Asymptotics of spectral quantities of Schrodinger operators

Kappeler, T ; Schaad, B ; Topalov, P

Abstract: In this paper we provide new asymptotic estimates of the Floquet exponents of Schrodinger operators on the circle. By the same techniques, known asymptotic estimates of various others spectral quantities are improved.

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Preface

These Proceedings are the outcome of an International Conference on Spectral Geometry held at Dartmouth College on July 19-23, 2010. Over eighty graduate students, postdoctoral researchers, and senior researchers participated in the conference, and many participants attended three minicourses on July 16-17 that gave needed background on semiclassical analysis, spectral theory on hyperbolic surfaces, and orbifold spectral geometry.

This volume contains these preparatory lectures together with a reprint of Peter Sarnak's article "Recent Progress on the Quantum Unique Ergodicity Conjecture" which will provide valuable background for researchers interested in forefront developments in spectral geometry. Research contributions include recent work on semiclassical measures, inverse spectral geometry, spectral properties of quantum graphs, statistics of nodal lines of eigenfunctions, spectral asymptotics, and many other developments of current interest in the field. We hope that researchers interested in spectral geometry and its interactions with number theory, physics, and applied mathematics will find this collection a valuable reference.

One of the principal themes in the conference and in this volume is the behavior of eigenfunctions. Peter Sarnak's survey article describes the very exciting recent progress on the Quantum Unique Ergodicity (QUE) conjecture of Zeev Rudnick and Sarnak, which addresses the behavior of the highly excited states in the quantization of ergodic Hamiltonian systems. Microlocal analysis allows one to associate functions on a phase space T^*M to functions on the configuration space, M, for example by the Wigner function construction. Weak limits of Wigner functions of sequences $\{\phi_j\}$ of eigenfunctions of the Laplacian on a Riemannian manifold measure whether the eigenfunctions tend to concentrate or become equidistributed (tend to Liouville measure) in the semi-classical limit. The beautiful Quantum Ergodicity Theorem says that if the classical flow is ergodic, there is a subsequence of density one that is uniformly distributed. This theorem leaves open, however, the possibility of "scarring": Could there exist subsequences of density zero that converge, say, to a measure concentrated along a periodic orbit of the Hamiltonian system?

The survey article by Nalini Anantharaman and Fabricio Macià provides an innovative companion to Sarnak's survey. The fundamental solution of the Schrödinger operator without potential on a compact manifold, $e^{it\Delta}$, is very dispersive because it propagates higher frequency waves faster than low frequency ones. It turns out however that time averages of Wigner distributions of suitable functions of the form $e^{it\Delta}(u_n)$, where n is a large frequency parameter, have a semiclassical or high-frequency limit. (Here the u_n's are quite general; they need not be eigenfunctions.) In their paper, Anantharaman and Macià review some of their results on the subtle relationship between these limits and dynamical properties of geodesic flow.

The numerical computation of eigenvalues and eigenfunctions has also played a pivotal role in advancing our understanding in areas such as scars and QUE. The paper of Andrew Hassell and Alex Barnett extends such a numerical method from the Dirichlet to the Neumann boundary condition, and gives rigorous error bounds on the eigenvalues.

Three articles address the patterns and volumes of nodal sets of eigenfunctions. In the past two years, dramatic improvements have been achieved, by both local and global methods, for the lower bound on volumes of nodal sets of eigenfunctions. The article by Hamid Hezari and Zuoqin Wang gives a further improvement with an elegantly simple proof and gives a nice exposition of the various methods used in the recent advances. Chris Judge's paper studies nodal sets of sums of Maass cusp forms corresponding to distinct eigenvalues on a finite-volume hyperbolic surface X: motivated by Cheng's well-known result that the nodal set of a Laplace eigenfunction on a compact surface is a topological graph, Judge shows that the closure of the nodal set of such a sum is the embedding of a graph into the closure of X. Numbers of nodal domains and the lengths of nodal lines of chaotic eigenfunctions on surfaces are very difficult to address; more progress has been made in the related case of random eigenfunctions of symmetric spaces (such as random spherical harmonics of fixed degree on S^2). Igor Wigman provides a clear survey of these latest results.

Grauert tubes are strictly pseudo-convex Stein manifolds, and in some ways are analogous to strictly pseudo-convex domains in $\mathbb{C}^m$ and to Hermitian unit bundles in negative line bundles. The purpose of Steven Zelditch's article is to extend to Grauert tubes some of the basic notions and results of PSH (pluri-subharmonic) function theory on stricty pseudo-convex domains in $\mathbb{C}^m$ and their recent generalization of this theory to Kähler manifolds. The basic theme is to use analytic continuations of eigenfunctions $\varphi_{\lambda_j}^{\mathbb{C}}$ in place of holomorphic polynomials of degree $\sim \lambda_j$ on $\mathbb{C}^m$ or holomorphic sections of line bundles of degree $\sim \lambda_j$ over a Kähler manifold. Zelditch also gives a detailed proof of a theorem stated by Boutet de Monvel that the Poisson kernel admits an analytic continuation as a Fourier integral operator with complex phase.

A second major theme is inverse spectral geometry, the study of the extent to which geometry is encoded in spectral data.

When a compact Riemannian manifold admits a Lie group of symmetries, the eigenspaces of the Laplacian become naturally representation spaces for the symmetry group. One can then raise an "equivariant inverse spectral problem" taking into account not only the eigenvalues of the Laplacian but also the representation of the symmetry group on the corresponding eigenspace. The pair of plenary lectures at the conference by Victor Guillemin and the expository article in this volume by Emily Dryden, Victor Guillemin and Rosa Sena-Dias, consider the case of a toric variety with a toric metric, and ask whether the equivariant spectrum determines the moment polytope. They show that this is indeed the case for generic toric orbifolds.

Geometric quantization associates to a classical Hamiltonian system a corresponding quantum system. In the case of magnetic geodesic flows on flat tori, Carolyn Gordon, William Kirwin, Dorothee Schueth, and David Webb address the dependence of the eigenvalue spectrum of the quantum Hamiltonian on the metric and symplectic structure.

The heat trace spectral invariants are the coefficients of the asymptotic expansion of the trace of $e^{-t\Delta}$ or of $e^{-t(\Delta+V)}$ as t tends to zero, and they have played a very important role in the inverse spectral problem for the metric or the potential V. Following a method of Hitrik and Polterovich, the article by Victor Guillemin, Alejandro Uribe and Zuoqin Wang establishes a symbolic method of computing the coefficients of the *semiclassical* expansion of the trace of e^{-tH} where H is a perturbation of the n-dimensional harmonic oscillator and t is a suitably rescaled time parameter. Since this is a semiclassical expansion the coefficients are functions of the rescaled time. Using these invariants the authors obtain several inverse spectral results on the perturbation.

A quantum graph is a metric graph together with a self-adjoint Schrödinger type operator with specified vertex conditions. Gregory Berkolaiko and Peter Kuchment analyze the dependence of the Hamiltonian, its resolvent, and its spectrum on the vertex conditions and edge lengths, obtaining interesting new analyticity results and addressing applications. The article also provides a well-written introduction to quantum graphs.

Spectral asymptotics play a crucial role in the analysis of completely integrable equations such as the periodic mKdV and KdV equations. The paper of Thomas Kappeler, Beat Schaad, and Peter Topalov derives asymptotics of Floquet exponents for Schrödinger operators on the circle, and uses these results to obtain improved asymptotics for the periodic/antiperiodic, Dirichlet, and Neumann spectra.

Many physical quantities depend on the spectral density; one is the Casimir force. The paper of Jeffrey Bouas, Stephen Fulling, Fernando Mera, Krishna Thapa, Cynthia Trendafilova, and Jef Wagner applies semiclassical methods to study this force in the case of a penetrable wall with various one-dimensional potential forms.

The editors (who together co-organized the conference) are grateful first of all to Dartmouth College and to the National Science Foundation (through grant DMS-1005360) for generous financial support that made this conference possible. The staff at Dartmouth's Department of Mathematics, and in particular Tracy Moloney, provided superb support and warm hospitality. Sarunas Burdulis was instrumental in building the online database that tracked participant registration and housing. Dartmouth's friendly atmosphere, New England's stunning natural beauty, and the high quality of all conference lectures combined for a memorable experience of what one participant called a "summer camp for mathematicians"! Finally, the editors are grateful to all those who contributed papers to this volume; to Peter Sarnak and to the AMS for arranging the reprinting of his Bulletin survey; and to Christine Thivierge at the American Mathematical Society, whose support and superhuman patience were instrumental in bringing the volume to fruition.

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