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THREE CIRCLES THEOREMS FOR SCHRÖDINGER OPERATORS ON CYLINDRICAL ENDS AND GEOMETRIC APPLICATIONS

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Abstract. We show that for a Schrödinger operator with bounded potential on a manifold with cylindrical ends the space of solutions which grows at most exponentially at infinity is finite dimensional and, for a dense set of potentials (or, equivalently for a surface, for a fixed potential and a dense set of metrics), the constant function zero is the only solution that vanishes at infinity. Clearly, for general potentials there can be many solutions that vanish at infinity.

These results follow from a three circles inequality (or log convexity inequality) for the Sobolev norm of a solution u to a Schrödinger equation on a product $N \times [0,T]$, where N is a closed manifold with a certain spectral gap. Examples of such N's are all (round) spheres $\mathbf{S}^n$ for $n \geq 1$ and all Zoll surfaces.

Finally, we discuss some examples arising in geometry of such manifolds and Schrödinger operators.

0. Introduction

Many problems in Geometric Analysis are about the space of solutions of non-linear PDE's, like solutions of the Yang-Mills equation, the Einstein equation, the Yamabe equation, the harmonic map equation, the minimal surface equation, etc. For such problems it is often of interest to estimate "how many" solutions there are and be able to say something about their properties. Infinitesimally, the space of nearby solutions to a given solution solve a linear PDE, which is often a Schrödinger equation. For this reason it is therefore very useful when one can say that the space of solutions (with some constraints at infinity) to a Schrödinger equation is finite dimensional and even more significant when one can say that the trivial solution, that is, the function that is identically zero, is the only such solution. The first case corresponds to that the "tangent space" is finite dimensional and the second case corresponds to that the space of solutions is infinitesimally rigid. We will return to some specific examples later in the introduction after stating our main results.

Let M be a complete non-compact $(n+1)$-dimensional Riemannian manifold with finitely many ends $E_1, \dots, E_k$. Suppose also that $M \setminus \bigcup_{j=1}^k E_j$ has compact closure and each end is cylindrical. By cylindrical we will mean different things depending on whether $n = 1$, in which case more general ends will be allowed, or $n \geq 2$. For $n \geq 2$ we assume that each end E_i is isometric to a product of a closed manifold N_i and a half-line $[0, \infty)$, whereas, for $n = 1$ we assume only that each end is bi-Lipschitz to $\mathbf{S}^1 \times [0, \infty)$ and has bounded geometry. Recall that a surface (or manifold) has bounded geometry if its sectional curvature is bounded above and below and the injectivity radius is bounded away from zero.

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We will consider Schrödinger operators $L = \Delta_M + V$ on the manifold M and on each cylindrical end use coordinates (θ, t). Given a constant α, let $H_\alpha(M) = H_\alpha(M, L)$ be the linear space of all solutions u of $Lu = 0$ that grow slower than $\exp(\alpha r)$, where r is the distance to a fixed point. That is, for any fixed point p

$$\limsup_{r \to \infty} \max_{\partial \mathcal{B}_r(p)} \mathrm{e}^{-\alpha\, r} |u| = 0 \,, \tag{0.1}$$

Where $\mathcal{B}_r(p)$ is the intrinsic ball of radius r and center p. Note that $H_0(M)$ is the set of solutions that vanish at infinity.

One of our main results is the next theorem about the solutions of Schrödinger operators on manifolds with cylindrical ends, where the cross-section of each end has a (infinite) sequence of eigenvalues λ_{m_i} for the Laplacian with

$$\lambda_{m_i} - \lambda_{m_i - 1} \to \infty \,. \tag{0.2}$$

This last condition on the spectral gaps is satisfied on any round sphere $\mathbf{S}^n$ for $n \geq 1$. On $\mathbf{S}^n$, the eigenvalues occur with multiplicity in clusters with the m-th cluster at $m^2 + (n-1)\, m$. The spectral gap condition is also satisfied on any Zoll surface (normalized so the closed geodesics have length 2π). The eigenvalues of a Zoll surface occur in clusters, where the eigenvalues in the m-th cluster all lie in the interval

$$J_m = \left[(m + \beta/4)^2 - K, (m + \beta/4)^2 + K \right] \tag{0.3}$$

for constants K and β; see Guillemin, [Gu], and Colin de Verdière, [Cv]. Notice that the gap between J_m and J_{m+1} grows linearly in m, as did the spectral gaps for $\mathbf{S}^n$, thus giving the required spectral gap.[1]

Theorem 0.4. Let M be a complete non-compact $(n+1)$-dimensional manifold with finitely many cylindrical ends satisfying (0.2).

(1) If V is a $C^{0,1}$ bounded[2] function[3] (potential) on M, then $H_\alpha(M, \Delta_M + V)$ is finite dimensional for every α; the bound for $\dim H_\alpha$ depends only on M, α, and $||V||_{C^{0,1}}$.
(2) For a dense set of $C^{0,1}$ bounded potentials H_0 contains only the constant function zero; for a surface this is equivalent to that, for a fixed potential, there is a dense set of metrics (with finitely many cylindrical ends) where $H_0 = \{0\}$.

For easy reference, we state also this theorem in the special case of surfaces.

Theorem 0.5. Let M be a complete non-compact surface with finitely many cylindrical ends.

(1) If V is a bounded function (potential) on M, then $H_\alpha(M, \Delta_M + V)$ is finite dimensional for every α; the bound for $\dim H_\alpha$ depends only on M, α, and $||V||_{L^\infty}$.

[1] Weyl's asymptotic formula gives for a general closed n-dimensional manifold that $\lambda_m \approx m^{n/2}$, so (0.2) does not hold in general for $n \geq 2$.

[2] A function f is in $C^{0,1}$ if it is both bounded and Lipschitz. The $C^{0,1}$ norm is

$$||f||_{C^{0,1}} = \sup_M |f| + \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{|x - y|} \,.$$

[3] We will prove that both parts (1) and (2) of the theorem also hold for bounded potentials V, whenever the cross-section of each end is a round $\mathbf{S}^n$, $n \geq 1$, or a Zoll surface.

(2) For a dense set of bounded potentials, H_0 contains only the constant function zero; or equivalently, for a fixed potential, there is a dense set of metrics (with finitely many cylindrical ends) where $H_0 = \{0\}$.

Even the special case of our theorem where $M = \mathbf{S}^1 \times \mathbf{R}$ is a flat cylinder is of interest. In that case we can define spaces H_+ and H_- of solutions to the Schrödinger equation where H_+ are the solutions that vanish at $+\infty$ and H_- the space that vanishes at $-\infty$ and thus H_0 is the interesection of the two. In this case both H_+ and H_- can be infinite dimensional, as can be seen when $V \equiv 0$ by considering separation of variable solutions:

$$\{\mathrm{e}^{kt}\, \cos(k\theta) \text{ and } \mathrm{e}^{kt}\, \sin(k\theta) \,|\, k \in \mathbf{Z}, k < 0\} \subset H_+\,, \tag{0.6}$$

$$\{\mathrm{e}^{kt}\, \cos(k\theta) \text{ and } \mathrm{e}^{kt}\, \sin(k\theta) \,|\, k \in \mathbf{Z}, k > 0\} \subset H_-\,. \tag{0.7}$$

In particular, one can easily construct (non-generic) compactly supported potentials V on the flat cylinder $\mathbf{S}^1 \times \mathbf{R}$ where H_0 is non-trivial by patching together exponentially decaying solutions on each end.

One of the key ingredients in the proof of Theorem 0.4 is a three circles inequality (or log convexity inequality) for the Sobolev norm of a solution u to a Schrödinger equation on a product $N \times [0, T]$, where N satisfies (0.2). We will state the first version of the three circles theorem next when N is a sphere or a Zoll surface and the dependence of the constants is cleanest; see Theorem 3.36 below for the statement for a general N satisfying (0.2).

Theorem 0.8. Let $N = \mathbf{S}^n$ for any $n \geq 1$ or a Zoll surface. There exists a constant $C > 0$ depending on N and $||V||_{C^{0,1}}$ so that if u is a solution to the Schrödinger equation $\Delta u + V\, u = 0$ on $N \times [0, T]$ and α satisfies

$$\alpha \geq \frac{1}{T} \left[\log \frac{I(T)}{I(0)} \right] \,, \tag{0.9}$$

then u's $W^{1,2}$ norm at $0 < t < T$ satisfies the following three circles type inequality (logarithmic convexity type inequality)

$$\log I(t) \leq C + (C + |\alpha|)\, t + \log I(0)\,. \tag{0.10}$$

Here

$$I(s) = \int_{N \times \{s\}} \left(u^2 + |\nabla u|^2 \right) \, d\theta\,. \tag{0.11}$$

Our argument actually gives a stronger bound than we record in Theorem 3.36, but we have tailored the statement to fit our geometric applications.

Even if the potential is merely bounded, and not Lipschitz, we get the following estimate:

Theorem 0.12. Let $N = \mathbf{S}^n$ for any $n \geq 1$ or a Zoll surface. There exists a constant $C > 0$ depending on N and $||V||_{L^\infty}$ so that if u is a solution to the Schrödinger equation $\Delta u + V\, u = 0$ on $N \times [0, T]$ and α satisfies

$$\alpha \geq \frac{1}{T} \left[\log \frac{\int_{N \times \{T\}} u^2}{\int_{N \times \{0\}} u^2} \right] \,, \tag{0.13}$$

then u's L^2 norm at $0 < t < T$ satisfies

$$\log \left(\int_{N \times \{t\}} u^2 \, d\theta \right) \leq C + (C + |\alpha|)\, t + \log I(0) \, . \tag{0.14}$$

One of the main reasons why such estimates are useful is that it shows that if a solution grows/decays initially with at least a certain rate (the constant C in (0.10) and (0.14) gives a threshold), then it will keep growing/decaying indefinitely.

As an immediate corollary of the general version of Theorem 0.8 where N is only assumed to satisfy (0.2), i.e., Theorem 3.36, (and Schauder estimates) we get the following:

Corollary 0.15. Let N be a closed n-dimensional manifold satisfying (0.2). Given $\alpha \in \mathbf{R}$, there exists a constant $\nu > 0$ depending on α, the $C^{0,1}$ norm of V, and N so that if $u \in H_\alpha(N \times [0, \infty))$, then its $W^{1,2}$ norm grows at most exponentially with the estimate

$$\int_{N \times \{t\}} \left(u^2 + |\nabla u|^2 \right) \, d\theta \leq \nu \, \mathrm{e}^{\nu \, t} \int_{N \times \{0\}} \left(u^2 + |\nabla u|^2 \right) \, d\theta \, . \tag{0.16}$$

Remark 0.17. The corollary also holds for bounded potentials V whenever N is an n-dimensional sphere or a Zoll surface; in this case, we apply Theorem 0.12.

0.1. **Examples from geometry.** Let $\Sigma \subset M^3$ be a smooth surface (possibly with boundary) in a complete Riemannian 3-manifold M and with orientable normal bundle. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{ x + \exp_x(t \, \phi(x) \, \mathbf{n}_\Sigma(x)) | x \in \Sigma \} \, . \tag{0.18}$$

Here $\mathbf{n}_\Sigma$ is the unit normal to Σ and exp is the exponential map on M.[4] The so-called first variation formula of area is the equation (integration is with respect to the area of Σ)

$$\left. \frac{d}{dt} \right|_{t=0} \mathrm{Area}(\Sigma_{t,\phi}) = \int_\Sigma \phi \, H \, , \tag{0.19}$$

where the *mean curvature* H of Σ is the sum of the principal curvatures κ_1, κ_2.[5] The surface Σ is said to be a *minimal* surface (or just minimal) if

$$\left. \frac{d}{dt} \right|_{t=0} \mathrm{Area}(\Sigma_{t,\phi}) = 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma) \tag{0.20}$$

or, equivalently by (0.19), if the mean curvature H is identically zero. Thus Σ is minimal if and only if it is a critical point for the area functional.

Since a critical point is not necessarily a minimum the term "minimal" is misleading, but it is time-honored. A computation shows that if Σ is minimal, then

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \mathrm{Area}(\Sigma_{t,\phi}) = - \int_\Sigma \phi \, L_\Sigma \phi \, , \tag{0.21}$$

where $L_\Sigma \phi = \Delta_\Sigma \phi + |A|^2 \phi + \mathrm{Ric}_M(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma) \phi$ is the second variational (or Jacobi) operator. Here Δ_Σ is the Laplacian on Σ, $\mathrm{Ric}_M(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma)$ is the Ricci curvature of M in the direction

[4]For instance, if $M = \mathbf{R}^3$, then $\exp_x(v) = x + v$.

[5]When Σ is non-compact, $\Sigma_{t,\phi}$ in (0.19) is replaced by $\Gamma_{t,\phi}$, where Γ is any compact set containing the support of ϕ.

of the unit normal to Σ, and A is the second fundamental form of Σ. So A is the covariant derivative of the unit normal of Σ and $|A|^2 = \kappa_1^2 + \kappa_2^2$.

For us, the key is that *the second variational operator is a Schrödinger operator* with potential $V = |A|^2 + \text{Ric}(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma)$.

A useful example to keep in mind is that of the catenoid. The catenoid is the complete embedded minimal surface in $\mathbf{R}^3$ that is given by conformally embedding the flat 2-dimensional cylinder into $\mathbf{R}^3$ by

$$(\theta, t) \to (-\cosh t \, \sin \theta, \cosh t \cos \theta, t) \, . \tag{0.22}$$

A calculation shows that pulling back the second variational operator to the flat cylinder gives a rotationally symmetric Schrödinger operator with potential

$$V(\theta, t) = V(t) = 2 \, \cosh^{-2}(t) \, . \tag{0.23}$$

Similarly, each of the singly-periodic minimal surfaces known as the Riemann examples is conformal to a flat cylinder with a periodic set of punctures. Pulling back the second variational operator to the flat cylinder gives a Schrödinger operator with bounded potential.

A minimal surface Σ is said to be stable if

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \text{Area}(\Sigma_{t,\phi}) \geq 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma) \, . \tag{0.24}$$

The *Morse index* of Σ is the index of the critical point Σ for the area functional, that is, the number of negative eigenvalues (counted with multiplicity) of the second derivative of area; i.e., the number of negative eigenvalues of L.[6] Thus Σ is *stable* if the index is zero. If $\lambda = 0$, then ϕ is said to be a Jacobi field.

Suppose that M^3 is a fixed closed 3-manifold with a bumpy[7] metric with positive scalar curvature and let Σ_i be a sequence without repeats, i.e., with $\Sigma_i \neq \Sigma_j$ for $i \neq j$, of embedded minimal surfaces of a given fixed genus. After possibly passing to a subsequence one expects that it converges to a singular lamination[8] that looks like one of the two illustrated below:

> One expects that any singular limit lamination has only finitely many leaves. Each closed leaf is a strictly stable 2-sphere. Each non-compact leaf has only finitely many ends and each end accumulates around exactly one of the closed leaves. The accumulation looks almost exactly as in either Figure 1 or Figure 2.

Indeed the lamination in Figure 1 can happen as a limit of fixed genus embedded minimal surfaces in a 3-manifold, see [CD] (even in a 3-manifold with positive scalar curvature); cf. also with B. White, [W2] and M. Calle and D. Lee, [CaL].

For us, the key is that (see Section 1):

Each non-compact leaf is conformally a Riemann surface with finitely many cylindrical ends,

and under the conformal change,

the second variational operator becomes a Schrödinger operator with Lipschitz bounded potential.

[6]By convention, an eigenfunction ϕ with eigenvalue λ of L is a solution of $L\,\phi + \lambda\,\phi = 0$.

[7]Bumpy means that no closed minimal surface Σ has 0 as an eigenvalue of L_Σ and the space of such metrics is of Baire category by a result of B. White, [W1].

[8]A lamination is a foliation except for that it is not assumed to foliate the entire manifold.

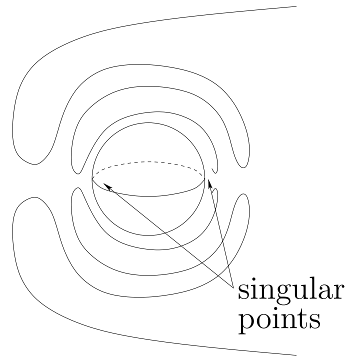

Figure 1. One of the two possible singular laminations in half of a neighborhood of a strictly stable 2-sphere. There are two leaves. Namely, the strictly stable 2-sphere and half of a cylinder. The cylinder accumulates towards the 2-sphere through catenoid type necks. In fact, the lamination has two singular points over which the necks accumulate.

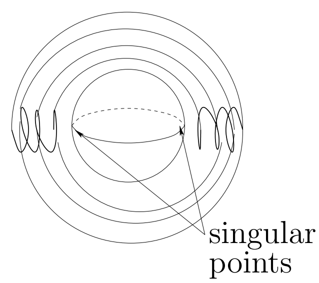

Figure 2. One of the two possible singular laminations in half of a neighborhood of a strictly stable 2-sphere. There are two leaves. Namely, the strictly stable 2-sphere and half of a cylinder. The cylinder accumulates towards the 2-sphere and is obtained by gluing together two oppositely oriented double spiral staircases. Each double spiral staircase winds tighter and tighter as it approaches the 2-sphere and, thus, never actually reaches the 2-sphere.

One would like to understand the moduli space of such non-compact minimal surfaces. Infinitesimally, the space of nearby non-compact minimal surfaces with finitely many ends, each as Figure 1 or 2, are solutions of the second variational equation on the initial surface. Thus, we are led to analyze the solutions of this Schrödinger equation.

0.2. **Schrödinger operators on $\mathbf{R}^{n+1}$.** Theorem 0.12 implies a three circles inequality, and a corresponding strong unique continuation theorem, for a Euclidean operator

$$L = \Delta_{\mathbf{R}^{n+1}} - (n-1)\,|x|^{-1}\,\partial_{|x|} + V(x)\,, \tag{0.25}$$

where $\partial_{|x|}$ is the radial derivative and the potential $V(x)$ satisfies

$$|V(x)| \leq C\,|x|^{-2}\,. \tag{0.26}$$

This unique continuation does not follow from the well-known sharp result for potentials $V \in L^{\frac{n+1}{2}}(\mathbf{R}^{n+1})$ of Jerison and Kenig, [JK]. It also does not follow from the unique continuation result of Garofalo and Lin, [GL], which holds when $|x|^2 \, |V(x)|$ goes to zero at a definite rate. To our knowledge, the sharpest unique continuation results for Euclidean operators of this general form are given in Pan and Wolff, [PW]. In that paper, they consider operators $\Delta_{\mathbf{R}^{n+1}} + W(x) \cdot \nabla_{\mathbf{R}^{n+1}} + V(x)$, where V satisfies (0.26) for some constant and W satisfies $|x| \, |W(x)| \leq C_0$ for a fixed small constant C_0.

To see why Theorem 0.12 applies to the operator L, it will be convenient to work in "exponential polar coordinates" ($\theta = x/|x|$, $t = \log |x|) \in \mathbf{S}^n \times \mathbf{R}$. In these coordinates, the chain rule gives

$$\partial_{|x|} = \mathrm{e}^{-t} \, \partial_t \, , \tag{0.27}$$

$$\partial_{|x|}^2 = \mathrm{e}^{-2t} \, \left(\partial_t^2 - \partial_t \right) \, . \tag{0.28}$$

Using this, we can rewrite the Euclidean Laplacian $\Delta_{\mathbf{R}^{n+1}}$ as

$$\Delta_{\mathbf{R}^{n+1}} = \partial_{|x|}^2 + \frac{n}{|x|} \, \partial_{|x|} + |x|^{-2} \, \Delta_{\mathbf{S}^n} = \mathrm{e}^{-2t} \, \Delta_{\mathbf{S}^n \times \mathbf{R}} + \mathrm{e}^{-2t} \, (n-1) \, \partial_t \, . \tag{0.29}$$

Therefore, the Euclidean operator L can be written

$$\mathrm{e}^{2t} \, L = \Delta_{\mathbf{S}^n \times \mathbf{R}} + \mathrm{e}^{2t} \, V(\mathrm{e}^t \, \theta) \, . \tag{0.30}$$

In particular, if V satisfies (0.26), then the operator $\mathrm{e}^{2t} \, L$ can be written as $\Delta_{\mathbf{S}^n \times \mathbf{R}} + \tilde{V}$, where the potential $\tilde{V}$ is bounded. It follows that Theorem 0.12 applies to an operator L satisfying (0.26).

0.3. **Outline of the paper.** In Section 2, on the half-cylinder $N \times [0, \infty)$ with coordinates (θ, t), we introduce notation for the Fourier coefficients (or spectral projections) of a function $f(\theta, t)$ on each cross-section $t =$ constant.

(In Appendix A, we specialize to the case of a cylinder $N \times \mathbf{R}$ and a rotationally symmetric potential $V(\theta, t) = V(t)$. This is meant only to explain some of the ideas in a simple case and the results will not be used elsewhere. Given a solution u of the Schrödinger equation, an easy calculation shows that the Fourier coefficients of u satisfy an ODE as a function of t. It follows from a Riccati comparison argument, that any sufficiently high Fourier coefficient of u grows exponentially at either plus infinity or minus infinity. In particular, if the solution u vanishes at both plus and minus infinity, then all sufficiently high Fourier coefficients vanish. It follows from this that the space H_0 is finite dimensional. Similarly for H_α when $\alpha > 0$.)

In Section 3, we prove the three circles theorem for Lipschitz potentials, i.e., Theorem 0.8. Unlike the case of rotationally symmetric potentials, the individual Fourier coefficients will no longer satisfy a useful ODE, but we will still be able to show that the simultaneous projection of a solution u onto all sufficiently large Fourier eigenspaces satisfies a useful differential inequality. To give a feel for the proof, we will now outline the argument. For each $t \in [0, T]$, let $[u]_j(t)$ be the j-th Fourier coefficient of a solution u restricted to the t-th slice. Define functions of t by $\mathcal{L}_m = \sum_{j=0}^{m-1} \left[([u]_j')^2 + (1 + \lambda_j) [u]_j^2 \right]$ and $\mathcal{H}_m = \sum_{j=m}^{\infty} \left[([u]_j')^2 + (1 + \lambda_j) [u]_j^2 \right]$ and note that the sum of the two is the Sobolev norm. A computation shows that they satisfy the two differential inequalities: $\mathcal{H}_m'' \geq (4\lambda_m - C) \, \mathcal{H}_m - C \mathcal{L}_m$ and $\mathcal{L}_m'' \leq (4\lambda_{m-1} + C) \, \mathcal{L}_m + C \mathcal{H}_m$, for some constant C depending only on the Lipschitz norm of the potential and in particular not on m. Subtracting the second inequality from the first and using the spectral gap yields that $[\mathcal{H}_m - \mathcal{L}_m]'' \geq (4\lambda_{m-1} + 2\kappa) \, [\mathcal{H}_m - \mathcal{L}_m]$ for some positive constant κ and m sufficiently large. We then use this differential inequality and the maximum principle applied to the function $f(t) = \mathrm{e}^{-\alpha t}[\mathcal{H}_m - \mathcal{L}_m]$, where α is the logarithmic growth rate of the Sobolev norm from $t = 0$ to $t = T$, to conclude that $\mathcal{H}_m(t)$ is bounded in terms of $\mathrm{e}^{\alpha t}\, I(0) + \mathcal{L}_m(t)$. Inserting this back into the first order differential inequality that $\mathcal{L}_m$ satisfies easily gives a bound for $\mathcal{L}_m(t)$ (and hence for $\mathcal{H}_m(t)$ and $I(t)$) in terms of $\mathrm{e}^{\alpha\, t}\, I(0)$. Unravelling it all yields the desired three circles inequality, i.e., Theorem 0.8. In Section 4, we prove a three circles inequality when the potential V is bounded, i.e., Theorem 0.12.

Using the results of Section 4, we will show in Section 5 that the space H_α is finite dimensional on a manifold with finitely many ends, each of which is isometric to a half-cylinder. In Section 6, we show that the space H_0 is zero dimensional for a dense set of potentials. Subsection 6.1 gives an example where the set of potentials with $H_0 = \{0\}$ is not open.

In Section 7, we prove a uniformization theorem that allows us to reduce the general case of surfaces with cylindrical ends to the case where the ends are isometric to flat half-cylinders. Together with the results of Sections 4, 5, and 6, this proves the main theorem.

1. Examples from geometry

In this section, we will show that for each non-compact leaf of the singular minimal lamination constructed in [CD] (see Figure 1) our main results, Theorem 0.4 and Theorem 0.8, apply. Namely, we show the following proposition.

Proposition 1.1. Each non-compact leaf of the singular minimal lamination constructed in [CD] is conformally a Riemann surface with finitely many cylindrical ends and, after this conformal change, the second variational operator becomes a Schrödinger operator with bounded potential. In fact, the conformal change of metric that we give below will directly make each end isometric to a flat half-cylinder.

Let M^3 be a closed 3-manifold with a Riemannian metric g and $\mathcal{L}$ a minimal lamination consisting of finitely many leaves, as constructed in [CD]. Each compact leaf is a strictly stable 2-sphere and each non-compact leaf has only finitely many ends, each end, a half infinite cylinder spiralling into one of the strictly stable 2-spheres as in Figure 1. To prove the proposition, it is enough to show that we can conformally change the metric on each end Σ to make it a flat cylinder and then show that, in this conformally changed metric, the second variational operator becomes a Schrödinger operator with bounded potential.

In this example, we can parametrize a neighborhood of the strictly stable 2-sphere by $\mathbf{S}^2 \times (-\varepsilon, \varepsilon)$ and on $\mathbf{S}^2$ use spherical coordinates (ϕ, θ); $r \in (-\varepsilon, \varepsilon)$ denotes the (signed) distance to the strictly stable 2-sphere. In these coordinates the metric g takes the form

$$dr^2 + \mu^2(r)(d\phi^2 + \sin^2 \phi\, d\theta^2) \tag{1.2}$$

(see equation (2) in [CD]). Moreover, μ is a smooth function with $\mu(0) = 1$, $\mu'(0) = 0$ and $\mu'' > 0$.

The minimal half-cylinder Σ is $\mathbf{S}^1$-invariant, i.e., it is the preimage of a curve γ_∞ on the strip $[0, \pi] \times (-\varepsilon, \varepsilon)$ under the projection map

$$(\phi, \theta, r) \;\mapsto\; (\phi, r)\,. \tag{1.3}$$

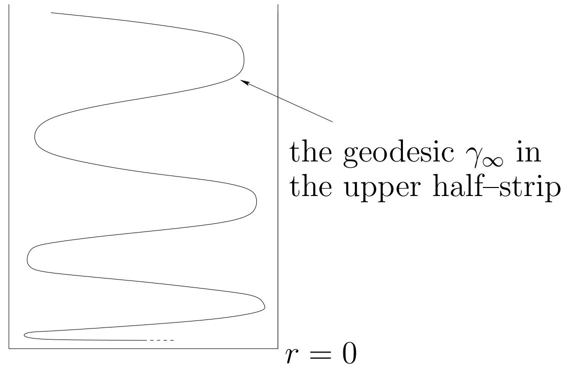

FIGURE 3. The projection of the half-infinite cylinder Σ in M is an infinite geodesic γ_∞ in the upper half-strip with the degenerate metric (1.5).

As first remarked by Hsiang and Lawson in [HsLa] (cf. with section 2 of [CD]), since Σ is a critical point for the area functional, γ_∞ is a critical point for the functional

$$F(\gamma_\infty) = \int_{\gamma_\infty} \mathrm{length}(\mathbf{S}^1 \times \{\gamma_\infty(t)\}) = \int_{\gamma_\infty} 2\pi\, \mu(r(t))\, \sin(\phi(t))\,. \tag{1.4}$$

Therefore, γ_∞ is an infinite geodesic for the degenerate metric

$$\mu^2(r) \sin^2\phi\, (dr^2 + \mu^2(r) d\phi^2)\,, \tag{1.5}$$

accumulating towards the geodesic segment $\{r = 0\}$; see Figure 3.

If we assume that $t \mapsto (\phi(t), r(t))$ is the parameterization of γ_∞ by arc-length ($t > 0$) in the degenerate metric (1.5), then

$$\left(\frac{dr}{dt}\right)^2 + \mu^2(r) \left(\frac{d\phi}{dt}\right)^2 \;=\; \mu^{-2}(r) \sin^{-2}\phi\,. \tag{1.6}$$

Therefore, if we parameterize Σ by $(t, \theta) \mapsto (\phi(t), r(t), \theta)$, the induced metric on Σ is

$$\begin{aligned} d\sigma^2 &= \left[\left(\frac{dr}{dt}\right)^2 + \mu^2(r)\left(\frac{d\phi}{dt}\right)^2\right] dt^2 + \mu^2(r)\sin^2\phi\, d\theta^2 \\ &= \mu^{-2}(r)\sin^{-2}\phi\, dt^2 + \mu^2(r)\sin^2\phi\, d\theta^2\,. \end{aligned} \tag{1.7}$$

Let τ be a new parameterization of γ_∞, so that

$$\frac{dt}{d\tau} = \mu^2(r(t))\sin^2(\phi(t))\,. \tag{1.8}$$

It follows that in the coordinates (τ, θ) the metric on Σ takes the form

$$\mu^2(r(\tau))\sin^2\phi(\tau)\big(d\tau^2 + d\theta^2\big)\,, \tag{1.9}$$

i.e., (τ, θ) is a conformal parameterization with conformal factor $h = \mu(r)\sin\phi$.

To complete the proof of Proposition 1.1, it only remains to show that the second variational operator $L = \Delta_{d\sigma^2} + (|A|^2 + \mathrm{Ric}_M(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma))$ on Σ has the same kernel as a Schrödinger operator $\tilde{L}$ with bounded potential in the conformally changed metric $ds^2 = h^{-2}d\sigma^2$. We will do this in the next lemma for the operator $\tilde{L} = h^2\, L$.

Lemma 1.10. In the conformally changed metric $ds^2 = h^{-2} d\sigma^2$ (i.e., the flat metric on the half cylinder), the operator $\tilde{L} = h^2 L$ is a Schrödinger operator with bounded potential.

Proof. Since $ds^2 = h^{-2} d\sigma^2$, we have $\Delta_{ds^2} = h^2 \Delta_{d\sigma^2}$ and, therefore, $\tilde{L} = \Delta_{ds^2} + h^2(|A|^2 + \mathrm{Ric}_M(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma))$ is a Schrödinger operator in the metric ds^2; cf. (7.10). Since both $\mathrm{Ric}_M(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma)$ and h are bounded, to prove the proposition, it suffices to show that $h^2|A|^2$ is bounded.

In what follows, we will denote by $\dot{r}$ and $\dot{\phi}$ the derivatives $\frac{dr}{d\tau}$ and $\frac{d\phi}{d\tau}$, respectively. According to (1.6) and (1.8) we have

$$(\dot{r})^2 + \mu^2(r)(\dot{\phi})^2 \; = \; \mu^2(r) \sin^2 \phi = h^2 \,. \tag{1.11}$$

Set

$$A_{\theta\theta} \; = \; -g\left(\mathbf{n}_\Sigma, \nabla_{\partial_\theta} \partial_\theta\right) , \qquad A_{\tau\theta} \; = \; -g\left(\mathbf{n}_\Sigma, \nabla_{\partial_\tau} \partial_\theta\right) , \qquad A_{\tau\tau} \; = \; -g\left(\mathbf{n}_\Sigma, \nabla_{\partial_\tau} \partial_\tau\right) . \tag{1.12}$$

By minimality, $A_{\theta\theta} = -A_{\tau\tau}$, and hence

$$h^2|A|^2 \; = \; h^2 \left[h^{-4}(A_{\theta\theta}^2 + A_{\tau\tau}^2 + 2A_{\tau\theta}^2)\right] \; = \; 2\, h^{-2} \left[A_{\theta\theta}^2 + A_{\tau\theta}^2\right] . \tag{1.13}$$

It can be readily checked that the normal $\mathbf{n} = \mathbf{n}_\Sigma$ is given by

$$\begin{aligned} \mathbf{n} \; &= \; (\dot{r}\, \partial_\phi - \mu^2(r)\, \dot{\phi}\, \partial_r)/[(\mu^2(r)\, (\dot{r})^2 + \mu^4(r)\, (\dot{\phi})^2)^{1/2}] \\ &= \; \mu^{-2}(r)\, \sin^{-1}\, \phi\, (\dot{r}\, \partial_\phi - \mu^2(r)\, \dot{\phi}\, \partial_r) \,. \end{aligned} \tag{1.14}$$

Moreover, since also ∂_τ lies in the linear span of ∂_r and ∂_ϕ and the level sets θ equal constant are totally geodesic in the metric g, it follows easily that $A_{\tau\theta} = 0$. Finally,

$$\begin{aligned} A_{\theta\theta} \; &= \; -\mu^{-2}(r) \sin^{-1} \phi \left[\dot{r} g\left(\partial_\phi, \nabla_{\partial_\theta} \partial_\theta\right) - \mu^2(r) \dot{\phi} g\left(\partial_r, \nabla_{\partial_\theta} \partial_\theta\right)\right] \\ &= \; -\mu^{-2}(r) \sin^{-1} \phi \left[-\frac{\dot{r}}{2} \partial_\phi (g(\partial_\theta, \partial_\theta)) + \frac{\mu^2(r)\dot{\phi}}{2} \partial_r (g(\partial_\theta, \partial_\theta))\right] \\ &= \; -\mu^{-2}(r) \sin^{-1} \phi \left(-\mu^2(r) \dot{r} \sin\phi \cos\phi + \mu^3(r) \dot{\phi} \mu'(r) \sin^2 \phi\right) \\ &= \; \dot{r} \cos\phi - \mu(r)\mu'(r)\dot{\phi} \sin\phi \,, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} h^4|A|^2 \; &= \; 2(\dot{r}\cos\phi - \mu(r)\mu'(r)\dot{\phi}\sin\phi)^2 \leq 4\left[(\dot{r})^2 \cos^2\phi + h^2(\mu'(r))^2(\dot{\phi})^2\right] \\ &\leq 4h^2 \left[1 + (\mu'(r))^2\right] , \end{aligned} \tag{1.16}$$

where the last inequality follows from (1.11). The desired bound on $h^2|A|^2$ now follows. □

Next, consider the Jacobi fields generated by sequences of spiralling cylinders $\{\Sigma_n\}$ of the form above. Then these Jacobi fields grow at most exponentially in τ.

Definition 1.17. We let M be the Riemannian manifold $\mathbf{S}^2 \times (-\varepsilon, \varepsilon)$ with the metric g of (1.2). Any isometry Φ of the standard $\mathbf{S}^2$ can be extended to an isometry of M in an obvious way, i.e. by mapping $(z, r) \in \mathbf{S}^2 \times (-\varepsilon, \varepsilon)$ to $(\Phi(z), r)$. We denote by G the set of such isometries. Finally, we denote by $\mathcal{S}$ the set of minimal $\mathbf{S}^1$–invariant cylinders spiraling into $\mathbf{S}^2 \times \{0\}$. That is, Γ is an element of $\mathcal{S}$ if and only if there exist a minimal cylinder Σ and a $\Phi \in G$ such that $\Gamma = \Phi(\Sigma)$ and Σ is the lifting of a curve γ_∞ under the projection map (1.3).

Loosely speaking, the set of Jacobi fields generated by sequences of elements of $\mathcal{S}$ gives the tangent space to $\mathcal{S}$. More precisely, let $\{\Sigma_k\}$ be a sequence of elements of $\mathcal{S}$ that converges to $\Sigma \in \mathcal{S}$. Consider a sequence of increasing compact domains $\Omega_0 \subset \Omega_1 \subset \ldots \subset \Sigma$ exhausting Σ. For each i we select ε_i sufficiently small and we consider the portion T_i of the ε_i–tubular neighborhood of Σ which is "lying above" Ω_i, that is

$$T_i \;=\; \left\{x + \exp_x(s\mathbf{n}_\Sigma(x)) \,|\, x \in \Omega_i, s \in (-\varepsilon_i, \varepsilon_i)\right\}. \tag{1.18}$$

Let i be given. By the standard regularity theory for minimal surfaces, for k large enough $\Sigma_k \cap T_i$ is a graph over Ω_i, i.e.

$$\Sigma_k \cap T_i \;=\; \left\{x + \exp_x(u_k(x)\mathbf{n}_\Sigma(x)) \,|\, x \in \Omega_i\right\} \tag{1.19}$$

for some smooth function u_k.

We normalize u_k to $f_k = u_k/\|u_k\|_{L^2(\Omega_0)}$. Then, a subsequence converges to a nontrivial smooth function f on Σ solving $\tilde{L}f = 0$, where $\tilde{L}$ is the operator of Lemma 1.10. We denote by $T_\Sigma\mathcal{S}$ the space of functions cf, where f is generated with the procedure above and c is a real number.

Lemma 1.20. There exists a constant α such that the following holds. Consider any $\Sigma \in \mathcal{S}$ with the rescaled flat metric ds^2 as in Lemma 1.10. Then $T_\Sigma\mathcal{S} \subset H_\alpha(\Sigma)$ for some $\alpha \geq 0$.

Proof. Without loss of generality we can assume that Σ is the lifting of a curve γ_∞ through the projection (1.3). Therefore, we use on Σ the coordinates (θ, τ) introduced in Lemma 1.10.

Let $\mathcal{G}$ be the Lie Algebra generating G and define the linear space $V = \{g(X, \mathbf{n}_\Sigma) | X \in \mathcal{G}\}$. Clearly, V is a space of bounded smooth functions on Σ. Moreover, V gives the Jacobi fields generated by minimal surfaces of the form $\{\Phi_n(\Sigma)\}$ for sequences $\{\Phi_n\} \subset G$ converging to the identity. Therefore, any element $f \in T_\Sigma\mathcal{S}$ can be written as $v + w$, where v belongs to V and w is a function of $T_\Sigma\mathcal{S}$ independent of the variable θ. We sketch a proof of this fact for the reader's convenience. Let f be a nontrivial element of $T_\Sigma\mathcal{S}$ generated by a sequence of $\mathbf{S}^1$–invariant minimal cylinders Σ_k as above. Then $\Sigma_k = \Phi_k(\Gamma_k)$, where

- $\{\Phi_k\}$ is a sequence of isometries converging to the identity;
- Γ_k are liftings of curves γ_k through the projection (1.3).

Let i be a given natural number. For k sufficiently large, $\Sigma_k \cap T_i$ has the form

$$\Sigma_k \cap T_i \;=\; \left\{x + \exp_x(u_k(x)\mathbf{n}_\Sigma(x)) \,|\, x \in \Omega_i\right\} \tag{1.21}$$

and $u_k/\|u_k\|_{L^2(\Omega_0)}$ converges to f.

On the other hand, by the standard theory of minimal surfaces, the Hausdorff distance between $\Gamma_k \cap T_i$ and $\Sigma \cap T_i$ and $\Phi_k(\Gamma_k) \cap T_i$ and $\Gamma_k \cap T_i$ converge to 0. Hence, for k sufficiently large, $\Gamma_k \cap T_i$ is a graph over Ω_i and $\Phi(\Gamma_k) \cap T_i$ is a graph over Γ_k. Thus we can find functions v_k and w_k such that

$$T_i \cap \Gamma_k \;=\; \left\{x + \exp_x(w_k \mathbf{n}_\Sigma(x)) \,|\, x \in \Omega_i\right\} \tag{1.22}$$

$$T_i \cap \Phi_k(\Gamma_k) \;=\; \left\{x + \exp_x(w_k\mathbf{n}_\Sigma(x)) + \exp_{x+\exp_x(w_k\mathbf{n}_\Sigma(x))}(v_k\mathbf{n}_{\Gamma_k}(x))\right\}. \tag{1.23}$$

Note that w_k is a function independent of θ. Moreover, up to subsequences we can assume that $w_k/\|w_k\|_{L^2(\Omega_0)}$ converges to a function w. Such a w belongs to $T_\Sigma\mathcal{S}$ and depends only on the variable τ. Finally, up to subsequences, we can assume that $v_k/\|v_k\|_{L^2(\Omega_0)}$ converges to an element v of V.

By the theory of minimal surfaces, the Hausdorff distances between $\Gamma_k \cap T_i$ and $\Sigma \cap T_i$ and $\Phi_k(\Gamma_k) \cap T_i$ and $\Gamma_k \cap T_i$ are controlled by $\|u_k\|_{L^2(\Omega_0)}$. Moreover, $u_k = w_k + v_k + o(\|u_k\|_{L^2(\Omega_0)})$. Since f is the limit of $u_k/\|u_k\|_{L^2(\Omega_0)}$, f must be a linear combination of v and w.

Having shown the desired decomposition for any element of $T_\Sigma \mathcal{S}$, since V is a space of bounded functions, it suffices to show the existence of $\alpha \geq 0$ such that every function $f \in T_\Sigma \mathcal{S}$ independent of θ belongs to $H_\alpha(\Sigma)$. For any such f we have, by Lemma 1.10, $f''(\tau) = -V(\tau) f(\tau)$. Since V is bounded, this gives the inequality

$$|f''| \;\leq\; \|V\|_\infty |f| \;=\; a|f|\,. \tag{1.24}$$

Consider the nonnegative locally Lipschitz function $g(\tau) = |f'(\tau)| + |f(\tau)|$ and set $\alpha = \max\{a, 1\}$. Then

$$g' \;\leq\; |f''| + |f'| \leq a|f| + |f'| \leq \alpha g\,. \tag{1.25}$$

Hence, from Gronwall's inequality, we get $|f(\tau)| \leq g(\tau) \leq g(0)e^{\alpha\tau}$ for $\tau \geq 0$, which is the desired bound. □

2. The spectral projection on a closed manifold

Suppose now that N^n is an n-dimensional closed Riemannian manifold and Δ_N is the Laplacian on N. We will generally use θ as a parameter on N. Fix an $L^2(N)$-orthonormal basis of Δ_N eigenfunctions $\phi_0, \phi_1, \dots$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots$, so that

$$\Delta_N \phi_j = -\lambda_j\, \phi_j\,. \tag{2.1}$$

Given an arbitrary L^2 function f on N, we will let $[f]_j$ denote the inner product of f with ϕ_j

$$[f]_j = \int_N f(\theta)\, \phi_j(\theta)\, d\theta\,. \tag{2.2}$$

In analogy to the special case where $N = \mathbf{S}^1$ (see below), we will often refer to this as the j-th Fourier coefficient, or j-th spectral projection. It follows that

$$f(\theta) = \sum_{j=0}^{\infty} [f]_j\, \phi_j(\theta)\,. \tag{2.3}$$

It will often be important to understand how the Fourier coefficients of a function $f(\theta, t)$ on the half-cylinder $N \times [0, \infty)$ vary as a function of t. To keep track of these coefficients, we define $[f]_j(t)$ by

$$[f]_j(t) = \int_N f(\theta, t)\, \phi_j(\theta)\, d\theta\,. \tag{2.4}$$

2.1. **The Fourier coefficients on a half-cylinder.** The simplest example of spectral projection is when N is the unit circle $\mathbf{S}^1$ with the standard orthonormal basis of eigenfunctions

$$\phi_0 = \frac{1}{\sqrt{2\pi}} \, , \, \left\{ \phi_{2k+1} = \frac{1}{\sqrt{\pi}} \, \sin(k\theta) \right\}_{k \geq 0} \, , \, \left\{ \phi_{2k} = \frac{1}{\sqrt{\pi}} \, \cos(k\theta) \right\}_{k \geq 1} \, , \tag{2.5}$$

with eigenvalues $\lambda_0 = 0$ and $\lambda_{2k+1} = \lambda_{2k} = k^2$. In this case, the $[f]_j$'s are the Fourier coefficients of the function f.

3. General Lipschitz bounded potentials: The three circles inequality

Throughout this section, u will be a solution of

$$\Delta u = -V u \tag{3.1}$$

on a product $N \times [0, t]$, where the potential V will be Lipschitz, but is no longer assumed to be rotationally symmetric.

The results of Appendix A in the rotationally symmetric case where $V = V(t)$ were stated on an entire cylinder, but the corresponding results for the half-cylinder motivate the general results of this section. Namely, the ODE (A.3) for the Fourier coefficients of u as a function of t implies that the j-th Fourier coefficient must either grow or decay exponentially if $\lambda_j > \sup V$. This same analysis holds even on a half-cylinder when V is rotationally symmetric. We will prove similar results in this section for a general bounded potential $V = V(\theta, t)$, but things are more complicated since multiplication by $V(\theta, t)$ does not preserve the eigenspaces of Δ_N (i.e., $\phi_j(\theta)$).[9]

The main result of this section is Theorem 3.36 below that shows a three circles inequality for the Sobolev norm of a solution of a Schrödinger equation on a product $N \times [0, T]$ where N has the required spectral gaps. This will give Theorem 0.8 in the special case where N is a round sphere or a Zoll surface. See the upshot to Section 3 in the introduction for an overview of the proof.

3.1. **The Fourier coefficients of u.** As in the rotationally symmetric case, it will be important to understand how the Fourier coefficients $[u]_j(t)$ and its derivatives grow or decay as a function of t.

The next lemma gives the ODE's that govern how the Fourier coefficients $[u]_j(t)$ grow or decay as functions of t; cf. the similar ODE (A.3) in the rotationally symmetric case.

Lemma 3.2. The Fourier coefficients $[u]_j(t)$ satisfy

$$[u]_j'(t) = [u_t]_j(t) \, , \tag{3.3}$$

$$[u]_j''(t) = \lambda_j \, [u]_j(t) - [Vu]_j(t) \, , \tag{3.4}$$

$$[u]_j'''(t) = \lambda_j \, [u]_j'(t) - [\partial_t (Vu)]_j(t) \, . \tag{3.5}$$

Proof. Differentiating $[u]_j(t)$ immediately gives the first claim. To get the second claim, first differentiate again to get

$$[u]_j''(t) = \int_N u_{tt}(\theta, t) \, \phi_j(\theta) \, d\theta \, . \tag{3.6}$$

[9]The reason that the ODE (A.3) is so simple is that the j-th Fourier coefficient of $V(t)u(\theta, t)$ is just $V(t)$ times the j-th Fourier coefficient of $u(\theta, t)$.

Next, bring in the equation $u_{tt} = -\Delta_N u - Vu$ and integrate by parts twice to get

$$[u]_j''(t) = -\int_{N\times\{t\}} \phi_j \, \Delta_N u \; d\theta - \int_{N\times\{t\}} V \, u \, \phi_j \, d\theta$$

$$= \lambda_j [u]_j(t) - [Vu]_j(t) \, . \tag{3.7}$$

Differentiating again gives

$$[u]_j'''(t) = \lambda_j \, [u]_j'(t) - [\partial_t(Vu)]_j(t) \, . \tag{3.8}$$

□

As mentioned above, (3.4) implies exponential growth (or decay) of $[u]_j(t)$ when V is rotationally symmetric and $\lambda_j > \sup V$. However, this is not the case for a general bounded V since the "error term" $[Vu]_j(t)$ need not be bounded by $[u]_j(t)$. We will get around this in the next subsection by considering all of the $[u]_j$'s above a fixed value at the same time. To get a well-defined quantity when we do this, we will have to sum the squares of the $[u]_j$'s. Unfortunately, the quantity $[u]_j^2$ does not satisfy as nice of an ODE, so we will have to consider a slightly different quantity. To see why, observe that when $V = 0$, then

$$\partial_t^2 \left[([u]_j')^2 + \lambda_j \, [u]_j^2 \right] = 4\lambda_j \, \left[([u]_j')^2 + \lambda_j \, [u]_j^2 \right] \, . \tag{3.9}$$

Equation (3.9) suggests looking at the quantity $([u]_j')^2 + \lambda_j \, [u]_j^2$, but it will be more convenient to look at the slightly different quantity $([u]_j')^2 + (1+\lambda_j) \, [u]_j^2$. This is because $([u]_j')^2 + \lambda_j \, [u]_j^2$ is a piece of the L^2 norm of ∇u, but $([u]_j')^2 + (1 + \lambda_j) \, [u]_j^2$ also includes part of the L^2 norm of u and, hence, corresponds to the full $W^{1,2}$ norm of u; see equation (3.30) below.

Lemma 3.10. The quantity $\left[([u]_j')^2 + (1 + \lambda_j) \, [u]_j^2 \right]$ satisfies the ODE's

$$\partial_t \left[([u]_j')^2 + (1 + \lambda_j) \, [u]_j^2 \right] = (4 \, \lambda_j + 2) \, [u]_j \, [u]_j' - 2[u]_j' \, [Vu]_j \, , \tag{3.11}$$

$$\partial_t^2 \left[([u]_j')^2 + (1 + \lambda_j) \, [u]_j^2 \right] = (4\lambda_j + 2) \, \left[([u]_j')^2 + (1 + \lambda_j) \, [u]_j^2 \right] - (4\lambda_j + 2) \, [u]_j^2 \tag{3.12}$$
$$- (6\lambda_j + 2) \, [u]_j [Vu]_j + 2 \, [Vu]_j^2 - 2 \, [u]_j' \, [\partial_t(Vu)]_j \, .$$

Proof. Using Lemma 3.2, we get

$$\frac{1}{2} \, ([u]_j^2)' = [u]_j \, [u]_j' \, , \tag{3.13}$$

$$\frac{1}{2} \, ([u]_j^2)'' = \lambda_j \, [u]_j^2 + ([u]_j')^2 - [u]_j \, [Vu]_j \, . \tag{3.14}$$

Similarly, differentiating $([u]_j')^2$ gives

$$\frac{1}{2} \, \partial_t([u]_j')^2 = \lambda_j \, [u]_j \, [u]_j' - [u]_j' \, [Vu]_j \, , \tag{3.15}$$

$$\frac{1}{2} \, \partial_t^2([u]_j')^2 = (\lambda_j [u]_j - [Vu]_j)^2 + [u]_j' \left(\lambda_j \, [u]_j' - [\partial_t(Vu)]_j \right) \, . \tag{3.16}$$

The lemma follows by combining (3.13) with (3.15) and then (3.14) with (3.16). □

The terms on the last line of (3.12) are the error terms that vanish when $V = 0$.

3.2. **Projecting onto high frequencies.** In contrast to the rotationally symmetric case, the ODE's in the previous subsection do not imply exponential growth or decay of the individual Fourier coefficients. This is because the error terms involve the Fourier coefficients of Vu and cannot be absorbed. To get around this, we will instead consider simultaneously all of the Fourier coefficients from some point on. To be precise, we fix a large non-negative integer m and let $\mathcal{H}_m(t)$ be the "high frequency" part of the norm of $u(t,\theta)$ given by

$$\mathcal{H}_m(t) = \sum_{j=m}^{\infty} \left(([u]_j')^2(t) + (1+\lambda_j)\, [u]_j^2(t) \right) . \tag{3.17}$$

Likewise, let $\mathcal{L}_m(t)$ be the left over "low frequency" part

$$\mathcal{L}_m(t) = \sum_{j=0}^{m-1} \left(([u]_j')^2(t) + (1+\lambda_j)\, [u]_j^2(t) \right) . \tag{3.18}$$

Note that $\mathcal{H}_m(t)$ is the contribution on the slice $N \times \{t\}$ to the square of the $W^{1,2}(N \times [0,T])$ norm of the $L^2(N)$-projection of the function u to the eigenspaces from m to ∞. Likewise, $\mathcal{L}_m(t)$ is the $N \times \{t\}$ part of the square of the $W^{1,2}$ norm of the $L^2(N)$-projection of the function u to the eigenspaces below m.

The next lemma gives the key differential inequalities for $\mathcal{H}_m(t)$ and $\mathcal{L}_m(t)$.

Lemma 3.19.

$$\mathcal{H}_m''(t) \geq (4\lambda_m - 6)\, \mathcal{H}_m(t) - 3 \int_{N \times \{t\}} \left[(Vu)^2 + |\nabla (Vu)|^2 \right] \, d\theta \, , \tag{3.20}$$

$$\mathcal{L}_m''(t) \leq (4\lambda_{m-1} + 6)\, \mathcal{L}_m(t) + 5 \int_{N \times \{t\}} \left[(Vu)^2 + |\nabla (Vu)|^2 \right] \, d\theta \, . \tag{3.21}$$

Proof. We will first prove the bound for $\mathcal{H}_m''(t)$ and then argue similarly for $\mathcal{L}_m''(t)$. Applying Lemma 3.10 and then summing over j gives

$$\begin{aligned} \mathcal{H}_m'' = & \sum_{j=m}^{\infty} (4\lambda_j + 2) \ \left[([u]_j')^2 + (1+\lambda_j)\, [u]_j^2 \right] - \sum_{j=m}^{\infty} \left[(4\lambda_j + 2)\, [u]_j^2 \right] \\ & - \sum_{j=m}^{\infty} \left[(6\,\lambda_j + 2) [u]_j [Vu]_j - 2\, [Vu]_j^2 + 2\, [u]_j'\, [Vu]_j' \right] \, . \end{aligned} \tag{3.22}$$

The first sum on the first line is at least $(4\lambda_m + 2)\, \mathcal{H}_m$, while the second is at least $-4\, \mathcal{H}_m$. We will now handle each of the three "error terms" in the second line. First, the Cauchy-Schwarz inequality gives

$$\begin{aligned} 2 \sum_{j=m}^{\infty} (1+\lambda_j) \ |[u]_j\, [Vu]_j| & \leq \sum_{j=m}^{\infty} (1+\lambda_j) \ \left[[u]_j^2 + [Vu]_j^2 \right] \\ & \leq \mathcal{H}_m + \int_{N \times \{t\}} \left[(Vu)^2 + |\nabla_N (Vu)|^2 \right] \, d\theta \, , \end{aligned} \tag{3.23}$$

where the second inequality used the standard relation between the Fourier coefficients of a function on N and those of its derivative. The second error term is clearly non-negative.

For the last error term, we again use the Cauchy-Schwarz inequality to get

$$2 \sum_{j=m}^{\infty} \left| [u]'_j \, [Vu]'_j \right| \leq \sum_{j=m}^{\infty} \left[([u]'_j)^2 + ([Vu]'_j)^2 \right] \leq \mathcal{H}_m + \int_{N \times \{t\}} (\partial_t (Vu))^2 \, d\theta \, . \tag{3.24}$$

Substituting the bounds (3.23) and (3.24) into (3.22) gives

$$\mathcal{H}''_m \geq (4\lambda_m - 6) \, \mathcal{H}_m - 3 \int_{N \times \{t\}} \left[(Vu)^2 + |\nabla_N (Vu)|^2 \right] \, d\theta - \int_{N \times \{t\}} (\partial_t (Vu))^2 d\theta \, , \tag{3.25}$$

giving the bound for $\mathcal{H}''_m$.

The bound for $\mathcal{L}''_m(t)$ follows similarly, except that the second term on the first line of (3.22) now has the right sign and the term corresponding to second error term for $\mathcal{H}_m$ now has the wrong sign. We bound this term by

$$2 \sum_{j=0}^{m-1} [Vu]_j^2 \leq 2 \int_{N \times \{t\}} (Vu)^2 \, d\theta \, . \tag{3.26}$$

□

Corollary 3.27. There is a constant C depending only on $||V||_{C^{0,1}}$ (but <u>not</u> on m) so that

$$\mathcal{H}''_m \geq (4\lambda_m - C) \, \mathcal{H}_m - C \, \mathcal{L}_m \, , \tag{3.28}$$

$$\mathcal{L}''_m \leq (4\lambda_{m-1} + C) \, \mathcal{L}_m + C \, \mathcal{H}_m \, . \tag{3.29}$$

Proof. Integrating by parts on the closed manifold N and using that $\nabla = \nabla_N + \partial_t$ gives

$$\int_{N \times \{t\}} \left[u^2 + |\nabla u|^2 \right] \, d\theta = \sum_{j=0}^{\infty} \left[(1 + \lambda_j) [u]_j^2 + ([u]'_j)^2 \right] = \mathcal{L}_m + \mathcal{H}_m \, . \tag{3.30}$$

It is easy to see that there is a constant c depending on $||V||_{C^{0,1}}$ so that

$$\int_{N \times \{t\}} \left[(Vu)^2 + |\nabla (Vu)|^2 \right] \, d\theta \leq c \int_{N \times \{t\}} \left[u^2 + |\nabla u|^2 \right] \, d\theta = c \, (\mathcal{L}_m + \mathcal{H}_m) \, , \tag{3.31}$$

where the equality used (3.30). The corollary follows from using this bound on the error terms in Lemma 3.19. □

3.3. Taking advantage of gaps in the spectrum.

The next proposition proves a differential inequality for an integer m where $\lambda_m - \lambda_{m-1}$ is large.

Proposition 3.32. There exists a constant $\kappa > 0$ depending on $||V||_{C^{0,1}}$ so that if m is an integer with $\lambda_m - \lambda_{m-1} > \kappa$, then

$$(\mathcal{H}_m - \mathcal{L}_m)'' \geq (4\lambda_{m-1} + 2\kappa) \, (\mathcal{H}_m - \mathcal{L}_m) \, . \tag{3.33}$$

Proof. To see this, apply Corollary 3.27 to get C depending only on $||V||_{C^{0,1}}$ so that

$$\begin{aligned} (\mathcal{H}_m - \mathcal{L}_m)'' &\geq (4\lambda_m - C) \, \mathcal{H}_m - C \, \mathcal{L}_m - (4\lambda_{m-1} + C) \, \mathcal{L}_m - C \, \mathcal{H}_m \\ &= (4\lambda_{m-1} + 2C) \, (\mathcal{H}_m - \mathcal{L}_m) + 4 \, (\lambda_m - \lambda_{m-1} - C) \, \mathcal{H}_m \, . \end{aligned} \tag{3.34}$$

□

3.4. **The three circles inequality.** We will next use Proposition 3.32 to prove the three circles inequality. In fact, we will prove a more general inequality than the one stated in Theorem 0.8. To state this, let N be any closed n-dimensional Riemannian manifold satisfying (0.2) and set

$$I(s) = \int_{N\times\{s\}} \left(u^2 + |\nabla u|^2\right)\, d\theta\,. \tag{3.35}$$

Theorem 3.36. There exists a constant $C > 0$ depending on $||V||_{C^{0,1}}$ so that if α satisfies

$$\alpha \geq \frac{1}{T}\left[\log\frac{I(T)}{I(0)}\right]\,, \tag{3.37}$$

then

$$\log I(t) \leq C + (c_3 + C + |\alpha|)\, t + \log I(0)\,, \tag{3.38}$$

where the constant c_3 is given by[10]

$$c_3 = \min_m \left\{2\lambda_{m-1}^{1/2} - |\alpha| \,\big|\, \lambda_m - \lambda_{m-1} > C \text{ and } 2\lambda_{m-1}^{1/2} > |\alpha|\right\}\,. \tag{3.39}$$

Before getting to the proof of Theorem 3.36, we will make a few remarks. First, when we have equality in (3.37), then Theorem 3.36 also applies to the reflected function $\bar{u}(t) = u(T-t)$ with $-\alpha$ in place of α. Next, observe that (3.38) simplifies considerably when

$$\alpha = \frac{1}{T}\left[\log\frac{I(T)}{I(0)}\right] \geq 0\,. \tag{3.40}$$

Namely, when (3.40) holds, then we get

$$\log I(t) \leq C + (c_3 + C)\, t + \frac{t}{T}\log I(T) + \frac{T-t}{T}\log I(0)\,. \tag{3.41}$$

Proof. (of Theorem 3.36). We will first use the spectral gap to bound $\mathcal{H}_m(t)$ in terms of $\mathcal{L}_m(t)$ and $\mathcal{H}_m(0)$ for some fixed m. The key for this is that Proposition 3.32 gives a constant $\kappa > 0$ depending on $||V||_{C^{0,1}}$ so that if m is an integer with

$$\lambda_m - \lambda_{m-1} \geq \kappa\,, \tag{3.42}$$

then we have

$$(\mathcal{H}_m - \mathcal{L}_m)'' \geq (4\lambda_{m-1} + 2\kappa)\, (\mathcal{H}_m - \mathcal{L}_m)\,. \tag{3.43}$$

Fix some m so that (3.42) holds and

$$4\lambda_{m-1} > \alpha^2\,. \tag{3.44}$$

On the interval $[0,T]$, we define a function f by

$$f(t) = \mathrm{e}^{-\alpha t}\, (\mathcal{H}_m - \mathcal{L}_m)\,(t)\,, \tag{3.45}$$

[10]The only place where we use the spectral gaps given by (0.2) is to get an m satisfying (3.39).

then

$$f(0) = (\mathcal{H}_m - \mathcal{L}_m)(0)\,, \tag{3.46}$$

$$f(T) \leq \frac{(\mathcal{H}_m + \mathcal{L}_m)(0)}{(\mathcal{H}_m + \mathcal{L}_m)(T)}\,(\mathcal{H}_m - \mathcal{L}_m)(T) \leq (\mathcal{H}_m + \mathcal{L}_m)(0)\,, \tag{3.47}$$

$$f' = \mathrm{e}^{-\alpha t}\left[(\mathcal{H}_m - \mathcal{L}_m)' - \alpha\,(\mathcal{H}_m - \mathcal{L}_m)\right]\,, \tag{3.48}$$

$$f'' = \mathrm{e}^{-\alpha t}\left[(\mathcal{H}_m - \mathcal{L}_m)'' - 2\alpha\,(\mathcal{H}_m - \mathcal{L}_m)' + \alpha^2\,(\mathcal{H}_m - \mathcal{L}_m)\right]\,. \tag{3.49}$$

By the maximum principle, at an interior maximum $t_0 \in (0,T)$ for f, $f'(t_0) = 0$ and $f''(t_0) \leq 0$. Hence, by (3.48) and (3.49)

$$(\mathcal{H}_m - \mathcal{L}_m)''(t_0) \leq \alpha^2\,(\mathcal{H}_m - \mathcal{L}_m)(t_0)\,. \tag{3.50}$$

However, this contradicts (3.43) and (3.44) if $f(t_0) > 0$, so we conclude that f does not have a positive interior maximum. Therefore, for all $t \in [0,T]$, we have that

$$f(t) \leq \max\{0, f(0), f(T)\} \leq (\mathcal{H}_m + \mathcal{L}_m)(0) = I(0)\,. \tag{3.51}$$

This implies that $(\mathcal{H}_m - \mathcal{L}_m)(t) \leq \mathrm{e}^{\alpha t}\,I(0)$ and, hence,

$$\mathcal{H}_m(t) \leq \mathcal{L}_m(t) + \mathrm{e}^{\alpha t}\,I(0)\,. \tag{3.52}$$

To complete the proof, we will substitute (3.52) into a differential inequality for $\mathcal{L}_m(t)$ and use this to prove an exponential upper bound for $\mathcal{L}_m(t)$. To get the differential inequality, recall that (3.11) in Lemma 3.10 gives

$$\begin{aligned} \left|\,\partial_t\left[([u]_j')^2 + (1+\lambda_j)\,[u]_j^2\right]\,\right| &= \left|\,(4\,\lambda_j + 2)\,[u]_j\,[u]_j' - 2[Vu]_j\,[u]_j'\right| \\ &\leq 2\,(1+\lambda_j)^{1/2}\,\left[([u]_j')^2 + (1+\lambda_j)\,[u]_j^2\right] + [Vu]_j^2 + ([u]_j')^2\,. \end{aligned} \tag{3.53}$$

Summing this up to $(m-1)$ and bounding the (Vu) terms as in (3.31) gives

$$|\mathcal{L}_m'(t)| \leq \left[2\,\lambda_{m-1}^{1/2} + C\right]\,\mathcal{L}_m(t) + C\mathcal{H}_m(t)\,, \tag{3.54}$$

where C depends only on $||V||_{C^{0,1}}$. Using the bound (3.52), we get

$$|\mathcal{L}_m'(t)| \leq c_1\mathcal{L}_m(t) + C\;\mathrm{e}^{\alpha t}\,I(0)\,, \tag{3.55}$$

where we set

$$c_1 = 2\,\left[\lambda_{m-1}^{1/2} + C\right] \tag{3.56}$$

to simplify notation. In particular, the function

$$\mathcal{L}_m(t)\,\mathrm{e}^{-c_1 t} + \frac{C}{c_1 - \alpha}\,I(0)\,\mathrm{e}^{(\alpha - c_1)\,t} \tag{3.57}$$

is non-increasing on $[0,T]$; we conclude that

$$\mathcal{L}_m(t) \leq \mathrm{e}^{c_1 t}\,\mathcal{L}_m(0) + \frac{C}{c_1 - \alpha}\,I(0)\,\left(\mathrm{e}^{c_1 t} - \mathrm{e}^{\alpha t}\right) \leq c_2\,\mathrm{e}^{c_1 t}\,I(0)\,, \tag{3.58}$$

where we have set $c_2 = \left(1 + \frac{C}{c_1 - |\alpha|}\right) \geq 1$. Substituting (3.58) into (3.52) gives a bound for $I(t) = \mathcal{H}_m(t) + \mathcal{L}_m(t)$

$$I(t) \leq 2\,\mathcal{L}_m(t) + \mathrm{e}^{\alpha t}\,I(0) \leq 2\,c_2\,\mathrm{e}^{c_1 t}\,I(0) + \mathrm{e}^{\alpha t}\,I(0) \leq (2\,c_2 + 1)\,\mathrm{e}^{c_1 t}\,I(0)\,, \tag{3.59}$$

where the last inequality used that $c_1 > |\alpha|$. The theorem follows from (3.59). □

We will next apply the three circles inequality of Theorem 3.36 to prove uniform estimates for the $W^{1,2}$ norm of an at most exponentially growing solution u on the half-cylinder $N \times [0, \infty)$, i.e., to prove Corollary 0.15.[11] As in the statement of Theorem 3.36, we will let $I(s)$ denote the $W^{1,2}$ of u on $N \times \{s\}$.

Proof. (of Corollary 0.15.) We will assume that $\alpha > 0$ (we can do this since $H_\alpha \subset H_{\bar{\alpha}}$ whenever $\alpha \leq \bar{\alpha}$). We will first use the definition of H_α to bound $I(T)$ for large values of T and then use the three circles inequality to bound $I(t)$ in terms of $I(0)$ and $I(T)$.

The interior Schauder estimates (theorem 6.2 in [GT]) give a constant C depending only on the C^β norm of V, where $\beta \in (0,1)$ is fixed, so that for all $t \geq 1$

$$I(t) = \int_{N \times \{t\}} \left(u^2 + |\nabla u|^2 \right) \, d\theta \leq C \sup_{N \times [t-1, t+1]} |u|^2 \, . \tag{3.60}$$

If we also bring in the definition of H_α, i.e., (0.1), then we get that

$$\limsup_{T \to \infty} \left(\mathrm{e}^{-2\,\alpha\,(T+1)} \, I(T) \right) = 0 \, . \tag{3.61}$$

Note that the bound (3.61) applies only in the limit as T goes to ∞ and, hence, does not give the Corollary. However, it does give a sequence $T_j \to \infty$ with

$$\frac{\log I(T_j) - \log I(0)}{T_j} \leq 2\alpha \, . \tag{3.62}$$

Applying the three circles inequality of Theorem 3.36 on $[0, T_j]$ gives

$$\log I(t) \leq C\,(1+t) + 2\alpha\, t + \log I(0) \, , \tag{3.63}$$

and exponentiating this gives the corollary. □

Note that ν in Corollary 0.15 has to also depend on the norm of V and not just on α. In particular, ν may have to be chosen positive even when α is zero. This can easily be seen by the following example for the one-dimensional Schrödinger equation. Suppose that $\Psi : \mathbf{R} \to \mathbf{R}$ is a smooth monotone non-decreasing function with

$$\Psi(x) = -1 \text{ for } x < -1, \; \Psi(x) = x \text{ on } [0, \ell], \; \Psi(x) = \ell + 1 \text{ for } x > \ell \, . \tag{3.64}$$

Then $u(x) = \mathrm{e}^{\Psi(x)}$ satisfies the Schrödinger equation $u'' = ((\Psi')^2 + \Psi'')\, u = V\, u$ for a bounded potential V with compact support. However, u is constant on each end, but grows exponentially on $[0, \ell]$. Similarly, one can easily construct a bounded (but no longer with compact support) potential so that the corresponding Schrödinger equation has a solution that grows exponentially on $[0, \ell]$, yet at infinity the solution vanishes.

[11] A similar argument, with Theorem 0.12 in place of Theorem 3.36, gives a corresponding result on spheres and Zoll surfaces even when V is just bounded.

3.5. **Unique continuation.** Rather than stating the most general three circles inequality possible, we have tailored the statement of Theorem 3.36 to fit our geometric applications. We will show here how to modify the proof to get strong unique continuation for the operator L on $N \times [0,\infty)$ since this is of independent interest:

Proposition 3.65. If u is a solution on $N \times [0,\infty)$, where N satisfies (0.2), and

$$\liminf_{T\to\infty} \frac{\log I(T)}{T} = -\infty \,, \tag{3.66}$$

then u is the constant solution $u = 0$.

Proof. Observe that Theorem 3.36 implies that if $I(0) = 0$, then u is the constant solution $u = 0$ (this also follows from [A]). Therefore, it suffices to show that $I(0) = 0$. We will argue by contradiction, so suppose that $I(0) > 0$. After replacing u by $I^{-1/2}(0)\, u$, we can assume that $I(0) = 1$.

Choose some m_0 so that $\mathcal{H}_m(0) < \mathcal{L}_m(0)$ for all $m \geq m_0$. Using the spectral gaps of N and Proposition 3.32, we can choose an arbitrarily large integer $m > m_0$ with

$$(\mathcal{H}_m - \mathcal{L}_m)'' \geq (4\lambda_{m-1} + 1)\, (\mathcal{H}_m - \mathcal{L}_m)\;. \tag{3.67}$$

The rapid decay given by (3.66) guarantees that we can find $T > 0$ with

$$\frac{\log I(T)}{T} < -4\, \lambda_{m-1}^{1/2}\,. \tag{3.68}$$

On the interval $[0,T]$, we define a function f by

$$f(t) = \mathrm{e}^{2\lambda_{m-1}^{1/2} t}\, (\mathcal{H}_m - \mathcal{L}_m)\,(t)\,. \tag{3.69}$$

Using first that $m \geq m_0$ and then using (3.68), we get that

$$f(0) < 0 \text{ and } f(T) \leq \mathrm{e}^{2\lambda_{m-1}^{1/2} t}\, I(T) < \mathrm{e}^{-2\lambda_{m-1}^{1/2} T}\,. \tag{3.70}$$

Using the maximum principle as in (3.43)–(3.50), we see that f cannot have a positive interior maximum and, hence, that

$$(\mathcal{H}_m - \mathcal{L}_m)\,(t) \leq \mathrm{e}^{-2\lambda_{m-1}^{1/2} (t+T)}\,. \tag{3.71}$$

Combining this with the bound (3.54) for $\mathcal{L}_m'$ gives

$$|\mathcal{L}_m'(t)| \leq \left[2\, \lambda_{m-1}^{1/2} + C\right]\, \mathcal{L}_m(t) + C\, \mathrm{e}^{-2\lambda_{m-1}^{1/2} (t+T)}\,, \tag{3.72}$$

where C depends only on $||V||_{C^{0,1}}$. It follows that the function

$$\mathrm{e}^{\left[2\,\lambda_{m-1}^{1/2} + C\right]\, t}\, \mathcal{L}_m(t) + \mathrm{e}^{Ct - 2\lambda_{m-1}^{1/2} T} \tag{3.73}$$

is non-decreasing on $[0,T]$. Evaluating this function at 0 and T gives

$$\mathcal{L}_m(0) \leq \mathrm{e}^{\left[2\,\lambda_{m-1}^{1/2} + C\right]\, T}\, \mathcal{L}_m(T) + \mathrm{e}^{CT - 2\lambda_{m-1}^{1/2} T} \leq 2\, \mathrm{e}^{CT - 2\lambda_{m-1}^{1/2} T}\,. \tag{3.74}$$

However, since λ_{m-1} can be arbitrarily large and C is fixed (i.e., does not depend on m), we conclude that $\mathcal{L}_m(0) = 0$. Finally, this gives the desired contradiction since $I(0) = \mathcal{L}_m(0) + \mathcal{H}_m(0) = 1$ and $\mathcal{L}_m(0) > \mathcal{H}_m(0)$. □

4. A three circles theorem for bounded potentials

We will show in this section that Theorem 0.8 holds even for potentials that are just bounded, i.e., the potential V does not have to be Lipschitz; this is Theorem 0.12. This result will require larger spectral gaps than were needed for the arguments in the Lipschitz case. Throughout this section, u will be a solution of

$$\Delta u = -V u \tag{4.1}$$

on a product $N \times [0,t]$, where the potential V is bounded, but is not assumed to be Lipschitz.

The main place where the Lipschitz bound entered previously was when we took second derivatives of $|\nabla u|^2$. To avoid doing this, we will work with the L^2 norm of the spectral projections of a solution u. Namely, we fix a large non-negative integer m and let $\bar{\mathcal{H}}_m(t)$ and $\bar{\mathcal{L}}_m(t)$ be the "high frequency" and "low frequency" parts, respectively, of $u(t,\theta)$ given by

$$\bar{\mathcal{H}}_m(t) = \sum_{j=m}^{\infty} [u]_j^2(t) \text{ and } \bar{\mathcal{L}}_m(t) = \sum_{j=0}^{m-1} [u]_j^2(t)\,. \tag{4.2}$$

Note that $\bar{\mathcal{H}}_m^{1/2}$ is the L^2 norm of the projection of u to the eigenspaces from m to ∞.

As in Section 3, we will derive a second order ODE for $\bar{\mathcal{H}}_m(t)$ and use this to control its growth. Unfortunately, the ODE (3.14) for the quantity $[u]_j^2$, and thus also for $\bar{\mathcal{H}}_m(t)$, is not as nice as for $[u_t]_j^2 + \lambda_j\,[u]_j^2$ because of the $[u_t]_j^2$ term on the right hand side. We will use the next lemma to get around this.

Corollary 4.3. Given $t_1 < t_2$, we get that

$$\left([u_t]_j^2 - \lambda_j\,[u]_j^2\right)(t_2) - \left([u_t]_j^2 - \lambda_j\,[u]_j^2\right)(t_1) = -2\int_{t_1}^{t_2} \left([Vu]_j\,[u_t]_j\right)\,dt\,. \tag{4.4}$$

Proof. Differentiating $\left([u_t]_j^2 - \lambda_j\,[u]_j^2\right)$ and then using Lemma 3.2 gives

$$\partial_t\left([u_t]_j^2 - \lambda_j\,[u]_j^2\right) = 2(\lambda_j\,[u]_j - [Vu]_j)\,[u_t]_j - 2\,\lambda_j\,[u]_j\,[u_t]_j = -2[Vu]_j\,[u_t]_j\,. \tag{4.5}$$

The corollary now follows from the fundamental theorem of calculus. □

The next lemma will give the key differential inequality for $\bar{\mathcal{H}}_m(t)$. To state this, it will be useful to define $J(t)$ to be the square of the L^2 norm of u on $N \times \{t\}$, i.e.,

$$J(t) = \int_{N\times\{t\}} u^2\,d\theta\,. \tag{4.6}$$

Lemma 4.7.

$$\begin{aligned}\bar{\mathcal{H}}_m''(t) \geq (4\lambda_m - 1)\,\bar{\mathcal{H}}_m(t) &- \int_{N\times\{t\}} (Vu)^2\,d\theta - 2\int_{N\times\{t_0\}} |\nabla u|^2\,d\theta - J'(t) + J'(t_0) \\ &- 2\int_{N\times(t_0,t)} \left[(V^2 + |V|)\,u^2\right]\,.\end{aligned} \tag{4.8}$$

Proof. Applying Lemma 3.2 and then summing over j gives

$$\bar{\mathcal{H}}_m'' = 2 \sum_{j=m}^{\infty} \left[[u_t]_j^2 + \lambda_j \, [u]_j^2 - [u]_j \, [Vu]_j \right] \tag{4.9}$$

$$= 2 \sum_{j=m}^{\infty} \left[2 \, \lambda_j \, [u]_j^2 - [u]_j \, [Vu]_j + \left([u_t]_j^2 - \lambda_j \, [u]_j^2 \right) (t_0) - 2 \int_{t_0}^t \left([Vu]_j \, [u_t]_j \right) \, ds \right] ,$$

where the second equality used Corollary 4.3. We will now handle each of the three "error terms" in the second line. First, the Cauchy-Schwarz inequality gives

$$2 \left| \sum_{j=m}^{\infty} [u]_j \, [Vu]_j \right| \leq \sum_{j=m}^{\infty} \left[[u]_j^2 + [Vu]_j^2 \right] \leq \bar{\mathcal{H}}_m + \int_{N \times \{t\}} (Vu)^2 \, d\theta \, , \tag{4.10}$$

where the second inequality used the standard relation between the Fourier coefficients of a function on N and its L^2 norm. The second error term is bounded by

$$2 \left| \sum_{j=m}^{\infty} \left([u_t]_j^2 - \lambda_j \, [u]_j^2 \right) \right| (t_0) \leq 2 \sum_{j=0}^{\infty} \left(\lambda_j \, [u]_j^2 + [u_t]_j^2 \right) (t_0) = 2 \int_{N \times \{t_0\}} |\nabla u|^2 \, d\theta \, , \tag{4.11}$$

where the equality used the standard relation between the Fourier coefficients of a function on N and those of its derivative. Similarly, for the last error term, the Cauchy-Schwarz inequality gives

$$4 \left| \sum_{j=m}^{\infty} \int_{t_0}^t \left([Vu]_j \, [u_t]_j \right) \, ds \right| \leq 2 \int_{t_0}^t \left(\int_{N \times \{s\}} \left[(Vu)^2 + (u_t)^2 \right] \, d\theta \right) \, ds \, . \tag{4.12}$$

The first term in (4.12) is of the right form, but it will be convenient to get a lower order bound for the $(u_t)^2$ term. To do this, we use Stokes' theorem to get

$$2 \int_{N \times (t_0, t)} \left[|\nabla u|^2 - V \, u^2 \right] = J'(t) - J'(t_0) \, , \tag{4.13}$$

so we get that

$$2 \int_{N \times (t_0,t)} (u_t)^2 \leq 2 \int_{N \times (t_0,t)} |\nabla u|^2 = J'(t) - J'(t_0) + 2 \int_{N \times (t_0,t)} V \, u^2 \, . \tag{4.14}$$

Finally, substituting the bounds (4.10)–(4.12) and (4.14) into (4.9) gives the lemma. □

We get the following immediate corollary of Lemma 4.7; note that the square of the $W^{1,2}$ norm of the projection to the low frequencies, i.e., $\mathcal{L}_m$, appears in the bound.

Corollary 4.15. There exists a constant C depending only on $\sup |V|$ so that

$$\bar{\mathcal{H}}_m'' \geq (4 \lambda_m - C) \, \bar{\mathcal{H}}_m - \bar{\mathcal{H}}_m' - 3 \, I(t_0) - C \, \bar{\mathcal{L}}_m - 2 \, (\bar{\mathcal{L}}_m \, \mathcal{L}_m)^{1/2}$$

$$- C \int_{t_0}^t \left[\bar{\mathcal{H}}_m(s) + \bar{\mathcal{L}}_m(s) \right] \, ds \, . \tag{4.16}$$

Proof. To bound the first "error term" from Lemma 4.7, bound V by $\sup |V|$ to get

$$\int_{N\times\{t\}} (Vu)^2 \, d\theta \leq \sup |V|^2 \int_{N\times\{t\}} u^2 \, d\theta = \sup |V|^2 \, \left[\bar{\mathcal{H}}_m(t) + \bar{\mathcal{L}}_m(t)\right] \, . \tag{4.17}$$

Similarly, the last error term is bounded by

$$2 \int_{N\times(t_0,t)} \left[(V^2 + |V|)\, u^2\right] \leq 2\,(\sup |V| + \sup |V|^2) \int_{t_0}^{t} \left[\bar{\mathcal{H}}_m(s) + \bar{\mathcal{L}}_m(s)\right] \, ds \, . \tag{4.18}$$

The second error term $2 \int_{N\times\{t_0\}} |\nabla u|^2 \, d\theta$ is trivially bounded by $2\, I(t_0)$. This leaves only the two J' terms. Use Cauchy-Schwarz to bound the second of these by

$$|J'(t_0)| = 2 \left| \int_{N\times\{t_0\}} u \, u_t \, d\theta \right| \leq \int_{N\times\{t_0\}} (u^2 + u_t^2) \, d\theta \leq I(t_0) \, . \tag{4.19}$$

To bound $J'(t)$, observe first that

$$\left|\bar{\mathcal{L}}_m'\right| = 2 \left| \sum_{j=0}^{m-1} [u]_j \, [u_t]_j \right| \leq 2 \, \left[\sum_{j=0}^{m-1} [u]_j^2 \right]^{1/2} \, \left[\sum_{j=0}^{m-1} [u_t]_j^2 \right]^{1/2} \leq 2\, (\bar{\mathcal{L}}_m \, \mathcal{L}_m)^{1/2} \, , \tag{4.20}$$

so we get

$$J' = \bar{\mathcal{H}}_m' + \bar{\mathcal{L}}_m' \leq \bar{\mathcal{H}}_m' + 2\, (\bar{\mathcal{L}}_m \, \mathcal{L}_m)^{1/2} \, . \tag{4.21}$$

The corollary now follows from Lemma 4.7. □

The next lemma gives the key differential inequality for $\mathcal{L}_m$ that will be used later to get an upper bound for $\mathcal{L}_m(t)$ (a similar, but slightly less sharp, bound was given in (3.54)).

Lemma 4.22. There exists a constant $C > 0$ depending only on $\sup |V|$ so that

$$|\mathcal{L}_m'| \leq 2(\lambda_{m-1} + 1)^{1/2} \mathcal{L}_m + C \, \sqrt{\bar{\mathcal{L}}_m + \bar{\mathcal{H}}_m} \, \sqrt{\mathcal{L}_m} \, . \tag{4.23}$$

Proof. To get the differential inequality, recall from (3.53) that Lemma 3.10 gives

$$\begin{aligned} \partial_t \left[([u]_j')^2 + (1 + \lambda_j)\, [u]_j^2\right] &= (4\, \lambda_j + 2)\, [u]_j \, [u]_j' - 2[u]_j' \, [Vu]_j \\ &\leq 2\, (\lambda_j + 1)^{1/2} \, \left[[u_t]_j^2 + (\lambda_j + 1)\, [u]_j^2\right] + 2\, |[Vu]_j \, [u_t]_j| \, . \end{aligned} \tag{4.24}$$

Summing this up to $(m-1)$ and then using the Cauchy-Schwarz inequality for series gives

$$|\mathcal{L}_m'| \leq 2\, (\lambda_{m-1} + 1)^{1/2} \, \mathcal{L}_m + C \, \sqrt{J} \, \sqrt{\mathcal{L}_m} \, , \tag{4.25}$$

where the constant C depends only on $\sup |V|$. □

4.1. **Exponentially weighted sup bounds for $\bar{\mathcal{H}}_m$, $\bar{\mathcal{L}}_m$ and $\mathcal{L}_m$.** We will next record an immediate consequence of Corollary 4.15 where the last three terms in (4.16) are bounded in terms of the sup norms of $\bar{\mathcal{H}}_m$, $\bar{\mathcal{L}}_m$ and $\mathcal{L}_m$ against an exponential weight. To make this precise, for each constant $\alpha > 0$, we define the exponentially weighted sup norm bounds $\bar{h}_{\alpha,m}$, $\bar{\ell}_{\alpha,m}$ and $\ell_{\alpha,m}$ by

$$\bar{h}_{\alpha,m} = \max_{[0,T]} \left[\bar{\mathcal{H}}_m(t)\, \mathrm{e}^{-\alpha\, t}\right] , \tag{4.26}$$

$$\bar{\ell}_{\alpha,m} = \max_{[0,T]} \left[\bar{\mathcal{L}}_m(t)\, \mathrm{e}^{-\alpha\, t}\right] , \tag{4.27}$$

$$\ell_{\alpha,m} = \max_{[0,T]} \left[\mathcal{L}_m(t)\, \mathrm{e}^{-\alpha\, t}\right] . \tag{4.28}$$

Clearly, by definition, we have that

$$\bar{\mathcal{H}}_m(t) \leq \bar{h}_{\alpha,m}\, \mathrm{e}^{\alpha\, t} \,,\; \bar{\mathcal{L}}_m(t) \leq \bar{\ell}_{\alpha,m}\, \mathrm{e}^{\alpha\, t} \,,\; \mathcal{L}_m(t) \leq \ell_{\alpha,m}\, \mathrm{e}^{\alpha\, t} . \tag{4.29}$$

Substituting these bounds into the differential inequality for $\bar{\mathcal{H}}_m$ gives:

Corollary 4.30. There exists a constant C depending only on $\sup |V|$ so that for $\alpha \geq 1$

$$\bar{\mathcal{H}}_m'' \geq (4\lambda_m - C)\, \bar{\mathcal{H}}_m - \bar{\mathcal{H}}_m' - 3I(t_0) - \left[C\, \left(\bar{\ell}_{\alpha,m} + \bar{h}_{\alpha,m}\right) + 2\, \left(\bar{\ell}_{\alpha,m}\, \ell_{\alpha,m}\right)^{1/2}\right] \mathrm{e}^{\alpha\, t} . \tag{4.31}$$

Proof. The corollary will follow directly from Corollary 4.15 by using (4.29) to bound the last three terms in (4.16). The bounds on $\bar{\mathcal{L}}_m$ and $(\bar{\mathcal{L}}_m\, \mathcal{L}_m)^{1/2}$ follow immediately from (4.29). Finally, to bound the last term in (4.16), note that

$$\int_{t_0}^{t} \left[\bar{\mathcal{H}}_m(s) + \bar{\mathcal{L}}_m(s)\right]\, ds \leq (\bar{h}_{\alpha,m} + \bar{\ell}_{\alpha,m}) \int_{t_0}^{t} \mathrm{e}^{\alpha\, s}\, ds \leq \frac{\bar{h}_{\alpha,m} + \bar{\ell}_{\alpha,m}}{\alpha}\, \mathrm{e}^{\alpha\, t} . \tag{4.32}$$

The corollary now follows from substituting these bounds into (4.16). □

4.2. **Taking advantage of gaps in the spectrum.** Fix a constant $\kappa \geq 1$ to be chosen (depending only on $\sup |V|$) and then choose a constant $\bar{\alpha} \geq \kappa$ with

$$\alpha \equiv \frac{1}{T} \left[\log \frac{I(T)}{I(0)}\right] \leq \bar{\alpha} , \tag{4.33}$$

and so that there exists m with

$$2\, (\lambda_{m-1} + 1)^{1/2} + 1 \leq \bar{\alpha} , \tag{4.34}$$

$$\bar{\alpha}^2 + \bar{\alpha} \leq 4\, \lambda_m - \kappa , \tag{4.35}$$

We will use the spectral gaps to show that such an $\bar{\alpha}$ always exists when N is a round sphere of a Zoll surface.

Proposition 4.36. If $\bar{\alpha} \geq 1$ satisfies (4.33), (4.34), and (4.35) for some constant $\kappa \geq 1$ depending only on $\sup |V|$, then for all $t \in [0,T]$

$$\int_{N \times \{t\}} u^2\, d\theta \leq C\, I(0)\, \mathrm{e}^{\bar{\alpha}\, t} , \tag{4.37}$$

where C depends only on $\sup |V|$.

The proof of Proposition 4.36 will be divided into four steps. First, we bound $\bar{\ell}_{\bar{\alpha},m}$ in terms of $\ell_{\bar{\alpha},m}$ and $I(0)$. Second, we use (4.34) to bound $\ell_{\bar{\alpha},m}$ in terms of $\bar{\ell}_{\bar{\alpha},m}$, $\bar{h}_{\bar{\alpha},m}$ and $I(0)$. Third, we combine these to bound both $\bar{\ell}_{\bar{\alpha},m}$ and $\ell_{\bar{\alpha},m}$ in terms of $\bar{h}_{\bar{\alpha},m}$ and $I(0)$. Finally, we substitute these bounds into the differential inequality for $\bar{\mathcal{H}}_m''$ to bound $\bar{h}_{\bar{\alpha},m}$ in terms of $I(0)$. In this last step, $\bar{h}_{\bar{\alpha},m}$ will show up on both sides of the inequality, but (4.35) will allow us to absorb the terms on the right hand side.

Proof. (of Proposition 4.36.)

Bounding $\bar{\ell}_{\bar{\alpha},m}$. To bound $\bar{\ell}_{\bar{\alpha},m}$, use (4.20) to get

$$\left|\bar{\mathcal{L}}_m'\right| \leq 2\,(\bar{\mathcal{L}}_m\,\mathcal{L}_m)^{1/2} \leq 2\,\bar{\mathcal{L}}_m^{1/2}\,\ell_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}\,t/2}\,. \tag{4.38}$$

On the interval $[0,T]$, we define a function $f_1(t) = \mathrm{e}^{-\bar{\alpha}t}\bar{\mathcal{L}}_m(t)$, so that

$$f_1(0) \leq I(0) \text{ and } f_1(T) \leq I(0)\,, \tag{4.39}$$

$$f_1' = \mathrm{e}^{-\bar{\alpha}t}\left[\bar{\mathcal{L}}_m' - \bar{\alpha}\,\bar{\mathcal{L}}_m\right]\,. \tag{4.40}$$

Observe that the maximum of f_1 on $[0,T]$ is precisely $\bar{\ell}_{\bar{\alpha},m}$. Hence, if the maximum of $f_1(t)$ occurs at a point s in the interior $(0,T)$, then we get

$$\bar{\alpha}\,\bar{\ell}_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}s} = \bar{\alpha}\,\bar{\mathcal{L}}_m(s) = \bar{\mathcal{L}}_m'(s) \leq 2\,\bar{\ell}_{\bar{\alpha},m}^{1/2}\,\ell_{\bar{\alpha},m}^{1/2}\,\mathrm{e}^{\bar{\alpha}\,s}\,. \tag{4.41}$$

Combining this with the fact that $f_1 \leq I(0)$ at both endpoints gives that

$$\bar{\ell}_{\bar{\alpha},m} \leq \frac{4}{\bar{\alpha}^2}\,\ell_{\bar{\alpha},m} + I(0)\,. \tag{4.42}$$

Bounding $\ell_{\bar{\alpha},m}$. Substituting the bound (4.29) for $\bar{\mathcal{H}}_m$ into Lemma 4.22 gives

$$|\mathcal{L}_m'(t)| \leq 2(\lambda_{m-1}+1)^{1/2}\mathcal{L}_m(t) + C\,\sqrt{\bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m}}\,\mathrm{e}^{\bar{\alpha}\,t/2}\,\sqrt{\mathcal{L}_m(t)}\,. \tag{4.43}$$

Consequently, if the maximum of $f_2(t) = \mathrm{e}^{-\bar{\alpha}\,t}\,\mathcal{L}_m(t)$ occurs at a point s in the interior $(0,T)$, then we get

$$\bar{\alpha}\,\ell_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}s} = \bar{\alpha}\,\mathcal{L}_m(s) = \mathcal{L}_m'(s) \leq 2(\lambda_{m-1}+1)^{1/2}\,\ell_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}s} + C(\bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m})^{1/2}\,\ell_{\bar{\alpha},m}^{1/2}\,\mathrm{e}^{\bar{\alpha}\,s}\,, \tag{4.44}$$

so we would get that

$$\left(\bar{\alpha} - 2(\lambda_{m-1}+1)^{1/2}\right)\,\ell_{\bar{\alpha},m} \leq C(\bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m})^{1/2}\,\ell_{\bar{\alpha},m}^{1/2}\,. \tag{4.45}$$

Using (4.34), we would then get that

$$\ell_{\bar{\alpha},m} \leq C\,\left(\bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m}\right)\,, \tag{4.46}$$

where the constant C depends only on $\sup|V|$. Combining this with the fact that $\mathrm{e}^{-\bar{\alpha}\,t}\,\mathcal{L}_m(t) \leq I(0)$ at the endpoints, we get that

$$\ell_{\bar{\alpha},m} \leq C\,\left(\bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m}\right) + I(0)\,. \tag{4.47}$$

Bounding both $\ell_{\bar{\alpha},m}$ and $\bar{\ell}_{\bar{\alpha},m}$ in terms of $\bar{h}_{\bar{\alpha},m}$ and $I(0)$. If we substitute the bound (4.42) into (4.47), then we get

$$\ell_{\bar{\alpha},m} \leq C\,I(0) + C\,\left(\bar{h}_{\bar{\alpha},m} + \frac{4}{\bar{\alpha}^2}\,\ell_{\bar{\alpha},m}\right)\,. \tag{4.48}$$

As long as $\bar{\alpha}^2 \geq 8\,C$, then we can absorb the $\ell_{\bar{\alpha},m}$ term on the right to get

$$\ell_{\bar{\alpha},m} \leq 2\,C\,I(0) + 2\,C\,\bar{h}_{\bar{\alpha},m}\,. \tag{4.49}$$

Finally, substituting this back into (4.42) gives

$$\bar{\ell}_{\bar{\alpha},m} \leq 2\,I(0) + \bar{h}_{\bar{\alpha},m}\,. \tag{4.50}$$

Bounding $\bar{h}_{\bar{\alpha},m}$ in terms of $I(0)$. The starting point is to substitute the bounds (4.49) and (4.50) into Corollary 4.30, to get

$$\begin{aligned}\bar{\mathcal{H}}_m'' &\geq (4\lambda_m - C)\,\bar{\mathcal{H}}_m - \bar{\mathcal{H}}_m' - 3I(0) - \left[C\,\left(\bar{\ell}_{\alpha,m} + \bar{h}_{\alpha,m}\right) + 2\,\left(\bar{\ell}_{\alpha,m}\,\ell_{\alpha,m}\right)^{1/2}\right]\,\mathrm{e}^{\bar{\alpha}\,t} \\ &\geq (4\lambda_m - C)\,\bar{\mathcal{H}}_m - \bar{\mathcal{H}}_m' - C\,\left[\bar{h}_{\alpha,m} + I(0)\right]\,\mathrm{e}^{\bar{\alpha}\,t}\,,\end{aligned} \tag{4.51}$$

where C depends only on $\sup|V|$ and we absorbed the $3\,I(0)$ term into the last term. Define a function $f_3(t) = \mathrm{e}^{-\bar{\alpha}t}\bar{\mathcal{H}}_m(t)$ on $[0,T]$, so that f_3 is bounded by $I(0)$ at 0 and T and

$$f_3' = \mathrm{e}^{-\bar{\alpha}t}\left[\bar{\mathcal{H}}_m' - \bar{\alpha}\,\bar{\mathcal{H}}_m\right]\,, \tag{4.52}$$

$$f_3'' = \mathrm{e}^{-\bar{\alpha}t}\left[\bar{\mathcal{H}}_m'' - 2\bar{\alpha}\bar{\mathcal{H}}_m' + \bar{\alpha}^2\,\bar{\mathcal{H}}_m\right]\,. \tag{4.53}$$

At an interior maximum $s \in (0,T)$ for f_3, we have $f_3'(s) = 0$ and $f_3''(s) \leq 0$. Hence, by (4.52) and (4.53)

$$\bar{\mathcal{H}}_m'(s) = \bar{\alpha}\,\bar{\mathcal{H}}_m(s) = \bar{\alpha}\,\bar{h}_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}\,s}\,, \tag{4.54}$$

$$\bar{\mathcal{H}}_m''(s) \leq \bar{\alpha}^2\,\bar{\mathcal{H}}_m(s) = \bar{\alpha}^2\,\bar{h}_{\bar{\alpha},m}\,\mathrm{e}^{\bar{\alpha}\,s}\,. \tag{4.55}$$

Combining these with (4.51) and multiplying through by $\mathrm{e}^{-\bar{\alpha}\,s}$ would give

$$(4\lambda_m - C)\,\bar{h}_{\bar{\alpha},m} - \bar{\alpha}\,\bar{h}_{\bar{\alpha},m} - C\,(\bar{h}_{\bar{\alpha},m} + I(0)) \leq \bar{\alpha}^2\,\bar{h}_{\bar{\alpha},m}\,. \tag{4.56}$$

If we now substitute (4.35) into this, then we would get that

$$\bar{h}_{\bar{\alpha},m} \leq \left(4\lambda_m - 2C - \bar{\alpha}^2 - \bar{\alpha}\right)\bar{h}_{\bar{\alpha},m} \leq C\,I(0)\,. \tag{4.57}$$

On the other hand, if the maximum of f_3 occurs at 0 or T, then we would get $\bar{h}_{\bar{\alpha},m} \leq I(0)$ so we conclude that (4.57) holds in either case. Combining all of this gives that

$$\max_{[0,T]}\left(\mathrm{e}^{-\bar{\alpha}\,t}\int_{N\times\{t\}} u^2\,d\theta\right) \leq \bar{h}_{\bar{\alpha},m} + \bar{\ell}_{\bar{\alpha},m} \leq C\,I(0)\,. \tag{4.58}$$

□

4.3. **Choosing $\bar{\alpha}$.** We will now show that Proposition 4.36 implies Theorem 0.12. The difference between the bounds in Proposition 4.36 and those in Theorem 0.12 is that the constant $\bar{\alpha}$ in Proposition 4.36 depends on the spectral gaps for the manifold N. On the other hand, when $N = \mathbf{S}^n$ (or a Zoll surface) we can use the explicit eigenvalue gaps to bound $(|\bar{\alpha}| - |\alpha|)$ uniformly. Namely, since the m-th cluster of eigenvalues on $\mathbf{S}^n$ occurs at

$$b_m = m^2 + (n-1)\,m\,, \tag{4.59}$$

we get that

$$b_{m-1} = m^2 + (n-3)\,m + 2 - n \text{ and } (b_{m-1}+1)^{1/2} = m + \frac{n-3}{2} + O(m^{-1})\,, \tag{4.60}$$

where $O(m^{-1})$ denotes a term that is bounded by $C\,m^{-1}$ for all $m \neq 0$. This gives

$$4\,b_m - \left(2\,(b_{m-1}+1)^{1/2}+1\right)^2 - \left(2\,(b_{m-1}+1)^{1/2}+1\right) = 2\,m + O(1)\,. \tag{4.61}$$

The key point is that the coefficient of the leading order term is positive, so there exists some m_0 depending only on κ and n so that both (4.34) and (4.35) hold for all $m \geq m_0$ with

$$\bar{\alpha} = 2\,(b_{m-1}+1)^{1/2} + 1\,. \tag{4.62}$$

Next, let m_1 be the smallest positive integer with

$$2\,(b_{m_1-1}+1)^{1/2}+1\geq\alpha\,. \tag{4.63}$$

Since $(b_{m-1}+1)^{1/2}$ grows linearly in m, there is a uniform bound for $2\,(b_{m_1-1}+1)^{1/2}+1-\alpha$. Finally, let m be the maximum of m_0 and m_1 and define $\bar{\alpha}$ by (4.62). It follows that we get a uniform bound for $|\bar{\alpha}|-|\alpha|$ that depends only on κ and n.

A similar argument applies for Zoll surfaces. In particular, this discussion shows that Proposition 4.36 gives Theorem 0.12.

4.4. **The frequency function.** The frequency function often gives an alternative approach to proving a three circles inequality for second order elliptic equations. This method is predicated upon having a function whose hessian is diagonal, such as $|x|^2$ on $\mathbf{R}^{n+1}$ or the function t on $N\times\mathbf{R}$. However, we will see that this method does not yield our three circles inequality, but would instead require some integrability of V as in [GL]. For simplicity and clarity, we will restrict to the case $\mathbf{S}^1\times\mathbf{R}$.

The frequency function $U(t)$ measures the logarithmic rate of growth of a function u. Namely, if we set $J(s)=\int_{\mathbf{S}^1\times\{s\}}u^2\,d\theta$, then the frequency is given by

$$U(t)=\partial_t\log J(t)=\frac{J'(t)}{J(t)}\,. \tag{4.64}$$

This is useful because $U(t)$ is a monotone non-decreasing function of t if u is harmonic and[12]

$$\lim_{t\to-\infty}\int_{\mathbf{S}^1\times\{t\}}|\nabla u|^2\,d\theta=0\,. \tag{4.65}$$

To see why U is monotone, first differentiate J to get that

$$J'(s)=2\int_{\mathbf{S}^1\times\{s\}}(uu_t)\;d\theta \tag{4.66}$$

$$J''(s)=2\int_{\mathbf{S}^1\times\{s\}}\left(u_t^2+u_\theta^2\right)\;d\theta\,, \tag{4.67}$$

where the second equation used that $u_{tt}=-u_{\theta\theta}$ and integration by parts on $\mathbf{S}^1$. To get this in a better form, observe that since $u_{tt}=-u_{\theta\theta}$

$$\partial_s\int_{\mathbf{S}^1\times\{s\}}(u_t^2-u_\theta^2)\,d\theta=-2\int_{\mathbf{S}^1\times\{s\}}\partial_\theta\;(u_tu_\theta)\;d\theta=0\,. \tag{4.68}$$

Since we assumed that ∇u vanishes at $-\infty$ in (4.65), it follows that

$$\int_{\mathbf{S}^1\times\{s\}}u_t^2\,d\theta=\int_{\mathbf{S}^1\times\{s\}}u_\theta^2\,d\theta\,. \tag{4.69}$$

Plugging this into the formula (4.67) for J'' gives

$$J''(s)=4\int_{\mathbf{S}^1\times\{s\}}u_t^2\,d\theta\,. \tag{4.70}$$

It now follows from the Cauchy-Schwarz inequality that $(J')^2\leq J\,J''$, so we conclude that $U'=\left[J''\,J-(J')^2\right]/J^2\geq 0$ as claimed.

[12]Equation (4.65) rules out functions like the linear function t where the frequency is *not* monotone.

Suppose now that u is no longer harmonic, but instead satisfies the Schrödinger equation $\Delta u = -Vu$. In this case, we get that

$$J''(s) = 2 \int_{\mathbf{S}^1 \times \{s\}} \left(u_t^2 + u_\theta^2 - V u^2 \right) \, d\theta \, , \tag{4.71}$$

introducing the "error term" $-\int_{\mathbf{S}^1 \times \{s\}} V u^2 \, d\theta$ in $J''(s)$ and giving an estimate of the form

$$(\log J)'' = U' \geq -C \, \sup |V| \, . \tag{4.72}$$

However, this lower bound is not integrable in t, so U can decrease by an arbitrarily large amount over a long enough stretch. We will see next that this method does not yield the three circles inequality of Theorem 0.8, i.e.,

$$\log I(t) \leq C(1+t) + \frac{t}{T} \left| \log \frac{I(T)}{I(0)} \right| + \log I(0) \, . \tag{4.73}$$

Namely, integrating (4.72) from s to T gives

$$U(s) \leq U(T) + C \, (T - s) \, , \tag{4.74}$$

and integrating this from 0 to t gives

$$\log J(t) = \log J(0) + \int_0^t U(s) \, ds \leq \log J(0) + t \, (U(T) + C \, T) \, . \tag{4.75}$$

To see why (4.73) is sharper than (4.75), suppose that $|U(T)|$ and $|\log I(T)|/T$ are uniformly bounded but let T go to infinity. In this case, the upper bound in (4.73) goes to $\log I(0) + C(1+t)$, whereas the upper bound in (4.75) goes to infinity.

It is interesting to note that N. Garofalo and F.H. Lin, [GL], proved unique continuation in a similar setting by using the frequency function under the stronger assumption that

$$\int_{\mathbf{R}} \left(\sup_{\mathbf{S}^1 \times \{s\}} |V| \right) \, ds < \infty \, . \tag{4.76}$$

5. Dimension bounds on a manifold with cylindrical ends

In this section, we consider functions u in H_0 that solve the Schrödinger equation $\Delta u = -Vu$ for a general bounded potential V on a manifold M with finitely many cylindrical ends, each of which is the product of a half-line with a round sphere or a Zoll surface.[13] In particular, M can be decomposed into a bounded region Ω together with a finite collection of ends $E_1, \dots, E_k$ where

- Ω has compact closure.
- Each E_j is isometric to $N_j \times [0, \infty)$, where N_j is either a sphere or a Zoll surface.

The main result of this section is that $H_\alpha(M)$ is finite dimensional for every $\alpha \in \mathbf{R}$.

Theorem 5.1. The linear space H_α has dimension at most

$$d = d \, (\alpha, \sup |V|, \Omega) \, . \tag{5.2}$$

Note that F. Hang and F.H. Lin, [HL], proved a similar result under the stronger hypothesis that $\sup |V| < \epsilon$ for some sufficiently small $\epsilon > 0$.

[13] A similar argument applies when the ends have spectral gaps as in (0.2) and V is Lipschitz.

5.1. **A consequence of unique continuation.** We will need an estimate that relates the $W^{1,2}$ norm of a solution u on the boundary of Ω to its $W^{1,2}$ norm inside Ω. This is given in the next lemma, where we will use $T_1(\Omega)$ to denote the tubular neighborhood of radius one about Ω.

Lemma 5.3. Given $\alpha \geq 0$, there exists a constant C depending on α, Ω, and $\sup |V|$ so that if $u \in H_\alpha$, then

$$\int_{T_1(\Omega)} \left(u^2 + |\nabla u|^2\right) \leq C \int_{\partial\Omega} \left(u^2 + |\nabla u|^2\right) . \tag{5.4}$$

Proof. We will argue by contradiction, so suppose instead that there is a sequence of functions u_j with $\Delta u_j = -V_j u_j$ where we have a uniform bound for $\sup |V_j|$, each u_j is in $H_\alpha(\Delta + V_j)$, and

$$\int_{T_1(\Omega)} \left(u_j^2 + |\nabla u_j|^2\right) > j \int_{\partial\Omega} \left(u_j^2 + |\nabla u_j|^2\right) . \tag{5.5}$$

The key to the compactness argument is that Corollary 0.15 gives a constant $\nu > 0$ (independent of j) so that

$$\int_{T_1(\Omega)\setminus\Omega} \left(u_j^2 + |\nabla u_j|^2\right) \leq \nu \int_{\partial\Omega} \left(u_j^2 + |\nabla u_j|^2\right) . \tag{5.6}$$

Therefore, after renormalizing the u_j's, we get that

$$\int_{T_1(\Omega)} \left(u_j^2 + |\nabla u_j|^2\right) = 1 \text{ and } \int_{T_1(\Omega)\setminus\Omega} \left(u_j^2 + |\nabla u_j|^2\right) < \nu/j . \tag{5.7}$$

The interior $W^{2,p}$ estimates (theorem 9.11 in [GT]) then give a uniform $W^{2,2}(T_{3/4}(\Omega))$ bound for the u_j's. By combining this with the Sobolev inequality (theorem 7.26 in [GT]), we get uniform higher L^p bounds on the u_j's, and hence on $V_j u_j$, and then elliptic theory again gives a higher $W^{2,p}$ bound on u_j's. After repeating this a finite number of times (depending on n), we will get a uniform $W^{2,p}$ bound for $p > (n+1)$. Once we have this, the Sobolev embedding (theorem 7.26 in [GT]) gives a uniform $C^{1,\mu}$ bound

$$||u_j||_{C^{1,\mu}(T_{1,2}(\Omega))} \leq \tilde{C} , \tag{5.8}$$

where $\mu > 0$ and $\tilde{C}$ does not depend on j. We will refer to this argument as "bootstrapping."

It follows from (5.8) that a subsequence of the u_j's converges uniformly in $C^1(T_{1/2}(\Omega))$ to a function u and thus, by (5.7), u satisfies

$$\int_{\Omega} \left(u^2 + |\nabla u|^2\right) = 1 \text{ and } \int_{T_{1/2}(\Omega)\setminus\Omega} \left(u^2 + |\nabla u|^2\right) = 0 . \tag{5.9}$$

We will see that this violates unique continuation of [A] since u vanishes on an open set, but is not identically zero. Namely, since the u_j's satisfy

$$|\Delta u_j| = |V_j|\, |u_j| \leq \left(\sup_j \sup_M |V_j|\right) |u_j| \leq C' \,|u_j| , \tag{5.10}$$

it follows that

$$|\Delta u| \leq C' \,|u| . \tag{5.11}$$

Finally, the differential inequality (5.11) allows us to directly apply [A]. □

5.2. **The proof of Theorem 5.1.** We will prove Theorem 5.1 by getting an upper bound for the number of $W^{1,2}(\partial\Omega)$-orthonormal functions in $H_\alpha(M)$; cf. [CM2].

Proof. (of Theorem 5.1.) Assume that $u_1, \dots, u_d$ are functions in $H_\alpha(M)$ that are $W^{1,2}(\partial\Omega)$-orthonormal, i.e., with

$$\int_{\partial\Omega} (u_i u_j + \langle \nabla u_i, \nabla u_j \rangle) = \delta_{ij} \, . \tag{5.12}$$

It follows from Corollary 0.15 that we can find a set of such functions for any finite d that is less than or equal to $\dim(H_\alpha)$. Therefore, the theorem will follow from proving an upper bound on d.

Let U denote the vector space spanned by the u_j's and define the projection kernel $K(x,y)$ to U on $\partial\Omega \times \partial\Omega$ by

$$K(x,y) = \sum_{j=1}^{d} (u_j(x)\, u_j(y) + \langle \nabla u_j(x), \nabla u_j(y) \rangle) \, . \tag{5.13}$$

Note that $\int_{\partial\Omega} K(x,x) = d$. We will also need the following standard estimate for $K(x,x)$

$$K(x,x) \leq (n+1) \sup_{u \in U \setminus \{0\}} \frac{u^2(x) + |\nabla u|^2(x)}{\int_{\partial\Omega} (u^2 + |\nabla u|^2)} \, . \tag{5.14}$$

To see this, observe first that $K(x,x)$ can be thought of the trace of a symmetric quadratic form on U and is therefore independent of the choice of $W^{1,2}(\partial\Omega)$-orthonormal basis for U. Since the map taking $u \in U$ to $(u(x), \nabla u(x))$ is a linear map from U to $\mathbf{R}^{n+1}$, we can choose a new $W^{1,2}(\partial\Omega)$-orthonormal basis $v_1, \dots, v_d$ for U so that $v_j(x)$ and $\nabla v_j(x)$ vanish for every $j > (n+1)$. Expressing $K(x,x)$ in this new basis gives

$$K(x,x) = \sum_{j=1}^{n+1} \left(v_j^2(x) + |\nabla v_j|^2(x) \right) \, , \tag{5.15}$$

and (5.14) follows.

We will use (5.14) to prove a pointwise estimate for $K(x,x)$. Namely, Lemma 5.3 implies for any $u \in U \setminus \{0\}$ that

$$\int_{T_1(\Omega)} \left(u^2 + |\nabla u|^2 \right) \leq C \int_{\partial\Omega} \left(u^2 + |\nabla u|^2 \right) \, , \tag{5.16}$$

where C depends only on α, Ω, and $\sup|V|$. Applying the bootstrapping argument of (5.8) to u, i.e., interior $W^{2,p}$ estimates and the Sobolev embedding (theorems 9.11 and 7.26 in [GT]), we get

$$u^2(x) + |\nabla u|^2(x) \leq C \int_{\partial\Omega} \left(u^2 + |\nabla u|^2 \right) \, , \tag{5.17}$$

where the new constant C still depends only on α, Ω, and $\sup|V|$. Substituting this back into (5.14) and integrating gives

$$d = \int_{\partial\Omega} K(x,x) \leq (n+1)\, C \, \mathrm{Volume}\,(\partial\Omega) \, , \tag{5.18}$$

giving the desired upper bound for d. □

6. Density of potentials with $H_0 = \{0\}$

As in the previous section, we will consider Schrödinger operators $\Delta + V$ where V is a bounded potential on a fixed manifold M with finitely many cylindrical ends, each of which is the product of a half-line with a round sphere or a Zoll surface.[14] In particular, M can be decomposed into a bounded region Ω together with a finite collection of ends $E_1, \dots, E_k$ where

- Ω has compact closure.
- Each E_j is isometric to $N_j \times [0, \infty)$, where N_j is either a sphere or a Zoll surface.

The main result of this section, Proposition 6.1, shows that the set of potentials V where $H_0 = \{0\}$ is dense.

Proposition 6.1. Suppose that $L = \Delta + V$ and f is a non-negative bounded function with compact support in Ω that is positive somewhere. There exists $\epsilon_0 > 0$ so that for all $\epsilon \in (0, \epsilon_0)$ we have

$$H_0(L + \epsilon\, f) = \{0\}\,. \tag{6.2}$$

One of the key properties that we will need in the proof is that if u and v are solutions of $Lu = Lv = 0$, then

$$\operatorname{div}\,(u\nabla v - v \nabla u) = 0\,. \tag{6.3}$$

Motivated by this, we define the skew-symmetric bilinear form $\omega(\cdot,\cdot)$ on functions that are in $L^2(\partial\Omega)$ and whose normal derivatives are in $L^2(\partial\Omega)$ by setting

$$\omega(u,v) = \int_{\partial\Omega} (u\partial_n v - v \partial_n u)\ . \tag{6.4}$$

The next lemma uses Stokes' theorem and (6.3) to prove that if u and v are solutions of $Lu = Lv = 0$ on $M \setminus \Omega$ that vanish at infinity, then $\omega(u,v) = 0$.

Lemma 6.5. If u, v are functions on $M \setminus \Omega$ that satisfy $Lu = Lv = 0$ and go to zero at infinity on each end E_j, then

$$\omega(u,v) = 0\,. \tag{6.6}$$

Proof. Since u and v go to zero at infinity, the bootstrapping argument of (5.8) implies that

$$\lim_{t\to\infty} \sum_{j=1}^m \int_{N_j \times \{t\}} \left(u^2 + v^2 + |\nabla u|^2 + |\nabla v|^2\right)\big|_{E_j}\, d\theta = 0\,. \tag{6.7}$$

The lemma now follows since Stokes' theorem and (6.3) imply that for any $t \geq 0$ we have

$$\omega(u,v) = \sum_{j=1}^m \int_{N_j \times \{t\}} (u\partial_n v - v\partial_n u)\big|_{E_j}\, d\theta\,. \tag{6.8}$$

□

[14] A similar argument applies when the ends have spectral gaps as in (0.2) and V is Lipschitz.

Proof. (of Proposition 6.1.) Fix a non-negative bounded function f with compact support in Ω. We will prove the existence of such a $\epsilon_0 > 0$ by contradiction. Suppose therefore that there exists a sequence $\epsilon_j \to 0$ and functions u_j with

$$u_j \in H_0(L + \epsilon_j\, f) \setminus \{0\}\,. \tag{6.9}$$

After dividing each u_j by its $W^{1,2}$ norm on $\partial\Omega$ (this is non-zero by Lemma 5.3 and unique continuation, [A]), we can assume that

$$\int_{\partial \Omega} \left(u_j^2 + |\nabla u_j|^2\right) = 1\,. \tag{6.10}$$

Lemma 5.3 then gives a constant C depending on Ω, $\sup |V|$ and $\sup |f|$ so that

$$\int_{T_1(\Omega)} \left(u_j^2 + |\nabla u_j|^2\right) \leq C\,. \tag{6.11}$$

The bootstrapping argument (i.e., $W^{2,p}$ estimates and Sobolev embedding; cf. (5.8)) then gives uniform $C^{1,\mu}$ estimates for the u_j's on the smaller tubular neighborhood for some $\mu > 0$. Therefore, there is a subsequence (which we will still denote u_j) so that u_j and ∇u_j converge uniformly in $T_{1/2}(\Omega)$ and the limiting function u satisfies the limiting equation[15]

$$L u = 0\,. \tag{6.12}$$

Since u_j and ∇u_j converge uniformly on $\partial\Omega$, we get that

$$\int_{\partial \Omega} \left(u^2 + |\nabla u|^2\right) = \lim_{j\to\infty} \int_{\partial \Omega} \left(u_j^2 + |\nabla u_j|^2\right) = 1\,, \tag{6.13}$$

$$\omega(u, u_k) = \lim_{j\to\infty}\, \omega(u_j, u_k) = 0\,, \tag{6.14}$$

where the last equality follows from Lemma 6.5 since $Lu_j = Lu_k = 0$ outside of Ω and both vanish at infinity on $E_1, \dots, E_k$ (recall that f has compact support in Ω). Note that (6.14) would have followed from Lemma 6.5 alone if we had known that u also vanishes at infinity.

To get a contradiction, we note that

$$\operatorname{div}\left(u \nabla u_k - u_k \nabla u\right) = -\epsilon_k\, f\, u_k\, u\,, \tag{6.15}$$

so that (6.14) and Stokes' theorem gives

$$0 = \omega(u, u_k) = \int_{\Omega} \operatorname{div}\left(u \nabla u_k - u_k \nabla u\right) = -\epsilon_k\, \int_{\Omega} f\, u\, u_k\,. \tag{6.16}$$

In particular, since $\epsilon_k > 0$, we must have

$$\int_{\Omega} f\, u\, u_k = 0\,. \tag{6.17}$$

Since $u_k \to u$ uniformly in Ω, these integrals converge to the integral of $f u^2$, so we get that

$$\int_{\Omega} f\, u^2 = 0\,. \tag{6.18}$$

[15]Initially, we only know that $Lu = 0$ weakly, but elliptic regularity then implies that u is a strong solution.

Since f is non-negative, but positive somewhere, we conclude that u vanishes on an open set and, by unique continuation, [A], that u is identically zero. However, this contradicts (6.13), so we conclude that no such sequence could have existed. The proposition follows. □

We can now sum up what we have proven so far as:

Theorem 6.19. Theorem 0.4 holds when each end is *isometric* to a half-cylinder.

Strictly speaking, we have shown the density of potentials where $H_0 = \{0\}$, but have not yet addressed the density of metrics where $H_0 = \{0\}$, namely, the more general case of the theorem that holds for $n = 1$. However, this is an easy consequence of what we have already shown. To see this, assume that M is 2-dimensional and change the metric g conformally by e^{2f}, where f is bounded, to get an equivalence between $H_0(\Delta_g + \mathrm{e}^{2f}\, V)$ and $H_0(\Delta_{\mathrm{e}^{2f}g} + V)$ (see (7.10)). So long as V is not identically zero, this allows us to perturb the potential to a nearby potential with $H_0 = \{0\}$. In the remaining case, where $V \equiv 0$ and the operator is the Laplacian, it follows from Stokes' theorem that $H_0 = \{0\}$. Namely, the gradient estimate implies that $|u| + |\nabla u| \to 0$ at infinity, so Stokes' theorem gives that $\int |\nabla u|^2 = 0$ and u must be constant; since the only constant that goes to zero at infinity is 0, we get $H_0 = \{0\}$.

6.1. The set of potentials where $H_0 = \{0\}$ is NOT open.

Example 6.20. $H_0 = \{0\}$ is not an open condition. Namely, it is easy to construct a sequence of potentials V_i on $\mathbf{R}$ with $|V_i|_{C^1} \to 0$ as $i \to \infty$ and so that $\dim H_0(V_i) > 0$ for each i (note that $H_0(0) = \{0\}$; the limiting Schrödinger equation has potential equal to 0).

To be precise, as we saw right after Theorem 0.4, it is easy to construct a potential V on the line (or the cylinder) such that there exists a solution to the corresponding Schrödinger equation that goes exponentially to zero at both plus and minus infinity. (On the cylinder the potential, as well as the solution, can be taken to be rotationally symmetric, i.e., independent of θ.) In fact, the potential can be taken to be constant (negative) outside a compact set. Pick such a potential and name it V. On the line look at the rescalings, $V_\epsilon(t) = \epsilon^2\, V(\epsilon t)$ (on the cylinder rescale just in the t direction, everything is rotationally symmetric). Each of the Schrödinger equations $u'' + V_\epsilon u = 0$ has solutions (namely, $u_\epsilon(t) = u(\epsilon t)$, where u is a solution to $u'' + Vu = 0$) that decay exponentially to zero at plus and minus infinity and clearly $|V_\epsilon|_{C^1} \to 0$ as $\epsilon \to 0$. Note also that in this example u_ϵ converges to the constant function $u(0)$ as ϵ goes to zero, which may be taken to be non-zero and is, of course, in any case, a solution to the limiting Schrödinger equation $u'' = 0$.

7. Surfaces with more general ends: The case $n = 1$ of Theorem 0.4

We will show in this section that our results apply to a more general class of surfaces, namely surfaces with bounded curvature, injectivity radius, and finitely many ends that are each bi-Lipschitz to a half-cylinder. We will prove this by finding a bi-Lipschitz conformal change of metric on such a surface that makes each end isometric to a (flat) half-cylinder and then apply our earlier results. For this, we will need the following proposition.

Proposition 7.1. Suppose that E is topologically a half-cylinder $\mathbf{S}^1 \times [0, \infty)$ with a Riemannian metric satisfying the following bounds:

(B1) The Gauss curvature K_E is bounded above and below by $|K_E| \leq 1$ and the injectivity radius of E is bounded below by $i_0 > 0$.

(B2) There is a bi-Lipschitz (bijection)

$$F = (\theta, r) : E \to \mathbf{S}^1 \times [0, \infty) \tag{7.2}$$

with $|dF| \leq \ell_0$ and $|dF^{-1}| \leq \ell_0$.

Then there exists a conformal map $\Phi : E \to \mathbf{S}^1 \times [0, \infty)$ satisfying $|d\Phi|_{C^1} \leq C$ and $|d\Phi^{-1}|_{C^1} \leq C$ for a constant C depending only on i_0 and ℓ_0.

Remark 7.3. There are two natural equivalent norms for the differential dF, depending on whether one thinks of $dF(x)$ as a vector in $\mathbf{R}^4$ (Hilbert-Schmidt norm) or as a linear operator on $\mathbf{R}^2$ (operator norm). We will use the Hilbert-Schmidt norm, but will often use that $|dF(x)(v)| \leq |dF(x)|\,|v|$. If $\Phi : (\Sigma, g) \to (\tilde{\Sigma}, \tilde{g})$ is a conformal map between surfaces, then this convention gives that

$$\Phi^*(\tilde{g}) = \frac{1}{2}\,|d\Phi|^2\, g \text{ and } |d\Phi^{-1}|^2 \circ \Phi = 4\,|d\Phi|^{-2}\,. \tag{7.4}$$

Assuming Proposition 7.1 for the moment, we will now complete the remaining case of Theorem 0.4 where $n = 1$ and the end merely has bounded geometry and is bi-Lipschitz to a flat half-cylinder (as opposed to being isometric to it). Namely, we prove the following theorem:

Theorem 7.5. Let Σ be a complete non-compact surface with finitely many cylindrical ends, each of which has bounded geometry and is bi-Lipschitz to a flat half-cylinder.

(1) If V is bounded (potential) on Σ, then $H_\alpha(\Sigma, \Delta_\Sigma + V)$ is finite dimensional for every α; the bound for $\dim H_\alpha$ depends only on Σ, α, and $||V||_{L^\infty}$.

(2) For a dense set of bounded potentials $H_0 = \{0\}$. For each fixed bounded V, there is a dense set of metrics (with finitely many cylindrical ends) where $H_0 = \{0\}$.

Proof. Using Proposition 7.1 we will first show that there exists another metric $\tilde{g}$ on Σ for which each end is a flat cylinder and a conformal diffeomorphism $\Phi : (\Sigma, g) \to (\Sigma, \tilde{g})$ with $|d\Phi|_{C^1} \leq C$ and $|d\Phi^{-1}|_{C^1} \leq C$. To do this, we first apply Proposition 7.1 to each end E_j of Σ to get conformal diffeomorphisms

$$\Phi_j : E_j \to \mathbf{S}^1 \times \mathbf{R}\,, \tag{7.6}$$

with a uniform C^1 bound for every j, i.e., a constant κ so that away from ∂E_j we have

$$|d\Phi_j|,\, |\nabla d\Phi_j|,\, |d\Phi_j^{-1}|,\, |\nabla d\Phi_j^{-1}| \leq \kappa\,. \tag{7.7}$$

Note that the pullback metric $|d\Phi_j|^2\, g$ makes the end E_j into a flat cylinder. It remains to patch these metrics together across the compact part $\Omega = \Sigma \setminus \bigcup_j E_j$ of Σ. Let ϕ be a smooth function on Σ that is identically one on the tubular neighborhood of radius one about Ω and has compact support in Σ and then set

$$\tilde{g} = \left(\phi + (1 - \phi)\, \chi_{E_j}\, |d\Phi_j|^2\right)\, g\,. \tag{7.8}$$

Here χ_{E_j} is the characteristic function of E_j.

To see how the operator $\Delta + V$ changes under a conformal change of metric, let Σ be a surface with a Riemannian metric g and f a smooth function on Σ. If u is a solution of the Schrödinger equation $\Delta_g u + V\, u = 0$ on Σ, then u also solves the equation $\Delta_{e^{2f}\, g} u + e^{-2f}\, V\, u = 0$ for the conformally changed metric. Namely,

$$\Delta_{e^{2f}\, g} u = \operatorname{div}_{e^{2f}\, g}(\nabla_{e^{2f}\, g} u) = \operatorname{div}_{e^{2f}\, g}(e^{-2f} \nabla_g u) = e^{-2f} \Delta_g u\,. \tag{7.9}$$

From this we see that if g and $\tilde{g} = \mathrm{e}^{2f}\, g$ are conformal metrics on a surface Σ, then

$$H_0(\Delta_g + V) = H_0(\Delta_{\mathrm{e}^{2f} g} + \mathrm{e}^{-2f}\, V)\,. \tag{7.10}$$

In particular, applying (7.10) to the surface Σ with cylindrical ends with $\tilde{g}$ given by (7.8), so that the ends of $(\Sigma, \tilde{g})$ are isometric to the cylinder, we get that $\dim(H_0(\Delta_g + V))$ is finite and is equal to zero for a generic potential V. More precisely, the finite dimensionality follows from applying Theorem 5.1 to the operator $\Delta_{\mathrm{e}^{2f} g} + \mathrm{e}^{-2f}\, V$, since the Lipschitz bounds on $|d\Phi|$ and $|d\Phi|^{-1}$ imply that

$$||\, |d\Phi|^{-2}\, V||_{C^1(\tilde{g})} \leq C\, ||V||_{C^1(g)}\,. \tag{7.11}$$

Arguing similarly, the zero dimensionality for a dense set of potentials V follows from (7.11) and the density for the metric $\tilde{g}$ proven in Theorem 6.19. □

It remains to use the bi-Lipschitz map F to find a bi-Lipschitz conformal map Φ from each end to a flat half-cylinder. This will be accomplished in the next two subsections. The first subsection proves the existence of a conformal diffeomorphism Φ and proves an L^∞ estimate, bounding the second component of Φ above and below in terms of r (see (7.2)). The second subsection proves the uniform Lipschitz estimates on the conformal factor $|d\Phi|^2$ and its inverse.

7.1. **Uniformization of a cylindrical end.** The next lemma constructs a harmonic function u on a cylindrical end E that is bounded above and below by the Lipschitz function r, i.e., the second component of the map F given by (7.2).

Lemma 7.12. There exist constants $C_0, C_1 > 0$ depending on i_0 and ℓ_0 so that if E satisfies (B1) and (B2), then there is a positive harmonic function u on E that vanishes on ∂E and satisfies

$$C_0^{-1} < \int_{\partial E} \partial_n u < C_0\,, \tag{7.13}$$

$$C_1^{-1}\, r(x) \leq u(x) \leq C_1\, r(x) \text{ for } r(x) \geq 1\,, \tag{7.14}$$

$$0 < |\nabla u|\,. \tag{7.15}$$

Remark 7.16. It follows that u and its harmonic conjugate u^* together give a proper conformal diffeomorphism

$$(u^*, u) : E \to \tau \mathbf{S}^1 \times [0, \infty)\,, \tag{7.17}$$

where the radius τ is given by

$$2\pi\, \tau = \int_{\partial E} \partial_n u\,. \tag{7.18}$$

To see this, observe that while u^* is not a globally well-defined function, it is well-defined up to multiples of $2\pi\tau$. Hence, u^* is a well-defined map to the circle of radius τ. Finally, note that (7.13) gives upper and lower bounds for τ.

Proof. (of Lemma 7.12). We will construct u as a limit of harmonic functions u_j on $\{r \leq j\} \subset E$ as $j \to \infty$. Namely, let u_j be the harmonic function on $\{r \leq j\}$ with $u_j = 0$ on ∂E and $u_j = j$ on the other boundary component $\{r = j\}$ (note that u_j exists and is $C^{2,\alpha}$ to the boundary by standard elliptic theory; cf. theorem 6.14 in [GT]).

We will repeatedly use the following consequences of Stokes' theorem for any s between 0 and j (Stokes' theorem is used in the first and last equality below)

$$s \int_{\partial E} \partial_n u_j = s \int_{\{u_j = s\}} \partial_n u_j = \int_{\{u_j=s\}} u \, \partial_n u_j = \int_{\{u_j \leq s\}} |\nabla u_j|^2 \, . \tag{7.19}$$

The first step will be to establish the bounds in (7.13) for the function u_j for a constant C_0 that does not depend on j. To get the upper bound, use Stokes' theorem to get

$$j \int_{\partial E} \partial_n u_j = \int_{\{r \leq j\}} |\nabla u_j|^2 \leq \int_{\{r \leq j\}} |\nabla r|^2 \leq \ell_0^2 \, \text{Area}\,(\{r \leq j\}) \leq 2\pi \, \ell_0^4 \, j \, , \tag{7.20}$$

where the first inequality above uses that u_j and r have the same boundary values and harmonic functions minimize energy for their boundary values. (The last two inequalities in (7.20) used the bi-Lipschitz bound for the map F. We will use this again several times in the proof without comment.) We conclude that

$$\int_{\partial E} \partial_n u_j \leq 2\pi \, \ell_0^4 \, . \tag{7.21}$$

To get the lower bound, note that it follows easily from the maximum principle that the level set $\{u_j = s\}$ cannot be contractible, so we must have

$$i_0 \leq \text{Length}(u_j^{-1}(s)) \, . \tag{7.22}$$

Here, length means the one-dimensional Hausdorff measure if $u_j^{-1}(s)$ is not a collection of smooth curves. However, we will integrate (7.22) with respect to s and Sard's theorem implies that almost every level set is smooth, so this is not an issue.[16] Integrating (7.22) from 0 to j and using the coarea formula gives

$$j \, i_0 \leq \int_0^j \text{Length}(u_j^{-1}(s)) \, ds = \int_{\{r \leq j\}} |\nabla u_j| \, . \tag{7.23}$$

Plugging this into Cauchy-Schwarz gives

$$j^2 \, i_0^2 \leq \left(\int_{\{r \leq j\}} |\nabla u_j| \right)^2 \leq 2\pi \, \ell_0^2 \, j \int_{\{r \leq j\}} |\nabla u_j|^2 = 2\pi \, \ell_0^2 \, j^2 \int_{\partial E} \partial_n u_j \, , \tag{7.24}$$

where the last equality comes from applying Stokes' theorem twice, first to $\text{div}(u_j \nabla u_j)$ and then to change the boundary integral of $\partial_n u_j$ on $\{r = j\}$ into the boundary integral on ∂E. We conclude that

$$\frac{i_0^2}{2\pi \, \ell_0^2} \leq \int_{\partial E} \partial_n u_j \, . \tag{7.25}$$

Hence, we have uniform upper and lower bounds for the flux of the u_j's; this will give (7.13) in the limit.

We will next establish the bounds (7.14) for the u_j's for a constant C_1 that does not depend on j. These uniform estimates will allow us to extract a limit u that also satisfies

[16] This is really not an issue here since the argument below for (7.15) also implies that $|\nabla u_j| \neq 0$, so every level set is smooth.

(7.14). We will first show the lower bound in (7.14). It will be convenient to let $m_j(s)$ and $M_j(s)$ denote the minimum and maximum of u_j on $\{r = s\}$, i.e.,

$$m_j(s) = \min_{\{r=s\}} u_j(s)\,, \tag{7.26}$$

$$M_j(s) = \max_{\{r=s\}} u_j(s)\,. \tag{7.27}$$

To get the lower bound, first use Stokes' theorem and the coarea formula to get that

$$s \int_{\partial E} \partial_n u_j = \int_0^s \left(\int_{\{r=t\}} \partial_n u_j \right) dt \leq \int_0^s \left(\int_{\{r=t\}} |\nabla u_j| \right) dt = \int_{\{r \leq s\}} |\nabla r|\, |\nabla u_j|\,. \tag{7.28}$$

Next, apply Cauchy-Schwarz to this and then use the bi-Lipschitz bound on F and the fact that $\{r \leq s\} \subset \{u_j \leq M_j(s)\}$ to get

$$\begin{aligned} s^2 \left(\int_{\partial E} \partial_n u_j \right)^2 &\leq \int_{\{r\leq s\}} |\nabla r|^2 \int_{\{r \leq s\}} |\nabla u_j|^2 \leq 2\pi\, \ell_0^4\, s \int_{\{u_j \leq M_j(s)\}} |\nabla u_j|^2 \\ &= 2\pi\, \ell_0^4\, s\, M_j(s) \int_{\partial E} \partial_n u_j\,, \end{aligned} \tag{7.29}$$

where the last equality follows from (7.19) with $M_j(s)$ in place of s. Combining this with (7.25) gives the desired linear lower bound for $M_j(s)$

$$s\, \frac{i_0^2}{2\pi\, \ell_0^2} \leq s \int_{\partial E} \partial_n u_j \leq 2\pi\, \ell_0^4\, M_j(s)\,. \tag{7.30}$$

To get the upper bound for $m_j(s)$, use the fact that the level sets of u_j cannot be contractible, see (7.22), and the coarea formula to get

$$i_0\, m_j(s) \leq \int_0^{m_j(s)} \text{Length}(u_j^{-1}(t))\, dt = \int_{\{u_j \leq m_j(s)\}} |\nabla u_j|\,. \tag{7.31}$$

Applying Cauchy-Schwarz and noting that $\{u_j \leq m_j(s)\} \subset \{r \leq s\}$ gives

$$\begin{aligned} i_0^2\, m_j^2(s) &\leq \text{Area}\,(\{r \leq s\}) \int_{\{u_j \leq m_j(s)\}} |\nabla u_j|^2 \leq 2\pi\, \ell_0^2\, s \int_{\{u_j \leq m_j(s)\}} |\nabla u_j|^2 \\ &= 2\pi\, \ell_0^2\, s\, m_j(s) \int_{\partial E} \partial_n u_j \leq 4\pi^2\, \ell_0^6\, s\, m_j(s)\,, \end{aligned} \tag{7.32}$$

where the last inequality uses (7.21). We conclude that

$$m_j(s) \leq 4\pi^2\, s\, \frac{\ell_0^6}{i_0^2}\,. \tag{7.33}$$

As long as we stay away from the boundary of $\{r \leq j\}$, we can apply the Harnack inequality to the positive harmonic function u_j. In particular, the lower bound for $M_j(s)$ gives a uniform lower bound for u_j and the upper bound for $m_j(s)$ gives a uniform upper bound for u_j.

We will now use these uniform bounds on the u_j's on each compact set to extract a limit u. Note first that the upper bounds for the u_j's in terms of r and standard elliptic theory (the boundary Schauder estimates; see theorem 6.6 in [GT]) give a $C^{2,\alpha}$ bound for the u_j's on each compact subset of E.[17] Therefore, Arzela-Ascoli gives a subsequence of the u_j's that converges uniformly in C^2 on compact subsets of E to a continuous non-negative harmonic function u. The uniform convergence implies that u vanishes on ∂E and u also satisfies (7.13) and (7.14); in particular, u is not identically zero.

We will prove (7.15) by contradiction. Suppose therefore that $|\nabla u|$ vanishes at some $x \in E$. Note that x cannot be in ∂E because of the Hopf boundary point lemma (see lemma 3.4 in [GT]). Note also that (7.14) implies that u is proper, so the nodal set $u^{-1}(u(x))$ must be compact. It follows from the standard structure of the nodal set of a harmonic function on a surface (see, e.g., lemma 4.28 in [CM3]) and the fact that E is a topological cylinder that there is at least one connected component of $\{y \,|\, u(y) \neq u(x)\}$ that both has compact closure and does not contain ∂E in its boundary. However, this violates the strong maximum principle and, hence, we conclude that $|\nabla u| \neq 0$. □

7.2. **A uniform Lipschitz bound on the conformal factor.** In this subsection, we will prove a uniform Lipschitz bound for $|d\Phi|$ and $|d\Phi^{-1}|$ for any conformal diffeomorphism Φ from E satisfying (B1) and (B2) to the flat half-cylinder $\mathbf{S}^1 \times [0, \infty)$. We will then apply this estimate to the conformal map constructed in the previous subsection to get Proposition 7.1. The desired Lipschitz estimate is given in the next lemma.

Lemma 7.34. There is a constant μ depending on i_0 and ℓ_0, so that if E satisfies (B1) and (B2) and $\Phi : E \to \mathbf{S}^1 \times [0, \infty)$ is a conformal diffeomorphism, then away from the i_0-tubular neighborhood of ∂E we get

$$|d\Phi|_{C^1} \leq \mu \text{ and } |d\Phi^{-1}|_{C^1} \leq \mu \,. \tag{7.35}$$

We will need two preliminary lemmas in the proof of Lemma 7.34. The first is the following result of Bloch (see Appendix D):

Lemma 7.36. Given $r_0 > 0$ and $\kappa > 0$, there exists a constant $B > 0$ so that if Σ is a surface with $|K| \leq \kappa$, the ball $B_{r_0}(p)$ is a topological disk in $\Sigma \setminus \partial\Sigma$, and f is a holomorphic function on $B_{r_0}(p)$, then the image $f(B_{r_0}(p))$ covers some disk of radius $B\,|df(p)|$ in $\mathbf{C}$.

We will also need a standard Bochner type formula:

Lemma 7.37. If f is a holomorphic function on a surface E and $|\nabla f| \neq 0$, then

$$\Delta \log |\nabla f| = K \,. \tag{7.38}$$

Proof. Let u and v be the real and imaginary parts of f, so that $f = u + i\,v$. The Cauchy-Riemann equations give $|\nabla f| = \sqrt{2}\,|\nabla u|$ and, hence,

$$\Delta \log |\nabla f| = \Delta \log |\nabla u| \,. \tag{7.39}$$

The Bochner formula for the harmonic function u gives

$$\Delta \log |\nabla u|^2 = 2\,K + \frac{2|\text{Hess}_u|^2}{|\nabla u|^2} - \frac{|\nabla |\nabla u|^2|^2}{|\nabla u|^4} \,. \tag{7.40}$$

[17] This bound grows with r. In the next subsection, we will come back and prove a Lipschitz bound for $|\nabla u|$ that does not grow with r.

Fixing a point x and working in geodesic normal coordinates that diagonalize the hessian of u at x (so that $u_{11}(x) = \lambda = -u_{22}(x)$ and $u_{12}(x) = u_{21}(x) = 0$), we get

$$\begin{aligned} 2|\mathrm{Hess}_u|^2\,|\nabla u|^2 - |\nabla|\nabla u|^2|^2 &= 4\lambda^2\,|\nabla u|^2 - 4\,u_j u_{jk} u_\ell u_{\ell k} \\ &= 4\,\left(\lambda^2|\nabla u|^2 - \lambda^2(u_1^2+u_2^2)\right) = 0\,, \end{aligned} \tag{7.41}$$

giving the lemma. □

We can now prove the Lipschitz estimate, i.e., Lemma 7.34.

Proof. (of Lemma 7.34.) We will first use Lemma 7.36 to get the upper bound for $|d\Phi|$. Given a point x with $B_{i_0/2}(x) \subset E \setminus \partial E$, let $\tilde{\Phi} : B_{i_0/2}(x) \to \mathbf{C}$ denote the composition of Φ with the covering map from the cylinder to $\mathbf{C}$. The fact that $B_{i_0/2}(x)$ is a disk implies that $\tilde{\Phi}$ is a well-defined holomorphic function. Furthermore, the fact that Φ is injective implies that the projection of $\tilde{\Phi}(B_{i_0/2}(x))$ to the cylinder is also an injection. However, Lemma 7.36 implies that $\tilde{\Phi}(B_{i_0/2}(x))$ covers a disk in $\mathbf{C}$ of radius

$$B\,|\nabla\tilde{\Phi}(x)| = B\,|d\Phi(x)|\,, \tag{7.42}$$

so we must have

$$B\,|d\Phi(x)| \leq \pi\,, \tag{7.43}$$

as desired.

We will next use the upper bound (7.43) together with the fact that $|d\Phi| \neq 0$ to get a lower bound for $\log|d\Phi|$, and hence an upper bound for $|d\Phi^{-1}|$. We will use that the map $F = (\theta, r)$ maps E to the cylinder with bi-Lipschitz constant ℓ_0. The key for getting the lower bound for $\log|d\Phi|$ is that the function $w = \log|d\Phi| = \log|\nabla\tilde{\Phi}|$ satisfies $|\Delta w| = |K| \leq 1$ (by Lemma 7.37) and

$$c_1 = \log(1/\ell_0) \leq \inf_s \max_{r=s} w \leq \sup w \leq \log(\pi/B) = c_2\,. \tag{7.44}$$

The first inequality in (7.44) follows from the fact that the curve $\Phi(\{r = s\})$ wraps around the cylinder and hence has length at least 2π, so that

$$2\pi \leq \mathrm{Length}\,(\Phi(\{r=s\})) = \int_{\{r=s\}} |d\Phi| \leq 2\pi\,\ell_0 \max_{\{r=s\}} |d\Phi|\,. \tag{7.45}$$

The last inequality in (7.44) comes from the upper bound (7.43) for $|d\Phi|$. Applying the Harnack inequality[18] to the non-negative function $c_2 - w$ centered on a point where $w \geq c_1$ gives (away from the i_0 tubular neighborhood of ∂E)

$$\sup(c_2 - w) \leq k_1\,(c_2 - c_1) + k_2\,\sup|\Delta w| \leq k_1\,(c_2 - c_1) + k_2\,, \tag{7.46}$$

where the constants k_1 and k_2 depend only on i_0 and ℓ_0. Here we have used that every point in E is a bounded distance from a point where $w \geq c_1$ (by (7.44)) to ensure that we apply

[18]The Harnack inequality that we use here is that if $w \geq 0$ on B_{2R}, then $\sup_{B_R} w \leq C_1 \inf_{B_R} w + C_2 \sup_{B_{2R}} |\Delta w|$ where C_1 and C_2 depend on R, $\sup|K|$, and i_0. Using standard estimates for the exponential map, it suffices to prove that if L is a uniformly elliptic second order operator on $B_2 \subset \mathbf{R}^2$ and $w \geq 0$ on B_2, then $\sup_{B_1} w \leq C_1 \inf_{B_1} w + C_2 \sup_{B_2} |L\,w|$ where C_1 and C_2 depend only on the bounds for the coefficients of L. This follows by combining theorems 9.20 and 9.22 in [GT].

the Harnack inequality on balls of a fixed bounded size. Rewriting (7.46) gives the desired lower bound for $|d\Phi|$

$$\log |d\Phi| = w \geq c_2 - k_1 (c_2 - c_1) - k_2 \, . \tag{7.47}$$

We have now established uniform bounds on $|d\Phi|$ and $|d\Phi^{-1}|$. To get the Lipschitz bounds, we will first work on the image $\Phi(E) = \mathbf{S}^1 \times [0, \infty)$ to estimate $|\nabla_{\mathbf{S}^1 \times [0,\infty)} \tilde{w}|$ where

$$\tilde{w} = \log |d\Phi^{-1}| = \log 2 - w \circ \Phi^{-1} \, , \tag{7.48}$$

where the last equality used Remark 7.3. The L^∞ estimates above for w imply that $\tilde{w}$ is bounded. In addition, applying the formula for the Laplacian of a conformally changed metric (see (7.9)) to $\Delta w = K$ gives

$$\Delta_{\mathbf{S}^1 \times [0,\infty)} \tilde{w} = -\frac{1}{2} |d\Phi^{-1}|^2 \, K \circ \Phi^{-1} \, . \tag{7.49}$$

(The factor $\frac{1}{2}$ comes from our choice of norm; see Remark 7.3.) In particular, both $|\tilde{w}|$ and $|\Delta_{\mathbf{S}^1 \times [0,\infty)} \tilde{w}|$ are uniformly bounded. Therefore, we can directly apply the Euclidean Cordes-Nirenberg estimate (see, e.g., theorem 12.4 in [GT]) to get

$$|\tilde{w}|_{C^{1,\alpha}} \leq C \left(|\tilde{w}|_{L^\infty} + |\Delta_{\mathbf{S}^1 \times [0,\infty)} \tilde{w}|_{L^\infty} \right) \leq C' \, . \tag{7.50}$$

This gives the desired bound on $|\nabla_{\mathbf{S}^1 \times [0,\infty)} d\Phi^{-1}|$ and then, using the chain rule, it also gives the desired bound on $|\nabla d\Phi|$. □

Finally, we can combine Lemma 7.12 and Lemma 7.34 to prove Proposition 7.1.

Proof. (of Proposition 7.1.) Let u be the positive harmonic function given by Lemma 7.12 and let u^* be its harmonic conjugate. As in Remark 7.16, we conclude that the map

$$\Phi = \tau^{-1} (u^*, u) : E \to \mathbf{S}^1 \times [0, \infty) \tag{7.51}$$

is the desired conformal diffeomorphism. Here τ is defined in (7.18) and bounded in (7.13). The Lipschitz bounds on $|d\Phi|$ and $|d\Phi^{-1}|$ follow immediately from Lemma 7.34. □

Appendix A. Dimension bounds for rotationally symmetric potentials on cylinders

In this appendix, we bound the dimension of the space H_α for a rotationally symmetric potential on a flat cylinder. In the rotationally symmetric case, things become particularly simple, but, never the less, it illustrates some of the ideas needed for the actual argument. We include some simple ODE comparison results that will also be used within the body of the paper.

In this appendix, we will assume that M is a cylinder $N \times \mathbf{R}$ with global coordinates (θ, t) and that the function V depends only on t, i.e., that $V(\theta, t) = V(t)$, and that $V(t)$ is bounded.

The first result is that the space of functions that vanish at infinity in the kernel of $\Delta + V$ is finite dimensional (we state and prove this only for H_0; arguing similarly gives dimension bounds for any H_α, where the bound depends also on α):

Proposition A.1. The linear space H_0 has dimension at most

$$2\left|\{j \mid \lambda_j \leq \sup V\}\right| . \tag{A.2}$$

In particular, when $N = \mathbf{S}^1$, the dimension is 0 if $\sup V < 0$ and is bounded by $4\sqrt{\sup V(t)}+2$ otherwise.

The key to prove this proposition is that the Fourier coefficients $[u]_j(t)$ of a solution u, defined in the previous section, satisfy the ODE

$$w''(t) = (\lambda_j - V(t))\, w(t) . \tag{A.3}$$

The proposition will follow by first showing that if u is in H_+ and the j-th Fourier coefficient for $\lambda_j > \sup V$ is non-zero[19], then u grows exponentially at $-\infty$ and likewise for H_-. Thus if u lies in H_0, so that it lies in the intersection of H_+ and H_-, then all j-th Fourier coefficients must be zero for $\lambda_j > \sup V$ and hence u lies in a finite dimensional space. The exponential growth will follow from Corollary A.9 below. This corollary records a consequence of the standard Riccati comparison argument in a convenient form that will also be needed within the body of the paper. The standard proof is included for completeness.

Lemma A.4. If w is a function on $[0,\infty)$ that satisfies the ODE inequality $w'' \geq K^2\, w$, $w(0) > 0$, and w_K is a positive solution to the ODE $w_K'' = K^2\, w_K$ with $(\log w)'(0) \geq (\log w_K)'(0)$, then w is positive and for all $t \geq 0$

$$(\log w)'(t) \geq (\log w_K)'(t) . \tag{A.5}$$

Proof. Fix some $b > 0$ so that w is positive on $[0,b)$. We will show that (A.5) holds for $t \in [0,b)$. Once we have shown this, we can integrate (A.5) from 0 to t to get

$$\log w(t) \geq \log w(0) + \int_0^t (\log w_K)'(t)\, dt = \log w(0) + \log w_K(t) - \log w_K(0) , \tag{A.6}$$

so that $w(b) = \lim_{t\to b} \exp(\log w(t)) > 0$. It follows that the set $\{t \mid w(t) > 0\}$ is both open and closed in $[0,\infty)$, so that $w(t) > 0$ for all $t \geq 0$. Consequently, (A.5) holds for all $t \geq 0$.

It remains to show that (A.5) holds for $t \in [0,b)$. To see this, set $v = (\log w)'$ and $v_K = (\log w_K)'$, so that v and v_K satisfy the Riccati equations

$$v' + v^2 - K^2 \geq 0 \text{ and } v_K' + v_K^2 - K^2 = 0 . \tag{A.7}$$

The claim now follows from the Riccati comparison argument. Namely, by (A.7) the function

$$(v - v_K) \exp\left(\int (v + v_K)\right) \tag{A.8}$$

is monotone non-decreasing. □

Corollary A.9. Let K be a positive constant. Suppose that w satisfies the ODE inequality $w'' \geq K^2\, w$ and $w(0) > 0$.
1). If $w'(0) \geq 0$ and w is defined on $[0,\infty)$, then $w(t) \geq w(0)\cosh(Kt)$ for $t \geq 0$.
2). If $w'(0) \leq 0$ and w is defined on $(-\infty, 0]$, then $w(t) \geq w(0)\cosh(Kt)$ for $t \leq 0$.

[19]The spaces H_+ and H_- were defined right after Theorem 0.4.

Moreover, we also have:
3). If $0 > w'(0) > -K\, w(0)$ and w is defined on $[0, \infty)$, then for $t \geq 0$ we have

$$w(t) \geq \frac{K\, w(0) + w'(0)}{2K}\, \mathrm{e}^{Kt} + \frac{K\, w(0) - w'(0)}{2K}\, \mathrm{e}^{-Kt}\,. \tag{A.10}$$

Proof. If we set $w_K = \cosh(Kt)$, then $w_K'' = K^2\, w_K$, w_K is positive everywhere, and $(\log w_K)'(0) = 0$. The first claim now follows from the lemma by integrating (A.5). The second claim follows from applying the first claim to the "reflected function" $w(-t)$.

To get the third claim, define the positive function w_K by

$$w_K = \frac{K\, w(0) + w'(0)}{2K}\, \mathrm{e}^{Kt} + \frac{K\, w(0) - w'(0)}{2K}\, \mathrm{e}^{-Kt}\,, \tag{A.11}$$

so that $w_K'' = K^2\, w_K$, $w_K(0) = w(0)$, and $w_K'(0) = w'(0)$. The last claim now also follows from the lemma by integrating (A.5). □

Proof. (of Proposition A.1.) Suppose that w is solution of (A.3) on $\mathbf{R}$ with $\lambda_j > \sup V$. If w is not identically zero, then we can apply either "1)." or "2)." in Corollary A.9 to get that w grows exponentially at either $+\infty$ or $-\infty$ (or both). In particular, the j-th Fourier coefficient $[u]_j(t)$ of a solution $u \in H_0$ must be zero for every $\lambda_j > \sup V$.

Since each Fourier coefficient of u satisfies a linear second order ODE as a function of t, it is determined by its value and first derivative at one point (say 0). It follows that any function $u \in H_0$ is completely determined by the values, and first derivatives, at 0 of its j-th Fourier coefficients for $\lambda_j \leq \sup V$. □

The next corollary is used in Appendix B, but not in the proof of our main theorem.

Corollary A.12. If $w(t)$ is a solution of (A.3) on $[0, \infty)$ with $\lambda_j > \sup V$, then either:

(1) $w(t)$ grows exponentially at $+\infty$ at least as fast as $\mathrm{e}^{t\sqrt{\lambda_j - \sup V}}$.
(2) $w(t)$ decays exponentially at $+\infty$ at least as fast as $\mathrm{e}^{-t\sqrt{\lambda_j - \sup V}}$.

Proof. It suffices to prove that (2) must hold whenever (1) does not. Assume therefore that $w(t)$ does not grow exponentially at $+\infty$. It follows from the first and third claims in Corollary A.9 that at any t with $w(t) > 0$ we must have

$$w'(t) \leq -\sqrt{\lambda_j - \sup V}\, w(t)\,, \tag{A.13}$$

since it would otherwise be forced to grow exponentially from t on. Integrating this gives (2) as long as we know that $w \neq 0$ from some point on. (If $w < 0$ from some point on, then we would apply the argument to $-w$.)

To complete the proof, recall that w can have only one zero unless, of course, w vanishes identically. This follows from integrating $(ww')' = (w')^2 + (\lambda_j - V)w^2 \geq (w')^2$ between any two zeros. □

A.1. **A geometric example.** We will next consider an example that illustrates the previous results. Namely, consider the rotationally symmetric potential $V(t)$ on the 2-dimensional flat cylinder $\mathbf{S}^1 \times \mathbf{R}$

$$V(t) = 2\, \cosh^{-2}(t)\,. \tag{A.14}$$

Since the potential V is rotationally symmetric, the space of solutions u of $\Delta u = -Vu$ can be written as linear combinations of separation of variables solutions $w_0(t)$, $\sin(k\theta)\, w_k(t)$ and $\cos(k\theta)\, w_k(t)$, where w_k is in the two-dimensional space of solutions to the ODE (A.3) with $\lambda_j = k^2$. Furthermore, Corollary A.9 implies that every w_k with $k^2 > 2 = \sup |V|$ must grow exponentially at plus or minus infinity. Hence, to find the space of bounded solutions, we need only check the solutions of (A.3) for $k = 0$ and $k = 1$. When $k = 0$, we get

$$\frac{\sinh(t)}{\cosh(t)} \text{ and } 1 - t\,\frac{\sinh(t)}{\cosh(t)}\,; \tag{A.15}$$

the first is bounded, while the second grows linearly. When $k = 1$, we get an exponentially growing solution together with an exponentially decaying solution

$$\frac{\sinh(2t) + 2t}{\cosh(t)} \text{ and } \frac{1}{\cosh(t)}\,. \tag{A.16}$$

It follows that the space of bounded solutions is spanned by

$$N_1 = \frac{\sin(\theta)}{\cosh(t)}\ ,\ N_2 = \frac{-\cos(\theta)}{\cosh(t)}\ ,\ N_3 = \frac{\sinh(t)}{\cosh(t)}\,, \tag{A.17}$$

while the space H_0 is spanned by N_1 and N_2.

This Schrödinger operator arises geometrically as a multiple of the Jacobi operator (i.e., the second variational operator) on the catenoid. The catenoid is the conformal minimal embedding of the cylinder into $\mathbf{R}^3$ given by

$$(\theta, t) \to (-\cosh t\, \sin\theta, \cosh t \cos\theta, t)\,. \tag{A.18}$$

It follows that the unit normal is given by

$$\mathbf{n} = \frac{(\sin\theta, -\cos\theta, \sinh t)}{\cosh t} = (N_1, N_2, N_3)\,, \tag{A.19}$$

so that N_1, N_2, and N_3 are the Jacobi fields that come from the coordinate vector fields. The other (linearly growing) $k = 0$ solution is the Jacobi field that comes from dilation.

The above discussion completely determined all polynomially growing functions in the kernel of the Schrödinger operator $L = \Delta + 2\cosh^{-2}(t)$ on the cylinder. Since the kernel of L is the 0 eigenspace of L, this leads naturally to ask what the entire spectrum of L is.[20] We will show that the spectrum[21] of L is

$$\{-1\} \cup [0, \infty)\,. \tag{A.20}$$

To see this, first use Weyl's theorem to see that the essential spectrum is $[0, \infty)$ since the potential V vanishes exponentially on both ends. Furthermore, we saw above that the positive function $\cosh^{-1}(t)$ is an eigenfunction of L with eigenvalue -1; this positivity implies that -1 is the lowest eigenvalue. It remains to show that there is no discrete spectrum

[20]The spectrum of L is the set of λ's such that $(L+\lambda) : W^{2,2} \to L^2$ does not have a bounded inverse (note the sign convention); the simplest way that this can occur is when λ is an eigenvalue of L, i.e, when there exists $u_\lambda \in W^{2,2} \setminus \{0\}$ with $Lu = -\lambda u$. We refer to Reed and Simon's *Methods of Modern Mathematical Physics*, volumes I through IV, for the definitions and results in spectral theory that we will use here.

[21]Note that this is not the same as the spectrum of the Jacobi operator on the catenoid since the two operators differ by multiplication by a positive function (which is why they have the same kernel).

between -1 and 0. This will follow from standard spectral theory once we show that the constant function $u = 0$ is the only polynomially growing solution u of

$$Lu = \lambda\, u\,, \tag{A.21}$$

for $0 < \lambda < 1$. It follows from Corollary A.12 that such a u must vanish exponentially at both plus and minus infinity. Consequently, every Fourier coefficient $[u]_j$ is an exponentially decaying solution of

$$[u]_j'' + 2\cosh^{-2}(t)\,[u]_j = (\lambda + \lambda_j)\,[u]_j\,, \tag{A.22}$$

where the λ_j's are the eigenvalues of $\mathbf{S}^1$. In particular, since the λ_j's are integers and λ is not, it follows that $\lambda + \lambda_j \neq 1$. A standard integration by parts argument[22] then shows that $[u]_j$ must be $L^2(\mathbf{R})$-perpendicular to the positive function $\cosh^{-2}(t)$; hence, $[u]_j$ must have a zero. After possibly reflecting in t, we can assume that $[u]_j(t_0) = 0$ for some $t_0 \geq 0$. Since $\tanh(t)$ satisfies the ODE (A.22) with $\lambda + \lambda_j = 0$ and vanishes only at 0, the lowest eigenvalue of the operator $\partial_t^2 + 2\cosh^{-2}(t)$ on any subdomain of the half-line $[0, \infty)$ must be non-negative. We will use two consequences of this:

- First, $[u]_j(t)$ cannot vanish for $t > t_0$ unless it vanishes identically; suppose therefore that $[u]_j(t) \geq 0$ for $t \geq t_0$.
- Second, the solution w of the ODE (A.22) with $\lambda + \lambda_j = 0$ and initial values $w(t_0) = 0$ and $w'(t_0) = 1$ must be positive for all $t \geq t_0$.

Note that we have already shown in (A.15) that any such w grows at most linearly in t. Hence, since $[u]_j$ vanishes exponentially, we know that

$$\lim_{t \to \infty} \left[w[u]_j' - w'[u]_j \right](t) \to 0 \tag{A.23}$$

Since $\left[w[u]_j' - w'[u]_j \right]' = (\lambda + \lambda_j)\, w[u]_j$, the fundamental theorem of calculus gives that

$$(\lambda + \lambda_j) \int_{t_0}^{\infty} w(t)[u]_j(t)\, dt = 0\,, \tag{A.24}$$

where we also used that $w(t_0) = [u]_j(t_0) = 0$. Since $w > 0$ and $[u]_j \geq 0$ on $[t_0, \infty)$, we conclude that $[u]_j$ vanishes identically as claimed, completing the proof of (A.20).

Appendix B. Growth and decay for generic rotationally symmetric and periodic potentials

In this appendix, we introduce the Poincaré map and use it to make some remarks about decay and growth for a generic rotationally symmetric potential on a cylinder; these are not needed elsewhere (nor are the results of Appendix C), but are included for completeness. This shows, in particular, that for an open and dense set of periodic potentials with positive operators any solution that vanishes at infinity decays exponentially.

For a bounded and rotationally symmetric potential on a cylinder $N \times \mathbf{R}$, Appendix A applies to all but a finite number of small eigenvalues of Δ_N. To understand the remaining small eigenvalues of Δ_N, we will need to understand the "Poincaré maps" associated to the ODE. We will define this next. For simplicity, we will assume throughout both this appendix and the next that V is smooth.

[22] The exponential decay guarantees that the integrals are well-defined and the boundary terms go to zero.

Given a non-negative number λ and $t_1 \leq t_2$, define the *Poincaré map* $P^\lambda_{t_1,t_2} : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$P^\lambda_{t_1,t_2}(a,b) = (u(t_2), u'(t_2)) \text{ where } u'' = (\lambda - V)\, u \text{ and } u(t_1) = a,\ u'(t_1) = b\,. \tag{B.1}$$

In general if f, g are functions on $\mathbf{R}$, not necessarily periodic, satisfying $f'' = (\lambda - V)f$ and $g'' = (\lambda - V)g$, then

$$\frac{d}{dt} \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = 0\,. \tag{B.2}$$

It follows from this and the fact that $P_{t,t}$ is the identity that $P_{t,t+s}$ is in $SL(2,\mathbf{R})$ for all t and $s \geq 0$.

We will below combine this with the simple fact that if A is a matrix in $SL(2,\mathbf{R})$, then either

(1) The absolute value of the trace of A is (strictly) greater than two, so the characteristic polynomial of A has two distinct real roots, $c \in \mathbf{R}$ and $1/c$ where $|c| > 1$. Such an A is said to be *hyperbolic* and can be diagonalized even over $\mathbf{R}$.
(2) The absolute value of the trace of A is (strictly) less than two, so the characteristic polynomial of A has two distinct complex roots, $\mathrm{e}^{i\,\phi}$ and $\mathrm{e}^{-i\,\phi}$ where $0 < \phi < \pi$.
(3) The absolute value of the trace of A is equal to two, so the characteristic polynomial of A has the root one, or the root minus one, with multiplicity two. Thus, there exists an orthonormal basis where A can be represented by (plus or minus)

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}. \tag{B.3}$$

Lemma B.4. For an open dense set of potentials V on $[0,\ell]$, the absolute value of the trace of the Poincaré map $P_{0,\ell}$ is not equal to two. In fact, if V is a potential with $|\mathrm{Trace}(P_{0,\ell})| = 2$, then there are potentials $V_j \to V$ with $V_j(0) = V(0)$ and $V_j(\ell) = V(\ell)$ so $P^{V_j}_{0,\ell}$ is hyperbolic.

To prove the lemma observe first that since trace is continuous on $SL(2,\mathbf{R})$ and the Poincaré map $P_{0,\ell}$ depends continuously on the potential, the set of potentials where the absolute value of the trace of $P_{0,\ell}$ is not two is clearly open. Consequently, to prove Lemma B.4, it is enough to prove density. The density is an easy consequence of the next lemma that allows us to perturb the Poincaré map.

We will need a few definitions before stating this perturbation lemma. Namely, given $\ell > 0$ and a function f on $[0,\ell]$ with $f(0) = f(\ell)$, let $P(f,s) = P_{0,\ell}(f,s)$ denote the Poincaré map from 0 to ℓ for the perturbed operator $\partial_t^2 + (V(t) + s\, f(t))$.[23]

Lemma B.5. The linear map from functions on $[0,\ell]$ with $f(0) = f(\ell) = 0$ to 2×2 matrices given by

$$f \to \frac{d}{ds}\Big|_{s=0} P(f,s) \tag{B.6}$$

is onto the three-dimensional space of matrices B such that $P^{-1}(f,0)\, B$ is trace free.

[23]Note that the perturbed potential agrees at 0 and ℓ if the original potential V does.

Proof. Let $u(s,t)$ and $v(s,t)$ be the solutions of $\partial_t^2 + (V(t) + s\, f(t))$ with initial conditions $(u, u_t)(s,0) = (1,0)$ and $(v, v_t)(s,0) = (0,1)$. It follows that

$$\frac{d}{ds}\Big|_{s=0} P(f,s) = \begin{pmatrix} u_s & v_s \\ u_{ts} & v_{ts} \end{pmatrix} (0,\ell)\,. \tag{B.7}$$

Note that when $s = 0$, we have $u_{tt} = -Vu$, $v_{tt} = -Vv$, $u_{stt} = -Vu_s - fu$, and $v_{stt} = -Vv_s - fv$. It follows that

$$(u_s v_t - v u_{st})(0,\ell) = \int_0^\ell \partial_t\, (u_s v_t - v u_{st})(0,t)\, dt = \int_0^\ell (fuv)(0,t)\, dt\,, \tag{B.8}$$

$$(u u_{st} - u_s u_t)(0,\ell) = \int_0^\ell \partial_t\, (u u_{st} - u_s u_t)(0,t)\, dt = -\int_0^\ell (fu^2)(0,t)\, dt\,, \tag{B.9}$$

$$(v_s v_t - v v_{st})(0,\ell) = \int_0^\ell \partial_t\, (v_s v_t - v v_{st})(0,t)\, dt = \int_0^\ell (fv^2)(0,t)\, dt\,, \tag{B.10}$$

$$(u v_{st} - v_s u_t)(0,\ell) = \int_0^\ell \partial_t\, (u v_{st} - v_s u_t)(0,t)\, dt = -\int_0^\ell (fuv)(0,t)\, dt\,. \tag{B.11}$$

These four quantities are the 11, 12, 21, and 22 coefficients, respectively, in the 2×2 matrix obtained by multiplying $P^{-1}(0,f)$ by $\frac{d}{ds}\big|_{s=0} P(f,s)$. Since $u^2(0,\cdot)$, $v^2(0,\cdot)$, and $(uv)(0,\cdot)$ are linearly independent[24] as functions on $[0,\ell]$ and composition by a linear map can only decrease the dimension of a vector space, we conclude that the image of $\frac{d}{ds}\big|_{s=0} P(f,s)$ must be at least three-dimensional. Finally, since it is contained in a three-dimensional space of matrices, the mapping must be onto. □

Proof. (of Lemma B.4.) As noted right after the statement of Lemma B.4, it is enough to show that if the absolute value of the trace of $P_{0,\ell}$ is two, then there is some function f with $f(0) = f(\ell) = 0$ so that for all $s > 0$ sufficiently small we have

$$\left| \text{Trace} P_{0,\ell}^{V+sf} \right| > 2\,, \tag{B.12}$$

where $P_{0,\ell}^{V+sf}$ is the Poincaré map for the potential $V + sf$. This follows immediately from two facts. First, Lemma B.5 says that we can choose f to arbitrarily perturb $P_{0,\ell}$ in $SL(2,\mathbf{R})$. Second, if P is a matrix in $SL(2,\mathbf{R})$ with $|\text{Trace}P| = 2$, then there are matrices $P_j \in SL(2,\mathbf{R})$ converging to P with $|\text{Trace}P_j| > 2$. Namely, if we consider $SL(2,\mathbf{R})$ as the hyper-surface $x_1 x_4 - x_2 x_3 = 1$ in $\mathbf{R}^4$, then the normal direction and the gradient of trace are

$$N = (x_4, -x_3, -x_2, x_1)\,, \tag{B.13}$$

$$\nabla \text{Trace} = (1,0,0,1)\,. \tag{B.14}$$

In particular, the projection of ∇Trace to the tangent space of $SL(2,\mathbf{R})$ vanishes only at the identity matrix and minus the identity matrix. It follows that we can perturb the trace as desired, at least away from (plus or minus) the identity matrix. This is all that we need, since it is easy to perturb (plus or minus) the identity matrix to a hyperbolic matrix. □

[24] To see this, note that the 3×3 matrix whose columns are $(u^2(0,0), (u^2)_t(0,0), (u^2)_{tt}(0,0))$, and similarly for v^2 and uv, is invertible.

In the next corollary, we will assume that the potential V is both periodic at $+\infty$ with period ℓ_+ and that the associated operator $\partial_t^2 + V$ is positive at infinity. That is, we will assume that there exists some $T > 0$ so that:

- For all $t > T$, we have that $V(t + \ell_+) = V(t)$.
- The only solution of $u'' = -Vu$ with at least two zeros on $[T, \infty)$ is the constant zero.

Note that the second condition is equivalent to the lowest eigenvalue[25] being positive on every compact subinterval of $[T, \infty)$; this follows from the domain monotonicity of eigenvalues.

Corollary B.15. For an open and dense set of ℓ_+ periodic at $+\infty$ potentials V on $\mathbf{R}$ that are also positive at infinity, any solution $u \in H_+$ to the Schrödinger equation $\Delta u = -V(t)\, u$ on the cylinder $N \times \mathbf{R}$ must decay exponentially to 0 at $+\infty$. Likewise for H_-.

Proof. We will use the positivity of $\partial_t^2 + V$, and hence also of $\partial_t^2 + V - \lambda_j$, to show first that the eigenvalues of $P^{\lambda_j}_{T,T+\ell_+}$ must be real for every j. To see this, suppose instead that the eigenvalues are $\mathrm{e}^{i\phi}$ and $-\mathrm{e}^{i\phi}$ with $0 < \phi < \pi$. In this case, we can choose some positive integer n to make both of the eigenvalues of

$$\left(P^{\lambda_j}_{T,T+\ell_+}\right)^n = P^{\lambda_j}_{T,T+n\,\ell_+} \tag{B.16}$$

as close as we want to -1. In particular, the solution of $f'' = (\lambda_j - V)f$ on $[T, T + 2n\,\ell_+]$ with initial values $f(T) = 1$ and $f'(T) = 0$ must be negative at $T + n\ell_+$ and then positive again at $T + 2n\ell_+$. This contradicts the positivity of the operator, so we conclude that the eigenvalues must be real.

Applying Lemma B.4, we may assume that $P^{\lambda_j}_{T,T+\ell_+}$ has two distinct real eigenvalues for $\lambda_j \leq \sup|V|$ and hence is hyperbolic. To complete the proof, we will prove the exponential decay of any $u \in H_+$ for such a potential.

By expanding a solution u into its Fourier series, it suffices to prove a uniform rate of exponential decay for bounded solutions f of $f'' = (\lambda_j - V)f$ on $[0, \infty)$. Here "uniform" means independent of j. Corollary A.12 gives this uniform exponential decay for every j with $\lambda_j > \sup|V|$; this does not use the periodicity at $+\infty$.

Assume now that $\lambda_j \leq \sup|V|$. It remains to show that if $P^{\lambda_j}_{T,T+\ell_+}$ is hyperbolic, f vanishes at $+\infty$, and $f'' = (\lambda_j - V)f$ on $[0, \infty)$, then f decays exponentially to zero at $+\infty$. For simplicity, we will assume that $j = 0$ and $T = 0$. The argument in the general case follows with obvious modifications. Let v_1 and v_2 be the two eigenvectors of the Poincaré map $P_{T,T+\ell_+}$ such that v_1 corresponds to the eigenvalue with norm larger than one. Let f_1 and f_2 be solutions on $\mathbf{R}$ to the equation $f'' = -V(t)\, f$ defined by $(f_i(0), f_i'(0)) = v_i$. It follows, that f_1 grows exponentially at $+\infty$ and f_2 decays exponentially to 0 at $+\infty$. Moreover, since the space of solutions is two dimensional and f_1 and f_2 are clearly linearly independent any solution f can be written as a linear combination of f_1 and f_2. Thus $f = a\, f_1 + b\, f_2$ for constants a and b. It follows that if f vanishes at $+\infty$, then $a = 0$ and hence f must decay exponentially at $+\infty$. □

[25] By convention, λ is an eigenvalue of $\partial_t^2 + V$ on $[a, b]$ if there is a (not identically zero) solution u of $u'' + Vu = -\lambda u$ with $u(a) = u(b) = 0$.

Example B.17. We will compute the Poincaré maps in the geometric example from subsection A.1, where $V(t)$ is a rotationally symmetric potential on $\mathbf{S}^1 \times \mathbf{R}$ given by

$$V(t) = 2\,\cosh^{-2}(t)\,. \tag{B.18}$$

Using the solutions in (A.15), we get that

$$P_{0,t} = \begin{pmatrix} 1 - t\tanh t & \tanh t \\ -\tanh t - t\,\cosh^{-2} t & \cosh^{-2} t \end{pmatrix}. \tag{B.19}$$

It follows that if $s < 0 < t$ are large, then $P_{s,t} = P_{0,t} \circ P_{0,s}^{-1}$ is approximately given by

$$\begin{pmatrix} -1 & 1 + s - t \\ 0 & -1 \end{pmatrix}. \tag{B.20}$$

Here, "approximately" means up to terms that decay exponentially in s or t.

Appendix C. The symplectic form and the symplectic Poincaré maps

Much of the discussion of the previous appendix generalizes to general bounded potentials that are no longer assumed to be rotationally symmetric. To explain this, we will need to recall some standard definitions. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let ω be the canonical symplectic form on $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. That is, if (v_1, v_2) and (w_1, w_2) are in $\mathcal{H}^2$, then $\omega((v_1, v_2), (w_1, w_2)) = \langle v_1, w_2 \rangle - \langle v_2, w_1 \rangle$. (The skew symmetric 2-form ω is symplectic since it is non-degenerate.) By definition a linear map from $\mathcal{H}^2$ to itself is said to be a *symplectic map* if it preserves ω. A linear subspace of $\mathcal{H}^2$ is said to be a *symplectic subspace* if ω restricted to the subspace is non-degenerate and a linear subspace of a symplectic subspace is said to be *isotropic* if the restriction of the symplectic form vanishes on the subspace. An isotropic subspace is said to be *Lagrangian* when it is maximal, i.e., is not strictly contained in a larger isotropic subspace. Finally, if W is a finite dimensional symplectic subspace of dimension $2n$, then it follows from Darboux's theorem that ω^n is a volume form on W and thus if A is a symplectic map from W to W, then A also preserves the volume form.

Consider now again solutions u of $\Delta u = -V u$ on the half-cylinder $N \times [0, \infty)$. The potential V will be smooth and bounded, and is now also allowed to depend on $\theta \in N$.

The Hilbert space will be $L^2(N)$ with the usual inner product whose norm is the L^2 norm. This is because the next lemma will allow us to identify a solution of $\Delta u = -V u$ with its Cauchy data $(u, \partial_t u)$ on a slice $N \times \{t_0\}$.

Lemma C.1. If $u(\cdot, t_0) = \partial_t u(\cdot, t_0) = 0$, then u is identically zero.

Proof. We will show first that u and all of its derivatives vanish on $N \times \{t_0\}$. Since u vanishes on $N \times \{t_0\}$ and ∂_t commutes with ∇_N, every partial derivative with at least one derivative in a direction tangent to N also vanishes. It remains to check that $\partial_t^n u(\cdot, t_0)$ vanishes for all $n \geq 2$. To get this for $n = 2$, use the equation $\Delta u = -V u$ to write

$$\partial_t^2 u(\theta, t_0) = -\Delta_N u(\theta, t_0) - V(\theta, t_0)\, u(\theta, t_0) = 0\,. \tag{C.2}$$

Similarly, differentiating the equation gives for $n > 2$ that

$$\partial_t^n u(\theta, t_0) = -\partial_t^{n-2}\, \Delta_N u(\theta, t_0) - \partial_t^{n-2}\, [V(\theta, t_0)\, u(\theta, t_0)]\,. \tag{C.3}$$

By induction, the terms on the right hand side of the equation all vanish, so we conclude that $\partial_t^n u(\cdot, t_0)$ also vanishes for all $n \geq 2$.

Finally, since the potential V is bounded and u vanishes to infinite order on $N \times \{t_0\}$, it follows from the theory of unique continuation, [A], that u must vanish everywhere. □

We conclude from the lemma that the linear map that takes a solution u of $\Delta u = -V\, u$ to its Cauchy data $(u, \partial_t u)$ is injective and hence we can identify u with its Cauchy data on an arbitrary but fixed slice $N \times \{t_0\}$.

Motivated by this, we define the skew-symmetric bilinear form $\omega(\cdot,\cdot)$ on solutions by

$$\omega(u,v) = \int_{N\times\{t_0\}} (u\partial_n v - v\partial_n u)\ . \tag{C.4}$$

Thus under the Cauchy data identification the space of solutions is identified with (a subspace of) $L^2(N) \times L^2(N)$ and the skew symmetric bilinear form is the pull back of the canonical symplectic form on $L^2(N) \times L^2(N)$.

As an immediate consequence of Stokes' theorem and that $\operatorname{div}(u\nabla v - v\nabla u) = 0$ if $Lu = Lv = 0$, the next lemma shows that the skew symmetric form ω does not depend on the choice of slice $N \times \{t_0\}$.

Lemma C.5. If u, v are functions on $N \times [t_0, t_1]$ that satisfy $Lu = Lv = 0$, then

$$\omega(u,v) \equiv \int_{N\times\{t_0\}} (u\partial_n v - v\partial_n u) = \int_{N\times\{t_1\}} (u\partial_n v - v\partial_n u)\ . \tag{C.6}$$

As an immediate consequence of Lemma C.5, we get that ω vanishes on the space of solutions of $Lu = 0$ that vanish at $+\infty$, i.e., the image of the map from H_0 to its Cauchy data is an isotropic subspace; cf. Lemma 6.5.

C.1. **The Poincaré map.** In the spirit of the previous appendix, we can use solutions of the equation $Lu = 0$ to define a Poincaré map which maps the Cauchy data at one time to the Cauchy data (of the same solution) at a later time. Namely, given $t_1 \leq t_2$, define the *Poincaré map*

$$P_{t_1,t_2} : L^2(N) \times L^2(N) \to L^2(N) \times L^2(N) \tag{C.7}$$

by

$$P_{t_1,t_2}(f,g) = (u(\cdot,t_2), \partial_t u(\cdot,t_2)) \text{ where } Lu = 0,\ u(\cdot,t_1) = f,\ \partial_t u(\cdot,t_1) = g\,. \tag{C.8}$$

Lemma C.5 then says that the linear map P_{t_1,t_2} preserves the skew symmetric form ω:

Corollary C.9. The linear Poincaré map P_{t_1,t_2} is symplectic, i.e.,

$$\omega(f,g) = \omega(P_{t_1,t_2}(f), P_{t_1,t_2}(g))\,. \tag{C.10}$$

C.2. **Perturbing the Poincaré map.** We will now consider a one-parameter family of Schrödinger operators

$$L + sf = \Delta + V(\theta,t) + s\, f(\theta,t)\,, \tag{C.11}$$

together with the associated one-parameter family of Poincaré maps

$$P = P_{0,\ell}(L+sf) : L^2(N) \times L^2(N) \to L^2(N) \times L^2(N)\,. \tag{C.12}$$

The next lemma will allow us to compute the derivative with respect to s of the Poincaré map P. In order to state the lemma, it will be convenient to define a map

$$L^{-1} : L^2(N) \times L^2(N) \to L^2(N \times [0,\ell]) \tag{C.13}$$

which takes a pair of functions (f_1, f_2) to the solution u of $Lu = 0$ with Cauchy data $u(\cdot, 0) = f_1$ and $\partial_t u(\cdot, 0) = f_2$. (Note that this is not defined for all f_1 and f_2 since the Cauchy problem in not solvable in general for elliptic equations.)

Lemma C.14. Given functions (f_1, f_2) and (g_1, g_2) in $L^2(N) \times L^2(N)$, we have

$$\omega\left(P(f_1, f_2), \frac{d}{ds}\Big|_{s=0} P(g_1, g_2)\right) = -\int_{N\times(0,\ell)} f\, L^{-1}(f_1, f_2),\, L^{-1}(g_1, g_2)\,. \tag{C.15}$$

Proof. Let $u(s, t, \theta)$ and $v(s, t, \theta)$ be solutions of

$$(L + sf)u = (L + sf)v = 0\,, \tag{C.16}$$

with initial conditions[26]

$$(u, \partial_t u)(s, 0, \theta) = (f_1, f_2)(\theta) \text{ and } (v, \partial_t v)(s, 0, \theta) = (g_1, g_2)(\theta)\,. \tag{C.17}$$

It follows that

$$\omega\left(P(f_1, f_2), \frac{d}{ds}\Big|_{s=0} P(g_1, g_2)\right) = \omega\left(\, (u, \partial_t u)(0, \ell, \cdot),\, (v_s, \partial_t v_s)(0, \ell, \cdot)\,\right)\quad. \tag{C.18}$$

The equation (C.16), and its s-derivative, implies that $Lu(0, \cdot, \cdot) = 0$ and

$$Lv_s(0, \cdot, \cdot) = -f(\cdot, \cdot)\, v(0, \cdot, \cdot)\,. \tag{C.19}$$

In particular,

$$\operatorname{div}\left(u\nabla v_s - v_s \nabla u\right)(0, \cdot, \cdot) = -(fuv)(0, \cdot, \cdot)\,. \tag{C.20}$$

Since $(v_s, \partial_t v_s)(0, 0, \cdot) = (0, 0)$, we can use Stokes' theorem and (C.20) to get

$$\omega\left(\, (u, \partial_t u)(0, \ell, \cdot),\, (v_s, \partial_t v_s)(0, \ell, \cdot)\,\right) = -\int_{N\times(0,\ell)} (fuv)(0, \cdot, \cdot)\quad. \tag{C.21}$$

□

Note that since ω is a non-degenerate form, Lemma C.14 completely determines the mapping $\frac{d}{ds}\big|_{s=0} P(g_1, g_2)$. Finally, observe that if the potential V is ℓ-periodic on $N \times [0, \infty)$, then the map $P_{\ell,0}$ maps the Cauchy data of H_α into itself.

Appendix D. Bloch's theorem

The classical Bloch theorem is usually stated for a disk in $\mathbf{C}$. We need a version of Bloch's theorem for a topological disk in a surface with bounded curvature. Since we were unable to find an exact reference for this, we will explain here how to get the needed version. The following lemma is an immediate consequence of the classical Bloch theorem (see [Ah]):

Lemma D.1. There exists a constant $B_0 > 0$, so that if f is a holomorphic function on the unit disk $D_1(0) \subset \mathbf{C}$, then the image $f(D_1(0))$ covers some disk of radius $B_0\, |f'(0)|$.

The case $|f'(0)| = 1$ appears in [Ah] and the general case follows from applying the case $|f'(0)| = 1$ to the function $g = f/|f'(0)|$. The version of Bloch's theorem that we used, i.e., Lemma 7.36, follows by combining Lemma D.1 with the following uniformization result:

[26]We must *assume* that these exist, since the Cauchy problem in not generally solvable.

Lemma D.2. Given $r_0 > 0$ and κ, there exists a constant $\mu > 0$ so that if Σ is a surface with $|K| \leq \kappa$ and the ball $B_{r_0}(p) \subset \Sigma$ is a topological disk, then there is a holomorphic diffeomorphism $F : D_1(0) \to B_{r_0}(p)$ with $F(0) = p$ and $|dF(p)| > \mu$.

Proof. The existence of the holomorphic diffeomorphism F follows immediately from the uniformization theorem. Namely, after extending a neighborhood of the disk $B_{r_0}(p) \subset \Sigma$, we can assume that it sits inside a closed Riemann surface. By the uniformization theorem, the universal cover of the closed Riemann surface is either the flat plane, the flat disk, or the round sphere. Hence, the topological disk $B_{r_0}(p)$ that sits inside the Riemann surface must be conformal to a topological disk in $\mathbf{C}$ (a topological disk in the round sphere is conformal to one in the plane by stereographic projection) and, by the Riemann mapping theorem, also conformal to the unit disk $D_1(0)$.

Therefore, the point is to get the lower bound on $|dF(p)|$. Note first that the inverse map F^{-1} is a holomorphic function on $B_{r_0}(p)$ that is bounded by one and vanishes at p. Since Σ has curvature bounded below by $-\kappa$ and each component of a holomorphic function on a surface is automatically also harmonic, the gradient estimate of [ChY] implies that

$$|\nabla F^{-1}(p)| \leq C \sup_{B_{r_0}(p)} |F^{-1}| = C \, , \tag{D.3}$$

for a fixed constant C depending only on r_0 and κ. This proves the lemma. □

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