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THE ZEROS OF RANDOM POLYNOMIALS CLUSTER UNIFORMLY NEAR THE UNIT CIRCLE

C.P. HUGHES AND A. NIKEGHBALI

Abstract. In this paper we deduce a universal result about the asymptotic distribution of roots of random polynomials, which can be seen as a complement to an old and famous result of Erdős and Turan. More precisely, given a sequence of random polynomials, we show that, under some very general conditions, the roots tend to cluster near the unit circle, and their angles are uniformly distributed. The method we use is deterministic: in particular, we do not assume independence or equidistribution of the coefficients of the polynomial.

1. Introduction

In this paper, we are interested in the uniform concentration near the unit circle of roots of polynomials.

Let $(P_N(Z))_{N\geq 1}$ be a sequence of polynomials. Denote the zeros of $P_N(Z)$ by $z_1, \dots, z_N$. Let

$$\nu_N(\rho) := \#\left\{z_k \;:\; 1-\rho \leq |z_k| \leq \frac{1}{1-\rho}\right\} \tag{1}$$

be the number of zeros of $P_N(Z)$ lying in the annulus bounded by $1-\rho$ and $\frac{1}{1-\rho}$, where $0 \leq \rho \leq 1$, and let

$$\nu_N(\theta,\phi) := \#\left\{z_k \;:\; \theta \leq \arg(z_k) < \phi\right\} \tag{2}$$

be the number of zeros of $P_N(Z)$ whose argument lies between θ and ϕ, where $0 \leq \theta < \phi \leq 2\pi$.

We shall say that the zeros cluster uniformly around the unit circle if for all fixed $0 < \rho < 1$,

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\rho) = 1 \tag{3}$$

and

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi} \tag{4}$$

The purpose of this paper is to find a general but simple condition for when the zeros cluster uniformly around the unit circle.

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Theorem 1. *Let $(P_N(Z))$ be a sequence of polynomials, with*

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k,$$

such that $a_{N,0}a_{N,N} \neq 0$ for all N. Let

$$L_N(P_N) = \log\left(\sum_{k=0}^{N} |a_{N,k}|\right) - \frac{1}{2}\log|a_{N,0}| - \frac{1}{2}\log|a_{N,N}|.$$

If

$$L_N(P_N) = o(N),$$

then the zeros of this sequence cluster uniformly near the unit circle, i.e. for all $0 < \rho < 1$,

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\rho) = 1,$$

and for all $0 \leq \theta < \phi \leq 2\pi$,

$$\lim_{N\to\infty} \frac{1}{N}\nu_N(\theta,\phi) = \frac{\phi-\theta}{2\pi}.$$

where $\nu_N(\rho)$ and $\nu_N(\theta,\phi)$ are defined in (1) and (2) respectively.

The second part of our theorem, on $\nu_N(\theta,\phi)$, follows from the celebrated result of Erdős and Turan [6] on the distribution of roots of polynomials:

Theorem 2 (Erdős-Turan)**.** *Let $(a_k)_{0\leq k\leq N}$ be a sequence of complex numbers such that $a_0 a_N \neq 0$, and let*

$$P(Z) = \sum_{k=0}^{N} a_k Z^K .$$

Define

$$L_N(P) = \log\sum_{k=0}^{N} |a_k| - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|. \tag{5}$$

Then

$$\left|\frac{1}{N}\nu_N(\theta,\phi) - \frac{\phi-\theta}{2\pi}\right|^2 \leq \frac{C}{N} L_N(P)$$

for some constant C, where $\nu_N(\theta,\phi)$ is defined in (2).

The above theorem shows that if $L_N(P)$ is small compared to the degree N, then the angles of the roots are nearly uniformly distributed, and that is precisely the reason why this theorem has been extensively used to prove asymptotic uniform concentration near the unit circle of the roots of some families of random polynomials.

In this paper, we prove a natural compliment to this result.

Theorem 3. *Let $(a_k)_{0\leq k\leq N}$ be a sequence of complex numbers such that $a_0 a_N \neq 0$, and let*

$$P(Z) = \sum_{k=0}^{N} a_k Z^K.$$

Let $L_N(P)$ be defined as in (5)*. Then for* $0<\rho<1$*,*

$$\left(1-\frac{1}{N}\nu_N(\rho)\right)\leq\frac{2}{N\rho}L_N(P)$$

where $\nu_N(\rho)$ is defined in (1)*.*

These two theorems on deterministic polynomials give a sufficient condition for the roots of random polynomials to cluster uniformly around the unit circle, and we will show how results of Šparo and Šur [10], Arnold [1], and Shmerling and Hochberg [9] follow as corollaries of this. Indeed, we show that some of their conditions on the coefficients of the polynomials can be dropped.

More precisely, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, on which a random array $(a_{N,k})_{\substack{N\geq 1\\ 0\leq k\leq N}}$ is defined. Consider now the sequence of random polynomials $(P_N(Z))$, with

$$P_N(Z)=\sum_{k=0}^{N}a_{N,k}Z^k. \tag{6}$$

The asymptotics for roots of random polynomials in the complex plane have already been studied in the special case where

$$P_N(Z)=\sum_{k=0}^{N}a_kZ^k;$$

which corresponds to the special array $(a_{N,k})_{k\leq N}=(a_0,a_1,\ldots,a_N)$, and it is not our intention to give a full historical account here (see [2], [7], [5] for more details and references), but we shall rather mention the papers of Šparo and Šur [10], Arnold [1], and Shmerling and Hochberg [9] which contain the most general results about uniform clustering near the unit circle. Šparo and Šur [10] have considered i.i.d. complex coefficients $(a_n)_{n\geq 0}$ and have shown that under some integrability conditions, the zeros of such sequences cluster uniformly near the unit circle, with convergence in (3) and (4) holding in probability. Arnold [1] improved this result and proved that the convergence holds in fact almost surely and in the p^{th} mean if the moduli of a_k are equidistributed (plus some integrability conditions). Recently, Shmerling and Hochberg [9] have obtained stronger results: they have shown that the condition on equidistribution can be dropped if $(a_n)_{n\geq 0}$ is a sequence of independent variables which have continuous densities f_n which are uniformly bounded in some neighborhood of the origin with finite means μ_n and standard deviations σ_n that satisfy the condition

$$\max\left\{\limsup_{n\to\infty}\sqrt[n]{|\mu_n|}\ ,\ \limsup_{n\to\infty}\sqrt[n]{|\sigma_n|}\right\}=1,$$
$$\mathbb{P}\left[a_0=0\right]=0$$

The authors above mentioned consider separately the cases of (3) and (4). They prove (4) using Theorem 2, and prove (3) using techniques from random power series. In particular, to prove (4), they are able to show that $\frac{L_N(P)}{N}\to 0$, as $N\to\infty$ and this is only a few lines proof, while the techniques used to prove (3) are more sophisticated. Thus a side benefit of Theorem 3 is that once one proves the uniform distribution of the angles using Theorem 2 of Erdős and Turan, then one actually proves uniform clustering near the unit circle considerably simplifying the arguments in the proofs on uniform clustering in [1] and [9]. Moreover, as we shall see, our results also give us an estimate for the rate of clustering (that is, how quickly $\rho \to 0$ as a function of N).

The results we have mentioned above do not apply to the more general case of sequences of random polynomials of the form (6) we are dealing with, and they also do not lead to anything interesting in the case of deterministic sequences of polynomials (the asymptotic study of roots sequences of deterministic polynomials occurs in problems of equidistribution of algebraic integers for example [3]). Moreover, these results do not cover the cases where the coefficients are dependent with different distributions: for example in the semiclassical approximations for multidimensional quantum systems, one needs to locate roots of high degree random polynomials with dependent and non identically distributed coefficients [4] (we should point out that the authors in [4] have observed the uniform clustering in the special case of self-reciprocal polynomials with complex Gaussian coefficients with finite variance).

The aim of this paper is to show that the phenomenon of uniform concentration of zeros around the unit circle is universal, in the sense that no independence or equidistribution on the coefficients is required, but only conditions on their size. Our method, based on elementary complex analysis, reduces both convergences (3) and (4) to the same problem, namely showing that $L_N(P)$, defined in (5), is small compared to the degree, N, of the polynomial P, thus complementing Theorem 2 of Erdős and Turan.

The structure of this paper is as follows: In section 2 we prove our main theorems and then in section 3 we use them to deduce clustering of zeros for general sequences of random polynomials.

2. Basic estimates

For $N \geq 1$, let $(a_k)_{0\leq k\leq N}$ be a sequence of complex numbers satisfying $a_0 a_N \neq 0$. From this sequence construct the polynomial

$$P_N(Z) = \sum_{k=0}^{N} a_k Z^k,$$

and denote its zeros by z_i (where i ranges from 1 to N). For $0 \leq \rho \leq 1$, we are interested in estimates for

$$\begin{aligned}
\widetilde{\nu}_N(\rho) &= \#\{z_j\ ,\ |z_j| < 1-\rho\} \\
\overline{\nu}_N(\rho) &= \#\left\{z_j\ ,\ |z_j| > \frac{1}{1-\rho}\right\} \\
\nu_N(\rho) &= \#\left\{z_j\ ,\ 1-\rho \leq |z_j| \leq \frac{1}{1-\rho}\right\}
\end{aligned}$$

which counts the number of zeros of the polynomial $P_N(Z)$ which lie respectively inside the open disc of radius $1-\rho$, outside the closed disc of radius $1/(1-\rho)$, and inside the closed annulus bounded by circles of radius $1-\rho$ and $1/(1-\rho)$.

Theorem 4. *For $N \geq 1$, let $(a_k)_{0\leq k\leq N}$ be an sequence of complex numbers which satisfy $a_0 a_N \neq 0$. Then, for $0 < \rho < 1$*

$$\frac{1}{N}\widetilde{\nu}_N(\rho) \leq \frac{1}{N\rho}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \log|a_0|\right), \tag{7}$$

$$\frac{1}{N}\overline{\nu}_N(\rho) \leq \frac{1}{N\rho}\left(\log\left(\sum_{k=0}^{n}|a_k|\right) - \log|a_N|\right) \tag{8}$$

and

$$\left(1 - \frac{1}{N}\nu_N(\rho)\right) \leq \frac{2}{N\rho}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right) \tag{9}$$

Proof. An application of Jensen's formula (see [8]) yields

$$\frac{1}{2\pi}\int_0^{2\pi}\log|P_N(e^{\imath\varphi})|\ d\varphi - \log|P_N(0)| = \sum_{|z_i|<1}\log\frac{1}{|z_i|}$$

where the sum on the right hand side is on zeros lying inside the open unit disk. We have the following minorization for this sum:

$$\begin{aligned}\sum_{|z_i|<1}\log\frac{1}{|z_i|} &\geq \sum_{|z_i|<1-\rho}\log\frac{1}{|z_i|}\\ &\geq \rho\widetilde{\nu}_N(\rho)\end{aligned}$$

since if $0 \leq \rho \leq 1$, then for all $|z_i| \leq 1-\rho$, $\log(1/|z_i|) \geq \rho$, and by definition there are $\widetilde{\nu}_N(\rho)$ such terms in the sum.

We also have the following trivial upper bound

$$\max_{\varphi\in[0,2\pi]}|P_N(e^{\imath\varphi})| \leq \sum_{k=0}^{N}|a_k|,$$

and so

$$\begin{aligned}\rho\widetilde{\nu}_N(\rho) &\leq \frac{1}{2\pi}\int_0^{2\pi}\log|P_N(e^{\imath\varphi})|\,\mathrm{d}\varphi - \log|a_0|\\ &\leq \log\left(\sum_{k=0}^{N}|a_k|\right) - \log|a_0|\end{aligned}$$

which gives equation (7).

To estimate the number of zeros lying outside the closed disc of radius $(1-\rho)^{-1}$, note that if z_0 is a zero of the polynomial $P_N(Z) = \sum_{k=0}^{N} a_k Z^k$, then $1/z_0$ is a zero of the polynomial $Q_N(Z) := Z^N P_N\left(\frac{1}{Z}\right) = a_N + a_{N-1}Z + \ldots + a_0 Z^N$. Therefore, the number of zeros of $P_N(Z)$ outside the closed disc of radius $1/(1-\rho)$ equals the number of zeros of $Q_N(Z)$ inside the open disc of radius $1-\rho$. Therefore, from (7) we get

$$\frac{1}{N}\overline{\nu}_N(\rho) \leq \frac{1}{N\rho}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \log|a_N|\right)$$

which gives equation (8).

Since

$$N - \nu_N(\rho) = \widetilde{\nu}_N(\rho) + \overline{\nu}_N(\rho)$$

we immediately get (9). □

Theorem 3 is merely a restatement of equation (9). Combining this with the theorem of Erős and Turan, Theorem 2, yields Theorem 1.

Remark. Note that if $a_k \mapsto \lambda a_k$ for some $\lambda \neq 0$, then the zeros of $P_N(Z)$ are unchanged, and

$$\log\left(\sum_{k=0}^{N} |\lambda a_k|\right) - \frac{1}{2}\log|\lambda a_0| - \frac{1}{2}\log|\lambda a_N| \\ = \log\left(\sum_{k=0}^{N} |a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|$$

so, in some sense, this is a natural function to control the location of the zeros.

3. Uniform clustering results for roots of random polynomials

We require no independence restriction on our random variables. We only assume that

$$\mathbb{P}\left[a_{N,0} = 0\right] = 0 \tag{10}$$

and

$$\mathbb{P}\left[a_{N,N} = 0\right] = 0, \tag{11}$$

for all N.

3.1. **The main theorem for random polynomials.**

Theorem 5. *For $N \geq 1$, let $(a_{N,k})_{0\leq k\leq N}$ be an array of random complex numbers such that $\mathbb{P}\left[a_{N,0} = 0\right] = 0$ and $\mathbb{P}\left[a_{N,N} = 0\right] = 0$ for all N. Let*

$$L_N = \log\left(\sum_{k=0}^{N} |a_{N,k}|\right) - \frac{1}{2}\log|a_{N,0}| - \frac{1}{2}\log|a_{N,N}| \tag{12}$$

If

$$\mathbb{E}\left[L_N\right] = o(N) \qquad \text{as } N \to \infty \tag{13}$$

then there exists a positive function α_N satisfying $\alpha_N = o(N)$ such that the zeros of the random polynomial

$$P_N(Z) = \sum_{k=0}^{N} a_{N,k} Z^k$$

satisfy

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)\right] = 1$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\nu_N\left(\theta, \phi\right)\right] = \frac{\phi - \theta}{2\pi}$$

In fact the convergence also holds in probability and in the $p^{\,th}$ mean, for all positive p.

Furthermore, if there exists a (deterministic) positive function α_N satisfying $\alpha_N \leq N$ for all N, such that

$$L_N = o(\alpha_N) \quad \textit{almost surely} \tag{14}$$

then both convergences hold almost surely (and also in the p th mean, for all positive p).

Proof. The convergence in mean for $\nu_N(\alpha_N/N)$ is a consequence of (9). We have

$$1 - \mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)\right] \leq \frac{2}{\alpha_N}\,\mathbb{E}\left[L_N\right]$$

Therefore we see that the result follows for any positive function α_N satisfying $\alpha_N \leq N$ for all N such that $\mathbb{E}\left[L_N\right]/\alpha_N \to 0$, and such a function exists by assumption (13).

Similarly from Theorem 2 and (13) we have that

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\nu_N\left(\theta,\phi\right) - \frac{\phi-\theta}{2\pi}\right|^2\right] &\leq \frac{C}{N}\,\mathbb{E}\left[L_N\right] \\ &= o(1)\end{aligned}$$

Note that the mean square convergence implies convergence in the mean, as in the theorem, and also convergence in probability. Note further, that since the random variables are uniformly bounded ($0 \leq \frac{1}{N}\nu_N\left(\theta,\phi\right) \leq 1$), mean convergence implies convergence in the p^{th} mean for all positive p.

In the same way, the almost sure convergence of $\frac{1}{N}\nu_N(\alpha_N/N)$ and $\frac{1}{N}\nu_N(\theta,\phi)$ follows immediately from (9) and Theorem 2, using (14). □

In the following subsections, we shall give some sufficient conditions, which are easy to check, for Theorem 5 to hold. We first consider the case of general sequences of random polynomials for which there exist no results to our knowledge; then we deal with the classical random polynomials.

3.2. **General sequences of random polynomials.**

Proposition 6. *Let $(a_{N,k})$ be an array of random complex numbers which satisfy (10) and (11). Assume that $\mathbb{E}\left[\log|a_{N,0}|\right] = o(N)$, and $\mathbb{E}\left[\log|a_{N,N}|\right] = o(N)$, and that there exists a fixed $0 < s \leq 1$ and a sequence ε_N tending to zero such that*

$$\sum_{k=0}^{N}\mathbb{E}\left[|a_{N,k}|^s\right] \leq \exp(\varepsilon_N N)$$

(or equivalently $\sum_{k=0}^{N}\mathbb{E}\left[|a_{N,k}|^s\right] = \exp\left(o\left(N\right)\right)$). Then, there exists an $\alpha_N = o(N)$ such that L_N, defined in (12), satisfies $\mathbb{E}\left[L_N\right] = o(\alpha_N)$, and so

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)\right] = 1$$

and

$$\lim_{N\to\infty}\mathbb{E}\left[\left|\frac{1}{N}\nu_N\left(\theta,\phi\right) - \frac{\phi-\theta}{2\pi}\right|\right] = 0$$

Proof. It is a consequence of Theorem 5 and the following concavity inequalities for $s > 0$:

$$\begin{aligned}\mathbb{E}\left[\log\left(\sum_{k=0}^{N}|a_{N,k}|\right)\right] &\leq \frac{1}{s}\log\left(\sum_{k=0}^{N}\mathbb{E}\left[|a_{N,k}|^s\right]\right)\\ &\leq \frac{1}{s}\log\left((N+1)\exp(\varepsilon_N N)\right) = o(N)\end{aligned}$$

since we assume $\varepsilon_N \to 0$ as $N \to \infty$. Therefore L_N, defined in (12), satisfies $L_N = o(N)$, and the result follows from Theorem 5. □

Remark. The Proposition shows that under some very general conditions (just some conditions on the size of the expected values of the modulus of the coefficients), without assuming any independence or equidistribution condition, the zeros of random polynomials tend to cluster uniformly near the unit circle. We can also remark that we do not assume that our coefficients must have density functions: they can be discrete-valued random variables. Finally, observe from the proof that we obtain an estimate for the rate of clustering: any α_N such that $(\log N + N\varepsilon_N)/\alpha_N \to 0$ suffices.

Now we give two examples which could not be dealt with the previous results available in the literature.

Example. Let $a_{N,k}$ be random variables distributed according to the Cauchy distribution with parameter $N(k+1)$. The first moment does not exist but some fractional moments do, and in particular we have for $0 \leq s < 1$

$$\begin{aligned}\mathbb{E}\left[|a_{N,k}|^s\right] &= \frac{N(k+1)}{\pi}\int_{-\infty}^{\infty}\frac{|x|^s}{x^2+N^2(k+1)^2}\,\mathrm{d}x\\ &= \frac{1}{\pi}N^s(k+1)^s\Gamma(\frac{1}{2}+\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})\end{aligned}$$

Moreover,

$$\mathbb{E}\left[\log|a_{N,k}|\right] = \log(N(k+1))$$

Hence we can apply Proposition 6 and deduce that the zeros of the sequence of random polynomials with coefficients $(a_{N,k})_{\substack{N\geq 1\\ 0\leq k\leq N}}$ where $a_{N,k}$ are chosen from the Cauchy distribution with parameter $N(k+1)$ cluster uniformly around the unit circle.

Example. For each N, let $(a_{N,k})$, $0 \leq k \leq N$, be discrete random variables taking values in $\{\pm 1, \ldots, \pm N\}$, not necessarily having the same distribution; then

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\nu_N\left(\frac{\log^{1+\gamma}(N)}{N}\right) &= 1, \quad a.s., \quad \forall\gamma > 0\\ \frac{1}{N}\nu_N(\theta,\phi) &\to \frac{\phi-\theta}{2\pi}, \quad a.s.\end{aligned}$$

As a special case, we have the well known random polynomials $\sum_{k=0}^{N}\mu_k Z^k$, where $\mu_k = \pm 1$, with probabilities p and $(1-p)$. Moreover, we have from the Markov inequality, the following rate for the convergence in probability:

$$\begin{aligned}\mathbb{P}\left[\left(1-\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)\right)>\varepsilon\right] &\leq \frac{1}{\varepsilon}\frac{C\log N}{\alpha_N}\\ \mathbb{P}\left[\left|\frac{1}{N}\nu_N\left(\theta,\phi\right)-\frac{\phi-\theta}{2\pi}\right|>\varepsilon\right] &\leq \frac{1}{\varepsilon^2}\frac{C\log N}{N}\end{aligned}$$

for any fixed $\varepsilon > 0$.

3.3. **Classical Random Polynomials.** Let us now consider the special, but important, case of the classical random polynomials as mentioned in the first section, that is

$$P_N\left(Z\right)=\sum_{k=0}^{N}a_k Z^k \tag{15}$$

These polynomials have been extensively studied (see, for example, [2] or [7] for a complete account).

The results of the previous section take a simpler form in the special case of random polynomials of the form (15). The conditions (10) and (11) become

$$\mathbb{P}\left[a_N=0\right]=0,\ \text{for all } N\geq 0$$

In this more special case, we can deal more easily with almost sure convergence, which in our framework is the strongest convergence.

Theorem 7. *Let $(a_k)_{k\geq 0}$ be a sequence of complex random variables. Assume that*

- *There exists some $s\in(0,1]$ such that for all k,*

  $$\lambda_k := \mathbb{E}\left[|a_k|^s\right]<\infty.$$

  Assume further that

  $$\limsup_{k\to\infty}\left(\lambda_k\right)^{1/k}=1$$

  or, equivalently, there exists a sequence (ε_N) tending to zero such that

  $$\sum_{k=0}^{N}\lambda_k=\exp(N\varepsilon_N).$$

- *For some $0<\delta\leq 1$ there exists $t>0$, such that for all N*

  $$\xi_N := \mathbb{E}\left[\frac{1}{\left|a_N\right|^t}\mathbb{1}_{\{|a_N|\leq\delta\}}\right]=O\left(N^q\right) \tag{16}$$

  for some $q>0$.

Then for any deterministic sequence α_N subject to $0<\alpha_N<N$ for all N, and satisfying $\frac{\alpha_N}{N\varepsilon_N+\log N}\to\infty$ as $N\to\infty$, we have

$$\lim_{N\to\infty}\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)=1,\quad a.s.$$

and

$$\lim_{N\to\infty}\frac{1}{N}\nu_N\left(\theta,\phi\right)=\frac{\phi-\theta}{2\pi},\quad a.s.$$

In fact the convergence also holds in the p^{th} mean for every positive p.

Proof. Note that (16) implies $\mathbb{P}\{|a_k| = 0\} = 0$. Therefore, from Theorem 5 it is sufficient to prove that for the choice of α_N given in the theorem,

$$\frac{1}{\alpha_N}\left(\log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N|\right) \to 0 \quad a.s.$$

For $0 < \delta \le 1$ we have

$$\begin{aligned}\log 2 &\le \log\left(\sum_{k=0}^{N}|a_k|\right) - \frac{1}{2}\log|a_0| - \frac{1}{2}\log|a_N| \\ &\le \frac{1}{s}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) + \frac{1}{2t}\left(\log\frac{1}{|a_0|}\right)\mathbb{1}_{\{|a_0|\le\delta\}} + \frac{1}{2t}\left(\log\frac{1}{|a_N|}\right)\mathbb{1}_{\{|a_N|\le\delta\}} \\ &\qquad + \log\frac{1}{\delta}\end{aligned}$$

so since $\alpha_N \to \infty$, it is sufficient to show that

$$\frac{1}{\alpha_N}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) = 0 \quad a.s$$

and

$$\frac{1}{\alpha_N}\left(\log\frac{1}{|a_N|^t}\right)\mathbb{1}_{\{|a_N|\le\delta\}} = 0 \quad a.s.$$

We are first going to prove that $\lim_{N\to\infty}\frac{1}{\alpha_N}\log\left(1+\sum_{k=0}^{N}|a_k|\right) = 0$, $a.s.$ for our sequence α_N.

Consider first the case when $\sum_{k=0}^{\infty}\lambda_k$ is finite. By the monotone convergence theorem, the sum $\sum_{k=0}^{N}|a_k|^s$ converges almost surely as $N \to \infty$ to an integrable random variable X. Therefore, since α_N tends to infinity as $N \to \infty$, we see that

$$\lim_{N\to\infty}\frac{1}{\alpha_N}\frac{1}{s}\log\left(1+\sum_{k=0}^{N}|a_k|^s\right) = 0 \quad a.s.$$

We can thus assume that $\sum_{k=0}^{\infty}\lambda_k = \infty$. Given $\varepsilon > 0$, take $\beta > 0$ such that $\log(1+\beta) \le \varepsilon/3$. As $\limsup_{k\to\infty}(\lambda_k)^{\frac{1}{k}} = 1$, and $\lambda_k < \infty$ for all k, there exists a constant $C = C(\beta)$ such that for all k we have $\lambda_k \le C(1+\beta)^k$. Hence, for N sufficiently large,

$$0 \le \log\left(\sum_{k=0}^{N}\lambda_k\right) \le \log C + (N+1)\log(1+\beta) - \log(\beta)$$

Thus,

$$0 \le \frac{1}{N+1}\log\left(\sum_{k=0}^{N}\lambda_k\right) \le \frac{1}{N+1}\log C + \log(1+\beta) - \frac{1}{N+1}\log(\beta)$$

There exists N' such that for $N \geq N'$,

$$\begin{aligned} \frac{1}{N+1}\log C &\leq \varepsilon/3 \\ \frac{1}{N+1}|\log\beta| &\leq \varepsilon/3 \end{aligned}$$

Hence, for all $\varepsilon > 0$, we found $N_0 = \max(N', k_0)$, such that for all $N \geq N_0$ we have $\frac{1}{N+1}\log\left(\sum_{k=0}^{N}\lambda_k\right) \leq \varepsilon$, which implies $\log\left(\sum_{k=0}^{N}\lambda_k\right) = o(N)$. We can thus write for $N \geq 0$:

$$\log\left(\sum_{k=0}^{N}\lambda_k\right) = \varepsilon_N N$$

with $\varepsilon_N \to 0$ and $\varepsilon_N N \to \infty$.

Since $(N+1)^2/(k+1)^2 \geq 1$ for all $0 \leq k \leq N$, we have

$$\begin{aligned} \log\left(1 + \sum_{k=0}^{N}|a_k|^s\right) &\leq \log\left(1 + (N+1)^2\exp(\varepsilon_N N)\sum_{k=0}^{N}\frac{|a_k|^s\exp(-\varepsilon_N N)}{(k+1)^2}\right) \\ &\leq 2\log(N+1) + \varepsilon_N N + \log\left(1 + \sum_{k=0}^{N}\frac{|a_k|^s\exp(-\varepsilon_N N)}{(k+1)^2}\right) \end{aligned}$$

Now, as

$$\sum_{k=0}^{N}\lambda_k = \exp(\varepsilon_N N),$$

we have

$$\sum_{k=0}^{\infty}\mathbb{E}\left[\frac{|a_k|^s\exp(-\varepsilon_N N)}{(k+1)^2}\right] \leq \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} < \infty$$

We deduce from the monotone convergence theorem that

$$\sum_{k=0}^{N}\frac{|a_k|^s\exp(-\varepsilon_N N)}{(k+1)^2}$$

converges almost surely to an integrable random variable. Hence, taking α_N to be any positive function such that

$$\frac{\alpha_N}{\varepsilon_N N + \log N} \to \infty$$

we have

$$\lim_{N\to\infty}\frac{1}{\alpha_N}\log\left(1 + \sum_{k=0}^{N}|a_k|^s\right) = 0, \quad a.s.$$

Now, let us show that for the same sequence α_N, we have

$$\frac{1}{\alpha_N}\left(\log\frac{1}{|a_N|^t}\right)\mathbb{1}_{\{|a_N|\leq\delta\}} = 0, \quad a.s.$$

From (16) we have

$$0 \leq \log\left(\frac{1}{|a_N|^t}\right) 1\!\!1_{\{|a_N|\leq\delta\}} \leq \log\left(1+\frac{1}{|a_N|^t} 1\!\!1_{\{|a_N|\leq\delta\}}\right)$$
$$\leq (q+2)\log(N+1) + \log\left(1+\frac{1}{(N+1)^{q+2}|a_N|^t} 1\!\!1_{\{|a_N|\leq\delta\}}\right)$$

From the Markov inequality, we have, for any $\varepsilon > 0$:

$$\mathbb{P}\left[\left(\frac{1}{(N+1)^{q+2}|a_N|^t} 1\!\!1_{\{|a_N|\leq\delta\}}\right) > \varepsilon\right] \leq \frac{1}{\varepsilon}\frac{\xi_N}{(N+1)^{q+2}}$$

As $\xi_N = O(N^q)$, for N large enough,

$$\mathbb{P}\left[\left(\frac{1}{(N+1)^{q+2}|a_N|^t} 1\!\!1_{\{|a_N|\leq\delta\}}\right) > \varepsilon\right] \leq \frac{1}{\varepsilon}\frac{C}{(N+1)^{2}}$$

for some positive constant C. Hence by the Borel-Cantelli lemma,

$$\frac{1}{(N+1)^{q+2}|a_N|^t} 1\!\!1_{\{|a_N|\leq\delta\}} \to 0 \quad a.s.$$

We can conclude that if α_N goes to infinity faster than $\log N$ (which our choice of α_N does), then

$$\lim_{N\to\infty}\frac{1}{\alpha_N}\left(\log\left(\frac{1}{|a_N|^t}\right) 1\!\!1_{\{|a_N|\leq\delta\}}\right) = 0, \quad a.s.$$

and the theorem follows. □

From theorem 7 we deduce the following corollary:

Corollary 7.1. *Let $(a_k)_{k\geq 0}$ be a sequence of (possibly dependent) complex random variables such that the moduli $(|a_k|)$ have densities which are uniformly bounded in a neighborhood of the origin. Assume that there exists some $s \in (0,1]$ such that if $\lambda_k := \mathbb{E}[|a_k|^s]$ then $\lambda_k < \infty$ for all k, and*

$$\limsup_{k\to\infty}(\lambda_k)^{1/k} = 1.$$

Then almost surely the zeros of the classical random polynomial

$$P(Z) = \sum_{k=0}^{N} a_k Z^k$$

cluster uniformly around the unit circle.

Proof. It suffices to notice that in this special case, $\sup_N \xi_N \leq C$ for some positive constant C, and so the conclusions of theorem 7 follow. These imply uniform clustering of the zeros (and even give an estimate on the rate of clustering). □

Example. Let $P_N(Z) = \sum_{k=0}^{N} a_k Z^k$, with a_k being distributed on $\mathbb{R}_+$ with Cauchy distribution with parameter $k^{-\sigma}$, $\sigma > 0$. This distribution has density

$$\frac{2}{\pi k^\sigma}\frac{1}{x^2+k^{-2\sigma}}$$

on the positive real line. The conditions of Theorem 7 are satisfied since $\lambda_k := \mathbb{E}\left[a_k^{1/2}\right] \leq \frac{C}{k^\sigma}$ and $\xi_N := \mathbb{E}\left[\frac{1}{a_N^{1/2}} \mathbb{1}_{\{|a_N| \leq 1\}}\right] \leq Ck^\sigma$. Therefore, if $\alpha_N = o(N)$ is such that $\alpha_N / \log N \to \infty$, then

$$\lim_{N\to\infty} \frac{1}{N} \nu_N\left(\frac{\alpha_N}{N}\right) = 1, \quad a.s.$$

and

$$\lim_{N\to\infty} \frac{1}{N} \nu_N\left(\theta, \phi\right) = \frac{\phi - \theta}{2\pi}, \quad a.s.$$

Again, the convergence also holds in the p^{th} mean for every positive p.

We can still weaken the hypotheses and still have mean convergence.

Proposition 8. *Let $(a_k)_{k\geq 0}$ be a sequence of complex random variables. Assume that there exists some $s \in (0,1]$ such that if $\lambda_k := \mathbb{E}\left[|a_k|^s\right]$ then for all k, $\lambda_k < \infty$, and*

$$\limsup_{k\to\infty} \left(\lambda_k\right)^{1/k} = 1.$$

Assume that there exists some $t > 0$ such that for all N,

$$\xi_N \equiv \mathbb{E}\left[\frac{1}{|a_N|^t} \mathbb{1}_{\{|a_N| \leq \delta\}}\right] < \infty$$

for any $\delta \in (0,1]$, and

$$\log\left(1 + \xi_N\right) = o\left(N\right)$$

Then, for some sequence $\alpha_N = o\left(N\right)$, $0 < \alpha_N < N$,

$$\lim_{N\to\infty} \mathbb{E}\left[\left(1 - \frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)\right)^p\right] = 0, \quad \forall p > 0$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\frac{1}{N}\nu_N\left(\theta, \phi\right) - \frac{\phi - \theta}{2\pi}\right|^p\right] = 0, \quad \forall p > 0$$

Proof. We first go through the same arguments as previously for the mean convergence and then conclude to the p^{th} mean convergence because of the boundedness of $\frac{1}{N}\nu_N\left(\frac{\alpha_N}{N}\right)$ and $\frac{1}{N}\nu_N\left(\theta, \phi\right)$. □

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