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# On 3-dimensional Asymptotically Harmonic Manifolds

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## Abstract

Let  $(M, g)$  be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. We show that  $M$  is a hyperbolic manifold of constant sectional curvature  $-\frac{h^2}{4}$ , provided  $M$  is asymptotically harmonic of constant  $h > 0$ .

## 1 Introduction

Let  $(M, g)$  be a complete, simply connected Riemannian manifold without conjugate points. Let  $SM$  be the unit tangent bundle of  $M$ . For  $v \in SM$ , let  $\gamma_v$  be the geodesic with  $\gamma_v'(0) = v$  and  $b_{v,t}(x) = \lim_{t \rightarrow \infty} (d(x, \gamma_v(t)) - t)$  the corresponding *Busemann function* for  $\gamma_v$ . The level sets  $b_v^{-1}(t)$  are called *horospheres*.

A complete, simply connected Riemannian manifold without conjugate points is called *asymptotically harmonic* if the mean curvature of its horospheres is a universal constant, that is if its Busemann functions satisfy  $\Delta b_v \equiv h$ ,  $\forall v \in SM$ , where  $h$  is a nonnegative constant. Then  $b_v$  is a smooth function on  $M$  for all  $v$  and all horospheres of  $M$  are smooth, simply connected hypersurfaces in  $M$  with constant mean curvature  $h$ .

For example, every simply connected, complete harmonic manifold without conjugate points is asymptotically harmonic.

For more details on this subject we refer to the discussion and to the references in [1]. In [1] the following result was proved:

Let  $M$  be a Hadamard manifold of dimension 3 whose sectional curvatures are bounded from above by a negative constant (i.e.  $K \leq -a^2$  for some  $a \neq 0$ ) and whose curvature tensor satisfies  $\|\nabla R\| \leq C$  for a suitable constant  $C$ . If  $M$  is asymptotically harmonic, then  $M$  is symmetric and hence of constant sectional curvature.

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We prove this result without any hypothesis on the curvature tensor.

**Theorem 1.1.** *Let  $(M, g)$  be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. If  $M$  is asymptotically harmonic of constant  $h > 0$ , then  $M$  is a manifold of constant sectional curvature  $\frac{-h^2}{4}$ .*

## 2 Proof of the Theorem

The first part of the proof (Lemma 2.1 to Lemma 2.3) is a modification of the results in [1]. Therefore we recall some notations which were already used in that paper. Our general assumption is that  $M$  is 3-dimensional, has no conjugate points and is asymptotically harmonic with constant  $h > 0$ . For  $v \in SM$  and  $x \in v^\perp$ , let

$$u^+(v)(x) = \nabla_x \nabla b_{-v} \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b_v.$$

Thus  $u^\pm(v) \in \text{End}(v^\perp)$ . With  $\lambda_1(v), \lambda_2(v)$  we denote the eigenvalues of  $u^\pm(v)$ . The endomorphism  $u^\pm$  satisfy the Riccati equation along the orbits of the geodesic flow  $\varphi^t : SM \rightarrow SM$ .

Thus if  $u^\pm(t) := u^\pm(\varphi^t v)$  and  $R(t) := R(\cdot, \gamma'_v(t))\gamma'_v(t) \in \text{End}(\gamma'_v(t)^\perp)$ , then

$$(u^\pm)' + (u^\pm)^2 + R = 0.$$

We define  $V(v) = u^+(v) - u^-(v)$  and correspondingly  $V(t) = V(\varphi^t(v))$  along  $\gamma_v(t)$ . We also define  $X(v) = \frac{-1}{2}(u^+(v) - u^-(v))$  and  $X(t) = X(\varphi^t(v))$ . Then the Riccati equation for  $u^\pm(t)$  yields

$$XV + VX = (u^-)^2 - (u^+)^2 = (u^+)' - (u^-)' = V'. \quad (1)$$

**Lemma 2.1.** *For fixed  $v \in SM$  the map  $t \mapsto \det V(\varphi^t v)$  is constant.*

*Proof.* Assume that  $V(t)$  is invertible, then

$$\frac{d}{dt} \log \det V(t) = \text{tr} V'(t)V^{-1}(t) = \text{tr} (XV + VX)V^{-1}(t) = 2 \text{tr} X = 0.$$

The last step follows as  $M$  is asymptotically harmonic. Thus as long as  $\det V(t) \neq 0$ , it is constant along  $\gamma_v$ . Therefore  $\det V(t)$  is constant along  $\gamma_v$  in any case.  $\square$

**Lemma 2.2.** *Let  $v \in SM$  be such that  $V(v) = \mu \text{Id}$ , for some  $\mu \in \mathbb{R}$ , then  $R(t) = \frac{-h^2}{4} \text{Id}$ ,  $\forall t$ .*

*Proof.* Note that if  $V(v) = \mu \text{Id}$ , then  $V(\gamma'_v(t)) = h \text{Id}$  for all  $t$ , as  $\text{tr } V \equiv 2h$  and by Lemma 2.1 the determinant of  $V$  is constant along  $\gamma_v(t)$ . Now by equation (1)  $V' = XV + VX$ . Hence, along  $\gamma_v$ ,  $V'(t) \equiv 0$ . Thus  $2hX = 0$  and since we assume  $h > 0$  we have  $X = 0$  along  $\gamma_v$ . Therefore,  $u^+(t) = -u^-(t)$ . But from the definition of  $V$ ,  $u^+(t) \equiv \frac{h}{2} \text{Id}$  i.e  $u^+$  is a scalar operator. By the Riccati equation  $(u^+(t))^2 + R(t) = 0$ , i.e.  $R(t) = \frac{-h^2}{4} \text{Id}$ .  $\square$

**Lemma 2.3.** *For every point  $p \in M$  there exists  $v \in S_p M$  such that  $R(x, v)v = \frac{-h^2}{4} x$ ,  $\forall x \in v^\perp$ . In particular,  $\text{Ric}(v, v) = \frac{-h^2}{2}$ .*

*Proof.* Since  $TS^2$  is nontrivial, an easy topological argument shows, that for every  $p \in M$  there exists  $v \in S_p M$  such that the two eigenvalues of  $V(v)$  coincide. Thus  $V(v) = \mu \text{Id}$ . The result now follows from Lemma 2.2  $\square$

**Lemma 2.4.** *For all  $v \in SM$  we have  $\text{Ric}(v, v) \leq \frac{-h^2}{2}$ .*

*Proof.* The Riccati equation for  $t \mapsto u^+(t)$  implies  $(u^+)' + (u^+)^2 + R = 0$ . Hence,  $\text{tr}(u^+)^2 + \text{tr } R = 0$ . Thus,  $\text{Ric}(v, v) = -(\lambda_1^2(v) + \lambda_2^2(v))$ . By hypothesis  $\lambda_1(v) + \lambda_2(v) = h$ , hence  $\lambda_1^2(v) + \lambda_2^2(v) \geq \frac{h^2}{2}$ . Consequently,  $\text{Ric}(v, v) \leq \frac{-h^2}{2}$ .  $\square$

**Lemma 2.5.** *The sectional curvature  $K$  of  $M$  satisfies  $K \leq -\frac{h^2}{4}$ .*

*Proof.* Let  $p \in M$ , and let  $v$  be the vector in Lemma 2.3. Take  $e_1 = v$ , and let  $e_2$  and  $e_3$  be unit vectors orthogonal to  $e_1$  so that  $\{e_1, e_2, e_3\}$  forms an orthonormal basis of  $T_p M$ . Then  $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$  forms an orthonormal basis of  $\Lambda^2 T_p M$ . We want to show that the curvature operator, considered as map  $R : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$ ,  $\langle R(X \wedge Y), V \wedge W \rangle = \langle R(X, Y)W, V \rangle$  is diagonal in this basis.

From Lemma 2.3 we see  $R(e_2, e_1)e_1 = \frac{-h^2}{4} e_2$ ,  $R(e_3, e_1)e_1 = \frac{-h^2}{4} e_3$ . Thus  $K(e_1, e_2) = K(e_1, e_3) = \frac{-h^2}{4}$  and  $K(e_2, e_3) \leq \frac{-h^2}{4}$  as  $\text{Ric}(e_3, e_3) \leq \frac{-h^2}{2}$ , where  $K(v, w)$  denotes the sectional curvature of the plane spanned by  $v$  and  $w$ . We will prove below that

$$\langle R(e_1, e_3)e_3, e_2 \rangle = 0 \quad \text{and} \quad \langle R(e_1, e_2)e_2, e_3 \rangle = 0. \quad (2)$$

Assuming this for a moment, it follows that  $R(e_1 \wedge e_3) \perp \text{span}\{e_1 \wedge e_2, e_2 \wedge e_3\}$  and  $R(e_1 \wedge e_2) \perp \text{span}\{e_1 \wedge e_3, e_2 \wedge e_3\}$ . Hence,

$$R(e_1 \wedge e_2) = \frac{-h^2}{4} e_1 \wedge e_2 \quad \text{and} \quad R(e_1 \wedge e_3) = \frac{-h^2}{4} e_1 \wedge e_3.$$

Since  $e_1 \wedge e_2$  and  $e_1 \wedge e_3$  are eigenvectors of  $R$ , also  $e_2 \wedge e_3$  is an eigenvector and we obtain

$$R(e_2 \wedge e_3) = K(e_2, e_3) e_2 \wedge e_3.$$

Thus the curvature operator is diagonal in the basis  $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$  and all eigenvalues are  $\leq \frac{-h^2}{4}$ , which proves the result.

It remains to show (2). Consider for  $t \in (-\varepsilon, \varepsilon)$  the vectors  $v_t = \cos t e_1 + \sin t e_2$ . Then,

$$\begin{aligned} f(t) &:= \text{Ric}(v_t, v_t) = K(v_t, e_3) + K(v_t, -e_1 \sin t + e_2 \cos t) \\ &= K(e_1, e_2) + \sin^2 t K(e_2, e_3) + \cos^2 t K(e_1, e_3) + \sin 2t \langle R(e_1, e_3) e_3, e_2 \rangle. \end{aligned}$$

By Lemma 2.4  $f(0) = \text{Ric}(v, v) = \frac{-h^2}{2}$  is maximal and hence  $f'(0) = 0$ . This implies the first equation in (2). If we replace  $e_2$  by  $e_3$  in the above computation we obtain the second equation. □

Finally we come to the

*Proof.* (of Theorem 1.1)

Lemma 2.5 implies that  $K_M \leq \frac{-h^2}{4}$ . By standard comparison geometry we obtain  $\lambda_1(v) \geq \frac{h}{2}$  and  $\lambda_2(v) \geq \frac{h}{2}$ . Now  $\lambda_1 + \lambda_2 = h$  implies that  $\lambda_1 = \lambda_2 = \frac{h}{2}$ . Hence,  $u^+(v)$  is a scalar operator and therefore  $R(x, v)v = \frac{-h^2}{4} x$ ,  $\forall v$  and  $\forall x \in v^\perp$ . Thus,  $K_M \equiv \frac{-h^2}{4}$ . □

### 3 Final Remark

We expect that the result holds also in the case  $h = 0$ , i.e. if all horospheres are minimal. Our argument, however, uses  $h > 0$  essentially in the proof of Lemma 2.2

### References

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