Analytical Option Pricing under an Asymmetrically Displaced Double Gamma Jump-Diffusion Model

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Abstract

We generalize the Kou (2002) double exponential jump-diffusion model in two directions. First, we independently displace the two tails of the jump size distribution away from the origin. Second, we allow for each of the displaced tails to follow a gamma distribution with an integer-valued shape parameter. Both extensions introduce additional flexibility in the tails of the corresponding return distribution. Our model is supported by an equilibrium economy and we obtain closed-form solutions for European plain vanilla options. Our valuation function is computationally fast to evaluate and robust across the full parameter space. We estimate the physical model parameters through maximum likelihood and for a diverse sample of equities, commodities and exchange rates. For all assets under consideration, the original Kou (2002) model can be rejected in favor of our newly introduced asymmetrically displaced double gamma dynamics.

Keywords: displaced tails, jump-diffusion, option pricing, maximum likelihood estimation, closed-form solution

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1 Introduction

One of the central trade-offs in the contingent claim literature is between the generality of the model dynamics for the risky underlying asset and the computational tractability of the associated valuation functions. The central contribution of this paper is to propose the most general jump-diffusion dynamics so far that admit closed-form solutions for European plain vanilla options. We consider an extension of the Kou (2002) double exponential (DE) jump-diffusion model, where each of the two tails of the jump size distribution is independently displaced away from the origin and follows a gamma distribution. The shifts are motivated by the intuition that jumps should represent relatively rare events with a significant absolute return. They further allow to better disentangle the diffusion and jump components based on the time series of logarithmic returns. The gamma distribution allows for more flexible shapes of the conditional probability density function (PDF) of positive and negative jumps. Our asymmetrically displaced double gamma (AD-DG) dynamics further nest the model recently proposed by Detering et al. (2013). They arise as the equilibrium logarithmic asset price process in a Naik and Lee (1990) type economy. This competitive equilibrium induces a risk-neutral probability measure which can be represented as an Esscher transform.

Our results are highly relevant for two main reasons. First, we demonstrate that the asymmetrically displaced jumps are not only academically interesting but also successfully capture some statistical properties of asset dynamics. To establish this, we estimate the model parameters under the physical probability measure through maximum likelihood (ML) and based on the historical time series of logarithmic returns. Our empirical analysis covers a diverse sample of assets, from equity indices over commodity indices to foreign exchange. Statistical tests confirm that for all assets, the special case of zero displacements can be rejected at the 1% level in favor of our AD-DG dynamics. Second, while the closed-form expressions for European plain vanilla option prices are tedious to derive in the first place, their computational speed is generally faster and they exhibit a superior numerical stability over the full range of model parameter values.

The remainder of this paper is organized as follows. Section 2 provides a review of the related literature and contrasts models with closed-form and quasi-analytical solutions. Section 3 defines the physical dynamics of the AD-DG jump-diffusion model and derives its statistical properties. Section 4 constructs and characterizes a risk-neutral probability measure. Section 5 derives closed-form solutions for European plain vanilla options. In Section 6, we discuss alternative estimation approaches based on the time series of logarithmic returns. Section 7 describes the data set and evaluates the empirical results. Finally, Section 8 summarizes and concludes the paper. The appendices provide the derivations of various technical results and present the parameter estimates.

2 Literature Review

Very few extensions of the Black and Scholes (1973) and Merton (1976) models admit closed-form solutions for European plain vanilla options. Merton (1976) extends the geometric Brownian motion by introducing normally distributed compound Poisson jumps to the logarithmic stock price process. The values of European plain vanilla options can be expressed through an infinite summation over Black and Scholes (1973) prices with exponentially decaying summands. Cox and Ross (1976) consider various alternative pure diffusion and pure jump models and obtain closed-form solutions in the special case of proportional jumps, constant volatility and proportional variance. Kou (2002) introduces asymmetry to the discontinuous return component by modeling the jump sizes to follow a DE distribution. He obtains closed-form solutions for the tail probabilities under the two relevant pricing measures; see also Kou and Wang (2003, 2004) for applications to weakly path-dependent options.

In a seminal paper, Heston (1993) models the stock price as a diffusion process whose variance itself follows a Cox et al. (1985) square-root process. Given closed-form expressions for the characteristic functions of the logarithmic terminal spot price under the two relevant numéraires, he obtains the corresponding exercise probabilities through a numerical Fourier inversion. These quasi-analytical Fourier inversion techniques have since been widely applied to contingent claim valuation since the characteristic function of the logarithmic asset prices is often tractable, either analytically or numerically, under more complex underlying dynamics; see Bates (1996), Schöbel and Zhu (1999) and Bakshi et al. (1997). Carr and Madan (1999) and Lewis (2001) and Attari (2004) obtain alternative pricing representations. Bakshi and Madan (2000) provide an economic foundation for valuation using the characteristic function, by showing that it represents an equivalent basis for spanning the payoff universe of most contingent claims. Duffie et al. (2000) unify much of the previous theoretical work on stochastic volatility jump-diffusion models. They show that an affine structure of the drift, covariance matrix, jump intensity and instantaneous interest rate allows us to obtain the Fourier transform of the relevant random variables in terms of a system of ordinary differential equations. Contingent claim valuation based on the characteristic function is the standard approach for time-changed and stochastic clock exponential Lévy models; see Madan et al. (1998), Barndorff-Nielsen (1998), Geman et al. (2001), Carr et al. (2003) and Carr and Wu (2004).

Despite its fairly general applicability and widespread use, a numerical implementation of the Fourier inversion technique that is both fast and stable across the full parameter space is very intricate. A first remark hinting at its inherent complexity can be found in Footnote 7 in Schöbel and Zhu (1999), p. 28. We outline three typical numerical issues encountered when implementing the representation proposed by Carr and Madan (1999), though similar problems arise in alternative formulations. First, the integrand often becomes highly oscillatory for options that far out-of-the-money relative to their maturity; see Carr and Madan (1999), Andersen and Andreasen (2002) and Joshi and Yang (2011) for remedies. Second, the lack of integrability of the European call option price as a function of the logarithmic strike prices requires the use of either a dampening factor or the generalized Fourier transform; see Carr and Madan (1999) and Lewis (2001). Optimal choices for this contour of integration are discussed in Lee (2004) and Lord and Kahl (2007). Third, always evaluating of the complex logarithms and square roots in the integrand at their principal branch might lead to discontinuities; see Kahl and Jäckel (2005), Lord and Kahl (2006), Albrecher et al. (2007) and Lord and Kahl (2010) for unconditionally stable approach.

Fang and Oosterlee (2008) propose the so-called COS method that approximates the whole integral in the valuation problem through its respective Fourier-cosine series expansion. Their approach exhibits exponential convergence and the authors demonstrate that for a given level of accuracy the COS method is computationally more efficient than, among others, the Carr (1995) representation. For many underlying asset dynamics, a caching technique allows for a simultaneous computation of the European option prices for a vector of strike prices similar to the fast Fourier transform (FFT); see Carr and Madan (1999) and Kilin (2011).

The preceding survey highlights the apparent trade-off between the generality of the model dynamics and the computational tractability of the associated valuation functions. Besides its preference independence, the availability of robust closed-form solutions is one of the major reasons for the success of the Black and Scholes (1973) option pricing model. This paper proposes a jump-diffusion model that captures additional statistical properties of the logarithmic return process while yielding valuation functions for European plain vanilla options that are similarly straightforward to implement.

The motivation for the generalizations introduced in this paper is based on the symmetrically displaced double exponential (SD-DE) jump-diffusion dynamics considered by Detering et al. (2013). These authors are interested in the performance of different investment strategies for capital guaranteed equity-linked retirement plans. They do not consider the valuation problem for European plain vanilla options but instead analyze path-dependent payoff structures under stochastic interest rates. This necessitates the valuation through Monte Carlo simulation. In contrast, one of our main contributions is to show that a closed-form solution for European plain vanilla options can be obtained even when considering two further generalizations of their dynamics, asymmetric displacements and gamma tails, both of which are novel. Furthermore, this paper provides the first empirical test of the nested SD-DE model specification.

3 Physical Spot Price Dynamics

Let $W = \{W_t : t \in [0, T^*]\}$ be a standard Brownian motion, $N = \{N_t : t \in [0, T^*]\}$ be a Poisson process and $(Y_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables on a complete filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$. We interpret $\mathbb{P}$ to be the physical or real-world probability measure, and consider continuous trading in the interval $[0, T^*]$ for a fixed terminal time $0 < T^* < \infty$. The filtration $\mathbb{F} = (\mathfrak{F}_t)_{t\in[0,T^*]}$ is the $\mathbb{P}$-augmentation of the natural filtration induced by the processes W and N and the sequence of random variables $(Y_i)_{i\in\mathbb{N}}$, that is

$$\mathfrak{F}_t = \sigma\left(W_u, N_u : u \in [0, t]; Y_i : i \in \{1, 2, \dots, N_t\}\right) \vee \mathcal{N},$$

where $\mathcal{N}$ are the corresponding $\mathbb{P}$-null sets. We further assume that the processes W and N as well as the sequence of random variables $(Y_i)_{i\in\mathbb{N}}$ are pairwise independent. The Poisson process N has a constant intensity of $\lambda \in \mathbb{R}_+$, and each random variable Y_i follows the PDF $f_Y(x)$.

The frictionless market consists of two assets. The first is a risky limited liability spot asset $S = \{S_t : t \in [0, T^*]\}$, which later serves as the underlying asset for contingent claims. We assume that S pays no holding returns, such as dividends. This assumption is not crucial and is later implicitly dropped when considering the valuation of European plain vanilla options on forwards. The second asset is a money market account $B = \{B_t : t \in [0, T^*]\}$ with non-random dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t,$$

where the risk-free interest rate $r \in \mathbb{R}$ is a constant and $B_0 = 1$. Imposing non-random interest rates is necessary to obtain a tractable closed-form solution. As shown by Scott (1997), the variability of actual interest rates has relatively little impact on the prices of short-term equity index options. Due to the presence of the randomly distributed jumps, this market is generally incomplete in the Harrison and Pliska (1981) sense. Consequently, contingent claims are not redundant assets and cannot be priced solely by no-arbitrage arguments.

Let $X = \{X_t : t \in [0, T^*]\}$ be the logarithmic return process defined as $X_t = \ln(S_t/S_0)$ with $S_0 \in \mathbb{R}_+$. We assume that X follows a time-homogeneous jump-diffusion process of the form

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

under $\mathbb{P}$, where the drift term $\gamma \in \mathbb{R}$ and the diffusion coefficient $\sigma \in \mathbb{R}_+$ are constants. We often decompose X as $X_t = X_t^c + X_t^j$, where $X^c = \{X_t^c : t \in [0, T^*]\}$ and $X^j = \{X_t^j : t \in [0, T^*]\}$ are the continuous and pure-jump components, respectively. Each Y_i is an AD-DG random variable with law

$$Y_i \sim \begin{cases} \xi^+ & \text{with probability } p \in [0,1] \\ -\xi^- & \text{with probability } 1-p \end{cases},$$

where $\xi^+ - \kappa_+ \sim \Gamma(\delta_+, \eta_+)$ and $\xi^- + \kappa_- \sim \Gamma(\delta_-, \eta_-)$ are gamma random variables with integer-valued shape parameters $\delta_\pm \in \mathbb{N}$. We require that the two rate parameters, which

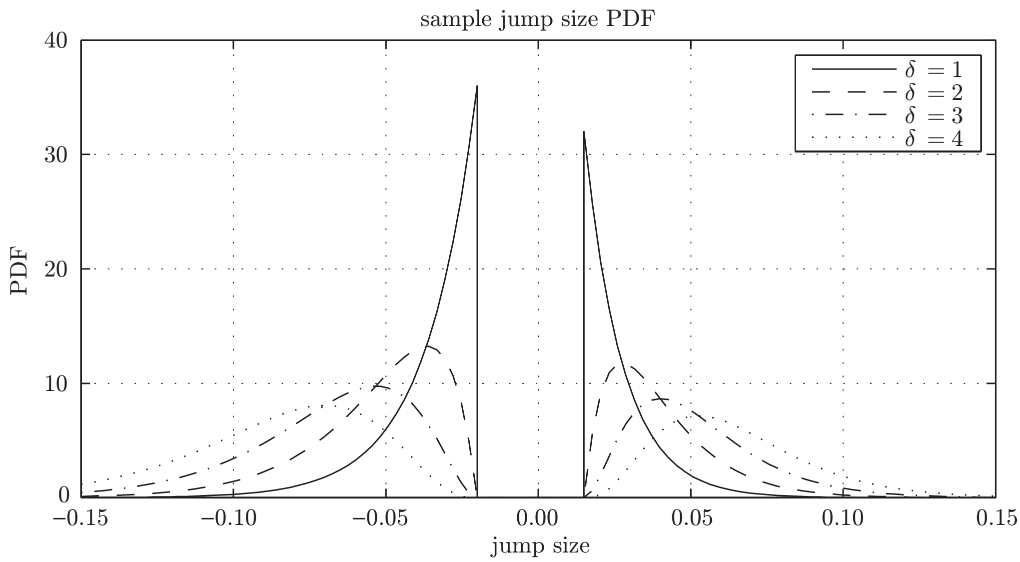

Figure 1: Sample AD-DG jump size PDF for $\delta_\pm \in \{1, 2, 3, 4\}$ $p = 40\%$, $\eta_+ = 80$, $\eta_- = 60$, $\kappa_+ = +1.50\%$ and $\kappa_- = -2.00\%$.

control the tail behavior of the jumps, satisfy $\eta_+ > 1$ and $\eta_- > 0$. The former condition is necessary to be able to compute the drift-compensator in Section 4. Since X models the logarithmic asset return, it corresponds to the mean size of an up-jump being less than $\delta_+ \geq 100\%$. Consequently, this restriction should not be binding for most real-world markets. The displacement terms satisfy $\kappa_- \leq 0 \leq \kappa_+$ and each jump is positive with probability $p \in [0, 1]$. The corresponding jump size PDF is given by

$$\begin{aligned} f_Y(x) &= \frac{p\eta_+^{\delta_+}}{\Gamma(\delta_+ - 1)} (x - \kappa_+)^{\delta_+ - 1} \mathrm{e}^{-\eta_+(x-\kappa_+)} 1\{x \geq \kappa_+\} \\ &\quad + \frac{(1-p)\eta_-^{\delta_-}}{\Gamma(\delta_- - 1)} (\kappa_- - x)^{\delta_- - 1} \mathrm{e}^{\eta_-(x-\kappa_-)} 1\{x \leq \kappa_-\}. \end{aligned}$$

This parametrization nests the original Kou (2002) DE jump size distribution as a special case when $\kappa_\pm = 0$ and $\delta_\pm = 1$; see also Ramezani and Zeng (1999). We obtain the Detering et al. (2013) SD-DE jump-diffusion model by imposing $\kappa_- = -\kappa_+$ and $\delta_\pm = 1$. Figure 1 shows how the AD-DG jump size PDF changes for different values of the shape parameters $\delta_\pm$.

Introducing the additional displacement terms facilitates the disentanglement of the price fluctuations caused by the diffusion and jump components, respectively, over discrete time intervals; see Detering et al. (2013). While jumps can be perfectly identified in continuous observations, the effect of the two driving sources of uncertainty blends when samples are discrete. In the DE model, jumps not only account for relatively large changes in the asset price but significantly contribute to the small noise in returns. Additional degrees of asymmetry between jumps corresponding to good and bad news, respectively, are introduced through both the displacements and the shape parameters. This is one of the main motivations for Ramezani and Zeng (1999) and Kou (2002) to consider two distinct exponential tails in the first place.

In the mathematical finance literature, the proposed asset dynamics are often considered as exogenously given. However, from an economic viewpoint, it is desirable to show that there exists a model economy with viable preferences that embeds the postulated price processes in equilibrium. Following Naik and Lee (1990), we can construct an infinite horizon continuous time Lucas (1978) type pure exchange economy in which a representative agent maximizes her iso-elastic expected lifetime utility of consumption. While the former authors focus on the Merton (1973) jump-diffusion model, their setup is sufficiently general to accommodate most exponential Lévy models. The necessary key assumption is to let the logarithmic dividend process of a single fully equity financed firm follow the same AD-DG jump-diffusion process that we previously postulated for the stock price. In equilibrium, the representative agent engages in exogenous production such that this yields her optimal consumption, the stock price and dividend processes are identical up to a scaling factor and the risk-free interest rate is constant. Kou (2002) considers a variation of this model economy where the logarithmic endowment of the representative agent follows a jump-diffusion process.

The following result will be used repeatedly throughout this paper.

Lemma 1 (Characteristic Exponent).
The characteristic exponent of X under $\mathbb{P}$ is given by

$$\psi_X(\omega) = \ln\left(\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\omega X_1}\right]\right) = \mathrm{i}\omega\gamma - \frac{1}{2}\omega^2\sigma^2 + \lambda\left(\phi_Y(\omega) - 1\right),$$

where

$$\phi_Y(\omega) = p\phi_{Y^+}(\omega) + (1-p)\phi_{Y^-}(\omega)$$

is the characteristic function of the sequence of random variables $(Y_i)_{i\in\mathbb{N}}$ under $\mathbb{P}$. Here, $\phi_{Y^\pm}(\omega)$ are the characteristic functions of the upper and lower tails of the jump size *distribution under $\mathbb{P}$ given by*

$$\phi_{Y^{\pm}}(\omega) = \left(\frac{\eta_{\pm}}{\eta_{\pm} \mp \mathrm{i}\omega}\right)^{\delta_{\pm}} \mathrm{e}^{\mathrm{i}\omega\kappa_{\pm}}.$$

Proof Given the characteristic function of the gamma distribution, the functional form of $\phi_Y(\omega)$ follows from elementary properties of the Fourier transform. The expression for $\psi_X(\omega)$ is then an immediate consequence of the Lévy-Khintchine representation for finite activity Lévy processes; see Theorems 1.2.14 and 1.3.3 in Applebaum (2004), pp. 28, 41. □

Having analytical expressions for the characteristic functions of logarithmic returns allows us to compute the corresponding cumulants.

Lemma 2 (Cumulants of the Logarithmic Return Process).
The n-th cumulant of the logarithmic return process X under $\mathbb{P}$ is given by

$$\begin{aligned} c_n(X_t) &= t\left(\gamma 1\{n=1\} + \sigma^2 1\{n=2\} + \lambda\left(p\sum_{i=0}^{n}\binom{n}{i}\frac{(\delta_+ + n - i - 1)!}{(\delta_+ - 1)!}\frac{\kappa_+^i}{\eta_+^{n-i}}\right.\right. \\ &\quad \left.\left. +(1-p)\sum_{i=0}^{n}\binom{n}{i}\frac{(\delta_- + n - i - 1)!}{(\delta_- - 1)!}\frac{\kappa_-^i}{(-\eta_-)^{n-i}}\right)\right). \end{aligned}$$

Proof The n-th cumulant is related to the n-th derivative of the cumulant generating function with respect to the transform parameter evaluated at zero; see Theorem 2.3.1 in Lukacs (1970), pp. 20–21. The result then follows from carefully differentiating $\psi_X(\omega)$. □

Given closed-form expressions for the cumulants of all orders, we can also compute all corresponding (central) moments; see Section 3.14 in Stuart and Ord (1994), pp. 85–89. In Section 6, we advocate to infer the physical model parameters through ML. However, Lemma 2 alternatively allows for an generalized method of moments (GMM) estimation by matching the empirical and model implied cumulants or (central) moments of the logarithmic returns. While the skewness can be both positive and negative, the excess kurtosis is always strictly positive for $\lambda > 0$. As for all Lévy processes with finite variance, the Lindeberg-Levy central limit theorem implies that X_t converges in distribution to a normal random variable as $t \to \infty$ with the i-th standardized moment decaying to zero at a rate of $t^{(i-1)/2}$. Consequently, the impact of the jumps is averaged out and asymptotically vanishes.

Next, we define the drift-compensated return process $\tilde{X} = \left\{ \tilde{X}_t : t \in [0, T^*] \right\}$ by

$$\tilde{X} = X_t - t\psi_X(-\mathrm{i}).$$

This expression is well-defined for $\eta_+ > 1$, as previously assumed. It then follows that the process $\tilde{S} = \left\{ \tilde{S}_t : t \in [0, T^*] \right\}$ given by $\tilde{S}_t = S_0 \mathrm{e}^{\tilde{X}_t}$ is a $(\mathbb{P}, \mathbb{F})$-martingale; see Proposition 2.1.3 in Applebaum (2004), p. 72–73. Consequently, the mean return $\mu = \ln\left(\mathbb{E}_{\mathbb{P}}\left[S_1/S_0\right]\right)$ of S under $\mathbb{P}$ is linked to the drift γ through

$$\gamma = \mu - \frac{1}{2}\sigma^2 - \lambda\left(\phi_Y(-\mathrm{i}) - 1\right).$$

While the dynamics proposed in this paper aim at providing a realistic model for the jump size distribution, they do not capture other empirical stylized features of asset returns. This is a deliberate modeling choice, as incorporating additional effects would prevent us from obtaining closed-form solutions for European plain vanilla options. In all cases, it presents no difficulty to augment the model dynamics, while at least preserving a closed-form solution for the corresponding characteristic function. Like in any pure jump-diffusion model, logarithmic returns are stationary and, in particular, do not exhibit volatility clustering. This induces constant market prices of diffusion and jump risk and the corresponding implied volatility smile flattens out for long times-to-maturity. However, the flexibility of the jump size distribution allows for a good calibration to the market prices of options with short times-to-maturity. The aforementioned effects prevent the model from fitting the term-structure of implied volatilities.

4 Risk-Neutral Spot Price Dynamics

Following Gerber and Shiu (1994), we characterize a risk-neutral probability measure through an Esscher transform of the logarithmic return process. Since the market is incomplete, this construction is not unique and seems arbitrary at first. However, the corresponding change of measure processes arises as the compounded equilibrium pricing kernel in a Naik and Lee (1990) model economy; see also the discussion following Gerber and Shiu (1994), pp. 175–177. Milne and Madan (1991) use this framework to define the risk-neutral probability measure in the Madan and Senata (1990) variance gamma model. In contrast to the pure drift change applied by Merton (1976), no strong assumptions about the idiosyncrasy of the jumps are made and both diffusion and jump risk are priced.

The Esscher transform probability measure (ETPM) $\hat{\mathbb{P}}(X,\beta)$ equivalent to $\mathbb{P}$ on $[0,T^*]$ is defined through

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left\{\beta X_{T^*} - T^*\psi_X(-\mathrm{i}\beta)\right\} \qquad \mathbb{P}\text{-a.s.},$$

with transform parameter $\beta \in \mathbb{R}$ and given that this expression is well-defined. The corresponding Radon-Nikodým derivative process $\nu\big(\mathbb{P},\hat{\mathbb{P}}\big) = \big\{\nu_t\big(\mathbb{P},\hat{\mathbb{P}}\big) : t \in [0,T^*]\big\}$ can be represented as

$$\nu_t\big(\mathbb{P},\hat{\mathbb{P}}\big) = \frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\bigg|\mathfrak{F}_t = \exp\left\{\beta\sigma W_t - \frac{1}{2}\beta^2\sigma^2 t\right\}\exp\left\{\beta\sum_{i=1}^{N_t} Y_i - t\psi_{X^j}(-\mathrm{i}\beta)\right\} \qquad \mathbb{P}\text{-a.s.},$$

where $\psi_{X^j}(\omega) = \lambda\left(\phi_Y(\omega) - 1\right)$ is the characteristic exponent of the pure-jump component X^j. Again, Proposition 2.1.3 in Applebaum (2004), pp. 72–73, ensures that each of the two factors is an independent $(\mathbb{P},\mathbb{F})$-martingale and thus $\nu\big(\mathbb{P},\hat{\mathbb{P}}\big)$ is as well.

Proposition 1 (Esscher Transform Dynamics).
For $\beta \in \mathcal{B} = (-\eta_-,\eta_+)$, the Esscher transform is well-defined. Under the new probability measure $\hat{\mathbb{P}}(X,\beta)$, X is also an AD-DG jump-diffusion process with parameters

$$\hat{\gamma} = \gamma + \beta\sigma^2, \qquad \hat{\lambda} = \lambda\phi_Y(-\mathrm{i}\beta), \qquad \hat{p} = \frac{p\phi_{Y^+}(-\mathrm{i}\beta)}{\phi_Y(-\mathrm{i}\beta)}, \qquad \hat{\eta}_\pm = \eta_\pm \mp \beta.$$

The diffusion coefficient σ, the shape parameters $\delta_\pm$ as well as the displacement terms $\kappa_\pm$ are invariant under the measure change.

Proof This can be shown by computing the characteristic function $\hat{\phi}_{X_t^j}(\omega)$ of the compound Poisson process X^j under $\hat{\mathbb{P}}(X,\beta)$. All details are given in Appendix A.1. □

Note, in particular, that the jump sizes still follow an AD-DG distribution under $\hat{\mathbb{P}}(X,\beta)$ with PDF

$$\begin{aligned}\hat{f}_Y(x) &= \frac{\hat{p}\hat{\eta}_+^{\delta_+}}{\Gamma(\delta_+-1)}(x-\kappa_+)^{\delta_+-1}\mathrm{e}^{-\hat{\eta}_+(x-\kappa_+)}1\{x\geq\kappa_+\} \\ &\quad+\frac{(1-\hat{p})\,\hat{\eta}_-^{\delta_-}}{\Gamma(\delta_--1)}(\kappa_--x)^{\delta_--1}\mathrm{e}^{\hat{\eta}_-(x-\kappa_-)}1\{x\leq\kappa_-\}.\end{aligned}$$

The invariance of the two displacement terms κ_+ and κ_- is necessary for the equivalence of the two probability measures.

Characterizing the dynamics of X under the ETPM has a second important application besides defining the risk-neutral probability measure. The Radon-Nikodým derivative process corresponding to the change of numéraire from the bank account to the spot asset corresponds to an Esscher transform of the logarithmic return process with a unit transform parameter; see Gerber and Shiu (1994) and Geman et al. (1995). This significantly simplifies the contingent claim valuation problem considered in Section 5. It further emphasizes why the closedness of the jump size distribution under this particular type of measure change is crucial.

We define the Esscher transform martingale measure (ETMM) $\mathbb{P}^*$ as the ETPM $\hat{\mathbb{P}}(X,\beta^*)$, where the transform parameter β^* is chosen such that the discounted asset price process is a $(\mathbb{P}^*,\mathbb{F})$-martingale. This condition is equivalent to $\psi_X^*(-\mathrm{i})=r$ or

$$g(\beta^*) := \gamma - r + \left(\beta^*+\frac{1}{2}\right)\sigma^2 + \lambda\int_{-\infty}^{+\infty}(\mathrm{e}^x-1)\,\mathrm{e}^{\beta^* x}f_Y(x)\mathrm{d}x = 0;$$

see the proof of Proposition 1 in Appendix A.1 for details.

Proposition 2 (Existence and Uniqueness). *The ETMM exists and is unique. That is, $g(\beta^*)=0$ has a unique solution $\beta^*\in\mathcal{B}$.*

Proof To see this, note that

$$\int_{-\infty}^{+\infty}(\mathrm{e}^x-1)\,\mathrm{e}^{\beta x}f_Y(x)\mathrm{d}x = \vartheta_Y(1+\beta)-\vartheta_Y(\beta),$$

where $\vartheta_Y(x) = \phi_Y(-\mathrm{i}x)$ is the moment generating function of $(Y_i)_{i\in\mathbb{N}}$ under $\mathbb{P}$. Since $\vartheta_Y(x)$ is strictly convex on $x \in \mathcal{B}$, it follows that $g(\beta)$ is strictly increasing in β; see Theorem 7.1.4 in Lukacs (1970), p. 197. We can further show that $\lim_{\beta\downarrow-\eta_-} g(\beta) = -\infty$ and $\lim_{\beta\uparrow\eta_+} g(\beta) = +\infty$ and Proposition 2 follows. □

The following corollary establishes that we find $\beta^* < 0$ for assets that bear a positive amount of systematic risk.

Corollary 1 (Sign of the Transform Parameter).
The sign of β^* is positive (negative) if the excess return $\mu - r$ is negative (positive). When the excess return is zero, then $\beta^* = 0$ and the dynamics of X under $\mathbb{P}^*$ and $\mathbb{P}$ coincide.

Proof By expressing the martingale condition $g(\beta^*) = 0$ in terms of the mean return, we immediately see that a zero excess return implies that $\beta^* = 0$. Thus, $\nu_t(\mathbb{P}, \mathbb{P}^*) = 1$ for all $t \in [0, T^*]$ and consequently the two probability measures $\mathbb{P}$ and $\mathbb{P}^*$ coincide. As shown in the proof of Proposition 2, the function $g(\beta)$ is strictly increasing. It follows that a positive value of the excess return has to be compensated by a negative value of β^* and *vice versa*. □

Corollary 2 (Comparative Statics for the Transform Parameter).
Compared to the physical probability measure $\mathbb{P}$, positive (negative) values for β^*

(i) dampen (increase) the lower tail of the jump size distribution,

(ii) increase (dampen) the upper tail,

(iii) decrease (increase) the probability p^* of an up-jump,

(iv) increase (decrease) the mean jump return $\mathbb{E}_{\mathbb{P}^*}[Y]$ and

(v) might either increase or decrease the intensity λ^*.

Proof By Proposition 1, we have $\eta_-^* - \eta_- = \beta^*$ $(\eta_+^* - \eta_+ = -\beta^*)$. Since the length of the lower (upper) tail under $\mathbb{P}^*$ is decreasing in η_-^* (η_+^*), Properties (i) and (ii) follow. Some tedious algebra shows that $\partial p^*/\partial\beta^* > 0$ thus proving Property (iii). Property (iv) is a direct consequence of Properties (i)–(iii). To establish Property (v), it is sufficient to show that the slope of the moment generating function $\vartheta_Y(x)$ in $x = 0$ might be both positive and negative, depending on the parameters of the AD-DG distribution. □

5 Option Pricing

Probably the main reason for the popularity of the Kou (2002) model is that it explicitly incorporates non-normal higher moments but still features closed-form solutions for European plain vanilla options. In Kou and Wang (2003), the authors furthermore obtain expressions for the Laplace transform of the first passage time density and use it in Kou and Wang (2004) to price path-dependent contingent claims such as lookback and barrier options. In this section, we show that analytical solutions for European plain vanilla options can also be attained when the jump sizes follow an AD-DG distribution.

5.1 Auxiliary Results for Exponential Tails

We first derive some auxiliary results regarding the distribution of a sum of asymmetrically displaced double exponential (AD-DE) distributed random variables, that is with shape parameters $\delta_\pm = 1$. The generalization to AD-DG jumps is discussed in Section 5.2. The main property of the Kou (2002) model that makes it possible to obtain analytical expressions for the tail probabilities is the memorylessness of the exponential distribution. For $\kappa_+ = \kappa_- = 0$, we have

$$\mathbb{P}\left\{\xi^+ - \xi^- \middle| \xi^+ > \xi^-\right\} \sim \xi^+.$$

This feature is retained in the SD-DE specification with $\kappa_+ = -\kappa_-$ but not in the AD-DE model. The key idea is to consider asymmetric displacements as being symmetrically displaced with respect to a different y-axis.

Let $\alpha = (\kappa_+ + \kappa_-)/2$ to be the midpoint of the interval $[\kappa_-, \kappa_+]$. Next, introduce two auxiliary random variables $\hat{\xi}^+ = \xi^+ - \alpha$ and $\hat{\xi}^- = \xi^- + \alpha$ and define $\kappa = \kappa_+ - \alpha = -(\kappa_- - \alpha)$. Then $\hat{\xi}^+ - \kappa \sim \mathcal{E}(\eta_+)$ and $\hat{\xi}^- - \kappa \sim \mathcal{E}(\eta_-)$ are exponentially

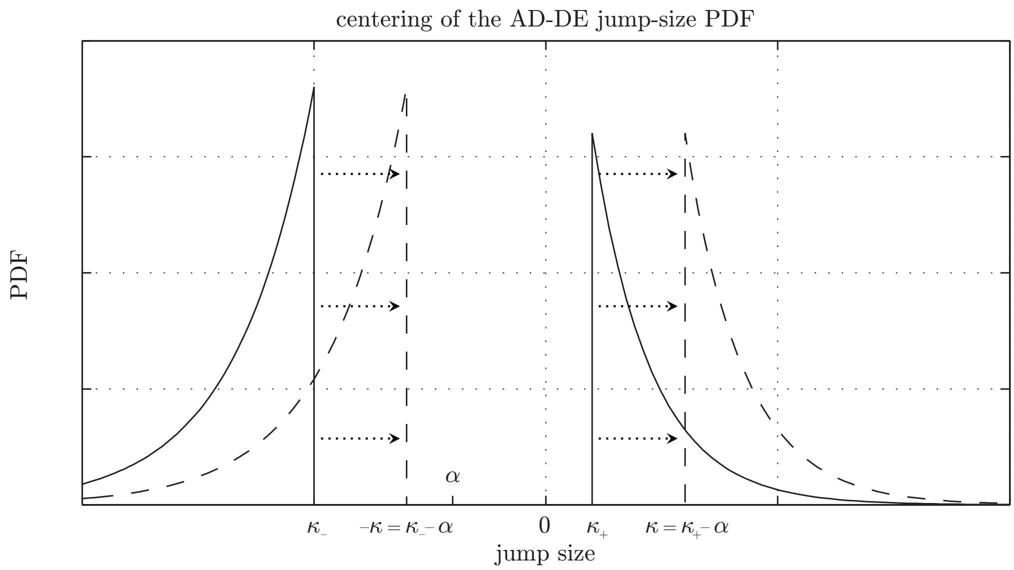

Figure 2: Centering of the AD-DE jump size PDF. The point $\alpha = (\kappa_+ - \kappa_-)/2$ is the midpoint of the interval $[\kappa_-, \kappa_+]$. Both tails are shifted by $-\alpha$ to have the origin as the new midpoint.

distributed. Consequently, the sequence of i.i.d. random variables $\left(\hat{Y}_i\right)_{i\in\mathbb{N}}$ given by

$$\hat{Y}_i \sim \begin{cases} \hat{\xi}^+ & \text{with probability } p \in [0,1] \\ -\hat{\xi}^- & \text{with probability } 1-p \end{cases}$$

follows an SD-DE distribution. Figure 2 illustrates that we obtain $\hat{Y}_i$ by adding the constant term $-\alpha$ to Y_i, thus centering the formerly asymmetric displacement around zero. Lemma 3 directly follows from the memorylessness property.

Lemma 3 (Distribution of $\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-$).
The distribution of $\hat{\xi}^+ - \hat{\xi}^-$ is given by

$$\hat{\xi}^+ - \hat{\xi}^- \sim \begin{cases} \xi^+ - \alpha - \kappa & \textit{with probability } \eta_-/\left(\eta_+ + \eta_-\right) \\ -\xi^- - \alpha + \kappa & \textit{with probability } \eta_+/\left(\eta_+ + \eta_-\right) \end{cases}.$$

Proof By explicitly computing the corresponding PDFs, we find that the distribution of $\hat{\xi}^+ - \hat{\xi}^-$ $(\hat{\xi}^+ - \hat{\xi}^-)$ conditional on $\hat{\xi}^+ > \hat{\xi}^-$ $(\hat{\xi}^+ < \hat{\xi}^-)$ is that of an exponential random variable with rate parameter η_+ (η_-). $\square$

Now let

$$A(n,m) = \sum_{i=1}^{n} \xi_i^+ - \sum_{j=1}^{m} \xi_j^-$$

for $n, m \in \mathbb{N}$. Lemma 4 characterizes the distribution of $A(n,m)$ in terms of a probability-weighted average over random variables of the form $A(k,0)$ or $A(0,l)$, that is, sums only involving either ξ^+ or ξ^-. This step is important as we can explicitly compute the joint distribution of a normal random variable and either $A(k,0)$ or $A(0,l)$, but in general not $A(n,m)$.

Lemma 4 (Distribution of $A(n,m)$).
The distribution of $A(n,m)$ admits the decomposition

$$A(n,m) \sim \begin{cases} A(k,0) + (n-k)(\alpha+\kappa) + m(\alpha-\kappa) & \text{with probability } \tilde{p}(n,m,k) \\ & \text{for } k = 1,2,\ldots,n \\ A(0,l) + n(\alpha+\kappa) + (m-l)(\alpha-\kappa) & \text{with probability } \tilde{q}(n,m,l) \\ & \text{for } l = 1,2,\ldots,m \end{cases},$$

where

$$\begin{aligned} \tilde{p}(n,m,k) &= \binom{n-k+m-1}{m-1} \left(\frac{\eta_+}{\eta_+ + \eta_-}\right)^{n-k} \left(\frac{\eta_-}{\eta_+ + \eta_-}\right)^{m}, \\ \tilde{q}(n,m,l) &= \binom{n-1+m-l}{n-1} \left(\frac{\eta_+}{\eta_+ + \eta_-}\right)^{n} \left(\frac{\eta_-}{\eta_+ + \eta_-}\right)^{m-l}. \end{aligned}$$

Proof Lemma 3 allows us to express $A(n,m)$ as a mixture over $A(n,m-1)-\alpha-\kappa$ and $A(n-1,m)-\alpha+\kappa$. This step is iteratively repeated until we are left with an expression of either the form $A(k,0)$ or $A(0,l)$ plus some deterministic function of n and m. All details are given in Appendix B.1. □

Let $\tau_n = \inf\{t \geq 0 : N_t = n\}$ for $n = 1,2,\ldots$ be the arrival time of the n-th jump and consider the random variable $X^j_{\tau_n} = \sum_{i=1}^{n} Y_i$. We can interpret $X^j_{\tau_j}$ as randomly taking the values $A(i, n-i)$ plus some constant for $i = 0,1,\ldots,n$, following a binomial $\mathcal{B}(n,p)$ distribution. Proposition 3, which generalizes Proposition B.1 in Kou (2002), p. 1098, to AD-DE distributed jump sizes, is the main result of this section.

Proposition 3 (Distribution of $X^j_{\tau_n}$).
The distribution of $X^j_{\tau_n}$ admits the decomposition

$$X^j_{\tau_n} \sim \begin{cases} \sum_{j=1}^{k} \xi_j^+ + (n-k)\alpha & \textit{with probability } \hat{p}(i,n)\tilde{p}(i,n-i,k) \\ +(2i-n-k)\kappa & \textit{for } k = 1,2,\ldots,n-1;\ i = k, k+1, \ldots, n-1 \\ \sum_{j=1}^{n} \xi_j^+ & \textit{with probability } \hat{p}(n,n) \\ -\sum_{j=1}^{l} \xi_j^- + (n-l)\alpha & \textit{with probability } \hat{p}(i,n)\tilde{q}(i,n-i,l) \\ +(2i-n+l)\kappa & \textit{for } l = 1,2,\ldots,n-1;\ i = 1,2,\ldots,n-l \\ -\sum_{j=1}^{n} \xi_j^- & \textit{with probability } \hat{p}(0,n) \end{cases},$$

where

$$\hat{p}(i,n) = \binom{n}{i} p^i (1-p)^{n-i}$$

is the binomial probability.

Proof This follows immediately from Lemma 4. We replace $A(n,m)$ by $A(i,n-i)$ and multiply by the probability $\hat{p}(i,n-i)$ of observing i up-jumps when the total number of jumps is n. Note that $A(k,0)$ ($A(0,l)$) appears only in the decompositions of $A(i,n-i)$ for $i = k, k+1, \ldots, n-1$ ($i = 1,2,\ldots,n-l$). $\square$

5.2 Auxiliary Results for Gamma Tails

In this section, we generalize the auxiliary results from Section 5.1 to a sum of random variables AD-DG distributed random variables.

Lemma 5 (Sums of Displaced Exponential Random Variables).
Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. displaced exponential random variables such that $A_i - \kappa/\delta \sim \mathcal{E}(\eta)$. Define

$$B = \sum_{i=1}^{\delta} A_i.$$

Then B follows a displaced gamma distribution, that is $B - \kappa \sim \Gamma(\delta, \eta)$.

Proof This follows immediately from the well-known special case result for zero displacements; see Chapter I.3 in Feller (1970), pp. 8–11. □

Now let $\xi_i^+ - \kappa_+/\delta_+ \sim \mathcal{E}(\eta_+)$ and $\xi_i^- + \kappa_-/\delta_- \sim \mathcal{E}(\eta_-)$ be sequences of i.i.d. exponential random variables. Then, by Lemma 5

$$Y_i \sim \begin{cases} \sum_{i=1}^{\delta_+} \xi_i^+ & \text{with probability } p \in [0,1] \\ -\sum_{i=1}^{\delta_-} \xi_i^- & \text{with probability } 1-p \end{cases} .$$

We can thus interpret $X_{\tau_n}^j$ as randomly taking the values $A(i\delta_+, (n-i)\delta_-)$ plus some constant for $i = 0, 1, \ldots, n$ following a binomial $\mathcal{B}(n,p)$ distribution. The following result then generalizes and replaces Proposition 3.

Proposition 3* (Distribution of $X_{\tau_n}^j$).
The distribution of $X_{\tau_n}^j$ admits the decomposition

$$X_{\tau_n}^j \sim \begin{cases} \sum_{j=1}^{k\delta_+} \xi_j^+ + (n-k)\alpha & \textit{with probability } \hat{p}(i,n)\tilde{p}(i,n-i,k) \\ +(2i-n-k)\kappa & \textit{for } k = 1,2,\ldots,n-1;\ i = k, k+1, \ldots, n-1 \\ \sum_{j=1}^{n\delta_+} \xi_j^+ & \textit{with probability } \hat{p}(n,n) \\ -\sum_{j=1}^{l\delta_-} \xi_j^- + (n-l)\alpha & \textit{with probability } \hat{p}(i,n)\tilde{q}(i,n-i,l) \\ +(2i-n+l)\kappa & \textit{for } l = 1,2,\ldots,n-1;\ i = 1,2,\ldots,n-l \\ -\sum_{j=1}^{n\delta_-} \xi_j^- & \textit{with probability } \hat{p}(0,n) \end{cases} ,$$

where the probabilities $\hat{p}(i,n)$ are as given in Proposition 3.

Proof The proof is fully analogous to the one given for Proposition 3. □

5.3 Tail Probabilities

We first define

$$Z_t(A,n) = \gamma t + \sigma W_t + \sum_{i=1}^{n} A_i,$$

where $(A_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with the same distribution as the random variable A. For example, $X_t = Z_t(Y, N_t)$ corresponds to a jump-diffusion process whose jump size distribution follows an AD-DG law. Throughout this section, we repeatedly need a slight generalization of Proposition B.3 in Kou (2002), p. 1100.

Lemma 6 (Distribution of $Z_t(A, n)$).
Let ξ be an exponential random variable with arrival rate η. Then for every $n \geq 1$, we have

$$\mathbb{P}\{Z_t(\pm\xi, n) \in \mathrm{d}x\} = \frac{(\sigma\eta\sqrt{t})^n}{\sigma\sqrt{2\pi t}} \exp\left\{\frac{1}{2}(\sigma\eta)^2 t \mp (x-\gamma t)\eta\right\} \mathrm{Hh}_{n-1}\left(\mp\frac{x-\gamma t}{\sigma\sqrt{t}} + \sigma\eta\sqrt{t}\right)\mathrm{d}x$$

and

$$\mathbb{P}\{Z_t(\pm\xi_1, n) \geq x\} = \frac{(\sigma\eta\sqrt{t})^n}{\sigma\sqrt{2\pi t}} \exp\left\{\frac{1}{2}(\sigma\eta)^2 t\right\} \mathrm{I}_{n-1}\left(x-\gamma t; \mp\eta, \mp\frac{1}{\sigma\sqrt{t}}, -\sigma\eta\sqrt{t}\right),$$

where

$$\begin{aligned}
\mathrm{Hh}_{-1}(x) &= \mathrm{e}^{-x^2/2}, \\
\mathrm{Hh}_0(x) &= \sqrt{2\pi}\Phi(-x), \\
\mathrm{Hh}_n(x) &= \frac{1}{n!}\int_x^\infty (t-x)^n \mathrm{e}^{-t^2/2}\mathrm{d}t \qquad n = 1, 2, \ldots, \\
\mathrm{I}_n(c; \alpha, \beta, \delta) &= \int_c^\infty \mathrm{e}^{\alpha x}\mathrm{Hh}_n(\beta x - \delta)\mathrm{d}x \qquad n = 0, 1, \ldots.
\end{aligned}$$

Proof See the proof of Proposition B.3 in Kou (2002), p. 1100. □

The Hh_n-function is a special function from mathematical physics. Its properties are discussed in detail in Abramowitz and Stegun (1972) and Kou (2002). Section 7.2 in Abramowitz and Stegun (1972), pp. 299-300, establishes the connection between the Hh_n-function and the error function and gives an iterative recurrence relation for the latter. In the same spirit, Proposition B.2 in Kou (2002), p. 1099, derives a closed-form expression for the I_n-function in terms of finite sums over the Hh_m-function for $m = 1, 2, \ldots, n$ evaluated at the same point. It is valid for all parameter combinations that are relevant for our purposes. For practical implementations, we choose these representations as they are both exact and can be implemented efficiently without the need for numerical quadrature.

The following theorem is an extension of Theorem B.1 in Kou (2002), p. 1098, to AD-DG distributed jump sizes. It represents our main result.

Theorem 1 (Tail Probability of X_t).
The upper tail probability of the AD-DG process $X_t = Z_t(Y, N_t)$ is given by

$$\begin{aligned}
&\mathbb{P}\{X_t \geq x\} \\
= \ &\mathbb{P}\{N_t = 0\}\,\mathbb{P}\{Z_t(\cdot, 0) \geq x\} \\
&+\sum_{n=1}^{\infty} \mathbb{P}\{N_t = n\} \Big(\mathbb{P}\left\{Z_t\left(\xi^+ - \kappa_+/\delta_+, n\delta_+\right) + n\kappa_+ \geq x\right\} \hat{p}(n,n) \\
&+\mathbb{P}\left\{Z_t\left(-\xi^- - \kappa_-/\delta_-, n\delta_-\right) + n\kappa_- \geq x\right\} \hat{p}(0,n) \\
&+\sum_{k=1}^{n-1}\sum_{i=k}^{n-1} \left(\mathbb{P}\left\{Z_t\left(\xi^+ - \kappa_+/\delta_+, k\delta_+\right) + i\kappa_+ + (n-i)\kappa_- \geq x\right\} \hat{p}(i,n)\tilde{p}(i, n-i, k)\right. \\
&\left.+\mathbb{P}\left\{Z_t\left(-\xi^- - \kappa_-/\delta_-, k\delta_-\right) + (n-i)\kappa_+ + i\kappa_- \geq x\right\} \hat{p}(n-i,n)\tilde{q}(n-i,i,k)\right)\Big),
\end{aligned}$$

where the probabilities on the right-hand side are given in Lemmata 4, 6 and Proposition 3, and

$$\mathbb{P}\{N_t = n\} = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}$$

is the Poisson probability mass function.

Proof The proof is given in Appendix B.2. □

While this expression looks quite formidable, it is composed entirely of elementary functions. Thus, it can be evaluated readily in standard programming languages. The corresponding PDF can be expressed in terms of Hh-functions and is immediately obtained by differentiating the upper tail probability. When implementing this formula, we need to truncate the infinite summation at some level $n_{\max}$ which has to be determined such that the truncation error does not exceed a predefined threshold. The summands quickly converge to zero, thanks to the factorial term in the denominator of the Poisson probability mass function. Lemma 7 provides the necessary results.

Lemma 7 (Truncation Error Bound).
The truncation error induced by computing the upper tail probability based on the first $n_{\max}$

terms only is bounded by

$$\sum_{n_{\max}+1}^{\infty} \mathbb{P}\left\{N_t = n\right\} \mathbb{P}\left\{Z_t\left(Y, n\right) \geq x\right\} \leq \frac{\gamma\left(n_{\max}+1, \lambda t\right)}{n_{\max}!},$$

where

$$\gamma(s, x) = \int_0^x t^{s-1} \mathrm{e}^{-t} \mathrm{d}t$$

is the lower incomplete gamma function.

Proof Since $\mathbb{P}\left\{Z_t\left(Y, n\right) \geq x\right\} \in [0, 1]$, we can bound the truncation error by $\mathbb{P}\left\{N_t > n_{\max}\right\}$, which yields the given expression. □

It should be noted that the error bound in Lemma 7 is of course not unique and there might exist better bounds. The inverse problem of finding the smallest $n_{\max}$ such that the truncation error is below a fixed absolute threshold can be solved by using the Poisson inverse cumulative distribution function.

5.4 European Plain Vanilla Options

Let $t \geq 0$ be the current point in time and $C = \{C_t : t \in [0, T]\}$ be the price process of a European plain vanilla call option on the spot asset S with maturity in $T \in [0, T^*]$ and terminal payoff $C_T = (S_T - K)^+$. $B(\cdot, T) = \{B(t, T) : t \in [0, T]\}$ is the price process of a zero-coupon bond. Denote the time-to-maturity by $\tau = T - t$.

Proposition 4 (European Plain Vanilla Call Options on Spot Assets).
Let

$$\Lambda\left(x; t, \gamma, \sigma, \lambda, p, \delta_+, \delta_-, \eta_+, \eta_-, \kappa_+, \kappa_-\right) := \mathbb{P}\left\{X_t \geq x\right\},$$

be the upper tail probability of the AD-DG jump-diffusion process X as given in Theorem 1. The value of the European plain vanilla call option is given by

$$C_t = S_t \Lambda_1 - B(t, T) K \Lambda_2,$$

where

$$\begin{aligned}
\Lambda_1 &:= \Lambda\left(\ln\left(\frac{K}{S_t}\right); \tau, \gamma^S, \sigma, \lambda^S, p^S, \delta_+, \delta_-, \eta_+^S, \eta_-^S, \kappa_+, \kappa_-\right), \\
\Lambda_2 &:= \Lambda\left(\ln\left(\frac{K}{S_t}\right); \tau, \gamma^*, \sigma, \lambda^*, p^*, \delta_+, \delta_-, \eta_+^*, \eta_-^*, \kappa_+, \kappa_-\right)
\end{aligned}$$

The risk-neutral drift γ^* is given by

$$\gamma^* = r - \frac{1}{2}\sigma^2 - \lambda^* \left(\phi_Y^*(-\mathrm{i}) - 1\right)$$

and

$$\gamma^S = \gamma^* + \sigma^2, \qquad \lambda^S = \lambda^* \phi_Y^*(-\mathrm{i}), \qquad p^S = \frac{p^* \phi_{Y+}^*(-\mathrm{i})}{\phi_Y^*(-\mathrm{i})}, \qquad \eta_\pm^S = \eta_\pm^* \mp 1.$$

All parameters under the risk-neutral probability measure are as determined in Section 4.

Proof This follows from applying the risk-neutral pricing formula performing a change of numéraire to obtain an expression in terms of two probabilities which can be readily computed using Theorem 1. See Appendix B.3 for all details. □

The resulting formula is similar in structure to the Black and Scholes (1973) formula with two terms representing the expected values of the asset and the strike price payment upon exercise. In the special case when $\delta_\pm = 1$ and $\kappa_\pm = 0$, the valuation formula reduces to the one given in Theorem 2 in Kou (2002), p. 1095. For the dynamic risk-management of derivative books, the availability of analytical solutions for the hedge ratio and other Greeks is just as important as the possibility to rapidly re-evaluate the option positions themselves.

Lemma 8 (Delta and Gamma of European Plain Vanilla Call Options).
The delta and gamma of the European plain vanilla call option are given by

$$\Delta_t^C = \Lambda_1 \qquad \text{and} \qquad \Gamma_t^C = -\frac{1}{S_t}\Lambda_1',$$

where

$$\Lambda'\left(x; t, \gamma, \sigma, \lambda, p, \delta_+, \delta_-, \eta_+, \eta_-, \kappa_+, \kappa_-\right) \mathrm{d}x := \mathbb{P}\left\{X_t \in \mathrm{d}x\right\}$$

is the PDF of the AD-DG jump-diffusion process X.

Proof While the expression for delta seems obvious given the call price formula, care has to be taken when taking the partial derivative of C_t with respect to S_t as both probabilities Λ_1 and Λ_2 are functions of the asset price as well. However, we do not need to explicitly apply the chain rule of differentiation but can instead use a homogeneity result; see Theorem 9 in Merton (1973), p. 149. □

Lemma 9 (European Plain Vanilla Put Options on Spot Assets).

The price of the corresponding European plain vanilla put option is given by

$$P_t = B(t,T)K\left(1-\Lambda_2\right) - S_t\left(1-\Lambda_1\right)$$

with Greeks

$$\Delta_t^P = \Lambda_1 - 1 \qquad \textit{and} \qquad \Gamma_t^P = -\frac{1}{S_t}\Lambda_1'.$$

Proof This follows immediately from the put-call parity relationship. □

Corollary 3 (European Plain Vanilla Options on Forwards).

Let $F_S(\cdot,U) = \{F_S(t,U) : t \in [0,T^*]\}$ be the price process of a forward contract on the asset S with maturity in $U \geq T$. The prices of the European plain vanilla call and put options on $F_S(\cdot,U)$ are given by

$$\begin{aligned} C_t &= B(t,T)\left(F_S(t,U)\Lambda_1 - K\Lambda_2\right), \\ P_t &= B(t,T)\left(K\left(1-\Lambda_2\right) - F_S(t,U)\left(1-\Lambda_1\right)\right). \end{aligned}$$

where

$$\gamma^* = -\frac{1}{2}\sigma^2 - \lambda^*\left(\phi_Y^*(-\mathrm{i}) - 1\right)$$

all remaining parameters are as given in Proposition 4 with $F_S(t,U)$ replacing S_t.

Proof We first note that the forward price $F_S(t,U) = S_t/B(t,U)$ is a $(\mathbb{P}^*,\mathbb{F})$-martingale and its logarithm has the drift $\gamma^* - r$. The proof is then fully analogous to that of Proposition 4. □

6 Parameter Estimation

This section proposes a methodology to estimate the physical parameters of the AD-DG jump-diffusion model based on the time series of logarithmic returns and briefly discusses alternative estimation approaches that have been proposed in the literature. We argue that, in addition to its desirable statistical properties, ML requires no ad-hoc decisions about the construction of the moment conditions as does the GMM. Furthermore, it can be evaluated efficiently through a vectorized implementation of the COS method.

6.1 Estimation Framework

The ML estimator is appealing, due to its consistency, asymptotic normality and efficiency; see Theorem 16.1 in Greene (2008), p. 487. Sørensen (1991) shows that these general properties continue to hold for the inference of jump-diffusion processes under mild regularity conditions. Aït-Sahalia (2002) considers the problem of estimating a general Itō diffusion process based on discrete samples. Aït-Sahalia (2004) establishes that the asymptotic variance of the diffusion coefficient estimator is not deteriorated by the presence of compound Poisson type jumps within the ML framework. He further confirms our intuition that in the limit of infinitely frequent sampling, the jumps and diffusion components can be perfectly disentangled. Bates (2006) adopts the ML approach to the estimation of latent affine processes which allow for both a time-varying volatility and jump intensity. Ramezani and Zeng (1999, 2007) apply ML to estimate the parameters of the DE model based on a two year long sample of daily stock returns. Their empirical results support the hypothesis that upward- and downward-jumps exhibit different characteristics.

A common alternative approach to estimating the physical model parameters of models with a known characteristic function is the Hansen (1982) GMM. The population orthogonality conditions can then be constructed from matching the empirical and model implied central moments and tail probabilities. The model considered in this paper admits closed-form solutions for both of these types of moment conditions. Within any iteration of the corresponding optimization problem, each of these computationally relatively expensive expressions has to be evaluated only once for the full sample. In contrast to this, the conditional likelihood has to be computed for every observation in the sample separately. This suggests that the GMM might be computationally more efficient.

There are three main arguments against using the GMM for estimation. First, for the GMM to be asymptotically efficient, a consistent estimator of the covariance matrix of the moment conditions is needed, which requires iterated estimations. In the presence of serial correlation in the time series of returns, an autocorrelation consistent estimator of the covariance matrix has to be computed using, for example, the method suggested by Newey and West (1987). Second, GMM is typically less efficient for finite samples than the ML. Third, the particular choice of the moment conditions and their total number is rather ad-hoc. It is not clear how to optimally choose the central moments and tail values to minimize the variance of the estimator.

Ball and Torous (1983) use a GMM procedure to estimate the Merton (1976) jump-diffusion model. They construct the moment conditions from the first six cumulants of the return process. Ramezani and Zeng (1999) estimate the Kou (2002) DE model using both the cumulant based GMM and ML approaches. In accordance with Press (1968) and Beckers (1981), they find that the cumulant method, unlike ML, sometimes yields economically unreasonable parameter estimates. Furthermore, using higher order moments might be problematic for small samples as the corresponding empirical moment estimates become increasingly noisy.

Singleton (2001) and Chacko and Viceira (2003) suggest a closely related approach based on the empirical characteristic function. They construct the moment conditions from the real and imaginary parts of the characteristic function evaluated at a predefined set of transform parameters. This is particularly appealing when even the cumulants cannot be computed in closed-form. However, their approach suffers from the same shortcomings as the one previously discussed. In particular, it is not clear how to optimally choose the set of transform parameter values.

Finally, we briefly discuss the estimation procedure proposed by Detering et al. (2013) for the AD-DE model. The authors start by setting the displacement term κ equal to the average of the absolute logarithmic returns corresponding to the α and $1-\alpha$ quantiles of the empirical distribution function. They suggest the use of a value of $\alpha = 1\%$ and thus implicitly categorize all absolute logarithmic returns greater than α as jumps. Next, λ is set to be equal to the total number of jumps divided by the total number of observations.

However, it is clear that once the level of α has been fixed, the estimate of λ is just equal to the average number of trading days per year times α. The tail parameters η_+ and η_- are chosen such that they fit the mean of the returns classified as jumps. Finally, the diffusive variance is given by the total sample variance minus the variance of the compound Poisson component, whose parameters have already been determined.

While it can be extended to AD-DG distributed jumps in a straightforward fashion, this approach seems rather ad-hoc. Its parameter estimates fully depend on the discretionary choice of the quantile α. Another important shortcoming is, that it provides no estimate of the covariance matrix of the parameter estimates. Thus, it is not possible to construct confidence intervals or to conduct further hypothesis tests.

6.2 Computational Aspects

As indicated in the previous section, the computational bottleneck of ML estimation is the high number of evaluations of the conditional likelihood function that it requires. To mitigate this problem and significantly accelerate the estimation, we employ a vectorized implementation of the Fang and Oosterlee (2008) COS method to simultaneously obtain the PDF on an equally spaced logarithmic return grid. Similar to the algorithm proposed by Kilin (2011), this approach caches the computationally expensive evaluations of the characteristic function. The grid values can then be interpolated to match the observations in the sample. A major advantage of the COS method over the Cooley and Tuckey (1965) FFT is that it allows to freely choose the spatial grid. As an alternative, we consider the Bailey and Swarztrauber (1991, 1994) fractional FFT. This algorithm also uncouples the grid sizes in the spacial and frequency domains but still requires them to be equally spaced. In agreement with Fang and Oosterlee (2008) we find that, for a given level of accuracy, the COS method is computationally more efficient than the fractional FFT.

When numerically maximizing the sample likelihood function, great care has to be taken in selecting an appropriate optimization routine to ensure convergence to the global minimum. Through numerical experiments, we find that the convergence of standard gradient-based algorithms strongly depends upon the set of starting values chosen. This suggests the existence of multiple local maxima, and consequently a non-convex nature of the optimization problem at hand. Kiefer (1978) shows within a similar mixture density setting, that the likelihood function may exhibit local optima when the sample size is finite. Differential evolution is a population based heuristic optimization routine developed by Storn and Price (1997) that does not rely on a set of strong assumptions about the underlying optimization problem. It uses a stochastic search strategy and is guaranteed to converge to the global optimum in the limit, though this comes at the cost of a high number of objective function evaluations. Ardia et al. (2011) estimate the Merton (1976) jump-diffusion model through ML using differential evolution and find that it outperforms all convex optimization routines considered. Gilli and Schumann (2010, 2012) provide further applications of differential evolution in financial econometrics.

7 Empirical Results

This section describes the data set used and discusses the estimation results. All tables can be found in Appendix C.

7.1 Data Set

We estimate the parameters of the AD-DG jump-diffusion model based on the 30 year historical daily logarithmic returns from January 1, 1982 to December 31, 2011. All data is obtained from Bloomberg. The assets fall into three main categories: (i) equity indices, (ii) commodity indices and (iii) foreign exchange (FX) rates and spot precious metals. Table 1 provides the summary statistics.

Category (i) consists of: DAX 30 (Germany), Dow Jones Industrial (USA), Hang Seng (Hong Kong), MSCI World (global), NASDAQ Composite (USA), Nikkei 225 (Japan), S&P 500 (USA) and TOPIX (Japan). All of these indices are market capitalization weighted and, except for the DAX 30, are calculated as price indices. The dividend payments can thus be well approximated by a continuous dividend yield since all of these indices are highly diversified and the payment dates are spread throughout the year. Our objective is to estimate the parameters of the different jump size distribution specifications but not the mean return. Consequently, the non-zero dividend yield is irrelevant for our purposes.

Category (ii) corresponds to the S&P GSCI Excess Return (ER) commodity index and three of its sub-indices. The basic index constituents are commodity futures contracts, weighted by their relative world production. The S&P GSCI ER index itself can be broken down into five sub-indices corresponding to the main classes of commodities: S&P GSCI Energy ER, S&P GSCI Industrial Metals ER, S&P GSCI Precious Metals ER, S&P GSCI Agriculture ER and S&P GSCI Livestock ER. This study excludes the S&P GSCI Energy ER and S&P GSCI Livestock ER indices, since no historical data is available on Bloomberg for the first years of the considered time span.

Finally, category (iii) contains the three major spot exchange rate pairs EUR/USD, GBP/USD and USD/JPY as well as the spot prices of silver and gold. We remark that, despite the naming convention adopted, USD is the domestic currency in the quotation of the EUR/USD and GBP/USD exchange rate pairs but the foreign currency in USD/JPY quotation.

7.2 Estimation Results

Table 2 shows the parameter estimates and hypotheses tests for equity indices. We make two key observations. First, without exception, both shape parameter estimates are equal to one. Consequently, the historical return distribution is consistent with the AD-DE model and the generalization to AD-DG distributed jump sizes provides no further improvement in the fit. Second, and again for all equity indices, both positive and negative displacement terms are individually significant at the 1% level. However, for some equity indices, such as the Hang Seng and the S&P 500, one of the two displacement terms is economically insignificant. Both the null hypothesis $\mathcal{H}_0^{(1)} : \kappa_+ + \kappa_- = 0$ of symmetric displacements as well as the null hypothesis $\mathcal{H}_0^{(2)} : \kappa_+ = \kappa_- = 0$ of jointly zero displacements can be rejected at the 1% level in all cases. We conclude that the asymmetric displacements succeed at capturing a statistical property that is consistently present in equity index returns. We can thus reject both the DE and the SD-DE model in favor of the AD-DE dynamics. We further note, that on average only 25.35% of the total historical variance can be attributed to the diffusion component. Except for the three Asian indices, downward jumps are significantly more frequent. For all equity indices

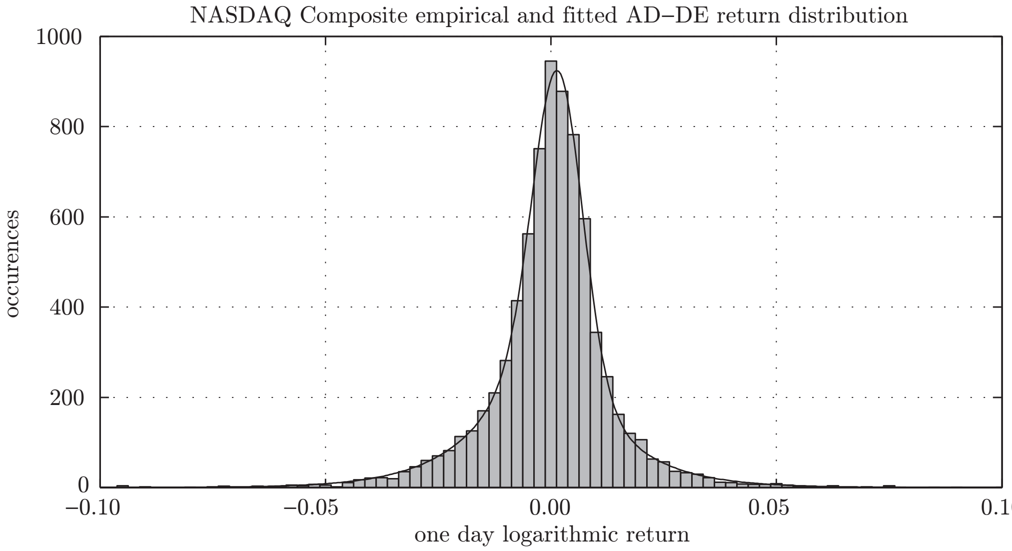

Figure 3: Fitted AD-DE density for the NASDAQ Composite index based on daily logarithmic returns from January 1, 1982 to December 31, 2011. The parameters are $\sigma = 8.59\%$, $\lambda = 174.18$, $p = 43.87\%$, $\eta_+ = 98.96$, $\eta_- = 104.36$, $\kappa_+ = 0.06\%$ and $\kappa_- = -0.29\%$.

except for the DAX 30 and the NASDAQ Composite, η_- is smaller compared to η_+, thus implying a slower decay of the lower tail the jump size distribution.

As shown in Tables 3 and 4, most of the above findings for equity indices also hold for commodity indices, exchange rates and precious metals. In particular, all displacement terms are individually significant and the 1% level for all assets and we can again reject the null hypotheses $\mathcal{H}_0^{(1)}$ and $\mathcal{H}_0^{(2)}$ at the 1% level in all cases. While the AD-DE model still provides the best fit to the times series of logarithmic returns for most assets, we find that the historical return distribution of the S&P GSCI Agriculture is consistent with the AD-DG dynamics. The upper tail now follows a displaced gamma distribution with shape parameter $\delta_+ = 4$. The corresponding displacement term is significant at the 1% level though economically insignificant. This is not surprising since the mode of the gamma distribution is strictly positive for shape parameters greater than one. A likelihood ratio test for the restriction $\delta_+ = 1$ yields a p-value of 34.29% so that we cannot reject the AD-DE model in favor of the AD-DG model at the 10% level.

In summary, we find very strong empirical support for the AD-DE model across three different asset classes. The AD-DG dynamics only improve the fit to the historical return distribution for one asset. Our results are robust with respect to other estimation horizons. In particular, we obtain the same qualitative results for the 20 year sub-period from January 1, 1992 to December 31, 2011.

Care has to be taken when comparing our results to the studies by Ramezani and Zeng (1999, 2007), which use a slightly different parametrization of the DE model. Instead of estimating the jump frequency and the probability of an up-jump, they estimate the frequency of the independent up- and down-jumps directly. Furthermore, their jump frequency is expressed in number of jumps per trading day instead of per year. When accounting for these difference, then the parameter estimates are of the same order of magnitude. Compared to Detering et al. (2013), we find that the jump frequency for the SD-DE model implied in the time series of logarithmic returns is much higher than the approximately five jumps per year that the authors postulate.

8 Conclusion

This paper proposes a novel jump-diffusion model in which the jump sizes follow an AD-DG distribution and thus generalizes the DE model in two directions. Our dynamics are supported by an equilibrium economy which also implies a risk-neutral pricing measure. One of our main contributions is to show that the valuation problem for European plain vanilla options still admits a closed-form solution. Our model constitutes the most general jump-diffusion dynamics so far with this property. Through empirical test, we demonstrate that introducing asymmetrically displaced jumps is not only academically interesting but also reflects the statistical properties of asset returns. We estimate the model parameters based on a diverse sample of assets across equity indices, commodity indices and foreign exchange. For all assets in the sample, both displacement terms are individually and jointly significant at the 1% level. We can reject both the DE as well as the SD-DE model in favor of the AD-DG dynamics. We further find that the special case of AD-DE distributed jump sizes provides the best fit for almost all assets.

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A Appendix for Section 4

A.1 Esscher Transform Logarithmic Return Dynamics

This appendix contains the detailed proof of Proposition 1. While the first factor in the Radon-Nikodým derivative process changes drift of the Brownian motion W, the second changes the intensity and jump size distribution of the compound Poisson process X^j. We define a new process $\hat{W} = \{\hat{W} : t \in [0, T^*]\}$ by

$$\hat{W}_t = W_t - \beta\sigma t.$$

It then follows by Girsanov's theorem that $\hat{W}$ is a standard Brownian motion under $\hat{\mathbb{P}}(X, \beta)$; see Theorem III.5.1 in Karatzas and Shreve (1991). By substituting in the dynamics of X, we thus find that $\hat{\gamma} = \gamma + \beta\sigma^2$. Next, the characteristic function $\hat{\phi}_{X_t^j}(\omega)$ of the compound Poisson process X^j under $\hat{\mathbb{P}}(X, \beta)$ is given by

$$\begin{aligned} \hat{\phi}_{X_t^j}(\omega) &= \mathbb{E}_{\mathbb{P}}\left[\nu_t\big(\mathbb{P}, \hat{\mathbb{P}}\big) \exp\left\{\mathrm{i}\omega \sum_{i=1}^{N_t} Y_i\right\}\right], \\ &= \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\mathrm{i}(\omega - \mathrm{i}\beta) \sum_{i=1}^{N_t} Y_i - t\psi_{X^j}(-\mathrm{i}\beta)\right\}\right]. \end{aligned}$$

Here, we change the measure from $\hat{\mathbb{P}}$ to $\mathbb{P}$ and use the independence and martingale properties of the first factor in the Radon-Nikodým derivative process. We obtain

$$\hat{\phi}_{X_t^j}(\omega) = \frac{\phi_{X_t^j}(\omega - \mathrm{i}\beta)}{\phi_{X_t^j}(-\mathrm{i}\beta)}.$$

Applying Lemma 1 then yields

$$\hat{\psi}_{X^j}(\omega) = \lambda \int_{-\infty}^{+\infty} \left(\mathrm{e}^{\mathrm{i}\omega x} - 1\right) \mathrm{e}^{\beta x} f_Y(x) \mathrm{d}x.$$

While this expression closely resembles the characteristic exponent of a compound Poisson process, the last term in the integrand needs to be re-normalized to represent a valid PDF. This yields $\hat{\lambda} = \lambda \phi_Y(-\mathrm{i}\beta)$ and $\hat{f}_Y(x) = \mathrm{e}^{\beta x} f_Y(x)/\phi_Y(-\mathrm{i}\beta)$. We observe that the jump size PDF under $\hat{\mathbb{P}}(X, \beta)$ corresponds to the Esscher transform of the one under $\mathbb{P}$ with transform parameter β. The β-th exponential moment of X under $\mathbb{P}$ exists if $\eta_+ > \beta$

and $\eta_- > -\beta$. The set of admissible values for the transform parameter is thus given by $\mathcal{B} = (-\eta_-, \eta_+)$. Next, we observe that $f_Y(x)$ is a mixture of two natural exponential families with parameters $-\eta_+$ and η_-, respectively. It follows that $\hat{f}_Y(x)$ is the same natural exponential mixture family with parameters $-\hat{\eta}_+$ and $\hat{\eta}_-$, where $\hat{\eta}_\pm = \eta_\pm \mp \beta$. Finally, the $\hat{\mathbb{P}}(X, \beta)$ probability of an up-jump is given by $\hat{p} = p\phi_{Y^+}(-\mathrm{i}\beta)/\phi_Y(-\mathrm{i}\beta)$, where the term $\phi_{Y^+}(-\mathrm{i}\beta)$ arises from re-normalizing the upper tail of $\hat{f}_Y(x)$. It is straightforward to check that $\hat{p} \in [0, 1]$ and thus constitutes a valid weight.

B Appendix for Section 5

B.1 Distribution of $A(n, m)$

This appendix contains the detailed proof of Lemma 4. It follows the same steps as the one given for Lemma B.1 in Kou (2002), pp. 1098–1099, carefully taking the additional terms into account. We first apply Lemma 3 to decompose the distribution of

$$A(n, m) = A(n-1, m-1) + \hat{\xi}_n^+ - \hat{\xi}_m^- + 2\alpha$$

into the mixture

$$A(n, m) \quad \sim \quad \begin{cases} A(n, m-1) + \alpha - \kappa & \text{with probability } \eta_- / (\eta_+ + \eta_-) \\ A(n-1, m) + \alpha + \kappa & \text{with probability } \eta_+ / (\eta_+ + \eta_-) \end{cases}.$$

We can iteratively repeat this step until we are left with an expression of either the form $A(k, 0)$ or $A(0, l)$ plus some deterministic function of α and κ. Consider the number of ξ^+ and ξ^- in the sums to be the position of a random walk on an integer lattice starting at $\{n, m\}$ in the first quadrant and stopping once it hits either of the two axes. Each step reduces the number of either ξ^+ or ξ^- in the sums by one and thus corresponds to moving either left or down in the lattice. Consequently, only the nodes $\{k, 0\}$ ($\{0, l\}$) for $k = 1, 2, \ldots, n$ ($l = 1, 2, \ldots, m$) can be reached on the x-axis (y-axis). In particular, the node $\{0, 0\}$ can never be reached. Immediately before hitting the node $\{k, 0\}$ ($\{0, l\}$), the random walk has to be at $\{k, 1\}$ ($\{1, l\}$) and then take a down (left) step. It has to take a total of $n - k$ ($n - 1$) left and $m - 1$ ($m - l$) down steps to move from $\{n, m\}$ to $\{k, 1\}$

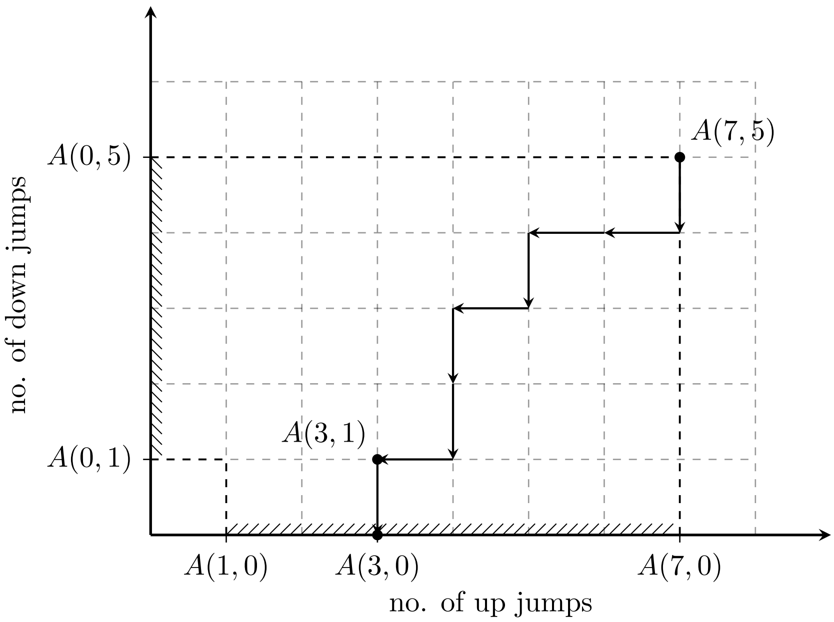

Figure 4: Sample path of the random walk starting at $A(7,5)$ and moving left four and down five steps to stop at $A(3,0)$. The thick dotted lines represent the boundary of the domain on which the random walk lives. The dashed lines correspond to the parts of the boundary on which the random walk is stopped.

$(\{1,l\})$ and there are $\binom{n-k+m-1}{m-1}$ $(\binom{n-1+m-l}{n-1})$ such paths. The factor multiplying κ is equal to the difference between the number of left- and down-steps. The term multiplying α is equal to the total number of iteration steps. Figure 4 illustrates the domain and a sample path of the random walk starting at $A(7,5)$.

B.2 Jump-Diffusion Upper Tail Probability

This appendix contains the detailed proof of Theorem 1. We start by expression the upper tail probability of the jump-diffusion process X as a probability weighted sum over the tail probabilities conditional on a fixed total number of jumps, that is

$$\mathbb{P}\left\{Z_t\left(Y,N_t\right)\geq x\right\}=\sum_{n=0}^{\infty}\mathbb{P}\left\{N_t=n\right\}\mathbb{P}\left\{Z_t(Y,n)\geq x\right\}.$$

Using Proposition 3 and Lemma 6, we can express the summands in terms of I_n-functions as

$$\begin{aligned}
\ldots \quad = \quad & \mathbb{P}\{N_t = 0\}\,\mathbb{P}\{Z_t(\cdot,0) \geq x\} \\
& + \sum_{n=1}^{\infty} \mathbb{P}\{N_t = n\} \Big(\mathbb{P}\left\{Z_t\left(\xi^+, n\delta_+\right) \geq x\right\} \hat{p}(n,n) + \mathbb{P}\left\{Z_t\left(-\xi^-, n\delta_-\right) \geq x\right\} \hat{p}(0,n) \\
& + \sum_{k=1}^{n-1} \Big(\sum_{i=k}^{n-1} \mathbb{P}\left\{Z_t\left(\xi^+, k\delta_+\right) + (n-k)\alpha + (2i-n-k)\kappa \geq x\right\} \hat{p}(i,n)\tilde{p}(i,n-i,k) \\
& + \sum_{i=1}^{n-k} \mathbb{P}\left\{Z_t\left(-\xi^-, k\delta_-\right) + (n-k)\alpha + (2i-n+k)\kappa \geq x\right\} \hat{p}(i,n)\tilde{q}(i,n-i,k) \Big) \Big).
\end{aligned}$$

We can merge the inner two summations by changing the order of summation which corresponds to replacing i by $n-i$ in the second sum and get

$$\begin{aligned}
\ldots \quad = \quad & \mathbb{P}\{N_t = 0\}\,\mathbb{P}\{Z_t(\cdot,0) \geq x\} \\
& + \sum_{n=1}^{\infty} \mathbb{P}\{N_t = n\} \Big(\mathbb{P}\left\{Z_t\left(\xi^+, n\delta_+\right) \geq x\right\} \hat{p}(n,n) + \mathbb{P}\left\{Z_t\left(-\xi^-, n\delta_-\right) \geq x\right\} \hat{p}(0,n) \\
& + \sum_{k=1}^{n-1} \sum_{i=k}^{n-1} \Big(\mathbb{P}\left\{Z_t\left(\xi^+, k\delta_+\right) + (n-k)\alpha + (2i-n-k)\kappa \geq x\right\} \hat{p}(i,n)\tilde{p}(i,n-i,k) \\
& + \mathbb{P}\left\{Z_t\left(-\xi^-, k\delta_-\right) + (n-k)\alpha - (2i-n-k)\kappa \geq x\right\} \hat{p}(n-i,n)\tilde{q}(n-i,i,k) \Big) \Big).
\end{aligned}$$

Also, note that the ξ^+ $(-\xi^-)$ are not standard (negative) exponential random variables. To make this explicit such that Lemma 6 can be directly applied, we write

$$\begin{aligned}
\ldots \quad = \quad & \mathbb{P}\{N_t = 0\}\,\mathbb{P}\{Z_t(\cdot,0) \geq x\} \\
& + \sum_{n=1}^{\infty} \mathbb{P}\{N_t = n\} \Big(\mathbb{P}\left\{Z_t\left(\xi^+ - \kappa_+/\delta_+, n\delta_+\right) + n\kappa_+ \geq x\right\} \hat{p}(n,n) \\
& + \mathbb{P}\left\{Z_t\left(-\xi^- - \kappa_-/\delta_-, n\delta_-\right) + n\kappa_- \geq x\right\} \hat{p}(0,n) \\
& + \sum_{k=1}^{n-1} \sum_{i=k}^{n-1} \Big(\mathbb{P}\left\{Z_t\left(\xi^+ - \kappa_+/\delta_+, k\delta_+\right) + i\kappa_+ + (n-i)\kappa_- \geq x\right\} \hat{p}(i,n)\tilde{p}(i,n-i,k) \\
& + \mathbb{P}\left\{Z_t\left(-\xi^- - \kappa_-/\delta_-, k\delta_-\right) + (n-i)\kappa_+ + i\kappa_- \geq x\right\} \hat{p}(n-i,n)\tilde{q}(n-i,i,k) \Big) \Big).
\end{aligned}$$

Here, we also substituted for $\alpha = \left(\kappa_+ + \kappa_-\right)/2$ and $\kappa = \left(\kappa_+ - \kappa_-\right)/2$.

B.3 European Plain Vanilla Call Options on Spot Assets

This appendix contains the detailed proof of Proposition 4. By the risk-neutral pricing formula, we have

$$C_t = B_t \mathbb{E}_{\mathbb{P}^*}\left[\left.\frac{C_T}{B_T}\right| \mathfrak{F}_t\right].$$

Here, we assume that $\mathbb{P}^*$ is a risk-neutral probability measure as defined in Section 4 such that the discounted asset price S_t/B_t is a $(\mathbb{P}^*, \mathbb{F})$-martingale. Expanding the payoff function yields

$$C_t = B_t \mathbb{E}_{\mathbb{P}^*}\left[\left.\frac{S_T}{B_T} 1\left\{S_T \geq K\right\}\right| \mathfrak{F}_t\right] - B(t,T) K \mathbb{P}^*\left\{S_T \geq K \mid \mathfrak{F}_t\right\}.$$

The second expression can be readily computed using Theorem 1. We obtain

$$\mathbb{P}^*\left\{S_T \geq K \mid \mathfrak{F}_t\right\} = \Lambda\left(\ln\left(\frac{K}{S_t}\right); \tau, \gamma^*, \sigma, \lambda^*, p^*, \delta_+, \delta_-, \eta_+^*, \eta_-^*, \kappa_+, \kappa_-\right).$$

Here, all parameters are specified under the risk-neutral probability measure. In particular,

$$\gamma^* = r - \frac{1}{2}\sigma^2 - \lambda^*\left(\phi_Y^*(-\mathrm{i}) - 1\right).$$

To compute the first expression, we change the numéraire from B to S. To this end, we define a new probability measure $\mathbb{P}^S$ equivalent to $\mathbb{P}^*$ on $[0, T^*]$ by

$$\frac{\mathrm{d}\mathbb{P}^S}{\mathrm{d}\mathbb{P}^*} = \frac{S_{T^*} B_0}{S_0 B_{T^*}} \qquad \mathbb{P}^*\text{-a.s.}$$

with associated Radon-Nikodým derivative process $\nu\left(\mathbb{P}^*, \mathbb{P}^S\right) = \left\{\nu_t\left(\mathbb{P}^*, \mathbb{P}^S\right) : t \in [0, T^*]\right\}$

$$\nu_t\left(\mathbb{P}^*, \mathbb{P}^S\right) = \left.\frac{\mathrm{d}\mathbb{P}^S}{\mathrm{d}\mathbb{P}^*}\right| \mathfrak{F}_t = \exp\left\{X_t - t\psi_X^*(-\mathrm{i})\right\} \qquad \mathbb{P}^*\text{-a.s.}$$

We recognize this expression as an Esscher transform of the risk process X with transform parameter $\beta = 1$ and $\mathbb{P}^*$ taking the role of the prior probability measure. From Proposition 1, it then immediately follows that X is an AD-DG jump-diffusion process under the new probability measure $\mathbb{P}^S$ with parameters

$$\gamma^S = \gamma^* + \sigma^2, \qquad \lambda^S = \lambda^*\phi_Y^*(-\mathrm{i}), \qquad p^S = \frac{p^*\phi_{Y+}^*(-\mathrm{i})}{\phi_Y^*(-\mathrm{i})}, \qquad \eta_\pm^S = \eta_\pm^* \mp 1.$$

Using the abstract Bayes rule, the first expression in the valuation equation becomes

$$B_t \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_T}{B_T} 1\left\{S_T \geq K\right\} \middle| \mathfrak{F}_t \right] = S_t \mathbb{P}^S \left\{ S_T \geq K \middle| \mathfrak{F}_t \right\}.$$

Again using Theorem 1 yields

$$\mathbb{P}^S \left\{ S_T \geq K \middle| \mathfrak{F}_t \right\} = \Lambda\left(\ln\left(\frac{K}{S_t}\right) \tau, \gamma^S, \sigma, \lambda^S, p^S, \delta_+, \delta_-, \eta_+^S, \eta_-^S, \kappa_+, \kappa_- \right).$$

C Appendix for Section 7

This appendix holds the detailed estimation results for Section 7.

Table 1: Summary statistics for the full data set used in the empirical analysis. Daily logarithmic returns are calculated on a close-to-close basis for the time span from January 1, 1982 to December 31, 2011. Means and standard deviations are annualized assuming 252 trading days per year.

Asset	Bloomberg Ticker	N	Mean	Stand. Dev.	Skew	Excess Kurt.	Min.	Max.
DAX 30	DAX	7,566	8.24%	22.48%	-0.30	6.22	-13.71%	10.80%
Dow Jones Industrial	INDU	7,568	8.75%	18.27%	-1.58	39.48	-25.63%	10.51%
Hang Seng	HSI	7,418	8.81%	28.45%	-2.07	46.35	-40.54%	17.25%
MSCI World	MXWO	7,822	6.73%	14.64%	-0.53	11.29	-10.36%	9.10%
NASDAQ Composite	CCMP	7,569	8.62%	22.30%	-0.24	8.26	-12.05%	13.25%
Nikkei 225	NKY	7,387	0.31%	22.55%	-0.30	8.83	-16.14%	13.23%
S&P 500	SPX	7,569	7.75%	18.61%	-1.21	26.89	-22.90%	10.96%
TOPIX	TPX	7,387	0.82%	20.12%	-0.36	9.94	-15.81%	12.86%
S&P GSCI	SPGSCIP	7,571	2.28%	20.04%	-0.64	9.87	-18.45%	7.53%
S&P GSCI Agriculture	SPGSAGP	7,573	-4.06%	17.26%	-0.11	3.51	-7.48%	7.16%
S&P GSCI Industrial Metals	SPGSINP	7,574	3.85%	23.05%	-0.22	4.02	-12.52%	8.40%
S&P GSCI Precious Metals	SPGSPMP	7,573	0.07%	18.28%	-0.15	5.71	-8.25%	8.76%
EUR/USD	EURUSD	7,689	0.50%	10.40%	0.05	1.96	-3.38%	4.72%
GBP/USD	GBPUSD	7,825	-0.67%	10.03%	-0.07	3.05	-4.13%	4.59%
USD/JPY	USDJPY	7,825	-3.38%	10.75%	-0.36	5.04	-6.95%	5.50%
Silver	XAG	7,707	4.05%	29.90%	-0.68	8.49	-20.39%	13.18%
Gold	XAU	7,693	4.51%	17.32%	-0.11	9.51	-12.90%	10.48%

Table 2: AD-DG model ML estimation results for equity indices based on daily logarithmic returns over the time span from January 1, 1982 to December 31, 2011. The standard errors are given in parenthesis below the parameter estimates. Black superscripts ***, **, and * denote significance at 1%, 5%, and 10%, respectively, for the displacements terms κ_+ and κ_-. Gray superscript indicate a lack of significance and are used to highlight the parameters that are tested. $\mathcal{H}_0^{(1)} : \kappa_+^{\text{AD-DG}} + \kappa_-^{\text{AD-DG}} = 0$ and $\mathcal{H}_0^{(2)} : \kappa_+^{\text{AD-DG}} = \kappa_-^{\text{AD-DG}} = 0$ are evaluated based on a Wald test.

Asset	DAX 30	Dow Jones Industrial	Hang Seng	MSCI World	NASDAQ Composite	Nikkei 225	S&P 500	TOPIX
σ	**13.43%**	**9.46%**	**14.95%**	**8.44%**	**8.58%**	**9.46%**	**8.84%**	**9.96%**
	(0.40%)	(0.00%)	(0.01%)	(0.28%)	(0.29%)	(0.35%)	(0.01%)	(0.45%)
λ	**107.70**	**142.46**	**137.18**	**98.45**	**175.55**	**235.41**	**187.33**	**180.89**
	(12.92)	(4.13)	(6.15)	(12.74)	(11.18)	(16.31)	(9.27)	(19.84)
p	**43.17%**	**44.32%**	**52.83%**	**38.66%**	**43.74%**	**56.99%**	**48.34%**	**58.56%**
	(2.16%)	(0.01%)	(0.06%)	(3.76%)	(2.29%)	(2.19%)	(0.07%)	(2.96%)
δ_+	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
	(–)	(–)	(–)	(–)	(–)	(–)	(–)	(–)
δ_-	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
	(–)	(–)	(–)	(–)	(–)	(–)	(–)	(–)
η_+	**90.20**	**133.71**	**83.36**	**143.31**	**99.05**	**129.92**	**133.35**	**134.80**
	(6.16)	(3.58)	(2.69)	(11.59)	(5.13)	(6.16)	(4.77)	(8.67)
η_-	**92.55**	**115.71**	**67.42**	**129.15**	**104.28**	**123.52**	**119.62**	**120.67**
	(6.00)	(2.73)	(1.96)	(9.29)	(4.54)	(5.89)	(4.15)	(7.15)
κ_+	**+0.10%*****	**+0.24%*****	**+0.00%*****	**+0.28%*****	**+0.05%*****	**+0.08%*****	**+0.03%*****	**+0.08%*****
	(0.01%)	(0.00%)	(0.00%)	(0.05%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)
κ_-	**−0.33%*****	**−0.01%*****	**−0.10%*****	**−0.06%*****	**−0.28%*****	**−0.43%*****	**−0.00%*****	**−0.36%*****
	(0.02%)	(0.00%)	(0.00%)	(0.01%)	(0.02%)	(0.02%)	(0.00%)	(0.02%)
$\mathcal{H}_0^{(1)}$	**−0.23%*****	**+0.23%*****	**−0.10%*****	**+0.22%*****	**−0.23%*****	**−0.35%*****	**+0.03%*****	**−0.29%*****
	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{H}_0^{(2)}$	–***	–***	–***	–***	–***	–***	–***	–***
	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{L}$	22,179.29	24,081.39	20,456.56	26,513.74	22,755.20	21,662.19	23,974.51	22,527.43
N	7,566	7,568	7,418	7,822	7,569	7,387	7,569	7,387

Table 3: AD-DG model ML estimation results for commodity indices based on daily logarithmic returns over the time span from January 1, 1982 to December 31, 2011. The standard errors are given in parenthesis below the parameter estimates. Black superscripts ***, **, and * denote significance at 1%, 5%, and 10%, respectively, for the displacements terms κ_+ and κ_-. Gray superscript indicate a lack of significance and are used to highlight the parameters that are tested. $\mathcal{H}_0^{(1)} : \kappa_+^{\text{AD-DG}} + \kappa_-^{\text{AD-DG}} = 0$ and $\mathcal{H}_0^{(2)} : \kappa_+^{\text{AD-DG}} = \kappa_-^{\text{AD-DG}} = 0$ are evaluated based on a Wald test.

Asset	S&P GSCI	S&P GSCI Agriculture	S&P GSCI Industrial Metals	S&P GSCI Precious Metals
σ	**10.06%**	**11.45%**	**12.72%**	**5.98%**
	(0.26%)	(0.28%)	(0.34%)	(0.23%)
λ	**164.98**	**80.08**	**159.20**	**319.74**
	(11.79)	(9.71)	(15.98)	(16.91)
p	**31.81%**	**34.33%**	**74.25%**	**59.30%**
	(1.16%)	(1.37%)	(2.31%)	(1.71%)
δ_+	**1**	**4**	**1**	**1**
	(−)	(−)	(−)	(−)
δ_-	**1**	**1**	**1**	**1**
	(−)	(−)	(−)	(−)
η_+	**163.29**	**264.04**	**117.96**	**184.76**
	(10.63)	(14.21)	(6.86)	(6.99)
η_-	**119.36**	**124.48**	**99.65**	**133.60**
	(5.03)	(8.41)	(6.39)	(5.22)
κ_+	**+0.83%*****	**+0.00%*****	**+0.04%*****	**+0.10%*****
	(0.03%)	(0.00%)	(0.00%)	(0.01%)
κ_-	**−0.01%*****	**−0.20%*****	**−0.89%*****	**−0.04%*****
	(0.00%)	(0.01%)	(0.04%)	(0.00%)
$\mathcal{H}_0^{(1)}$	**+0.81%*****	**−0.22%*****	**−0.85%*****	**+0.06%*****
	(0.00%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{H}_0^{(2)}$	−***	−***	−***	−***
	(0.01%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{L}$	23,017.86	23,935.71	21,860.41	23,903.33
N	7,571	7,573	7,574	7,573

Table 4: AD-DG model ML estimation results for FX and precious metals based on daily logarithmic returns over the time span from January 1, 1982 to December 31, 2011. The standard errors are given in parenthesis below the parameter estimates. Black superscripts ***, **, and * denote significance at 1%, 5%, and 10%, respectively, for the displacements terms κ_+ and κ_-. Gray superscript indicate a lack of significance and are used to highlight the parameters that are tested. $\mathcal{H}_0^{(1)} : \kappa_+^{\text{AD-DG}} + \kappa_-^{\text{AD-DG}} = 0$ and $\mathcal{H}_0^{(2)} : \kappa_+^{\text{AD-DG}} = \kappa_-^{\text{AD-DG}} = 0$ are evaluated based on a Wald test.

Asset	EUR/USD	GBP/USD	USD/JPY	Silver	Gold
σ	**4.64**%	**4.45**%	**5.72**%	**11.79**%	**5.34**%
	(0.31%)	(0.21%)	(0.22%)	(0.36%)	(0.22%)
λ	**482.98**	**323.15**	**221.74**	**241.71**	**291.37**
	(51.21)	(25.66)	(22.87)	(14.37)	(14.72)
p	**41.74**%	**47.30**%	**51.63**%	**52.46**%	**47.63**%
	(1.95%)	(2.07%)	(2.38%)	(2.05%)	(1.62%)
δ_+	**1**	**1**	**1**	**1**	**1**
	(−)	(−)	(−)	(−)	(−)
δ_-	**1**	**1**	**1**	**1**	**1**
	(−)	(−)	(−)	(−)	(−)
η_+	**362.27**	**370.14**	**301.27**	**103.93**	**186.44**
	(20.09)	(19.31)	(18.58)	(4.67)	(7.13)
η_-	**387.92**	**305.52**	**230.68**	**76.73**	**157.65**
	(20.93)	(14.49)	(12.39)	(3.18)	(5.77)
κ_+	**+0.09**%***	**+0.16**%***	**+0.11**%***	**+0.29**%***	**+0.24**%***
	(0.01%)	(0.01%)	(0.01%)	(0.02%)	(0.01%)
κ_-	**−0.04**%***	**−0.04**%***	**−0.05**%***	**−0.04**%***	**−0.04**%***
	(0.00%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{H}_0^{(1)}$	**+0.05**%***	**+0.12**%***	**+0.06**%***	**+0.25**%***	**+0.19**%***
	(0.00%)	(0.00%)	(0.00%)	(0.01%)	(0.00%)
$\mathcal{H}_0^{(2)}$	–***	–***	–***	–***	–***
	(0.01%)	(0.00%)	(0.00%)	(0.00%)	(0.00%)
$\mathcal{L}$	27,967.61	28,903.52	28,482.43	20,595.44	24,778.66
N	7,689	7,825	7,825	7,707	7,693