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UNIQUENESS AND COMPARISON THEOREMS FOR SOLUTIONS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH NONSTANDARD GROWTH CONDITIONS

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(To the memory of Professor I. V. Skrypnik)

Abstract. The paper addresses the Dirichlet problem for the doubly nonlinear parabolic equation with nonstandard growth conditions:

$$u_t = \operatorname{div}\left(a(x,t,u)|u|^{\alpha(x,t)}|\nabla u|^{p(x,t)-2}\nabla u\right) + f(x,t)$$

with given variable exponents $\alpha(x,t)$ and $p(x,t)$. We establish conditions on the data which guarantee the comparison principle and uniqueness of bounded weak solutions in suitable function spaces of Orlicz-Sobolev type.

1. **Introduction.** We study the Dirichlet problem for the doubly nonlinear parabolic equation

$$\begin{cases} u_t = \operatorname{div}\left(a(z,u)|u|^{\alpha(z)}|\nabla u|^{p(z)-2}\nabla u\right) + f(z) \\ \qquad\qquad \text{for } z=(x,t) \in Q = \Omega \times (0,T], \\ u(x,0) = u_0(x) \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma = \partial\Omega \times [0,T]. \end{cases} \tag{1}$$

Equation (1) is formally parabolic, but may degenerate or become singular at the points where u and/or $|\nabla u|$ vanish or become infinite. Let us introduce the functions

$$\begin{aligned} &\gamma(z) = \frac{\alpha(z)}{p(z)-1}, \quad v(z) = \int_0^u |s|^{\gamma(z)}\,ds = \frac{u|u|^{\gamma(z)}}{\gamma(z)+1}, \\ &u(z) = \Phi_0(z,v) = (1+\gamma)^{\frac{1}{1+\gamma}}|v|^{\frac{-\gamma}{1+\gamma}}v, \end{aligned} \tag{2}$$

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and rewrite problem (1) in the form:

$$\left\{\begin{array}{l}\partial_t \Phi_0(z,v) = \operatorname{div}\big(b(z,v)|\nabla v + \mathcal{B}(v)|^{p(z)-2}(\nabla v + \mathcal{B}(v)\big) + f \quad \text{in } Q,\\ v = 0 \text{ on } \Gamma,\\ v(x,0) \equiv v_0(x) = \dfrac{u_0|u_0|^{\gamma(x,0)}}{1+\gamma(x,0)} \text{ in } \Omega,\end{array}\right. \tag{3}$$

with

$$b(z,v) \equiv a(z,\Phi_0(z,v)), \qquad \mathcal{B}(v) = -\nabla\gamma \cdot \int_0^u |s|^{\gamma(z)} \ln|s|\,ds.$$

Problem (3) will be the subject of the further study. Equations of the types (1) and (3) with constant exponents α and p arise in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology - see [5, 6, 16, 17, 22, 33] and further references therein. The questions of existence and uniqueness of solutions to equations like (1) and (3) with constant exponents of nonlinearity α and p were studied by many authors - see [6, 14, 15, 16, 24, 28, 29] for equations of the type (1) and [17, 21] for the equations of the type (3) with the prescribed function $\mathcal{B} \equiv \mathcal{B}(x,t)$ independent of the solution v. Existence, uniqueness, and qualitative properties of solutions for parabolic equations with variable nonlinearity corresponding to the special cases $\alpha(x,t) = 0$, or $p(x,t) = 2$ were studied in [1, 2, 3, 4, 8, 9, 10], see also [7] for a review of results concerning elliptic equations with variable nonlinearity, and [12] for elliptic equations with triple variable nonlinearity. The Cauchy problem for doubly nonlinear parabolic equations with constant exponents of nonlinearity is studied in [30, 31, 32].

In the present work we prove comparison principle and uniqueness of weak solutions for the Dirichlet problem (3) in which the exponents α and p are allowed to be variable.

The paper is organized as follows. In Section 2 we prove several auxiliary assertions and collect some known facts from the theory of Orlicz–Sobolev spaces. The precise assumptions on the data and main results are given in Section 3. Besides, in this section we recall the known existence theorem for problem (3) published in [11]. In Section 4 we derive formulas of integration by parts for the elements of the main function spaces used throughout the paper. In Sections 5, 6 we give the proofs of the main comparison theorems. The comparison principle and uniqueness are proved for the solutions subject to some additional restrictions, but under weaker assumptions on the data, and are independent of the proof of the existence theorem. To be precise, the comparison principle and uniqueness are true for the weak solutions with $\partial_t \Phi_0(z,v) \in L^1(Q)$. In order to ensure that this class of solutions is nonempty, in the final Section 7 we give a sketch of the proof of the existence theorem from [11], formulated in Section 3, and show that the already constructed solution belongs to the class of uniqueness, provided that the data of the problem satisfy some additional conditions.

2. The function spaces.

2.1. **Spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$.** The definitions of the function spaces used throughout the paper and a brief description of their properties follow [18, 19, 23, 25]. Further references can be found in the review papers [20, 26]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ be Lipschitz-continuous, and let $p(x)$ be log-continuous in Ω: $\forall\, x, y \in \Omega$ such that $|x-y| < \frac{1}{2}$

$$|p(x)-p(y)| \le \omega(|x-y|) \quad \text{with} \quad \overline{\lim}_{\tau\to 0+}\left(\omega(\tau)\ln\frac{1}{\tau}\right) = C < \infty. \tag{4}$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)}\,dx < \infty.$$

The set $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}\left(f/\lambda\right) \le 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,\,p(\cdot)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,\,p(\cdot)}(\Omega) = \left\{f \in L^{p(\cdot)}(\Omega) :\ |\nabla f|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \displaystyle\sum_i \|D_i u\|_{p(\cdot),\Omega} + \|u\|_{p(\cdot),\Omega}. \end{cases} \tag{5}$$

Throughout the paper we use the following properties of the functions from the spaces $W_0^{1,p(\cdot)}(\Omega)$:

- if condition (4) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$, and the space $W_0^{1,\,p(\cdot)}(\Omega)$ can be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5) – see [27, 34, 35, 36];
- if $p(x) \in C^0(\overline{\Omega})$, the the space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive;
- if $1 < q(x) \le \sup\limits_{\Omega} q(x) < \inf\limits_{\Omega} p_*(x)$ with

$$p_*(x) = \begin{cases} \dfrac{p(x)n}{n-p(x)} & \text{if } p(x) < n, \\ \infty & \text{if } p(x) > n, \end{cases} \tag{6}$$

then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact;

- it follows directly from the definition that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \le A_{p(\cdot)}(f) \le \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right); \tag{7}$$

- for all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with $p(x) \in (1,\infty)$, $\quad p' = \frac{p}{p-1}$ Hölder's inequality holds,

$$\int\limits_{\Omega} |f\,g|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)} \le 2\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)}\,. \tag{8}$$

2.2. **Parabolic spaces $L^{p(\cdot,\cdot)}(Q)$ and $\mathbf{W}(Q)$.** Let $p(z)$, $z = (x,t) \in Q$, satisfy condition (4) in the cylinder Q. For every fixed $t \in [0,T]$ we introduce the Banach space

$$\mathbf{V}_t(\Omega) = \left\{u(x) :\ u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), \quad |\nabla u(x)|^{p(x,t)} \in L^1(\Omega)\right\},$$
$$\|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(\cdot,t),\Omega},$$

and denote by $\mathbf{V}_t'(\Omega)$ its dual. By $\mathbf{W}(Q)$ we denote the Banach space

$$\left\{\begin{array}{l} \mathbf{W}(Q)=\left\{u:\,[0,T]\mapsto \mathbf{V}_t(\Omega)|\; u\in L^2(Q),\, |\nabla u|^{p(z)}\in L^1(Q)\right\},\\ \|u\|_{\mathbf{W}(Q)}=\|\nabla u\|_{p(\cdot),Q}+\|u\|_{2,Q}. \end{array}\right. \tag{9}$$

$\mathbf{W}'(Q)$ is the dual of $\mathbf{W}(Q)$ (the space of linear functionals over $\mathbf{W}(Q)$):

$$w\in \mathbf{W}'(Q) \iff \left\{\begin{array}{l} w=(w_0,w_1,\dots,w_n), \quad w_0\in L^2(Q), \quad w_i\in L^{p'(\cdot)}(Q),\\ \forall \phi\in \mathbf{W}(Q) \quad \langle\langle w,\phi\rangle\rangle=\displaystyle\int_{Q_T}\left(w_0\phi+\sum_i w_iD_i\phi\right)dz. \end{array}\right.$$

The norm in $\mathbf{W}'(Q)$ is defined by

$$\|v\|_{\mathbf{W}'(Q)}=\sup\left\{\langle\langle v,\phi\rangle\rangle|\,\phi\in\mathbf{W}(Q),\,\|\phi\|_{\mathbf{W}(Q)}\le 1\right\}.$$

Set

$$\mathbf{V}_+(\Omega)=\left\{u(x)|\;u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\;|\nabla u|\in L^{p^+}(\Omega)\right\}.$$

Since $\mathbf{V}_+(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\psi_k(x)\}\subset \mathbf{V}_+(\Omega)$.

We will need two elementary inequalities.

Proposition 1 ([16])**.** *For every $p\ge 2$, $|a|\ge|b|\ge 0$*

$$||a|^{p-2}a-|b|^{p-2}b|\le C(p)|a-b|(|a|+|b|)^{p-2}.$$

This proposition is an immediate byproduct of the easily verified relation

$$1-t^{p-1}\le C(p)(1-t)(1+t)^{p-2} \quad \forall\, p\ge 2, \quad t\in[0,1].$$

Proposition 2 ([16])**.** *For $2-p<\beta<1$ and $|a|\ge|b|\ge 0$*

$$||a|^{p-2}a-|b|^{p-2}b|\le C(p)|a-b|^{1-\beta}(|a|+|b|)^{p-2+\beta}.$$

The assertion follows from the inequality

$$1-t^{p-1}\le C(p)(1-t)^{1-\beta}(1+t)^{p-2+\beta}, \quad t\in[0,1]$$

with the same p and β.

3. Assumptions and results.

The existence result is established for the problem

$$\left\{\begin{array}{l} \partial_t\Phi_0(z,v)=\operatorname{div}\left(b(z,v)|\nabla v+\mathcal{B}(v)|^{p(z)-2}(\nabla v+\mathcal{B}(v)\right)+f \quad \text{in } Q,\\ v(x,0) \text{ in } \Omega, \quad v=0 \text{ on } \Gamma \end{array}\right. \tag{10}$$

with b, Φ_0, $\mathcal{B}$ defined in (2), (3). Problem (10) is formally equivalent to problem (1). Throughout the paper we assume that the coefficient $a(z,r)$ and the exponents on nonlinearity $p(z)$, $\alpha(z)$ satisfy the following conditions:

- $a(z,r)$ is a Carathéodory function such that there exist constants $a^{\pm}$ such that
$$\forall\, z\in Q,\; r\in\mathbb{R} \qquad a^-\le a(z,r)\le a^+<\infty, \tag{11}$$
- $\alpha(z)$, $p(z)$ are measurable and bounded in Q, there exist constants $\alpha^{\pm}$, $p^{\pm}$ such that
$$-1<\alpha^-\le\alpha(z)\le\alpha^+<\infty, \quad 1<p^-\le p(z)\le p^+<\infty, \quad \alpha^-+p^->1, \tag{12}$$
- the exponent $\gamma(z)=\frac{\alpha(z)}{p(z)-1}$ satisfies
$$|\nabla\gamma(z)|^{p(z)}\in L^1(Q), \quad \partial_t\gamma(z)\in L^2(Q). \tag{13}$$

The solution of problem (10) is understood in the following sense.

Definition 3.1. A function $v(z)$ is called weak solution of problem (10) if

1. $v \in \mathbf{W}(Q) \cap L^\infty(Q)$, $\partial_t \Phi_0(z,v) \in \mathbf{W}'(Q)$,
2. for every $\phi \in \mathbf{W}(Q)$

$$\int_Q \left(\phi\, \partial_t \Phi_0(z,v) + b(z,v)|\nabla v + \mathcal{B}(v)|^{p-2}(\nabla v + \mathcal{B}(v)) \cdot \nabla \phi - f\, \phi\right) dz = 0, \tag{14}$$

3. $\forall\, \phi(x) \in C_0^\infty(\Omega)$

$$\int_\Omega \Phi_0(z, v(z))\, \phi(x)\, dx \to \int_\Omega \Phi_0((x,0), v_0(x))\, \phi(x)\, dx \quad \text{as } t \to 0.$$

The main existence result is given in the following theorem.

Theorem 3.2 ([11])**.** *Let conditions (11), (12), (13), (4) be fulfilled. Then for every $f \in L^1(0,T; L^\infty(\Omega))$, u_0, $v_0 \in L^\infty(\Omega)$ problem (10) has at least one weak solution $v(z)$ in the sense of Definition 3.1.*

The uniqueness result is proved for the solutions satisfying the additional restriction, not included into Definition 3.1: it is required that $\partial_t \Phi_0(z, v(z)) \in L^1(Q)$. In Section 7 we review the proof of Theorem 3.2 given in [11] and show that the class of uniqueness is nonempty, provided that the problem data possess some additional regularity. Another restriction is that either $a(z,v) \equiv 1$, or $\alpha(z) \equiv 0$. In the latter case $\Phi_0(z,v) \equiv v$ and the equation transforms into the evolutional $p(z)$-Laplacian equation.

Theorem 3.3. *Let us assume that the data of problem (10) satisfy the conditions*

$$a(z,u) \equiv 1, \quad \Phi_0(z,s) \in C^1(Q \times \mathbb{R}).$$

Let conditions (12), (13) be fulfilled. Then for every weak solutions v_1, v_2, such that $\partial_t \Phi_0(z, v_i) \in L^1(Q)$, and $t \in (0,T)$

$$\begin{aligned} &\|\Phi_0(z, v_1(z)) - \Phi_0(z, v_2(z))\|_{L^1(\Omega)} \\ &\qquad\qquad \leq \|\Phi_0(x,0,v_{01}) - \Phi_0(x,0,v_{02})\|_{L^1(\Omega)} + \|f_1 - f_2\|_{L^1(Q)}. \end{aligned}$$

Theorem 3.4. *Let v_1, v_2 be two weak solutions of problem (10) with $\alpha(z) \equiv 0$. Let the coefficient $a(z,s)$ be Hölder-continuous with respect to s,*

$$|a(z,s) - a(z,r)| \leq C\,|s-r|^\beta, \quad C = const, \quad \beta \in [1/2, 1].$$

If conditions conditions (11), (12) are fulfilled and $\partial_t u_i \in L^1(Q)$, then for every $t \in (0,T)$

$$\|v_1(x,t) - v_2(x,t)\|_{L^1(\Omega)} \leq \|v_{01} - v_{02}\|_{L^1(\Omega)} + \|f_1 - f_2\|_{L^1(Q)}.$$

The uniqueness is proved in a narrower class of functions than the existence, but since the proofs of Theorems 3.3, 3.4 are practically independent on the proof of Theorem 3.2, the conditions on the exponents $\alpha(z)$, $p(z)$ are less restrictive. For the sake of completeness of presentation, in the end of the paper we present the conditions on the data of problem (10) which guarantee that the corresponding solution satisfy the conditions of the comparison and uniqueness theorems.

4. **Formulas of integration by parts.** Let ρ be the Friedrich's mollifying kernel

$$\rho(s)=\begin{cases}\kappa\,\exp\left(-\frac{1}{1-|s|^2}\right) & \text{if } |s|<1,\\ 0 & \text{if } |s|>1,\end{cases}\qquad \kappa=const:\quad \int_{\mathbb{R}^{n+1}}\rho(z)\,dz=1.$$

Given a function $v\in L^1(Q_T)$, we extend it to the whole $\mathbb{R}^{n+1}$ by a function with compact support (keeping the same notation for the continued function) and then define

$$v_h(z)=\int_{\mathbb{R}^{n+1}}v(s)\rho_h(z-s)\,ds\qquad \text{with } \rho_h(s)=\frac{1}{h^{n+1}}\rho\left(\frac{s}{h}\right),\quad h>0.$$

Lemma 4.1. *If $u\in\mathbf{W}(Q_T)$ with the exponent $p(z)$ satisfying (4) in Q, then*

$$\|u_h\|_{\mathbf{W}(Q)}\le C\,\left(1+\|u\|_{\mathbf{W}(Q)}\right)\ \textit{ and }\ \|u_h-u\|_{\mathbf{W}(Q)}\to 0\ \textit{ as } h\to 0.$$

Lemma 4.1 is an immediate byproduct of [36, Theorem 2.1].

Lemma 4.2 ([10])**.** *Let in the conditions of Proposition 4.1 $u_t\in\mathbf{W}'(Q)$. Then $(u_h)_t\in\mathbf{W}'(Q)$, and for every $\psi\in\mathbf{W}(Q)$ $\langle\langle(u_h)_t,\psi\rangle\rangle\to\langle\langle u_t,\psi\rangle\rangle$ as $h\to 0$.*

Lemma 4.3 (Integration by parts)**.** *Let $v,w\in\mathbf{W}(Q)$ and $v_t,w_t\in\mathbf{W}'(Q)$ with the exponent $p(z)$ satisfying (4) in Q. Then*

$$\forall\,\text{a.e. } t_1,t_2\in(0,T]\qquad \int_{t_1}^{t_2}\int_\Omega v\,w_t\,dz+\int_{t_1}^{t_2}\int_\Omega v_t\,w\,dz=\int_\Omega v\,w\,dx\Big|_{t=t_1}^{t=t_2}.$$

Proof. Let $t_1<t_2$. Take

$$\chi_k(t)=\begin{cases}0 & \text{for } t\le t_1,\\ k(t-t_1) & \text{for } t_1\le t\le t_1+\frac{1}{k},\\ 1 & \text{for } t_1+\frac{1}{k}\le t\le t_2-\frac{1}{k},\\ k(t_2-t) & \text{for } t_2-\frac{1}{k}\le t\le t_2,\\ 0 & \text{for } t\ge t_2.\end{cases}\tag{15}$$

For every $k\in\mathbb{N}$ and $h>0$

$$0=\int_Q(v_hw_h\chi_k)_t\,dz\equiv\int_Q(v_hw_h)_t\chi_k\,dz-k\int_{\theta-\frac{1}{k}}^{\theta}\int_\Omega v_hw_h\,dz\Big|_{\theta=t_1+\frac{1}{k}}^{\theta=t_2}.$$

The last two integrals on the right-hand side exist because $v_h,w_h\in L^2(Q)$. Letting $h\to 0$, we obtain the equality

$$\lim_{h\to 0}\int_Q\left(v_h\,(w_h)_t+(v_h)_tw_h\right)\chi_k(t)\,dz=k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega v\,w\,dz-k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega v\,w\,dz.$$

According to Lemmas 4.1, 4.2 $v_h\to v$ in $\mathbf{W}(Q)$, $(w_h)_t=(w_t)_h\to w_t$ weakly in $\mathbf{W}'(Q)$ as $h\to 0$, and $\|v\|_{\mathbf{W}}$, $\|(w_h)_t\|_{\mathbf{W}'}$ are uniformly bounded. It follows that

$$\begin{aligned}\lim_{h\to 0}\int_Q v_h\,(w_h)_t\,\chi_k(t)\,dz&=\lim_{h\to 0}\int_Q(v_h-v)(w_h)_t\chi_k(t)\,dz\\&+\lim_{h\to 0}\int_Q v\,((w_h)_t-w_t)\chi_k(t)\,dz+\int_Q v\,w_t\,\chi_k(t)\,dz=\int_Q v\,w_t\chi_k(t)\,dz.\end{aligned}$$

In the same way we check that

$$\lim_{h\to 0}\int_Q(v_h)_t\,w_h\,\chi_k(t)\,dz=\int_Q v\,w_t\chi_k(t)\,dz.$$

By the Lebesgue differentiation theorem

$$\forall\,\text{a.e.}\,\theta>0 \qquad \lim_{k\to 0} k\int_{\theta-\frac{1}{k}}^{\theta}\left(\int_\Omega v\,w\,dx\right)dt=\int_\Omega v\,w\,dx,$$

whence for almost every $t_1,\,t_2\in[0,T]$

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega (v\,w_t+v_t w)\,dz&=\lim_{k\to\infty}\int_Q (v\,w_t+v_t w)\chi_k(t)\,dz\\&=\lim_{k\to\infty} k\int_{\theta-\frac{1}{k}}^{\theta}\int_\Omega v\,w\,dx\Big|_{\theta=t_1}^{t=t_2}=\int_\Omega v\,w\,dx\Big|_{\theta=t_1}^{t=t_2}.\end{aligned}$$

□

Corollary 1. *Let $u\in\mathbf{W}(Q)$ and $u_t\in\mathbf{W}'(Q)$ with the exponent $p(z)$ satisfying (4). Then*

$$\forall\,\text{a.e.}\,t_1,t_2\in(0,T]\qquad \int_{t_1}^{t_2}\int_\Omega u\,u_t\,dz=\frac{1}{2}\|u\|^2_{2,\Omega}\Big|_{t=t_1}^{t=t_2}.$$

Lemma 4.4. *Let $u\in\mathbf{W}(Q)\cap L^\infty(Q)$, $u_t\in\mathbf{W}'(Q)$, and let the exponent $p(z)$ satisfy (4). Introduce the function*

$$v=\int_0^u(\epsilon+|s|)^{\gamma(z)}\,ds,\qquad \epsilon>0,$$

with the exponent $\gamma(z)\ge\gamma^->-1$ such that $\gamma_t\in L^2(Q)$ and $|\nabla\gamma(z)|^{p(z)}\in L^1(Q)$. For a.e. $t_1,\,t_2\in[0,T]$

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega u_t\,v\,dz&=\int_\Omega\frac{u\,v}{\gamma+2}\,dx\Big|_{t=t_1}^{t=t_2}+\int_{t_1}^{t_2}\int_\Omega\frac{u\,v}{\gamma+2}\,\gamma_t\,dz+\epsilon\int_\Omega\frac{v}{\gamma+2}\,dx\Big|_{t=t_1}^{t=t_2}\\&\quad+\epsilon\int_{t_1}^{t_2}\int_\Omega\frac{\gamma_t}{\gamma+2}\int_0^u(\epsilon+|s|)^\gamma\,\ln(\epsilon+|s|)\,ds\,dz\\&\quad-\epsilon\int_{t_1}^{t_2}\int_\Omega v\,\frac{\gamma_t}{(\gamma+2)^2}\,dz\equiv\mu_\epsilon(u,v).\end{aligned}\tag{16}$$

Proof. Let $u_h\in C^\infty(Q)$ be the mollification of $u\in\mathbf{W}(Q)$ and

$$v_h=\int_0^{u_h}(\epsilon+|s|)^{\gamma(z)}\,ds\equiv\frac{\text{sign}\;u_h}{\gamma+1}\left((\epsilon+|u_h|)^{\gamma+1}-\epsilon^{\gamma+1}\right).$$

Since u and u_h are bounded by a constant $1+K_0$, and $\gamma(z)\ge\gamma^->-1$, it follows from Propositions 1, 2 that

$$|v_h-v|\le C\,\max\left\{|v_h-v|,\,|v_h-v|^{1+\min\{0,\,\gamma^-\}}\right\},\quad C\equiv C(\epsilon,\,p^\pm,\,\alpha^\pm,\,K_0).$$

The inclusion $u\in L^\infty(Q)$ entails the convergence $\|v_h-v\|_{L^r(Q)}\to 0$ as $h\to 0$ for every $r>1$. Explicitly calculating the primitive, in the same way we check that for every $r>1$

$$\left\|\int_u^{u_h}(\epsilon+|s|)^{\gamma(z)}\,\ln(\epsilon+|s|)\,ds\right\|_{L^r(Q)}\to 0\quad\text{as } h\to 0.$$

Let $\psi_k(z) = \frac{\chi_k(t)}{\gamma+2}$ with the function χ_k introduced in (15). Following the proof of Lemma 4.3, we find:

$$
\begin{aligned}
k \int_{\theta-\frac{1}{k}}^{\theta} dt \int_\Omega \frac{u_h v_h}{\gamma+2}\, dx\Big|_{\theta=t_1}^{\theta=t_2} = & \int_{t_1}^{t_2}\int_\Omega \chi_k(t)(u_h)_t\, v_h\, dz \\
& - \int_{t_1}^{t_2}\int_\Omega \frac{u_h\, v_h}{\gamma+2}\,\gamma_t\,\chi_k(t)\, dz \\
& - \epsilon \int_{t_1}^{t_2}\int_\Omega \chi_k(t)\,\frac{\gamma_t}{\gamma+2}\int_0^{u_h}(\epsilon+|s|)^\gamma\,\ln(\epsilon+|s|)\, ds\, dz \\
& + \epsilon \int_{t_1}^{t_2}\int_\Omega \chi_k(t)\, v_h\,\frac{\gamma_t}{(\gamma+2)^2}\, dz + \epsilon\, k \int_{\theta-\frac{1}{k}}^{\theta} dt \int_\Omega \frac{v_h}{\gamma+2}\, dx\Big|_{\theta=t_1}^{\theta=t_2}.
\end{aligned}
\tag{17}
$$

Since $u \in \mathbf{W}(Q)\cap L^\infty(Q)$ and $\gamma^- > -1$, $v \in \mathbf{W}(Q)$ for every $\epsilon > 0$. Indeed: since $\|u\|_{L^\infty(Q)} \le M$, we have the estimates

$$
\|v\|_{L^\infty(Q)} \le M_1(\gamma^\pm,\, M), \qquad \left\|\int_0^{|u|}(\epsilon+s)^\gamma\,|\ln(\epsilon+s)|\, ds\right\|_{L^\infty(Q)} \le M_2(\gamma^\pm,\, M),
$$

which provide the inequality

$$
|\nabla v| \le (\epsilon+|u|)^{\gamma(z)}|\nabla u| + |\nabla\gamma| \int_0^{|u|}(\epsilon+s)^{\gamma(z)}|\ln(\epsilon+s)|\, ds \quad \text{a.e. in } Q
$$

and the inclusion $|\nabla v(z)|^{p(z)} \in L^1(Q)$. By Lemma 4.1

$$
\|v_h\|_{\mathbf{W}(Q)} \le C\,(1+\|v\|_{\mathbf{W}(Q)}) \quad \text{and} \quad \|v_h - v\|_{\mathbf{W}(Q)} \to 0 \quad \text{as } h \to 0.
$$

We may now pass to the limit as $h \to 0$ in every term of (17), following the proof of Lemma 4.3:

$$
\begin{aligned}
k \int_{\theta-\frac{1}{k}}^{\theta} dt \int_\Omega \frac{u\, v}{\gamma+2}\, dx\Big|_{\theta=t_1}^{\theta=t_2} = & \int_{t_1}^{t_2}\int_\Omega \chi_k(t) u_t\, v\, dz - \int_{t_1}^{t_2}\int_\Omega \frac{u\, v}{\gamma+2}\,\gamma_t\,\chi_k(t)\, dz \\
& - \epsilon \int_{t_1}^{t_2}\int_\Omega \chi_k(t)\,\frac{\gamma_t}{\gamma+2}\int_0^{u}(\epsilon+|s|)^\gamma\,\ln(\epsilon+|s|)\, ds\, dz \\
& + \epsilon \int_{t_1}^{t_2}\int_\Omega \chi_k(t)\, v\,\frac{\gamma_t}{(\gamma+2)^2}\, dz + \epsilon\, k \int_{\theta-\frac{1}{k}}^{\theta} dt \int_\Omega \frac{v}{\gamma+2}\, dx\Big|_{\theta=t_1}^{\theta=t_2}.
\end{aligned}
$$

Letting $k \to \infty$ and applying the Lebesgue differentiation theorem, we arrive at (16). $\square$

Remark 1. Let $\epsilon = 0$, $u \in \mathbf{W}(Q)$, $u_t \in \mathbf{W}'(Q)$, and let $v = \frac{u\,|u|^\gamma}{\gamma+1} \in \mathbf{W}(Q)$. Under the foregoing conditions on the exponents $p(z)$ and $\gamma(z)$ the following formula of integration by parts holds: $\forall$ a.e. $t_1,\, t_2 \in [0,T]$

$$
\int_{t_1}^{t_2}\int_\Omega u_t\, v\, dz = \int_\Omega \frac{u\, v}{\gamma+2}\, dx\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega \frac{u\, v}{\gamma+2}\,\gamma_t\, dz \equiv \mu(u,v).
$$

Let us introduce the function space

$$
\mathcal{V}(Q) \equiv \left\{v(z) :\, v \in \mathbf{W}(Q)\cap L^\infty(Q),\, \partial_t\Phi_0(z,v) \in L^1(Q)\cap \mathbf{W}'(Q)\right\}.
$$

with Φ_0 defined in (2) and define the functions

$$
T_\delta(s) = \frac{s}{\sqrt{\delta^2+s^2}}, \quad \delta > 0,
$$

and

$$\phi_{k,\delta,\theta}(z) = \chi_{k,\theta}(t)\, T_\delta(v(z)) \tag{18}$$

with

$$\chi_{k,\theta}(t) = \begin{cases} 0 & \text{for } t \le 0, \\ k\,t & \text{for } 0 \le t \le \frac{1}{k}, \\ 1 & \text{for } \frac{1}{k} \le t \le \theta - \frac{1}{k}, \\ k\,(\theta - t) & \text{for } \theta - \frac{1}{k} \le t \le \theta, \\ 0 & \text{for } t \ge \theta, \end{cases} \qquad \frac{1}{k} < \theta \le T.$$

It is easy to see that

$$T_\delta(s) \to \text{sign }\ s \text{ as } \delta \to 0, \quad T'_\delta(s) = \frac{\delta^2}{(\delta^2 + s^2)^{\frac{3}{2}}} > 0, \quad -1 \le s T'_\delta(s) \le 1 \text{ for } s \in \mathbb{R}.$$

Lemma 4.5. *Let $v_i \in \mathcal{V}(Q)$, $v = v_1 - v_2$ and $w = w_1 - w_2 \equiv \Phi_0(z, v_1) - \Phi_0(z, v_2)$. For a.e. $\theta \in (0, T)$ there exists the limit*

$$\lim_{\delta \to 0} \lim_{k \to \infty} \int_Q \phi_{k,\delta,\theta}\, \partial_t w\, dz = \int_\Omega |w|\, dx \Big|_{t=0}^{t=\theta}.$$

Proof. From now on, we will denote

$$Q_\tau = Q \cap \{t < \tau\}, \qquad \tau \in [0, T).$$

Since $w \in L^\infty(Q)$ and $\phi = \phi_{k,\delta,\theta}$ are uniformly bounded, it follows from the dominated convergence theorem that

$$\int_Q \chi_{k,\theta}(t)\, T_\delta(v)\, \partial_t w\, dz \to \int_{Q_\theta} T_\delta(v)\, \partial_t w\, dz \quad \text{as } k \to \infty,\ Q_\theta = Q \cap \{t < \theta\},$$

and, because sign $v =$ sign w,

$$\begin{aligned} \lim_{k \to \infty} \int_Q \phi_{k,\delta,\theta}\, \partial_t w\, dz &= \int_Q T_\delta(v)\, \partial_t w\, dz \\ &\to \int_{Q_\theta} \text{sign }\ v\, \partial_t w\, dz \equiv \int_{Q_\theta} \text{sign }\ w\, \partial_t w\, dz = J \quad \text{as } \delta \to 0. \end{aligned}$$

On the other hand, repeating the same arguments with the test-function $\phi_{k,\delta,\theta} \equiv \chi_{k,\theta}(t)\, T_\delta(w)$, we find that

$$J = \lim_{\delta \to 0} \lim_{k \to \infty} \int_Q T_\delta(w)\, \chi_{k,\theta}(t)\, \partial_t w\, dz.$$

The straightforward computation shows that

$$\begin{aligned} \int_Q T_\delta(w)\, \chi_{k,\theta}(t)\, \partial_t w\, dz &= \int_Q \chi_{k,\theta}(t)\, \partial_t \left(\int_0^w T_\delta(s)\, ds \right) dz \\ &= k \int_{\theta - 1/k}^{\theta} dt \int_\Omega \left(\int_0^w T_\delta(s)\, ds \right) dx \\ &\quad - k \int_0^{1/k} dt \int_\Omega \left(\int_0^w T_\delta(s)\, ds \right) dx, \end{aligned}$$

where

$$\int_0^w T_\delta(s)\, ds = \sqrt{\delta^2 + w^2} - \delta \to \sqrt{w^2} = |w| \quad \text{as } \delta \to 0.$$

Letting $k \to \infty$, $\delta \to 0$ and applying the Lebesgue differentiation theorem, we find that for a.e. $\theta \in (0,T)$

$$\int_Q T_\delta(w)\,\chi_{k,\theta}(t)\,\partial_t w\,dz = \int_\Omega \left(\int_0^w T_\delta(s)\,ds\right)\,dx\Big|_{t=0}^{t=\theta} = \int_\Omega \sqrt{\delta^2+w^2}\,dx\Big|_{t=0}^{t=\theta}$$
$$\to J = \int_\Omega |w|\,dx\Big|_{t=0}^{t=\theta}.$$

□

5. **Proof of Theorem 3.3.** Let $v_i \in \mathcal{V}(Q)$ be two bounded weak solutions of problem (3) with the data (f_i, v_{0i}), $i = 1,2$. Introduce the functions

$$w = \Phi_0(z,v_1) - \Phi_0(z,v_2), \quad v = v_1 - v_2, \quad F(s) = |\mathcal{B}(v) + \nabla v|^{p(z)-2}\,(\mathcal{B}(v) + \nabla v)\,.$$

By (14) for every test-function $\phi \in \mathbf{W}(Q)$

$$\int_Q \Big(\phi\,\partial_t w + (F(v_1) - F(v_2)) \cdot \nabla \phi\Big)\,dz = \int_Q (f_1 - f_2)\,\phi\,dz. \tag{19}$$

Taking for the test-function $\phi_{k,\delta,\theta}$ defined in (18) and applying Lemma 4.5 we have that for a.e. $\theta \in (0,T)$ there exists the limit of the first term on the left-hand side of (19):

$$\int_Q \phi_{k,\delta,\theta}\partial_t w\,dz \to \int_\Omega |w|\,dx\Big|_{t=0}^{t=\theta} \qquad \text{as } k \to \infty,\ \delta \to 0. \tag{20}$$

On the other hand, the rest of the terms in (19) are continuous functions of θ because of the property of absolute continuity of the integral. It follows that (20) is true for all $t \in [0,T]$. The second term on the left-hand side of (19) with $\phi(z) = \chi_{k,\theta}(t)\,T_\delta(v(z))$ is represented in the form

$$\begin{aligned} I_2 &= \int_Q (F(v_1) - F(v_2)) \cdot \nabla\phi\,dz = \int_Q \chi_{k,\theta}(F(v_1) - F(v_2))\,\nabla T_\delta(v)\,dz \\ &= \int_Q \chi_{k,\theta}\,T'_\delta(v)\,(F(v_1) - F(v_2))\,\nabla v\,dz. \end{aligned} \tag{21}$$

Let us denote

$$\zeta_i = \nabla v_i + \mathcal{B}(v_i), \quad i = 1,2,$$

so that

$$\nabla v_i = \zeta_i - \mathcal{B}(v_i), \qquad F(v_i) = |\zeta_i|^{p(z)-2}\zeta_i, \qquad \zeta_i = |F(v_i)|^{p'(z)-2}F(v_i)$$

(recall that $\mathcal{B}(s)$ is defined in (3)). Passing to the limit as $k \to \infty$, for every fixed δ and θ we obtain the equality

$$\begin{aligned} \lim_{k\to\infty} I_2 &= \int_{Q_\theta} T'_\delta(v)\Big(F(v_1) - F(v_2)\Big) \cdot \nabla v\,dz \\ &= \int_{Q_\theta} T'_\delta(v)\Big(F(v_1) - F(v_2)\Big)\,(\zeta_1 - \zeta_2)\,dz \\ &\quad - \int_{Q_\theta} T'_\delta(v)\Big(F(v_1) - F(v_2)\Big)\,(\mathcal{B}(v_1) - \mathcal{B}(v_2))\,dz \\ &\equiv J_1(\delta) - J_2(\delta). \end{aligned}$$

Making use of the well-known inequality

$$\forall\,\xi,\eta\in\mathbb{R}^n \quad \left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\geq\begin{cases}2^{-p}|\xi-\eta|^p & \text{if } 2\leq p<\infty,\\ (p-1)\dfrac{|\xi-\eta|^2}{\left(|\xi|^p+|\eta|^p\right)^{\frac{2-p}{p}}} & \text{if } 1<p<2,\end{cases} \tag{22}$$

we may write

$$\begin{aligned}J_1(\delta)&=\int_{Q_\theta}T'_\delta(v)\left(|\zeta_1|^{p(z)-2}\zeta_1-|\zeta_2|^{p(z)-2}\zeta_2\right)(\zeta_1-\zeta_2)\,dz\\&\geq 2^{-(p^-)'}\int\limits_{Q_\theta\cap\{p(z)\geq 2\}}T'_\delta(v)\,|F(v_1)-F(v_2)|^{p(z)}\,dz\\&\quad+(p^--1)\int\limits_{Q_\theta\cap\{p(z)\in(1,2)\}}T'_\delta(v)\,|F(v_1)-F(v_2)|^2\\&\qquad\times\left(|F(v_1)|^{p(z)}+|F(v_2)|^{p(z)}\right)^{\frac{p(z)-2}{p(z)}}\,dz.\end{aligned}$$

Next,

$$\begin{aligned}J_2(\delta)&\leq\int_{Q_\theta\cap\{z:\,p(z)\geq 2\}}T'_\delta(v)\,|F(v_1)-F(v_2)|\,|\mathcal{B}(v_1)-\mathcal{B}(v_2)|\,dz\\&\quad+\int_{Q_\theta\cap\{z:\,1<p(z)<2\}}\dots\equiv\mathcal{J}^{(1)}(\delta)+\mathcal{J}^{(2)}(\delta).\end{aligned}$$

To estimate $\mathcal{J}^{(1)}(\delta)$ we make use of the following elementary lemma.

Lemma 5.1. *For every $p(z)\in[p^-,p^+]\subset(1,\infty)$ and $\epsilon\in(0,1)$*

$$a\,b\leq\epsilon\,a^{p'(z)}+\frac{\epsilon^{1-p^+}}{p^-}\left(\frac{p^+}{p^+-1}\right)^{1-p^-}b^{p(z)}\qquad\forall\,a,\,b\geq 0.$$

Proof. The assertion follows from Young's inequality

$$\begin{aligned}a\,b&=(\epsilon\,p'(z))^{\frac{1}{p'(z)}}\,a\,(\epsilon\,p'(z))^{-\frac{1}{p'(z)}}\,b\\&\leq\frac{1}{p'(z)}\left((\epsilon\,p'(z))^{\frac{1}{p'(z)}}\,a\right)^{p'(z)}+\frac{1}{p(z)}\left((\epsilon\,p'(z))^{-\frac{1}{p'(z)}}\,b\right)^{p(z)}\end{aligned}$$

and the inequalities

$$\begin{aligned}&\epsilon^{-\frac{p(z)}{p'(z)}}=\epsilon^{-(p(z)-1)}=\mathrm{e}^{-(p(z)-1)\,\ln\epsilon}\leq\mathrm{e}^{-(p^+-1)\,\ln\epsilon}=\epsilon^{1-p^+},\\&(p'(z))^{-\frac{p(z)}{p'(z)}}=\left(1-\frac{1}{p(z)}\right)^{p(z)-1}=\mathrm{e}^{(p(z)-1)\ln\left(1-\frac{1}{p(z)}\right)}\\&\qquad\leq\mathrm{e}^{(p^--1)\ln\left(1-\frac{1}{p^+}\right)}=\left(\frac{p^+}{p^+-1}\right)^{1-p^-}.\end{aligned}$$

$\square$

Applying Lemma 5.1 we have:

$$\begin{aligned}\mathcal{J}^{(1)}(\delta) \equiv & \int\limits_{Q_\theta\cap\{p(z)\geq 2\}} (T'_\delta(v))^{\frac{1}{p(z)}}\,|F(v_1)-F(v_2)|\,(T'_\delta(v))^{\frac{1}{p(z)}}\,|\mathcal{B}(v_1)-\mathcal{B}(v_2)|\,dz \\ & \leq \epsilon \int\limits_{Q_\theta\cap\{p(z)\geq 2\}} T'_\delta(v)\,|F(v_1)-F(v_2)|^{p(z)}\,dz \\ & \qquad + C(\epsilon,p^-,p^+) \int\limits_{Q_\theta\cap\{p'(z)\geq 2\}} T'_\delta(v)\,|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^{p(z)}\,dz \end{aligned} \tag{23}$$

By Young's inequality

$$\begin{aligned}\mathcal{J}^{(2)}(\delta) \equiv & \int\limits_{Q_\theta\cap\{p(z)<2\}} \left(\sqrt{T'_\delta(v)}|F(v_1)-F(v_2)|\,(|F(v_1)|^p+|F(v_2)|^p)^{\frac{p-2}{2p}}\right) \\ & \times\left(\sqrt{T'_\delta(v)}|\mathcal{B}(v_1)-\mathcal{B}(v_2)|\,(|F(v_1)|^p+|F(v_2)|^p)^{\frac{2-p}{2p}}\right) dz \\ & \leq \epsilon \int\limits_{Q_\theta\cap\{p(z)<2\}} T'_\delta(v)|F(v_1)-F(v_2)|^2\,(|F(v_1)|^p+|F(v_2)|^p)^{\frac{p-2}{p}}\,dz \\ & \quad + \frac{1}{4\epsilon} \int\limits_{Q_\theta\cap\{p(z)<2\}} T'_\delta(v)|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^2\,(|F(v_1)|^p+|F(v_2)|^p)^{\frac{2-p}{p}}\,dz \end{aligned} \tag{24}$$

Gathering (22), (23) and (24) we arrive at the inequality

$$\begin{aligned} J_1(\delta)-J_2(\delta) & \geq J_1(\delta)-\mathcal{J}^{(1)}(\delta)-\mathcal{J}^{(2)}(\delta) \\ & \geq (2^{-(p^-)'}-\epsilon) \int\limits_{Q_\theta\cap\{p(z)\geq 2\}} T'_\delta(v)|F(v_1)-F(v_2)|^{p(z)}\,dz \\ & \quad - C(\epsilon,p^-,p^+) \int\limits_{Q_\theta\cap\{p(z)\geq 2\}} T'_\delta(v)\,|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^{p(z)}\,dz \\ & \quad + (p^- - 1 - \epsilon) \int\limits_{Q_\theta\cap\{p(z)\in(1,2)\}} T'_\delta(v)\,|F(v_1)-F(v_2)|^2 \\ & \quad \times\left(|F(v_1)|^{p(z)}+|F(v_2)|^{p(z)}\right)^{\frac{p(z)-2}{p(z)}}\,dz \\ & \quad - \frac{1}{4\epsilon} \int\limits_{Q_\theta\cap\{p(z)<2\}} T'_\delta(v)|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^2\,(|F(v_1)|^p+|F(v_2)|^p)^{\frac{2-p}{p}}\,dz. \end{aligned}$$

Choosing $\epsilon \equiv \epsilon(p^-)$ sufficiently small we then have

$$\begin{aligned}\lim_{k\to\infty} I_2 \geq & - C\int_{Q_\theta\cap\{p(z)\geq 2\}} T'_\delta(v)\,|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^{p(z)}\,dz \\ & - \frac{1}{4\epsilon}\int_{Q_\theta\cap\{p(z)<2\}} T'_\delta(v)|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^2 \\ & \times(|F(v_1)|^p+|F(v_2)|^p)^{\frac{2-p}{p}}\,dz \end{aligned} \tag{25}$$

with a positive constant $C \equiv C(p^{\pm})$. It remains to show that the right-hand side of the last inequality tends to zero as $\delta \to 0$.

We will use the following Lemmas.

Lemma 5.2. *For every $\eta \in (0,1)$*

$$|\mathcal{B}(v_1) - \mathcal{B}(v_2)| \leq |\nabla\gamma(z)|\, |v_1 - v_2|^{1-\eta} \left| \int_{u_1}^{u_2} |s|^{\gamma(z)} \; |\ln|s||^{\frac{1}{\eta}} \; ds \right|^{\eta}.$$

Proof. By the definition

$$\mathcal{B}(v) = -\nabla\gamma \cdot \int_0^u |s|^{\gamma(z)} \ln|s|\, ds, \qquad v = \int_0^u |s|^{\gamma(z)}\, ds.$$

Not loosing generality we may assume that $u_1 \geq u_2$ and, thus, $v_1 \geq v_2$. Then for every $\eta \in (0,1)$

$$\begin{aligned} |\mathcal{B}(v_1) - \mathcal{B}(v_2)| &= |\nabla\gamma(z)| \left| \int_{u_1}^{u_2} |s|^{\gamma(z)} \ln|s|\, ds \right| \\ &\leq |\nabla\gamma(z)| \int_{u_1}^{u_2} |s|^{\gamma(z)(1-\eta)} |s|^{\gamma(z)\eta} |\ln|s||\, ds \end{aligned}$$

and the assertion follows by Young's inequality with $p = \frac{1}{1-\eta}$, $q = \frac{1}{\eta}$. □

Lemma 5.3. *For every $p, r > 0$, $q \geq \frac{r}{2}$ and all $v \in \mathbb{R}$*

$$\frac{\delta^p |v|^r}{(\delta^2 + v^2)^q} \leq \delta^{p+r-2q}.$$

Proof. It suffices to notice that

$$\frac{\delta^p |v|^r}{(\delta^2 + v^2)^q} \leq \frac{\delta^p (\delta^2 + v^2)^{\frac{r}{2}}}{(\delta^2 + v^2)^q} = \frac{\delta^p}{(\delta^2 + v^2)^{q - \frac{r}{2}}} = \left(\frac{\delta^2}{\delta^2 + v^2} \right)^{q - \frac{r}{2}} \delta^{p - 2\left(q - \frac{r}{2}\right)} \leq \delta^{p+r-2q}.$$

□

End of the proof of Theorem 3.3. Fix some $\eta \in (0,1)$ and denote

$$K(u_1, u_2, \eta) = \left| \int_{u_1}^{u_2} |s|^{\gamma(z)} \; |\ln|s||^{\frac{1}{\eta}} \; ds \right|^{\eta}.$$

Since u_i are bounded, so is $K(u_1, u_2, \eta)$. By virtue of Lemmas 5.2, 5.3 the first term on the right-hand side of (25) is estimated as follows: for $\eta \in (0, 1 - 1/p^-)$

$$\begin{aligned} &\int_{Q_\theta \cap \{p(z) \geq 2\}} T'_\delta(v)\, |\mathcal{B}(v_1) - \mathcal{B}(v_2)|^{p(z)}\; dz \\ &\qquad \leq \int_{Q_\theta \cap \{p(z) \geq 2\}} \frac{\delta^2}{(\delta^2 + \eta^2)^{\frac{3}{2}}} |v|^{p(1-\eta)} |\nabla\gamma|^p K^p\, dz \\ &\qquad \leq \int_{Q_\theta \cap \{p(z) \geq 2\}} \delta^{p(z) - 1 - p(z)\eta} |\nabla\gamma|^{p(z)} K^{p(z)}\, dz \\ &\qquad \leq \delta^{p^- - 1 - p^- \eta} \int_{Q_\theta \cap \{p'(z) \geq 2\}} |\nabla\gamma|^{p(z)} K^{p(z)}\, dz \to 0 \quad \text{as } \delta \to 0 \end{aligned} \tag{26}$$

because $K^{p(z)}$ is uniformly bounded and $|\nabla\gamma(z)|^{p(z)} \in L^1(Q)$. The second term of (25) is estimated in a similar way:

$$\begin{aligned}
&\int\limits_{Q_\theta\cap\{p(z)<2\}} T'_\delta(v)|\mathcal{B}(v_1)-\mathcal{B}(v_2)|^2\left(|F(v_1)|^p+|F(v_2)|^p\right)^{\frac{2-p}{p}}\,dz\\
&\le \int\limits_{Q_\theta\cap\{p(z)<2\}} \frac{\delta^2}{(\delta^2+\eta^2)^{\frac{3}{2}}}|v|^{2(1-\eta)}\left(|F(v_1)|^p+|F(v_2)|^p\right)^{\frac{2-p}{p}}|\nabla\gamma|^2K^2\,dz\\
&\le \delta^{1-2\eta}\int\limits_{Q_\theta\cap\{p(z)<2\}} \left(|F(v_1)|^p+|F(v_2)|^p\right)^{\frac{2-p}{p}}|\nabla\gamma|^2K^2\,dz\\
&\le \delta^{1-2\eta}\int\limits_{Q_\theta\cap\{p(z)<2\}} \left[\frac{2}{p}|\nabla\gamma|^pK^p+\left(1-\frac{2}{p}\right)\left(|F(v_1)|^p+|F(v_2)|^p\right)\right]dz\to 0
\end{aligned}\tag{27}$$

as $\delta\to 0$, because $|F(v_i)|^{p'(z)} = |\nabla v_i + \mathcal{B}(v_i)|^{p(z)} \in L^1(Q)$. Plugging (26)-(27) to (25), we obtain, letting $\delta\to 0$ in (21): for a.e. $\theta\in(0,T)$

$$\int_\Omega |w(x,\theta)|\,dx \le \int_\Omega |w(x,0)|\,dx + \int_Q \operatorname{sign} w\,(f_1-f_2)\,dz,$$

whence the assertion of Theorem 3.3. $\square$

6. **Proof of Theorem 3.4: $\alpha = 0$.** Let us consider the Dirichlet problem for the evolutional $p(z)$-Laplace equation

$$\begin{cases} v_t = \operatorname{div}\left(a(z,v)|\nabla v|^{p(z)-2}\nabla v\right) + f(z) & \text{in } Q,\\ v(x,0) = v_0(x) \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma. \end{cases}\tag{28}$$

This equation is a particular case of equation (1) with $\alpha = 0$. Given two weak solutions of problem (28) v_1 and v_2, we denote $v = v_1 - v_2$. Following the proof of Theorem 3.3 we see that to prove Theorem 3.4 amounts to show that $\lim\limits_{\delta\to 0}(I_1+I_2) \ge 0$, where

$$\begin{aligned}
I_1 &= \int_Q a(z,v_1)\,T'_\delta(v)\,\left(\left(|\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2\right)\cdot\nabla v\right)\,dz,\\
I_2 &= \int_Q T'_\delta(v)(a(z,v_1)-a(z,v_2))|\nabla v_2|^{p-2}\nabla v_2\cdot\nabla v\,dz.
\end{aligned}$$

Proposition 3 ([13]). *Let $1<p<\infty$. There is a constant $C(p^-,p^+)$ such that*

$$\left(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right)\cdot(\xi-\zeta) \ge C\,|\xi-\zeta|^2\left(|\xi|+|\zeta|\right)^{p-2}.$$

Proof. For $p\in(1,2]$ the assertion is a byproduct of (22). Let $p\ge 2$. Take some $\xi,\zeta\in\mathbb{R}^n$ and assume, without loss of generality, that $|\xi|\ge|\zeta|$. Denote

$$A = \left(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right)\cdot(\xi-\zeta) = |\xi|^p + |\zeta|^p - \left\{|\xi|^{p-2}+|\zeta|^{p-2}\right\}(\xi,\zeta).$$

Since $(\xi,\zeta) = \dfrac{1}{2}\left(|\xi|^2+|\zeta|^2-|\xi-\zeta|^2\right)$, we have

$$\begin{aligned}
A &= \frac{1}{2}\left\{|\xi|^{p-2}+|\zeta|^{p-2}\right\}|\xi-\zeta|^2 + |\xi|^p + |\zeta|^p - \frac{1}{2}\left(|\xi|^{p-2}+|\zeta|^{p-2}\right)\left(|\xi|^2+|\zeta|^2\right)\\
&= \frac{1}{2}\left\{|\xi|^{p-2}+|\zeta|^{p-2}\right\}|\xi-\zeta|^2 + \frac{1}{2}\left(|\xi|^{p-2}-|\zeta|^{p-2}\right)\left(|\xi|^2-|\zeta|^2\right).
\end{aligned}$$

For $p \geq 2$ the mapping $X \mapsto X^{p-2}$ is nondecreasing. The second term of the last inequality is then nonnegative and can be dropped. Moreover, $|\xi| \geq \frac{1}{2}(|\xi| + |\zeta|)$ by assumption. It follows that for $p \geq 2$

$$A \geq \frac{1}{2}\left\{|\xi|^{p-2} + |\zeta|^{p-2}\right\}|\xi - \zeta|^2 \geq \frac{1}{2}|\xi|^{p-2}|\xi - \zeta|^2 \geq \frac{1}{2^{p-1}}\left(|\xi| + |\zeta|\right)^{p-2}|\xi - \zeta|^2.$$

□

Let us make the convention to denote by C different constants, depending only on the known parameters, but independent of v_i and w. Applying Proposition 3 we have

$$\begin{aligned}
I_1 &\geq C\int_{Q_\theta} \frac{\delta^2}{(\delta^2+v^2)^{3/2}}\left|\nabla(v_1 - v_2)\right|^2 \left(|\nabla v_1| + |\nabla v_2|\right)^{p-2} dz,\\
&= C\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2}{(\delta^2+v^2)^{3/2}}\left|\nabla(v_1 - v_2)\right|^2 \left(|\nabla v_1| + |\nabla v_2|\right)^{p-2} dz,\\
I_2 &\leq C\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2}{(\delta^2+v^2)^{3/2}}|v|^\beta \left|\nabla v_2\right|^{p-1} \left|\nabla(v_1 - v_2)\right| dz\\
&\leq C\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2}{(\delta^2+v^2)^{3/2}}|v|^\beta \left(|\nabla v_1| + |\nabla v_2|\right)^{p-1} \left|\nabla(v_1 - v_2)\right| dz\\
&= C\int_{Q_\theta\cap\{v\neq 0\}} \left(\frac{\delta^2}{(\delta^2+v^2)^{3/2}}\right)^{1/2} \left|\nabla(v_1 - v_2)\right| \left(|\nabla v_1| + |\nabla v_2|\right)^{\frac{p-2}{2}} \left(|\nabla v_1| + |\nabla v_2|\right)^{\frac{p}{2}}\\
&\qquad \times \left(\frac{\delta^2}{(\delta^2+v^2)^{3/2}}\right)^{1/2} |v|^\beta\, dz\\
&\leq C\,\epsilon\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2}{(\delta^2+v^2)^{3/2}}\left|\nabla(v_1 - v_2)\right| \left(|\nabla v_1| + |\nabla v_2|\right)^{p-2} dz\\
&\qquad + \frac{C}{4\varepsilon}\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2\,|v|^{2\beta}}{(\delta^2+v^2)^{3/2}} \left(|\nabla v_1| + |\nabla v_2|\right)^{p} dz.
\end{aligned}$$

For all sufficiently small ϵ these inequalities yield

$$I_1 + I_2 \geq -\frac{C}{4\varepsilon}\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta^2\,|v|^{2\beta}}{(\delta^2+v^2)^{3/2}} \left(|\nabla v_1| + |\nabla v_2|\right)^{p} dz.$$

For $\beta > 1/2$ we have

$$I_1 + I_2 \geq -\frac{C\delta^{2\beta-1}}{4\varepsilon}\int_{Q_\theta\cap\{v\neq 0\}} \left(|\nabla v_1| + |\nabla v_2|\right)^{p} dz \to 0.$$

For $\beta = 1/2$ one has

$$\frac{\delta^2\,|v|^{2\beta}}{(\delta^2+v^2)^{3/2}} = \frac{\delta^2\,|v|}{(\delta^2+v^2)^{3/2}} \leq \frac{1}{2}\frac{\delta}{(\delta^2+v^2)^{1/2}}$$

and

$$I_1 + I_2 \geq -\frac{C}{8\varepsilon}\int_{Q_\theta\cap\{v\neq 0\}} \frac{\delta}{(\delta^2+v^2)^{1/2}} \left(|\nabla v_1| + |\nabla v_2|\right)^{p} dz \to 0$$

by the Lebesgue Theorem.

7. **Existence of solutions $u \in \mathcal{V}(Q)$: L^1-estimate for $\partial_t \Phi(z,v)$.** Let us check that problem (10) indeed admits solutions in $\mathcal{V}(Q)$, which means that the class of uniqueness is nonempty. Following [11], we construct a solution as the limit of the sequence of solutions of the regularized problems

$$\begin{cases} \partial_t u_\epsilon = \operatorname{div}\left(\mathcal{A}_{\epsilon,K}(z,u_\epsilon)|\nabla u_\epsilon|^{p(z)-2}\nabla u_\epsilon\right) + f(z) \quad \text{in } Q, \\ u_\epsilon(x,0) = u_0 \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \Gamma \end{cases} \tag{29}$$

with the coefficient

$$\mathcal{A}_{\epsilon,K}(z,u_\epsilon) = a(z,u_\epsilon)(\epsilon + \min\{K, |u_\epsilon|\})^{\alpha(z)},$$

depending on the given parameters $\epsilon > 0$, $K > 0$. For every $\epsilon \in (0,1)$ and $1 < K < \infty$ the coefficient $\mathcal{A}_{\epsilon,K}(z,u_\epsilon)$ is separated away from zero and infinity, so that problem (29) can be regarded as the Dirichlet problem for the evolutional $p(z)$-Laplacian.

Theorem 7.1 ([10])**.** *For every $u_0 \in L^2(\Omega)$, $f \in L^2(Q)$, $\epsilon > 0$, $K > 0$ problem (29) has at least one weak solution $u_\epsilon \in L^\infty(0,T;L^2(\Omega)) \cap \mathbf{W}(Q)$ such that $\partial_t u_\epsilon \in \mathbf{W}'(Q)$ and for every test-function $\phi \in L^\infty(0,T;L^2(\Omega)) \cap \mathbf{W}(Q)$ with $\phi_t \in \mathbf{W}'(Q)$ and arbitrary $t_1, t_2 \in [0,T]$*

$$\int_\Omega u_\epsilon\, \phi\, dx\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2}\int_\Omega \left[u_\epsilon \phi_t - \mathcal{A}_{\epsilon,K}(z,u_\epsilon)|\nabla u_\epsilon|^{p(z)-2}\nabla u_\epsilon \cdot \nabla\phi + f\phi\right] dz.$$

Moreover, if $u_0 \in L^\infty(\Omega)$, $f \in L^1(0,T;L^\infty(\Omega))$, this solution belongs to $L^\infty(Q)$ and obeys the estimate

$$\|u_\epsilon\|_{\infty,Q} \le \|u_0\|_{\infty,\Omega} + \int_0^T \|f(\cdot,s)\|_{\infty,\Omega}\, ds \equiv K_0. \tag{30}$$

As a byproduct we also have that for every $\phi \in \mathbf{W}(Q)$ (see [10])

$$\int_Q \left[\phi\, \partial_t u_\epsilon + \mathcal{A}_{\epsilon,K}(z,u_\epsilon)|\nabla u_\epsilon|^{p(z)-2}\nabla u_\epsilon \cdot \nabla\phi - f\phi\right] dz = 0. \tag{31}$$

The solution of problem (29) is obtained as the limit as $m \to \infty$ of the sequence of Galerkin's approximations,

$$u_\epsilon^{(m)}(z) = \sum_{i=1}^m c_{i,m,\epsilon}(t)\psi_i(x), \tag{32}$$

where the family $\{\psi_i(x)\}$ is dense in $\mathbf{V}_+(\Omega)$ and forms an orthogonal basis of $L^2(\Omega)$. Estimate (30) makes the coefficient $\mathcal{A}_{\epsilon,K}(z,u_\epsilon)$ independent of K, provided that $K \ge K_0 + 1$:

$$\mathcal{A}_{\epsilon,K}(z,u_\epsilon) \equiv \mathcal{A}_\epsilon(z,u_\epsilon) = a(z,u_\epsilon)\,(\epsilon + |u_\epsilon|)^{\alpha(z)}.$$

Problem (29) is considered then as a problem with the unique regularization parameter ϵ. Passage to the limit as $\epsilon \to 0$ is justified in [11, Sec.5] in the proof of Theorem 3.1. To this end problem (29) is substituted by the formally equivalent problem

$$\begin{cases} \partial_t \Phi_\epsilon(z,v_\epsilon) = \operatorname{div}\left(b(z,v_\epsilon)|\nabla v_\epsilon + \mathcal{B}(v_\epsilon)|^{p(z)-2}(\nabla v_\epsilon + \mathcal{B}(v_\epsilon))\right) + f \quad \text{in } Q, \\ v_\epsilon = 0 \text{ on } \Gamma, \quad v_\epsilon(x,0) = v_0(x) \text{ in } \Omega, \end{cases} \tag{33}$$

in which

$$\begin{aligned} &v_\epsilon(z) = \int_0^{u_\epsilon(z)} (\epsilon + |s|)^{\gamma(z)}\, ds, \qquad u_\epsilon = \Phi_\epsilon(z, v_\epsilon),\\ &\gamma(z) = \frac{\alpha(z)}{p(z)-1} \geq \gamma^- > -1. \end{aligned} \tag{34}$$

and

$$\mathcal{B}(v_\epsilon) = -\nabla\gamma \int_0^{u_\epsilon} (\epsilon + |s|)^{\gamma(z)} \ln(\epsilon + |s|)\, ds, \qquad b(z, v_\epsilon) \equiv a(z, u_\epsilon).$$

The proof is based on the uniform a priori estimates for the functions v_ϵ, ∇v_ϵ and $\nabla v_\epsilon + \mathcal{B}(v_\epsilon)$ in the variable Lebesgue spaces $L^{p(z)}(Q)$, the integration-by-parts formulas (see Lemma 4.4), and the monotonicity of the elliptic part of equation (33).

The proof of integrability of $\partial_t \Phi_0(z, v) \equiv \partial_t u$ is thus reduced to checking that for the solutions $v_\epsilon^{(m)}$ of the regularized problems (33) the norms $\|\partial_t \Phi_\epsilon(z, v_\epsilon^{(m)})\|_{1,Q}$ are bounded uniformly with respect to ϵ and m. By virtue of (32) and (34), the coefficients $c_{i,m,\epsilon}(t)$ are defined as the solutions of the system of the ordinary nonlinear differential equations

$$\begin{cases} c'_{i,m,\epsilon}(t) = -\displaystyle\int_\Omega \left|\nabla v_\epsilon^{(m)} + \mathcal{B}_\epsilon(v_\epsilon^{(m)})\right|^{p(z)-2} \left(\nabla v_\epsilon^{(m)} + \mathcal{B}_\epsilon(v_\epsilon^{(m)})\right) \cdot \nabla\psi_i\, dx + f_i(t),\\ c_i(0) = u_{i0}, \qquad i = 1, \dots, m, \end{cases}$$

where u_{0i} and $f_i(t)$ are the Fourier coefficients of the functions $u_0(x)$ and $f(z)$ in the basis $\{\psi_i\}$:

$$u_0^{(m)} = \sum_{i=1}^m u_{0i}\psi_i(x) \to u_0, \qquad f^{(m)} = \sum_{i=1}^m f_i(t)\psi_i(x) \to f.$$

The function $u_\epsilon^{(m)} = \Phi_\epsilon(z, v_\epsilon^{(m)})$ defined by (32) is a weak solution of problem (33) with the data $u_0^{(m)}$, $f^{(m)}$ and satisfies (31) with an arbitrary $\phi \in \mathbf{W}(Q)$. Let us fix some $\epsilon > 0$, $m \in \mathbb{N}$, and introduce the function

$$V = \partial_t \Phi_\epsilon(z, v_\epsilon^{(m)}) \equiv \sum_{i=1}^m c'_{i,m,\epsilon}(t)\psi_i(x).$$

Set $\Psi = \nabla v_\epsilon^{(m)} + \mathcal{B}_\epsilon(v_\epsilon^{(m)})$, $\mathcal{F} = \left(|\Psi|^{p-2}\Psi\right)_t$. Differentiating equation (33) for $u_\epsilon^{(m)}$ in t, we write the equation for V in the form

$$V_t = \operatorname{div} \mathcal{F} + f_t^{(m)}. \tag{35}$$

This equation is fulfilled in the following sense: for every test-function $\phi \in \mathbf{W}(Q)$

$$\int_Q \left[\phi\, V_t + \mathcal{F} \cdot \nabla\phi - f_t^{(m)}\, \phi\right] dz = 0.$$

The straightforward calculation gives the equalities

$$\begin{aligned} &V = (\Phi_\epsilon)_v v_t + (\Phi_\epsilon)_t,\\ &\mathcal{F} = (p-1)|\Psi|^{p-2}\Psi_t + |\Psi|^{p-2}\Psi \ln|\Psi|\, p_t,\\ &\Psi_t = \nabla v_t + (\mathcal{B}_\epsilon)_v\, v_t + (\mathcal{B}_\epsilon)_t. \end{aligned}$$

Combining these formulas we conclude that

$$
\begin{aligned}
v_t &= \frac{V-(\Phi_\epsilon(z,v))_t}{(\Phi_\epsilon(z,v))_v},\\
\Psi_t &= \nabla\left(\frac{V-(\Phi_\epsilon(z,v))_t}{(\Phi_\epsilon(z,v))_v}\right)+(\mathcal{B}_\epsilon(v))_v\frac{V-(\Phi_\epsilon(z,v))_t}{(\Phi_\epsilon(z,v))_v}+(\mathcal{B}_\epsilon(v))_t,\\
\mathcal{F} &= (p-1)|\Psi|^{p-2}\left[\nabla\left(\frac{V-(\Phi_\epsilon(z,v))_t}{(\Phi_\epsilon(z,v))_v}\right)\right.\\
&\qquad\left.+(\mathcal{B}_\epsilon(v))_v\frac{V-(\Phi_\epsilon(z,v))_t}{(\Phi_\epsilon(z,v))_v}+(\mathcal{B}_\epsilon(v))_t\right]+|\Psi|^{p-2}\Psi\,\ln|\Psi|\,p_t.
\end{aligned}
$$

Let us introduce the functions

$$
h_\mu(\sigma)=\begin{cases}\dfrac{2}{\mu}\left(1-\dfrac{|\sigma|}{\mu}\right) & \text{if } |\sigma|<\mu,\\ 0 & \text{if } |\sigma|\ge\mu,\end{cases}
$$

$$
H_\mu(\sigma)=\int_0^\sigma h_\mu(s)ds,\qquad \mathbf{H}_\mu(\sigma)=\int_0^\sigma\int_0^q h_\mu(s)\,dsdq.
$$

According to the definition

$$
\begin{cases} h_\mu(\sigma)\ge 0, & \lim_{\eta\to 0}\sigma h_\mu(\sigma)=0,\\ |H_\mu(\sigma)|\le 1, & \lim_{\eta\to 0}H_\mu(\sigma)=\operatorname{sign}\sigma,\quad \lim_{\mu\to 0}\mathbf{H}_\mu(\sigma)=|\sigma|.\end{cases} \tag{36}
$$

Multiplying (35) by $H_\mu(V)$ and integrating by parts in t, we arrive at the equality

$$
\begin{aligned}
\int_Q H_\mu(V)\,V_t\,dz &= \int_Q \partial_t\mathbf{H}(V(z))\,dz\\
&= \int_\Omega \mathbf{H}_\mu(V(z))\,dx-\int_\Omega \mathbf{H}_\mu(V(x,0))\,dx\\
&= -\int_Q \mathcal{F}\,\nabla H_\mu(V)\,dz+\int_Q f_t^{(m)}\,H_\mu(V)\,dz\\
&= -\int_Q \mathcal{F}\,h_\mu(V)\,\nabla V\,dz+\int_Q f_t^{(m)}\,H_\mu(V)\,dz.
\end{aligned} \tag{37}
$$

Let us consider the simple case: $p_t=0$, $\gamma_t=0$, $\Phi_\epsilon\equiv\Phi_\epsilon(x,v)$. In this case

$$
\begin{aligned}
\mathcal{F} &= (p-1)|\Psi|^{p-2}\left[\nabla\left(\frac{V}{(\Phi_\epsilon)'_v}\right)+(\mathcal{B}_\epsilon(v))'_v\frac{V}{(\Phi_\epsilon)'_v}\right]\\
&= (p-1)\frac{|\Psi|^{p-2}}{(\Phi_\epsilon)'_v}\left[\nabla V-\frac{V\,\nabla v}{(\Phi_\epsilon)'_v}+V\,(\mathcal{B}_\epsilon(v))'_v\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
&(\Phi_\epsilon)'_v=(\gamma+1)^{-\frac{\gamma}{\gamma+1}}v^{-\frac{\gamma}{\gamma+1}},\\
&(\mathcal{B}_\epsilon(v))'_v=\nabla\gamma\cdot(\epsilon+|\Phi_\epsilon(x,v)|)^\gamma\,\ln(\epsilon+|\Phi_\epsilon(x,v)|)(\Phi_\epsilon)'_v,
\end{aligned}
$$

the previous equality becomes

$$
\mathcal{F}=(p-1)\,|\Psi|^{p-2}\left[\frac{\nabla V}{(\Phi_\epsilon)'_v}-\frac{V\,\nabla v}{((\Phi_\epsilon)'_v)^2}+V\,\nabla\gamma\cdot(\epsilon+|\Phi_\epsilon|)^\gamma\,\ln(\epsilon+|\Phi_\epsilon|)\right].
$$

Let us write (37) in the form

$$
\int_\Omega \mathbf{H}_\mu(V)\,dx\Big|_{t=0}^{t=T}=I_1+I_2+I_3+\int_Q f_t^{(m)}\,H_\mu(V)\,dz \tag{38}
$$

with

$$I_1 = -\int_Q (p-1)\,|\Psi|^{p-2}\,(\gamma+1)^{\frac{\gamma}{\gamma+1}}\,v^{\frac{\gamma}{\gamma+1}}\,|\nabla V|^2\,h_\mu(V)\,dz \le 0,$$

$$I_2 = \int_Q (p-1)\,|\Psi|^{p-2}\,(\gamma+1)^{\frac{2\gamma}{\gamma+1}}\,v^{\frac{2\gamma}{\gamma+1}}\,\nabla V\cdot\nabla v\,\left(V\,h_\mu(V)\right)\,dz,$$

$$I_3 = -\int_Q (p-1)\,|\Psi|^{p-2}\,(\nabla\gamma\cdot\nabla V)\,(\epsilon+|\Phi_\epsilon|)^{\gamma}\,\ln(\epsilon+|\Phi_\epsilon|)\,\left(V\,h_\mu(V)\right)\,dz.$$

Dropping the nonpositive term I_1 on the right-hand side of (38), letting $\mu\to 0$ and using (36) we finally obtain:

$$\int_\Omega |V(x,t)|\,dx \le \int_\Omega |V(x,0)|\,dx + \int_Q |f_t|\,dz, \qquad V = \partial_t\Phi_\epsilon(x,v). \tag{39}$$

Since the right-hand side of this inequality is independent of m and ϵ, the needed estimate follows by passing to the limit as $m\to\infty$ and $\epsilon\to 0$. These arguments are summarized in the following assertion.

Proposition 4. *Let the conditions of Theorem 3.2 be fulfilled and, additionally, $f_t\in L^1(Q)$, $p\equiv p(x)$, $\alpha\equiv\alpha(x)$ and*

$$\operatorname{div}\Big(b((x,0),v_0)|\nabla v_0+\mathcal{B}(v_0)|^{p(x)-2}(\nabla v_0+\mathcal{B}(v_0))\Big)\in L^1(\Omega).$$

Then $V=\partial_t\Phi(x,v)\in L^1(Q)$ and satisfies inequality (39).

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