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Default times, no-arbitrage conditions and changes of probability measures

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Abstract In this paper, we give a financial justification, based on no-arbitrage conditions, of the (**H**)-hypothesis in default time modeling. We also show how the (**H**)-hypothesis is affected by an equivalent change of probability measure. The main technique used here is the theory of progressive enlargements of filtrations.

Keywords Default modeling · Credit risk models · Random times · Enlargements of filtrations · Immersed filtrations · No-arbitrage conditions · Equivalent change of measure

Mathematics Subject Classification (2010) 60G07 · 91G40

JEL Classification C60 · G12 · G14

1 Introduction

In this paper, we study the stability of the (**H**)-hypothesis (or immersion property) under equivalent changes of probability measures. Given two filtrations $\mathbb{F} \subset \mathbb{G}$, we

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say that $\mathbb{F}$ is immersed in $\mathbb{G}$ if all $\mathbb{F}$-local martingales are $\mathbb{G}$-local martingales. In the default risk literature, the filtration $\mathbb{G}$ is obtained by the progressive enlargement of $\mathbb{F}$ with a random time (the default time), and the immersion property under a risk-neutral measure appears to be a suitable no-arbitrage condition (see [4]). Because immersion in general is not preserved under equivalent changes of probability measures (see [24] and [3]), reduced-form default models are usually specified directly under a given risk-neutral measure.

However, it seems crucial to understand how the immersion property is modified under an equivalent change of probability measure. This is important not only because credit markets are highly incomplete, but also because the physical default probability appears to play an important role in the presence of incomplete information. This role is emphasized by a more recent body of literature, initiated by [12] (see also [7, 15, 17, 20], among others), which proposes to rely on accounting information, and to incorporate the imperfect information about the accounting indicators, in computing credit spreads. The default intensities are computed endogenously, using the available observations about the firm. Some of the constructions do not satisfy the immersion property [16, 26]. It is therefore important to understand the role of the immersion property for pricing.

More generally, our goal in this paper is to provide efficient and precise tools from martingale theory and the general theory of stochastic processes to model default times. We wish to justify on economic grounds the default models which use the technique of progressive enlargements of filtrations, and to explain the reasons why such an approach is useful. We provide and study (necessary and) sufficient conditions for a market model to be arbitrage-free in the presence of default risk. More precisely, the paper is organized as follows.

In Sect. 2, we describe the financial framework which uses enlargements of filtrations techniques and introduce the corresponding no-arbitrage conditions. In Sect. 3, we present useful tools from the theory of progressive enlargements of filtrations. Subsequently, we study how the immersion property is affected under equivalent changes of probability measures. In Sect. 4, we give a simple proof of the not well-known fact (due to Jeulin and Yor [24]) that immersion is preserved under a change of probability measure whose Radon–Nikodým density is $\mathcal{F}_\infty$-measurable. Using this result, we show that a sufficient no-arbitrage condition is that the immersion property should hold under an equivalent change of measure (not necessarily risk-neutral). Then, using a general representation property for $\mathbb{G}$-martingales (Sect. 5), we characterize the class of equivalent changes of probability measures which preserve the immersion property when the random time τ avoids the $\mathbb{F}$-stopping times (Sect. 6), thus extending results of Jeulin and Yor [24] to our setting. Finally, we show how the Azéma supermartingale is computed for a large class of equivalent changes of measures.

2 No-arbitrage conditions

In this section, we briefly comment on some no-arbitrage conditions appearing in default models that use the progressive enlargement of a reference filtration (for further discussion in the case of complete default-free markets, see [4]). All notions from the theory of enlargements of filtrations used in this section are gathered in the next section. In default modeling, the technique of progressive enlargements of filtrations has been introduced by Kusuoka [26] and further developed in Elliott et al. [13]. It consists of a two-step construction of the market model, as follows.

Let $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. For us, the probability $\mathbf{P}$ stands for the physical measure under which financial events and prices are observed. Let τ be a random time; this is a $\mathcal{G}$-measurable nonnegative random variable which usually represents the default time of a company. It is not an $\mathbb{F}$-stopping time.

Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration obtained by progressively enlarging the filtration $\mathbb{F}$ with the random time τ. Obviously, $\forall t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{G}$.

Usually, the filtration $\mathbb{G}$ plays the role of the market filtration (and is sometimes called the full market filtration), meaning that the price processes are $\mathbb{G}$-adapted, and the pricing of defaultable claims is performed with respect to this filtration. On the other hand, the definition of the filtration $\mathbb{F}$ (called the reference filtration) is not always clear in the literature so far, and several interpretations can be given.

Let us now suppose that the reference filtration $\mathbb{F}$ contains the *market price* information which an investor is using for evaluating some defaultable claims. Typically, this is the natural filtration of a vector of semimartingales $S = (S_t)_{t\geq 0}$, with $S := (S^1, \dots, S^n)$. The vector S is recording the prices of observable default-free (with respect to τ) assets which are sufficiently liquid to be used for calibrating the model. Here, we may include assets without default risk, as well as assets with a different default time than τ, typically issued by other companies than the one we are analyzing. We shall call τ-default-free assets the components of S, since these are not necessarily assets without default risk.

As usual, we let S^0 stand for the locally risk-free asset (i.e., the money market account); the remaining assets are risky. We denote by $\mathcal{M}(\mathbb{F}, \mathbf{P})$ the set of all equivalent local martingale measures for the numéraire S^0, i.e.,

$$\mathcal{M}(\mathbb{F}, \mathbf{P}) = \left\{ \mathbf{Q} \sim \mathbf{P} \text{ on } \mathcal{G} \,\middle|\, \frac{S}{S^0} = \left(\frac{S^1}{S^0}, \dots, \frac{S^n}{S^0}\right) \text{ is an } (\mathbb{F}, \mathbf{Q})\text{-local martingale} \right\},$$

and we suppose that $\mathcal{M}(\mathbb{F}, \mathbf{P})$ is not empty in order to ensure absence of arbitrage opportunities (see e.g. Delbaen and Schachermayer [8]). Notice that because we shall work with different filtrations, we prefer to always define the probability measures on the sigma-algebra $\mathcal{G}$. In this way, we avoid dealing with extensions of a probability measure. When the $\mathbb{F}$-market is complete, all measures belonging to $\mathcal{M}(\mathbb{F}, \mathbf{P})$ have the same $\mathbb{F}$-restriction.

In practice, investors might use different information sets than $\mathbb{F}$, say $\mathbb{G}$. In this case, they can construct $\mathbb{G}$-portfolios and $\mathbb{G}$-strategies. Then, from the viewpoint of arbitrage theory, one needs to understand what the relevant prices become in a different filtration.

In particular, some investors may use more than the information in $\mathbb{F}$ for constructing portfolios. For instance, they might take into account the macroeconomic environment, or firm-specific accounting information which is not directly seen in prices. In this case $\mathbb{F} \subset \mathbb{G}$. Denote

$$\mathcal{M}(\mathbb{G}, \mathbf{P}) = \left\{ \mathbf{Q} \sim \mathbf{P} \text{ on } \mathcal{G} \,\middle|\, \frac{S}{S^0} = \left(\frac{S^1}{S^0}, \dots, \frac{S^n}{S^0} \right) \text{ is a } (\mathbb{G}, \mathbf{Q})\text{-local martingale} \right\}.$$

Are there (local) martingale measures for $\mathbb{G}$-informed traders? One has to understand what $\mathbb{F}$-martingales become in a larger filtration. There is no general answer to this question; in general, martingales of a given filtration are not semimartingales in a larger filtration [23]. However, from a purely economic point of view, if one assumes that the information contained in $\mathbb{G}$ is available for all investors without cost (i.e., this is public information), then the no-arbitrage condition becomes

$$\mathcal{M}(\mathbb{G}, \mathbf{P}) \neq \emptyset.$$

This is coherent with the semi-strong form of market efficiency, which says that a price process fully reflects all relevant information that is publicly available to investors. This means that publicly available information cannot be used in order to obtain arbitrage profits.

Let us now come to the particular case of default models, where $\mathbb{F}$ stands for the information about the prices of τ-default-free assets. In general, τ is not an $\mathbb{F}$-stopping time and for the purpose of pricing defaultable claims, the progressively enlarged filtration $\mathbb{G}$ has to be introduced. As an illustration, let us take the filtering model introduced by Kusuoka.

Example 2.1 (Kusuoka's filtering model [26]) Let $(B_t^1, B_t^2)_{t\in[0,T]}$ be a 2-dimensional Brownian motion. The default event is triggered by the process (for instance the cash flow balance of the firm, or assets' value)

$$\mathrm{d}X_t := \sigma^1(t, X_t)\,\mathrm{d}B_t^1 + b(t, X_t)\,\mathrm{d}t, \qquad X_0 = x_0.$$

Let $\tau := \inf\{t \in [0, T] | X_t = 0\}$ be the default time. Suppose that market investors do not observe X, but instead the process

$$\mathrm{d}Y_t := \sigma^2(t, Y_t)\,\mathrm{d}B_t^2 + \mu(t, X_{t\wedge\tau}, Y_t)\,\mathrm{d}t, \qquad Y_0 = y_0.$$

The process Y might be a τ-default-free asset price that is correlated with the defaultable asset value. For instance, suppose X is the asset value of an oil company. Then the oil price is an important piece of information to take into account when estimating the default risk of the company. Then Y can be the spot price of oil. The reference filtration is $\mathcal{F}_t := \sigma(Y_s, s \le t)$, and the market filtration is constructed as $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge s, s \le t)$.

As Kusuoka pointed out, the above example does not fulfill the immersion property. It is natural to investigate if such a model is arbitrage-free.

Let us assume that $\mathcal{M}(\mathbb{F}, \mathbf{P})$ is not empty, i.e., the τ-default-free market is arbitrage-free, and let us introduce the following alternative no-arbitrage conditions:

(1) There exists $\mathbf{Q} \in \mathcal{M}(\mathbb{F}, \mathbf{P})$ such that every $(\mathbb{F}, \mathbf{Q})$-(local) martingale is a $(\mathbb{G}, \mathbf{Q})$-(local) martingale, i.e., the immersion property holds under a risk-neutral measure.
(2) There exists a measure $\mathbf{Q} \sim \mathbf{P}$ such that every $(\mathbb{F}, \mathbf{P})$-(local) martingale is a $(\mathbb{G}, \mathbf{Q})$-(local) martingale.

The idea behind both conditions is that since default events are public information, an investor who uses this information to decide on his trading strategy should not be able to make arbitrage profits. Condition (1) says that there is (at least) one local martingale measure in common for an investor who uses information from default (filtration $\mathbb{G}$) in his trading and a less informed one, who is only concerned with τ-default-free prices levels when trading (filtration $\mathbb{F}$). Condition (2) looks at first sight less restrictive, by only saying that for each such type of investor, there exists a local martingale measure (but which could a priori be different). A closer inspection tells us that the two conditions are in fact equivalent. This equivalence will be proved in Sect. 4, where we also show that these conditions are equivalent to:

(3) There exists $\mathbf{Q} \sim \mathbf{P}$ such that the immersion property holds under $\mathbf{Q}$.

In other words, as soon as the immersion property holds under an equivalent probability measure, immersion holds as well under (at least) one $\mathbb{F}$-risk-neutral measure. Furthermore, $\mathcal{M}(\mathbb{G}, \mathbf{P})$ is not empty, i.e., absence of arbitrage holds for the defaultable market. Hence the immersion property is an important no-arbitrage condition to study.

Note also that the conditions listed above are sufficient for $\mathcal{M}(\mathbb{G}, \mathbf{P})$ to be not empty, but not necessary. One only needs that the martingale invariance property holds for the discounted price processes S/S^0, not for all $\mathbb{F}$-local martingales. Thus, when the $\mathbb{F}$-market is incomplete, weaker conditions can be stated. We now recall some important facts from the theory of progressive enlargements of filtrations which are relevant for our study.

3 Basic facts about random times and progressive enlargements of filtrations

In this section, we recall some important facts from the general theory of stochastic processes which we shall need in the sequel. We assume we are given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual assumptions. We do not assume that $\mathcal{G} = \mathcal{F}_\infty$.

Definition 3.1 A *random time* τ is defined to be a nonnegative random variable $\tau : (\Omega, \mathcal{G}) \to ([0, \infty], \mathcal{B})$. With a random time τ, we associate the sigma-field

$$\mathcal{F}_\tau = \sigma\left\{z_\tau;\ (z_t) \text{ any } \mathbb{F}\text{-optional process}\right\}.$$

The theory of progressive enlargements of filtrations was introduced to study properties of random times which are not stopping times; it originated in a paper by Barlow [2] and was further developed by Yor and Jeulin [21–23, 32]. For further details, the reader can also refer to Jeulin and Yor [25] which is written in French or to Mansuy and Yor [27] or Protter [31], Chap. VI, for an English text. This theory gives the decomposition of local martingales in the initial filtration $\mathbb{F}$ as semimartingales in the progressively enlarged one $\mathbb{G}$. More precisely, one enlarges the initial filtration $\mathbb{F}$ with the one generated by the process $(\tau \wedge t)_{t\geq 0}$, so that the new enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the smallest filtration (satisfying the usual assumptions) containing $\mathbb{F}$ and making τ a stopping time, i.e.,

$$\mathcal{G}_t = \mathcal{K}^o_{t+}, \quad \text{where } \mathcal{K}^o_t = \mathcal{F}_t \vee \sigma(\tau \wedge t).$$

Let X be a $\mathbb{G}$-adapted process. We denote by ${}^{(o,\mathbf{P})}X$ (resp. ${}^{(p,\mathbf{P})}X$) the optional (resp. predictable) projection of the process X onto the filtration $\mathbb{F}$, under the measure $\mathbf{P}$. When there is no ambiguity about the probability measure, we simply write ${}^oX$ or ${}^pX$.

A few processes play a crucial role in our discussion. These are the following:

- The $\mathbb{F}$-supermartingale, called *Azéma supermartingale*,

$$Z^\tau_t = \mathbf{P}[\tau > t \mid \mathcal{F}_t] \tag{3.1}$$

  chosen to be càdlàg, associated with τ by Azéma [1] (note that $Z^\tau_t > 0$ on the set $\{t < \tau\}$).
- The $\mathbb{F}$-dual optional and predictable projections of the process $\mathbf{1}_{\{\tau\leq t\}}$, denoted, respectively, by A^τ_t and a^τ_t. We recall that by definition, the $\mathbb{F}$-dual optional (resp. $\mathbb{F}$-dual predictable) projection of the increasing process $\mathbf{1}_{\{\tau\leq t\}}$ is the $\mathbb{F}$-optional (resp. $\mathbb{F}$-predictable) increasing process A^τ (resp. a^τ) that satisfies

$$\mathbf{E}\bigg[\int_0^\infty {}^oX_s\, \mathrm{d}\mathbf{1}_{\{\tau\leq s\}}\bigg] = \mathbf{E}\bigg[\int_0^\infty X_s\, \mathrm{d}A^\tau_s\bigg],$$

  respectively,

$$\mathbf{E}\bigg[\int_0^\infty {}^pX_s\, \mathrm{d}\mathbf{1}_{\{\tau\leq s\}}\bigg] = \mathbf{E}\bigg[\int_0^\infty X_s\, \mathrm{d}a^\tau_s\bigg],$$

  for any bounded and measurable process X.
- The càdlàg $\mathbb{F}$-martingale

$$\mu^\tau_t = \mathbf{E}\big[A^\tau_\infty \mid \mathcal{F}_t\big] = A^\tau_t + Z^\tau_t.$$

- The Doob–Meyer decomposition of the supermartingale (3.1) as

$$Z^\tau_t = m^\tau_t - a^\tau_t, \tag{3.2}$$

  where m^τ is an $\mathbb{F}$-martingale.

In the credit risk literature, the hazard process is very often used:

Definition 3.2 Let τ be a random time such that $Z^\tau_t > 0$ for all $t \geq 0$ (in particular τ is not an $\mathbb{F}$-stopping time). The nonnegative stochastic process $(\Gamma_t)_{t\geq 0}$ defined by

$$\Gamma_t = -\log Z^\tau_t$$

is called the *hazard process*.

The Azéma supermartingale in (3.1) is the main tool for computing the $\mathbb{G}$-predictable compensator of $\mathbf{1}_{\{\tau \le t\}}$.

Theorem 3.3 (Jeulin/Yor [23]) *Let H be a bounded $\mathbb{G}$-predictable process. Then*

$$H_\tau \mathbf{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \frac{H_s}{Z^\tau_{s-}}\,\mathrm{d}a^\tau_s$$

is a $\mathbb{G}$-martingale. In particular, taking $H \equiv 1$, we find that

$$N_t := \mathbf{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \frac{1}{Z^\tau_{s-}}\,\mathrm{d}a^\tau_s$$

is a $\mathbb{G}$-martingale.

It is important to know how $\mathbb{F}$-local martingales are affected under a progressive enlargement of filtrations; in general, for an arbitrary random time, an $\mathbb{F}$-local martingale is not always a $\mathbb{G}$-semimartingale (see [22, 23]). However, we have the following general result.

Theorem 3.4 (Jeulin/Yor [23]) *Every $\mathbb{F}$-local martingale (M_t) stopped at τ is a $\mathbb{G}$-semimartingale, with canonical decomposition*

$$M_{t\wedge\tau} = \widetilde{M}_t + \int_0^{t\wedge\tau} \frac{\mathrm{d}\langle M, \mu^\tau\rangle_s}{Z^\tau_{s-}},$$

where $(\widetilde{M}_t)$ is a $\mathbb{G}$-local martingale.

The following assumptions are often encountered in the literature on enlargements of filtrations or on the modeling of default times:

- The (**H**)-hypothesis: Every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. One says that the filtration $\mathbb{F}$ is *immersed* in $\mathbb{G}$, or that the *immersion property* holds.
- Assumption (**A**): The random time τ avoids every $\mathbb{F}$-stopping time T, in the sense that $\mathbf{P}[\tau = T] = 0$.

A property weaker than the (**H**)-hypothesis is when every $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale. In this situation, τ is called a pseudo-stopping time.

Definition 3.5 (Nikeghbali/Yor [30]) A random time τ is a *pseudo-stopping time* if $\mathbf{E}[m_\tau] = m_0$ for any bounded $\mathbb{F}$-martingale m.

When one assumes that the random time τ avoids $\mathbb{F}$-stopping times, then one further has

Lemma 3.6 (Jeulin/Yor [23], Jeulin [22]) *If τ avoids $\mathbb{F}$-stopping times (i.e., condition* (A) *is satisfied), then $A^\tau = a^\tau$ and A^τ is continuous. Therefore, the compensator of the process $\mathbf{1}_{\{\tau \le t\}}$ is continuous.*

We now recall several useful equivalent characterizations of the (**H**)-hypothesis in the next theorem. Note that except for the last equivalence, the results are true for any filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathcal{F}_t \subset \mathcal{G}_t$. The theorem is a combination of results by Brémaud and Yor [5], Theorem 3, and also by Dellacherie and Meyer [10], Résultat 3, in the special case when the larger filtration is obtained by progressively enlarging the smaller one with a random time.

Theorem 3.7 (Dellacherie/Meyer [10] and Brémaud/Yor [5]) *The following assertions are equivalent*:

1. (**H**): *Every* $\mathbb{F}$*-martingale is a* $\mathbb{G}$*-martingale.*
2. *For all bounded* $\mathcal{F}_\infty$*-measurable random variables* $\mathbf{F}$ *and all bounded* $\mathcal{G}_t$*-measurable random variables* $\mathbf{G}_t$, *we have*

$$\mathbf{E}[\mathbf{F}\mathbf{G}_t \mid \mathcal{F}_t] = \mathbf{E}[\mathbf{F} \mid \mathcal{F}_t]\,\mathbf{E}[\mathbf{G}_t \mid \mathcal{F}_t].$$

3. *For all bounded* $\mathcal{F}_\infty$*-measurable random variables* $\mathbf{F}$,

$$\mathbf{E}[\mathbf{F} \mid \mathcal{G}_t] = \mathbf{E}[\mathbf{F} \mid \mathcal{F}_t].$$

4. *For all* $s \le t$,

$$\mathbf{P}[\tau \le s \mid \mathcal{F}_t] = \mathbf{P}[\tau \le s \mid \mathcal{F}_\infty].$$

Let us give, as a consequence of Theorem 3.7, an invariance property for the Azéma supermartingale associated with τ for a particular class of equivalent changes of measure.

Proposition 3.8 *Suppose that the* (**H**)*-hypothesis holds under the measure* $\mathbf{P}$ *and let* $\mathbf{Q}$ *be a probability measure which is equivalent to* $\mathbf{P}$ *on* $\mathcal{G}$. *If* $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$ *is* $\mathcal{F}_\infty$*-measurable*, *then*

$$\mathbf{Q}[\tau > t \,|\, \mathcal{F}_t] = \mathbf{P}[\tau > t \,|\, \mathcal{F}_t] = Z_t^\tau.$$

Consequently, the predictable compensator of $\mathbf{1}_{\{\tau \le t\}}$ *is unchanged under such equivalent changes of probability measures*, *i.e.*,

$$N_t = \mathbf{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \frac{\mathrm{d}a_s^\tau}{Z_{s-}^\tau}$$

is a $\mathbb{G}$*-martingale under* $\mathbf{P}$ *and* $\mathbf{Q}$. *Moreover, the* (**H**)*-hypothesis holds under the measure* $\mathbf{Q}$.

Proof Let $\rho = \mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$. We have for $s \le t$ that

$$\mathbf{Q}[\tau > s \,|\, \mathcal{F}_t] = \frac{\mathbf{E}[\rho \mathbf{1}_{\{\tau > s\}} \,|\, \mathcal{F}_t]}{\mathbf{E}[\rho \,|\, \mathcal{F}_t]},$$

and from Theorem 3.7(2), we have

$$\mathbf{E}[\rho \mathbf{1}_{\{\tau > s\}} \,|\, \mathcal{F}_t] = \mathbf{E}[\rho \,|\, \mathcal{F}_t]\mathbf{E}[\mathbf{1}_{\{\tau > s\}} \,|\, \mathcal{F}_t] = \mathbf{E}[\rho \,|\, \mathcal{F}_t]\mathbf{P}[\tau > s \,|\, \mathcal{F}_t],$$

and hence

$$\mathbf{Q}[\tau > s \mid \mathcal{F}_t] = \mathbf{P}[\tau > s \mid \mathcal{F}_t] = \mathbf{P}[\tau > s \mid \mathcal{F}_\infty] = \mathbf{Q}[\tau > s \mid \mathcal{F}_\infty].$$

The result then follows from Theorem 3.7(4). □

We now indicate some consequences of the immersion property and (**A**).

Corollary 3.9 *Suppose that the immersion property holds. Then $Z^\tau = 1 - A^\tau$ is a decreasing process. If, in addition, τ avoids $\mathbb{F}$-stopping times, then Z^τ is continuous.*

Proof This is an immediate consequence of Theorem 3.7 and Lemma 3.6. □

Remark 3.10 (i) It is known that if τ avoids $\mathbb{F}$-stopping times, then Z^τ is continuous and decreasing if and only if τ is a pseudo-stopping time (see [30] and [6]).

(ii) When the immersion property holds and τ avoids $\mathbb{F}$-stopping times, we have from the above corollary and Theorem 3.3 that the compensator of $\mathbf{1}_{\{\tau \le t\}}$ is given by $\log \frac{1}{Z^\tau_{t\wedge\tau}} = \Gamma_{t\wedge\tau}$.

4 Immersion property and equivalent changes of probability measures: first results

Let $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t\ge 0}, \mathbf{P})$ be a filtered probability space satisfying the usual assumptions, and $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be such that $\mathbb{F} \subset \mathbb{G}$.

Notations

- We write $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ if $\mathbb{F}$ is immersed in $\mathbb{G}$ under the probability measure $\mathbf{P}$. Let $\mathcal{I}(\mathbf{P})$ be the set of all probability measures $\mathbf{Q}$ which are equivalent to $\mathbf{P}$ on $\mathcal{G}$ and such that $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$.
- Since we deal with different probability measures, we write $\mathbf{E}^{\mathbf{P}}$ (resp. $\mathbf{E}^{\mathbf{Q}}$) to emphasize that the expectation is under the measure $\mathbf{P}$ (resp. $\mathbf{Q}$). Whenever there is no ambiguity, $\mathbf{E}$ is used for $\mathbf{E}^{\mathbf{P}}$.

We now want to see how the immersion property is affected by equivalent changes of probability measures. Let $\mathbf{Q}$ be a probability measure which is equivalent to $\mathbf{P}$ on $\mathcal{G}$, with $\rho = \mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$. Define

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = e_t \quad \text{and} \quad \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{G}_t} = E_t. \tag{4.1}$$

We always consider càdlàg versions of the martingales e and E.

What can one say about the $(\mathbb{F}, \mathbf{Q})$-martingales when considered in the filtration $\mathbb{G}$? A simple application of Girsanov's theorem yields

Proposition 4.1 *Assume that $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$. Let $\mathbf{Q}$ be a probability measure which is equivalent to $\mathbf{P}$ on $\mathcal{G}$. Then every $(\mathbb{F}, \mathbf{Q})$-semimartingale is a $(\mathbb{G}, \mathbf{Q})$-semimartingale.*

The decomposition of the $(\mathbb{F}, \mathbf{Q})$-martingales in the larger filtration can be found by applying twice Girsanov's theorem, first in the filtration $\mathbb{F}$ and then in the filtration $\mathbb{G}$.

Theorem 4.2 (Jeulin/Yor [24]) *Assume that* $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$. *With the notation introduced in* (4.1), *if* (X_t) *is an* $(\mathbb{F}, \mathbf{Q})$-*local martingale*, *then the stochastic process*

$$I_t^X := X_t + \int_0^t \frac{E_{s-}}{E_s}\left(\frac{1}{e_{s-}}\,\mathrm{d}[X,e]_s - \frac{1}{E_{s-}}\,\mathrm{d}[X,E]_s\right),$$

is a $(\mathbb{G}, \mathbf{Q})$-*local martingale. Moreover*,

$$I_t^X = X_t + \int_0^t \frac{1}{\eta_{s-}}\,\mathrm{d}[X,\eta]_s,$$

where $\eta = e/E$ *is a* $(\mathbb{G}, \mathbf{Q})$-*martingale.*

The next lemma emphasizes the fact that a change of measure using an $\mathcal{F}_\infty$-measurable Radon–Nikodým derivative preserves the $(\mathbf{H})$-hypothesis. This can be seen as a consequence of Theorem 4.2, but we give below the proof, which is elementary.

Lemma 4.3 *Consider a probability space* $(\Omega, \mathcal{G}, \mathbf{P})$ *with filtrations* $\mathbb{F} \subset \mathbb{G}$. *Assume that* $\tilde{\mathbf{Q}} \in \mathcal{I}(\mathbf{P})$ *and* $\mathbf{Q} \sim \tilde{\mathbf{Q}}$ *is such that* $\mathrm{d}\mathbf{Q}/\mathrm{d}\tilde{\mathbf{Q}}$ *is* $\mathcal{F}_\infty$-*measurable. Then* $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$.

Proof Let (M_t) be an $(\mathbb{F}, \mathbf{Q})$-martingale. We must show that it is also a $(\mathbb{G}, \mathbf{Q})$-martingale, that is, $M\eta$ is a $(\mathbb{G}, \tilde{\mathbf{Q}})$-martingale, where $\eta_t := \mathrm{d}\mathbf{Q}/\mathrm{d}\tilde{\mathbf{Q}}|_{\mathcal{G}_t}$ is by assumption $\mathcal{F}_t$-measurable. Since $\mathbb{F} \overset{\tilde{\mathbf{Q}}}{\hookrightarrow} \mathbb{G}$ and $M\eta$ is $\mathbb{F}$-adapted, it suffices to show that $M\eta$ is an $(\mathbb{F}, \tilde{\mathbf{Q}})$-martingale. For this purpose, we note that η is an $\mathbb{F}$-adapted $(\mathbb{G}, \tilde{\mathbf{Q}})$-martingale, hence an $(\mathbb{F}, \tilde{\mathbf{Q}})$-martingale. Hence the Bayes formula yields for $t < s$

$$\mathbf{E}^{\tilde{\mathbf{Q}}}[M_s\eta_s \mid \mathcal{F}_t] = \mathbf{E}^{\mathbf{Q}}[M_s \mid \mathcal{F}_t]\mathbf{E}^{\tilde{\mathbf{Q}}}[\eta_s \mid \mathcal{F}_t] = M_t\eta_t,$$

as required. By localization, the proof can be extended to local martingales. □

Let us now state a necessary and sufficient condition for $\mathcal{I}(\mathbf{P}) \neq \emptyset$.

Proposition 4.4 *Consider a probability space* $(\Omega, \mathcal{G}, \mathbf{P})$ *with filtrations* $\mathbb{F} \subset \mathbb{G}$. *The following conditions are equivalent*:

(a) $\mathcal{I}(\mathbf{P}) \neq \emptyset$.
(b) *There exists* $\mathbf{Q} \sim \mathbf{P}$ *such that every* $(\mathbb{F}, \mathbf{P})$-*martingale is a* $(\mathbb{G}, \mathbf{Q})$-*martingale.*

We first prove a lemma which is interesting on its own.

Lemma 4.5 *Consider a probability space* $(\Omega, \mathcal{G}, \mathbf{P})$ *with filtrations* $\mathbb{F} \subset \mathbb{G}$ *and such that* $\mathcal{I}(\mathbf{P}) \neq \emptyset$. *Assume that* $\tilde{\mathbf{Q}} \in \mathcal{I}(\mathbf{P})$. *Then there exists* $\mathbf{Q} \sim \mathbf{P}$ *such that*:

(i) $\mathbf{Q} = \mathbf{P}$ *on* $\mathbb{F}$;
(ii) $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$;
(iii) *the Radon–Nikodým density* $\mathrm{d}\mathbf{Q}/\mathrm{d}\tilde{\mathbf{Q}}$ *is* $\mathcal{F}_\infty$*-measurable.*

Proof Suppose $\tilde{\mathbf{Q}} \in \mathcal{I}(\mathbf{P})$ with $\tilde{\rho} = \mathrm{d}\mathbf{P}/\mathrm{d}\tilde{\mathbf{Q}}$. Since $\tilde{\mathbf{Q}} \in \mathcal{I}(\mathbf{P})$, the process $\mathbf{E}^{\tilde{\mathbf{Q}}}[\tilde{\rho} \,|\, \mathcal{F}_t]$ is a positive $(\mathbb{G}, \tilde{\mathbf{Q}})$-martingale. We define the probability measure $\mathbf{Q} \sim \mathbf{P}$ by

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\tilde{\mathbf{Q}}}\right|_{\mathcal{G}_t} := \mathbf{E}^{\tilde{\mathbf{Q}}}[\tilde{\rho} \,|\, \mathcal{F}_t], \quad \forall t \geq 0,$$

so that (iii) holds. Since $\mathrm{d}\mathbf{Q}/\mathrm{d}\tilde{\mathbf{Q}}$ is $\mathcal{F}_\infty$-measurable, we have from Lemma 4.3 that $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$; hence (ii) is fulfilled. We now check that $\mathbf{Q}$ satisfies (i) as well. Indeed,

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\tilde{\mathbf{Q}}}\right|_{\mathcal{F}_t} \left.\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \frac{\mathbf{E}^{\tilde{\mathbf{Q}}}[\tilde{\rho} \,|\, \mathcal{F}_t]}{\mathbf{E}^{\tilde{\mathbf{Q}}}[\tilde{\rho} \,|\, \mathcal{F}_t]} = 1. \qquad \square$$

Proof of Proposition 4.4 (a) $\Rightarrow$ (b). We assume $\mathcal{I}(\mathbf{P}) \neq \emptyset$. We consider a measure $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$ which satisfies the requirements of Lemma 4.5. It follows that all $(\mathbb{F}, \mathbf{P})$-martingales are $(\mathbb{F}, \mathbf{Q})$-martingales, since $\mathbf{Q} = \mathbf{P}$ on $\mathbb{F}$, and also $(\mathbb{G}, \mathbf{Q})$-martingales, since $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$. We conclude that any $(\mathbb{F}, \mathbf{P})$-martingale is a $(\mathbb{G}, \mathbf{Q})$-martingale, as required.

(b) $\Rightarrow$ (a). We assume there exists $\mathbf{Q} \sim \mathbf{P}$ such that every $(\mathbb{F}, \mathbf{P})$-martingale is a $(\mathbb{G}, \mathbf{Q})$-martingale. It suffices to show that $\mathbf{Q} = \mathbf{P}$ on $\mathbb{F}$, so that any $(\mathbb{F}, \mathbf{Q})$-martingale is an $(\mathbb{F}, \mathbf{P})$-martingale, hence, by assumption, a $(\mathbb{G}, \mathbf{Q})$-martingale, i.e., $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$.

If m is any $(\mathbb{F}, \mathbf{P})$-martingale, by statement (b), m is a $(\mathbb{G}, \mathbf{Q})$-martingale, which is $\mathbb{F}$-adapted. Therefore m is an $(\mathbb{F}, \mathbf{Q})$-martingale. In particular, the $(\mathbb{F}, \mathbf{P})$-martingale $e_t = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}|_{\mathcal{F}_t}$ is an $(\mathbb{F}, \mathbf{Q})$-martingale, which is equivalent to saying that e^2 is an $(\mathbb{F}, \mathbf{P})$-martingale. Since e and e^2 are $(\mathbb{F}, \mathbf{P})$-martingales, it follows that $e \equiv 1$ and $\mathbf{Q} = \mathbf{P}$ on $\mathbb{F}$. $\square$

Let us now go back to the financial framework of Sect. 2, where $\mathbf{P}$ stands for the physical measure, and let us analyze the no-arbitrage conditions introduced there. We suppose that $\mathcal{M}(\mathbb{F}, \mathbf{P})$ is not empty, i.e., the $\mathbb{F}$-market is arbitrage-free. Now we show that if there exists an equivalent probability measure such that immersion holds, then there exists as well a risk-neutral one such that immersion holds.

Corollary 4.6 *If* $\mathcal{M}(\mathbb{F}, \mathbf{P})$ *and* $\mathcal{I}(\mathbf{P})$ *are not empty, then* $\mathcal{M}(\mathbb{F}, \mathbf{P}) \cap \mathcal{I}(\mathbf{P}) \neq \emptyset$. *Therefore*, $\mathcal{M}(\mathbb{G}, \mathbf{P}) \neq \emptyset$, *i.e., the market model is arbitrage-free.*

Proof Suppose that $\tilde{\mathbf{Q}} \in \mathcal{I}(\mathbf{P})$ and $\mathbf{P}' \in \mathcal{M}(\mathbb{F}, \mathbf{P})$. Notice that any probability measure that has the same $\mathbb{F}$-restriction as $\mathbf{P}'$ will also belong to $\mathcal{M}(\mathbb{F}, \mathbf{P})$. Let us define a probability measure $\mathbf{Q}$ as in Lemma 4.5, with $\mathbf{P}$ replaced by $\mathbf{P}'$ (of course, $\mathcal{I}(\mathbf{P}) = \mathcal{I}(\mathbf{P}')$ since $\mathbf{P} \sim \mathbf{P}'$). In particular, $\mathbf{Q} = \mathbf{P}'$ on $\mathbb{F}$ by (i) in Lemma 4.5; hence $\mathbf{Q} \in \mathcal{M}(\mathbb{F}, \mathbf{P})$ and also $\mathbf{Q} \in \mathcal{I}(\mathbf{P})$ by (ii) in Lemma 4.5. Therefore, we have precisely $\mathbf{Q} \in \mathcal{M}(\mathbb{F}, \mathbf{P}) \cap \mathcal{I}(\mathbf{P}')$. $\square$

The above corollary tells us that a sufficient no-arbitrage condition for the financial market introduced in Sect. 2 is $\mathcal{I}(\mathbf{P}) \neq \emptyset$. This result is very useful. The model by Kusuoka [26] presented in Example 2.1 is arbitrage-free, because there exists an equivalent change of measure such that τ is independent from $\mathcal{F}_T$, and hence immersion holds (see [26], pp. 79–80 for details). In this setting where the filtration $\mathbb{G}$ is obtained by progressively enlarging $\mathbb{F}$ in order to make a random time τ a stopping time, one can show that the $\mathcal{F}_\infty$-measurable random times which are not stopping times do not fulfill this no-arbitrage condition.

Lemma 4.7 *Let τ be a random time which is $\mathcal{F}_\infty$-measurable. Then $\mathcal{I}(\mathbf{P}) \neq \emptyset$ if and only if τ is an $\mathbb{F}$-stopping time* (*in this case $\mathbb{G} = \mathbb{F}$*).

Proof Suppose that $\mathbf{P}^* \in \mathcal{I}(\mathbf{P})$. Then $\mathbf{P}^*[\tau > t \,|\, \mathcal{F}_t] = \mathbf{P}^*[\tau > t \,|\, \mathcal{F}_\infty]$ for all $t \geq 0$; see Theorem 3.7(4). Since τ is $\mathcal{F}_\infty$-measurable, we have $\mathbf{P}^*[\tau > t \,|\, \mathcal{F}_\infty] = \mathbf{1}_{\{\tau > t\}}$, and hence $\mathbf{P}^*[\tau > t \,|\, \mathcal{F}_t] = \mathbf{1}_{\{\tau > t\}}$. This is possible if and only if $\{\tau > t\} \in \mathcal{F}_t$ for all t, that is, if and only if τ is an $\mathbb{F}$-stopping time. The converse is obvious. □

Remark 4.8 So-called honest times (which are ends of predictable sets) are examples of random times which are $\mathcal{F}_\infty$-measurable. In the financial literature, they are encountered in models with insider information, where insiders are shown to obtain free lunches with vanishing risks (see [18]). In the default risk literature, $\mathcal{F}_\infty$-measurable random times which are not stopping times appear in models with delayed information (see [16]). In this case, it follows from Lemma 4.7 that $\mathcal{I}(\mathbf{P}) = \emptyset$.

In the rest of the paper, we shall only consider the particular setting where the filtration $\mathbb{F}$ as well as a random time τ are initially given and $\mathbb{G}$ is obtained by progressively enlarging $\mathbb{F}$ in order to make τ a stopping time, as explained in Sect. 3. We should like to answer the following questions: Are there more general changes of probability measures that preserve the immersion property? More generally, how is the predictable compensator of τ modified under an equivalent change of probability measure? Indeed, it is known that market-implied (i.e., risk-neutral) default intensities are very different from the ones computed using historical data from defaults (i.e., under the physical measure). Hence for financial applications, it is crucial to understand how the predictable compensator is modified under general changes of probability measures. Note also the recent paper [14] where the particular case is studied where the $\mathbb{F}$-conditional distribution of τ admits a density with respect to some non-atomic positive measure.

For the sake of completeness, we state a general result due to Jeulin and Yor [24] which is unfortunately not easy to use in practical applications.

Proposition 4.9 (Jeulin/Yor [24]) *Let $\mathbf{Q}$ be a probability measure which is equivalent to $\mathbf{P}$ on $\mathcal{G}$, with $\rho = \mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$ on $\mathcal{G}_\infty$. Define the processes E and e as in* (4.1) *and suppose that $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$. Then $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$ if and only if*

$$\frac{\mathbf{E}^{\mathbf{P}}[X\rho \,|\, \mathcal{G}_t]}{E_t} = \frac{\mathbf{E}^{\mathbf{P}}[X\rho \,|\, \mathcal{F}_t]}{e_t} \quad \textit{for all } t \geq 0 \textit{ and } \mathcal{F}_\infty\textit{-measurable } X. \tag{4.2}$$

In particular, if ρ is $\mathcal{F}_\infty$-measurable, then $e = E$ and $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$.

Proof Using the Bayes formula, (4.2) is equivalent to

$$\mathbf{E}^{\mathbf{Q}}[X \mid \mathcal{G}_t] = \mathbf{E}^{\mathbf{Q}}[X \mid \mathcal{F}_t] \quad \text{for all } t \geq 0 \text{ and } \mathcal{F}_\infty\text{-measurable } X,$$

which is equivalent to the immersion property under $\mathbf{Q}$ by Theorem 3.7. □

Remark 4.10 Proposition 4.9 holds for general filtrations (i.e., $\mathbb{G}$ need not be obtained by progressively enlarging $\mathbb{F}$ with a random time). Moreover, although this is not mentioned in [24], the necessary and sufficient condition (4.2) is valid even if $\mathbb{F}$ is not immersed in $\mathbb{G}$ under $\mathbf{P}$. However, it will not directly help us to find a larger class than the change of probability measures for which the density ρ is $\mathcal{F}_\infty$-measurable.

5 Some martingale representation properties

In the remainder of the paper, $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is a filtered probability space satisfying the usual assumptions, τ is a random time and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the progressively enlarged filtration which makes τ a stopping time. Moreover, we suppose that τ is such that condition (A) holds and that the immersion property holds under $\mathbf{P}$. Recall from Sect. 3 that these assumptions imply that the Azéma supermartingale (Z_t^τ) is a decreasing and continuous process. Recall also that the notation $\mathbf{E}$ stands for the expectation under the measure $\mathbf{P}$.

Under these assumptions, we prove in this section several general martingale representation theorems for martingales of the larger filtration $\mathbb{G}$. These results will allow us to construct in Sect. 6 yet larger classes of equivalent probability measures that preserve the immersion property.

We begin with a few useful lemmas.

Lemma 5.1 *Assume that* (A) *and* $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ *hold. Let H be a $\mathbb{G}$-predictable process and let N be the $\mathbb{G}$-martingale $N_t = \mathbf{1}_{\{\tau \leq t\}} - \Gamma_{t\wedge\tau}$. If $\mathbf{E}[|H_\tau|] < \infty$, then*

$$\mathbf{E}\left[\int_0^t H_s \, \mathrm{d}N_s \,\middle|\, \mathcal{F}_t\right] = 0.$$

Proof First we note that because (A) holds, any $\mathbb{G}$-predictable process equals an $\mathbb{F}$-predictable process on the stochastic interval $[\![0,\tau]\!]$ (see [22], Sect. 4.2, and [23], Lemma 1). Therefore, using the definition of N, there exists an $\mathbb{F}$-predictable process $\tilde{H}$ such that for all $t \geq 0$, we have $\int_0^t H_s \, \mathrm{d}N_s = \int_0^t \tilde{H}_s \, \mathrm{d}N_s$. It follows that we can assume without loss of generality that the process H is $\mathbb{F}$-predictable; this will be assumed in the rest of the proof.

Let us first show that if $\mathbf{E}[|H_\tau|] < \infty$, then $\mathbf{E}[\int_0^\infty |H_s| \, \mathrm{d}N_s] < \infty$, so that all quantities under consideration are well defined. It is enough to check that both $\mathbf{E}[\int_0^\infty |H_s| \, \mathrm{d}\mathbf{1}_{\{\tau\leq s\}}]$ and $\mathbf{E}[\int_0^\tau |H_s| \frac{\mathrm{d}A_s^\tau}{Z_s^\tau}]$ are finite. The first quantity equals $\mathbf{E}[|H_\tau|]$

and is hence finite. For the second quantity, using the fact that A^τ is continuous and hence predictable and using properties of predictable projections, we have

$$\mathbf{E}\left[\int_0^\tau |H_s|\frac{\mathrm{d}A_s^\tau}{Z_s^\tau}\right]=\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{\tau>s\}}|H_s|\frac{\mathrm{d}A_s^\tau}{Z_s^\tau}\right]=\mathbf{E}\left[\int_0^\infty {}^p\left(\mathbf{1}_{\{\tau>s\}}\frac{|H_s|}{Z_s^\tau}\right)\mathrm{d}A_s^\tau\right],$$

where ${}^p(\cdot)$ denotes the $(\mathbb{F},\mathbf{P})$-predictable projection. Now we use the fact that ${}^p(\mathbf{1}_{\tau>s})=Z_s^\tau$ because τ avoids $\mathbb{F}$-stopping times to conclude that

$$\mathbf{E}\left[\int_0^\tau |H_s|\frac{\mathrm{d}A_s^\tau}{Z_s^\tau}\right]=\mathbf{E}\left[\int_0^\infty |H_s|\,\mathrm{d}A_s^\tau\right]=\mathbf{E}[|H_\tau|],$$

and consequently $\mathbf{E}[\int_0^\tau |H_s|\frac{\mathrm{d}A_s^\tau}{Z_s^\tau}]$ is also finite.

Since N is a local martingale of finite variation, it is purely discontinuous. Now let (M_t) be any square-integrable $\mathbb{F}$-martingale. Since $\mathbb{F}\overset{\mathbf{P}}{\hookrightarrow}\mathbb{G}$, (M_t) is also a $\mathbb{G}$-martingale. We also have $[M,N]_t=0$, because N is purely discontinuous and has a single jump at τ which avoids $\mathbb{F}$-stopping times. Consequently, N is strongly orthogonal to all square-integrable $\mathbb{F}$-martingales, and hence $\mathbf{E}[M_tN_t]=0$ for all t and all square-integrable $\mathbb{F}$-martingales M. This proves the lemma. □

Lemma 5.2 (Brémaud/Yor [5]) *Assume that* $\mathbb{F}\overset{\mathbf{P}}{\hookrightarrow}\mathbb{G}$. *Let H be a bounded $\mathbb{G}$-predictable process and m an $\mathbb{F}$-local martingale. Then*

$$\mathbf{E}\left[\int_0^t H_s\,\mathrm{d}m_s\,\middle|\,\mathcal{F}_t\right]=\int_0^t {}^pH_s\,\mathrm{d}m_s,$$

where ${}^pH$ is the $(\mathbb{F},\mathbf{P})$*-predictable projection of the process H.*

We now easily deduce from Lemma 5.1 the following projection formula.

Lemma 5.3 *Let τ be any random time and (z_t) an $\mathbb{F}$-predictable process such that* $\mathbf{E}[|z_\tau|]<\infty$.

(i) *Assume that* (A) *holds. Then*

$$\mathbf{E}[z_\tau\mathbf{1}_{\{\tau>t\}}\,|\,\mathcal{F}_t]=\mathbf{E}\left[\int_t^\infty z_s\,\mathrm{d}A_s^\tau\,\middle|\,\mathcal{F}_t\right].$$

(ii) *Assume further that* $\mathbb{F}\overset{\mathbf{P}}{\hookrightarrow}\mathbb{G}$. *Then*

$$\mathbf{E}[z_\tau\,|\,\mathcal{F}_t]=\mathbf{E}\left[\int_0^\infty z_s\,\mathrm{d}A_s^\tau\,\middle|\,\mathcal{F}_t\right].$$

If moreover the hazard process Γ is defined for all $t\geq 0$, that is, if $Z_t^\tau>0$ for all $t\geq 0$, then

$$\mathbf{E}[z_\tau\,|\,\mathcal{F}_t]=\mathbf{E}\left[\int_0^\infty z_se^{-\Gamma_s}\,\mathrm{d}\Gamma_s\,\middle|\,\mathcal{F}_t\right].$$

Proof (i) This is a consequence of the projection formulae in Theorem V.25 of [9]; see also [29].

(ii) It is enough to check the result for $z_s = H_r \mathbf{1}_{(r,u]}(s)$, with $r < u$ and H_r an integrable $\mathcal{F}_r$-measurable random variable. But in this case, the result is an immediate consequence of Theorem 3.7. □

We now state and prove a first representation theorem result for some $\mathbb{G}$-martingales under the assumption that (Z_t^τ) is continuous and decreasing, that is, τ is a pseudo-stopping time that avoids stopping times (the pseudo-stopping time assumption is an extension of the (**H**)-hypothesis framework; see [30] and [6]). This result appears in [4], Proposition 3, where it is derived under the (**H**)-hypothesis. We give here a simpler proof which easily extends to any random time. But before, we state a lemma which we shall use in the proof.

Lemma 5.4 [4] *Let τ be an arbitrary random time. Define*

$$L_t = \mathbf{1}_{\{\tau > t\}} e^{\Gamma_t}.$$

Then $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-martingale, which is well defined for all $t \geq 0$. If τ is a pseudo-stopping time and (A) *holds* (*or equivalently (Z_t^τ) is continuous and decreasing*), *then*

$$L_t = 1 - \int_0^t \frac{\mathrm{d}N_s}{Z_s^\tau},$$

where (N_t) is the $\mathbb{G}$-martingale $N_t = \mathbf{1}_{\{\tau \leq t\}} - \Gamma_{t\wedge\tau}$.

Theorem 5.5 *Let τ be a pseudo-stopping time and z an $\mathbb{F}$-predictable process such that $\mathbf{E}[|z_\tau|] < \infty$*:

(i) *If* (A) *holds*, *then*

$$\mathbf{E}[z_\tau \mid \mathcal{G}_t] = m_0 + \int_0^{t\wedge\tau} \frac{\mathrm{d}m_s}{Z_s^\tau} + \int_0^t (z_s - h_s)\,\mathrm{d}N_s,$$

where $m_t = \mathbf{E}[\int_0^\infty z_s\,\mathrm{d}A_s^\tau \mid \mathcal{F}_t]$ and $h_t = (Z_t^\tau)^{-1}(m_t - \int_0^t z_s\,\mathrm{d}A_s^\tau)$.

(ii) *If in addition $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ and there exists a constant c such that $\mathbf{E}[z_\tau \mid \mathcal{F}_\infty] = c$, then*

$$\mathbf{E}[z_\tau \mid \mathcal{G}_t] = c + \int_0^t (z_s - h_s)\,\mathrm{d}N_s.$$

Proof (i) It is well known (see [11], formula XX.(75.2) or [28], Proposition 9.12) that

$$\mathbf{E}[z_\tau \mid \mathcal{G}_t] = L_t \mathbf{E}[z_\tau \mathbf{1}_{\{\tau > t\}} \mid \mathcal{F}_t] + z_\tau \mathbf{1}_{\{\tau \leq t\}}.$$

Furthermore, from Lemma 5.3, with the notation of Theorem 5.5, we have

$$\mathbf{E}[z_\tau \mathbf{1}_{\{\tau > t\}} \mid \mathcal{F}_t] = m_t - \int_0^t z_s\,\mathrm{d}A_s^\tau.$$

Consequently,

$$\mathbf{E}[z_\tau \mid \mathcal{G}_t] = L_t m_t - L_t \int_0^t z_s \, \mathrm{d}A_s^\tau + z_\tau \mathbf{1}_{\{\tau \le t\}}.$$

Now noting that (L_t) is a purely discontinuous martingale with a single jump at τ, we obtain the result that (L_t) is orthogonal to any $\mathbb{F}$-martingale. An integration by parts combined with Lemma 5.4 yields the desired result.

(ii) From Lemma 5.3(ii), we have under the assumption $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ that

$$m_t = \mathbf{E}\left[\int_0^\infty z_s \, \mathrm{d}A_s^\tau \,\middle|\, \mathcal{F}_t \right] = \mathbf{E}[z_\tau \mid \mathcal{F}_t].$$

Since it is assumed that $\mathbf{E}[z_\tau \mid \mathcal{F}_t] = \mathbf{E}[\mathbf{E}[z_\tau]\mathcal{F}_\infty \mid \mathcal{F}_t] = c$, the result follows at once. □

Remark 5.6 The proof of Theorem 5.5(i) can be adapted so that the result holds for an arbitrary random time that avoids stopping times. The only thing to modify is Lemma 5.4: For an arbitrary random time τ that avoids stopping times, (Z_t^τ) is continuous but not of finite variation any more, so that an extra term must be added when expressing (L_t) as a sum of stochastic integrals.

We now combine Theorem 5.5(ii) with Proposition 3.8 to obtain a representation theorem for a larger class of $\mathbb{G}$-martingales.

Proposition 5.7 *Let τ be a random time such that* (A) *and* $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ *hold. Let $G = F z_\tau$, where F is an integrable, $\mathcal{F}_\infty$-measurable random variable such that $F \neq 0$ a.s. and z is an $\mathbb{F}$-predictable process such that $z_\tau F$ is integrable. Then*

$$\mathbf{E}[G \mid \mathcal{G}_t] = \mathbf{E}[G] + \int_0^t \left(\mathbf{E}[G] + Y_s - L_s \frac{m_s^G}{m_s^F} + \int_0^s k_u \, \mathrm{d}N_u \right) \mathrm{d}m_s^F$$
$$+ \int_0^t L_s \, \mathrm{d}m_s^G + \int_0^t m_s^F k_s \, \mathrm{d}N_s,$$

where

$$m_t^F := \mathbf{E}[F \mid \mathcal{F}_t], \qquad m_t^G := \mathbf{E}[G \mid \mathcal{F}_t], \qquad Y_t = \int_0^t L_s \, \mathrm{d}\left(\frac{m_t^G}{m_t^F} \right),$$

and where (k_t) is an $\mathbb{F}$-predictable process (*which can be given explicitly*).

Proof Without loss of generality, we can assume that F is strictly positive and that $\mathbf{E}[F] = 1$ (the general case would follow by writing $F = F^+ - F^-$). Then we define $\mathrm{d}\tilde{\mathbf{Q}}|_{\mathcal{G}_\infty} = F \, \mathrm{d}\mathbf{P}|_{\mathcal{G}_\infty}$. By Proposition 3.8, the $(\mathbf{H})$-hypothesis holds under $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}[\tau > t \mid \mathcal{F}_t] = \mathbf{P}[\tau > t \mid \mathcal{F}_t]$. We then obtain

$$\mathbf{E}[G \mid \mathcal{G}_t] = \mathbf{E}[z_\tau F \mid \mathcal{G}_t] = \mathbf{E}[F \mid \mathcal{G}_t] \mathbf{E}^{\tilde{\mathbf{Q}}}[z_\tau \mid \mathcal{G}_t] = m_t^F \mathbf{E}^{\tilde{\mathbf{Q}}}[z_\tau \mid \mathcal{G}_t].$$

Using the decomposition from Theorem 5.5(i), we get

$$\mathbf{E}^{\tilde{\mathbf{Q}}}[z_\tau \mid \mathcal{G}_t] = \mathbf{E}^{\tilde{\mathbf{Q}}}[z_\tau] + Y_t + \int_0^t k_s \, \mathrm{d}N_s,$$

where $Y_t = \int_0^t L_s \, \mathrm{d}\widetilde{m}_s$. Here, $\tilde{m}$ is the $\tilde{\mathbf{Q}}$-martingale defined by

$$\tilde{m}_t := \mathbf{E}^{\tilde{\mathbf{Q}}}[z_\tau \mid \mathcal{F}_t] = \mathbf{E}^{\mathbf{P}}[z_\tau F \mid \mathcal{F}_t]\big(m_t^F\big)^{-1} = \frac{m_t^G}{m_t^F}$$

and

$$k_t = z_t - \big(Z_t^\tau\big)^{-1}\left(\widetilde{m}_t - \int_0^t z_s \, \mathrm{d}A_s^\tau\right).$$

Consequently,

$$\mathbf{E}[G \mid \mathcal{G}_t] = m_t^F\left(\mathbf{E}^{\mathbf{P}}[G] + \int_0^t L_s \, \mathrm{d}\left(\frac{m_s^G}{m_s^F}\right) + \int_0^t k_s \, \mathrm{d}N_s\right).$$

Now, an integration by parts and some tedious computations lead to

$$\begin{aligned}\mathbf{E}[G \mid \mathcal{G}_t] = \mathbf{E}[G] m_t^F &+ \int_0^t \left(Y_s - L_s \frac{m_s^G}{m_s^F} + \int_0^s k_u \, \mathrm{d}N_u\right) \mathrm{d}m_s^F \\ &+ \int_0^t L_s \, \mathrm{d}m_s^G + \int_0^t m_s^F k_s \, \mathrm{d}N_s,\end{aligned}$$

which completes the proof of our theorem. □

As a corollary, we obtain the following generalization of a representation result by Kusuoka [26], which was obtained in the Brownian filtration.

Corollary 5.8 *Let τ be a random time such that* (A) *and $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ hold. Then any $\mathbb{G}$-locally square-integrable local martingale (M_t) can be written as*

$$M_t = M_0 + V_t + \int_0^t h_s \, \mathrm{d}N_s, \tag{5.1}$$

where (V_t) is in the closed subspace of $\mathbb{G}$-locally square-integrable local martingales generated by the stochastic integrals of the form $\int_0^t R_s \, \mathrm{d}m_s$, where (m_t) is an $\mathbb{F}$-locally square-integrable local martingale, (R_t) is a $\mathbb{G}$-predictable process such that $\int_0^t R_s^2 \, \mathrm{d}\langle m, m\rangle_s$ is locally integrable and (h_t) is an $\mathbb{F}$-predictable process such that h_τ^2 is integrable.

Proof The result follows from Proposition 5.7 and the fact that any $\mathcal{G}_\infty$-measurable random variable can be written as a limit of finite linear combinations of functions of the form $Ff(\tau)$ where F is an $\mathcal{F}_\infty$-measurable random variable and f a Borel function such that $Ff(\tau)$ is integrable. □

Remark 5.9 Since any element V in the closed subspace of $\mathbb{G}$-locally square-integrable local martingales generated by the stochastic integrals of the form $\int_0^t R_s \, \mathrm{d}m_s$ is strongly orthogonal to the purely discontinuous martingales of the form $\int_0^t h_s \, \mathrm{d}N_s$, it follows that the decomposition (5.1) is unique.

Corollary 5.10 (Kusuoka [26]) *Assume that $\mathbb{F}$ is the natural filtration of a one-dimensional Brownian motion (W_t). Let τ be a random time such that* (A) *and $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ hold. Then any $\mathbb{G}$-locally square-integrable local martingale M can be written as*

$$M_t = M_0 + \int_0^t R_s \, \mathrm{d}W_s + \int_0^t h_s \, \mathrm{d}N_s,$$

where (R_t) is a $\mathbb{G}$-predictable process such that $\int_0^t R_s^2 \, \mathrm{d}s$ is locally integrable and (h_t) is an $\mathbb{F}$-predictable process such that h_τ^2 is integrable.

Remark 5.11 A result similar to the representation of Corollary 5.10 would hold if the filtration $\mathbb{F}$ has the predictable representation property with respect to a family of locally square-integrable local martingales.

Combining Lemma 5.1 and Corollary 5.8, one gets

Corollary 5.12 *Let τ be a random time such that* (A) *and $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ hold. Assume that (M_t) is a locally square-integrable local $\mathbb{G}$-martingale. Then (M_t) is strongly orthogonal to all locally square-integrable local $\mathbb{F}$-martingales if and only if there exists an $\mathbb{F}$-predictable process (h_t) such that h_τ^2 is integrable and such that*

$$M_t = M_0 + \int_0^t h_s \, \mathrm{d}N_s.$$

6 Equivalent changes of probability measures: further results

In this section, we prove two important results. We first characterize the Radon–Nikodým derivative $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$ of two equivalent probability measures under which the (**H**)-hypothesis holds. Then we generalize Proposition 3.8: We compute the Azéma supermartingale $\mathbf{Q}[\tau > t \,|\, \mathcal{F}_t]$ for a very large class of probability measures $\mathbf{Q}$ which are equivalent to $\mathbf{P}$ but do not necessarily preserve the immersion property.

Let us first emphasize a multiplicative decomposition of the Radon–Nikodým derivative which is very simple and general (we only need $\mathcal{F}_\infty \subset \mathcal{G}_\infty$). We can always write on $\mathcal{G}_\infty$ that $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} = FH$, where F is positive $\mathcal{F}_\infty$-measurable with $\mathbf{E}^{\mathbf{P}}[F] = 1$ and H is a positive random variable such that $\mathbf{E}^{\mathbf{P}}[H \,|\, \mathcal{F}_\infty] = 1$. Indeed, one has

$$\mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{G}_\infty] = \mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{F}_\infty] \frac{\mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{G}_\infty]}{\mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{F}_\infty]} =: FH \tag{6.1}$$

with $F = \mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{F}_\infty]$ and $H = \frac{\mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{G}_\infty]}{\mathbf{E}^{\mathbf{P}}[\rho \,|\, \mathcal{F}_\infty]}$ satisfying the claimed properties.

The following theorem relates the multiplicative decomposition (6.1) to the immersion property.

Theorem 6.1 *Let* τ *be a random time such that* (A) *and* $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ *hold. Let* $\mathbf{Q}$ *be a probability measure which is equivalent to* $\mathbf{P}$ *with Radon–Nikodým density given by* (6.1):

(i) *If* H *is* $\mathcal{F}_\tau$*-measurable* (*see Definition* 3.1), *then* $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$.

(ii) *If* $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$ *and if*

$$\mathbf{E}^{\mathbf{P}}\left[\frac{e_\infty^2}{E_\infty}\right] < \infty, \tag{6.2}$$

which is equivalent to $\mathbf{E}^{\mathbf{Q}}[\frac{1}{H^2}] < \infty$, *then* H *is* $\mathcal{F}_\tau$*-measurable.*

Proof (i) Assume first that $F = 1$. Since H is $\mathcal{F}_\tau$-measurable and $\mathbf{E}^{\mathbf{P}}[H \mid \mathcal{F}_\infty] = 1$, it follows from Theorem 5.5(ii) that we have $E_t := \mathbf{E}^{\mathbf{P}}[H \mid \mathcal{G}_t] = 1 + \int_0^t h_s \, \mathrm{d}N_s$ and $\mathbf{E}^{\mathbf{P}}[H \mid \mathcal{F}_t] = 1$. In addition, since τ avoids $\mathbb{F}$-stopping times and since E is a purely discontinuous martingale, $[M, E] = 0$ for any $(\mathbb{F}, \mathbf{P})$-martingale (M_t). Hence by Girsanov's theorem, the immersion property holds under $\mathbf{Q}$.

For the general case, introduce $\mathrm{d}\tilde{\mathbf{Q}} = F \, \mathrm{d}\mathbf{P}$. This gives $\mathrm{d}\mathbf{Q} = H \, \mathrm{d}\tilde{\mathbf{Q}}$ and $\mathbf{E}^{\tilde{Q}}[H \mid \mathcal{F}_\infty] = 1$. From Proposition 3.8, we know that the immersion property holds under $\tilde{\mathbf{Q}}$; so using the case $F = 1$, it follows that the immersion property also holds under $\mathbf{Q}$.

(ii) Recall that the decomposition (6.1) holds and further assume that $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$ also holds. Assumption (6.2) is easily seen to mean that (η_t) is an $L^2(\mathbb{G}, \mathbf{Q})$-bounded martingale. It follows from the Bayes formula that if (m_t) is any $L^2(\mathbb{F}, \mathbf{Q})$-bounded martingale, then $(m_t \eta_t)$ is a $(\mathbb{G}, \mathbf{Q})$-uniformly integrable martingale. Indeed, if (m_t) is an $(\mathbb{F}, \mathbf{Q})$-martingale, then $(m_t e_t)$ is an $(\mathbb{F}, \mathbf{P})$-martingale. Since $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ holds, we also find that $(m_t e_t)$ is a $(\mathbb{G}, \mathbf{P})$-martingale. Now another application of the Bayes formula yields that $(m_t \frac{e_t}{E_t})$, which is (by definition) $(m_t \eta_t)$, is a $(\mathbb{G}, \mathbf{Q})$-martingale. Put differently, the $(\mathbb{G}, \mathbf{Q})$-martingale η is strongly orthogonal to all $(\mathbb{F}, \mathbf{Q})$-martingales viewed as $(\mathbb{G}, \mathbf{Q})$-martingales (recall that by assumption $\mathbb{F} \overset{\mathbf{Q}}{\hookrightarrow} \mathbb{G}$). Then by Corollary 5.12, η_∞ is $\mathcal{F}_\tau$-measurable, and so is $H = (\eta_\infty)^{-1}$. □

As we have seen, the Azéma supermartingale plays an important role in credit risk modeling, in particular for the construction of the predictable compensator of the default arrival, or the intensity process when it exists. Now we should like to display the form of the $\mathbf{Q}$-Azéma supermartingale, denoted $Z^{\mathbf{Q}}$, under a large class of equivalent changes of probability measures. This will also shed new light on some of the previous results. We shall see for instance that the changes of measure appearing in Theorem 6.1 with $F = 1$ affect only the compensator of the default arrival and leave unchanged the dynamics of the τ-default-free assets in both filtrations $\mathbb{F}$ and $\mathbb{G}$. We shall also identify a class of changes of measure, larger that the one introduced in Proposition 3.8, which do not affect the compensator.

Before doing so, we should like to state a very useful, though somehow forgotten, result by Itô and Watanabe [19] on multiplicative decompositions of supermartingales. In particular, the multiplicative decomposition turns out to be useful in the study of the intensity of the default time, as we shall see.

Theorem 6.2 (Itô/Watanabe [19]) *Let (Z_t) be a nonnegative càdlàg supermartingale, and define*

$$T_0 = \inf\{t : Z_t = 0\}.$$

Suppose $\mathbb{P}[T_0 > 0] = 1$. Then Z admits a multiplicative decomposition as

$$Z_t = Z_t^{(0)} Z_t^{(1)}$$

with a positive local martingale $(Z_t^{(0)})$ and a decreasing process $(Z_t^{(1)})$, with $Z_0^{(1)} = 1$. If there are two such factorizations, then they are identical on $[\![0, T_0[\![$.

If Z^τ is continuous, then so are $(Z_t^{(0)})$ and $(Z_t^{(1)})$, as well as (m_t^τ) and (a_t^τ) appearing in the additive (i.e., Doob–Meyer) decomposition (3.2) of Z^τ. It can be easily shown by using Itô's lemma and the Doob–Meyer decomposition (3.2) that if $Z_t^\tau > 0$ for all t, a.s., and Z^τ is continuous, then there exist a unique local martingale (m_t^τ) and a unique predictable increasing process (Λ_t) such that the multiplicative decomposition of Z^τ is

$$Z_t^\tau = \mathcal{E}_t \left(\int_0^\cdot \frac{\mathrm{d}m_s^\tau}{Z_s} \right) e^{-\Lambda_t},$$

where the process Λ is given by

$$\Lambda_t = \int_0^t \frac{1}{Z_s^\tau} \mathrm{d}a_s^\tau$$

and $\mathcal{E}(\cdot)$ is the stochastic exponential. From Theorem 3.3, we know that the process $N_t := \mathbf{1}_{\{\tau \le t\}} - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.

Theorem 6.3 *Let τ be a random time such that* (A) *and $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ hold. Assume further that $Z_t^\tau > 0$ for all $t \ge 0$, so that the process $\Gamma = -\ln Z^\tau$ is well defined. Let (m_t) be an $(\mathbb{F}, \mathbf{P})$-martingale and F a $\mathbb{G}$-predictable process such that $\mathcal{E}(\int_0^\cdot F_s \,\mathrm{d}m_s)$ is a uniformly integrable $\mathbb{G}$-martingale. Let H be an $\mathbb{F}$-predictable process such that $\mathcal{E}(\int_0^\cdot H_s \,\mathrm{d}N_s)$ is a uniformly integrable $\mathbb{G}$-martingale. Let*

$$E_t = \mathcal{E}_t \left(\int_0^\cdot F_s \,\mathrm{d}m_s \right) \mathcal{E}_t \left(\int_0^\cdot H_s \,\mathrm{d}N_s \right).$$

Assume further that (E_t) is a uniformly integrable $\mathbb{G}$-martingale (this is the case for example if $\mathcal{E}_t(\int_0^\cdot H_s \,\mathrm{d}N_s)$ is bounded in L^2, since $\int_0^t F_s \,\mathrm{d}m_s$ and $\int_0^t H_s \,\mathrm{d}N_s$ are orthogonal). Define

$$\mathrm{d}\mathbf{Q} = E_t \,\mathrm{d}\mathbf{P} \quad \text{on } \mathcal{G}_t, \text{ for all } t \ge 0.$$

Then the $\mathbf{Q}$*-Azéma supermartingale associated with* τ *has the multiplicative decomposition*

$$Z_t^{\mathbf{Q}} = \mathbf{Q}[\tau > t \,|\, \mathcal{F}_t] = \mathcal{E}_t\left(\int_0^{\cdot} (\tilde{F}_s - {}^{(\mathbf{Q},p)}F_s)\, \mathrm{d}\widetilde{m}_s\right) e^{-\int_0^t (1+H_s)\,\mathrm{d}\Gamma_s},$$

where:

- ${}^{(p,\mathbf{Q})}F$ *is the* $\mathbb{F}$*-predictable projection of the process* F *under the probability* $\mathbf{Q}$;
- $\tilde{F}$ *is an* $\mathbb{F}$*-predictable process such that* $\mathbf{1}_{\{\tau>t\}}F_t = \mathbf{1}_{\{\tau>t\}}\tilde{F}$; *and*
- $\widetilde{m}_t = m_t - \int_0^t \frac{\mathrm{d}[m,e]_s}{e_{s-}}$ *is an* $(\mathbb{F}, \mathbf{Q})$*-martingale.*

It follows that the process

$$N_t^{\mathbf{Q}} := \mathbf{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} (1 + H_s)\,\mathrm{d}\Gamma_s$$

is a $(\mathbb{G}, \mathbf{Q})$*-martingale. In particular, if the process* F *is* $\mathbb{F}$*-predictable, then*

$$Z_t^{\mathbf{Q}} = \mathbf{Q}[\tau > t \,|\, \mathcal{F}_t] = e^{-\int_0^t (1+H_s)\,\mathrm{d}\Gamma_s}$$

and the immersion property holds under $\mathbf{Q}$.

Remark 6.4 The process H above is taken to be $\mathbb{F}$-predictable to simplify the notations. Indeed, since the martingale N is constant after τ and since a $\mathbb{G}$-predictable process before τ is equal to an $\mathbb{F}$-predictable process, we could as well take H to be $\mathbb{G}$-predictable.

Proof First, we need to compute $e_t := \mathbf{E}^{\mathbf{P}}[E_t \,|\, \mathcal{F}_t]$. When applying Lemma 5.1 to

$$E_t = 1 + \int_0^t E_{s-}F_s\,\mathrm{d}m_s + \int_0^t E_{s-}H_s\,\mathrm{d}N_s,$$

one obtains that

$$e_t = 1 + \mathbf{E}^{\mathbf{P}}\left[\int_0^t E_{s-}F_s\,\mathrm{d}m_s \,\middle|\, \mathcal{F}_t\right] = 1 + \int_0^t {}^{(p,\mathbf{P})}(E_{s-}F_s)\,\mathrm{d}m_s$$

(see [5], Proposition 7, for the second equality above). Next we want to show that ${}^{(p,\mathbf{P})}(E_{t-}F_t) = e_{t-}{}^{(p,\mathbf{Q})}F_t$. Let T be a predictable stopping time. The martingales E and e satisfy

$$\mathbf{E}^{\mathbf{P}}[E_T \,|\, \mathcal{G}_{T-}] = E_{T-} \quad \text{and} \quad \mathbf{E}^{\mathbf{P}}[e_T \,|\, \mathcal{F}_{T-}] = e_{T-}.$$

Therefore, for any predictable stopping time T, we have

$$\begin{aligned}\mathbf{E}^{\mathbf{P}}[E_{T-}F_T\mathbf{1}_{\{T<\infty\}}] &= \mathbf{E}^{\mathbf{P}}\big[E_T F_T\mathbf{1}_{\{T<\infty\}}\big] = \mathbf{E}^{\mathbf{Q}}\big[{}^{(p,\mathbf{Q})}F_T\mathbf{1}_{\{T<\infty\}}\big]\\ &= \mathbf{E}^{\mathbf{P}}\big[e_T{}^{(p,\mathbf{Q})}F_T\mathbf{1}_{\{T<\infty\}}\big] = \mathbf{E}^{\mathbf{P}}\big[e_{T-}{}^{(p,\mathbf{Q})}F_T\mathbf{1}_{\{T<\infty\}}\big].\end{aligned}$$

Hence

$$e_t = \mathcal{E}_t\left(\int_0^{\cdot} {}^{(p,\mathbf{Q})}F_s \,\mathrm{d}m_s\right).$$

Putting this into the formula $Z_t^{\mathbf{Q}} = \mathbf{E}^{\mathbf{P}}[\mathbf{1}_{\{\tau>t\}} E_t \mid \mathcal{F}_t]/e_t$ leads us to

$$\begin{aligned} Z_t^{\mathbf{Q}} &= e^{-\int_0^t (1+H_s)\,\mathrm{d}\Gamma_s} \frac{\mathcal{E}_t(\int_0^{\cdot} \widetilde{F}_s \,\mathrm{d}m_s)}{\mathcal{E}_t(\int_0^{\cdot} {}^{(p,\mathbf{Q})}F_s \,\mathrm{d}m_s)} \\ &= e^{-\int_0^t (1+H_s)\,\mathrm{d}\Gamma_s} \\ &\quad \times \exp\left\{\int_0^t \left(\widetilde{F}_s - {}^{(p,\mathbf{Q})}F_s\right)\mathrm{d}m_s - \frac{1}{2}\int_0^t \left(\widetilde{F}_s^2 - \left({}^{(p,\mathbf{Q})}F_s\right)^2\right)\mathrm{d}[m,m]_s\right\}. \end{aligned}$$

Using Girsanov's theorem, $\widetilde{m}_t = m_t - \int_0^t \frac{\mathrm{d}[m,e]_s}{e_{s-}} = m_t - \int_0^t {}^{(p,\mathbf{Q})}F_s \,\mathrm{d}[m,m]_s$ is an $(\mathbb{F},\mathbf{Q})$-martingale. The result follows when replacing m_t in the above expression of $Z_t^{\mathbf{Q}}$ by $\widetilde{m}_t + \int_0^t {}^{(p,\mathbf{Q})}F_s \,\mathrm{d}[m,m]_s$. □

Corollary 6.5 *Suppose that* $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ *and* (A) *hold. Assume further that* $Z_t^{\tau} > 0$ *for all* $t \geq 0$. *Define* $\mathbf{Q}$ *by*

$$e_t = 1 + \mathbf{E}^{\mathbf{P}}\left[\int_0^t E_{s-}F_s \,\mathrm{d}m_s \,\middle|\, \mathcal{F}_t\right] = 1 + \int_0^t {}^{(p,\mathbf{P})}(E_{s-}F_s)\,\mathrm{d}m_s$$

with F *a* $\mathbb{G}$*-predictable process such that* E *is a uniformly integrable martingale. Then, under* $\mathbf{Q}$, *the process* $N_t = \mathbf{1}_{\{\tau\leq t\}} - \Gamma_{t\wedge\tau}$ *remains a* $\mathbb{G}$*-martingale.*

Proof It suffices to take $H = 0$ in Theorem 6.3. □

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