcal{R}$-filtration $\mathcal{N}$ of N induced by ϕ given by $\mathrm{F}_i\mathcal{N} = \phi(\mathrm{F}_i\mathcal{M})$ such that there exists a graded $\mathrm{G}\mathcal{R}$-isomorphism $\mathrm{G}\phi : \mathrm{G}\mathcal{M} \to \mathrm{G}\mathcal{N}$ induced by ϕ, see 2.1.5. Moreover, if $\mathcal{M}$ is good, then $\mathcal{N}$ is good, as one checks easily.

Proposition 2.1.31. *Let* R *be a commutative ring and* $\mathcal{R}$ *be a filtration of* R *such that induced* $\mathcal{R}$*-filtrations on submodules and quotient modules of* R *are good. Let* M *be a finitely generated* R*-module and* $\mathcal{M}$ *be an* $\mathcal{R}$*-filtration such that induced* $\mathcal{R}$*-filtrations on submodules and quotient modules of* M *are good. Consider the annihilator* $(0:M)$ *of* M *in* R *provided with its induced* $\mathcal{R}$*-filtration, which we denote by* $(0:\mathcal{M})$*. Then* $\sqrt{\mathrm{G}(0:\mathcal{M})} = \sqrt{(0:\mathrm{G}\mathcal{M})}$ *as ideals of the commutative ring* $\mathrm{G}\mathcal{R}$.

Proof. We find $t \in \mathbb{N}$ such that M is generated by t elements. If $t = 1$, there exists an R-module isomorphism $\phi : M \to R/I$ for some ideal I of R. We furnish the R-module R/I with the induced $\mathcal{R}$-filtration $\mathcal{R}/\mathcal{I}$, good by hypothesis, and with the ϕ-induced $\mathcal{R}$-filtration, denoted $\phi(\mathcal{M})$, which is good by 2.1.30 since $\mathcal{M}$ is good by hypothesis. By 2.1.30, $(0:\mathrm{G}\mathcal{M}) = (0:\mathrm{G}\phi(\mathcal{M}))$. By 2.1.26 and 2.1.27, $\sqrt{(0:\mathrm{G}\phi(\mathcal{M}))} = \sqrt{(0:\mathrm{G}\mathcal{R}/\mathcal{I})}$. As $(0:M) = (0:R/I) = I$, $(0:\mathcal{M})$ is precisely the induced $\mathcal{R}$-filtration of I, hence by 2.1.7 we have $(0:\mathrm{G}\mathcal{R}/\mathcal{I}) = \mathrm{G}(0:\mathcal{M})$. Thus $\sqrt{(0:\mathrm{G}\mathcal{M})} = \sqrt{\mathrm{G}(0:\mathcal{M})}$.

Now let $t > 1$. Assume inductively that the statement holds for all R-modules generated by less than t elements. We find a cyclic submodule N of M such that M/N is generated over R by $t-1$ elements. We provide N and M/N by the respective induced filtrations $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$, which are good, and provide the ideals $(0 : N)$ and $(0 : M/N)$ of R by the respective induced filtrations, denoted $(0 : \mathcal{N})$ and $(0 : \mathcal{M}/\mathcal{N})$, which are good by hypothesis. By the case with $t = 1$, we have $\sqrt{\mathrm{G}(0 : \mathcal{N})} = \sqrt{(0 : \mathrm{G}\mathcal{N})}$. By the induction hypothesis, we have $\sqrt{\mathrm{G}(0 : \mathcal{M}/\mathcal{N})} = \sqrt{(0 : \mathrm{G}\mathcal{M}/\mathcal{N})}$. The short exact sequence $N \rightarrowtail M \twoheadrightarrow M/N$ of filtered R-modules induces the short exact sequence $\mathrm{G}\mathcal{N} \rightarrowtail \mathrm{G}\mathcal{M} \twoheadrightarrow \mathrm{G}\mathcal{M}/\mathcal{N}$ of graded $\mathrm{G}\mathcal{R}$-modules, see 2.1.5. Thus $\sqrt{(0 : \mathrm{G}\mathcal{M})} = \sqrt{(0 : \mathrm{G}\mathcal{N})} \cap \sqrt{(0 : \mathrm{G}\mathcal{M}/\mathcal{N})}$, whence $\sqrt{(0 : \mathrm{G}\mathcal{M})} = \sqrt{\mathrm{G}(0 : \mathcal{N})} \cap \sqrt{\mathrm{G}(0 : \mathcal{M}/\mathcal{N})}$. So, by 2.1.29, $\sqrt{(0 : \mathrm{G}\mathcal{M})} = \sqrt{\mathrm{G}(0 : \mathcal{M})}$. $\square$

Remark 2.1.32. We finish this section with a remark that will be useful later on. Let R be a commutative ring and $\mathcal{R}$ be a filtration of R, so that $\mathcal{R}$ trivially is commutative. Let I be an ideal of R and provide I with its induced $\mathcal{R}$-filtration, denoted $\mathcal{I}$, and provide $\sqrt{I}$ with its induced $\mathcal{R}$-filtration, denoted $\sqrt{\mathcal{I}}$. Then $\sqrt{\mathrm{G}\sqrt{\mathcal{I}}} = \sqrt{\mathrm{G}\mathcal{I}}$. Indeed let $\overline{x} \in \mathrm{G}\sqrt{\mathcal{I}}$ be a homogeneous element of degree $i \in \mathbb{Z}$. So $\overline{x} = x + \mathrm{F}_{i-1}\mathcal{R}$ for some $x \in \mathrm{F}_i\sqrt{\mathcal{I}} = \mathrm{F}_i\mathcal{R} \cap \sqrt{I}$. We find $k \in \mathbb{N}$ such that $x^k \in I$, and so $x^k \in \mathrm{F}_{ki}\mathcal{R} \cap I = \mathrm{F}_{ki}\mathcal{I}$, thus $\overline{x}^k = x^k + \mathrm{F}_{ki-1}\mathcal{R} \in \mathrm{G}\mathcal{I}$, hence $\overline{x} \in \sqrt{\mathrm{G}\mathcal{I}}$. We have shown that $\mathrm{G}\sqrt{\mathcal{I}} \subseteq \sqrt{\mathrm{G}\mathcal{I}}$. On the other hand, by 2.1.6, we have $\mathrm{G}\mathcal{I} \subseteq \mathrm{G}\sqrt{\mathcal{I}}$. Passing to the radicals, the claim follows.

2.2 Weyl algebras

In this section let $n \in \mathbb{N}$ and K be a field of characteristic 0. We write $K[X, Y]$ for the commutative polynomial ring $K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ and denote its subring $K[X_1, \ldots, X_n]$ by $K[X]$.

For all $(r, s) \in \mathbb{N}_0 \times \mathbb{N}_0$ we write $(r \,|\, s)$ for the vector $\omega \in \mathbb{N}_0^{2n}$ such that $\omega_i = r$ and $\omega_{n+i} = s$ for all $i \in \{1, \ldots, n\}$. For all $\alpha, \beta \in \mathbb{N}_0^n$ we write $(\alpha \,|\, \beta)$ for the vector $\omega \in \mathbb{N}_0^{2n}$ with $\omega_i = \alpha_i$ and $\omega_{n+i} = \beta_i$ for $1 \le i \le n$. For all $t \in \mathbb{N}$ and all $\alpha, \beta \in \mathbb{N}_0^t$ we denote the sum $\sum_{i=1}^t \alpha_i\beta_i$ by $\alpha \cdot \beta$. For all $i \in \{1, \ldots, n\}$ we put $\varepsilon^i = (\delta_{ij})_{j=1}^n \in \mathbb{N}_0^n$, where $\delta_{ij} \in \mathbb{N}_0$ is the Kronecker symbol.

We introduce Weyl algebras over K and state some facts about them. In doing this, we generalize certain well known results that are proved for instance in [19].

Definition 2.2.1. The n^{th} *Weyl algebra* W over the field K is defined as the K-subalgebra

$K\langle \xi_1, \ldots, \xi_n, \partial_1, \ldots, \partial_n \rangle$ of $\mathrm{End}_K(K[X])$, where multiplication is given by composition, generated by the K-endomorphisms $\xi_1, \ldots, \xi_n$ and $\partial_1, \ldots, \partial_n$ of $K[X]$ given by $\xi_i(p) = X_i p$ and $\partial_i(p) = \frac{\partial p}{\partial X_i}$ for all $p \in K[X]$. These generators fulfill the relations: (a) $[\xi_i, \xi_j] = 0$, (b) $[\partial_i, \partial_j] = 0$, (c) $[\xi_i, \partial_j] + \delta_{ij} = 0$, where $[a, b] = ab - ba$ for all $a, b \in \mathrm{End}_K(K[X])$ and $\delta_{ij} \in K$ is the Kronecker symbol.

Remark 2.2.2. As a K-module, W has a canonical basis $\{\xi^\lambda \partial^\mu \mid (\lambda, \mu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n\}$ consisting of *normal monomials* $\xi^\lambda \partial^\mu$, see for instance [16, Satz 2.7] or [19, Proposition 1.2.1]. Hence for each $w \in W$ there exists a unique function $c_w : \mathbb{N}_0^n \times \mathbb{N}_0^n \to K$ of finite *support* $\mathrm{supp}(w) = \{(\lambda, \mu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid c_w(\lambda, \mu) \neq 0\}$ such that $w = \sum c_w(\lambda, \mu) \xi^\lambda \partial^\mu$, where the sum is taken over all $(\lambda, \mu) \in \mathrm{supp}(w)$. We write $c_{\lambda\mu}$ for $c_w(\lambda, \mu)$ and say that $\sum c_{\lambda\mu} \xi^\lambda \partial^\mu$ is the *canonical form* of w.

Definition 2.2.3. By 2.2.2 we may define $\deg^\omega(w) = \sup\{\omega \cdot (\lambda \,|\, \mu) \mid (\lambda, \mu) \in \mathrm{supp}(w)\}$ for all $\omega \in \mathbb{N}_0^{2n}$ and all $w \in W$, the *ω-degree of w* with values in $\mathbb{Z} \cup \{-\infty\}$.

Lemma 2.2.4. *Let $\omega \in \mathbb{N}_0^{2n}$. Let $u, v \in W$. Then* $\deg^\omega(u + v) \leq \max\{\deg^\omega(u), \deg^\omega(v)\}$. *Moreover, equality holds if* $\deg^\omega(u) \neq \deg^\omega(v)$.

Proof. Writing u and v in canonical form and adding up similar monomials we obtain the canonical form of $u + v$, and now the claim is clear. □

Lemma 2.2.5. *Let $\omega \in \mathbb{N}_0^{2n}$. Let $c \in K$ and $u \in W$. Then* $\deg^\omega(cu) \leq \deg^\omega(u)$, *and equality holds if $c \neq 0$.*

Proof. Writing u in canonical form, the statement is clear by the definition of ω-degree. □

Lemma 2.2.6. *Let $\omega \in \mathbb{N}_0^{2n}$. Let $u, v \in W$. Then* $\deg^\omega(uv) \leq \deg^\omega(u) + \deg^\omega(v)$.

Proof. First we prove our claim for monomials in canonical form, that is, we show that $\deg^\omega(\xi^\lambda \partial^\mu \xi^\rho \partial^\sigma) \leq \deg^\omega(\xi^\lambda \partial^\mu) + \deg^\omega(\xi^\rho \partial^\sigma)$ for all $\lambda, \mu, \rho, \sigma \in \mathbb{N}_0^n$.

We proceed by induction over $|\mu| = \sum_{i=1}^n \mu_i$. If $|\mu| = 0$, then $\mu = 0$, and hence $\xi^\lambda \partial^\mu \xi^\rho \partial^\sigma = \xi^{\lambda+\rho} \partial^\sigma$. Hence we have $\deg^\omega(\xi^\lambda \partial^\mu \xi^\rho \partial^\sigma) = (\lambda + \rho \,|\, \sigma) \cdot \omega = (\lambda \,|\, 0) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega = (\lambda \,|\, \mu) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega = \deg^\omega(\xi^\lambda \partial^\mu) + \deg^\omega(\xi^\rho \partial^\sigma)$.

If $|\mu| > 0$, then $\mu_i > 0$ for some $i \in \{1, \ldots, n\}$. So, $\xi^\lambda \partial^\mu \xi^\rho \partial^\sigma = \xi^\lambda \partial^{\mu - \varepsilon^i} \partial_i \xi^\rho \partial^\sigma$. In the case when $\rho_i = 0$, we have $\xi^\lambda \partial^\mu \xi^\rho \partial^\sigma = \xi^\lambda \partial^{\mu - \varepsilon^i} \xi^\rho \partial^{\sigma + \varepsilon^i}$. Since $|\mu - \varepsilon^i| < |\mu|$, by the induction hypothesis we get $\deg^\omega(\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma) \le (\lambda \,|\, \mu - \varepsilon^i) \cdot \omega + (\rho \,|\, \sigma + \varepsilon^i) \cdot \omega = (\lambda + \rho \,|\, \mu + \sigma) \cdot \omega = (\lambda \,|\, \mu) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega = \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma)$. Otherwise, if $\rho_i > 0$, we have $\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma = \xi^\lambda\partial^{\mu-\varepsilon^i}(\rho_i\xi^{\rho-\varepsilon_i} + \xi^\rho\partial_i)\partial^\sigma = \rho_i\xi^\lambda\partial^{\mu-\varepsilon^i}\xi^{\rho-\varepsilon_i}\partial^\sigma + \xi^\lambda\partial^{\mu-\varepsilon^i}\xi^\rho\partial^{\sigma+\varepsilon^i}$. In virtue of 2.2.4 and by the induction hypothesis we have $\deg^\omega(\rho_i\xi^\lambda\partial^{\mu-\varepsilon^i}\xi^{\rho-\varepsilon_i}\partial^\sigma) \le (\lambda \,|\, \mu - \varepsilon^i) \cdot \omega + (\rho - \varepsilon_i \,|\, \sigma) \cdot \omega = (\lambda \,|\, \mu) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega - \omega_{n+i} - \omega_i \le \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma)$. By the induction hypothesis we have also $\deg^\omega(\xi^\lambda\partial^{\mu-\varepsilon^i}\xi^\rho\partial^{\sigma+\varepsilon^i}) \le (\lambda \,|\, \mu - \varepsilon^i) \cdot \omega + (\rho \,|\, \sigma + \varepsilon^i) \cdot \omega = (\lambda + \rho \,|\, \mu + \sigma) \cdot \omega = (\lambda \,|\, \mu) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega = \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma)$. So, by 2.2.4, we obtain $\deg^\omega(\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma) \le \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma)$.

Now we write u and v in canonical form as $\sum a_{\lambda\mu}\xi^\lambda\partial^\mu$ and $\sum b_{\rho\sigma}\xi^\rho\partial^\sigma$, respectively. Hence it follows $uv = \sum a_{\lambda\mu}b_{\rho\sigma}\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma$, and the claim is clear by what we have just shown for the monomials $\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma$ and by 2.2.5 and 2.2.4. □

Definition 2.2.7. We define $\Omega = \{\omega \in \mathbb{N}_0^{2n} \mid \omega_i + \omega_{n+i} > 0 \text{ whenever } 1 \le i \le n\}$, the *natural polynomial region of* W. The reason for this name will be clear by Theorem 2.2.16.

Lemma 2.2.8. *It holds* $\deg^\omega([u,v]) \le \deg^\omega(u) + \deg^\omega(v) - \min_{1\le i\le n}\{\omega_i + \omega_{n+i}\}$ *for all* $\omega \in \mathbb{N}_0^{2n}$ *and all* $u, v \in W$. *In particular, if* $\omega \in \Omega$, *then* $\deg^\omega([u,v]) \le \deg^\omega(u) + \deg^\omega(v) - 1$.

Proof. We write u and v in canonical form as $\sum a_{\lambda\mu}\xi^\lambda\partial^\mu$ and $\sum b_{\rho\sigma}\xi^\rho\partial^\sigma$, respectively. Since the commutator $[-,-]$ is K-bilinear, it holds $[u,v] = \sum a_{\lambda\mu}b_{\rho\sigma}[\xi^\lambda\partial^\mu, \xi^\rho\partial^\sigma]$. By 2.2.5 and 2.2.4, it is sufficient to show that $\deg^\omega([\xi^\lambda\partial^\mu, \xi^\rho\partial^\sigma]) \le \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma) - m$ for all $\lambda, \mu, \rho, \sigma \in \mathbb{N}_0^n$, where $m = \min_{1\le i\le n}\{\omega_i + \omega_{n+i}\}$. We proceed by induction over $|\mu| = \sum \mu_i$.

If $|\mu| = 0$, then $\mu = 0$, so we have to prove $\deg^\omega([\xi^\lambda, \xi^\rho\partial^\sigma]) \le \deg^\omega(\xi^\lambda) + \deg^\omega(\xi^\rho\partial^\sigma) - m$. We do it by induction over $|\sigma| = \sum \sigma_i$. If $|\sigma| = 0$, then $\sigma = 0$, so $[\xi^\lambda, \xi^\rho\partial^\sigma] = [\xi^\lambda, \xi^\rho] = 0$, and the claim is clear. If $|\sigma| > 0$, then $\sigma_i > 0$ for some i, and we can write $[\xi^\lambda, \xi^\rho\partial^\sigma] = \xi^\rho[\xi^\lambda, \partial^\sigma] + [\xi^\lambda, \xi^\rho]\partial^\sigma = \xi^\rho[\xi^\lambda, \partial^\sigma] = \xi^\rho[\xi^\lambda, \partial_i\partial^{\sigma-\varepsilon^i}] = \xi^\rho\partial_i[\xi^\lambda, \partial^{\sigma-\varepsilon^i}] + \xi^\rho[\xi^\lambda, \partial_i]\partial^{\sigma-\varepsilon^i}$. As for the first term, by 2.2.6 and by the induction hypothesis, we obtain $\deg^\omega(\xi^\rho\partial_i[\xi^\lambda, \partial^{\sigma-\varepsilon^i}]) \le \deg^\omega(\xi^\rho\partial_i) + \deg^\omega(\xi^\lambda) + \deg^\omega(\partial^{\sigma-\varepsilon^i}) - m = \deg^\omega(\xi^\lambda) + (\rho \,|\, \varepsilon^i) \cdot \omega + (0 \,|\, \sigma - \varepsilon^i) \cdot \omega - m = \deg^\omega(\xi^\lambda) + (\rho \,|\, \sigma) \cdot \omega - m = \deg^\omega(\xi^\lambda) + \deg^\omega(\xi^\rho\partial^\sigma) - m$. As for the second term, if $\lambda_i = 0$, then $[\xi^\lambda, \partial_i] = 0$, thus we can assume that $\lambda_i > 0$. In this case we have $[\xi^\lambda, \partial_i] = -\lambda_i\xi^{\lambda-\varepsilon^i}$, so $\xi^\rho[\xi^\lambda, \partial_i]\partial^{\sigma-\varepsilon^i} = \lambda_i\xi^{\lambda+\rho-\varepsilon^i}\partial^{\sigma-\varepsilon^i}$, hence $\deg^\omega(\xi^\rho[\xi^\lambda, \partial_i]\partial^{\sigma-\varepsilon_i}) = (\lambda + \rho - \varepsilon^i \,|\, \sigma - \varepsilon^i) \cdot \omega = (\lambda \,|\, 0) \cdot \omega + (\rho \,|\, \sigma) \cdot \omega - \omega_i - \omega_{n+i} \le \deg^\omega(\xi^\lambda) + \deg^\omega(\xi^\rho\partial^\sigma) - m$. Thus, by 2.2.4, we obtain $\deg^\omega([\xi^\lambda, \xi^\rho\partial\sigma]) \le \deg^\omega(\xi^\lambda) + \deg^\omega(\xi^\rho\partial^\sigma) - m$.

Now let $|\mu| > 0$. One has $[\xi^\lambda\partial^\mu, \xi^\rho\partial^\sigma] = \xi^\rho[\xi^\lambda, \partial^\sigma]\partial^\mu - \xi^\lambda[\xi^\rho, \partial^\mu]\partial^\sigma$. By the previously treated case with $|\mu| = 0$, we know that $\deg^\omega([\xi^\lambda, \partial^\sigma]) \leq \deg^\omega(\xi^\lambda) + \deg^\omega(\partial^\sigma) - m$. Hence, similarly, we obtain $\deg^\omega([\xi^\rho, \partial^\mu]) \leq \deg^\omega(\xi^\rho) + \deg^\omega(\partial^\mu) - m$. In virtue of 2.2.6, we have $\deg^\omega(\xi^\rho[\xi^\lambda, \partial^\sigma]\partial^\mu) \leq \deg^\omega(\xi^\rho) + \deg^\omega(\xi^\lambda) + \deg^\omega(\partial^\sigma) - m + \deg^\omega(\partial^\mu)$, and also $\deg^\omega(\xi^\lambda[\xi^\rho, \partial^\mu]\partial^\sigma) \leq \deg^\omega(\xi^\lambda) + \deg^\omega(\xi^\rho) + \deg^\omega(\partial^\mu) - m + \deg^\omega(\partial^\sigma)$. By 2.2.4, it follows $\deg^\omega([\xi^\lambda\partial^\mu, \xi^\rho\partial^\sigma]) \leq (\lambda\,|\,0)\cdot\omega + (0\,|\,\mu)\cdot\omega + (\rho\,|\,0)\cdot\omega + (0\,|\,\sigma)\cdot\omega - m = (\lambda\,|\,\mu)\cdot\omega + (\rho\,|\,\sigma)\cdot\omega - m = \deg^\omega(\xi^\lambda\partial^\mu) + \deg^\omega(\xi^\rho\partial^\sigma) - m$. □

Lemma 2.2.9. *Let $\omega \in \Omega$. Let $u, v \in W$. Then* $\deg^\omega(uv) = \deg^\omega(u) + \deg^\omega(v)$.

Proof. We write u and v in canonical form as $\sum a_{\lambda\mu}\xi^\lambda\partial^\mu$ and $\sum b_{\rho\sigma}\xi^\rho\partial^\sigma$, respectively. Thus $uv = \sum a_{\lambda\mu}b_{\rho\sigma}\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma = \sum a_{\lambda\mu}b_{\rho\sigma}\xi^\lambda[\partial^\mu, \xi^\rho]\partial^\sigma + \sum a_{\lambda\mu}b_{\rho\sigma}\xi^{\lambda+\rho}\partial^{\mu+\sigma}$. By 2.2.8, 2.2.6, 2.2.5 and 2.2.4, we have $\deg^\omega(\sum a_{\lambda\mu}b_{\rho\sigma}\xi^\lambda[\partial^\mu, \xi^\rho]\partial^\sigma) \leq \deg^\omega(u) + \deg^\omega(v) - 1$.

Now put $d = \sup\{(\lambda + \rho\,|\,\mu + \sigma) \cdot \omega \mid (\lambda, \mu) \in \operatorname{supp}(u), (\rho, \sigma) \in \operatorname{supp}(v)\}$ and choose $(\lambda', \mu') \in \operatorname{supp}(u)$ and $(\rho', \sigma') \in \operatorname{supp}(v)$ such that $(\lambda' + \rho'\,|\,\mu' + \sigma') \cdot \omega = d$. Also put $e = \sup\{(\lambda\,|\,\mu) \cdot \omega \mid (\lambda, \mu) \in \operatorname{supp}(u)\}$ and choose $(\lambda'', \mu'') \in \operatorname{supp}(u)$ such that $(\lambda''\,|\,\mu'') \cdot \omega = e$. Finally put $f = \sup\{(\rho\,|\,\sigma) \cdot \omega \mid (\rho, \sigma) \in \operatorname{supp}(v)\}$ and choose $(\rho'', \sigma'') \in \operatorname{supp}(u)$ such that $(\rho''\,|\,\sigma'') \cdot \omega = f$. If $(\lambda''\,|\,\mu'') \cdot \omega > (\lambda'\,|\,\mu') \cdot \omega$, then $d = (\lambda' + \rho'\,|\,\mu' + \sigma') \cdot \omega = (\lambda'\,|\,\mu') \cdot \omega + (\rho'\,|\,\sigma') \cdot \omega < (\lambda''\,|\,\mu'') \cdot \omega + (\rho'\,|\,\sigma') \cdot \omega = (\lambda'' + \rho'\,|\,\mu'' + \sigma') \cdot \omega$, a contradiction. Therefore $(\lambda''\,|\,\mu'') \cdot \omega \leq (\lambda'\,|\,\mu') \cdot \omega$. Analogously, $(\rho''\,|\,\sigma'') \cdot \omega \leq (\rho'\,|\,\sigma') \cdot \omega$. It follows $d = (\lambda' + \rho'\,|\,\mu' + \sigma') \cdot \omega = (\lambda'\,|\,\mu') \cdot \omega + (\rho'\,|\,\sigma') \cdot \omega \geq (\lambda''\,|\,\mu'') \cdot \omega + (\rho''\,|\,\sigma'') \cdot \omega = e + f$.

Because K is an integral domain, we have $d = \deg^\omega(\sum a_{\lambda\mu}b_{\rho\sigma}\xi^{\lambda+\rho}\partial^{\mu+\sigma})$. Since clearly $e = \deg^\omega(u)$ and $f = \deg^\omega(v)$, by what we have shown above and by 2.2.4 we conclude that $\deg^\omega(uv) = d$, and therefore $\deg^\omega(uv) \geq \deg^\omega(u) + \deg^\omega(v)$. The claim follows now from 2.2.6. □

Proposition 2.2.10. *Summing up the previous lemmas, for all $\omega \in \Omega$ and all $u, v \in W$ one has:* (a) $\deg^\omega(u+v) \leq \max\{\deg^\omega(u), \deg^\omega(u)\}$, (b) $\deg^\omega([u, v]) \leq \deg^\omega(u) + \deg^\omega(v) - 1$, *and* (c) $\deg^\omega(uv) = \deg^\omega(u) + \deg^\omega(v)$. *Equality holds in* (a) *if* $\deg^\omega(u) \neq \deg^\omega(v)$. □

Corollary 2.2.11. *Weyl algebras are domains.*

Proof. Clear by 2.2.10(c). □

Definition 2.2.12. Let $\omega \in \mathbb{N}_0^{2n}$. Consider the family $\mathrm{F}^\omega W = (\mathrm{F}_i^\omega W)_{i\in\mathbb{Z}}$ given by $\mathrm{F}_i^\omega W = \{w \in W \mid \deg^\omega(w) \leq i\}$. Then $\mathrm{F}^\omega W$ is a filtration of W by 2.2.10. We denote by $\mathrm{G}^\omega W$

the associated graded ring of W with respect to $\mathrm{F}^\omega W$, and by $\mathrm{G}^\omega_i W$ the i^{th} homogeneous component $\mathrm{F}^\omega_i W/\mathrm{F}^\omega_{i-1} W$ of $\mathrm{G}^\omega W$.

Given any *ω-filtration* $\mathrm{F}^\omega W$-filtration $\mathrm{F}^\omega M = (\mathrm{F}^\omega_i M)_{i\in\mathbb{Z}}$ of a left W-module M, we denote by $\mathrm{G}^\omega M$ the associated graded left $\mathrm{G}^\omega W$-module associated to M with respect to $\mathrm{F}^\omega M$, and by $\mathrm{G}^\omega_i M$ the i^{th} homogeneous component of $\mathrm{G}^\omega M$.

We denote the symbol map $W \to \mathrm{G}^\omega W$ by σ^ω and the i^{th} symbol map $\mathrm{F}^\omega_i W \to \mathrm{G}^\omega_i W$ by σ^ω_i. Thus $\sigma^\omega(w) = \sigma^\omega_{\deg^\omega(w)}(w)$ for all $w \in W$.

Remark 2.2.13. Let $\omega \in \Omega$ and $v, w \in W$. As $\deg^\omega(uv) = \deg^\omega(u) + \deg^\omega(v)$ by 2.2.10(c), it holds $\sigma^\omega(uv) = \sigma^\omega(u)\sigma^\omega(v)$.

Remark 2.2.14. For all $\omega \in \Omega$ the filtration $\mathrm{F}^\omega W$ of W is commutative by 2.2.10(b), so that the ring $\mathrm{G}^\omega W$ is commutative.

Reminder 2.2.15. We recall the universal property of commutative polynomial rings, see for instance [15, Satz 2.6.5]. Let $\rho : R \to S$ be a homomorphism of commutative rings. Let $b_1, \ldots, b_u$ be finitely many elements of S. Then there exists a unique homomorphism of commutative rings $\chi : R[X_1, \ldots, X_u] \to S$ such that $\chi\restriction_R = \rho$ and $\chi(X_l) = b_l$ for $1 \leq l \leq u$.

Remarks 2.2.13 and 2.2.14, the canonical injection $K \rightarrowtail \mathrm{G}^\omega W$, and the universal property 2.2.15 imply the following theorem.

Theorem 2.2.16. *For each $\omega \in \Omega$ one has an isomorphism of commutative K-algebras $\psi^\omega : K[X, Y] \to \mathrm{G}^\omega W$, $\sum_{(\lambda,\mu)\in\mathbb{N}_0^n\times\mathbb{N}_0^n} c_{\lambda\mu} X^\lambda Y^\mu \mapsto \sum_{(\lambda,\mu)\in\mathbb{N}_0^n\times\mathbb{N}_0^n} c_{\lambda\mu}\sigma^\omega(\xi^\lambda)\sigma^\omega(\partial^\mu)$, which is graded if we put $\deg(X_i) = \omega_i$ and $\deg(Y_i) = \omega_{n+i}$ for $1 \leq i \leq n$.*

Proof. Let $\omega \in \Omega$. Since $\mathrm{G}^\omega_0 W$ contains a homomorphic image of K, there exists a homomorphism of rings $\kappa : K \to \mathrm{G}^\omega W$. Let us consider the distinct elements $\sigma^\omega(\xi_1), \ldots, \sigma^\omega(\xi_n)$ and $\sigma^\omega(\partial_1), \ldots, \sigma^\omega(\partial_n)$ of $\mathrm{G}^\omega W$. The ring $\mathrm{G}^\omega W$ is commutative as the filtration $\mathrm{F}^\omega W$ is commutative, see 2.2.14, thus we can apply 2.2.15 and so we get a homomorphism of rings $\chi : K[X, Y] \to \mathrm{G}^\omega W$ with $\chi\restriction_K = \kappa$ and $\chi(X_i) = \sigma^\omega(\xi_i)$ and $\chi(Y_i) = \sigma^\omega(\partial_i)$ for $1 \leq i \leq n$. As κ is injective, χ is indeed a homomorphism of K-algebras. As σ^ω is multiplicative, see 2.2.13, it is immediate to see that $\chi = \psi^\omega$.

The K-algebra $\mathrm{G}^\omega W$ is generated by $\sigma^\omega(\xi_1), \ldots, \sigma^\omega(\xi_n)$ and $\sigma^\omega(\partial_1), \ldots, \sigma^\omega(\partial_n)$. Indeed, if $\overline{u}$ is a homogeneous element of $\mathrm{G}^\omega W$ of degree $d \in \mathbb{Z}$, then $\overline{u} = u + \mathrm{F}^\omega_{d-1} W$ for some element $u \in \mathrm{F}^\omega_d W \setminus \mathrm{F}^\omega_{d-1} W$. We write u in canonical form as $\sum_{\lambda,\mu} c_{\lambda\mu}\xi^\lambda\partial^\mu$, and we can assume that $(\lambda\,|\,\mu)\cdot\omega = d$ for all $(\lambda,\mu) \in \mathrm{supp}(u)$. It follows $\overline{u} = \sum_{\lambda,\mu}\sigma^\omega(c_{\lambda\mu}\xi^\lambda\partial^\mu)$, and so $\overline{u} = \sum_{\lambda,\mu} c_{\lambda\mu}\prod_i \sigma^\omega(\xi_i)^{\lambda_i}\sigma^\omega(\partial_i)^{\mu_i}$, whence our claim. This shows that ψ^ω is surjective.

Let $u \in \mathrm{Ker}\,\psi^\omega$. We uniquely write u as $\sum_{\lambda,\mu} c_{\lambda\mu}\xi^\lambda\partial^\mu$, and so $\sum_{\lambda,\mu} c_{\lambda\mu}\sigma^\omega(\xi^\lambda)\sigma^\omega(\partial^\mu) = 0$, hence $\sum_k\sum_{(\lambda\,|\,\mu)\cdot\omega=k} c_{\lambda\mu}\sigma^\omega(\xi^\lambda\partial^\mu) = 0$. As the ring $\mathrm{G}^\omega W$ is graded, for all $k \in \mathbb{Z}$ we have $\sum_{(\lambda\,|\,\mu)\cdot\omega=k} c_{\lambda\mu}\sigma^\omega(\xi^\lambda\partial^\mu) = 0$. As $\{\xi^\lambda\partial^\mu \mid (\lambda\,|\,\mu)\cdot\omega \le k\}$ is a K-basis of $\mathrm{F}^\omega_k W$ for each $k \in \mathbb{Z}$, it follows that $\{\xi^\lambda\partial^\mu + \mathrm{F}^\omega_{k-1}W \mid (\lambda\,|\,\mu)\cdot\omega = k\}$ is a K-basis of $\mathrm{G}^\omega_k W$ for each $k \in \mathbb{Z}$. Since $\sigma^\omega(\xi^\lambda\partial^\mu) = \xi^\lambda\partial^\mu + \mathrm{F}^\omega_{k-1}W$ whenever $(\lambda\,|\,\mu)\cdot\omega = k$, so we get $c_{\lambda\mu} = 0$ for all $k \in \mathbb{Z}$ and all $(\lambda,\mu) \in \mathrm{supp}(u)$ with $(\lambda\,|\,\mu)\cdot\omega = k$. We conclude that $\mathrm{supp}(u) = \varnothing$, thus $u = 0$, and ψ^ω is injective.

Because it holds $\deg^\omega(\xi_i) = \omega_i$ and $\deg^\omega(\partial_i) = \omega_{n+i}$, the symbols $\sigma^\omega(\xi_i)$ and $\sigma^\omega(\partial_i)$ respectively have degree ω_i and ω_{n+i} in $\mathrm{G}^\omega W$, so that the homomorphism ψ^ω is graded if the ring $K[X,Y]$ is provided with the grading induced by setting $\deg(X_i) = \omega_i$ and $\deg(Y_i) = \omega_{n+i}$. □

Corollary 2.2.17. *Weyl algebras are left noetherian.*

Proof. Fix any $\omega \in \Omega$. The ring $\mathrm{G}^\omega W$ is commutative by 2.2.14. Let L be any left ideal of W. The ideal $\mathrm{G}^\omega L$ of $\mathrm{G}^\omega W$ is generated by $\sigma^\omega(L)$. Because $\mathrm{G}^\omega W$ is noetherian by 2.2.16, there exists a finite subset F of L such that $\mathrm{G}^\omega L$ is generated by $\sigma^\omega(F)$. Thus L is generated by F in virtue of 2.1.12. □

Remark 2.2.18. All what we have defined and said in this section about Weyl algebras can be done and proved in the same way for the commutative polynomial ring $K[X,Y]$, too. In this situation we may even drop the hypothesis that the field K be of characteristic 0 and may consider whole $\mathbb{N}_0^{2n}$ instead of Ω. We shall use a similar notation as introduced above for Weyl algebras, with one exception: given any $\nu \in \mathbb{N}_0^{2n}$, we shall write τ^ν_i for the i^{th} symbol map $\mathrm{F}^\nu_i K[X,Y] \to \mathrm{G}^\nu_i K[X,Y]$ and τ^ν for the symbol map $K[X,Y] \to \mathrm{G}^\nu K[X,Y]$, in order to distinguish them from the symbol maps of the n^{th} Weyl algebra.

2.3 Gröbner bases in Weyl algebras

We keep the notation of the previous section, and denote by M the canonical K-basis $\{X^\lambda Y^\mu \mid (\lambda,\mu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n\}$ of $K[X,Y]$ consisting of the monomials $X^\lambda Y^\mu$, and by N the canonical K-basis $\{\xi^\lambda\partial^\mu \mid (\lambda,\mu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n\}$ of W consisting of the normal monomials $\xi^\lambda\partial^\mu$.

For each $\omega \in \Omega$ we shall tacitly identify the ring $\mathrm{G}^\omega W$ with $K[X,Y]$ by means of the K-algebra isomorphism ψ^ω of 2.2.16 and hence for each left ideal L consider $\mathrm{G}^\omega L$ as an ideal of $K[X,Y]$. Similarly for each $\nu \in \mathbb{N}_0^{2n}$ we shall identify $\mathrm{G}^\nu K[X,Y]$ with $K[X,Y]$ and thus for each ideal I of $K[X,Y]$ consider $\mathrm{G}^\nu I$ as an ideal of $K[X,Y]$.

Remark 2.3.1. There exists a K-module isomorphism $\Phi : W \to K[X,Y]$ which maps the canonical basis N of W to the canonical basis M of $K[X,Y]$ by the rule $\xi^\lambda\partial^\mu \mapsto X^\lambda Y^\mu$. In particular, $W_K^{2n,\Phi}$ is an algebra of countable type.

Theorem 2.3.2. *Let L be a left ideal of W and $\preceq$ be a total ordering on N. Then L admits a Gröbner basis with respect to $\preceq$.*

Proof. Clear by 2.3.1, 2.2.17, and 1.9.4. □

Definition 2.3.3. Let ϕ denote the homeomorphism of $\mathrm{TO}(N)$ in $\mathrm{TO}(M)$ induced by Φ according to 1.4.2.

Lemma 2.3.4. *Let $\preceq\, \in \mathrm{AO}(N)$ and let $\leq\, = \phi(\preceq)$. Then for all $u, v \in W^+$ it holds:* (a) $\mathrm{LM}_\preceq(u+v) \leq \max_\leq\{\mathrm{LM}_\preceq(u), \mathrm{LM}_\preceq(v)\}$ *whenever $u + v \neq 0$ with equality holding if* $\mathrm{LM}_\preceq(u) \neq \mathrm{LM}_\preceq(v)$, (b) $\mathrm{LM}_\preceq(uv) = \mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v)$, (c) $\mathrm{LM}_\preceq([u,v]) < \mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v)$ *whenever* $[u,v] \neq 0$.

Proof. Statement (a) clearly follows from the inclusion $\mathrm{Supp}(u+v) \subseteq \mathrm{Supp}(u) \cup \mathrm{Supp}(v)$. It also follows from the analogous result in $K[X,Y]$ because Φ is K-linear.

Since $M = \{\mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v) \mid u, v \in W^+\}$, we may prove statements (b) and (c) by noetherian induction over $\mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v)$ in the well-ordered set $(M, \leq)$.

Let $u, v \in W^+$. If $\mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v) = 1$, then $\mathrm{LM}_\preceq(u) = 1 = \mathrm{LM}_\preceq(v)$, hence $u \in K^+$ and $v \in K^+$, so that (b) is clear and (c) is trivially true as $[u,v] = 0$.

Let $\mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v) > 1$ and assume that statements (b) and (c) hold for all $u', v' \in W^+$ such that $\mathrm{LM}_\preceq(u')\,\mathrm{LM}_\preceq(v') < \mathrm{LM}_\preceq(u)\,\mathrm{LM}_\preceq(v)$.

Choose any $(\lambda,\mu) \in \mathrm{supp}(u)$ and any $(\rho,\sigma) \in \mathrm{supp}(v)$. If there exists $i \in \{1,\ldots,n\}$ such that $\mu_i > 0$, we can write $[\xi^\lambda\partial^\mu, \xi^\rho\partial^\sigma] = \xi^\lambda\partial^{\mu-\varepsilon_i}[\partial_i, \xi^\rho\partial^\sigma] + [\xi^\lambda\partial^{\mu-\varepsilon_i}, \xi^\rho\partial^\sigma]\partial_i$ with $\varepsilon_i = (\delta_{ih})_{1\leq h\leq n}$ where $\delta_{ih} \in \mathbb{N}_0$ is the Kronecker delta. Since ∂_i and ∂^σ commute, it holds $[\partial_i, \xi^\rho\partial^\sigma] = [\partial_i, \xi^\rho]\partial^\sigma$. It follows that $[\partial_i, \xi^\rho\partial^\sigma] = 0$ if $\rho_i = 0$, whereas $[\partial_i, \xi^\rho\partial^\sigma] = \rho_i\xi^{\rho-\varepsilon_i}\partial^\sigma$ if $\rho_i > 0$. If $\rho_i > 0$, we get $\mathrm{LM}_\preceq(\xi^\lambda\partial^{\mu-\varepsilon_i}[\partial_i, \xi^\rho\partial^\sigma]) = X^{\lambda+\rho-\varepsilon_i}Y^{\mu+\sigma-\varepsilon_i}$ by the induction hypothesis. By the induction hypothesis, $\mathrm{LM}_\preceq([\xi^\lambda\partial^{\mu-\varepsilon_i}, \xi^\rho\partial^\sigma]) < X^{\lambda+\rho}Y^{\mu+\sigma-\varepsilon_i}$.

Thus $\mathrm{LM}_{\preceq}([\xi^\lambda\partial^{\mu-\varepsilon_i},\xi^\rho\partial^\sigma])\,\mathrm{LM}_{\preceq}(\partial_i) < X^{\lambda+\rho}Y^{\mu+\sigma}$ and hence we may appeal again to the induction hypothesis to get $\mathrm{LM}_{\preceq}([\xi^\lambda\partial^{\mu-\varepsilon_i},\xi^\rho\partial^\sigma]\partial_i) = \mathrm{LM}_{\preceq}([\xi^\lambda\partial^{\mu-\varepsilon_i},\xi^\rho\partial^\sigma])\,\mathrm{LM}_{\preceq}(\partial_i) < X^{\lambda+\rho}Y^{\mu+\sigma}$. We conclude by (a) that $\mathrm{LM}_{\preceq}([\xi^\lambda\partial^\mu,\xi^\rho\partial^\sigma]) < X^{\lambda+\rho}Y^{\mu+\sigma}$. Furthermore we have $\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma = \xi^{\lambda+\rho}\partial^{\mu+\sigma} + \xi^\lambda[\partial^\mu,\xi^\rho]\partial^\sigma$. As $X^\rho Y^\mu \leq X^{\lambda+\rho}Y^{\mu+\sigma}$, one shows as above that $\mathrm{LM}_{\preceq}([\partial^\mu,\xi^\rho]) < X^\rho Y^\mu$. Hence, using the induction hypothesis and the compatibility property twice, we get $\mathrm{LM}_{\preceq}(\xi^\lambda[\partial^\mu,\xi^\rho]\partial^\sigma) = \mathrm{LM}_{\preceq}(\xi^\lambda)\,\mathrm{LM}_{\preceq}([\partial^\mu,\xi^\rho])\,\mathrm{LM}_{\preceq}(\partial^\sigma) < X^{\lambda+\rho}Y^{\mu+\sigma}$. Because clearly $\mathrm{LM}_{\preceq}(\xi^{\lambda+\rho}\partial^{\mu+\sigma}) = X^{\lambda+\rho}Y^{\mu+\sigma}$, it follows $\mathrm{LM}_{\preceq}(\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma) = X^{\lambda+\rho}Y^{\mu+\sigma}$.

If $\mu = 0$ and there exists $j \in \{1,\ldots,n\}$ such that $\sigma_j > 0$, we reduce immediately to the previous case since $[\xi^\lambda\partial^\mu,\xi^\rho\partial^\sigma] = -[\xi^\rho\partial^\sigma,\xi^\lambda\partial^\mu]$, whereas if $\mu = 0$ and $\sigma = 0$, then $[\xi^\lambda\partial^\mu,\xi^\rho\partial^\sigma] = 0$ and clearly $\mathrm{LM}_{\preceq}(\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma) = X^{\lambda+\rho}Y^{\mu+\sigma}$.

We uniquely write u and v in their canonical form as $u = \sum_{(\lambda,\mu)\in\mathrm{supp}(u)} a_{(\lambda,\mu)}\xi^\lambda\partial^\mu$ and $v = \sum_{(\rho,\sigma)\in\mathrm{supp}(v)} b_{(\rho,\sigma)}\xi^\rho\partial^\sigma$ where $a_{(\lambda,\mu)} \in K^+$ for all $(\lambda,\mu) \in \mathrm{supp}(u)$ and $b_{(\rho,\sigma)} \in K^+$ for all $(\rho,\sigma) \in \mathrm{supp}(v)$. We find a unique $(\overline{\lambda},\overline{\mu}) \in \mathrm{supp}(u)$ such that $\mathrm{lm}_{\preceq}(u) = \xi^{\overline{\lambda}}\partial^{\overline{\mu}}$ and a unique $(\overline{\rho},\overline{\sigma}) \in \mathrm{supp}(v)$ such that $\mathrm{lm}_{\preceq}(v) = \xi^{\overline{\rho}}\partial^{\overline{\sigma}}$. Thus $\mathrm{LM}_{\preceq}(u)\,\mathrm{LM}_{\preceq}(v) = X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$.

If $(\lambda,\mu) \in \mathrm{supp}(u) \setminus \{(\overline{\lambda},\overline{\mu})\}$, say $\lambda \neq \overline{\lambda}$, then $X^\lambda < X^{\overline{\lambda}}$. Indeed, if $X^\lambda \geq X^{\overline{\lambda}}$, then $X^\lambda Y^{\overline{\mu}} \geq X^{\overline{\lambda}}Y^{\overline{\mu}}$ by compatibility, thus $X^\lambda Y^{\overline{\mu}} = X^{\overline{\lambda}}Y^{\overline{\mu}}$ as $X^{\overline{\lambda}}Y^{\overline{\mu}} = \mathrm{LM}_{\preceq}(u)$, hence $\lambda = \overline{\lambda}$, a contradiction. Similarly, $Y^\mu < Y^{\overline{\mu}}$ if $\mu \neq \overline{\mu}$. Clearly, an analogous result holds for all $(\rho,\sigma) \in \mathrm{supp}(v) \setminus \{(\overline{\rho},\overline{\sigma})\}$. By compatibility it follows $X^{\lambda+\rho}Y^{\mu+\sigma} < X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$ for all $((\lambda,\mu),(\rho,\sigma)) \in \mathrm{supp}(u)\times\mathrm{supp}(v) \setminus \{((\overline{\lambda},\overline{\mu}),(\overline{\rho},\overline{\sigma}))\}$.

It holds $[u,v] = \sum_{((\lambda,\mu),(\rho,\sigma))\in\mathrm{supp}(u)\times\mathrm{supp}(v)} a_{(\lambda,\mu)}b_{(\rho,\sigma)}[\xi^\lambda\partial^\mu,\xi^\rho\partial^\sigma]$. By (a) and by the shown inequalities $\mathrm{LM}_{\preceq}([\xi^\lambda\partial^\mu,\xi^\rho\partial^\sigma]) < X^{\lambda+\rho}Y^{\mu+\sigma}$ and $X^{\lambda+\rho}Y^{\mu+\sigma} \leq X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$ for all $((\lambda,\mu),(\rho,\sigma)) \in \mathrm{supp}(u)\times\mathrm{supp}(v)$, we get $\mathrm{LM}_{\preceq}([u,v]) < X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$.

As $uv = a_{(\overline{\lambda},\overline{\mu})}b_{(\overline{\rho},\overline{\sigma})}\xi^{\overline{\lambda}}\partial^{\overline{\mu}}\xi^{\overline{\rho}}\partial^{\overline{\sigma}} + \sum_{((\lambda,\mu),(\rho,\sigma))\in\mathrm{supp}(u)\times\mathrm{supp}(v)\setminus\{((\overline{\lambda},\overline{\mu}),(\overline{\rho},\overline{\sigma}))\}} a_{(\lambda,\mu)}b_{(\rho,\sigma)}\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma$, by (a) and by the shown equalities $\mathrm{LM}_{\preceq}(\xi^\lambda\partial^\mu\xi^\rho\partial^\sigma) = X^{\lambda+\rho}Y^{\mu+\sigma}$ for all $((\lambda,\mu),(\rho,\sigma)) \in \mathrm{supp}(u)\times\mathrm{supp}(v)$, and because for all $((\lambda,\mu),(\rho,\sigma)) \in \mathrm{supp}(u)\times\mathrm{supp}(v)\setminus\{((\overline{\lambda},\overline{\mu}),(\overline{\rho},\overline{\sigma}))\}$ one has $X^{\lambda+\rho}Y^{\mu+\sigma} < X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$, we conclude that $\mathrm{LM}_{\preceq}(uv) = X^{\overline{\lambda}+\overline{\rho}}Y^{\overline{\mu}+\overline{\sigma}}$. □

Proposition 2.3.5. *Weyl algebras are admissible algebras.*

Proof. Clear by 2.3.1 and 2.3.4(b). □

Theorem 2.3.6. *Each left ideal L of W admits a universal Gröbner basis.*

Proof. Clear by 2.3.5 and 1.10.5. A direct proof for Weyl algebras is in A.2.4. □

Remark 2.3.7. Let I be a graded ideal of a $\mathbb{Z}$-graded commutative polynomial ring R over a field. Let $\preceq$ be a total ordering on the set of all monomials of R. Let $x \in I \setminus \{0\}$ and for each $i \in \mathbb{Z}$ let x_i denote the homogeneous component of x of degree i. Then, of course, $x_i \in I$ for all i. Moreover, $\mathrm{LM}_{\preceq}(x) = \mathrm{LM}_{\preceq}(x_i)$ for some i with $x_i \neq 0$. Indeed, $\mathrm{supp}(x) = \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(x_i)$ and $\mathrm{supp}(x_j) \cap \mathrm{supp}(x_k) = \varnothing$ whenever $j \neq k$.

Theorem 2.3.8. *Let $\omega \in \Omega$, let $\preceq \in \mathrm{AO}(N)$, let $\leq = \phi(\preceq)$, let $L \subseteq W$ be a left ideal, and let B be a $\preceq_\omega$-Gröbner basis of L. Then $\sigma^\omega(B)$ is a $\leq$-Gröbner basis of $\mathrm{G}^\omega L$, thus $\mathrm{G}^\omega L = \langle \sigma^\omega(b) \mid b \in B \rangle$ and $\mathrm{LM}_{\leq}(\mathrm{G}^\omega L) = \langle \mathrm{LM}_{\leq}(\sigma^\omega(b)) \mid b \in B \rangle$ as ideals of $K[X,Y]$.*

Proof. Let $\overline{x} \in \mathrm{G}^\omega L \setminus \{0\}$ be a ω-homogeneous element. Then there exists $x \in L$ with $\overline{x} = \sigma^\omega(x)$. Thus $\mathrm{LM}_{\leq}(\overline{x}) = \mathrm{LM}_{\leq}(\sigma^\omega(x)) = \mathrm{LM}_{\preceq_\omega}(x)$. Because $\mathrm{G}^\omega L$ is ω-graded, from 2.3.7 it follows $\mathrm{LM}_{\leq}(\mathrm{G}^\omega L) \subseteq \mathrm{LM}_{\preceq_\omega}(L)$. Conversely, the hypothesis about B implies $\mathrm{LM}_{\preceq_\omega}(L) = \langle \mathrm{LM}_{\preceq_\omega}(b) \mid b \in B \rangle = \langle \mathrm{LM}_{\leq}(\sigma^\omega(b)) \mid b \in B \rangle \subseteq \mathrm{LM}_{\leq}(\mathrm{G}^\omega L)$.

Hence $\langle \mathrm{LM}_{\leq}(\sigma^\omega(b)) \mid b \in B \rangle = \mathrm{LM}_{\leq}(\mathrm{G}^\omega L)$. Since $\leq$ is an admissible ordering on M, this implies $\langle \sigma^\omega(b) \mid b \in B \rangle = \mathrm{G}^\omega L$ as it is well known from the theory of Gröbner bases in commutative polynomial rings over a field with respect to admissible orderings, see for instance [20, Corollary 2.5.6]. □

Remark 2.3.9. A proof of 2.3.8 is also given in [27, Propositions V.7.2 & II.4.2] and is sketched in [35, Theorem 1.1.6(1)].

Remark 2.3.10. Analogously as in 2.3.8, if $\nu \in \mathbb{N}_0^{2n}$, $\preceq \in \mathrm{AO}(N)$, $I \subseteq K[X,Y]$ is an ideal, B is a $\phi(\preceq_\nu)$-Gröbner basis of I, then $\tau^\nu(B)$ is a $\phi(\preceq)$-Gröbner basis of $\mathrm{G}^\nu I$. Notice that the orderings $\phi(\preceq_\nu)$ and $\phi(\preceq)_\nu$ are equal.

Corollary 2.3.11. *For every left ideal L of W the set $\{\mathrm{G}^\omega L \mid \omega \in \Omega\}$ is finite. Similarly, for every ideal I of $K[X,Y]$ the set $\{\mathrm{G}^\nu I \mid \nu \in \mathbb{N}_0^{2n}\}$ is finite.*

Proof. By 2.3.6 we can find a universal Gröbner basis $U \supseteq \{0\}$ of L. By 2.3.8 we have $\mathrm{G}^\omega L = \langle \sigma^\omega(u) \mid u \in U \rangle$. So $\#\{\mathrm{G}^\omega L \mid \omega \in \Omega\} \leq \prod_{u \in U} \sum_{0 \leq k \leq \#\,\mathrm{supp}(u)} \binom{\#\,\mathrm{supp}(u)}{k} < \infty$. □

Remark 2.3.12. Another proof of 2.3.11 by homogenization is in [4, Theorem 3.6].

2.4 Stability of characteristic varieties

We encounter the notion of characteristic variety and critical cone and prove our main result, from which a relation between characteristic varieties and critical cones follows. We keep the notation of the previous section.

Remark 2.4.1. Fix any $\omega \in \Omega$. By 2.2.16, $\mathrm{G}^\omega W \cong K[X,Y]$ as K-algebras. Let M be a finitely generated left W-module. By 2.1.16 we can provide M with a good ω-filtration $\mathrm{F}^\omega M$. By 2.1.17 the $K[X,Y]$-module $\mathrm{G}^\omega M$ is finitely generated, and by 2.2.14, 2.1.26, 2.1.27 the ideal $\sqrt{(0:\mathrm{G}^\omega M)}$ of $K[X,Y]$ is independent of the choice of $\mathrm{F}^\omega M$.

Definition 2.4.2. Let $\omega \in \Omega$ and let M be a finitely generated left W-module. By 2.4.1 we may define the *ω-characteristic variety $\mathcal{V}^\omega(M)$ of M* as the closed set $\mathrm{Var}(0:\mathrm{G}^\omega M)$ of $\mathrm{Spec}(K[X,Y])$. In particular we consider $\mathcal{V}^{(1\,|\,1)}(M)$ and $\mathcal{V}^{(0\,|\,1)}(M)$, the characteristic variety of M *by degree* and *by order*.

We define the *ω-critical cone $\mathcal{C}^\omega(M)$ of M* as $\mathrm{Var}(\mathrm{G}^{(1\,|\,1)}\sqrt{(0:\mathrm{G}^\omega M)})$, which is equal to $\mathrm{Var}(\mathrm{G}^{(1\,|\,1)}(0:\mathrm{G}^\omega M))$ and $\mathrm{Var}(0:\mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M)$ by and 2.1.32 and 2.1.31, a closed set of $\mathrm{Spec}(K[X,Y])$. In particular we consider $\mathcal{C}^{(1\,|\,1)}(M)$ and $\mathcal{C}^{(0\,|\,1)}(M)$, the critical cone of M *by degree* and *by order*.

Remark 2.4.3. Let M be a finitely generated left W-module and let N be a submodule of M. If M is provided with a good filtration, then by 2.2.16 and by 2.1.22 and 2.1.23 the induced ω-filtrations of N and M/N are good. Therefore what said in 2.4.1 and 2.4.2 applies also to N and M/N.

Theorem 2.4.4. *Given any finitely generated left W-module M, there are only finitely many distinct characteristic varieties $\mathcal{V}^\omega(M)$ for ω varying in Ω.*

Proof. Given a submodule N of M, by 2.1.5 one has $\mathcal{V}^\omega(M) = \mathcal{V}^\omega(N) \cup \mathcal{V}^\omega(M/N)$ for all $\omega \in \Omega$. Hence by induction over the number of generators of M, the claim follows from 2.3.11 and 2.1.7. □

Lemma 2.4.5. *Let $w \in W$, $\nu \in \mathbb{N}_0^{2n}$, $\omega \in \Omega$. Let $l \in \mathbb{N}_0$ with $l \geq \deg^\nu(w)$ in W, let $m \in \mathbb{N}_0$ with $m \geq \deg^\omega(w)$ in W, let $p \in \mathbb{N}_0$ with $p \geq \deg^\nu(\sigma_m^\omega(w))$ in $K[X,Y]$. Then in $K[X,Y]$ for all $s \in \mathbb{N}$ such that $s > l-p$ it holds $\tau_p^\nu(\sigma_m^\omega(w)) = \sigma_{p+sm}^{\nu+s\omega}(w)$.*

Proof. Let us write w in canonical form as $\sum_{(\lambda,\mu)\in\mathbb{S}} c_{\lambda\mu}\xi^\lambda\partial^\mu$, where $\mathbb{S} = \operatorname{supp}(w)$ and $c_{\lambda\mu} \in K \setminus \{0\}$ for all $(\lambda,\mu) \in \mathbb{S}$. By definition, we have $\omega \cdot (\lambda \,|\, \mu) \le m$ for all $(\lambda,\mu) \in \mathbb{S}$. It follows $\sigma^\omega_m(w) = \sum_{(\lambda,\mu)\in\mathbb{S}_m} c_{\lambda\mu}X^\lambda Y^\mu$, where $\mathbb{S}_m = \{(\lambda,\mu) \in \mathbb{S} \mid \omega\cdot(\lambda\,|\,\mu) = m\}$. Similarly we have $\nu\cdot(\lambda\,|\,\mu) \le p$ for all $(\lambda,\mu) \in \mathbb{S}_m$. Therefore $\tau^\nu_p(\sigma^\omega_m(w)) = \sum_{(\lambda,\mu)\in\mathbb{S}_{m,p}} c_{\lambda\mu}X^\lambda Y^\mu$, where $\mathbb{S}_{m,p} = \{(\lambda,\mu) \in \mathbb{S}_m \mid \nu\cdot(\lambda\,|\,\mu) = p\}$.

Let $(\lambda,\mu) \in \mathbb{S}$. As just observed, $\omega\cdot(\lambda\,|\,\mu) \le m$, and moreover if $\omega\cdot(\lambda\,|\,\mu) = m$, then $\nu\cdot(\lambda\,|\,\mu) \le p$. Thus we have the following three cases.

If $\omega\cdot(\lambda\,|\,\mu) = m$ and $\nu\cdot(\lambda\,|\,\mu) = p$, then $(\nu+s\omega)\cdot(\lambda\,|\,\mu) = \nu\cdot(\lambda\,|\,\mu) + s\omega\cdot(\lambda\,|\,\mu) = p+sm$, hence $\xi^\lambda\partial^\mu \in \mathrm{F}^{\nu+s\omega}_{p+sm}W \setminus \mathrm{F}^{\nu+s\omega}_{p+sm-1}W$ for all $s \in \mathbb{N}$.

If $\omega\cdot(\lambda\,|\,\mu) = m$ and $\nu\cdot(\lambda\,|\,\mu) < p$, then $(\nu+s\omega)\cdot(\lambda\,|\,\mu) = \nu\cdot(\lambda\,|\,\mu) + s\omega\cdot(\lambda\,|\,\mu) < p+sm$, hence $\xi^\lambda\partial^\mu \in \mathrm{F}^{\nu+s\omega}_{p+sm-1}W$ for all $s \in \mathbb{N}$.

If $\omega\cdot(\lambda\,|\,\mu) < m$, then $(\nu+s\omega)\cdot(\lambda\,|\,\mu) = \nu\cdot(\lambda\,|\,\mu) + s\omega\cdot(\lambda\,|\,\mu) \le l+sm-s < p+sm$ as soon as $s > l-p$, hence $\xi^\lambda\partial^\mu \in \mathrm{F}^{\nu+s\omega}_{p+sm-1}W$ for all $s \in \mathbb{N}$ with $s > l-p$.

So, putting $\mathbb{S}'_{m,p} = \{(\lambda,\mu) \in \mathbb{S} \mid \omega\cdot(\lambda\,|\,\mu) = m,\ \nu\cdot(\lambda\,|\,\mu) = p\}$, we obtain $\sigma^{\nu+s\omega}_{p+sm}(w) = \sum_{(\lambda,\mu)\in\mathbb{S}'_{m,p}} c_{\lambda\mu}X^\lambda Y^\mu$ for all $s \in \mathbb{N}$ with $s > l-p$. Since $\mathbb{S}_{m,p} = \mathbb{S}'_{m,p}$, we are done. □

Lemma 2.4.6. *Let $w \in W$, and let $\nu \in \mathbb{N}^{2n}_0$ and $\omega \in \Omega$. Then for all $s \in \mathbb{N}$ such that $s > \deg^\nu(w) - \deg^\nu(\sigma^\omega(w))$ it holds $\deg^\nu(\sigma^\omega(w)) + s\deg^\omega(w) = \deg^{\nu+s\omega}(w)$.*

Proof. If $w = 0$, then the statement holds for all $s \in \mathbb{N}$. Hence let $w \ne 0$, and put $l = \deg^\nu(w)$, $m = \deg^\omega(w)$ and $p = \deg^\nu(\sigma^\omega_m(w))$. Let $s \in \mathbb{N}$ with $s > l-p$ and put $d = \deg^{\nu+s\omega}(w)$. As in the proof of 2.4.5 we obtain $(\nu+s\omega)\cdot(\lambda\,|\,\mu) \le p+sm$ for all $(\lambda,\mu) \in \operatorname{supp}(w)$, hence $d = \sup\{(\nu+s\omega)\cdot(\lambda\,|\,\mu) \mid (\lambda,\mu) \in \operatorname{supp}(w)\} \le p+sm$. If it held $d < p+sm$, then we would have $\sigma^{\nu+s\omega}_{p+sm}(w) = 0$, whereas $\tau^\nu_p(\sigma^\omega_m(w)) \ne 0$, in contradiction to 2.4.5. Hence $p+sm = d$, our claim. □

Lemma 2.4.7. *Let $w \in W$, and let $\nu \in \mathbb{N}^{2n}_0$ and $\omega \in \Omega$. Then for all $s \in \mathbb{N}$ such that $s > \deg^\nu(w) - \deg^\nu(\sigma^\omega(w))$ it holds $\tau^\nu(\sigma^\omega(w)) = \sigma^{\nu+s\omega}(w)$.*

Proof. Clear by 2.4.5 with $l = \deg^\nu(w)$, $m = \deg^\omega(w)$, $p = \deg^\nu(\sigma^\omega_m(w)) = \deg^\nu(\sigma^\omega(w))$, and by 2.4.6. □

Theorem 2.4.8 extends a result published in 1971 by Bernstein as a part of the proof of [10, Theorem 3.1], namely that $\mathrm{G}^{(1\,|\,1)}\mathrm{G}^{(0\,|\,1)}L \subseteq \mathrm{G}^{(1\,|\,s)}L$ for $s \gg 0$. In greater generality we prove also the converse inclusion.

Theorem 2.4.8. *Let L be a left ideal of W. For all $\nu \in \mathbb{N}_0^{2n}$ it exists $s_\nu \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$ it holds $\mathrm{G}^\nu \mathrm{G}^\omega L = \mathrm{G}^{\nu+s\omega} L$ as ideals of $K[X,Y]$.*

Proof. Let $\nu \in \mathbb{N}_0^{2n}$. We can choose a universal Gröbner basis U of L by 2.3.6, and we can fix an admissible ordering $\preceq \in \mathrm{AO}(N)$, see 1.6.6, 2.3.1, 1.4.2. Thus U is a $(\preceq_\nu)_\omega$-Gröbner basis of L for all $\omega \in \Omega$, see 1.6.7, 2.3.1, 1.4.2.

Let denote $\phi(\preceq)$ by $\leq$, so that $\phi(\preceq_\nu)$ equals $\leq_\nu$. By 2.3.8, $\sigma^\omega(U)$ is a $\leq_\nu$-Gröbner basis of $\mathrm{G}^\omega L$ for all $\omega \in \Omega$. Hence, by 2.3.10, $\tau^\nu(\sigma^\omega(U))$ is a $\leq$-Gröbner basis of $\mathrm{G}^\nu\mathrm{G}^\omega L$ for all $\omega \in \Omega$. Therefore, in particular, it holds $\mathrm{G}^\nu\mathrm{G}^\omega L = \langle \tau^\nu(\sigma^\omega(u)) \mid u \in U\rangle$ for all $\omega \in \Omega$. Putting $s_\nu = \max\{\deg^\nu(u) \mid u \in U,\, u \neq 0\}$ if $U \nsubseteq \{0\}$, and $s_\nu = 0$ if $U \subseteq \{0\}$, by 2.4.7 we get $\mathrm{G}^\nu\mathrm{G}^\omega L = \langle \sigma^{\nu+s\omega}(u) \mid u \in U\rangle$ for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$.

On the other hand, U is a Gröbner basis of L with respect to $\preceq_{\nu+s\omega}$ for all $\omega \in \Omega$ and all $s \in \mathbb{N}$. Therefore, by 2.3.8, $\sigma^{\nu+s\omega}(U)$ is a Gröbner basis of $\mathrm{G}^{\nu+s\omega}L$ with respect to $\leq$, whence $\langle \sigma^{\nu+s\omega}(u) \mid u \in U\rangle = \mathrm{G}^{\nu+s\omega}L$, for all $\omega \in \Omega$ and all $s \in \mathbb{N}$. □

Main Theorem 2.4.9. *Let M be a finitely generated left W-module. For all $\nu \in \mathbb{N}_0^{2n}$ it exists $s_\nu \in \mathbb{N}_0$ with the property that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$ it holds $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^{\nu+s\omega} M)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} M)}$ as ideals of $K[X,Y]$.*

Proof. We fix any $\nu \in \mathbb{N}_0^{2n}$. We find $r \in \mathbb{N}$ such that M is generated over R by r of its elements.

First, by induction over r, we prove the existence of $s_\nu \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$ it holds $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} M)}$.

If $r = 1$, then $M \cong W/L$ for a left ideal L of W. By 2.1.5, 2.1.7, 2.4.8 we find $s_\nu \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$ it holds $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega W/L)} = \sqrt{\mathrm{G}^\nu\mathrm{G}^\omega L} = \sqrt{\mathrm{G}^{\nu+s\omega} L} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} W/L)}$.

If $r > 1$, we find a cyclic submodule N of M such that $P = M/N$ is generated by $r-1$ elements. As before, by 2.4.8 we find $s'_\nu \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s'_\nu$ it holds $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega N)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} N)}$. By induction we find $s''_\nu \in \mathbb{N}_0$ such that $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega P)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} P)}$ for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s''_\nu$. By 2.1.5 we get $\sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega N)} \cap \sqrt{(0 : \mathrm{G}^\nu\mathrm{G}^\omega P)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} N)} \cap \sqrt{(0 : \mathrm{G}^{\nu+s\omega} P)} = \sqrt{(0 : \mathrm{G}^{\nu+s\omega} M)}$ for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$, where $s_\nu = \max\{s'_\nu, s''_\nu\}$, so that s_ν is independent of ω. This completes the induction step.

Now, by 2.1.31, 2.1.32, 2.1.13, it follows $\sqrt{(0 : \mathrm{G}^\nu \mathrm{G}^{\nu+s\omega} M)} = \sqrt{\mathrm{G}^\nu}\sqrt{(0 : \mathrm{G}^{\nu+s\omega} M)} = \sqrt{\mathrm{G}^\nu}\sqrt{(0 : \mathrm{G}^\nu \mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^\nu \mathrm{G}^\nu \mathrm{G}^\omega M)} = \sqrt{(0 : \mathrm{G}^\nu \mathrm{G}^\omega M)}$ for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_\nu$. ☐

Corollary 2.4.10. *There exists $s_{(1\,|\,1)} \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ and all $s \in \mathbb{N}$ with $s > s_{(1\,|\,1)}$ one has $\mathcal{C}^\omega(M) = \mathcal{V}^{(1\,|\,1)+s\omega}(M) = \mathcal{C}^{(1\,|\,1)+s\omega}(M)$.*

Proof. Immediately clear by 2.4.9. ☐

Corollary 2.4.11. *It holds $\mathcal{C}^{(0\,|\,1)}(M) = \mathcal{V}^{(1\,|\,s)}(M) = \mathcal{C}^{(1\,|\,s)}(M)$ for $s \gg 0$, whereas $\mathcal{C}^{(1\,|\,1)}(M) = \mathcal{V}^{(1\,|\,1)}(M)$.*

Proof. The first statement is clear by 2.4.10, the second follows from 2.1.13. ☐

Example 2.4.12. Let L be the left ideal of the 1st Weyl algebra over $\mathbb{C}$ generated by $\xi^2\partial^2 - \partial^2 + \xi^3\partial + 2\xi^4\partial + 3\xi^5$. Then we have:

$$\begin{aligned}
\mathcal{V}^{(0\,|\,1)}(W/L) &= \mathcal{V}(X^2Y^2 - Y^2), & \mathcal{C}^{(0\,|\,1)}(W/L) &= \mathcal{V}(X^2Y^2),\\
\mathcal{V}^{(1\,|\,1)}(W/L) &= \mathcal{V}(2X^4Y + 3X^5), & \mathcal{C}^{(1\,|\,1)}(W/L) &= \mathcal{V}(2X^4Y + 3X^5),\\
\mathcal{V}^{(1\,|\,2)}(W/L) &= \mathcal{V}(X^2Y^2 + 2X^4Y), & \mathcal{C}^{(1\,|\,2)}(W/L) &= \mathcal{V}(2X^4Y),\\
\mathcal{V}^{(1\,|\,3)}(W/L) &= \mathcal{V}(X^2Y^2), & \mathcal{C}^{(1\,|\,3)}(W/L) &= \mathcal{V}(X^2Y^2).
\end{aligned}$$

Observe in Figure 2.1 that $\mathcal{C}^{(0\,|\,1)}(W/L) = \mathcal{V}^{(1\,|\,1+s)}(W/L) = \mathcal{C}^{(1\,|\,1+s)}(M)$ for all $s > 1$.

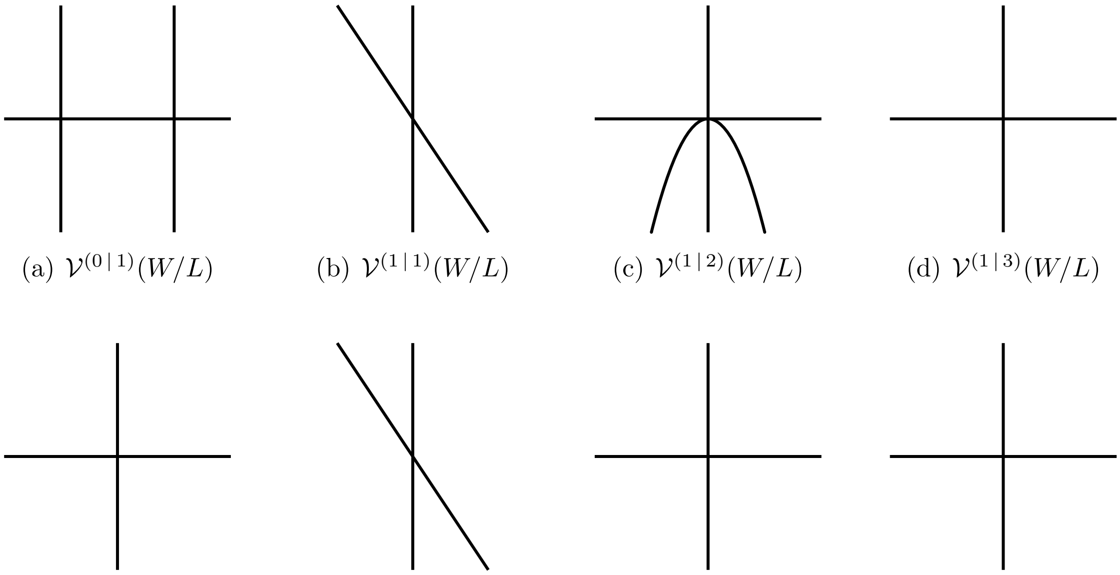
2.5 Dimension of characteristic varieties

In this section, as an application of Theorem 2.4.9, we aim to furnish a new proof of a classical result: for a fixed finitely generated left W-module M, the characteristic varieties $\mathcal{V}^\omega(M)$, $\omega \in \Omega$, all have the same Krull dimension.

This is usually proved, as exposed by Ehlers in [14, Chapter V], by not trivial homological methods. It turns out indeed that $\mathrm{Kdim}_{K[X,Y]}\, \mathrm{G}^\omega M = 2n - \mathrm{j}_W(M)$ for all $\omega \in \Omega$, where $\mathrm{j}_W(M) = \inf\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_W(M, W) \neq 0\}$.

Bernstein provided in 1971 a proof that $\mathcal{V}^{(1\,|\,1)}(M)$ and $\mathcal{V}^{(0\,|\,1)}(M)$ have the same Krull dimension, see [10, Theorem 3.1].

Our proof descends (1) from the *equality of annihilators* obtained in 2.4.9, which in particular allows, so to say, to make a transition from nonfinite to finite filtrations, (2) from



(a) $\mathcal{V}^{(0\,|\,1)}(W/L)$ (b) $\mathcal{V}^{(1\,|\,1)}(W/L)$ (c) $\mathcal{V}^{(1\,|\,2)}(W/L)$ (d) $\mathcal{V}^{(1\,|\,3)}(W/L)$

(e) $\mathcal{C}^{(0\,|\,1)}(W/L)$ (f) $\mathcal{C}^{(1\,|\,1)}(W/L)$ (g) $\mathcal{C}^{(1\,|\,2)}(W/L)$ (h) $\mathcal{C}^{(1\,|\,3)}(W/L)$

Figure 2.1: Some characteristic varieties and their critical cones

the *preservation of the Gelfand–Kirillov dimension* when passing from finitely filtered rings and modules to their associated graded rings and modules, see 2.5.12, and (3) from the *equality of Krull and Gelfand–Kirillov dimension* in the category of noetherian modules over a noetherian commutative F-algebra, F a field, see 2.5.7.

We begin with presenting some necessary results about the notion of Gelfand–Kirillov dimension, which can be found in [25] or [32].

Reminder 2.5.1. Let F be a field and B be a finitely generated F-algebra. Then there exists a *generating space of* B, that is, an F-module V of finite length such that F is contained in V and B is generated as an F-algebra by V. By V^i, $i \in \mathbb{N}_0$, we denote the F-module generated by all products of at most i elements of V, so that $V^0 = F$, $V^1 = V$, $V^i \subseteq V^{i+1}$ and $B = \bigcup_{i\in\mathbb{N}_0} V^i$. The *Gelfand–Kirillov dimension* $\operatorname{GKdim} B$ *of* B is defined to be the limes superior $\overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F V^i) \in [0,\infty]$.

$\operatorname{GKdim} B$ is indeed independent of the choice of V. Let W be another generating space of B. The F-modules V^i and W^i have finite length for all $i \in \mathbb{N}$ as V and W are finitely generated F-modules. We claim that $\overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F V^i) = \overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F W^i)$ holds in $\mathbb{R} \cup \{\infty\}$. Because V is a finitely generated F-submodule of B and $B = \bigcup_{i\in\mathbb{N}} W^i$, there exists $j \in \mathbb{N}$ such that $V \subseteq W^j$. Hence $V^i \subseteq W^{ij}$ for all $i \in \mathbb{N}$, therefore $\log_i(\operatorname{len}_K V^i) \leq$

$\log_i(\operatorname{len}_K W^{ij}) = (1+\log_i(j))\log_{ij}(\operatorname{len}_K W^{ij})$ for all $i \in \mathbb{N}$, thus $\overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_K V^i) \le \overline{\lim}_{i\to\infty}\log_{ij}(\operatorname{len}_K W^{ij}) \le \overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_K W^i)$. A similar argument shows the converse inequality.

If A is any F-algebra, we define $\operatorname{GKdim} A = \sup_B \operatorname{GKdim} B$, the supremum being taken over all finitely generated F-subalgebras B of A. For finitely generated F-algebras the two definitions clearly are equivalent.

Let N be a finitely generated left B-module. There exists a *generating space of* N, that is, an F-module G of finite length such that N is generated as a B-module by G. The *Gelfand–Kirillov dimension of* N is $\operatorname{GKdim}_B N = \overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_F V^iG) \in [0,\infty]$ and is independent of the choice of V and of G by a similar argument as above.

For any A-module M let $\operatorname{GKdim}_A M = \sup_B \sup_N \operatorname{GKdim}_B N$, where the suprema are taken over all finitely generated F-subalgebras B of A and all finitely generated B-submodules of M. For finitely generated modules over finitely generated F-algebras the two definitions clearly are equivalent.

Lemma 2.5.2. *Let F be a field, B be a finitely generated F-algebra, and V be a generating space of B. Then* $\operatorname{GKdim} B = \inf\{\alpha \in \mathbb{R} \mid \operatorname{len}_F V^i \le \beta i^\alpha$ *for all integers* $i \gg 0\}$ *for all* $\beta \in \mathbb{R}$ *with* $\beta > 0$.

Proof. Without loss of generality we may restrict to treating only the case when $\beta = 1$. Put $\Lambda = \{\lambda \in \mathbb{R} \mid \exists\, i_0 \in \mathbb{N}\, \forall\, i > i_0 : \operatorname{len}_F V^i \le i^\lambda\}$. If $\Lambda = \varnothing$, then for all $\lambda \in \mathbb{R}$ there exists $i > 1$ such that $\log_i(\operatorname{len}_F V^i) > \lambda$, and so $\operatorname{GKdim} B = \overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_F V^i) = \infty = \inf \Lambda$ in this case.

Assume that $\Lambda \ne \varnothing$. Let $\lambda \in \Lambda$. We find $i_0 \in \mathbb{N}$ such that for all $i > i_0$ it holds $\operatorname{len}_F V^i \le i^\lambda$. It follows $\log_i(\operatorname{len}_F V^i) \le \lambda$ for all $i > i_0$, hence $\overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_F V^i) \le \lambda$. Since λ was chosen arbitrarily from Λ, it follows $\operatorname{GKdim} B \le \inf \Lambda$.

Now let $M = \{\mu \in \mathbb{R} \mid \exists\, i_0 \in \mathbb{N}\ \forall\, i > i_0 : \log_i(\operatorname{len}_F V^i) \le \mu\}$. Then, by definition, $\overline{\lim}_{i\to\infty}\log_i(\operatorname{len}_F V^i) = \inf M$. Suppose that $\inf M < \inf \Lambda$. Then we can find $\mu \in M$ with $\mu < \inf \Lambda$. But then we find also $i_0 \in \mathbb{N}$ such that for all $i > i_0$ it holds $\log_i(\operatorname{len}_F V^i) \le \mu$. It follows $\operatorname{len}_F V^i \le i^\mu$ for all $i > i_0$, thus $\mu \in \Lambda$. Hence $\inf \Lambda \le \mu$, a contradiction. Thus $\inf M \ge \inf \Lambda$, that is, $\operatorname{GKdim} B \ge \inf \Lambda$. ☐

Example 2.5.3. Let F be a field, A be an F-algebra, $t \in \mathbb{N}$, and $X_1, \dots, X_t$ be indeterminates commuting with each other and with A. Then $\operatorname{GKdim} A[X_1, \dots, X_t] = \operatorname{GKdim} A + t$.

It is sufficient to prove the statement for any finitely generated F-subalgebra B of A. By induction we may assume without restriction that $t = 1$, and we write here X for X_1. Let V be a generating space of B. Put $W = V + FX$. Then W is a generating space of $B[X]$.

As X commutes with B, $W^i \subseteq V^i \oplus V^iX \oplus V^iX^2 \oplus \ldots \oplus V^iX^i$ for all $i \in \mathbb{N}$, and hence $\operatorname{len}_F W^i \leq (i+1)\operatorname{len}_F V^i \leq 2i \operatorname{len}_F V^i$ for all $i \in \mathbb{N}$. Therefore, by 2.5.2, we get $\operatorname{GKdim} B[X] = \inf\{\alpha \in \mathbb{R} \mid \operatorname{len}_F W^i \leq i^\alpha,\ i \gg 0\} \leq \inf\{\alpha \in \mathbb{R} \mid 2i \operatorname{len}_F V^i \leq i^\alpha,\ i \gg 0\} = \inf\{\alpha \in \mathbb{R} \mid \operatorname{len}_F V^i \leq 2^{-1} i^{\alpha-1},\ i \gg 0\} = 1 + \operatorname{GKdim} B$.

On the other hand, one has also $W^i \supseteq V^j \oplus V^jX \oplus V^jX^2 \oplus \ldots \oplus V^jX^j$ for all $i \in \mathbb{N}$ with $i \geq 2$ and all $j \in \mathbb{N}$ with $j \leq i/2$. In particular it follows $\operatorname{len}_F W^i \geq (\lfloor i/2 \rfloor + 1)\operatorname{len}_F V^{\lfloor i/2 \rfloor}$ for all $i \in \mathbb{N}$ with $i \geq 2$, where $\lfloor i/2 \rfloor$ is the greatest integer less or equal to $i/2$, and therefore we obtain $\operatorname{GKdim} B[X] = \overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F W^i) \geq \overline{\lim}_{i\to\infty} \log_i((\lfloor i/2 \rfloor + 1)\operatorname{len}_F V^{\lfloor i/2 \rfloor}) \geq \lim_{i\to\infty} \frac{\log(i/2)}{\log(i/2)+\log(2)} + \overline{\lim}_{i\to\infty} \frac{\log(\operatorname{len}_F V^{\lfloor i/2 \rfloor})}{\log(\lfloor i/2 \rfloor)+\log(3)} = 1 + \operatorname{GKdim} B$.

Remark 2.5.4. Let F be a field and A be an F-algebra. It holds $\operatorname{GKdim} A = \operatorname{GKdim}_A A$, as one immediately sees from the definitions.

Moreover, as stated in [25, Proposition 5.1(c)], the Gelfand–Kirillov dimension is independent of the base ring, that is, $\operatorname{GKdim}_{A/I} M = \operatorname{GKdim}_A M$ for all left A-modules M and all two-sided ideals I of A such that $IM = 0$.

Indeed, let B any finitely generated F-subalgebra of A and V be a generating space of B, let N be a finitely generated B-submodule of M and G be a generating space of N. Then $V + I/I$ is a generating space of the finitely generated F-subalgebra $B + I/I$ of A/I, and G clearly is a generating space of the finitely generated $B + I/I$-submodule N of M, and we have an isomorphism $V^iG \cong (V + I/I)^iG$ of F-modules for each $i \in \mathbb{N}$. This shows that $\operatorname{GKdim}_A M \leq \operatorname{GKdim}_{A/I} M$.

Conversely, each finitely generated F-subalgebra of A/I is of the form $B + I/I$ for some finitely generated F-subalgebra B of A, and for any fixed such B each finitely generated $B + I/I$-submodule of M is a finitely generated B-submodule of M, so that we obtain $\operatorname{GKdim}_{A/I} M \leq \operatorname{GKdim}_A M$.

In particular, when $M = A/I$, we have $\operatorname{GKdim} A/I = \operatorname{GKdim}_{A/I} A/I = \operatorname{GKdim}_A A/I$ for all two-sided ideals I of A.

Theorem 2.5.5. *Let $\mathcal{A}$ be a discrete finite filtration of an F-algebra A such that the associated F-algebra $\mathrm{G}\mathcal{A}$ is finitely generated and left noetherian. Under these assumptions* *the Gelfand–Kirillov dimension is exact in the category of finitely generated left A-modules, that is, it holds* $\operatorname{GKdim}_A M = \max\{\operatorname{GKdim}_A N, \operatorname{GKdim}_A P\}$ *whenever* $N \rightarrowtail M \twoheadrightarrow P$ *is an exact sequence of finitely generated left A-modules.*

Proof. See [25, Theorem 6.14]. □

Theorem 2.5.6. *Let F be a field. Let A and B be finitely generated F-algebras such that B is a F-subalgebra of A and A is a finitely generated B-module. Then* $\operatorname{GKdim} A = \operatorname{GKdim} B$.

Proof. We report here essentially the proof of [9, Proposition 1.1.12]. A second proof can be found in [25, Proposition 5.5].

By 2.5.5 $\operatorname{GKdim} B \leq \operatorname{GKdim} A$. Conversely, we can write $A = Ba_1 + \ldots + Ba_t$ for some $t \in \mathbb{N}$ and $a_1, \ldots, a_t \in A$. Let W be a generating space of B. Then $V = W + Fa_1 + \ldots + Fa_t$ is a generating space of A. We note that $BV \supseteq BW + Ba_1 + \ldots + Ba_t \supseteq A \supseteq V^2$. So, as the F-module V^2 is finitely generated, we can find a finitely generated F-submodule X of B such that $XV \supseteq V^2$. The F-module $Y = X + W$ is then a generating space of B such that $YV \supseteq V^2$. Thus $Y^2V \supseteq YV^2 \supseteq V^3$, and we inductively obtain $Y^iV \supseteq V^{i+1}$ for all $i \in \mathbb{N}$. Thus $Y^iV \supseteq V^i$, and hence $\operatorname{len}_F V^i \leq \operatorname{len}_F Y^iV \leq \operatorname{len}_F Y^i \cdot \operatorname{len}_F V$, for all $i \in \mathbb{N}$. Therefore $\operatorname{GKdim} A = \overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F V^i) \leq \overline{\lim}_{i\to\infty}(\log_i(\operatorname{len}_F Y^i) + \log_i(\operatorname{len}_F V)) = \overline{\lim}_{i\to\infty} \log_i(\operatorname{len}_F Y^i) = \operatorname{GKdim} B$. □

Reminder 2.5.7. Let F be a field, A be a finitely generated commutative F-algebra, and M be a finitely generated A-module. The *Krull dimension* $\operatorname{Kdim}_A M$ of M is defined as the supremum of the lengths of chains of prime ideals of the commutative ring $A/(0 : M)$.

Theorem 2.5.8. *Let F be a field, A be a finitely generated commutative F-algebra, and M be a finitely generated A-module. It holds* $\operatorname{GKdim}_A M = \operatorname{Kdim}_A M \in \mathbb{N}_0 \cup \{-\infty\}$.

Proof. One has $\operatorname{GKdim} M = -\infty$ if and only if $M = 0$, and $\operatorname{Kdim} M = -\infty$ if and only if $M = 0$. Thus we may assume that $M \neq 0$, so that $\operatorname{Kdim} M \in \mathbb{N}_0$. In our hypotheses both dimensions are exact, see 2.5.5 for the Gelfand-Kirillov dimension, and hence we may assume that $M = A/I$ for some ideal I. As both dimensions are preserved when changing the base ring from A to A/I, see 2.5.4 for the Gelfand–Kirillov dimension, it is sufficient to compare $\operatorname{Kdim} A/I$ to $\operatorname{GKdim} A/I$. As both dimensions are preserved when passing to integral extensions, see 2.5.6 for the Gelfand–Kirillov dimension, by Emmy

Noether's Normalization Lemma we may replace the finitely generated F-algebra A/I by the polynomial ring $F[X_1, \dots, X_d]$ having the same Krull dimension d as A/I. By 2.5.3 one has $\operatorname{GKdim} F[X_1, \dots, X_d] = d$. □

Remark 2.5.9. As an alternative proof, one can apply the result shown in 2.5.2, namely, $\operatorname{GKdim} A = \inf \{\alpha \in \mathbb{R} \mid \exists\, i_0 \in \mathbb{N}_0 \,\forall\, i > i_0 : \operatorname{len}_K V^i \leq i^\alpha\}$. It follows that $\operatorname{GKdim} A$ is indeed equal to the degree of the Hilbert polynomial of A, which in turn is equal to $\operatorname{Kdim} A$, and one concludes again by the exactness of both dimensions and by changing the base ring.

Definition 2.5.10. Let F be a field, A be an F-algebra, $\mathcal{A}$ be a filtration of A, M be a left A-module, and $\mathcal{M}$ be an $\mathcal{A}$-filtration of M. We say that $\mathcal{M}$ is *finite* if $\operatorname{len}_F(\mathrm{F}_i\mathcal{M}) < \infty$ for all $i \in \mathbb{Z}$.

Remark 2.5.11. In the notation of 2.5.10, if $\mathcal{A}$ is finite and M is finitely generated and $\mathcal{M}$ is good, then $\mathcal{M}$ is finite and discrete. Indeed, $\mathcal{M}$ is equivalent to a standard good filtration $\mathcal{S}$ of M, see 2.1.26 and 2.1.16. Now, $\mathcal{S}$ is finite whenever $\mathcal{A}$ is finite, and $\mathcal{S}$ is always discrete.

Lemma 2.5.12. *Let F be a field, A be a F-algebra, $\mathcal{A}$ be a filtration of A, M be a left A-module, and $\mathcal{M}$ be an $\mathcal{A}$-filtration of M. Then* $\operatorname{GKdim}_{\mathrm{G}\mathcal{A}} \mathrm{G}\mathcal{M} \leq \operatorname{GKdim}_A M$.

Furthermore, if the filtration $\mathcal{A}$ is finite and is such that the F-algebra $\mathrm{G}\mathcal{A}$ is finitely generated, and if the $\mathcal{A}$-filtration $\mathcal{M}$ is finite and discrete and is such that the $\mathrm{G}\mathcal{A}$-module $\mathrm{G}\mathcal{M}$ is finitely generated, then $\operatorname{GKdim}_{\mathrm{G}\mathcal{A}} \mathrm{G}\mathcal{M} = \operatorname{GKdim}_A M$.

Proof. By arguments of Linear Algebra, see [25, Lemma 6.5 & Proposition 6.6]. □

Now we come to the promised proof of the fact that the characteristic varieties $\mathcal{V}^\omega(M)$, $\omega \in \Omega$, all have the same Krull dimension, which is actually equal to the Gelfand–Kirillov dimension of M.

Theorem 2.5.13. *As in the previous section, let $n \in \mathbb{N}$, let K be a field of characteristic 0, let W be the n^{th} Weyl algebra over K, let M be a finitely generated left W-module, and let $K[X, Y]$ be the commutative polynomial ring over K in $2n$ indeterminates. Then* $\operatorname{Kdim}_{K[X,Y]} \mathrm{G}^\omega M = \operatorname{GKdim}_{K[X,Y]} \mathrm{G}^\omega M = \operatorname{GKdim}_W M$, *thus* $\operatorname{Kdim} \mathcal{V}^\omega(M) = \operatorname{GKdim}_W M$, *for all $\omega \in \Omega$.*

Proof. Let $\omega \in \Omega$. As the $(1\,|\,1)$-filtration of $K[X,Y]$ is finite, any good $(1\,|\,1)$-filtration of $\mathrm{G}^\omega M$ is finite and discrete by 2.5.11. Therefore by 2.5.12 we have $\mathrm{GKdim}_{K[X,Y]}\,\mathrm{G}^\omega M = \mathrm{GKdim}_{K[X,Y]}\,\mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M$. By 2.1.17, $\mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M$ is finitely generated over $K[X,Y]$, and so by 2.5.7 we get $\mathrm{GKdim}_{K[X,Y]}\,\mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M = \mathrm{GKdim}\,K[X,Y]/\sqrt{(0 : \mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M)}$. By 2.4.9 it holds $\mathrm{GKdim}\,K[X,Y]/\sqrt{(0 : \mathrm{G}^{(1\,|\,1)}\mathrm{G}^\omega M)} = \mathrm{GKdim}\,K[X,Y]/\sqrt{(0 : \mathrm{G}^{(1\,|\,1)+s\omega} M)}$, $s \gg 0$. By 2.5.7, $\mathrm{GKdim}\,K[X,Y]/\sqrt{(0 : \mathrm{G}^{(1\,|\,1)+s\omega} M)} = \mathrm{GKdim}_{K[X,Y]}\,\mathrm{G}^{(1\,|\,1)+s\omega} M$, $s \in \mathbb{N}$. As the $(1\,|\,1) + s\omega$-filtrations of W are finite, and hence by 2.5.11 the good $(1\,|\,1) + s\omega$-filtrations of M are finite and discrete, by 2.5.12 and 2.2.16 we obtain $\mathrm{GKdim}_{K[X,Y]}\,\mathrm{G}^{(1\,|\,1)+s\omega} M = \mathrm{GKdim}_W M$, $s \in \mathbb{N}$. As for the Krull dimension, we conclude by 2.5.7. □

2.6 Classification of characteristic varieties

As before, let K be a field of characteristic 0. For an arbitrary left ideal L of the 1st Weyl algebra W over K we aim to classify the characteristic varieties of W/L. More precisely, we aim to partition $\Omega = \mathbb{N}_0^2 \smallsetminus \{(0,0)\}$ into regions corresponding to equivalence classes $[\omega]_{\sim_L}$ of weights $\omega \in \Omega$ such that $\omega' \sim_L \omega''$ if and only if $\mathrm{G}^{\omega'} L = \mathrm{G}^{\omega''} L$. This would permit us to determine the number $\chi(L)$ of distinct ideals $\mathrm{G}^\omega L$, $\omega \in \Omega$, which we know to be finite by 2.3.11. Hence, because $\mathrm{G}^{\omega'} L = \mathrm{G}^{\omega''} L$ implies $\mathcal{V}^{\omega'}(W/L) = \mathcal{V}^{\omega''}(W/L)$ by 2.1.7, $\chi(L)$ would be an upper bound for the number of distinct ω-characteristic varieties of W/L.

We do not succeed in this but by a computer experiment we approximate $\Omega/{\sim_L}$ and this allows us to conjecture an upper bound for $\chi(L)$ in terms of total degrees of universal Gröbner bases of L.

Remark 2.6.1. Let $n \in \mathbb{N}$. For each finitely generated left module M over the n^{th} Weyl algebra over K and for each $\nu \in \mathbb{N}_0^{2n}$ there exists a *minimal* number $\kappa_\nu(M) \in \mathbb{N}_0$ such that for all $\omega \in \Omega$ the characteristic varieties $\mathcal{V}^{\nu+s\omega}(M)$ stabilize to $\mathrm{Var}(0 : \mathrm{G}^\nu \mathrm{G}^\omega M)$ as soon as $s > \kappa_\nu(M)$.

In particular, $\mathcal{V}^{(1\,|\,1)+s\omega}(M)$ becomes precisely the critical cone $\mathcal{C}^\omega(M)$ for all $\omega \in \Omega$ as soon as $s > \kappa(M) = \kappa_{(1\,|\,1)}(M)$.

Remark 2.6.2. Let $n \in \mathbb{N}$. For each left ideal L of the n^{th} Weyl algebra over K and for each $\nu \in \mathbb{N}_0^{2n}$ we put $\gamma_\nu(L) = \inf_U \sup_{u \in U \smallsetminus \{0\}} \deg^\nu(u)$, where the infimum is taken over all universal Gröbner bases U of L. By the proof of 2.4.8, (a) $\kappa_\nu(W/L) \le \gamma_\nu(L) \in \mathbb{N}_0$. Clearly, (b) $\gamma_{\nu'}(L) \le \gamma_{\nu''}(L)$ whenever $|\nu'| \le |\nu''|$. Finally, (c) $\gamma_{k\nu}(L) = k\gamma_\nu(L)$ for all $k \in \mathbb{N}_0$.

Experiment 2.6.3. Let L be any left ideal of the 1^{st} Weyl algebra W over K. By 2.4.8 we can compute an *approximation* of $\Omega/{\sim_L}$ if we know $\kappa_\nu(W/L)$ for all $\nu \in \mathbb{N}_0^2$. By the relations (a), (b), (c) of 2.6.2 we have $\kappa_\nu(W/L) \leq \gamma_\nu(L) \leq \gamma_{\|\nu\|(1\,|\,1)}(L) = \|\nu\|\gamma(L)$ for all $\nu \in \mathbb{N}_0^2$, where we put $\gamma(L) = \gamma_{(1\,|\,1)}(L)$ and $\|\nu\| = \max\{\nu_1, \nu_2\}$. Thus, by 2.4.8, knowing the upper bound $\gamma(L)$ of $\kappa(W/L)$ is sufficient for computing a (coarser) approximation of $\Omega/{\sim_L}$.

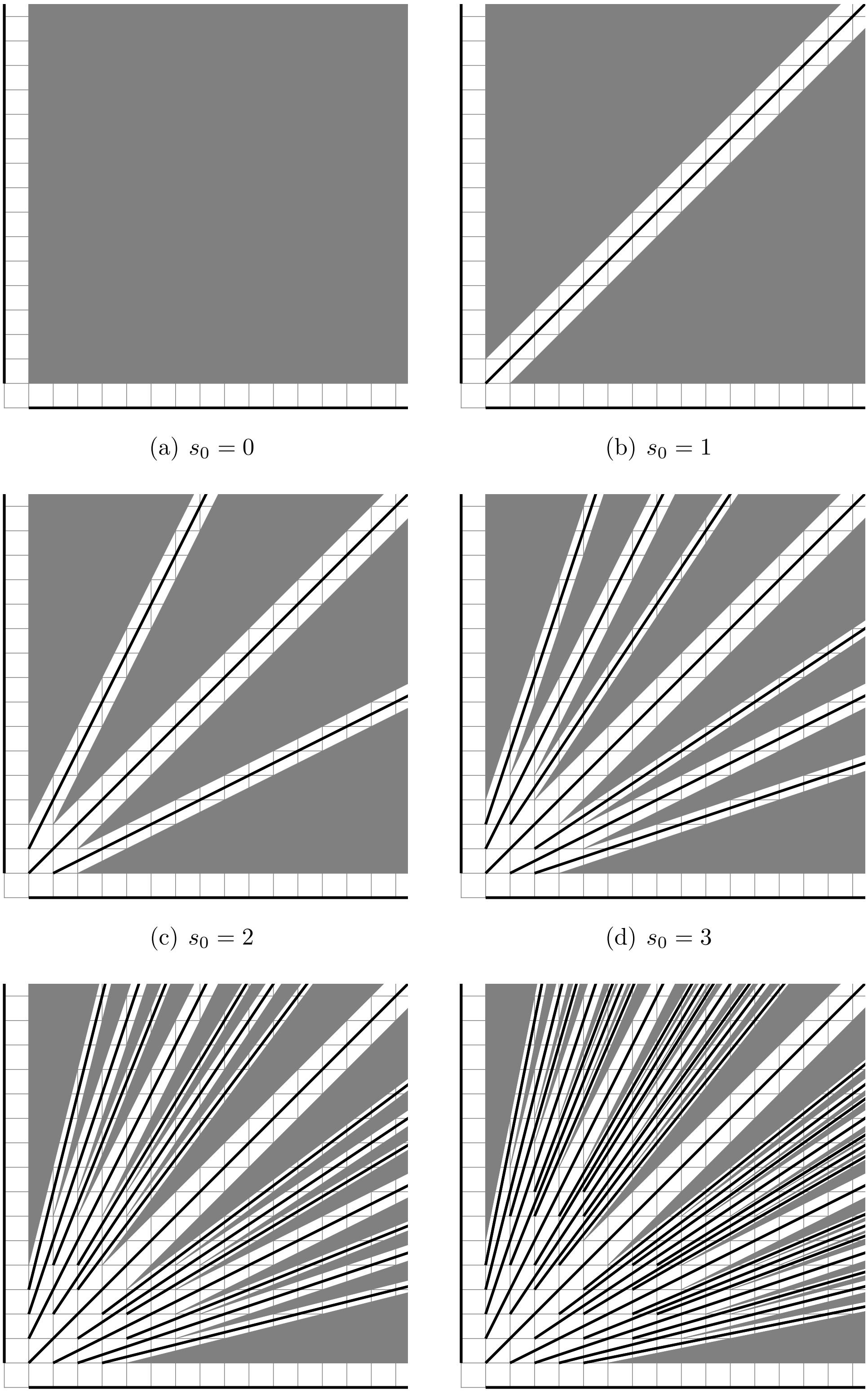
For some numbers $s_0 \in \mathbb{N}_0$ we repeatedly do an experiment parametrized by s_0 as follows. A computer calculates for us the intersection points among the half-lines $\ell_{\nu,\omega} \subseteq \Omega$ of the form $\ell_{\nu,\omega}(s) = \nu + s\omega$, $\nu \in \mathbb{N}_0^2$, $\omega \in \Omega$, for $s > s_0$, and paints incident half-lines by a common colour. The points of Ω having got the same colour turn out to build cones in Ω. For instance, for $s_0 = 3$ the computer program painted 17 differently coloured cones, among which 9 are degenerate, that is, half-lines. For typographical reasons, in Figure 2.2 we depict the so obtained cones by connected regions in $\mathbb{R}^2$, alternately in black and gray. For $s_0 = 3$ the 9 degenerate cones are filled in black, whereas the 8 non-degenerate cones are filled in gray, and similarly in the other pictures of Figure 2.2.

By 2.4.8, as soon as $s_0 \geq \gamma(L)$, each of these cones is a subset of precisely one equivalence class of $\Omega/_{\sim_L}$. Thus the results of our experiment let us conjecture an upper bound for $\chi(L)$ in terms of $\gamma(L)$, namely, $\chi(L) \leq 2^{1+\gamma(L)} + 1$.

Our experiment also indicates that the coordinates $(x_1, x_2) \in \mathbb{N}_0^2$ of the vertices of the cones lying in the lower semiquadrant without the diagonal satisfy precisely the conditions (a) $F(1) \leq x_1 \leq F(2 + s_0)$, (b) $F(0) \leq x_2 \leq F(1 + s_0)$, (c) $\gcd(x_1, x_2) = 1$, and (d) $x_1 > x_2$, where $F(s)$ is the s^{th} Fibonacci number, that is, $F(0) = 0$, $F(1) = 1$, and $F(s) = F(s-1) + F(s-2)$ for all $s \geq 2$. For instance, if $s_0 = 3$, these coordinates are $(1, 0)$, $(2, 1)$, $(3, 1)$, $(4, 1)$, $(3, 2)$, $(5, 2)$, $(4, 3)$, $(5, 3)$, as one can read from Figure 2.2.

So $2^{\gamma(L)}$ is equal to the number of the points $(x_1, x_2) \in \mathbb{N}_0^2$ satisfying the conditions (a)–(d) with $s_0 = \gamma(L)$, and the experiment indicates that $\chi(L) \leq \#\{(x_{\sigma(1)}, x_{\sigma(2)}) \in \mathbb{N}_0^2 \mid \sigma \in \Sigma_2, F(1) \leq x_1 \leq F(2 + \gamma(L)), F(0) \leq x_2 \leq F(1 + \gamma(L)), \gcd(x_1, x_2) = 1, x_1 \geq x_2\} = \#\Sigma_2 \cdot (2^{\gamma(L)} + 1) - (\#\Sigma_2 - 1) = 2^{1+\gamma(L)} + 1$, where Σ_2 is the 2^{nd} symmetric group.

Remark 2.6.4. Weyl algebras are the prototype of algebras of solvable type, see [23], and, similarly as in the case of commutative polynomial rings over a field, a universal Gröbner basis of L can be constructed as a union of reduced Gröbner bases of L.



(a) $s_0 = 0$ (b) $s_0 = 1$

(c) $s_0 = 2$ (d) $s_0 = 3$

(e) $s_0 = 4$ (f) $s_0 = 5$

Figure 2.2: Equality regions of characteristic varieties

In [2, Corollary 0.2], an upper bound is given for the total degree of elements of reduced Gröbner bases of a left ideal of an algebra of solvable type in terms of the total degree of generators of the ideal, thus in particular an upper bound for $\gamma(L)$. Therefore if our conjecture is true, one obtains an upper bound for the cardinality of $\Omega/{\sim_L}$ in terms of the total degree of generators of L.

Question 2.6.5. We may ask whether similar upper bounds for $\chi(L)$ as in 2.6.3 exist when considering a left ideal L of the n^{th} Weyl algebra for $n > 1$, namely: (1) a bound in terms of n and $\gamma(L)$, and (2) a bound in terms of Fibonacci numbers.

2.7 Localization of characteristic varieties

In this section we summarize a part of the interesting work of Škoda [38] about non-commutative localization and apply it to describe the characteristic variety of the localization at a polynomial of a left module over a Weyl algebra.

Definition 2.7.1. Let R be a ring. A subset S of R is a *multiplicative set of* R if it holds (a) $0 \notin S$, (b) $1 \in S$, and (c) $\forall\, s_1, s_2 \in S : s_1 s_2 \in S$.

We say that a multiplicative set S of R is a *left denominator set of* R if S satisfies (d) the left Ore condition: $\forall\, s \in S\ \forall\, r \in R\ \exists\, s' \in S\ \exists\, r' \in R : r's = s'r$, and (e) the left reversibility condition: $\forall\, s \in S\ \forall\, p, q \in R : (ps = qs \Rightarrow \exists\, s' \in S : s'p = s'q)$.

Reminder 2.7.2. Let R be a ring and let S be a left denominator set of R. Analogously as in the commutative case, there exists an equivalence relation $\sim$ on $S \times R$ such that $(s_1, r_1) \sim (s_2, r_2) \Leftrightarrow (\exists\, s \in S\ \exists\, r \in R : rs_1 = ss_2 \wedge rr_1 = sr_2)$. The quotient set $(S \times R)/{\sim}$ is denoted $S^{-1}R$, and the equivalence class $[(s, r)]_\sim$ of $(s, r) \in S \times R$ is written $s^{-1}r$ or r/s.

The set $S^{-1}R$ carries a ring structure with addition and multiplication respectively defined by $(r_1/s_1) + (r_2/s_2) = (ur_1 + pr_2)/(us_1)$ and $(r_1/s_1)(r_2/s_2) = (qr_2)/(vs_1)$, where $(u, p) \in S \times R$ is such that $ps_2 = us_1$ and $(v, q) \in S \times R$ is such that $qs_2 = vr_1$. The existence of such (u, p) and (v, q) is guaranteed by the left Ore condition applied respectively for s_2 and s_1 and for s_2 and r_1.

There exists a homomorphism of rings $\eta_S : R \to S^{-1}R$ given by the assignment $r \mapsto r/1$. If R is a domain, then η_S is injective, so that R can be seen as a subring of $S^{-1}R$.

Remark 2.7.3. Let $(R, \mathcal{R})$ be a filtered ring. Then the following statements are equivalent: (i) $\mathrm{G}\mathcal{R}$ is a domain, (ii) $\deg^{\mathcal{R}}(r_1 r_2) = \deg^{\mathcal{R}}(r_1) + \deg^{\mathcal{R}}(r_2)$ for all $r_1, r_2 \in R$, and (iii) $\sigma^{\mathcal{R}}(r_1 r_2) = \sigma^{\mathcal{R}}(r_1)\sigma^{\mathcal{R}}(r_2)$ for all $r_1, r_2 \in R$. If (ii) holds, then by abuse of language we say that $\deg^{\mathcal{R}}$ is *additive*.

Remark 2.7.4. Let $(R, \mathcal{R})$ be a filtered ring such that $\deg^{\mathcal{R}}$ is additive. Let S be a left denominator set of R. Then for all $(s, r) \in S \times R$ and all $(s', r') \in r/s$ one has $\deg^{\mathcal{R}}(r') - \deg^{\mathcal{R}}(s') = \deg^{\mathcal{R}}(r) - \deg^{\mathcal{R}}(s)$. Therefore there exists a filtration $\mathcal{S}^{-1}\mathcal{R}$ of $S^{-1}R$ given by $\mathrm{F}_i\mathcal{S}^{-1}\mathcal{R} = \{r/s \in S^{-1}R \mid \deg^{\mathcal{R}}(r) - \deg^{\mathcal{R}}(s) \leq i\}$, which by abuse of language we call the *localization of* $\mathcal{R}$ *at* S. Thus one has $\deg^{\mathcal{S}^{-1}\mathcal{R}}(r/s) = \deg^{\mathcal{R}}(r) - \deg^{\mathcal{R}}(s)$. Moreover $\deg^{\mathcal{S}^{-1}\mathcal{R}}$ is additive, so that $\sigma^{\mathcal{S}^{-1}\mathcal{R}}$ is multiplicative and $\mathrm{G}^{\mathcal{S}^{-1}\mathcal{R}}$ is a domain.

Now let $(M, \mathcal{M})$ be an $\mathcal{R}$-filtered left R-module. Consider the left $S^{-1}R$-module $S^{-1}R \otimes_R M$, in the following denoted by $S^{-1}M$. Similarly as above, there exists an $\mathcal{S}^{-1}\mathcal{R}$-filtration $\mathcal{S}^{-1}\mathcal{M}$ of $S^{-1}M$ given by $\mathrm{F}_i\mathcal{S}^{-1}\mathcal{M} = \{m/s \in S^{-1}M \mid \deg^{\mathcal{M}}(m) - \deg^{\mathcal{R}}(s) \leq i\}$, where we write m/s for $(1/s) \otimes m$. One has $\deg^{\mathcal{S}^{-1}\mathcal{M}}(m/s) = \deg^{\mathcal{M}}(m) - \deg^{\mathcal{R}}(s)$. Moreover, in the obvious meaning, $\deg^{\mathcal{S}^{-1}\mathcal{M}}$ is additive and $\sigma^{\mathcal{S}^{-1}\mathcal{M}}$ is multiplicative.

Remark 2.7.5. Let R be a ring provided with a filtration $\mathcal{R}$ such that $\mathrm{G}\mathcal{R}$ is a domain. Let S be a left denominator set of R. The set of homogeneous elements of the graded ring $\mathrm{G}\mathcal{S}^{-1}\mathcal{R}$ associated to $\mathrm{F}\mathcal{S}^{-1}\mathcal{R}$ is precisely $\{\sigma^{\mathcal{S}^{-1}\mathcal{R}}(r/s) \mid r/s \in S^{-1}R\}$.

Proposition 2.7.6. *Let R be a ring provided with a filtration $\mathcal{R}$ such that* $\mathrm{G}\mathcal{R}$ *is a domain. Let S be a left denominator set of R. Then* (a) $\sigma^{\mathcal{R}}(S)$ *is a left denominator set of* $\mathrm{G}\mathcal{R}$, (b) *there is an isomorphism* $\mathrm{G}\mathcal{S}^{-1}\mathcal{R} \to \sigma^{\mathcal{R}}(S)^{-1}\mathrm{G}\mathcal{R}$ *of rings given by* $\sigma^{\mathcal{S}^{-1}\mathcal{R}}(r/s) \mapsto \sigma^{\mathcal{R}}(r)/\sigma^{\mathcal{R}}(s)$*, which is graded as soon as one imposes* $\deg^{\mathcal{R}}(r) - \deg^{\mathcal{R}}(s)$ *to be the degree of* $\sigma^{\mathcal{R}}(r)/\sigma^{\mathcal{R}}(s)$*,* (c) $\mathrm{G}\mathcal{S}^{-1}\mathcal{R}$ *is a graded domain containing* $\mathrm{G}\mathcal{R}$ *as a graded subring.*

Proof. For (a) see [38, 12.3]. For (b) see [38, 12.5] and conclude by 2.7.5. Now (c) is clear by 2.7.2. □

Remark 2.7.7. Let R be a ring provided with a filtration $\mathcal{R}$ such that $\mathrm{G}\mathcal{R}$ is a domain. Let S be a left denominator set of R. Let M be a left R-module filtered by an $\mathcal{R}$-filtration $\mathcal{M}$. The set of homogeneous elements of the graded left $\mathrm{G}\mathcal{S}^{-1}\mathcal{R}$-module $\mathrm{G}\mathcal{S}^{-1}\mathcal{M}$ associated to $\mathrm{F}\mathcal{S}^{-1}\mathcal{M}$ is precisely $\{\sigma^{\mathcal{S}^{-1}\mathcal{M}}(m/s) \mid m/s \in S^{-1}M\}$.

Proposition 2.7.8. *Let R be a ring provided with a filtration $\mathcal{R}$ such that $\mathrm{G}\mathcal{R}$ is a domain. Let M be a left R-module filtered by an $\mathcal{R}$-filtration $\mathcal{M}$. Let S be a left denominator set of R. Then there is an isomorphism $\mathrm{G}\mathcal{S}^{-1}\mathcal{M} \to \sigma^{\mathcal{R}}(S)^{-1}\mathrm{G}\mathcal{M}$ of left modules over $\mathrm{G}\mathcal{S}^{-1}\mathcal{R} \cong \sigma^{\mathcal{R}}(S)^{-1}\mathrm{G}\mathcal{R}$ given by $\sigma^{\mathcal{S}^{-1}\mathcal{M}}(m/s) \mapsto \sigma^{\mathcal{M}}(m)/\sigma^{\mathcal{R}}(s)$, which is graded as soon as one imposes $\deg^{\mathcal{M}}(m) - \deg^{\mathcal{R}}(s)$ to be the degree of $\sigma^{\mathcal{M}}(m)/\sigma^{\mathcal{R}}(s)$.*

Proof. See [38, 12.8]. □

In the following let K be a field of characteristic 0, let $n \in \mathbb{N}$, let W be the n^{th} Weyl algebra over K, let M be a finitely generated left W-module, let $\omega \in \Omega$, let $f \in K[\xi] \subseteq W$, and let $T = \{f^k \mid k \in \mathbb{N}_0\}$. Then T is a left denominator set of W, and sometimes we write W_f for $T^{-1}W$ and M_f for $T^{-1}M$. By the above results we aim to study the ω-characteristic variety of M_f constructed by localizing a good ω-filtration of M. By 2.7.15 this construction is indeed independent of the chosen good ω-filtration of M.

Notation 2.7.9. We write $\mathrm{F}^\omega(W_f)$ for the localization of the ω-filtration $\mathrm{F}^\omega W$ of W, and we denote by $\mathrm{G}^\omega(W_f)$ the graded ring $\bigoplus_{i\in\mathbb{Z}} \mathrm{F}^\omega_i(W_f)/\,\mathrm{F}^\omega_{i-1}(W_f)$ associated to $\mathrm{F}^\omega(W_f)$.

Lemma 2.7.10. *There exists an isomorphism $\mathrm{G}^\omega(W_f) \cong (\mathrm{G}^\omega W)_{\sigma^\omega(f)}$ of graded rings. In particular, $\mathrm{G}^\omega(W_f)$ is a commutative.*

Proof. As $\mathrm{G}^\omega W$ is a domain, the claim immediately follows from 2.7.3 and 2.7.6. □

Notation 2.7.11. We write $\mathrm{F}^\omega(M_f)$ for the localization of an ω-filtration $\mathrm{F}^\omega M$ of M, and we denote by $\mathrm{G}^\omega(M_f)$ the graded $\mathrm{G}^\omega(W_f)$-module $\bigoplus_{i\in\mathbb{Z}} \mathrm{F}^\omega_i(M_f)/\,\mathrm{F}^\omega_{i-1}(M_f)$ associated to $\mathrm{F}^\omega(M_f)$.

Lemma 2.7.12. *There exists an isomorphism $\mathrm{G}^\omega(M_f) \cong (\mathrm{G}^\omega M)_{\sigma^\omega(f)}$ of graded modules over the commutative rings $\mathrm{G}^\omega(W_f) \cong (\mathrm{G}^\omega W)_{\sigma^\omega(f)}$.*

Proof. As $\mathrm{G}^\omega W$ is a domain, the claim immediately follows from 2.7.3 and 2.7.8. □

Reminder 2.7.13. Let C be a commutative ring, G be a finitely generated C-module, and S be a denominator set of C. Then $\sqrt{(0 :_{S^{-1}C} S^{-1}G)} = S^{-1}\sqrt{(0 :_C G)}$.

Reminder 2.7.14. Let C be a commutative domain, G be a finitely generated C-module, and S be a denominator set of C. Then $\sqrt{(0 :_C S^{-1}G)} = C \cap \sqrt{(0 :_{S^{-1}C} S^{-1}G)}$, where the ideal restriction is taken with respect to the canonical ring homomorphism $\eta_S : C \to S^{-1}C$, $c \mapsto c/1$. Notice that η_S is injective as C is a domain.

Lemma 2.7.15. *Let $\mathcal{W}$ denote the ω-filtration $\mathrm{F}^\omega W$ of W. Let $\mathcal{M}'$ and $\mathcal{M}''$ be equivalent $\mathcal{W}$-filtrations of M. Then their respective localizations $\mathcal{T}^{-1}\mathcal{M}'$ and $\mathcal{T}^{-1}\mathcal{M}''$ at T are equivalent $\mathcal{T}^{-1}\mathcal{W}$-filtrations of $T^{-1}M$.*

Proof. There exists $r \in \mathbb{N}$ such that for all $i \in \mathbb{Z}$ it holds $\mathrm{F}_{i-r}\mathcal{M}'' \subseteq \mathrm{F}_i\mathcal{M}' \subseteq \mathrm{F}_{i+r}\mathcal{M}''$. We claim that for all $i \in \mathbb{Z}$ it holds $\mathrm{F}_{i-r}\mathcal{T}^{-1}\mathcal{M}'' \subseteq \mathrm{F}_i\mathcal{T}^{-1}\mathcal{M}' \subseteq \mathrm{F}_{i+r}\mathcal{T}^{-1}\mathcal{M}''$. Indeed, let $i \in \mathbb{Z}$ and let $\overline{m} \in \mathrm{F}_i\mathcal{T}^{-1}\mathcal{M}'$. Then $\overline{m} = m/f^k$ for some $m \in M$ and $k \in \mathbb{N}_0$ such that $\deg^{\mathcal{M}'}(m) - \deg^{\mathcal{W}}(f^k) \leq i$. Because $m \in \mathrm{F}_{\deg^{\mathcal{M}'}(m)}\mathcal{M}' \subseteq \mathrm{F}_{\deg^{\mathcal{M}'}(m)+r}\mathcal{M}''$, we have $\deg^{\mathcal{M}''}(m) \leq \deg^{\mathcal{M}'}(m)+r$, thus $\deg^{\mathcal{M}''}(m)-\deg^{\mathcal{W}}(f^k) \leq \deg^{\mathcal{M}'}(m)+r-\deg^{\mathcal{W}}(f^k) \leq i+r$, whence $\overline{m} \in \mathrm{F}_{i+r}\mathcal{T}^{-1}\mathcal{M}''$. Similarly, $\mathrm{F}_{i-r}\mathcal{T}^{-1}\mathcal{M}'' \subseteq \mathrm{F}_i\mathcal{T}^{-1}\mathcal{M}'$ for all $i \in \mathbb{Z}$. □

Corollary 2.7.16. *The ideal $\sqrt{(0 :_{\mathrm{G}^\omega(W_f)} \mathrm{G}^\omega(M_f))} \cong (\sqrt{(0 :_{\mathrm{G}^\omega W} \mathrm{G}^\omega M)})_{\sigma^\omega(f)}$ is independent of the choice of a good ω-filtration of a finitely generated left W-module M.* □

Corollary 2.7.17. *The ideal $\sqrt{(0 :_{\mathrm{G}^\omega(W)} \mathrm{G}^\omega(M_f))} \cong (\mathrm{G}^\omega W) \cap (\sqrt{(0 :_{\mathrm{G}^\omega W} \mathrm{G}^\omega M)})_{\sigma^\omega(f)}$ is independent of the choice of a good ω-filtration of a finitely generated left W-module M.* □

Definition 2.7.18. By 2.7.17 we may define the *localized ω-characteristic variety $\mathcal{V}_f^\omega(M)$ of M at f* as $\mathrm{Var}(0 :_{\mathrm{G}^\omega(W)} \mathrm{G}^\omega(M_f))$, so that $\mathcal{V}_f^\omega(M)$ is independent of the choice of a good ω-filtration of M.

Remark 2.7.19. By definition, $\mathcal{V}^\omega(M_f) = \mathrm{Var}(0 :_{\mathrm{G}^\omega(W)} \mathrm{G}^\omega(M_f))$ is constructed by means of a good $\mathrm{F}^\omega W$-filtration of M_f. On the other hand, $\mathcal{V}_f^\omega(M) = \mathrm{Var}(0 :_{\mathrm{G}^\omega(W)} \mathrm{G}^\omega(M_f))$ is constructed by localizing a good $\mathrm{F}^\omega W$-filtration of M. We do not know whether these two constructions always yield the same variety. In any case, $\mathcal{V}^\omega(M_f) = \mathcal{V}_f^\omega(M)$ whenever one can choose a good $\mathrm{F}^\omega W$-filtration of M_f that is equivalent to the localization of a good ω-filtration $\mathrm{F}^\omega M$ of M.

Theorem 2.7.20. *$\mathcal{V}_f^\omega(M) = \mathcal{V}^\omega(M) \smallsetminus \mathcal{V}^\omega(W/Wf)$.*

Proof. In view of 2.7.17 it holds $\mathcal{V}_f^\omega(M) = \mathrm{Spec}(\eta_{\sigma^\omega(f)})(\mathrm{Var}((\sqrt{(0 :_{\mathrm{G}^\omega W} \mathrm{G}^\omega M)})_{\sigma^\omega(f)}))$, where $\eta_{\sigma^\omega(f)}$ is the canonical ring homomorphism $\mathrm{G}^\omega W \to (\mathrm{G}^\omega W)_{\sigma^\omega(f)}$. Therefore $\mathcal{V}_f^\omega(M) = \mathrm{Var}(0 :_{\mathrm{G}^\omega W} \mathrm{G}^\omega M) \cap \{P \in \mathrm{Spec}(\mathrm{G}^\omega W) \mid \sigma^\omega(f) \notin P\} = \mathcal{V}^\omega(M) \smallsetminus \mathrm{Var}(\mathrm{G}^\omega W \sigma^\omega(f)) = \mathcal{V}^\omega(M) \smallsetminus \mathrm{Var}(0 : \mathrm{G}^\omega W/\mathrm{G}^\omega Wf) = \mathcal{V}^\omega(M) \smallsetminus \mathrm{Var}(0 : \mathrm{G}^\omega(W/Wf)) = \mathcal{V}^\omega(M) \smallsetminus \mathcal{V}^\omega(W/Wf)$. □

Question 2.7.21. If M is *holonomic*, that is, it holds $\mathrm{GKdim}_W M = n$, then M_f is holonomic as a left W-module, that is, $\mathrm{GKdim}_W M_f = n$, see [19, Theorem 12.5.4]. To show this fact, one uses a criterion on filtrations, see [19, Lemma 10.3.1], and a particular filtration of M_f, here denoted $\mathrm{F}^{(1\,|\,1)}(M_f)$, given by $\mathrm{F}_i^{(1\,|\,1)}(M_f) = \{m/f^i \mid m \in \mathrm{F}_{(d+1)i}^{(1\,|\,1)}M\}$, where $d = \deg^{(1\,|\,1)}(f)$ and $\mathrm{F}^{(1\,|\,1)}M$ is a good filtration of M over the Bernstein filtration $\mathrm{F}^{(1\,|\,1)}W$ of W.

An interesting question is whether the radical ideal $I(M, f) = \sqrt{(0 :_{\mathrm{G}^{(1\,|\,1)}W} \mathrm{G}^{(1\,|\,1)}(M_f))}$ is independent of the choice of a good $(1\,|\,1)$-filtration of M. If this is the case, can one describe $\mathrm{Var}(I(M, f))$ in terms of the characteristic variety $\mathcal{V}^{(1\,|\,1)}(M)$ and f?

Appendix A

Division properties and universal Gröbner bases in Weyl algebras

In this appendix, as above, let n be a positive integer and K be a field of characteristic 0, let W be the n^{th} Weyl algebra over K and N be the set of normal monomials $\xi^\lambda \partial^\mu$ of W, let $K[X, Y]$ denote the commutative polynomial ring over K in the indeterminates $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ and M be the set of monomials $X^\lambda Y^\mu$ of $K[X, Y]$. Let Φ be the K-module isomorphism of W in $K[X, Y]$ that maps $\xi^\lambda \partial^\mu$ to $X^\lambda Y^\mu$, and let ϕ be the homeomorphism of $\mathrm{TO}(N)$ in $\mathrm{TO}(M)$ induced by Φ. For each subset B of any additive monoid $(A, +, 0)$ we write B^+ for $B \setminus \{0\}$.

We aim to give a slightly more direct proof than 2.3.6 of the existence of universal Gröbner bases of left ideals of Weyl algebras which involves some division properties of Weyl algebras beside the compactness of the space of all admissible orderings.

A.1 Division properties of Weyl algebras

Like commutative polynomial rings, Weyl algebras have nice division properties.

Notation A.1.1. Given any total ordering $\preceq$ of N, for all $w \in W^+$ we write $\mathrm{ls}_\preceq(w)$ for $\mathrm{lc}_\preceq(w)\,\mathrm{lm}_\preceq(w)$ and $\mathrm{LS}_\preceq(w)$ for $\mathrm{LC}_\preceq(w)\,\mathrm{LM}_\preceq(w)$.

Proposition A.1.2. *Let $w \in W$. Let $F \subseteq W$ be finite, $\preceq\, \in \mathrm{AO}(N)$, and $\leq\, = \phi(\preceq)$. Then there exist $r \in W$ and $(q_f)_{f \in F}$ with $q_f \in W$ for all $f \in F$ enjoying the following properties:* (a) $w = \sum_{f \in F} q_f f + r$, (b) $\forall f \in F : (f \neq 0 \Rightarrow \forall s \in \mathrm{Supp}(r) : \mathrm{LM}_\preceq(f) \nmid \Phi(s))$, (c) $w \neq 0 \Rightarrow \forall f \in F : (q_f f \neq 0 \Rightarrow \mathrm{LM}_\preceq(q_f f) \leq \mathrm{LM}_\preceq(w))$.

Proof. In the case when $w = 0$, we put $r = 0$ and $(q_f)_{f\in F} = (0)_{f\in F}$. Thus let $w \neq 0$. Since $M = \{\mathrm{LM}_{\preceq}(v) \mid v \in W^+\}$, we may proceed by noetherian induction in $(M, \leq)$ assuming that the statement holds for all $v \in W^+$ such that $\mathrm{LM}_{\preceq}(v) < \mathrm{LM}_{\preceq}(w)$. We distinguish between two cases: (i) if there exists $f^+ \in F^+$ such that $\mathrm{LM}_{\preceq}(f^+) \mid \mathrm{LM}_{\preceq}(w)$, then we put $w' = w - \frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)} f^+$; (ii) otherwise we set $w' = w - \mathrm{ls}_{\preceq}(w)$.

In the case (i) we obtain $\mathrm{LS}_{\preceq}(\frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)} f^+) = \mathrm{LS}_{\preceq}(\frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)})\,\mathrm{LS}_{\preceq}(f^+) = \frac{\mathrm{LS}_{\preceq}(w)}{\mathrm{LS}_{\preceq}(f^+)}\,\mathrm{LS}_{\preceq}(f^+) = \mathrm{LS}_{\preceq}(w)$ by 2.3.4(b). So, provided that $w' \neq 0$, by 2.3.4(a) we have $\mathrm{LM}_{\preceq}(w') < \mathrm{LM}_{\preceq}(w)$. This last relation clearly holds also in the case (ii) when $w' \neq 0$. Either by the induction hypothesis or by the preliminarily treated case when $w' = 0$, we find $r' \in W$ and $(q'_f)_{f\in F}$ with all $q'_f \in W$ such that properties (a), (b), (c) hold for w' with respect to r' and $(q'_f)_{f\in F}$. In the case (i) we put $r = r'$ and assign $q_{f^+} = q'_{f^+} + \frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)}$ and $q_f = q'_f$ for all $f \in F \smallsetminus \{f^+\}$. In the case (ii) we set $r = r' + \mathrm{ls}_{\preceq}(w)$ and $q_f = q'_f$ for all $f \in F$.

We now verify that in either case properties (a), (b), (c) are fulfilled by r and $(q_f)_{f\in F}$. Property (a) is clearly satisfied. As for property (b), we may assume that $w' \neq 0$. In the case (i) we have $\mathrm{Supp}(r) = \mathrm{Supp}(r')$, so that the statement holds by the induction hypothesis. In the case (ii) we have $\mathrm{Supp}(r) \subseteq \mathrm{Supp}(r') \cup \{\mathrm{lm}_{\preceq}(w)\}$, thus (b) holds by the induction hypothesis and by our assumption that $\mathrm{LM}_{\preceq}(f) \nmid \mathrm{LM}_{\preceq}(w)$ for all $f \in F^+$.

Let us consider property (c). In the case (i), when $w' = 0$, then we have $q_f = 0$ for all $f \in F \smallsetminus \{f^+\}$ and $q_{f^+} = \frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)}$, so that $q_{f^+} f^+ = w$ and hence $\mathrm{LM}_{\preceq}(q_{f^+} f^+) = \mathrm{LM}_{\preceq}(w)$. When $w' \neq 0$, by the induction hypothesis and by what we have said above, for all $f \in F \smallsetminus \{f^+\}$ with $q_f f \neq 0$ we obtain $\mathrm{LM}_{\preceq}(q_f f) = \mathrm{LM}_{\preceq}(q'_f f) \leq \mathrm{LM}_{\preceq}(w') < \mathrm{LM}_{\preceq}(w)$, whereas as for f^+, whenever $q_{f^+} f^+ \neq 0$, using in addition 2.3.4 we get $\mathrm{LM}_{\preceq}(q_{f^+} f^+) \leq \max_{\leq}\{\mathrm{LM}_{\preceq}(q'_{f^+} f^+), \mathrm{LM}_{\preceq}(\frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)} f^+)\} \leq \max_{\leq}\{\mathrm{LM}_{\preceq}(w'), \mathrm{LM}_{\preceq}(w)\} = \mathrm{LM}_{\preceq}(w)$ if $q'_{f^+} f^+ \neq 0$, and similarly $\mathrm{LM}_{\preceq}(q_{f^+} f^+) = \mathrm{LM}_{\preceq}(\frac{\mathrm{ls}_{\preceq}(w)}{\mathrm{ls}_{\preceq}(f^+)} f^+) = \mathrm{LM}_{\preceq}(w)$ if $q'_{f^+} f^+ = 0$.

In the case (ii), when $w' = 0$, then $q_f = 0$ for all $f \in F$, so that (c) holds trivially. When $w' \neq 0$, then by induction we have $\mathrm{LM}_{\preceq}(q_f f) = \mathrm{LM}_{\preceq}(q'_f f) \leq \mathrm{LM}_{\preceq}(w') < \mathrm{LM}_{\preceq}(w)$ whenever $q_f f \neq 0$. ☐

A.2 Universal Gröbner bases in Weyl algebras

The division properties A.1.2 of a Weyl algebra imply a sort of stability property of its Gröbner bases, see A.2.1, which in turn allows to construct a particular covering of the space of all admissible orderings on its normal monomials, see A.2.2 and A.2.3, whence the existence of universal Gröbner bases follow, see A.2.4.

A very similar topological approach as here to proving the existence of universal Gröbner bases was pioneered by Becker [6], [7], [8] to show the existence of universal standard bases in power series rings over a field, extended by Aschenbrenner [1] to power series rings over an arbitrary commutative ring, and used by Sikora [37] in the context of commutative polynomial rings over a field.

Proposition A.2.1. *Let L be an ideal of W, let $\preceq$ be an admissible ordering of W, and let B be a Gröbner basis of L with respect to $\preceq$. Let $\preceq'$ be an admissible ordering of W such that $\preceq'\restriction_{\mathrm{Supp}(B)} = \preceq\restriction_{\mathrm{Supp}(B)}$. Then B is a Gröbner basis of L with respect to $\preceq'$.*

Proof. Let $\leq\, = \phi(\preceq)$ and $\leq'\, = \phi(\preceq')$ be the induced monomial orderings of $K[X,Y]$. Let $x \in L^+$. In view of A.1.2, we can write $x = \sum_{b\in B} q_b b + r$ for some $r \in W$ and some $(q_b)_{b\in B} \in W^{\oplus B}$ enjoying the properties: (i) $\mathrm{LM}_{\preceq'}(q_b b) \leq' \mathrm{LM}_{\preceq'}(x)$ whenever $q_b b \neq 0$, and (ii) $\mathrm{LM}_{\preceq'}(b) \nmid \Phi(s)$ for all $s \in \mathrm{Supp}(r)$ whenever $b \neq 0$.

Clearly, $r \in L$. Suppose that $r \neq 0$. Then $\mathrm{LM}_{\preceq}(r) \in \mathrm{LM}_{\preceq}(L)$, thus the monomial $\mathrm{LM}_{\preceq}(r)$ lies in the monomial ideal $\langle \mathrm{LM}_{\preceq}(b) \mid b \in B^+\rangle$ of $K[X,Y]$. Hence there exists $b \in B^+$ such that $\mathrm{LM}_{\preceq}(b) \mid \mathrm{LM}_{\preceq}(r)$, see [20, Lemma 2.4.2]. Since $\preceq$ and $\preceq'$ agree on $\mathrm{Supp}(B)$, we have $\mathrm{LM}_{\preceq}(b) = \mathrm{LM}_{\preceq'}(b)$, and it follows $\mathrm{LM}_{\preceq'}(b) \mid \mathrm{LM}_{\preceq}(r) \in \Phi(\mathrm{Supp}(r))$, in contradiction to (ii).

Hence $r = 0$. So, by (i), $\Phi(x) = \sum_{b\in B} \Phi(q_b b)$ with $\mathrm{LM}_{\leq'}(\Phi(q_b b)) \leq' \mathrm{LM}_{\leq'}(\Phi(x))$ whenever $q_b b \neq 0$. Thus there exists $b' \in B$ with $q_{b'} b' \neq 0$ such that $\mathrm{LM}_{\leq'}(\Phi(x)) = \mathrm{LM}_{\leq'}(\Phi(q_{b'} b'))$, that is, $\mathrm{LM}_{\preceq'}(x) = \mathrm{LM}_{\preceq'}(q_{b'} b')$. We get $\mathrm{LM}_{\preceq'}(x) = \mathrm{LM}_{\preceq'}(q_{b'})\, \mathrm{LM}_{\preceq'}(b')$ by 2.3.4(b), and so $\mathrm{LM}_{\preceq'}(x) \in \langle \mathrm{LM}_{\preceq'}(b')\rangle$.

We have shown that $\mathrm{LM}_{\preceq'}(L) = \langle \mathrm{LM}_{\preceq'}(b) \mid b \in B^+\rangle$. Since clearly $L = \sum_{b\in B} Wb$, we conclude that B is a Gröbner basis of L with respect to $\preceq'$. □

Lemma A.2.2. *Let L be a left ideal of W and F be a finite subset of L. Then the set $\mathfrak{V}_L(F)$ of all admissible orderings $\preceq$ of W such that F is a Gröbner basis of L with respect to $\preceq$ is open in* $\mathrm{AO}(N)$.

Proof. Without restriction we assume that $\mathfrak{V}_L(F) \neq \varnothing$. Let $\preceq\, \in \mathfrak{V}_L(F)$. So F is a Gröbner basis of L with respect to $\preceq$. Let $(S_i)_{i\in\mathbb{N}_0}$ be a filtration of N consisting of finite sets

S_i. We find $r \in \mathbb{N}_0$ such that the finite subset $\mathrm{Supp}(F)$ of N lies in S_{r+1}. In the notation of 1.1.2, consider the open neighbourhood $\mathfrak{N}_r(\preceq) \cap \mathrm{AO}(N)$ of $\preceq$ in $\mathrm{AO}(N)$ and let $\preceq' \in \mathfrak{N}_r(\preceq) \cap \mathrm{AO}(N)$. So $\preceq'$ and $\preceq$ agree on S_{r+1} and in particular on $\mathrm{Supp}(F)$. From A.2.1 it follows that F is a Gröbner basis of L with respect to $\preceq'$, thus $\preceq' \in \mathfrak{V}_L(F)$. Therefore $\mathfrak{N}_r(\preceq) \cap \mathrm{AO}(N) \subseteq \mathfrak{V}_L(F)$, and hence $\mathfrak{V}_L(F)$ is open in $\mathrm{AO}(N)$. □

Remark A.2.3. Let L be a left ideal of W. For each $\preceq \in \mathrm{AO}(N)$ we can choose a Gröbner basis $B_\preceq$ of L with respect to $\preceq$ by 2.3.2. Of course $\preceq \in \mathfrak{V}_L(B_\preceq)$. Hence $(\mathfrak{V}_L(B_\preceq))_{\preceq \in \mathrm{AO}(N)}$ is an open covering of $\mathrm{AO}(N)$ by A.2.2.

Theorem A.2.4. *Each left ideal L of W admits a universal Gröbner basis.*

Proof. By A.2.3 we can choose an open covering $(\mathfrak{V}_L(B_\preceq))_{\preceq \in \mathrm{AO}(N)}$ of $\mathrm{AO}(N)$ where each $B_\preceq$ is a Gröbner basis of L with respect to $\preceq$. Since $\mathrm{AO}(N)$ is compact, see 1.6.8, we find a finite subcovering $(\mathfrak{V}_L(B_{\preceq_k}))_{1\le k\le t}$ with $t \in \mathbb{N}$. We claim that $V = \bigcup_{1\le k\le t} B_{\preceq_k}$ is a universal Gröbner basis of L. Indeed, let $\preceq_0 \in \mathrm{AO}(N)$. As $\mathrm{AO}(N) = \bigcup_{1\le k\le t} \mathfrak{V}_L(B_{\preceq_k})$, we have $\preceq_0 \in \mathfrak{V}_L(B_{\preceq_k})$ for some $k \in \{1, \dots, t\}$. Thus $B_{\preceq_k}$ is a Gröbner basis of L with respect to $\preceq_0$. It follows that V is a Gröbner basis of L with respect to $\preceq_0$. As the choice of $\preceq_0$ in $\mathrm{AO}(N)$ was arbitrary, we conclude that V is a universal Gröbner basis of L. □

Appendix B

Computing equivalence regions of characteristic varieties

In this appendix we present the C++ computer program that we created to draw Figure 2.2 in Section 2.6. Given any left ideal L of the 1^{st} Weyl algebra W, sometimes we speak of the *equivalence region* of $\mathrm{G}^{\omega}L$ meaning the equivalence class of $\omega \in \Omega$ with respect to $\sim_L$.

```
#define max(a,b) ((a) > (b) ? (a) : (b))
#define min(a,b) ((a) < (b) ? (a) : (b))
#define congruent(a,b,m) ((m) == 0 ? (a) == (b) : ((a) - (b)) % (m) == 0)
```

$\max(a, b)$ — Returns the maximum of two values a and b.

$\min(a, b)$ — Returns the minimum of two values a and b.

$\mathrm{congruent}(a, b, m)$ — Answers whether the integers a and b are congruent modulo m.

```
#define M 64
#define N 64
#define S_min 0
#define S_max 9
```

M, N — The region considered for computing is $(\{0, \ldots, M-1\} \times \{0, \ldots, N-1\}) \cap \Omega$.

$S_{\min}, S_{\max}$ — The equivalence regions are computed for all $s_0 \in \{S_{\min}, \ldots, S_{\max}\}$.

```
color graph[M][N];
color line_color[M][N][M][N];
```

$\mathrm{graph}[p][q]$ — Entry of an $M \times N$-matrix which records the equivalence class (color) of the weight $(p\,|\,q) \in \Omega$ with respect to $\sim_L$ for a fixed s_0. The matrix $\mathrm{graph}[M][N]$ is actually the output of the program.

line_color[x][y][u][v] — The equivalence class of the weights $\omega \in \ell_{(x\,|\,y),(u\,|\,v)} \subseteq \Omega$. The matrix line_color[$M$][$N$][$M$][$N$] is an auxiliary data structure.

```
void clear_all()
{
  for (int p = 0; p < M; ++p)
    for (int q = 0; q < N; ++q)
      graph[p][q] = -1;
  for (int x = 0; x < M; ++x)
    for (int y = 0; y < N; ++y)
      for (int u = 0; u < M; ++u)
        for (int v = 0; v < N; ++v)
          line_color[x][y][u][v] = -1;
}
```

clear_all() — Initializes graph and line_color to 'undefined color'. These data structures are recycled as s_0 varies.

```
int gcd(int a, int b)
{
  return b == 0 ? a : gcd(b, a % b);
}

bool normalized(int u, int v)
{
  return gcd(u, v) == 1;
}
```

gcd(a, b) — Computes the greatest common divisor of the integers a and b.

normalized(u, v) — Answers whether the direction vector $(u\,|\,v) \in \Omega$ is normalized, that is, its components u and v are coprime.

```
bool intersect(int x, int y, int u, int v, int  s, int xx, int yy, int uu, int vv, int ss)
{
  bool f = false;
  int d = v * uu - u * vv;
  if (d != 0)
  {
    int  t = (yy - y) * uu - (xx - x) * vv;
    int tt = u * (yy - y) - v * (xx - x);
    f = congruent(t, 0, d) && congruent(tt, 0, d) && (t/d > s) && (tt/d > ss);
  }
  else if (uu == 0)
    f = (xx == x) && congruent(yy, y, vv);
  else if (vv == 0)
    f = (yy == y) && congruent(xx, x, uu);
  else
    f = congruent(xx, x, uu) && congruent(yy, y, vv) && (uu * (yy - y) == (xx - x) *vv);
  return f;
}
```

intersect($x, y, u, v, s, x', y', u', v', s'$) — Answers whether the half-lines $\ell_{(x\,|\,y),(u\,|\,v)}(t)\restriction_{t>s} = \{(x+tu, y+tv) \mid t > s\}$ and $\ell_{(x'\,|\,y'),(u'\,|\,v')}(t')\restriction_{t'>s'} = \{(x'+t'u', y'+t'v') \mid t' > s'\}$ have nonempty intersection in Ω.

```
void trace_line(int x, int y, int u, int v, int s, color c)
{
  line_color[x][y][u][v] = c;
  for (int t = s + 1; x + t * u < M && y + t * v < N; ++t)
    graph[x+t*u][y+t*v] = c;
}
```

trace_line(x, y, u, v, s, c) — The half-line $\ell = \ell_{(x\,|\,y),(u\,|\,v)}(t)\restriction_{t>s} = \{(x+tu, y+tv) \mid t > s\}$ is drawn with the color c, thus assigning all weights $\omega \in \ell$ to the equivalence class c.

```
color choose_color(int s0, color c, int x, int y, int u, int v)
{
  int s = max(x, y) * s0;
  for (int uu = 0; uu <= u; ++uu)
    for (int vv = 0; vv <= (uu == u ? v : N-1); ++vv)
    {
      if (uu == 0 && vv == 0)
        continue;
      if (!normalized(uu, vv))
        continue;
      for (int xx = 0; xx <= (uu == u && vv == v ? x : M-1); ++xx)
        for (int yy = 0; yy <= (uu == u && vv == v && xx == x ? y : N-1); ++yy)
        {
          if (x == xx && y == yy && u == uu && v == vv)
            continue;
          int ss = max(xx, yy) * s0;
          bool f = intersect(x, y, u, v, s, xx, yy, uu, vv, ss);
          if (f)
          {
            color cc = line_color[xx][yy][uu][vv];
            return cc;
          }
        }
    }
  return c;
}
```

choose_color(s_0, c, x, y, u, v) — For a new half-line $\ell = \{(x+tu, y+tv) \mid t > s\} \subseteq \Omega$, $s = s_0\|(x\,|\,y)\| = s_0 \max\{x, y\}$, a new color is chosen. The routine checks whether ℓ intersects any previously drawn half-line $\ell' = \{(x'+t'u', y'+t'v') \mid t' > s'\} \subseteq \Omega$, $s' = s_0\|(x'\,|\,y')\| = s_0 \max\{x', y'\}$. In this case ℓ inherits the color c' of ℓ' because the weights $\omega \in \ell$ lie in the same equivalence class as any $\omega' \in \ell'$. Otherwise ℓ gets a new color c, and therefore the weights $\omega \in \ell$ are (at least temporarily) put in a new equivalence class. Notice how the considered half-lines are ordered by a total ordering $\prec$ implicitly defined by the four nested for-loops. This ordering $\prec$ is necessarily the only one used in the program.

```
void diffuse_color(int s0, int c, int x, int y, int u, int v)
{
  int s = max(x, y) * s0;
  trace_line(x, y, u, v, s, c);
  for (int uu = 0; uu <= u; ++uu)
```

```
    for (int vv = 0; vv <= (uu == u ? v : N-1); ++vv)
    {
      if (uu == 0 && vv == 0)
        continue;
      if (!normalized(uu, vv))
        continue;
      for (int xx = 0; xx <= (uu == u && vv == v ? x : M-1); ++xx)
        for (int yy = 0; yy <= (uu == u && vv == v && xx == x ? y : N-1); ++yy)
        {
          if (x == xx && y == yy && u == uu && v == vv)
            continue;
          int ss = max(xx, yy) * s0;
          bool f = intersect(x, y, u, v, s, xx, yy, uu, vv, ss);
          if (f)
          {
            color cc = line_color[xx][yy][uu][vv];
            assert(c <= cc);
            if (c != cc)
              diffuse_color(s0, c, xx, yy, uu, vv);
          }
        }
    }
}
```

diffuse_color(s_0, c, x, y, u, v) — The half-line $\ell = \{(x+tu, y+tv) \mid t > s\} \subseteq \Omega$, where $s = s_0\|(x \,|\, y)\| = s_0 \max\{x, y\}$, is colored by the color c. The half-lines $\ell' \prec \ell$ such that $\ell' \cap \ell \neq \varnothing$ recursively get and diffuse the same color c as ℓ.

```
void fill_graph(int s0)
{
  color nc = 0;
  for (int u = 0; u < M; ++u)
    for (int v = 0; v < N; ++v)
    {
      if (u == 0 && v == 0)
        continue;
      if (!normalized(u, v))
        continue;
      for (int x = 0; x < M; ++x)
        for (int y = 0; y < N; ++y)
        {
          int s = max(x, y) * s0;
          color c = choose_color(s0, nc, x, y, u, v);
          assert(c <= nc);
          diffuse_color(s0, c, x, y, u, v);
          if (nc <= c)
            nc = c + 1;
        }
    }
}
```

fill_graph(s_0) — The equivalence classes of $\Omega/\!\sim_L$ are computed by tracing the half-lines $\ell = \{(x+tu, y+tv) \mid t > s\} \subseteq \Omega$, where $s = s_0\|(x \,|\, y)\| = s_0 \max\{x, y\}$ and $(x, y) \in \{0, \ldots, M-1\} \times \{0, \ldots, N-1\}$ and $(u, v) \in (\{0, \ldots, M-1\} \times \{0, \ldots, N-1\}) \cap \Omega$. The half-lines ℓ are ordered by a total ordering $\prec$ implicitly defined by the four nested for-loops. Following this ordering, a color c for each half-line ℓ is then chosen: if there exists a half-line ℓ' such that $\ell' \prec \ell$ and $\ell' \cap \ell \neq \varnothing$, then ℓ inherits the color c' of the least such ℓ' with respect to $\prec$, otherwise ℓ gets a new color c. The color of ℓ is then diffused to all ℓ' such that $\ell' \prec \ell$ and $\ell' \cap \ell \neq \varnothing$. Since colors $c \in \mathbb{N}_0$ are progressively supplied in the canonical ordering of $\mathbb{N}_0$, this guarantees that for all half-lines ℓ, ℓ' and all colors c, c' such that c is the color of ℓ and c' is the color of ℓ' one has: $c' < c \Leftrightarrow \ell' \prec \ell \wedge \ell' \cap \ell = \varnothing$.

```
int main()
{
  for (int s0 = S_min; s0 <= S_max; ++s0)
  {
    clear_all();
    fill_graph(s0);
    print_ps_code(s0);
  }
  return 0;
}
```

main() — For some elements $s_0 \in \mathbb{N}_0$ the main routine initializes the data structures, fills the graph and prints the output in PSTricks code. We omit here the listing of the straightforward routine print_ps_code(s_0).

Notation

A short list of most of the used symbols:

$\mathrm{AO}(M)$, $\mathrm{AO}(N)$	Space of admissible orderings, a.k.a. monoid orderings, on the set M or N.
$\mathcal{C}^\omega(M)$	Critical cone of $\mathcal{V}^\omega(M)$.
$\mathrm{CO}(M)$, $\mathrm{CO}(N)$	Space of compatible orderings, a.k.a. semigroup orderings, on the set M or N.
$\mathrm{DCO}(M)$, $\mathrm{DCO}(N)$	Space of degree-compatible orderings on the set M or N.
$\mathrm{DO}(M)$, $\mathrm{DO}(N)$	Space of degree orderings on the set M or N.
$\mathrm{FO}_a(S)$	Space of a-founded orderings on the set S with $a \in S$.
$\mathrm{F}^\omega M$	Filtration of a left W-module M with respect to a weight ω.
$\mathrm{G}^\omega M$	Graded module associated to an ω-filtered left W-module M.
$\mathrm{G}\mathcal{M}$	Graded module associated to a filtration $\mathcal{M}$ of a left module M.
GKdim	Gelfand–Kirillov dimension.
Kdim	Krull dimension.
K	Field, when specified of characteristic 0.
$K[X]$	Commutative ring $K[X_1, \ldots, X_t]$ of polynomials over K.
$\mathrm{len}_R\, M$	Length of an R-module M.
$\mathrm{LM}_\preceq(H)$	Leading monomial ideal of H with respect to $\preceq$.
$\mathscr{lm}_\mathfrak{T}(H)$	Set $\{\mathrm{LM}_\preceq(H) \mid \preceq \in \mathfrak{T}\}$ of leading monomial ideals of H from $\mathfrak{T}$.
$\mathscr{min}_\mathfrak{T}(H)$	Set $\{\mathrm{LM}_\preceq(H) \mid \preceq \in \min_H(\mathfrak{T})\}$ of minimal leading monomial ideals of H from $\mathfrak{T}$.
$\min_H(\mathfrak{T})$	Set $\{\preceq \in \mathfrak{T} \mid \mathrm{LM}_\preceq(H)$ is $\subseteq$-minimal in $\mathscr{lm}_\mathfrak{T}(H)\}$.
$\mathcal{M}$	Filtration of a module M.
$\mathcal{M}/\mathcal{N}$	Canonical filtration of a quotient module M/N with respect to a filtration $\mathcal{M}$ of M.
M	Canonical basis of monomials of $K[X]$, or a module over a ring.
N	Canonical basis $\Phi^{-1}(M)$ of V or W, or a module over a ring.
$\mathbb{N}$	Set $\{1, 2, 3, \ldots\}$ of strictly positive integers.
$\mathbb{N}_0$	Set $\{0, 1, 2, \ldots\}$ of nonnegative integers.
$\mathrm{TO}(S)$	Space of total orderings on the set S.
V	K-module isomorphic to $K[X]$ through Φ, or a generating space of a K-algebra.

$\mathcal{V}^{\omega}(M)$	Characteristic variety of a left W-module M with respect to a weight $\omega \in \Omega$.
W	Weyl algebra over K, or a generating space of a module over a K-algebra.
$\mathrm{WO}(S)$	Space of total well-orderings on the set S.
$\mathbb{Z}$	Set of integers.
$\chi(L)$	Number of distinct ω-characteristic varieties $\mathcal{V}^{\omega}(W/L)$ of a left ideal L of W with ω varying in Ω.
ϕ	Homeomorphism of $\mathrm{TO}(N)$ in $\mathrm{TO}(M)$ induced by Φ.
Φ	K-module isomorphism of V in $K[X]$.
Ω	Natural polynomial region of W.

Rings are in general noncommutative and we always explicitly indicate when we are in the commutative case.

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