Badly approximable elements in diophantine approximation: Schmidt games, Jarník-type inequalities and f-aperiodic points

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Badly Approximable Elements in Diophantine Approximation: Schmidt Games, Jarník-Type Inequalities and F-Aperiodic Points

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Zusammenfassung

Diese Arbeit erzielt Resultate aus der metrischen Zahlentheorie und der Dynamischen Systeme, wobei der Schwerpunkt vor allem auf der Dynamik des geodätischen Flusses in Räumen negativer Krümmung (im metrischen Sinne) liegt. Diese Dynamik studieren wir mit Hilfe der Theorie der diophantischen Approximation und sind im Wesentlichen an 'schlecht approximierbaren' Elementen interessiert. Gegeben eine Sammlung $\mathcal{C}$ konvexer Mengen in einem negativ gekrümmten Raum Z (beispielsweise eine Sammlung von disjunkten Horobällen oder Geodätischen im hyperbolischen Raum), lässt sich ein geeignetes Modell der Diophantischen Approximation auf dem Rand von Z induzieren. Dieses Modell nutzt das tiefgreifende Zusammenspiel zwischen dem 'Rand' und dem 'Inneren' des Raumes Z aus und erlaubt es, gewisse diophantische Eigenschaften eines Randpunktes ξ mit dynamischen Eigenschaften des entsprechenden geodätischen Strahls (startend in einem Basispunkt o und asymptotisch zu ξ) bezüglich der Sammlung $\mathcal{C}$ zu interpretieren. So entspricht unter angemessenen Annahmen ein schlecht approximierbarer Randpunkt einem Strahl, der beschränkte 'Eindringungen' in die Umgebungen der konvexen Mengen von $\mathcal{C}$ hat, beziehungsweise diesen ausweicht. Zudem ermöglicht das Modell, im Hinblick auf die klassische Theorie der diophantischen Analysis, bekannte Konzepte und Resultate zu übertragen. So wenden wir das Konzept der sogenannten Schmidt Spiele an und verallgemeinern 'Jarník's Ungleichung'. Wir bestimmen abstrakte Bedingungen an ein allgemeines Rahmenwerk unter welchen die Anwendung dieser Konzepte möglich ist. Schliesslich verifizieren wir diese Bedingungen für zahlreiche Beispiele.

Als konkrete Anwendung sei beispielsweise M eine geschlossene hyperbolische Mannigfaltigkeit und α eine geschlossene Geodätische in M. Wir erhalten den Spezialfall, dass Z der hyperbolische Raum und $\mathcal{C}$ durch die Lifts von α gegeben ist, überprüfen unsere Bedingungen und leiten damit folgende Resultate her:
Sei $\bar{o} \in M$ ein fester Punkt und $SM_{\bar{o}}$ die Einheitstangetialvektoren in $\bar{o}$. Ein Vektor $v \in SM_{\bar{o}}$ bestimmt einen eindeutigen geodätischen Strahl γ_v in M durch v. Wir nennen v 'beschränkt', wenn eine Schranke $t(v)$ (abhänging von v selbst) existiert, sodass die Zeit jeder Eindringung von γ_v in eine ε-Umgebung von α beschränkt durch $t(v)$ ist. Wir zeigen dann, dass die Menge der beschränkten Vektoren $v \in SM_{\bar{o}}$ eine Schmidt-Gewinnmenge ist. Zudem, in Abhängigkeit von einer gegebenen Schranke T, bestimmen wir nicht triviale Schranken für die Hausdorff-Dimension der Menge der Vektoren $v \in SM_{\bar{o}}$ mit $t(v) \leq T$. Schliesslich zeigen wir die Existenz von aperiodischen Geodätischen in M, welche gewisse quantitative Eigenschaften hinsichtlich der Aperiodizität erfüllen. Diese stellen spezielle Beispiele von 'schlecht approxierbaren' Geodätischen dar.

Abstract

In this work we obtain results from metric number theory and dynamical systems, where the primary focus lies on the dynamics of the geodesic flow in spaces of negative curvature (in the metrical sense). We investigate this dynamics with the help of Diophantine approximation and are mainly interested in 'badly approximable' elements. Given a collection $\mathcal{C}$ of convex sets in a negatively curved space Z (for instance, a collection of disjoint horoballs or geodesics in the hyperbolic space), a suitable model of Diophantine approximation can be induced on the boundary of Z. This model exploits the deep interplay between the 'boundary' and the 'interior' of Z and enables to relate certain Diophantine properties of a boundary point ξ with dynamical properties, with respect to $\mathcal{C}$, of the corresponding geodesic ray (starting in a base point o and asymptotic to ξ). In fact, under reasonable requirements, a badly approximable boundary point corresponds to a ray which has bounded penetrations in the neighborhoods of the convex sets of $\mathcal{C}$ or avoids them respectively. Furthermore, using this model, we are able to transfer known concepts and results from the classical theory of Diophantine analysis. In fact, we apply the concept of the so-called Schmidt games and generalize 'Jarník's inequality'. We establish abstract conditions on a general framework which guarantee that these concepts are applicable. Finally we verify our conditions for various examples.

For an explicit application let for instance M be a closed hyperbolic manifold and α a closed geodesic in M. In this particular setting, Z is given by the hyperbolic space and $\mathcal{C}$ by the lifts of α, and, by checking our conditions, we deduce the following results:
Let $\bar{o} \in M$ be a fixed point and let $SM_{\bar{o}}$ denote the unit tangent vectors at $\bar{o}$. A vector $v \in SM_{\bar{o}}$ determines a unique geodesic ray γ_v in M through v and we call v 'bounded' if there exists a bound $t(v)$ (depending on v itself) such that the time of each penetration of γ_v in a ε-neighborhood of α is bounded by $t(v)$. Then, we show that the set of bounded vectors $v \in SM_{\bar{o}}$ is a wining set for Schmidt's game. Moreover, in dependence on a given bound T, we determine nontrivial lower and upper bounds on the Hausdorff-dimension of the set of vectors $v \in SM_{\bar{o}}$ with $t(v) \leq T$. Finally, we show the existence of aperiodic geodesics in M satisfying certain quantitative properties which measure the aperiodicity. These geodesics turn out to represent special examples of 'badly approximable' geodesics.

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Contents

Chapter 1

Introduction

1.1 Structure of the Thesis and Remarks

Before we start with the motivation and the statement of the results, some remarks on the rough structure of the thesis are necessary. The thesis consists of three parts, already published in

Ch.2: S. Weil, *Schmidt Games and Conditions on Resonant sets*, arXiv:1210.1152, 2012; [60]

Ch.3: S. Weil, *Jarník-type inequalities*, arXiv:1306.1314, 2013; [61]

Ch.4: V. Schroeder, S. Weil, *Aperiodic Sequences and Aperiodic Geodesics*, Ergodic Theory and Dynamical Systems, 2013, which is joint work with Viktor Schroeder; [51]

As can be seen in the outline below, Chapter 4 is independent from but motivates the respective questions in Chapters 2 and 3, and was in fact the starting point for these works. The Chapters 2 and 3 are similar in their settings but distinct in their motivation and intention. We therefore want to present each of the chapters in a self-contained way and independently from the other ones but need to point out the resulting overlap in Chapters 2 and 3, where partially basic settings and even Lemmata are shared.

The detailed outlines, motivations and main results of the three parts are given in the respective chapters itself. In the following overview, we want to state and connect several selected results of the thesis. The results are stated in their simplest settings and we try to avoid giving too many definitions. Note that the setting and results of Chapters 2 and 3 apply to various examples from number theory and dynamical systems. However, in the overview below, we focus on the investigation of the dynamics of the geodesic flow on negatively curved spaces Z/Γ - where Z denotes a CAT(-1) space and Γ a discrete subgroup of the isometry group of Z acting on Z - since this was the initial starting point of the thesis. This subject has been studied extensively and the investigation leads to a deeper understanding of the interplay between the isometric action of Γ on Z and the structure of its limit set $\Lambda\Gamma$ as well as the distribution of certain subsets in $\Lambda\Gamma$. Moreover, there is a connection to the theory of Diophantine approximation in negatively curved spaces due to models of Patterson [47] as well as Hersonsky, Parkkonen, Paulin (HPP) [25, 26, 46]. In particular, we are mainly concerned with 'badly approximable' orbits of the geodesic flow - in the following we will try to clarify what we mean by 'badly approximable' without giving a precise definition, which requires introducing the model of (HPP) (for the latter we refer to Subsection 2.3.6).

We start with the very special geodesics of Chapter 4 which we want to use as a descriptive example to motivate the questions in Chapters 2 and 3.

1.2 Chapter 4: Quantitative Recurrence and φ-Aperiodicity.

The study of recurrence and of the distribution of periodic orbits for a measure-preserving transformation on a finite measure, metric space is a fundamental question in ergodic theory and dynamical systems. In a suitable setting, Boshernitzan [7] gave a quantitative result on the recurrence of almost every point which is related to the Hausdorff-dimension of the space. Somewhat orthogonally, in a joint work with V. Schroeder, we study the recurrence of *whole* orbits which are very singular with respect to a quantitative condition and which avoid a neighborhood of every periodic point of the system. We discuss it for the case of the geodesic flow and remark that a similar result holds for the existence of words in the Bernoulli-shift (see Theorem 4.1).

Let M be a closed n-dimensional hyperbolic manifold, where $n \geq 2$. Let $i_M > 0$ denote the injectivity radius of M and let d be the Riemannian distance function on M. For a geodesic[1] $\gamma : \mathbb{R} \to M$ we define the *recurrence time* $R_\gamma^{t_0} : [0, \infty) \to [i_M/2, \infty]$ at time $t_0 \in \mathbb{R}$ by

$$R_\gamma^{t_0}(l) \equiv \inf\{s > i_M/2 : d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \frac{i_M}{2} \text{ for all } 0 \leq t \leq l\},$$

and the recurrence time independent of all times t_0, $R_\gamma(l) \equiv \inf\{R_\gamma^{t_0}(l) : t_0 \in \mathbb{R}\}$. If γ is a periodic geodesic, then R_γ is bounded by its period. One can therefore view the growth rate of R_γ as a measure for the aperiodicity of γ.

We prove the existence of special geodesics for which this growth rate is as near as possible to an optimal bound.

Theorem 1.1 (Theorem 4.2). *Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that*

$$\limsup_{l \to \infty} \frac{1}{l} \log(\varphi(l)) \leq \delta(n-1)$$

for some $0 < \delta < 1$. If $i_M > 2\log(2)$ then there exist $l_0 = l_0(\varphi, \delta, n, i_M) \geq 0$ and a geodesic $\gamma : \mathbb{R} \to M$ such that for all $l \geq l_0$, we have

$$R_\gamma(l) \geq \varphi(l). \tag{1.2.1}$$

A geodesic satisfying (1.2.1) for some $l_0 \geq 0$ is called *φ-aperiodic*. The result is optimal in the sense that there exists a constant $c = c(i_M, n, \operatorname{diam}(M))$ such that no φ-aperiodic geodesic can exist for the function $\varphi(l) = c \cdot e^{(n-1)l}$. The proof of the theorem involves mainly the geometry of negatively curved spaces as well as counting and coding arguments. Furthermore, the proofs of existence and optimality in Theorem 1.1 both essentially require the positivity of the volume and hence the topological entropy of the geodesic flow on SM. More precisely, the topological entropy is bounded below by the exponential growth rate of φ so that the existence of a φ-aperiodic geodesic, for a suitable function φ, implies that the topological entropy is positive.

We now connect the notion of φ-aperiodicity to the language of Diophantine approximation in negatively curved spaces in the (HPP)-model. Details can be found in Subsection 2.3.6 and

[1] Here and in the following, all geodesics are assumed to be unit speed.

the Appendix 4.6. First let α be a closed geodesic in M and let $\mathcal{N}_{\varepsilon_0}(\alpha)$ be the (closed) ε_0-neighborhood of α, where $\varepsilon_0 > 0$ is sufficiently small. Let $\mathfrak{p}_\alpha(\gamma, t)$ be the *penetration length* of γ in $\mathcal{N}_{\varepsilon_0}(\alpha)$ at time t_0[2] . Then by Hersonsky, Paulin [26], if μ denotes the Liouville measure on SM, for μ-almost every v in the unit tangent bundle SM of M we have for γ_v[3] ,

$$\limsup_{t\to\infty} \frac{\mathfrak{p}_\alpha(\gamma_v, t)}{\log(t)} = \frac{1}{n-1}. \tag{1.2.2}$$

For a vector v satisfying (1.2.2), γ_v is 'well approximable' with respect to the closed geodesic α in the (HPP)-model. Conversely, fix a point $x \in M$ and consider the set

$$S_\alpha \equiv \{v \in SM_x : \text{ there exists a length } L = L(v) \text{ such that } \mathfrak{p}_\alpha(\gamma_v, t) \le L,\ \forall t \ge 0\}, \tag{1.2.3}$$

which corresponds to the set of 'badly approximable' geodesics (starting in x) with respect to the closed geodesic α in the (HPP)-model. Clearly, (1.2.2) fails to hold for geodesics γ_v, where $v \in S_\alpha$. For $l \ge 0$, denote moreover

$$S_\alpha(l) \equiv \{v \in S_\alpha : L(v) \le l\} \subset SM_x,$$

the subset of S_α for which the lengths $L(v)$ are bounded by l. Let γ_v, $v \in SM_x$, be a φ-aperiodic geodesic. One can show that for *every* closed geodesic α in M of period p the penetration length $L(v)$ of γ_v in $\mathcal{N}_{i_M/8}(\alpha)$ is bounded; hence, $v \in S_\alpha$. In fact, if $p \ge l_0$ and $\varepsilon_0 \le i_M/8$, then $L(v)$ is bounded by $p + \varphi^{-1}(p) + i_M$[4]. It follows that the approximation constant of a φ-aperiodic geodesic is positive and can be estimated in terms of φ and p. Moreover, we have that $v \in \cap_\alpha S_\alpha$ where the intersection is over every closed geodesic α in M.

It is natural to ask for properties of the sets S_α, $S_\alpha(l)$, $\cap_\alpha S_\alpha$ and of the set of φ-aperiodic geodesics. This question leads to the theory of Schmidt games and to Jarník's inequality.

1.3 Chapters 2 and 3: Badly approximable elements in Diophantine Approximation

In the following, let $(\bar{X}, d)$ be a complete metric space. For a countable index set Λ, let $\{R_\lambda \subset \bar{X} : \lambda \in \Lambda\}$ be a collection of *resonant sets*, where to each R_λ we assign a *size* $s_\lambda \ge 0$ and the contraction $\psi_\lambda(c) \equiv \mathcal{N}_{e^{-(s_\lambda + c)}}(R_\lambda)$,[5] for $c \ge 0$. Suppose that the family $\mathcal{F} = (\Lambda, R_\lambda, f_\lambda)$ is *nested* and *discrete*, that is, if $s_\lambda \le s_\beta$ then $R_\lambda \subset R_\beta$, and, the sizes $\{s_\lambda\} \subset \mathbb{R}^+$ are discrete. For a subset $X \subset \bar{X}$ we then define the set of *badly approximable points with respect to* $\mathcal{F}$ in X by

$$\mathbf{Bad}_X(\mathcal{F}) := \{x \in X : \exists\, c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda\in\Lambda} \psi_\lambda(c)\}. \tag{1.3.1}$$

For the basic example, let $\Lambda = \mathbb{N}$, $R_n \equiv \{p/q \in \mathbb{Q} : p \in \mathbb{Z}, q \in \mathbb{N}, 0 < q \le n\}$, $s_n \equiv 2\log(n)$, and note that, for $X = \mathbb{R}$, $\mathbf{Bad}_\mathbb{R}(\mathcal{F})$ is the set of *badly approximable numbers* $\mathbf{Bad}^1_\mathbb{R}$; that is the

[2] That is, set $\mathfrak{p}_\alpha(\gamma, t) = 0$ if $\gamma(t) \notin \mathcal{N}_{\varepsilon_0}(\alpha)$. Otherwise, set it to be the maximal length $L \in [0, \infty]$ of an interval I, $t \in I$, such that $\gamma(s) \in \mathcal{N}_{\varepsilon_0}(\alpha)$ for all $s \in I$.

[3] We identify a vector $v \in SM$ with the unique geodesic $\gamma_v : \mathbb{R} \to M$ such that $\dot\gamma_v(0) = v$.

[4] We may assume that φ is strictly increasing.

[5] Here, $B(x, r) \equiv \{y \in \bar{X} : d(x, y) \le r\}$, $r > 0$, is the closed ball around $x \in \bar{X}$ and $\mathcal{N}_\varepsilon(A)$ is the ε-neighborhood of a set $A \subset \bar{X}$.

set of $x \in \mathbb{R}$ for which there exists $c = c(x) > 0$ such that, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$|x - \frac{p}{q}| \geq \frac{c}{q^2}.$$

Note that in a similar setting and under a suitable framework, Kristensen, Thorn, Velani [35] showed $\mathbf{Bad}_X(\mathcal{F})$ to be of 'full' Hausdorff-dimension, that is the dimension of the space X.

We want to determine simple conditions for this framework under which we can generalize and strengthen this result in two directions: One direction focuses on properties of the set $\mathbf{Bad}_X(\mathcal{F})$ and the other one rather on the spectrum of approximation constants, that is on the set

$$\mathcal{S}_X(\mathcal{F}) \equiv \{c(x) : x \in \mathbf{Bad}_X(\mathcal{F})\} \subset \mathbb{R};$$

assume here that $c(x)$ is the infimum of the constants as in (1.3.1). In fact, in Chapter 2 we consider conditions when $\mathbf{Bad}_X(\mathcal{F})$ is a winning set for Schmidt's game where we remark that this implies $\mathbf{Bad}_X(\mathcal{F})$ to be of full Hausdorff-dimension if X is a 'nice' fractal set (see Subsection 2.2.4 for details and definitions). In Chapter 3 we consider a more precise analysis on the spectrum of badly approximable elements and give estimates in terms of the Hausdorff-dimension with respect to a given upper bound on the approximation constants.

1.3.1 Chapter 2: Schmidt Games and Conditions on Resonant Sets

Schmidt's game is designed to capture information and properties of the set of badly approximable points in a suitable framework of Diophantine approximation and a concept initially coming from metric number theory (see Schmidt [50]). Winning sets of Schmidt's game enjoy a remarkable rigidity; for instance, winning sets in $\mathbb{R}^n$ are invariant under countable intersections and bi-Lipschitz homeomorphisms as well as of Hausdorff-dimension n. Because of this reason and due to a work of Dani [12, 13], Schmidt's game (and modifications of it) were applied to various examples from number theory and dynamical systems and gained attention even in the last few years. The goal in Schmidt's game is to determine a winning strategy by looking on a very local situation and, in most works, strategies are deduced that are strongly adapted to the specific example.

We introduce a modification of Schmidt's game which combines and generalizes the ones of Kleinbock, Weiss [34] and McMullen [40]. We then axiomatize conditions on a framework under which we can show $\mathbf{Bad}_X(\mathcal{F})$, the set of badly approximable points with respect to the given family $\mathcal{F}$ of resonant sets, to be a winning set for the modified game. The conditions address the local structure of the space X and distribution and structure of the resonant sets in X. While in general X is a 'nice' fractal set, let here $X = \mathbb{R}^n$ and $\mathcal{S}$ be the set of metric spheres and affine hyperplanes in $\mathbb{R}^n$. The main theorem in its simplest form can then be stated as follows:

We say that the family $\mathcal{F}$ is *locally contained in metric spheres* if for all closed metric balls $B = B(x, e^{-r})$, and for all $\lambda \in \Lambda$ with $s_\lambda \leq r$, there exists a generalized metric sphere $S \in \mathcal{S}$ such that $B \cap R_\lambda \subset S$. In particular, $\mathcal{F}$ is locally contained in metric spheres if for every $\lambda \in \Lambda$ and any two distinct points $x, y \in R_\lambda$, we have

$$d(x, y) > 2e^{-s_\lambda}. \tag{1.3.2}$$

Theorem 1.2 (Theorem 2.1). *Let $\mathcal{F} = (\Lambda, R_\lambda, f_\lambda)$ be a nested and discrete family. If $\mathcal{F}$ is locally contained in metric spheres, then $\mathbf{Bad}_{\mathbb{R}^n}(\mathcal{F})$ is a winning set for Schmidt's game.*

In a more general framework, we adopt the notion of *absolutely decaying measures* due to Kleinbock, Lindenstrauss, Weiss [31] and require that X supports a measure which is absolutely decaying with respect to the family $\mathcal{F}$. A similar result then holds and shows winningness (for the modified game) for several new examples in dynamical systems and number theory (see Section 2.3 for details); for instance for the set of badly approximable p-adic (integer) vectors $\mathbb{Z}_p^2$ with weights.

Note that the requirements of the theorem hold for the set of badly approximable vectors $\mathbf{Bad}^n_{\mathbb{R}^n} \subset \mathbb{R}^n$, in particular for $\mathbf{Bad}^1_{\mathbb{R}}$ which satisfies (1.3.2).[6] We here focus on the following example that is treated in detail in Section 2.3.6. Let Z be a proper geodesic CAT(-1) metric space and consider a family $\{C_n\}_{n\in\mathbb{N}}$ of convex sets in Z which are T-embedded for some $T \geq 0$; that is, the intersection of two distinct C_n and C_m has finite diamenter bounded by T. For simplicity, assume that each C_n is either a horoball or a ε-neighborhood of a geodesic line. For instance, the standard cusp neighborhood of a cusped finite-volume hyperbolic manifold or the $i_M/8$-neighborhood of a closed geodesic in a closed hyperbolic manifold determines such a collection in the universal covering $Z = \mathbb{H}^m$ (the hyperbolic space). Fix a base point $x \in Z$ and let $X = \partial_\infty Z$ denote the visual boundary of Z. Note that the family $\{C_n\}_{n\in\mathbb{N}}$ determines a nested, discrete family $\mathcal{F} = (\mathbb{N}, R_n, n + \bar{c})$ in (X, d_x) as above, where d_x denotes the visual distance with respect to x and $\bar{c} = \bar{c}(T)$ is a suitable constant. The key point is to show that (1.3.2) is satisfied (see Proposition 2.29). If X is a uniformly perfect space, it follows then that $\mathbf{Bad}_X(\mathcal{F})$ is a (Schmidt) winning set. Moreover, a point $\xi \in X$ belongs to $\mathbf{Bad}_X(\mathcal{F})$ if and only if $\xi \in S(\{C_n\}_{n\in\mathbb{N}})$; that is, there exists $c = c(\xi) < \infty$ such that the sequence of penetration lengths[7] $L(\gamma_{x,\xi}(\mathbb{R}^+) \cap C_n)$, $n \in \mathbb{N}$, is bounded by c. Here, $\gamma_{x,\xi}$ denotes the unique geodesic ray starting in x, asymptotic to ξ and L stands for the length of a geodesic segment.

As a corollary we obtain the following.

Corollary 1.3. *The sets S_α (defined in (1.2.3)) as well as $\cap_\alpha S_\alpha \subset SM_x$, where the intersection is over every closed geodesic α in M, are winning sets for Schmidt's game. The same is true for the set S_H of endpoints in $\partial_\infty\mathbb{H}^n$ of lifts of bounded geodesic rays (starting at a fixed point) in a single-cusped finite-volume hyperbolic manifold. In particular, the sets have Hausdorff-dimension $n-1$.*

An evident question is to ask about the Hausdorff-dimension and further properties of the set $S_\alpha(l)$ in dependence on $l \geq 0$ as well as the set $S_H(l)$ of geodesic rays in S_H which do not enter in a given shrinked (by the factor l) cusp neighborhood; this leads to Jarník's inequality.

1.3.2 Chapter 3: Jarník-type Inequalities

Before Schmidt [50] showed that the set $\mathbf{Bad}^1_{\mathbb{R}}$ of badly approximable numbers in $\mathbb{R}$ is a winning set, Jarník [29] proved it to be of full Hausdorff-dimension. In fact, Jarník was more precise and gave nontrivial lower and upper estimates on the Hausdorff-dimension of the set of badly approximable numbers with an approximation constant bounded below (see [29]): If M_N denotes the set of irrational numbers for which the entries of the continued fraction expansion are bounded by $N \in \mathbb{N}$, where $N > 8$, then

$$1 - \frac{4}{N\log(2)} \leq \dim(M_N) \leq 1 - \frac{1}{8N\log(N)}; \tag{1.3.3}$$

[6] In fact, for different rational points $p/q \neq \bar{p}/\bar{q}$, we have $|p/q - \bar{p}/\bar{q}| \geq 1/(q\bar{q})$.

[7] Note that since C_n is convex, $\gamma_{x,\xi}(\mathbb{R}^+) \cap C_n$ is the image of a connected geodesic segment.

here and in the following, 'dim' stands for the Hausdorff-dimension. We remark that an irrational number $x \in \mathbb{R}$ belongs to $\mathbf{Bad}^1_{\mathbb{R}}$ if and only $x \in M_N$ for some N and a small approximation constant corresponds to a large N.

The aim of Chapter 3 is to generalize this inequality to the abstract setting. More precisely, consider the set $\mathbf{Bad}_X(\mathcal{F}, c)$ which is the subset of $x \in \mathbf{Bad}_X(\mathcal{F})$ for which the *approximation constant* $c(x)$ is bounded above by c. We will determine simple conditions on a general framework under which we give an abstract formalism such that an inequality of the form (1.3.3) - which we call a *Jarník-type inequality* - for $\mathbf{Bad}_X(\mathcal{F}, c)$ holds.

With respect to our motivating example $S_\alpha(l)$, defined after (1.2.3), we obtain the following.

Theorem 1.4 (Theorem 3.28)**.** *Let α be a closed geodesic in a closed hyperbolic manifold M. There exist positive constants*[8] c_l, c_u *and* l_0 *such that for all lengths* $l > l_0$, *we have*

$$(n-1) - \frac{c_l}{l \cdot e^{(n-1)l/2}} \leq dim(S_\alpha(l)) \leq (n-1) - \frac{c_u}{l \cdot e^{(n-1)l}}.$$

In fact, we show Jarník-type inequalities for several examples from number theory and dynamical systems in a reasonable setting. A very similar result can be given for words in the Bernoulli-shift which avoid a given periodic word, as well as for the set $S_H(l)$, even in a geometrically finite setting, and for the set $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$ of badly approximable vectors with weight vector $\bar{r}$; see Section 3.3.

It is worth pointing out that a positive lower bound on the dimension of $\mathbf{Bad}(\mathcal{F}; c)$ yields an upper bound for the *Hurwitz constant*, that is, the infimum of the approximation constants in $\mathcal{S}_X(\mathcal{F})$. This constant seems to be of number theoretical or geometric significance (and is achieved at the golden ratio in $\mathbf{Bad}^1_{\mathbb{R}}$ or given in terms of the infimum of heights of closed geodesics in a single cusped finite volume manifold $M = \mathbb{H}^n/\Gamma$ (see [25])).

[8] The constants are geometric, i.e. they depend for instance on the diameter of M, the length of the systole of M and the period of α.

Chapter 2

Schmidt Games and Conditions on Resonant Sets

Large parts of this chapter are published in [60].

Abstract of Chapter 2. Winning sets of Schmidt's game enjoy a remarkable rigidity. Therefore, this game (and modifications of it) have been applied to many examples of complete metric spaces (X, d) to show that the set of 'badly approximable points' $\mathbf{Bad}_X(\mathcal{F})$, with respect to a given family $\mathcal{F}$ of resonant sets in X, is a winning set. For these examples, strategies were deduced that are, in most cases, strongly adapted to the specific dynamics and properties of the underlying setting. We introduce a new modification of Schmidt's game which combines and generalizes the ones of [34] and [40]. We then axiomatize conditions on the collection of resonant sets and the set X under which we can show $\mathbf{Bad}_X(\mathcal{F})$ to be a winning set for the modification. Moreover, we discuss properties of winning sets of this modification and verify our conditions for several examples - among them, the set $\mathbf{Bad}^{\bar{r}}$ of badly approximable vectors in $\mathbb{R}^n$, $\mathbb{C}^2$ and $\mathbb{Z}_p^2$ with weights $\bar{r}$ and the set of geodesic rays in proper geodesic CAT(-1) spaces which avoid a suitable collection of convex subsets.

Outline of Chapter 2. In the introduction, we begin with a motivation (Subsection 2.1.1) and the statement of the main results in their simplest settings (Subsection 2.1.2).

In Section 2.2, we first recall the ψ-modified Schmidt game due to [34] and its properties (Subsection 2.2.1). We introduce our modified version of the game in this setting and deduce properties for this game (Subsection 2.2.2). Moreover, we consider different conditions on the collection of resonant sets and on the metric space under which the set of badly approximable points is a winning set for the respective versions of the game (Subsection 2.2.3). Finally, we discuss on diffusion properties of the space X, on suitable (absolutely decaying) measures supported on X and on the structure and distribution of the resonant sets under which the deduced conditions are satisfied (Subsection 2.2.4).

In Section 2.3, we verify the conditions for several examples, where we distinguish between examples coming from number theory and the ones coming from dynamical systems: For the first part, we consider the set of badly approximable vectors in $\mathbb{R}^n$, $\mathbb{C}^2$ and $\mathbb{Z}_p^2$ with weights (see Subsections 2.3.1, 2.3.2, 2.3.3 respectively).
For the second part, we consider the set of sequences in the Bernoulli-shift which avoid periodic

sequences (Subsection 2.3.4) and the set of orbits of a sequence of matrices avoiding a sequence of separated sets (Subsection 2.3.5). Moreover, in more details, we consider the set of geodesics in a proper geodesic CAT(-1)-space which avoid certain convex subsets such as a collection of disjoint horoballs or neighborhoods of geodesic lines or of a separated set (see Subsection 2.3.6).

2.1 Introduction and Main Result

2.1.1 Introduction

We begin with a motivation. Let (X,d) be a metric space, μ a Borel probability measure and $T : X \to X$ an ergodic measure-preserving transformation. Let $A \subset X$ be a set of positive μ-measure. Then for μ-almost every point $x \in X$, the orbit $\mathcal{T}(x)$ of x hits A infinitely many times. The *shrinking target problem*, due to Hill and Velani [27], considers sets shrinking in time. More precisely, one considers a sequence of nested measurable sets $A_n \subset X$ and is interested in the properties of the points $x \in X$ whose orbit hit A_n for infinitely many times n. Such points are called *well approximable* in analogy with classical Diophantine approximation.

For instance, identify the one point compactification $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with the unit tangent space at a suitable point of the modular surface $\mathbb{H}^2/SL_2(\mathbb{Z})$. Then the well-approximable real numbers (in the classical sense) correspond to geodesics which penetrate a shrinking neighborhood of the only cusp of $\mathbb{H}^2/SL_2(\mathbb{Z})$ infinitely often. This is a set of full Lebesgue-measure. Conversely, a badly-approximable real number corresponds to a geodesic which avoids (i.e. does not enter) a certain neighborhood of the cusp. The set of badly-aproximable numbers is of Lebesgue-measure zero, yet of full Hausdorff-dimension and in fact a winning set for Schmidt's game.

Considering the lifts of the cusp neighborhood of $\mathbb{H}^2/SL_2(\mathbb{Z})$ to $\mathbb{H}^2$ - or rather their shadows to $\bar{\mathbb{R}}$ with respect to a given base point - this motivates the following question. Given a countable index set Λ, consider a family of sets $\{R_\lambda \subset X : \lambda \in \Lambda\}$, called *resonant sets*, together with a family of contractions $\{\psi_\lambda : \mathbb{R}^+ \to \mathcal{P}(X) : \lambda \in \Lambda\}$, where $R_\lambda \subset \psi_\lambda(t_1) \subset \psi_\lambda(t_2)$ for all $t_2 \leq t_1 < \infty$. Denote this family by $\mathcal{F} = (\Lambda, R_\lambda, \psi_\lambda)$ and define the set of *badly approximable points* (with respect to the family $\mathcal{F}$) by

$$\mathbf{Bad}(\mathcal{F}) \equiv \{x \in X : \exists\, c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda\in\Lambda} \psi_\lambda(c)\};$$

the set of points x in X which are not contained in the uniformly shrinked neighborhoods, depending on x, of the family $\mathcal{F}$. What kind of properties does the set $\mathbf{Bad}(\mathcal{F})$ admit?

In a suitable framework, Kristensen, Thorn, Velani [35] already showed that $\mathbf{Bad}(\mathcal{F})$ is of 'full' Hausdorff-dimension, that is the one of the space X. In this chapter, we strengthen this result and use a different approach via modified Schmidt games where, at least in a reasonably nice setting, full Hausdorff-dimension is a property of winning sets (among others, see Subsection 2.2.1). In fact, winning sets of Schmidt's game (and modifications of it, called Schmidt games) enjoy a remarkable rigidity which has been exploited by many authors. This can be seen from the probably incomplete list [1, 2, 3, 12, 13, 14, 17, 19, 20, 32, 34, 28, 40, 42, 11, 10, 9, 50, 55]. However, in most cases, strategies are deduced which are strongly adapted to the specific dynamics and properties of the considered example. The purpose of this chapter is the following. First, we introduce a modification of Schmidt's game which combines and generalizes the ones of Kleinbock, Weiss [34] and McMullen [40] (as well as Broderick et al. [11]). Second, we abstractize conditions on a given collection of resonant sets and the metric space X, under which we determine explicit winning strategies with respect to the set $\mathbf{Bad}(\mathcal{F})$ for this modified game. The conditions concern mainly the (local) structure of both, the space X and the resonant sets, and the (local) distribution of the resonant sets in X (both with respect to their "size"). Third, we verify our conditions and improve several known or obtain new examples.

We emphasize that the obtained axiomatization of a winning strategy is, of course, *not* applicable to every example. Nevertheless, confirmed by the applications in Section 3.3, at least in suitable settings it yields a significant simplification of the proofs and, by separating into and focussing on the conditions rather than determining a winning strategy, leads to new results.

2.1.2 Main Results

Although our main results, Theorem 2.6 and Theorem 2.11, will be stated in the setting of general metric spaces, we first illustrate it for the case that $X = \mathbb{R}^n$ is the Euclidean space. In fact, we point out that in the Euclidean setting already Dani in [12, 13], Dani and Shah [14], as well as Fishman [19] deduced conditions under which $\mathbf{Bad}(\mathcal{F})$ is a winning set. Their conditions also concern on the one hand, the (local) structure of the space X and the resonant sets, and on the other hand, their distribution in $\mathbb{R}^n$; for the precise statement see Theorem 3.2 in [12] and Theorem 2.2 in [19].

We now give a first version of our main result in its simplest form. Fix $\sigma > 0$. For a countable index set Λ, let $\{R_\lambda \subset \mathbb{R}^n : \lambda \in \Lambda\}$ be a collection of resonant sets, where to each R_λ we assign a *size* $s_\lambda \geq 0$ and the contraction $\psi_\lambda(c) \equiv \mathcal{N}_{e^{-\sigma(s_\lambda + c)}}(R_\lambda)$,[1] for $c \geq 0$. Suppose that the resonant sets are *nested* with respect to their sizes, that is, if $s_\lambda \leq s_\beta$ then $R_\lambda \subset R_\beta$, and, that the sizes $\{s_\lambda\} \subset \mathbb{R}^+$ are discrete. Let $\mathcal{S}$ be the collection of metric spheres and affine hyperplanes in $\mathbb{R}^n$. We say that the family $\mathcal{F} = (\Lambda, R_\lambda, \psi_\lambda)$ is *locally contained in metric spheres* if for all closed metric balls $B = B(x, e^{-\sigma r})$, and for all $\lambda \in \Lambda$ with $s_\lambda \leq r$, there exists a (generalized) metric sphere $S \in \mathcal{S}$ such that

$$B \cap R_\lambda \subset S.$$

In particular, $\mathcal{F}$ is locally contained in metric spheres if for every $\lambda \in \Lambda$ and for any two distinct points x, $y \in R_\lambda$, we have

$$d(x, y) > 2e^{-\sigma s_\lambda}. \tag{2.1.1}$$

Theorem 2.1. *Let the family $\mathcal{F} = (\Lambda, R_\lambda, s_\lambda)$ be as above. If the resonant sets are nested, their sizes are discrete and $\mathcal{F}$ is locally contained in metric spheres, then $\mathbf{Bad}(\mathcal{F})$ is a winning set for Schmidt's game. If moreover (2.1.1) is satisfied, then $\mathbf{Bad}(\mathcal{F})$ is a winning set for McMullen's game.*

Note that the above theorem already applies to the following examples. First, for $k \in \Lambda \equiv \mathbb{N}_{\geq 2}$ we define the set of rational vectors R_k with size s_k by

$$R_k \equiv \{\bar{p}/q : \bar{p} \in \mathbb{Z}^n, 0 < q < k\}, \qquad s_k \equiv \log(k) + \log(n! \cdot 2^n),$$

which gives a nested and discrete family $\mathcal{F}$. It is readily checked that, for $\sigma = 1 + 1/n$, $\mathbf{Bad}(\mathcal{F})$ equals the set of badly approximable vectors $\mathbf{Bad}^n_{\mathbb{R}^n}$ in $\mathbb{R}^n$ (see Subsection 3.18 for details). The 'Simplex-Lemma' (see Lemma 2.21) implies that, given $B = B(x, e^{-(1+1/n)r})$ and $k \in \mathbb{N}$ with $s_k \leq r$, $B \cap R_\lambda$ is contained in an affine hyperplane. Hence, Theorem 2.1 shows that $\mathbf{Bad}^n_{\mathbb{R}^n}$ is a winning set for Schmidt's game. In particular, if $n = 1$, then (2.1.1) holds and $\mathbf{Bad}^1_{\mathbb{R}}$ is a winning set for McMullen's game.
Similar arguments apply to the sets of badly approximable vectors in $\mathbb{R}^n$, $\mathbb{C}^2$, $\mathbb{Z}_p^2$ with weights for the modified game (see Section 3.3).

[1] Here and in the following, given a metric space X, $B(x, r) \equiv \{y \in \bar{X} : d(x, y) \leq r\}$, $r > 0$, is the closed ball around $x \in \bar{X}$ and $\mathcal{N}_\varepsilon(A) \equiv \cup_{x \in A} B(x, r)$ is the ε-neighborhood of a set $A \subset \bar{X}$.

Second, given a countable collection of pairwise disjoint horoballs H_l in the real hyperbolic upper half space $\mathbb{H}^{n+1}$ tangent to the points $x_l \in \mathbb{R}^n$ and of Euclidean radius $r_l > 0$, define

$$R_k \equiv \{x_l \in \mathbb{R}^n : r_l \geq e^{-k}\}, \qquad s_k \equiv k + \log(2),$$

which again gives a nested and discrete family $\mathcal{F}$ in $X = \mathbb{R}^n$. For $\sigma = 1$, clearly, (2.1.1) holds by disjointness of the horoballs and Theorem 2.1 implies that $\mathbf{Bad}(\mathcal{F})$ is a winning set for McMullen's game. Moreover, $\mathbf{Bad}(\mathcal{F})$ corresponds to the set of vertical geodesic lines in $\mathbb{H}^{n+1}$, for each of which the sequence of penetration lengths in the horoballs H_l is bounded or, in other words, avoids the same collection of uniformly shrinked horoballs (for further details and background, see Subsection 2.3.6).
This already simplifies and shortens the proof of McMullen (compare with [40]) significantly. For the motivating example from the introduction, consider the collection of pairwise disjoint horoballs given by the collection $B_{p/q} \subset \mathbb{H}^2$, where $B_{p/q}$ is an Euclidean ball tangent to $p/q \in \mathbb{Q}$ (with p, q coprime) of radius $1/2q^{-2}$. Note that each such ball is a cover of the only cusp of the modular surface $\mathbb{H}^2/SL_2(\mathbb{Z})$. Moreover, we have $\mathbf{Bad}(\mathcal{F}) = \mathbf{Bad}^1_{\mathbb{R}}$.

In Section 3.3, these and further examples are discussed in more details and in greater generality. In particular, we discuss intersections of $\mathbf{Bad}(\mathcal{F})$ with 'nice fractal sets' which are, for instance in the case of $\mathbf{Bad}^n_{\mathbb{R}^n}$, supports of absolutely decaying measures (see (3.15) for a definition).

2.2 Schmidt Games on Paramater Spaces

In this section, we combine two versions of Schmidt's game due to [34] and [40] in order to introduce a new modification. We first introduce but modify the setting of this section which is the terminology of [34]. Let (X, d) be a complete metric space. Fix $t_* \in \mathbb{R} \cup \{-\infty\}$ and define $\Omega \equiv X \times (t_*, \infty)$, the set of *formal balls* in X. Let $\mathcal{C}(X)$ be the set of nonempty compact subsets of X and assume we are given a function $\psi : \Omega \to \mathcal{C}(X)$ such that, for all $(x, t) \in \Omega$ and for all $s \geq 0$, we have

$$\psi(x, t + s) \subset \psi(x, t). \tag{2.2.1}$$

We can hence view Ω as parameter space for the function ψ which we call *monotonic*.

For instance, if X is proper, set $t_* = -\infty$ and for $x \in X$, $r > 0$, let $B(x, r) \equiv \{y \in X : d(x, y) \leq r\} \in \mathcal{C}(X)$. For $\sigma > 0$, the *standard function* $\bar{\psi}_\sigma \equiv B_\sigma$ is given by the monotonic function

$$B_\sigma : X \times (-\infty, \infty) \to \mathcal{C}(X), \qquad B_\sigma(x, t) \equiv B(x, e^{-\sigma t}). \tag{2.2.2}$$

Moreover, for a subset $Y \subset X$ and $t > t_*$, we call $(Y, t) \equiv \{(y, t) : y \in Y\}$ a *formal neighborhood*, and define $\mathcal{P} = \mathcal{P}(X) \times (t_*, \infty)$ to be the set of formal neighborhoods. Define the *ψ-neighborhood* of $(Y, t) \in \mathcal{P}$ by

$$\psi(Y, t) \equiv \bigcup_{y \in Y} \psi(y, t).$$

Note that by (2.2.1), $\psi(Y, t + s) \subset \psi(Y, t)$ for all $s \geq 0$.

2.2.1 The ψ-modified Schmidt game

We recall the (ψ, a_*)-modified Schmidt game due to [34], where $a_* \geq 0$. Two players, A and B, pick numbers a and b both bigger than a_*. Player B starts with his first move by choosing a formal ball $\omega_1 = (x_1, t) \in \Omega$. Given a choice $\omega_k = (x_k, t_k)$ of B, due to (2.2.1), player A can (and must) choose a formal ball $\bar{\omega}_k = (\bar{x}_k, t_k + a) \in \Omega$ such that $\psi(\bar{\omega}_k) \subset \psi(\omega_k)$. Also player B continues by choosing a formal ball $\omega_{k+1} = (x_{k+1}, t_k + a + b) \in \Omega$ such that $\psi(\omega_{k+1}) \subset \psi(\bar{\omega}_k)$. The game continues in this manner and we obtain a nested sequence of compact sets

$$B_1 \equiv \psi(\omega_1) \supset A_1 \equiv \psi(\bar{\omega}_1) \supset B_2 \equiv \psi(\omega_2) \supset \cdots \supset B_k \equiv \psi(\omega_k) \supset A_k \equiv \psi(\bar{\omega}_k) \supset \ldots,$$

where $\omega_k = (x_k, t_k)$ and $\bar{\omega}_k = (\bar{x}_k, \bar{t}_k)$ satisfy

$$t_k = t_1 + (k-1)(a+b), \text{ and } \quad \bar{t}_k = t_1 + (k-1)(a+b) + a.$$

The intersection of compact nested sets, given by

$$\bigcap_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} A_k,$$

is nonempty and compact. A subset $S \subset X$ is called (ψ, a_*, a, b)-*winning*, if player A can find a *strategy* which guarantees that $\cap_{k \geq 1} B_k$ intersects S, no matter what B's choices are. The set S is called (ψ, a_*, a)-*winning* if S is (ψ, a_*, a, b)-*winning* for every $b > a_*$. S is (ψ, a_*)-*winning* if it is (ψ, a_*, a)-*winning* for some $a > a_*$ and ψ-*winning* if it is (ψ, a_*)-winning for some $a_* \geq 0$.

With respect to the standard monotonic function $\psi = B_1$, the game described above coincides with the original (α, β)-Schmidt game for the choice

$$a = -\log(\alpha), \quad b = -\log(\beta), \quad a_* = 0, \quad t_* = -\infty.$$

If moreover $X = \mathbb{R}^n$ is the Euclidean space, then a winning set S enjoys the following properties (see [13, 50, 11]).

1. A winning set is dense and thick; a subset Y of a metric space X is called *thick*, if for any nonempty open set $U \subset X$, $Y \cap U$ has the same Hausdorff-dimension as X,

2. a countable intersection of α-winning sets is α-winning,

3. winning sets are preserved by bi-Lipschitz homeomorphisms, and,

4. winning sets are *incompressible*; that is, given a nonempty open set $U \subset \mathbb{R}^n$ and a countable sequence of uniformly[2] bi-Lipschitz maps $F_i : U \to \mathbb{R}^n$, then $\cap_{i=1}^{\infty} F_i^{-1}(S)$ has Hausdorff-dimension n.

Unfortunately, these properties are not satisfied in general; in fact, see [34], Proposition 5.2, for a ψ-winning set which is of Hausdorff-dimension zero in a space of positive dimension. However, the following (and further) properties for the ψ-modified Schmidt game can be found in [34][3] .

[2] That is, the Lipschitz constants L_i of F_i are bounded.
[3] Note that [34] uses a slightly different setting. Nevertheless, the properties hold true with the same arguments.

1. Let $S_i \subset X$, $i \in \mathbb{N}$, be a sequence of (ψ, a_*, a)-winning sets. Then, $\cap_{i\geq 1} S_i$ is also (ψ, a_*, a)-winning.

2. Let $\Omega_i = X_i \times (t_*, \infty)$, and ψ_i be given for $i = 1, 2$. Suppose that $S_i \subset X_i$ is a (ψ_i, a_*)-winning set for $i = 1, 2$. Then $S_1 \times S_2$ is a $(\psi_1 \times \psi_2, a_*)$-winning set in $X_1 \times X_2$ with the product metric, where $\psi_1 \times \psi_2(x_1, x_2, t) \equiv \psi_1(x_1, t) \times \psi_2(x_2, t)$.

Moroever, let μ be a locally finite Borel measure on X. Denote by $O(x, r) \equiv \{y \in X : d(x, y) < r\}$ the open metric ball around x. The *lower pointwise dimension* of μ at $x \in \text{supp}(\mu)$ is defined by

$$d_\mu(x) \equiv \liminf_{r\to 0} \frac{\log(\mu(O(x, r))}{\log r}.$$

For every open $U \subset X$ with $\mu(U) > 0$,

$$d_\mu(U) \equiv \inf_{x\in U\cap\, \text{supp}(\mu)} d_\mu(x),$$

which is known to be a lower bound for the Hausdorff-dimension of $\text{supp}(\mu)\cap U$ (see [18], Proposition 4.9 (a)). The measure μ is called *Federer* if there are $K > 0$ and $R > 0$ such that for all $x \in \text{supp}(\mu)$ and $0 < r < R$,

$$\mu(O(x, 3r)) \leq K\mu(O(x, r)).$$

In the case that we consider the standard function ψ_1, i.e., we focus on the classical Schmidt-game, the following lower estimate on the Hausdorff-dimension is given.

Proposition 2.2 ([34], Proposition 5.1)**.** *If S is a winning set (in the sense of Schmidt) in a complete metric space X which supports a Federer measure μ with $X = \text{supp}(\mu)$, then for every nonempty open set $U \subset X$, we have $\dim(S \cap U) \geq d_\mu(U)$, where 'dim' stands for the Hausdorff-dimension.*

If μ satisfies a *power law*, that is, there exist δ, c_1, c_2 and $R > 0$ such that for every $0 < r < R$ and $x \in \text{supp}(\mu)$ we have

$$c_1 r^\delta \leq \mu(O(x, r)) \leq c_2 r^\delta,$$

then μ is Federer and we have $d_\mu(x) = \delta$.

2.2.2 The weak ψ-modified Schmidt game

For $b_* > 0$ consider the following modification of rules for the players A and B. Fix a parameter $b \geq b_*$. Player B starts again with a formal ball $\omega_1 = (x_1, t_1) \in \Omega$. Then, given a formal ball $\omega_k = (x_k, t_k) \in \Omega$ of player B, player A can choose a nonempty set $\mathcal{L}_b(\omega_k) \subset \Omega$ of *legal moves*,

$$\mathcal{L}_b(\omega_k) \equiv \{\omega = (x, t_k + \bar{b}) : b_* \leq \bar{b} \leq m_k b,\ \psi(\omega) \subset \psi(\omega_k),\ \mathcal{C}(\omega_k)\}, \tag{2.2.3}$$

where $\mathcal{C}(\omega_k)$ denotes possible further conditions which A requires and $m_k \in \mathbb{N}$ is an integer which A chooses at each step. B then chooses a formal ball $\omega_{k+1} \in \mathcal{L}_b^\psi(\omega_k)$ and the game continues in this manner. Since $\psi(\omega_k) \supset \psi(\omega_{k+1})$, we obtain a nested sequence

$$B_1 \supset B_2 \supset \cdots \supset B_k \supset \ldots,$$

where $B_k = \psi(x_k, t_k)$ satisfies condition $\mathcal{C}(\omega_k)$. If the nonempty compact set $\cap_{k\geq 1} B_k$ intersects a given set $S \subset X$, then A *wins* this game. The set S is called *weakly* (ψ, b_*, b)*-winning* if player A finds a strategy such that A wins for every possible game, given the parameter b. S is called *weakly* (ψ, b_*)*-winning* if it is *weakly* (ψ, b_*, b)*-winning* for every $b \geq b_*$ and *weakly* ψ*-winning* if it is weakly (ψ, b_*)-winning for some $b_* > 0$.

Remark 2.3. Note that to leave $\mathcal{L}_b(\omega_k)$ nonempty is always possible by (2.2.1). Moreover, the conditions that $\psi(\omega_{k+1}) \subset \psi(\omega_k)$ and $\bar{b} \leq m_k b$ seemed to be the least suitable conditions to already assume for player A (and for our purpose) but can of course be weakened as well. The requirement that $b_* > 0$ implies that $t_k \to \infty$, which can be avoided if we say that A wins when $t_k \not\to \infty$.[4]

The difference to the original ψ-modified Schmidt game is that, rather than forcing B in a certain direction, A can precisely determine B's choices in the next move. Since A might leave B only one choice in each step, the weak ψ-modified Schmidt game loses in some sense the character of a game. Moreover, the conditions $C(\omega_k)$ determine the 'control' player A chooses and the more conditions A requires, the less properties S might enjoy. Therefore, player A also has an interest in leaving B as much choices and freedom as possible, with respect to a winning strategy. In particular, we are interested in conditions on strategies for player A such that a weakly ψ-winning set S satisfies similar or even the same properties than winning sets for Schmidt's, McMullen's or the ψ-game.

We want to point out the following special cases of modifications of Schmidt's game, where, given a choice $\omega_k = (x_k, t_k) \in \Omega$ of B, A chooses a set $A_k \subset X$ and requires for the condition $\mathcal{C}(\omega_k)$ that

$$\psi(\omega) \subset \psi(\omega_k) - A_k \quad \text{and} \quad m_k = m_* = 1. \tag{2.2.4}$$

First, let $\mathcal{S} = \{S \subset X\}$ be a given collection of subsets of $\bar{X}$. Assume then that for each of the sets A_k is either empty or a ψ-neighborhood

$$A_k = \psi(S_k, t_k + a_k), \quad S_k \in \mathcal{S}, \quad a_k \geq b, \tag{2.2.5}$$

and call a winning set under these requirements *absolute* ψ*-winning with respect to* $\mathcal{S}$; compare with [20] for the case that $\psi = B_1$ is the standard function.

Consider the standard case that

$$X = \mathbb{R}^n, \quad \psi = B_1, \quad b_* = \log(3), \quad t_* = -\infty.$$

Clearly, if $\mathcal{S}$ is the set of points in $\mathbb{R}^n$, this modification corresponds to the one of McMullen [40], called *absolute winning game* and a winning set is called *absolute winning*. Note that an absolute winning set in $\mathbb{R}^n$ is in particular a Schmidt winning set and in fact satisfies stronger properties (see [40]).

In the case that $\mathcal{S}$ denotes the set of affine hyperplanes in $\mathbb{R}^n$ (or in a vector space), then this modification corresponds to the one of Broderick et al. [11], called *hyperplane absolute winning game* and a winning set is called *hyperplane absolute winning* (short HAW set). Again, note that a HAW-set in $\mathbb{R}^n$ is in particular a Schmidt winning set and in fact satisfies stronger properties (see [11]).

[4] This alternative rule was chosen, for instance, by [20]. Note that, if S is dense in X and $\psi = B_1$, then B looses as soon as $t_k \not\to \infty$.

Second, let $b_* > a \geq a_* \geq 0$. Assume the sets A_k to be the complements of ψ-balls

$$A_k = \psi(y_k, t_k + a)^C, \quad (y_k, t_k + a) \in \Omega \quad \text{with} \quad \psi(y_k, t_k + a) \subset B_k = \psi(x_k, t_k),$$

If $\mathcal{C}(\omega_k)$ moreover requires that $\bar{b} = b$ in (2.2.3), this modification corresponds to the $(\psi, a_*, a, b - a)$-game and in particular to Schmidt's game for $X = \mathbb{R}^n$ and $\psi = B_1$.

Now in general, if $\mathcal{C}(\omega_k)$ requires for all sets $A_k \subset X$ which A chooses in (2.2.4) that there exists a formal ball $\bar{\omega} = (\bar{x}, t_k + b_*) \in \Omega$ such that

$$\psi(\bar{\omega}) \subset \psi(\omega_k) - A_k, \tag{2.2.6}$$

then a weakly ψ-winning set is ψ-winning.

Lemma 2.4. *If (2.2.6) is satisfied, then a weakly (ψ, b_*)-winning set S is (ψ, a_*)-winning for all $a_* \geq b_*$.*

Proof. Given $a \geq a_* \geq b_*$, $b > 0$, set $\tilde{b} = a + b \geq b_*$. Let player A play the (ψ, a_*, a, b)-modified Schmidt game and consider a further player $\bar{A}$ who plays the weak $(\psi, b_*, \tilde{b})$-modified Schmidt game. Suppose that player B has chosen his k-th move $\omega_k = (x_k, t_k) \in \Omega$. By (2.2.6), $\bar{A}$ chooses a set $A_k \subset X$ such that there exists a formal ball $\bar{\omega} = (\bar{x}, t_k + b_*) \in \Omega$ with

$$\psi(\bar{\omega}) \subset \psi(\omega_k) - A_k.$$

By (2.2.1) and since $a \geq b_*$, there exists a formal ball $\bar{\omega}_{k+1} = (\bar{x}_{k+1}, t_k + a) \in \Omega$ such that $\psi(\bar{\omega}_{k+1}) \subset \psi(\bar{\omega})$ which we take as A's choice. Note that any move $\omega_{k+1} = (x_{k+1}, t_k + k(a+b)) = (x_{k+1}, t_k + \tilde{b}) \in \Omega$ such that $\psi(\omega_{k+1}) \subset \psi(\bar{\omega}_{k+1})$ of B is a legal move for both games. Since $\bar{A}$ has a weak winning strategy, we see that

$$\bigcap_{k \geq 1} \psi(\bar{\omega}_k) = \bigcap_{k \geq 1} \psi(\omega_k)$$

intersects S. Hence, A wins and S is also a (ψ, a_*, a, b)-winning set. □

Hence, in view of the properties of ψ-winning sets (see Subsection 2.2.1), we will consider conditions which ensure that (2.2.6) is satisfied so that that the weak ψ-modified Schmidt game is at least as strong as the ψ-modified Schmidt game. However, some of the properties of ψ-winning sets can still be true in the weaker setting.

In fact, let S be a (ψ, b_*, b)-weakly-winning set. In order to estimate the lower bound for the Hausdorff-dimension of S, we consider the conditions given by [34] and only need to modify $(\mu 2)$ below:

(MSG1) For any open set $\emptyset \neq U \subset X$ there is $\omega \in \Omega$ such that $\psi(\omega) \subset U$.

(MSG2) There exist C, $\sigma > 0$ such that $\operatorname{diam}(\psi(x,t)) \leq Ce^{-\sigma t}$ for all $(x,t) \in \Omega$.

Note that if (MSG1) is satisfied, a weakly ψ-winning set is dense. Let moreover μ be a locally finite Borel measure on X such that:

$(\mu 1)$ For every formal ball $\omega \in \Omega$ we have $\mu(\psi(\omega)) > 0$.

($\mu 2$) There exist constants $c = c(b) > 0$ and $m_* = m_*(b) \in \mathbb{N}$ with the following property: If $\omega_k \in \Omega$ is a choice of B in the (ψ, b_*, b)-game, there exist legal moves $\omega^1_{k+1}, \dots, \omega^n_{k+1} \in \mathcal{L}_b(\omega_k)$, $\omega^i_{k+1} = (x^i_{k+1}, t_k + m_k b)$, $m_k = m_*$, with respect to the (ψ, b_*, b)-strategy of A, which satisfy $\mu(\psi(\omega^i_{k+1}) \cap \psi(\omega^j_{k+1})) = 0$ when $i \neq j$ as well as

$$\mu(\bigcup_{i=1\dots n} \psi(\omega^i_{k+1})) \geq c \cdot \mu(\psi(\omega_k)). \tag{2.2.7}$$

Note that from (MSG1) and ($\mu 1$), μ must have full support, i.e. $\operatorname{supp}(\mu) = X$.

Proposition 2.5. *Suppose that X, Ω, ψ and the measure μ satisfy (MSG1-2) and (μ1-2) with respect to a weakly (ψ, b_*, b)-winning set S. Then for every nonempty open set $U \subset X$ we have that*

$$\mathit{dim}(S \cap U) \geq d_\mu(U) + \frac{\log(c)}{\sigma m_* b},$$

where σ, $c = c(b)$ and m_* are the constants of (MSG2) and ($\mu 2$).

Proof. Similarly to the proof of [34], Theorem 2.7 (see Subsection 3.2.18 for details), one constructs a strongly treelike countable family of compact subsets of X whose limit set $A_\infty \cap U$ is a subset of $S \cap U$. We start with a formal ball $\omega_1 \in \Omega$ such that $\psi(\omega_1) \subset U$. The difference is that, instead of using the choices of A, we use the choices of B given in ($\mu 2$) in order to obtain that

$$\dim(A_\infty \cap U) \geq d_\mu(U) + \frac{\log(c)}{\sigma m_* b}.$$

The proof follows. □

2.2.3 The framework, conditions on the resonant sets and strategies

Let $\bar{X}$ be a proper metric space and X a closed subset of $\bar{X}$ which is, with the induced metric, a complete metric space. In many applications, we are interested in playing the ψ-game on X but do not require the resonant sets to be contained in X but in $\bar{X}$. Therefore, let $\bar{\Omega} = \bar{X} \times (t_*, \infty)$ and $\Omega = X \times (t_*, \infty) \subset \bar{\Omega}$. Let $\bar{\psi} : \bar{\Omega} \to \mathcal{C}(\bar{X})$ be a monotonic function on $\bar{\Omega}$, which induces a monotonic function ψ on Ω, defined by

$$\psi(\omega) \equiv \bar{\psi}(\omega) \cap X, \quad \omega \in \Omega.$$

Now, let Λ be a countable index set and $\{R_\lambda \subset \bar{X} : \lambda \in \Lambda\}$ be a family of *resonant sets* in $\bar{X}$, where we assign a *size* $s_\lambda \geq s_*$ to every R_λ with $t_* < s_* \in \mathbb{R}$. We consider the contractions of the $(\bar{\psi}, s_\lambda)$-neighborhoods of R_λ,

$$\psi_\lambda(s) \equiv \bar{\psi}(R_\lambda, s_\lambda + s) \subset \bar{\psi}(R_\lambda, s_\lambda), \quad s \geq 0.$$

Denote this family by

$$\mathcal{F} = (\Lambda, R_\lambda, s_\lambda).$$

Assume that the family $\mathcal{F}$ satisfies the following conditions.

(N) The resonant sets $\{R_\lambda\}$ are *nested* with respect to their sizes, that is, for $\lambda, \beta \in \Lambda$ we have

$$s_\lambda \leq s_\beta \implies R_\lambda \subset R_\beta.$$

(D) The sizes $\{s_\lambda\}$ are *discrete*, that is, for all $t > t_*$ we have

$$|\{\lambda \in \Lambda : s_\lambda \leq t\}| < \infty.$$

We then define the set of *badly approximable points* with respect to $\mathcal{F}$ by

$$\mathbf{Bad}_X^{\bar{\psi}}(\mathcal{F}) = \{x \in X : \exists\, c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda\in\Lambda} \psi_\lambda(c)\},$$

or simply by $\mathbf{Bad}(\mathcal{F})$ if there is no confusion about the parameter spaces under consideration.

For a parameter $t \geq s_1$, let $\lambda_t \in \Lambda$ such that s_{λ_t} is the maximal size with $s_\lambda \leq t$ (see (D)). We define the *relevant resonant set* with respect to the parameter t by

$$R(t) \equiv \bigcup_{s_\lambda \leq t} R_\lambda = R_{\lambda_t},$$

(see (N)), and we call s_{λ_t} the *relevant size*. Moreover, for $t \geq s_1$ and $b > 0$, we let

$$R(t, b) \equiv R(t) - R(t - b) \tag{2.2.8}$$

be the set of resonant points for which the 'minimal size' belongs to the spectrum $(r - b, r]$.

For $b_* > 0$, $n_* \in \mathbb{N}$ and $L_* \geq 0$, we consider two conditions, a strong and a weak one, on the space X and a nested and discrete family $\mathcal{F}$.

(b_*) (Ω, ψ) is *strongly* b_**-diffuse with respect to the family* $\mathcal{F}$, if there exists $n \in \mathbb{N}$ such that for all formal balls $\omega = (x, r) \in \Omega$ there exists a formal ball $\omega' = (x', r + b_*) \in \Omega$ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(R(r), r + nb_*). \tag{2.2.9}$$

(b_*, n_*, L_*) (Ω, ψ) is (b_*, n_*, L_*)*-diffuse with respect to the family* $\mathcal{F}$, if for all $b > b_*$ there exists a $n = n(b) \in \mathbb{N}$ such that, for all formal balls $\omega = (x, r) \in \Omega$, there exists a formal ball $\omega' = (x', r + b) \in \Omega$ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(R(r, n_*(b + L_*)), r + nb). \tag{2.2.10}$$

Condition (b_*) is too strong in general (see Subsections 2.3.5 and 2.3.6, Case 3.) but implies (b_*, n_*, L_*) for all $n_* \in \mathbb{N}$, $L_* \geq 0$ and is sufficient to guarantee that if $\mathbf{Bad}(\mathcal{F})$ is weakly (ψ, b_*)-winning it is also (ψ, b_*)-winning by Lemma 2.4.

In fact, under these conditions we can define the following strategies for the set $S = \mathbf{Bad}(\mathcal{F})$: Fix a parameter $b > b_*$ and assume B chose the formal ball $\omega_1 = (x_1, t_1) \in \Omega$.

The strategy for player A under the condition $(b_*, 1, 0)$. Let $m_* \in \mathbb{N}$ be the minimal integer such that $m_* b \geq t_1 - s_1$ and let $l_* = n(m_* b)$ be as in (2.2.10). Given the times t_k, define the relevant resonant sets $R_k \equiv R(t_k, m_* b)$. For $k \geq 1$, assume that B chose the formal ball $\omega_k = (x_k, t_k) \in \Omega$. Note that if we set

$$A_k \equiv \bar{\psi}(R_k, t_k + l_*(m_* b)) \cap X, \tag{2.2.11}$$

then, by (2.2.10), there exists a formal ball $\omega' = (x'_k, t_k + m_* b) \in \Omega$ such that

$$\psi(\omega') \subset \psi(x_k, t_k) - \bar{\psi}(R_k, t_k + l_*(m_* b)) = \psi(x_k, t_k) - A_k. \tag{2.2.12}$$

Thus, we define the strategy of player A to choose the nonempty set of legal moves

$$\mathcal{L}_b(\omega_k) \equiv \{\omega = (x, t_k + \bar{b}) : b_* \leq \bar{b} \leq m_* b,\ \psi(x, t_k + \bar{b}) \subset \psi(\omega_k) - A_k\}. \tag{2.2.13}$$

The strategy for player A under the condition (b_*). Let now $R_k = R(t_k)$, $m_* = 1$ and $l_* = n(b_*)$ as in (2.2.9) and set

$$A_k \equiv \bar{\psi}(R(t_k), t_k + l_* b_*) \cap X. \tag{2.2.14}$$

We define the strategy of player A with respect to ω_k to choose the set of legal moves

$$\mathcal{L}_b(\omega_k) \equiv \{\omega = (x, t_k + \bar{b}) : b_* \leq \bar{b} \leq b,\ \psi(x, t_k + \bar{b}) \subset \psi(\omega_k) - A_k\}, \tag{2.2.15}$$

which is nonempty by (2.2.9).

With respect to these strategies, we show our first main result.

Theorem 2.6. *Let $\mathcal{F}$ be a nested and discrete family.*

*If (Ω, ψ) is (b_*, n_*, L_*)-diffuse with respect to $\mathcal{F}$, then (2.2.13) defines a weakly (ψ, b_*, b)-winning strategy for the set **Bad**$(\mathcal{F})$.*

*If (Ω, ψ) is strongly b_*-diffuse with respect to $\mathcal{F}$, then **Bad**$(\mathcal{F})$ is in particular (ψ, a_*)-winning for every $a_* \geq b_*$.*

Proof of Theorem 2.6. We first show that the induced strategy is winning under the condition $(b_*, 1, 0)$. Hence, let $x_0 \in \cap_{k\geq 1}\psi(\omega_k)$. Assume that $x_0 \in \bar{\psi}(R_{\lambda_0}, s_{\lambda_0})$ for some $\lambda_0 \in \Lambda$ (if no such λ_0 exists, then A has already won). Since $t_1 - m_* b \leq s_1$ and $t_k \to \infty$ as $t_{k+1} \geq t_k + b_*$, we know that R_{λ_0} is covered by $R_{\lambda_0} \subset \cup_{k=1}^N R_k$ by finitely many sets $R_k = R(t_k, m_* b)$. (where we let N be the minimal such integer). Thus, there exists $1 \leq k \leq N$ such that $x_0 \in \bar{\psi}(R_k, s_{\lambda_0})$. Note that from the definition of R_k and the minimality of N we have $s_{\lambda_0} > t_k - m_* b$. Thus, (2.2.12) and the induced strategy (2.2.13) imply that

$$x_0 \in \psi(\omega_{k+1}) \subset \psi(\omega_k) - \bar{\psi}(R_k, t_k + l_* m_* b),$$

and in particular,

$$\begin{aligned} x_0 &\notin \bar{\psi}(R_k, t_k + l_* m_* b) \\ &= \bar{\psi}(R_k, t_k - m_* b + (l_* + 1) m_* b) \supset \bar{\psi}(R_k, s_{\lambda_0} + (l_* + 1) m_* b), \end{aligned} \tag{2.2.16}$$

by (2.2.1). This shows that

$$x_0 \notin \cup_{k=1}^N \bar{\psi}(R_k, s_{\lambda_0} + (l_* + 1) m_* b) \supset \bar{\psi}(R_{\lambda_0}, s_{\lambda_0} + (l_* + 1) m_* b).$$

Therefore, $x_0 \in$ **Bad**$(\mathcal{F})$, since

$$x_0 \notin \bigcup_{\lambda \in \Lambda} \bar{\psi}(R_\lambda, s_\lambda + (l_* + 1) m_* b).$$

Hence, A wins and we defined a winning strategy for the parameter $b > b_*$.

Now, assume that (b_*) is satisfied and note that in particular (2.2.6) is satisfied with respect to the sets A_k in (2.2.14). Hence, since (b_*) implies $(b_*, 1, 0)$, the first part of the theorem and Lemma 2.4 finish the proof. □

We want to show that the conditions are preserved under maps which satisfy some kind of bi-Lipschitz-property and by finite intersections.

First, let $(\bar{X}, \bar{\Omega}_{\bar{X}}, \psi_{\bar{X}})$ and $(\bar{Y}, \bar{\Omega}_{\bar{Y}}, \psi_{\bar{Y}})$ be two parameter spaces with monotonic functions. For a given constant $L_* \geq 0$, consider a map $F : \bar{X} \to \bar{Y}$ such that

$$\psi_{\bar{Y}}(F(x), r + 2L_*) \subset F(\psi_{\bar{X}}(x, r + L_*)) \subset \psi_{\bar{Y}}(F(x), r), \tag{2.2.17}$$

for all formal balls $(x, r) \in \Omega_{\bar{X}}$. If both $\psi_{\bar{X}} = B_1^{\bar{X}}$ and $\psi_{\bar{Y}} = B_1^{\bar{Y}}$, then F is a L_*-bi-Lipschitz map. Given a nested, discrete family of resonant sets $\mathcal{F}_{\bar{X}} = (\Lambda, R_\lambda, s_\lambda)$, consider the induced nested, discrete family in $\bar{Y}$,

$$\mathcal{F}_Y \equiv F(\mathcal{F}_X) \equiv (\Lambda, F(R_\lambda), s_\lambda - L_*).$$

If F is bijective, $X \subset \bar{X}$, then it is readily checked that $F(\mathbf{Bad}_X^{\psi_{\bar{X}}}(\mathcal{F}_X)) = \mathbf{Bad}_{F(X)}^{\psi_{\bar{Y}}}(\mathcal{F}_{\bar{Y}})$.

Proposition 2.7. *Let $(\bar{X}, \bar{\Omega}_{\bar{X}}, \psi_{\bar{X}})$, $(\bar{Y}, \bar{\Omega}_{\bar{Y}}, \psi_{\bar{Y}})$ and let $F : \bar{X} \to \bar{Y}$ be a bijective map which satisfies (2.2.17). If (Ω_X, ψ_X) is [strongly b_*-diffuse] $(b_*, n_*, 2L_*)$-diffuse with respect to $\mathcal{F}_X$, then, for $Y \equiv F(X)$, (Ω_Y, ψ_Y) is [strongly $(b_* + 2L_*)$-diffuse] $(b_* + 2L_*, n_*, 0)$-diffuse with respect to $\mathcal{F}_Y$.*

Proof. Assume that (Ω_X, ψ_X) is $(b_*, n_*, 2L_*)$-diffuse with respect to $\mathcal{F}_X$. Let $(y, r) \in \Omega_Y$ and $b > b_*$. There exists $n \in \mathbb{N}$ and $\omega' = (\bar{x}, r + L_* + b) \in \Omega_X$ such that

$$\psi_X(\omega') \subset \psi_X(F^{-1}(y), r + L_*) - \psi_{\bar{X}}(R_{\bar{X}}(r + L_*, n_*(b + 2L_*)), r + L_* + nb). \tag{2.2.18}$$

From (2.2.17) we have

$$\begin{aligned} \psi_Y(F(\bar{x}), r + (b + 2L_*)) &\subset F(\psi_X(\bar{x}, r + L_* + b)) \\ &= F(\psi(\omega')) \\ &\subset F(\psi_X(F^{-1}(y), r + L_*)) \subset \psi_Y(y, r). \end{aligned}$$

Note that $F(R_X(r + L_*, t)) = R_Y(r, t)$. We obtain

$$\begin{aligned} \psi_{\bar{Y}}(R_{\bar{Y}}(r, n_*(b + 2L_*)), r + n(b + 2L_*)) &\subset \psi_{\bar{Y}}(R_{\bar{Y}}(r, n_*(b + 2L_*)), r + 2L_* + nb) \\ &\subset F(\psi_{\bar{X}}(R_{\bar{X}}(r + L_*, n_*(b + 2L_*)), r + L_* + nb). \end{aligned}$$

By (2.2.18) we know that $F(\psi_X(\omega'))$ is disjoint to $F(\psi_{\bar{X}}(R(r + L_*, n_*(b + 2L_*)), r + L_* + nb)$ and hence we see that (Ω_Y, ψ_Y) is $(b_* + 2L_*, n_*, 0)$-diffuse with respect to $\mathcal{F}_Y$.

The case when (Ω_X, ψ_X) is strongly b_*-diffuse with respect to $\mathcal{F}_X$ follows similarly. □

Now consider finitely many families $\mathcal{F}_i = (\Lambda^i, R^i_{\lambda^i}, s^i_{\lambda^i})$, $i = 1, \ldots, n_*$, of nested and discrete families in $\bar{X}$. When (Ω, ψ) is strongly b_*-diffuse with respect to each $\mathcal{F}_i$, we know from Theorem 2.6 and properties of ψ-modified Schmidt games that $\cap_{i=1}^{n_*} \mathbf{Bad}(\mathcal{F}_i)$ is (ψ, b_*)-winning (and the same is true for countable intersections). In the weaker setting, we show the following.

Proposition 2.8. *If (Ω, ψ) is (b_*, n_*, L_*)-diffuse with respect to each family $\mathcal{F}_i$, then $\cap_{i=1}^{n_*} \boldsymbol{Bad}(\mathcal{F}_i)$ is weakly (ψ, b_*)-winning.*

Proof. Assume that X is (b_*, n_*, L_*)-diffuse with respect to each family $\mathcal{F}_i$ and let $b > b_*$. We only need to modify the strategy for player A with respect to the sets A_k in (2.2.11).In fact, if $\omega_1 = (x_1, t_1) \in \Omega$ is the first move of B, we let again $m_* \in \mathbb{N}$ such that $\tilde{b} = m_* b \geq t_1 - s_1$. Let $k = ln_* + s$ for $l \in \mathbb{N}_0$ and $1 \leq s \leq n_*$. Denote by $R_l^s = R^s(t_k, n_*\tilde{b})$, where R^s is the subset of the resonant sets with respect to $\mathcal{F}_s$. Moreover, let $l_* = l(\tilde{b}) = l_1 + \cdots + l_{n_*}$, where $l_i = n_i(\tilde{b})$ is the constant in (2.2.10) with respect the family $\mathcal{F}_i$. We therefore define

$$A_k = \bar{\psi}(R_l^s, t_k + l_*\tilde{b}) \cap X.$$

By (2.2.10), there exists a formal ball $\omega_{k+1} = (x_{k+1}, t_k + \tilde{b}) \in \Omega$ such that

$$\psi(\omega_{k+1}) \subset \psi(\omega_k) - \bar{\psi}(R_l^s, t_k + l_*\tilde{b}) = \psi(\omega_k) - A_k,$$

which shows that the set $\mathcal{L}_b^{\psi}(\omega_k)$ in (2.2.13) modified with respect to the set A_k is nonempty.

Thus, for $s = 1, \ldots, n_*$ and $x_0 \in \cap_{k\geq 1}\psi(\omega_k)$, we deduce similarly to (3.9) that $x_0 \in \mathbf{Bad}(\mathcal{F}_s)$. In particular, $x_0 \in \cap_s \mathbf{Bad}(\mathcal{F}_s)$ which is thus a (ψ, b_*)-weakly-winning set. □

Given $\bar{Y}_i, Y_i, \bar{\psi}_i$, $i = 1, \ldots, n_*$, assume that $\mathcal{F}_i = (\Lambda_i, R^i_{\lambda^i}, s_{\lambda^i})$ is a nested discrete family in $\bar{Y}_i$ and that $F_i : \bar{Y}_i \to \bar{X}$ is a bijective map satisfying (2.2.17) for a constant L_* with $F(Y_i) = X$. As a corollary, if each (Ω_i, ψ_i) is (b_*, n_*, L_*)-diffuse with respect to $\mathcal{F}_i$, then

$$\cap_{i=1}^{n_*} F_i(\mathbf{Bad}_{Y_i}^{\bar{\psi}_i}(\mathcal{F}_i)) \subset X \tag{2.2.19}$$

is a weakly $(\psi_X, b_* + 2L_*)$-winning set.

Remark 2.9. Let $\bar{\Omega}_i = \bar{X}_i \times (t_*, \infty)$ and $\bar{\psi}_i$ be given for $i = 1, 2$, where $\bar{\psi}_1 \times \bar{\psi}_2(x_1, x_2, t) = \bar{\psi}_1(x_1, t) \times \bar{\psi}_2(x_2, t)$. Moreover, let $\mathcal{F}_i = (\Lambda, R^i_\lambda, s_\lambda)$ be nested and discrete with the same index set and the same sizes. If (b_*) or (b_*, n_*, L_*) respectively is satisfied for both $X_i \subset \bar{X}_i$ and $\mathcal{F}_i$, then (b_*) or (b_*, n_*, L_*) respectively is satisfied for $X_1 \times X_2$ with respect to $\mathcal{F} = (\Lambda, R^1_\lambda \times R^2_\lambda, s_\lambda)$ and $\bar{\psi}_1 \times \bar{\psi}_2$.

2.2.4 Diffuse spaces and absolutely decaying measures.

In this subsection we first discuss diffusion properties of the subspace X in $\bar{X}$, or rather of the parameter spaces (Ω, ψ) in $(\bar{\Omega}, \bar{\psi})$, and then relate these properties to the (local) structure and distribution of the resonant sets of a given family $\mathcal{F}$ in $\bar{X}$.

In the following, let X be a nonempty closed subset of a proper metric space $\bar{X}$ with a given monotonic function $\bar{\psi}$. We give a special class of diffuse spaces X in which the resonant sets might be more general than points but are still nicely structured and distributed. More precisely, let $\mathcal{S} = \{S \subset \bar{X}\}$ be a given collection of subsets of $\bar{X}$. For instance, let $\mathcal{S}$ be the set of metric spheres $S(\bar{\omega}) \equiv \{y \in \bar{X} : d(\bar{x}, y) = e^{-t}\}$, where $\bar{\omega} = (\bar{x}, t) \in \bar{\Omega}$, or the set of affine hyperplanes in $\mathbb{R}^n$.

For $b_* > 0$, (Ω, ψ) is called *b_*-diffuse with respect to $\mathcal{S}$*, if for any formal ball $\omega = (x, t) \in \Omega$ and any set $S \in \mathcal{S}$ there exists a formal ball $\omega' = (x', t + b_*) \in \Omega$ such that

$$\psi(\omega') \subset \psi(\omega) - \bar{\psi}(S, t + b_*). \tag{2.2.20}$$

For the standard function $\psi = B_1$, our definition above is similar to the following special cases.

1. When $\bar{X} = \mathbb{R}^n$ is the Euclidean space and $\mathcal{S}$ is the set of k-dimensional affine hyperplanes in $\mathbb{R}^n$ $(0 \le k < n)$, then $X \subset \mathbb{R}^n$ is called *k-dimensionally hyperplane diffuse*; see [11].

2. When $k = 0$, that is, $\mathcal{S}$ is the set of points in a metric space $\bar{X}$, and $\beta = b_*$, then $X \subset \bar{X}$ is called *β-diffuse*; see [38].

For a class of β-diffuse spaces, let X be a *uniformly perfect* metric space, that is, there exists $r_* \in \mathbb{R} \cup \{-\infty\}$ and a constant $0 < \nu < \infty$ such that for any metric ball $B(x, e^{-r})$, $x \in X$, $r > r_*$ with $X - B(x, e^{-r}) \neq \emptyset$, we have

$$(B(x, e^{-r}) - B(x, e^{-(\nu + r)})) \cap X \neq \emptyset.$$

Similar to [38], Lemma 2.4, we show the following.

Lemma 2.10. *If X is uniformly perfect with respect to $\nu > 0$, then X is β-diffuse for any $\beta \ge \nu + \log(4) + \log(4/3)$.*

Proof. Let $x \in X$, $r > r_*$ and $\bar{x} \in \bar{X}$. If $d(x, \bar{x}) > 2e^{-(r+\beta)}$ then for $x' = x$ we have $B(x', e^{-(r+\beta)}) \subset B(x, e^{-r}) - B(\bar{x}, e^{-(r+\beta)})$. On the other hand, if $d(x, \bar{x}) \le 2e^{-(r+\beta)}$ then $B(\bar{x}, e^{-(r+\beta)}) \subset B(x, 3e^{-(r+\beta)})$. Let $c = \beta - \nu - \log(4) \ge \log(3/4)$. Since X is uniformly perfect, there exists $x' \in (B(x, e^{-(r+c)}) - B(x, e^{-(\nu + r + c)})) \cap X$. Hence,

$$4e^{-(r+\beta)} \le e^{-(r+\nu+c)} < d(x, x') \le e^{-(r+c)} \le \tfrac{3}{4} e^{-r} \le e^{-r} - e^{-(r+\beta)}.$$

Again we have $B(x', e^{-(\beta+r)}) \subset B(x, e^{-r}) - B(\bar{x}, e^{-(\beta+r)})$. □

Consider the following examples of b_*-diffuse spaces $X \subset \bar{X}$.

1. If Γ is a non-elementary finitely generated Kleinian group acting on the hyperbolic space $\mathbb{H}^{n+1}$ (the unit ball model), then the limit set $X = \Lambda\Gamma \subset S^n = \bar{X}$ of Γ is uniformly perfect by [30]. For the definitions see Subsection 2.3.6.

2. Let $n \ge 1$. If $\Sigma^+ = \{0, \ldots, n\}^{\mathbb{N}}$ denotes the set of one-sides sequences in the symbols $\{0, 1, \ldots, n\}$, together with the metric $d^+(w, \bar{w}) \equiv e^{-\min\{i \ge 1 : w(i) \neq \bar{w}(i)\}}$ for $w \neq \bar{w}$ and $d(w, w) \equiv 0$, then (Σ^+, d) is compact and β-diffuse for $\beta = 1$.

3. Let T be a tree of valence at least 3 with the path metric such that every edge is of length 1. For a vertex point $o \in T$, let d_o be the visual metric (see Section 2.3.6 for the definition) on the set ∂T of ends of T. Then $(\partial T, d_o)$ is compact and 1-diffuse.

4. If X is the support of a locally finite Borel measure on $\bar{X} = \mathbb{R}^n$ which is absolutely δ-decaying, then there exists $b_* = b_*(\delta) > 0$ such that X is $(n-1)$-dimensionally b_*-diffuse. For the definition and the proof see below. Moreover, the following result is due to [31]. Let $\{S_1, \ldots, S_k\}$ be an irreducible family of contracting self-similarity maps of $\mathbb{R}^n$ satisfying the open set condition and let X be the attractor. If μ is the restriction of the δ-dimensional Hausdorff-measure to X, $\delta = \dim(X)$, then μ is absolutely α-decaying and satisfies a power law with respect to the exponent δ. Particular examples of such sets are regular Cantor-sets, Koch's curve and the Sierpinski gasket.

In the following, consider a nested and discrete family $\mathcal{F} = (\Lambda, R_\lambda, s_\lambda)$ of resonant sets in $\bar{X}$. We are interested in properties of $\mathcal{F}$ such that condition (b_*) is 'inherited' from a given structure of the parameter space. The family $\mathcal{F}$ is called *locally contained in* $\mathcal{S}$ (with respect to $(\bar{\Omega}, \bar{\psi})$) if there exists $l_* \geq 0$ and a number $n_* \in \mathbb{N}$ such that for all $(x,t) \in \Omega$ we have

$$\bar{\psi}(x, t + l_*) \cap R(t) \subset \bigcup_{i=1}^{n_*} S_i \tag{2.2.21}$$

is contained in at most n_* sets S_i of $\mathcal{S}$.

For a constant $d_* > 0$, we say that the parameter space (Ω, ψ) is d_**-separating* if for all formal balls $(x,t) \in \Omega$ and for any set M disjoint to $\bar{\psi}(x,t)$, we have

$$\bar{\psi}(x, t + d_*) \cap \bar{\psi}(M, t + d_*) = \emptyset. \tag{2.2.22}$$

Clearly, the standard function B_σ is $\log(3)/\sigma$-separating in a proper metric space $\bar{X}$.

Theorem 2.11. *Let (Ω, ψ) be b_*-diffuse with respect to $\mathcal{S}$, d_*-separating and $\mathcal{F}$ be locally contained in $\mathcal{S}$ with $n_* = 1$.*

*Then (Ω, ψ) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}$ where $\bar{b}_* = l_* + d_* + b_*$. Hence, **Bad**$(\mathcal{F})$ is $(\psi, \bar{b}_*)$-winning and moreover absolute $(\psi, \bar{b}_*)$-winning with respect to $\mathcal{S}$.*[5]

Since $(\mathbb{R}^n \times \mathbb{R}, B_1)$ is $\log(3)$-diffuse with respect to the collection of affine hyperplanes and metric spheres, Theorem 2.11 (and Proposition 2.12 below) implies Theorem 2.1.

Proof. Given $(x,t) \in \Omega$ and l_*, as well as $S \in \mathcal{S}$ from the definition of (2.2.21), we claim that, for $s \geq 0$,

$$\psi(x, t + l_* + d_*) \cap \bar{\psi}(R(t), t + l_* + d_* + s) \subset \psi(x, t + l_* + d_*) \cap \bar{\psi}(S, t + l_* + d_* + s). \tag{2.2.23}$$

In fact, let M be the set $R(t) - S$ which is disjoint to $\bar{\psi}(x, t + l_*)$ by (3.2.31). The $\bar{\psi}$-ball $\bar{\psi}(x, t + l_* + d_*)$ is, by (3.2.32), disjoint to

$$\bar{\psi}(M, t + l_* + d_*) \supset \bar{\psi}(M, t + l_* + d_* + s),$$

for $s \geq 0$. This shows the above claim.

Set $\bar{b}_* = l_* + d_* + b_*$. Since (Ω, ψ) is b_*-diffuse with respect to $\mathcal{S}$, applied to the formal ball $\omega = (x, t + l_* + d_*)$, there exists $\omega' = (x', t + l_* + d_* + b_*) = (x', t + \bar{b}_*) \in \Omega$ as in (2.2.20). In particular, by monotonicy $\bar{\psi}$, we obtain for every $\lambda \in \Lambda$ with $s_\lambda \leq t$ that

$$\begin{aligned} \psi(\omega') &\subset \psi(x, t + l_* + d_*) - \bar{\psi}(S, t + l_* + d_* + b_*) \\ &\subset \psi(x, t + l_* + d_*) - \bar{\psi}(R(t), t + l_* + d_* + b_*) \subset \psi(x,t) - \bar{\psi}(R_\lambda, t + \bar{b}_*). \end{aligned} \tag{2.2.24}$$

This shows that (Ω, ψ) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}$.

In fact, (2.2.24) shows that we can even choose, for a parameter $b \geq \bar{b}_*$,

$$\bar{\psi}(S, t_k + \bar{b}_*) \supset A_k \equiv \bar{\psi}(S, t_k + b) \cap X \supset \bar{\psi}(R(t_k), t_k + b) \cap X$$

in (2.2.14) and (2.2.15) respectively. Following the proof of Theorem 2.6 shows that **Bad**$(\mathcal{F})$ is an absolute $(\psi, \bar{b}_*)$-winning set with respect to $\mathcal{S}$ (as define in (2.2.5)). $\square$

[5] We remark that in (2.2.5) we considered a collection $\mathcal{S}$ of sets in X instead of $\bar{X}$ since the supspace $\bar{X}$ was not yet introduced.

As a special case, let $\bar{\psi} = B_\sigma$ be the standard function and $\bar{X}$ be a proper metric space. Recall that $d_* \le \log(3)/\sigma$, and assume that for all distinct points x, $y \in R_\lambda$ we have

$$d(x, y) > \bar{c} \cdot e^{-\sigma s_\lambda}, \tag{2.2.25}$$

for some constant $\bar{c} > 0$. It is readily checked that, setting $l_* = -\log(\bar{c}) + \log(2)$ and $\mathcal{S}$ to be the set of points, the following is a corollary of Theorem 2.11 with $\sigma = 1$.

Proposition 2.12. *Assume that X is β-diffuse. If (2.2.25) is satisfied for $\sigma = 1$, then (Ω, B_1) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}$, where $\bar{b}_* = -\log(\bar{c}) + \log(2) + d_* + \beta$. In particular, **Bad**$(\mathcal{F})$ is absolute-winning (in the sense of McMullen).*

Remark 2.13. Note that Condition (2.2.25) is similar to, but in fact weaker than the condition

$$d(x, y) \ge \sqrt{e^{-s_\lambda} e^{-s_{\lambda'}}},$$

for $x \in R_\lambda$, $y \in R_{\lambda'}$. For $X = \mathbb{R}^n$, this condition was considered in a similar setting by [13] and recently by [14] where it was called $\mathcal{B}$-set.

For another class of examples of diffuse spaces, we extend the notion of absolutely decaying measures on X, introduced in [31], to the setting of parameter spaces (Ω, ψ) and collections $\mathcal{S}$. Note that in the Euclidean setting, already [19] and [11] used absolutely decaying measures in relation with Schmidt games.

A subset $S \subset \bar{X}$ is called *$\bar{\psi}$-Borel*, if $\bar{\psi}(S, t)$ is a Borel set for all $t > t_*$. Assume that every Borel set in X is $\bar{\psi}$-Borel.

Given a locally finite Borel measure μ with $\mathrm{supp}(\mu) = X$ and a collection $\mathcal{S} \equiv \{S \subset \bar{X}\}$ of $\bar{\psi}$-Borel sets, (Ω, ψ, μ) is said to be *absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$*, where δ, $c_\delta > 0$, if for all $(x, t) \in \Omega$ and $S \in \mathcal{S}$ we have for all $s \ge 0$ that

$$\mu(\psi(x, t) \cap \bar{\psi}(S, t + s)) \le c_\delta e^{-\delta s}\ \mu(\psi(x, t)). \tag{2.2.26}$$

The function $f(s) = c_\delta e^{-\delta s}$ determines the rate of the decay of the measure of $\bar{\psi}(S, t + c)$ in $\psi(x, t)$ in terms of the relative size s of the $\bar{\psi}$-neighborhood of S.

Clearly, if $\psi = B_1$ and $\mathcal{S}$ denotes the collection of affine hyperplanes in $\bar{X} = \mathbb{R}^n$, μ corresponds to an *absolutely δ-decaying* measure in the classical sense (see [31]).

We say that (Ω, ψ) is *d_*-separating with respect to $\mathcal{S}$*, if for all formal neighborhoods $(S, t) \in \mathcal{P}$, $S \in \mathcal{S}$ and all $x, y \in X$,

$$\begin{aligned} x \notin \bar{\psi}(S, t) \quad &\Longrightarrow \quad \psi(x, t + d_*) \cap \bar{\psi}(Y, t + d_*) = \emptyset \\ x \in \psi(y, t + d_*) \quad &\Longrightarrow \quad \psi(x, t + d_*) \subset \psi(y, t). \end{aligned} \tag{2.2.27}$$

Clearly, if $\bar{X} = \mathbb{R}^n$ and S is an affine hyperplane, then B_σ is $\log(2)/\sigma$-separating with respect to $\mathcal{S}$.

Proposition 2.14. *Let (Ω, ψ, μ) be absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$ and (Ω, ψ) be d_*-separating with respect to $\mathcal{S}$. Then (Ω, ψ) is b_*-diffuse with respect to $\mathcal{S}$ for all $b_* > \log(c_\delta)/\delta + 2d_*$.*

Proof. We only sketch the proof since it is very similar to the proof of Proposition 2.16 below. Given a formal ball $\omega = (x, t) \in \Omega$ and $S \in \mathcal{S}$, condition (2.2.26) applied to $\omega' = (x, t + 2d_*)$ implies the existence of a point $x' \in \psi(\omega') - \bar{\psi}(S, t + d_* + s)$, for all $s \ge s_0 > \log(c_\delta)/\delta$. Hence, (2.2.27) shows for the formal ball $\bar{\omega} = (x', t + 2d_* + s_0) \in \Omega$ that $\psi(\bar{\omega})$ is contained in $\psi(\omega)$ and disjoint to $\bar{\psi}(S, t + 2d_* + s_0)$. ☐

As a further tool to show that a parameter space satisfies (b_*) or (b_*, n_*, L_*) with respect to a given family $\mathcal{F}$, we want to extend the notion of absolutely decaying measures. Let X be the support of a locally finite Borel measure μ. Moreover, let $f : [0, \infty] \times \mathbb{R} \to [0, \infty)$ be a function, non-decreasing in the first and non-increasing in the second argument, where we denote $f_b(\cdot) \equiv f(b, \cdot)$. If every resonant set R_λ is $\bar{\psi}$-Borel,[6] we call the family $\mathcal{F}$ *measurable* and, for a function f as above, consider the following conditions.

(μ^s) (Ω, ψ, μ) is called *strongly (absolutely) f-decaying with respect to $\mathcal{F}$*, if for all formal balls $\omega = (x, r) \in \Omega$ and for all $s \in \mathbb{R}$ we have

$$\mu(\psi(\omega) \cap \bar{\psi}(R(r), r + s)) \leq f_\infty(s)\ \mu(\psi(\omega)).$$

(μ) (Ω, ψ, μ) is called *(absolutely) f-decaying with respect to $\mathcal{F}$*, if for all formal balls $\omega = (x, r) \in \Omega$, for all $b \geq 0$ and $s \in \mathbb{R}$ we have

$$\mu(\psi(\omega) \cap \bar{\psi}(R(r, b), r + s)) \leq f_b(s)\ \mu(\psi(\omega)). \tag{2.2.28}$$

Again, the function f determines the rate of decay of the measure of the relative neighborhood of the resonant set in $\psi(x, r)$. For constants $n_* \in \mathbb{N}$, and $d_*, L_* \geq 0$, $b_* > 2d_*$, we say that f is *(d_*, b_*, n_*, L_*)-decaying* if there exists a constant $c_0 < 1$ such that

$$f(n_*(b + L_*) + d_*, b - 2d_*) \leq c_0 \qquad \text{for all } b > b_*, \tag{2.2.29}$$

and *strongly (d_*, b_*)-decaying* if $f_\infty(b_* - 2d_*) \leq c_0$.

Proposition 2.15. *Let (Ω, ψ, μ) be absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$, and (Ω, ψ) be d_*-separating. If moreover $\mathcal{F} = (\Lambda, R_\lambda, s_\lambda)$ is locally contained in $\mathcal{S}$ for $n_* \in \mathbb{N}$ and $l_* \geq 0$, then (Ω, ψ, μ) is (absolutely) f-decaying with respect to $\mathcal{F}^* = (\Lambda, R_\lambda, s_\lambda + l_* + d_*)$ where $f_\infty(s) = n_* c_\delta e^{-\delta s}$ is (d_*, b_*)-decaying for $b_* > \log(n_* c_\delta e^{-2\delta d_*})/\delta$.*

Proof. Using the argument of Claim (2.2.23), the result is readily checked. We will give the proof in Section 3.2.4. □

We say that the parameter space (Ω, ψ) is *d_*-separating with respect to $\mathcal{F}$*, if there exists a constant $d_* > 0$ such that for all formal neighborhoods $(Y, t) = (R(r, b), t) \in \mathcal{P}$, or formal balls $(Y, t) = (y, t) \in \Omega$ and for all $x \in X$,

$$\begin{aligned} x \notin \bar{\psi}(Y, t) \quad &\Longrightarrow \quad \psi(x, t + d_*) \cap \bar{\psi}(Y, t + d_*) = \emptyset. \\ x \in \psi(y, t + d_*) \quad &\Longrightarrow \quad \psi(x, t + d_*) \subset \psi(y, t). \end{aligned} \tag{2.2.30}$$

Clearly, if $\bar{X}$ is a proper metric space and every R_λ is a discrete set, then the standard function B_σ is $\log(3)/\sigma$-separating with respect to $\mathcal{F}$.

Proposition 2.16. *Let (Ω, ψ) be d_*-separating with respect to $\mathcal{F}$ and μ be a locally finite Borel measure with $X = supp(\mu)$. If (Ω, ψ, μ) is [strongly] absolutely f-decaying with respect to $\mathcal{F}$ and a function f which is [(d_*, b_*)-decaying] (d_*, b_*, n_*, L_*)-decaying, Then (Ω, ψ) is [strongly $\bar{b}_*$-diffuse] $(\bar{b}_*, n_*, L_*)$-diffuse with respect to $\mathcal{F}$, where $\bar{b}_* = b_* + 2d_*$.*

[6] In this case, also $R(r, b) = R(r) - R(r - b)$ is $\bar{\psi}$-Borel for every $r \in \mathbb{R}$, $b > 0$.

Proof. Assume that (Ω, ψ, μ) is f-decaying with respect to $\mathcal{F}$ and f is (d_*, b_*, n_*, L_*)-decaying. For $\bar{b}_* = b_* + 2d_*$ and $b > \bar{b}_*$ note that $R(r, n_* b) \subset R(r + d_*, n_* b + d_*)$ and $b - 2d_* \geq b_*$. Let $\omega = (x, r) \in \Omega$ with $r > r_*$. We have

$$\begin{aligned} & \mu(\psi(x, r + d_*) \cap \bar{\psi}(R(r, n_*(b + L_*)), r + b - d_*)) \\ \leq \; & \mu(\psi(x, r + d_*) \cap \bar{\psi}(R(r + d_*, n_*(b + L_*) + d_*), r + d_* + (b - 2d_*))) \\ \leq \; & f(n_*(b + L_*) + d_*, b - 2d_*)\mu(\psi(x, r + d_*)). \end{aligned}$$

Since for $b > \bar{b}_* = b_* + 2d_*$ we have $f(n_*(b + L_*) + d_*, b - 2d_*) \leq c_0 < 1$ by (2.2.29), there exists a point $\bar{x} \in \psi(x, r + d_*) \cap \bar{\psi}(R(r, n_*(b + L_*)), r + b - d_*)^C$. By (2.2.22) and since $b > d_*$, we have for $\omega' = (\bar{x}, r + b) \in \Omega$ that $\psi(\omega') \subset \psi(\bar{x}, r + d_*) \subset \psi(x, r)$. Furthermore, (2.2.30) implies that $\psi(\bar{x}, r + b)$ is disjoint from $\bar{\psi}(R(r, n_*(b + L_*)), r + b)$. This shows that (Ω, ψ) is $(\bar{b}_*, n_*, L_*)$-diffuse with respect to $\mathcal{F}$.

The case when (Ω, ψ, μ) is strongly f-decaying follows similarly. □

We say that for a locally finite Borel measure μ on $X = \mathrm{supp}(\mu)$, (Ω, ψ, μ) satisfies a power law, if there are parameters τ, c_1, $c_2 > 0$, such that for all $\omega = (x, t) \in \Omega$ we have

$$c_1 e^{-\tau t} \leq \mu(\psi(x, t)) \leq c_2 e^{-\tau t}.$$

Note that τ might differ from the lower pointwise dimension of μ at a point.

Theorem 2.17. *Let (Ω, ψ) be d_*-separating with respect to $\mathcal{F}$ and let (Ω, ψ, μ) satisfy a power law. Assume that either (Ω, ψ, μ) is f-decaying with respect to $\mathcal{F}$ where f is $(d_*, b_*, 1, 0)$-decaying or that (Ω, ψ) is strongly b_*-diffuse with respect to $\mathcal{F}$. If moreover (MSG1-2) are satisfied, then for all nonempty open sets $U \subset X$, we have*

$$\dim(\boldsymbol{Bad}(\mathcal{F}) \cap U) \geq d_\mu(U).$$

Proof. Let first (Ω, ψ, μ) is f-decaying with respect to $\mathcal{F}$ where f is $(d_*, b_*, 1, 0)$-decaying. Let $b > \bar{b}_* = b_* + d_*$ and $\omega_1 = (x_1, t_1) \in \Omega$ be the first move of B such that, by (MSG1), $\psi(\omega_1) \subset U$. Let again $m_* \in \mathbb{N}$ with $\tilde{b} = m_* b \geq t_1$. For $k \geq 1$, let $\omega_k = (x_k, t_k)$ be a choice of B. As in the proof of Proposition 2.16 (with $n_* = 1$, $L_* = 0$), let $x^1 \in \psi(x_k, t_k + d_*) \cap \bar{\psi}(R(t_k, \tilde{b}), t_k + \tilde{b} - d_*)^C$. We moreover see that

$$\begin{aligned} & \mu(\psi(x_k, t_k + d_*) \cap (\psi(x^1, t_k + \tilde{b} - d_*) \cup \bar{\psi}(R(t_k + d_*, \tilde{b} + d_*), t + \tilde{b} - d_*))) \\ \leq \; & c_2 e^{-\tau(t_k + \tilde{b} - d_*)} + f(\tilde{b} + d_*, \tilde{b} - 2d_*) \cdot \mu(\psi(x_k, t_k + d_*) \\ \leq \; & (\tfrac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \tilde{b}} + c_0)\mu(\psi(x_k, t_k + d_*)). \end{aligned}$$

Since $c_0 < 1$, for $\tilde{b}$ sufficiently large such that $\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \tilde{b}} + c_0 < 1$, there exists a point

$$x^2 \in \psi(x_k, t_k + d_*) \cap \bar{\psi}(R(t_k + d_*, \tilde{b} + d_*), t_k + \tilde{b} - d_*)^C \cap \psi(x^1, t_k + \tilde{b} - d_*)^C.$$

With the same arguments as above, $\psi(x^2, r + \tilde{b})$ is contained in $\psi(x_k, t_k)$ and disjoint from both, $\psi(x^1, t_k + \tilde{b})$ and $\bar{\psi}(R(t_k + d_*, \tilde{b} + d_*), t_k + \tilde{b})$. Iterating this argument until

$$(N + 1)\tfrac{c_2}{c_1} e^{2\tau d_*} e^{-\tau \tilde{b}} + c_0 > 1,$$

we obtain N points $x^1, \dots, x^N$ such that $\psi(x^i, t_k + \tilde{b}) \subset \psi(x_k, t_k)$, $i = 1, \dots, N$, are disjoint and also disjoint to $\bar{\psi}(R(t_k + d_*, \tilde{b} + d_*), t_k + \tilde{b})$. Moreover, we have

$$\begin{aligned} \mu(\bigcup_{i=1}^{N} \psi(x^i, t_k + \tilde{b})) &\geq N c_1 e^{-\tau(t_k+\tilde{b})} \\ &\geq \tfrac{(N+1)c_1}{2} e^{-\tau(t_k+\tilde{b})} \\ &\geq \tfrac{(1-c_0)c_1^2 e^{-2\tau d_*}}{2c_2^2} \mu(\psi(x_k, t_k)) \equiv \bar{c}_0 \cdot \mu(\psi(x_k, t_k)), \end{aligned}$$

Furthermore, each of the formal balls $\omega^i = (x^i, t_{k+1}) = (x^i, t_k + \tilde{b}) \in \mathcal{L}_b^{\psi}(\omega_k)$ is a legal move according to the $(\psi, \bar{b}_*, b)$-winning-strategy of A defined in (2.2.13). This shows $(\mu 2)$ for the parameter b with $c = c(b) \geq \bar{c}_0$ and m_*. Finally, Proposition 2.5 implies that

$$\dim(\mathbf{Bad}(\mathcal{F}) \cap U) \geq d_\mu(U) + \frac{\log(\bar{c}_0)}{\sigma m_* b}, \tag{2.2.31}$$

and the proof follows since (2.2.31) is true for every $b > \bar{b}_*$.

If (Ω, ψ) is strongly b_*-diffuse with respect to $\mathcal{F}$, there exists $\psi(\bar{x}, t + b_*) \subset \psi(\omega) - \bar{\psi}(R(t), t + nb_*)$. With similar arguments, we can choose disjoint formal balls $\psi(x_i, t + b)$, $i = 1, \dots, N$, contained in $\psi(\bar{x}, t + b_*)$, where N is such that $(N+1)\frac{c_2}{c_1} e^{2\tau d_*} e^{-\tau b} > 1$ and each of the formal balls is a legal move according the to the (ψ, b_*, b)-winning strategy of A. The proof then follows similarly. □

Remark 2.18. If we modify the requirements and the proof of Theorem 2.17 (similar to the proof of Proposition 2.8) with respect to finitely many families $\mathcal{F}_i$, $i = 1, \dots, n_*$, where in particular (Ω, ψ, μ) is f_i-decaying with respect to $\mathcal{F}_i$, f_i is $(d_*, b_*, n_*, 0)$-decaying, then we can show the result for $\mathbf{Bad}(\mathcal{F})$ replaced by $\cap_{i=1}^{n_*} \mathbf{Bad}(\mathcal{F}_i)$. Moreover, if actually $X = F_i(Z)$ for bijective maps $F_i : \bar{Z} \to \bar{X}$ satisfying (2.2.17) and $\mathcal{F}_i = F_i(\mathcal{F}_Z^i)$ with families $\mathcal{F}_Z^i$ in $\bar{Z}$, we obtained that

$$\dim(\cap_{i=1}^{n_*} F(\mathbf{Bad}(\mathcal{F}_Z^i)) \cap U) \geq d_\mu(U),$$

for any nonempty open set $U \subset X$. This is a weaker version of the property that winning sets for Schmidt's game are incompressible; compare with (2.2.19).

2.3 Applications

In order to discuss our conditions, we consider several examples from metric number theory (Part I.) and from dynamical systems (Part II.). Given a complete metric space X in $\bar{X}$ with a monotonic function $\bar{\psi}$ on $\bar{\Omega}$, we are left with defining a suitable nested discrete family of resonant sets $\mathcal{F}$, verifying the Conditions (b_*) or (b_*, n_*, L_*) respectively as well as finding suitable measures for the purpose of determining the Hausdorff-dimension of $\mathbf{Bad}(\mathcal{F})$.

In both parts, we start with well known examples and results, where we either weaken the assumptions to our weaker setting or improve them, and end each of the parts with new examples.

I. Examples from Number Theory.

2.3.1 $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$

For $n \geq 1$, let $\bar{r} \in \mathbb{R}^n$ with $r^1, \ldots, r^n \geq 0$ such that $\sum r^i = 1$. Let $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$ be the set of points $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1,\ldots,n} |qx_i - p_i|^{1/r^i} \geq c(\bar{x})/q,$$

for every $q \in \mathbb{N}$ and $\bar{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. The set $\mathbf{Bad}^{n}_{\mathbb{R}^n} \equiv \mathbf{Bad}^{(1/n,\ldots,1/n)}_{\mathbb{R}^n}$ agrees with the set of badly approximable vectors.

Let $\bar{\Omega} = \mathbb{R}^n \times (0, \infty)$ and define the monotonic function $\bar{\psi} = \bar{\psi}_{\bar{r}}$ by

$$\bar{\psi}(\bar{x}, t) \equiv B(x_1, e^{-(1+r^1)t}) \times \cdots \times B(x_n, e^{-(1+r^n)t}).$$

While [11] showed that $\mathbf{Bad}^{n}_{\mathbb{R}^n} \cap X$, where X is the support of an absolutely decaying measure, is hyperplane absolute winning, [34] showed that $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$ is a winning set for the ψ-modified game. We want to combine these results and improve them to the following, where we set $\mathbf{Bad}^{\bar{r}}_{X} \equiv \mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n} \cap X$.

Theorem 2.19. *Let X be the support of a locally finite Borel measure μ such that (Ω, ψ, μ) is absolutely (δ, c_δ)-decaying with respect to the collection $\mathcal{S}$ of affine hyperplanes in $\mathbb{R}^n$. Then $\boldsymbol{Bad}^{\bar{r}}_{X}$ is absolute ψ-winning with respect to $\mathcal{S}$.*

Before we proof the Theorem, let μ be the Lebesgue-measure on $\mathbb{R}^n$. Note that (Ω, ψ, μ) is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$, for $\delta = 1 + \min\{r^1, \ldots, r^n\}$ and $c_\delta > 0$. Moreover, (Ω, ψ, μ) satisfies a power law with respect to the exponent $n + 1$; in fact, for all $(x, t) \in \Omega$ we have $\mu(\psi(x, t)) = 2^n e^{-(n+1)t}$.

More precisely, let μ_i on $X_i \subset \mathbb{R}$ such that $(\Omega_i, B_{\sigma_i}, \mu_i)$ satisfies a power law with respect to the exponent τ_i, $i = 1, \ldots, n$. In particular, $(\Omega_i, B_{\sigma_i}, \mu_i)$ is absolutely τ_i-decaying and $((\times_{i=1}^n X_i) \times \mathbb{R}^+, \times_{i=1}^n B_{\sigma_i}, \times_{i=1}^n \mu_i)$ satisfies a power law with respect to the exponent $\tau = \sum_{i=1}^n \tau_i$. Moreover, using the arguments of [31], Lemma 9.1, the following Lemma can be shown.

Lemma 2.20. *Assume that $(X_i \times \mathbb{R}^+, \psi_i, \mu_i)$, $X_i \subset \mathbb{R}^{n_i}$, is absolutely (δ_i, c_{δ_i})-decaying with respect to affine hyperplanes in $\mathbb{R}^{n_i}$, $i = 1, 2$. Then $(X_1 \times X_2 \times \mathbb{R}^+, \psi_1 \times \psi_2, \mu_1 \times \mu_2)$ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$ for $\delta = \min\{\delta_i\}$, $c_\delta = \max\{c_{\delta_i}\}$.*

Sketch of the proof. Given the box $\psi((x_0, y_0), t) = \psi_1(x_0, t) \times \psi_2(y_0, t)$ and an affine hyperplane S in $\mathbb{R}^{n_1+n_2}$, we may, up to interchanging the role of the indices, assume that each slice $S_x \equiv S \cap \{x\} \times \mathbb{R}^{n_2}$ is an affine hyperplane in $\mathbb{R}^{n_2}$. Hence, write

$$\psi((x_0, y_0), t) \cap \bar{\psi}(S, t + s) = \bigcup_{x \in \psi_1(x_0,t)} \{x\} \times (\psi_2(y_0, t) \cap \bar{\psi}_2(S_x, t + s)).$$

Disintegrating into the slices parallel to $\mathbb{R}^{n_2}$ and using that μ_2 is absolutely (δ_2, c_{δ_2})-decaying, we obtain

$$\begin{aligned} \mu(\psi((x_0, y_0), t) \cap \bar{\psi}(S, t + s)) &\leq \mu_1(\psi(x_0, t)) \cdot c_{\delta_2} e^{-\delta_2 s} \mu_2(\psi_2(y_0, t)) \\ &\leq c_\delta e^{-\delta s} \mu_1(\psi(x_0, t)) \mu_2(\psi(y_0, t)) = c_\delta e^{-\delta s} \mu(\psi((x_0, y_0), t)), \end{aligned}$$

showing the claim. ☐

So let X be a product space as above, and note that conditions (MSG1-2) are satisfied. By Theorem 2.17 (which we will see is applicable), for any nonempty open set $U \subset X$, we have

$$\dim(\mathbf{Bad}_X^{\bar{r}} \cap U) \geq d_\mu(U);$$

this strengthens [35], Theorem 11.

Proof of Theorem 3.18. For $k \in \Lambda \equiv \mathbb{N}_{\geq 2}$ we define the set of rational vectors

$$R_k \equiv \{\bar{p}/q : \bar{p} \in \mathbb{Z}^n, 0 < q < k\}$$

as resonant set and define its size by $s_k \equiv \log(k)$. The family $\mathcal{F} = (\mathbb{N}_{\geq 2}, R_k, s_k)$ is nested and discrete and we want to show that (Ω, ψ) is strongly b_*-diffuse with respect to $\mathcal{F}$.

We use the following version of the 'Simplex Lemma' due to Davenport and Schmidt.

Lemma 2.21 ([35], Lemma 4)**.** *Let $D \subset \mathbb{R}^n$ be a box of Euclidean volume vol$(D) < 1/(n!k^{n+1})$. Then there exists an affine hyperplane L such that $R_k \cap D \subset L$.*

Assuming the lemma for the moment, choose any resonant set R_k and let $\omega = (x, r) \in \Omega$ be a formal ball such that $s_k \leq r$. Note that, for $l_* > \log(n! \cdot 2^n)$, $\bar{\psi}(x, r + l_*)$ is a box of Euclidean volume

$$2e^{-(1+r^1)(r+l_*)} \cdots 2e^{-(1+r^n)(r+l_*)} = 2^n e^{-(1+n)(r+l_*)} < \tfrac{1}{n!k^{n+1}}.$$

The Simplex Lemma implies that $\bar{\psi}(x, r + l_*) \cap R_k \subset L$, where $L \in \mathcal{S}$, which shows that $\mathcal{F}$ is locally contained in $\mathcal{S}$ for $n_* = 1$.

It is readily checked that (Ω, ψ) is d_*-separating as well as d_*-separating with respect to $\mathcal{S}$, for $d_* = \log(3)/(1 + \min\{r^i\})$. Since (Ω, ψ, μ) is (δ, c_δ)-decaying, Proposition 2.14 implies that (Ω, ψ) is b_*-diffuse with respect to $\mathcal{S}$, for $b_* > 2d_* + \log(c_\delta)/\delta$. Thus, Theorem 2.11 shows that (Ω, ψ) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}$ where $\bar{b}_* = l_* + d_* + b_*$ and that $\mathbf{Bad}(\mathcal{F})$ is an absolute $(\psi, \bar{b}_*)$-winning set with respect to $\mathcal{S}$.

Finally, if $\bar{x} \in \mathbf{Bad}_X^{\psi}(\mathcal{F})$, there exists a constant $c = c(\bar{x}) < \infty$ such that for all $\bar{p}/q$, where $\bar{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$,

$$\bar{x} \notin \bar{\psi}(R_{q+1}, s_{q+1} + c) \supset \bar{\psi}(\bar{p}/q, s_{q+1} + c).$$

Hence, for some $i \in \{1, \ldots, n\}$, we have

$$|x_i - p_i/q| \geq e^{-(1+r^i)(s_{q+1}+c)} \geq \tfrac{e^{-(1+\max\{r^i\})c}}{2^{2n+2}n!} q^{-(1+r^i)},$$

and we see that $\mathbf{Bad}(\mathcal{F}) \subset \mathbf{Bad}_X^{\bar{r}}$. Remarking that a supset of a winning set is also a winning set finishes the proof. □

Although the Simplex Lemma is folklore, we want to give the proof of [35] for the sake of completeness.

Proof of the Simplex Lemma 2.21. Let $D \subset \mathbb{R}^n$ be a convex subset of volume less than $1/(n!k^{n+1})$. Assume by contradiction that there are $n+1$ rational vectors $\bar{p}_i/q_i \in D \cap R_k$ which are not contained in an affine hyperplane. These vectors span a simplex S which is contained in D by convexity of D. Moreover, the volume of S, vol$(S) \neq 0$, is given by

$$\text{vol}(D) \geq \text{vol}(S) = \frac{1}{n!}|\det(\begin{pmatrix} 1 & \bar{p}_1^T/q_1 \\ 1 & \bar{p}_2^T/q_2 \\ 1 & \bar{p}_3^T/q_3 \end{pmatrix})| \geq \frac{1}{n!}\frac{1}{q_1 \cdots q_{n+1}} > \frac{1}{n!k^{n+1}},$$

which is a contradiction and finishes the proof. □

2.3.2 $\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^n}$

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers in $\mathbb{C}$. For $n \geq 1$, let again $\bar{r} \in \mathbb{R}^n$ with $r^1, \ldots, r^n \geq 0$ such that $\sum r^i = 1$. Denote by $\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^n}$ the set of points $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$ for which there is a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1,\ldots,n} |qx_i - z_i|^{1/r^i} \geq c(\bar{x}) \cdot |q|^{-1},$$

for every $z_1, ..., z_n, q \in \mathbb{Z}[i]$, $q \neq 0$.

Let $\Omega = \mathbb{C}^n \times (0, \infty)$ and define the monotonic function $\psi = \psi_{\bar{r}}$ by the box

$$\psi(\bar{x}, t) \equiv B(x_1, e^{-(1+r^1)t}) \times \cdots \times B(x_n, e^{-(1+r^n)t}).$$

Note that [15] showed $\mathbf{Bad}^{(1/n,\ldots,1/n)}_{\mathbb{C}^2}$ to be a winning set for Schmidt's game. We want to show the following stronger result.

Theorem 2.22. *$\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^2}$ is absolute ψ-winning with respect to the collection $\mathcal{S}$ of affine complex lines.*

By Theorem 2.17 (which we will see is applicable), for any nonempty open set $U \subset \mathbb{C}^2$, we have for the Lebesgue measure μ on $\mathbb{C}^2$,

$$\dim(\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^2} \cap U) \geq d_\mu(U). \tag{2.3.1}$$

Let μ_i satisfy a power law on $X_i \subset \mathbb{C}$, $i = 1, 2$, and set $X = X_1 \times X_2 \subset \mathbb{C}^2$ with the product measure $\mu = \mu_1 \times \mu_2$. Then [35] showed that $\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^2} \cap X$ is of Hausdorff-dimension $\dim(X)$. In fact, in this case, μ is an absolutely decaying measure (compare with Lemma 2.20, modified with respect to complex affine subspaces), we can modify the proof below and show that $\mathbf{Bad}^{\bar{r}}_{\mathbb{C}^2} \cap X$ is absolute ψ-winning with respect to $\mathcal{S}$ in X. Moreover, (2.3.1) holds for sets $U \subset X$ and with respect to the product measure μ.

For simplicity and since all the arguments can be carried out analogously to the proof of Theorem 3.18 with respect to the complex setting, we restrict to the full space $X = \mathbb{C}^2$ and only sketch the proof.

Sketch of the Proof. For $n \in \Lambda \equiv \mathbb{N}_{\geq 2}$ define the resonant set

$$R_n \equiv \{(z_1/q, z_2/q) \in \mathbb{C}^2 : z_1, z_2, q \in \mathbb{Z}[i], 0 < |q| < n\}$$

with size $s_n \equiv \log(n)$, which gives a nested and discrete family $\mathcal{F}$.

We remark that implicitly in the proof of [35], Theorem 17, the following analogue of the Simplex Lemma is contained.

Lemma 2.23. *There exists $\bar{l}_* > 0$ such that, if $D = B(x_1, r_1) \times B(x_2, r_2)$ is a box with $r_1 r_2 < e^{-\bar{l}_*} n^{-3}$, then $D \cap R_n$ is contained in an affine complex line L.*

Thus, for any $l_* > \bar{l}_*/3$ with $\bar{l}_*$ as above, we have that $\psi(\bar{x}, \log(n) + l_*)$ is a box with radii r_1, r_2 satisfying

$$r_1 r_2 = e^{-(1+r^1)(\log(n)+l_*)} \cdot e^{-(1+r^2)(\log(n)+l_*)} < e^{-\bar{l}_*} n^{-3},$$

and we see that $\psi(\bar{x}, s_n + l_*) \cap R_n$ is contained in a complex line. This shows that $\mathcal{F}$ is locally contained in $\mathcal{S}$, the set of affine complex lines, with $n_* = 1$.

Moreover (Ω, ψ) is b_*-diffuse with respect to $\mathcal{S}$ and d_*-separating for some $b_* > 0$ and $d_* = \log(3)/(1 + \min\{r^1, r^2\})$. Thus, Theorem 2.11 implies that (Ω, ψ) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}$, where $\bar{b}_* = l_* + b_* + d_*$, as well as that $\mathbf{Bad}^{\psi}_{\mathbb{C}^2}(\mathcal{F})$ is absolute $(\psi, \bar{b}_*)$-winning with respect to $\mathcal{S}$. Finally, it is readily checked that $\mathbf{Bad}^{\psi}_{\mathbb{C}^2}(\mathcal{F}) \subset \mathbf{Bad}^{\bar{r}}_{\mathbb{C}^2}$. □

2.3.3 $\mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^n}$

Let p be a prime number , $|\cdot|_p$ the p-adic absolute value and $\mathbb{Z}_p$ be the p-adic integers in the p-adic field $\mathbb{Q}_p$. For $n \geq 1$, let again $\bar{r} \in \mathbb{R}^n$ with $r^1, \ldots, r^n \geq 0$ such that $\sum r^i = 1$. Because of the different properties of the p-adic field, we need to adjust the definition of badly approximable p-adic vectors. For further details, we refer to [35] and references therein. Let $\mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^n}$ be the set of points $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that

$$\max_{i=1,\ldots,n} |x_i - \tfrac{z_i}{q}|_p^{1/(1+r^i)} \geq c(\bar{x}) \max\{|z_1|, \ldots, |z_n|, |q|\}^{-1},$$

for all $(z_1, ..., z_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$. Let $d(x, y) \equiv |x - y|_p$ be the p-adic metric on $\mathbb{Z}_p$. For $(\bar{x}, t) \in \mathbb{Q}_p^n \times (0, \infty)$ consider the box

$$\bar{\psi}(\bar{x}, t) \equiv B(x_1, e^{-(1+r^1)t}) \times \cdots \times B(x_n, e^{-(1+r^n)t}).$$

For $n = 2$, it was already shown by [35] that (a slightly different version of) $\mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^2}$ is of Hausdorff-dimension 2. We show the following stronger result.

Theorem 2.24. *$\mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^2}$ is absolute ψ-winning with respect to the collection $\mathcal{S}$ of p-adic lines in $\mathbb{Z}_p^2$ and thick; that is, for any nonempty open set $U \subset \mathbb{Z}_p^2$, we have*

$$dim(\mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^2} \cap U) = 2.$$

Sketch of the proof. As previously, let $\Lambda \equiv \mathbb{N}_{\geq 2}$ and for $n \in \Lambda$ define the resonant set

$$R_n \equiv \{(z_1/q, z_2/q) \in \mathbb{Q}_p^2 : z_1, z_2 \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } \max\{|z_1|, |z_2|, |q|\} < n\}$$

with the size $s_n \equiv \log(n)$. For the nested discrete family $\mathcal{F} = (\Lambda, R_n, s_n)$ we show that for $X = \mathbb{Z}_p^2$, (Ω, ψ) is strongly b_*-diffuse with respect to $\mathcal{F}$.

Let $m \equiv \mu \times \mu$, where μ is the normalized Haar-measure on $\mathbb{Q}_p$. Hence, $\mu(\mathbb{Z}_p) = 1$ and $m(B(x_1, r_1) \times B(x_2, r_2)) = p^{-(t_1+t_2)}$ for $p^{-t_i} \leq r_i \leq p^{-t_i+1}$ and $t_i \in \mathbb{N}$, $i = 1, 2$. In particular, for $\omega = (\bar{x}, t) \in \Omega$ we have $p^{-4}e^{-3t} \leq m(\psi(\omega)) \leq e^{-3t}$. Thus, (Ω, ψ, μ) satisfies a power law with respect to the exponent $\tau = 3$.

Again, we remark that implicitly in the proof of [35], Theorem 18, the following analogue of the Simplex Lemma is contained.

Lemma 2.25. *Let $D \subset \mathbb{Z}_p^2$ be a box of measure $m(D) < 1/(6n^3)$. Then there exists an affine p-adic line L such that $R_n \cap D \subset L$.*

Thus, let $l_* > \log(6)/3$. For $\omega = (\bar{x}, t) \in \Omega$ and s_n with $s_n \leq t$ we have $m(\psi(x, t + l_*)) < 1/(6n^3)$ and Lemma 2.25 implies that $R_n \cap \psi(x, t + l_*) \subset L$, for an affine p-adic line L. This shows that $\mathcal{F}$ is locally contained in $\mathcal{S}$, the set of p-adic lines, with $n_* = 1$.

Next, we claim that (Ω, ψ) is b_*-diffuse with respect to $\mathcal{S}$ for $b_* > 0$ sufficiently large. Therefore, note that, as shown in [35], for $b_* > 0$ sufficiently large, a geometric argument implies that any number of disjoint boxes $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, intersecting a p-adic L is bounded above by $C \cdot e^{b_*(1+\max\{r^1, r^2\})}$, where C is independent of b_* and t. Using that (Ω, ψ, μ) satisfies a power law with respect to the exponent $\tau = 3$, for $b_* > 0$ sufficiently large, there exists a collection of disjoint boxes $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, whose number exceeds the one of its boxes intersecting L (independently from t). If we take such a box $\psi(\bar{x}_i, t + b_*) \subset \psi(\omega)$, $\bar{x}_i \in \mathbb{Z}_p^2$, not intersecting L, then $\psi(\bar{x}_i, t + 2b_*)$ is disjoint from $\psi(L, t + 2b_*)$ (if b_* is sufficiently large). This shows the above claim.

Since (Ω, ψ) is moreover d_*-separating for $d_* \leq \log(3)/(1 + \min\{r^1, r^2\})$, Theorem 2.11 shows that (Ω, ψ) is strongly $(2b_* + l_* + d_*)$-diffuse with respect to $\mathcal{F}$ and, moreover, that $\mathbf{Bad}(\mathcal{F})$ is absolute ψ-winning with respect to $\mathcal{S}$. Furthermore, by Theorem 2.17 and since (MSG1-2) is satisfied, for any open set $U = B(z_1, e^{-t_1}) \times B(z_2, e^{-t_2}) \cap \mathbb{Z}_p^2$, $z_1, z_2 \in \mathbb{Z}_p$, we have

$$\dim(\mathbf{Bad}(\mathcal{F}) \cap U) = d_\mu(U) = 2.$$

Finally, let $\bar{x} \in \mathbf{Bad}^{\psi}_{\mathbb{Z}_p^2}(\mathcal{F})$ and $(z_1/q, z_2/q) \in \mathbb{Q}^2$ with $\max\{|z_1|, |z_3|, |q|\} = n$. There exists $c(x) < \infty$ such that $\bar{x} \notin \bar{\psi}(R_n, s_n + c(x)) \supset \bar{\psi}((z_1/q, z_2/q), s_n + c(x))$. Hence, for some $i \in \{1, 2\}$ we have

$$|x_i - z_i/q|_p > e^{-(1+r^i)(s_n + c(x))} \geq e^{-(1+\max\{r^1, r^2\})(c_* + \log(4) + c(x))} n^{-(1+r^i)}.$$

Therefore, $\mathbf{Bad}(\mathcal{F}) \subset \mathbf{Bad}^{\bar{r}}_{\mathbb{Z}_p^2}$ which finishes the proof. □

II. Examples from Dynamical Systems.

2.3.4 The Bernoulli shift Σ^+

For $n \geq 1$, let $\Sigma^+ = \{0, \ldots, n\}^{\mathbb{N}}$ be the set of one-sided sequences in symbols from $\{0, \ldots, n\}$. Let T denote the shift and let d^+ be the metric given by $d^+(w, \bar{w}) \equiv e^{-\min\{i \geq 1 : w(i) \neq \bar{w}(i)\}}$ for $w \neq \bar{w}$ and $d(w, w) \equiv 0$.

Fix a periodic word $\bar{w} \in \Sigma^+$ of period $p \in \mathbb{N}$ and consider the set

$$S_{\bar{w}} = \{w \in \Sigma^+ : \exists\, c = c(w) < \infty \text{ such that } T^k w \notin B(\bar{w}, 2^{-(p+c+1)}) \text{ for all } k \in \mathbb{N}\}.$$

Theorem 2.26. *$S_{\bar{w}}$ is absolute winning (in the sense of McMullen) and of Hausdorff-dimension* $\log(n)$ *(and in fact thick).*

Remark 2.27. In particular, the intersection $\bigcap S_{\bar{w}}$ over all periodic words $\bar{w} \in \Sigma^+$ is $(B_1, 1)$-absolute winning. Note that the Morse-Thue sequence w in $\{0, 1\}^{\mathbb{N}}$ is a particular example of a word in $\bigcap_{\bar{w}} S_{\bar{w}}$. In fact, w does not contain any subword of the form WWa where a is the first letter of the subword W; for details and more general words in $\bigcap_{\bar{w}} S_{\bar{w}}$, we refer to Section 4.3.

Proof. For $k \in \mathbb{N}$ and $w_k \in \{0, .., n\}^k$, let $\bar{w}_k \in \Sigma^+$ denote the word $\bar{w}_k = w_k\bar{w}$. Let $\Lambda \equiv \mathbb{N}_0$ and consider the resonant sets

$$R_0 = \{\bar{w}\}, \qquad R_k = \{\bar{w}_l \in \Sigma^+ : w_l \in \{1, .., n\}^l, l \leq k\}) \cup R_0, \text{ for } k \in \mathbb{N}$$

which we give the size $s_k = p + k + 1$. Then, $\mathcal{F} = (\mathbb{N}_0, R_k, s_k)$ is nested and discrete.

Let $\bar{w}_m$ and $\tilde{w}_m \in R_m$ be distinct. By definition of $\bar{w}_m$ and $\tilde{w}_m$ there exists $i \in \{1, \ldots, m+p\}$ such that $\bar{w}_m(i) \neq \tilde{w}_m(i)$; hence

$$d^+(\bar{w}_m, \tilde{w}_m) \geq e^{-(p+m+1)} = e^{-s_m}$$

and we are given the special case (2.2.25). Moreover, (Σ^+, d^+) is β-diffuse for $\beta = 1$. Proposition 2.12 shows that $\mathbf{Bad}(\mathcal{F})$ is absolute-winning.

Moreover the probability measure $\mu = \{1/n, \ldots, 1/n\}^{\mathbb{N}}$ satisfies

$$\mu(B(w, e^{-(t+1)})) = n^{-t} = ne^{-\log(n)(t+1)},$$

where $t \in \mathbb{N}$. Hence, $(\Sigma^+ \times \mathbb{N}, B_1, \mu)$ satisfies a $(\log(n), n, n)$-power law and we see that $\mathbf{Bad}(\mathcal{F})$ is of Hausdorff-dimension $\log(n)$ (and thick) by Theorem 2.17.

Finally, we have $\mathbf{Bad}(\mathcal{F}) = S_{\bar{w}}$. In fact, $d^+(T^{k-1}w, \bar{w}) \leq e^{-(p+c+1)}$ for some $c \in \mathbb{N}$ if and only if $w(k) \ldots w(k+p+c) = \bar{w}(1) \ldots \bar{w}(p+c)$. Thus, for $w_k = w(1) \ldots w(k)$ and $\bar{w}_k = w_k\bar{w}$ we have $d^+(w, \bar{w}_k) \leq e^{-(p+k+c+1)}$ if and only if $w \in B(\bar{w}_k, e^{-(s_k+c)}) \subset \psi_1(R_k, s_k + c)$. □

2.3.5 Toral Endomorphisms

For the motivation, further generalizations and consequences of the following result, we refer to [10] and references therein. For $n \in \mathbb{N}$, let $\mathcal{M} = (M_k)$ be a sequence of real matrices $M_k \in GL(n, \mathbb{R})$ and $\mathcal{Z} = (Z_k)$ be a sequence of τ_k-separated[7] subsets of $\mathbb{R}^n$. Define

$$E_{\mathcal{M},\mathcal{Z}} \equiv \{x \in \mathbb{R}^n : \exists\ c = c(x) > 0 \text{ such that } d(M_k x, Z_k) \geq c \cdot \tau_k \text{ for all } k \in \mathbb{N}\},$$

where d is the Euclidean distance. The sequence $\mathcal{M}$ is *lacunary*, if for $t_k = \|M_k\|_{op}$ (the operator norm) we have $\inf_{k\in\mathbb{N}} \frac{t_{k+1}}{t_k} \equiv \lambda > 1$. The sequence $\mathcal{Z}$ is *uniformly discrete*, if there exists $\tau_0 > 0$ such that every set Z_k is τ_0-separated. Under the assumption that $\mathcal{M}$ is lacunary and $\mathcal{Z}$ is uniformly discrete, [10] showed that if X is the support of an absolutely δ-decaying measure, then $E_{\mathcal{M},\mathcal{Z}} \cap X$ is a winning set in X for Schmidt's game.

Using similar arguments for the proof, we want to consider the following weaker condition in our weaker setting: In fact, assume that, independently of $t \in \mathbb{R}^+$, we have

$$|\{k \in \mathbb{N} : \log(t_k/\tau_k) \in (t-b, t]\}| \leq \varphi(b), \quad \text{for all } b > 0, \tag{2.3.2}$$

for some function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Note that if $\mathcal{M}$ is lacunary and $\mathcal{Z}$ is uniformly discrete, then (2.3.2) holds for the function $\varphi(b) \leq b/\log(\lambda)$.

Let again $\mathcal{S}$ denote the set of affine hyperplanes in $\mathbb{R}^n$ and recall that the Lebesgue measure is absolutely $(1, c_0)$-decaying (see Lemma 2.20).

[7] That is, for every $y_1, y_2 \in Z_k$ we have $d(y_1, y_2) \geq \tau_k > 0$.

Theorem 2.28. *Let $X \subset \mathbb{R}^n$ be the support of an absolutely δ-decaying measure μ, let $\mathcal{M}$ and $\mathcal{Z}$ be as above satisfying (2.3.2) with a function $\varphi(b) \leq e^{\bar{\delta}b}$, with $\bar{\delta} < \delta$. Then, for every $n_* \in \mathbb{N}$ and $L_* \geq 0$ there is $b_* = b_*(n_*, L_*, \delta, \bar{\delta})$ such that (Ω, B_1) is (b_*, n_*, L_*)-diffuse with respect to $\mathcal{F}$ defined below.*

In particular, $F(E_{\mathcal{M},\mathcal{Z}} \cap X)$ is a B_1-weakly-winning set in $F(X)$ for every bi-Lipschitz map $F : \mathbb{R}^n \to F(\mathbb{R}^n)$.

If μ satisfies moreover a power law with respect to the exponent τ, then $E_{\mathcal{M},\mathcal{Z}} \cap X$ and hence $F(E_{\mathcal{M},\mathcal{Z}} \cap X)$ are of Hausdorff-dimension τ (and in fact thick) by Theorem 2.17.

Proof. Let $v_k \in \mathbb{R}^n$ be the unit vector such that $\|M_k v_k\| = t_k$ and if $V_k \equiv \{M_k v_k\}^\perp$ is the subspace orthogonal to $M_k v_k$, let $W_k \equiv M_k^{-1}(V_k)$. Then, for $k \in \mathbb{N}$ and $z \in Z_k$ we define the subsets

$$Y_k(z) \equiv (M_k^{-1}(z) + W_k) \cap M_k^{-1}(B(z, \tau_k/4)).$$

Set $s_k \equiv \log(\tau_k/t_k) + \log(12)$, which we reorder such that $s_k \leq s_{k+1}$, so that we obtain a discrete set of sizes. For $k \in \Lambda \equiv \mathbb{N}$ let the resonant set R_k be given by

$$\begin{aligned} R_k &\equiv \{x \in Y_l(z_l) : z_l \in Z_l \text{ and } \log(t_l/\tau_l) \leq s_k\} \\ &= \{x \in Y_l(z_l) : z_l \in Z_l \text{ and } \frac{\tau_l}{t_l} \geq \frac{\tau_k}{t_k}\}, \end{aligned}$$

which gives a nested and discrete family $\mathcal{F} = \{\mathbb{N}, R_k, s_k\}$.

Note that for all $x \in \mathbb{R}^n$ we have $\|x\| \geq \|M_k x\|/t_k$. Hence, for distinct points z_1, $z_2 \in Z_k$, $Y_k(z_1)$ and $Y_k(z_2)$ are subsets of parallel affine hyperplanes and we have

$$\begin{aligned} \|Y_k(z_1) - Y_k(z_2)\| &\geq \|M_k^{-1}(B(y_1, \tau_k/4)) - M_k^{-1}(B(y_2, \tau_k, /4))\| \\ &\geq \frac{\tau_k - 2\tau_k/4}{t_k} = \frac{\tau_k}{2t_k} \geq 6e^{-s_k}, \end{aligned} \tag{2.3.3}$$

since Z_k is τ_k-separated. Given a closed ball $B = B(x, 3e^{-t}) \subset \mathbb{R}^n$ with $x \in X$, for every $k \in \mathbb{N}$ with $s_k \leq t$, it follows from (2.3.3) that at most one of the sets $Y_k(y)$, $y \in Z_k$, can intersect B. Moreover, for $b > 0$, the number of $k \in \mathbb{N}$ with $s_k \in (t - b, t]$ is bounded by $\varphi(b)$ by (2.3.2). Thus, there exist at most $N = \lfloor \varphi(b) \rfloor$ affine hyperplanes $L_1, \ldots, L_N \in \mathcal{S}$ such that

$$B \cap R(t, b) \subset \bigcup_{i=1}^{N} L_i.$$

Since μ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$, and (Ω, B_1) is $\log(3)$-separating, we have for $B = B(x, e^{-t})$ and $s \geq 0$ that

$$\begin{aligned} \mu(B \cap \mathcal{N}_{e^{-(t+s)}}(R(t,b))) &\leq \sum_{i=1}^{N} \mu(B \cap \mathcal{N}_{e^{-(t+s)}}(L_i)) \\ &\leq \varphi(b) \cdot c_\tau e^{-\delta s} \mu(B) \equiv f(b, s)\mu(B). \end{aligned}$$

Note that, since $\varphi(b) \leq e^{\bar{\delta}b}$ with $\bar{\delta} < \delta$, for all $n_* \in \mathbb{N}$ and $L_* \geq 0$, there exists a $b_* = b_*(n_*, L_*, \delta, \bar{\delta})$ such that $f(n_*(b+L_*)+\log(3), b-2\log(3)) \leq c_0 < 1$ for all $b > \bar{b}_*$. Thus, we showed that (Ω, B_1, μ) is f-decaying with respect to $\mathcal{F}$ and f is $(\log(3), b_*, n_*, L_*)$-decaying.

Moreover, (Ω, B_1) is $\log(3)$-separating with respect to $\mathcal{F}$. By Proposition 2.16, (Ω, B_1) is $(b_* + \log(3), n_*, L_*)$-diffuse with respect to $\mathcal{F}$. Hence, by Theorem 2.6 and Proposition 2.7, $F(\mathbf{Bad}(\mathcal{F}))$ is B_1-weakly-winning for any $L_*/2$-bi-Lipschitz map.

Finally, let $x \in \mathbf{Bad}(\mathcal{F})$, that is, there exists $c < \infty$ such that

$$d(x, Y_k(y)) \geq e^{-(s_k+c)} \geq e^{-c-\log(12)}\tau_k/t_k \equiv \bar{c}\tau_k/t_k$$

for every $k \in \mathbb{N}$ and $y \in Z_k$. Assume that $M_k x \in B(y, \tau_k/4)$. Then,

$$x \in \mathcal{N}_{\bar{c}\tau_k/t_k}(M_k^{-1}(y) + W_k)^C \cap M_k^{-1}(B(y, \tau_k/4))$$

and we can write the vector $v = x - M_k^{-1}(y)$ as $v = w + \tilde{c}\tau_k/t_k v_k$ with $w \in W_k$ and $\tilde{c} \geq \bar{c}$. Hence, since $M_k W_k$ is orthogonal to $M_k v_k$,

$$d(M_k x, y) = \|M_k v\| = \|M_k w + \tilde{c}\ \tfrac{\tau_k}{t_k} M_k v_k\| \geq \tilde{c}\ \tfrac{\tau_k}{t_k}\|M_k v_k\| \geq \bar{c}\tau_k,$$

so that $M_k x \notin B(y_k, \bar{c}\tau_k)$. This shows that $\mathbf{Bad}(\mathcal{F}) \subset E_{\mathcal{M},\mathcal{Z}} \cap X$ which implies that $E(\mathcal{M}, \mathcal{Z}) \cap X$ is B_1-weakly-winning. □

2.3.6 The geodesic flow in CAT(-1)-spaces

We discuss this example in more details. If GZ denotes the space of geodesic rays in a proper geodesic CAT(-1) metric space Z, then the semigroup $\mathbb{R}^+$ acts on GZ via the geodesic flow (g^s) which itself acts by reparameterization,

$$g^s(\gamma)(t) = \gamma(t + s).$$

Given a family of (convex) resonant sets $\tilde{R}_\lambda \subset Z, \lambda \in \Lambda$, we can ask about the rays which avoid contractions of or have bounded penetrations in neighborhoods of the resonant sets. The behavior of penetration lengths of geodesic rays in convex subsets of Z leads to a model of Diophantine approximation in CAT(-1)-spaces, developed by Hersonsky, Parkkonen and Paulin in [25, 26, 46], as well as [38]. With respect to the visual metric d_o (where o is a base point), we thereby translate our problem to the compact metric space $(\partial_\infty Z, d_o)$ and, since d_o is a metric on the set of asymptotic rays, we induce suitable resonant sets R_λ in $\partial_\infty Z$ related to the resonant sets $\tilde{R}_\lambda$.

We begin by introducing the setting and stating the main results of this subsection. In Subsubsection 2.3.6-2 we introduce the model of Diophantine approximation and relate the model to our setting and results. In Subsubsection 2.3.6-3, we discuss on the question of the Hausdorff-dimension and on the required conditions. In order to keep the exposition readable, we will skip all of the main proofs until Subsubsection 2.3.6-4.

2.3.6-1 Main Results

For a general reference and further details we refer to [8]. In the following, (Z, d) denotes a proper geodesic CAT(-1) metric space and, for a convex subset $Y \subset Z$, $\partial_\infty Y$ its visual boundary, that is, the set of equivalence classes of asymptotic rays in Y. Equip $\bar{Z} \equiv Z \cup \partial_\infty Z$ with the cone topology. Given two points $x, y \in \bar{Z}$ we denote by $[x, y]$ the unique geodesic segment from x to y. For three points $o, x, y \in \bar{Z}$, let

$$(x, y)_o \equiv \frac{1}{2}(d(o, x) + d(o, y) - d(x, y))$$

be the Gromov-product at o and for $\xi, \eta \in \partial_\infty Z$, let $(\xi,\eta)_o \equiv \lim_{t\to\infty}(\gamma_{o,\xi}(t), \gamma_{o,\eta}(t))_o$ be the extended Gromov-product at o, where $\gamma_{o,\xi} \equiv [o,\xi]$. For $o \in Z$, we define $d_o : \partial_\infty Z \times \partial_\infty Z \to [0,\infty)$ by $d_o(\xi,\xi) \equiv 0$ and for $\xi \neq \eta$ by

$$d_o(\xi,\eta) \equiv e^{-(\xi,\eta)_o},$$

called the *visual metric* at o. Then $(\partial_\infty Z, d_o)$ is a compact metric space.

For $\xi \in \partial_\infty Z$ and $y \in Z$, the *Busemann function* $\beta = \beta_{\xi,y} : Z \to \mathbb{R}$ (with respect to y) is defined by

$$\beta(x) \equiv \lim_{t\to\infty} d(x, \gamma_{y,\xi}(t)) - t,$$

which is continuous and convex on Z and $\beta(y) = 0$. The level sets of $\beta_{\xi,y}$ are called *horospheres* at ξ and the sublevel sets are called *horoballs* at ξ (with respect to y).

For technical reasons, let $t_0 > 0$ be a sufficiently large constant determined below. Now, given a base point $o \in Z$, assume we are given a countable collection of closed convex sets $\mathcal{C} = \{C_m \subset Z : m \in \mathbb{N}\}$ such that the collection of distances $\{d_m \equiv d(o, C_m) : m \in \mathbb{N}\} \subset (t_0, \infty)$ is a discrete set. Remarking that Z is a δ_0-hyperbolic space for some $\delta_0 > 0$ (see (2.3.5)), we will consider the following three cases simultaneously:

1. $\mathcal{C}_1 = \{C_m\}$ is a collection of pairwise disjoint horoballs based at $\partial_\infty C_m \equiv \xi_m$.
2. $\mathcal{C}_2 = \{C_m\}$ is a collection of convex sets with $|\partial_\infty C_m| \geq 1$ which is $(2\delta_0, T)$-*embedded*; that is, we have that $\operatorname{diam}(\mathcal{N}_{2\delta_0}(C_i) \cap \mathcal{N}_{2\delta_0}(C_j)) \leq T$ for $i \neq j$.
3. $\mathcal{C}_3 = \{x_m\}$ is a collection of τ_0-separated points x_m in the hyperbolic space $Z = \mathbb{H}^{n+1}$.

Note that Case 1. is in fact covered by Case 2. but treated explicitly as an interesting special case.

In the first two cases, we obtain a collection of nonempty sets $\mathcal{C}_i^\infty \equiv \{\partial_\infty C_m\}$ in $\partial_\infty Z$ which we will see is disjoint. For the third case, let $\mathcal{C}_3^\infty \equiv \{\xi_m^\infty\}$ be the collection of the boundary projections of x_m with respect to o, that is $\xi_m^\infty \equiv \gamma_{o,x_m}(\infty) \in S^n = \partial_\infty \mathbb{H}^{n+1}$. By abuse of notation, ξ_m^∞ is also denoted by $\partial_\infty C_m$ in the following. The following result on the distribution of $\mathcal{C}_\infty$ in $\bar{X}$ is essential.

Proposition 2.29. *Let $l_1 \equiv \delta_0$ and $l_2 \equiv T + 2\delta_0$. Then, for the respective cases, we have*

1. $d_o(\xi_i, \xi_j) > e^{-l_1} e^{-\max\{d_i, d_j\}}$,
2. $d_o(\partial_\infty C_i, \partial_\infty C_j) > e^{-l_2} e^{-\max\{d_i, d_j\}}$,

for $i \neq j$. Moreover, there exists a constant $c_0 = c_0(\tau_0)$ such that, for every $b > 0$ and every ball $B = B_{d_o}(\xi, 2e^{-t})$,

3. $|\{\xi_m^\infty \in B : d_m \in (t-b, t]\}| \leq c_0\, b$.

Given the three collections $\{d_m^i \equiv d(o, C_m) : m \in \mathbb{N}\}$, $i = 1,2,3$, which we relabel to define the set of sizes $s_m^i \equiv d_m^i$ and reorder such that $s_m^i \leq s_{m+1}^i$. For $m \in \Lambda_i \equiv \mathbb{N}$ let

$$\bar{R}_m^i \equiv \{\xi \in \partial_\infty C_j : \partial_\infty C_j \in \partial_\infty \mathcal{C} \text{ such that } e^{-d_j} \geq e^{-s_m}\},$$

which gives a nested and discrete family $\mathcal{F}_i = (\mathbb{N}, R_m^i, s_m^i)$. Given a closed subset $X \subset \bar{X} \equiv \partial_\infty Z$, set $\Omega \equiv X \times (t_0, \infty)$. If moreover every set $\partial_\infty C_m \in \mathcal{C}_2^\infty$ is closed (hence compact), note that a point $\xi \in \mathbf{Bad}_X^{B_1}(\mathcal{F}_i)$ if and only if there exists a constant $c = c(\xi) > 0$ such that, for every $\partial_\infty C_m \in \mathcal{C}_i^\infty$,

$$d_o(\xi, \partial_\infty C_m) > c\, e^{-d(o,C_m)}. \tag{2.3.4}$$

Assuming that X satisfies suitable diffusion properties in $\bar{X}$ with respect to the collections $\mathcal{C}_i^\infty$, we obtain our main result using Proposition 2.29.

Theorem 2.30. *For Case 1. and for Case 2. if every set $C_i \in \mathcal{C}_2$ is a geodesic line, assume that X is β-diffuse. Then $\mathbf{Bad}_X^{B_1}(\mathcal{F}_i)$ is absolute-winning (in the sense of McMullen).*

For Cases $1, 2.$ assume that (Ω, B_1) is b_*-diffuse with respect to $\mathcal{C}_i^\infty$. Then $\mathbf{Bad}_X^{B_1}(\mathcal{F}_i)$ is absolute B_1-winning with respect to $\mathcal{C}_i^\infty$ (in particular Schmidt winning).

If X is the support of a locally finite Borel measure such that (Ω, B_1, μ) satisfies a power law with respect to the exponent τ, then, for every $n_* \in \mathbb{N}$ and $L_* \geq 0$, there is $b_* = b_*(n_*, L_*, \tau, \tau_0)$ such that (Ω, B_1) is (b_*, n_*, L_*)-diffuse with respect to $\mathcal{F}_3$. In particular, $F(\mathbf{Bad}_X^{B_1}(\mathcal{F}_3))$ is B_1-weakly-winning in $F(X)$ and of Hausdorff-dimension τ (and in fact thick), for any bi-Lipschitz map $F : S^n \to F(S^n)$.

We remark that the first case has been considered by [38] (as well as [2, 12, 13, 40, 50] in stronger and more specific settings than ours) in the setting of proper geodesic δ-hyperbolic metric spaces where they used a similar definition of badly approximable points using the size of the shadows of the disjoint horoballs. In fact, note that our proof of Case 1. works equally well in their weaker setting.
Moreover, Case 3. also holds if Z is a manifold of pinched negative curvature.

Remark 2.31. Note that the visual distance at a point $o \in X$ is comparable to the Hamenstädt metric with respect to a horoball H_0: For every compact subset K of $\partial_\infty X - \partial_\infty H_0$, there exists a constant $c_K > 0$ such that for all $\xi, \eta \in K$,

$$c_K^{-1} d_o(\xi, \eta) \leq d_{H_0}(\xi, \eta) \leq c_K d_o(\xi, \eta);$$

see [25], Lemma 2.3. We therefore focus only on the visual distance in our settings, which can however, up to further requirements, be replaced by the Hamenstädt metric.

Before we relate our setting to the model of Diophantine approximation due to Hersonsky, Parkkonen and Paulin, we want to point out the following dynamical interpretation.

Lemma 2.32. *Let $t_0, l_0 > 0$ be sufficiently large constants. Given $C = C_m \in \mathcal{C}_i$ with $d(o, C) \geq t_0$ and $\xi \in \partial_\infty Z$, we have*

1. $\gamma_{o,\xi}([t, t+l]) \subset C$,
2. $\gamma_{o,\xi}([t, t+l]) \subset \mathcal{N}_{2\delta_0}(C)$,
3. $\gamma_{o,\xi}(t) \in B(x_m, e^{-l})$,

for a suitable time $t > t_0$ and length $l > l_0$, if and only if

1. $d_o(\xi, \xi_m) \leq \bar{c}\, e^{-l_0/2} \cdot e^{-d(o,C)}$,
2. $d_o(\xi, \partial_\infty C) \leq \bar{c}\, e^{-l} \cdot e^{-d(o,C)}$,

3. $d_o(\xi, \xi_m^\infty) \leq \bar{c}\, e^{-l} \cdot e^{-d(o,C)}$,

where $\bar{c} > 0$ *is a universal constant.*

Hence, in view of (2.3.4), for $i = 1, 2, 3$, if every $\partial_\infty C_m \in \mathcal{C}_i^\infty$ is closed, we obtain that

$$\mathbf{Bad}_X^{B_1}(\mathcal{F}_i) = S_i,$$

for the following sets S_i.

1. $S_1 \equiv \{\xi \in X : \exists\, l = l(\xi) < \infty$ such that the lengths[8] $L(\gamma_{o,\xi}(\mathbb{R}^+) \cap C_m) \leq l$ for all horoballs $C_m \in \mathcal{C}_1\}$,
2. $S_2 \equiv \{\xi \in X : \exists\, l = l(\xi) < \infty$ such that the lengths $L(\gamma_{o,\xi}(\mathbb{R}^+) \cap \mathcal{N}_{2\delta_0}(C_m)) \leq l$ for all convex sets $C_m \in \mathcal{C}_2\}$,
3. $S_3 \equiv \{\xi \in X : \exists\, c = c(\xi) > 0$ such that $\gamma_{o,\xi}(\mathbb{R}^+) \cap B(x_m, c) = \emptyset$ for all points $x_m \in \mathcal{C}_3\}$.

Remark 2.33. In view of Lemma 2.43, we can in fact consider the ε-neighborhoods $\mathcal{N}_\varepsilon(C_m)$ of C_m in S_2, for $\varepsilon > 0$. Moreover, a geodesic $\gamma_{o,\xi}$, $\xi \in S_1$, has bounded penetration lengths in the collection of horoballs $\mathcal{C}_1$ if and only if it avoids the same collection of uniformly shrinked horoballs; see Lemma 2.45 for a precise statement.

2.3.6-2 A model of Diophantine approximation in negatively curved spaces.

Let $\Gamma \subset I(Z)$ be a discrete subgroup of the isometry group $I(Z)$ of Z. Note that every isometry $\varphi \in I(Z)$ extends to a homeomorphism on $\partial_\infty Z$. The *limit set* $\Lambda\Gamma$ of Γ is the compact subset $\overline{\Gamma.x} \cap \partial_\infty Z$ of $\partial_\infty Z$, for any $x \in Z$. If $\Lambda\Gamma$ contains at least two points, then $\mathcal{C}\Gamma$ denotes the convex hull of $\Lambda\Gamma$.

Recall that a subgroup Γ_0 of Γ is called *convex cocompact* if $\Lambda\Gamma_0$ contains at least two points and the action of Γ_0 on the convex hull $\mathcal{C}\Gamma_0$ has compact quotient.
We call Γ_0 *bounded parabolic* if Γ_0 is the stabilizer of a parabolic fixed point $\xi_0 \in \Lambda\Gamma$, and if there exists a horoball C_0 at ξ_0 such that the action of Γ_0 on ∂C_0 has compact quotient. Note that up to considering the CAT(-1)-space $Z \cap \Lambda\Gamma$ instead of Z, our definition agrees with the classical definition of bounded parabolic fixed points in the hyperbolic space; see [49].
Finally, we call Γ_0 *almost malnormal* if $\Lambda\Gamma_0$ is precisely invariant, that is, if for all $\varphi \in \Gamma - \Gamma_0$ we have $\varphi.\Lambda\Gamma_0 \cap \Lambda\Gamma_0 = \emptyset$.

Now, let $\Gamma_i \subset \Gamma$, $i = 1, 2$, be almost malnormal subgroups in Γ of infinite index and without elliptic elements. We treat the following three cases simultaneously:

1. Let Γ_1 be bounded parabolic and let C_1 be the horoball as in the definition.
2. Let Γ_2 be convex-cocompact and let $C_2 = \mathcal{C}\Gamma_2$ be the convex hull of Γ_2, where we assume that either
   a) C_2 is a geodesic line, or,
   b) every image $\varphi.\Lambda\Gamma_2 \subset \Lambda\Gamma$, $[\varphi] \in \Gamma/\Gamma_2$, is contained in a metric sphere in $\partial_\infty Z$ (with respect to d_o).

[8] Note that since C_m is convex, $\gamma_{o,\xi}(\mathbb{R}^+) \cap C_m$ is the image of a connected geodesic segment.

3. Let Γ_3 be the identity element of Γ, and, for $x \in Z = \mathbb{H}^{n+1}$, take $C_3 = \{x\}$ in the following.

Note that since Γ_i is almost malnormal for the Cases 1, 2. and Γ is without elliptic elements for Case 3., we have $\Gamma_i = \mathrm{Stab}_\Gamma(C_i)$.

Example 2.34. Let C_2 be a totally geodesic submanifold of dimension $m+1$ in $\mathbb{H}^{n+1}$, the hyperbolic ball model, and $o = 0$ be the center of $\mathbb{H}^{n+1}$. Hence, C_2 is a subspace isometric to the hyperbolic space $\mathbb{H}^{m+1}$ and the boundary of this subspace is a metric sphere (with respect to the angle metric d_o) of dimension m. Since Γ_2 is almost malnormal and convex cocompact, we have $m < n$. Hence, $\partial_\infty C_2 = \Lambda\Gamma_2$ and every image $\varphi.\Lambda\Gamma_2$ are contained in metric spheres.

For the respective cases, $i = 1, 2, 3$, given again $t_0 > 0$, denote the data by

$$\mathcal{D}_i = (Z, \Gamma, C_i, o, t_0).$$

For $r = [\varphi] \in \Gamma/\Gamma_i$ we define

$$D_i(r) = d(o, \varphi(C_i))$$

which does not depend on the choice of the representative φ of r.

Remark 2.35. For $r = [\varphi] \in \Gamma/\Gamma_i$, let $\mathcal{S}_o(\varphi(C_i)) \subset \partial_\infty Z$ be the shadow of the set $\varphi(C_i)$ with respect to the base point o. Using (2.3.6), Lemma 2.42 and 2.44 below, one can show that if $D_i(r)$ is sufficiently large, the size (diameter with respect to d_o) of the shadows $\mathcal{S}_o(\varphi C_i)$ is comparable to the quantitiy $e^{-D_i(r)}$. We therefore consider the *approximation function* $f_i(r) = e^{D_i(r)}$ as a renormalization of the size of the shadows.

Note that the set $\{D_i(r) : r \in \Gamma/\Gamma_i\}$ is discrete and unbounded:

Lemma 2.36. *For every $D \geq 0$ there are only finitely many elements $r \in \Gamma/\Gamma_i$ such that $D_i(r) \leq D$ and there exists an $r \in \Gamma/\Gamma_i$ such that $D_i(r) > D$.*

Proof. For the second case, the proof follows from Lemma 3.1 and 3.2 in [46] with the difference that we do not consider the stabilizer of o in Γ (which is trivial in our assumption and only a finite subgroup in general). The arguments of the proof also work for the first case. The third case follows since Γ is discrete and Γ_3 is of infinite index in Γ. □

Now, for $i = 1, 2, 3$ and for $\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i$ define the *approximation constant*

$$c_i(\xi) = \liminf_{r=[\varphi]\in\Gamma/\Gamma_i : D_i(r) > t_0} e^{D_i(r)} d_o(\xi, \varphi.\Lambda\Gamma_i),$$

where we replace $\varphi.\Lambda\Gamma_i$ by $\varphi(x)_\infty \equiv \gamma_{o,\varphi(x)}(\infty)$ in the third case. If $c_i(\xi) = 0$ then ξ is called *well approximable*, otherwise it is called *badly approximable* (with respect to $\mathcal{D}_i$). Define the set of badly approximable limit points by

$$\mathbf{Bad}(\mathcal{D}_i) = \{\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i : c_i(\xi) > 0\} \subset \Lambda\Gamma.$$

Consider the collections $\mathcal{C}_i \equiv \{\varphi(C_i) : r = [\varphi] \in \Gamma/\Gamma_i \text{ with } D_i(r) > t_0\}$, $i = 1, 2, 3$, and note that $\mathcal{C}_i$ is $(2\delta_0, T)$-*embedded*. In fact, this follows easily for Case 3. since, by discreteness of Γ, $\mathcal{C}_3$ is in fact τ_0-separated for some $\tau_0 > 0$. For Case 2. we refer to [26] and remark that the proof works similarly for Case 1. In the first case, we will therefore assume, after shrinking C_1, that the images $\varphi(C_1)$, $[\varphi] \in \Gamma/\Gamma_1$ are pairwise disjoint. Using Lemma 2.36, we thus established the setting of the previous subsubsection for the corresponding cases. Again, in view of (2.3.4), we have for $X = \Lambda\Gamma$ and $\bar{X} = \partial_\infty Z$ that

$$\mathbf{Bad}(\mathcal{D}_i) = \mathbf{Bad}^{B_1}_{\Lambda\Gamma}(\mathcal{F}_i) = S_i.$$

Thus, as a corollary of Theorem 2.30, we obtain the following.

Corollary 2.37. *If the limit set $\Lambda\Gamma$ of Γ is β-diffuse for the Cases 1. and 2a), then $\boldsymbol{Bad}(\mathcal{D}_i)$ is absolute-winning (in the sense of McMullen).*

For the Case 2b), if $(\Lambda\Gamma \times (t_0, \infty), B_1)$ is b_*-diffuse with respect to the collection $\mathcal{S}$ of metric spheres in $\Lambda\Gamma$, then $\boldsymbol{Bad}(\mathcal{D}_2)$ is absolute B_1-winning with respect to $\mathcal{S}$.

If $\Lambda\Gamma$ is the support of a locally finite Borel measure satisfying a power law with respect to the exponent τ, then for any bi-Lipschitz map $F : S^n \to F(S^n)$, $F(\boldsymbol{Bad}(\mathcal{D}_3))$ is weakly B_1-winning in $F(\Lambda\Gamma)$ and of Hausdorff-dimension τ (and in fact thick in $F(\Lambda\Gamma)$).

Remark 2.38. Recall that if $X = \mathbb{H}^{n+1}$ and Γ is a non-elementary finitely generated Kleinian group, then $\Lambda\Gamma \subset S^n$ is uniformly perfect; see [30]. In particular, $\Lambda\Gamma$ is β-diffuse for some $\beta > 0$ by Lemma 2.10.
For Case 2. we refer to Corollary 2.40 below, and for Case 3. to the next subsubsection.

2.3.6-3 A measure on $\Lambda\Gamma$.

Let $X = \mathbb{H}^{n+1}$ be the hyperbolic ball model and let $o = 0$ be the center. Note that the visual distance d_o is bi-Lipschitz equivalent to the angle metric on the unit sphere $S^n = \partial_\infty \mathbb{H}^{n+1}$. Hence, if Γ is of the *first kind*, that is $\Lambda\Gamma = \partial_\infty \mathbb{H}^{n+1}$, then $\Lambda\Gamma = S^n$ is β-diffuse and b_*-diffuse with respect to $\mathcal{S}$, for $\beta = b_* > \log(3)$, and the Lebesgue measure on S^n satisfies a power law with respect to the visual metric d_0 and the exponent n. More generally, recall that the *critical exponent* of a discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ is given by

$$\delta(\Gamma) \equiv \inf\{s > 0 : \sum_{\varphi \in \Gamma} e^{-sd(x, \varphi(x))} < \infty\},$$

for any $x \in \mathbb{H}^{n+1}$. Associated to Γ, there is a canonical measure, the *Patterson-Sullivan* measure μ_Γ, which is a $\delta(\Gamma)$-conformal probability measure supported on $\Lambda\Gamma$. For a precise definition we refer to [43]. If Γ is non-elementary and convex-cocompact, then $\delta(\Gamma)$ equals the Hausdorff-dimension of $\Lambda\Gamma$; in particular, the Patterson-Sullivan measure $\mu_{\Gamma,o}$ (at o) satisfies a power law with respect to the exponent $\delta(\Gamma)$. There are various further results concerning the Patterson-Sullivan measure. Here, we point out the following.

Regarding Case 1., [38] showed that if Γ is a non-elementary geometrically finite Kleinian group, the set of limit points which correspond to geodesics starting in o and projecting to bounded geodesics in $\mathbb{H}^{n+1}/\Gamma$ has dimension $\delta(\Gamma)$. In particular, $S_1 = \mathbf{Bad}(\mathcal{D}_1)$ contains this set and is thus of dimension $\delta(\Gamma)$.

For the second case, let $\mathcal{H}(\Gamma) \equiv \{S \cap \Lambda\Gamma : S$ is a sphere in S^n of codimension at least $1\}$ which contains the set $\mathcal{S}$. A finite Borel measure ν on S^n is called $\mathcal{H}(\Gamma)$-*friendly*, if ν is Federer and if $(\Lambda\Gamma \times (t_0, \infty), B_1, \nu)$ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{H}(\Gamma)$.

Theorem 2.39 ([53], Theorem 2)**.** *For every non-elementary convex cocompact discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ (without elliptic elements), such that $\Lambda\Gamma$ is not contained in a finite union of elements of $\mathcal{H}(\Gamma)$, the Patterson-Sullivan measure $\mu_{\Gamma,o}$ is $\mathcal{H}(\Gamma)$-friendly.*

The Theorem is in fact true for a set of $\mu_{\Gamma,o}$-neglectable subsets (for details and the definition we refer to [53]) and the requirements of Case 2. can be weakened.

As a corollary of Proposition 2.14 we obtain the following.

Corollary 2.40. *Let Γ be as in Theorem 3.32. Then $(\Lambda\Gamma \times (t_0, \infty), B_1)$ is b_*-diffuse with respect to $\mathcal{H}(\Gamma)$ and hence with respect to $\mathcal{S}$ for some $b_* > 0$ sufficiently large.*

Hence, for Case 2., $S_2 = \mathbf{Bad}(\mathcal{D}_2)$ is absolute B_1-winning with respect to $\mathcal{S}$ by Corollary 2.37. Moreover, since $\mu_{\Gamma,o}$ satisfies a power law, we see that S_2 is thick, by Theorem 2.17.

Summarizing, we have the following.

Corollary 2.41. *In our setting of Subsubsection 2.3.6, if Γ is non-elementary geometrically finite Kleinian group in Case 1. or as in Theorem 3.32 in Case 2., then $\mathbf{Bad}(\mathcal{D}_i)$ is of Hausdorff-dimension $\delta(\Gamma)$ for $i = 1, 2$. In fact, for any nonempty open set $U \subset \Lambda\Gamma$, $\mathbf{Bad}(\mathcal{D}_2) \cap U$ is of Hausdorff-dimension $d_\mu(\Lambda\Gamma) = \delta(\Gamma) = dim(\Lambda\Gamma)$.*

2.3.6-4 Proofs.

First, note that every CAT(-1) space is a (tripod) δ_0-hyperbolic space for some $\delta_0 > 0$, which implies that for all $o, x, y \in Z$ we have

$$p \in [o, x], q \in [o, y] \text{ with } d(o, p) = d(o, q) \leq (x, y)_o \implies d(p, q) \leq \delta_0, \tag{2.3.5}$$

and (2.3.5) also holds for $x, y \in \partial_\infty Z$ as well. Moreover, Z is a (Gromov) δ_0-hyperbolic space (we may assume the same δ_0) so that, given a geodesic triangle with vertices $x, y, z \in \bar{Z}$, every edge of the triange lies in the δ_0-neighborhood of the two other ones. Finally, there exists a $\kappa > 0$ (depending only on δ_0), such that for all $o \in Z$ and $\xi, \eta \in \partial_\infty Z$,

$$0 \leq d(o, [\xi, \eta]) - (\xi, \eta)_o \leq \kappa. \tag{2.3.6}$$

We start with the proofs of Proposition 2.29 and Theorem 2.30.

Proof of Proposition 2.29. We start with Cases 1. and 2. Given $\eta \in \partial_\infty C_1$, $\bar{\eta} \in \partial_\infty C_2$, where C_1 and $C_2 \in \mathcal{C}_i$, set $D \equiv \max\{d(o, C_1), d(o, C_2)\}$, and assume that

$$d_o(\eta, \bar{\eta}) = e^{-(\eta, \bar{\eta})_o} \leq e^{-(D + l_i)}. \tag{2.3.7}$$

Equivalently, we have $(\eta, \bar{\eta})_o \geq D + l_i$ and by (2.3.5), we see that

$$d(\gamma_{o,\eta}(D + l_i), \gamma_{o,\bar{\eta}}(D + l_i)) \leq \delta_0. \tag{2.3.8}$$

Case 1: By definition, $l_1 = \delta_0$ and hence,

$$d(\gamma_{o,\eta}(D + \delta_0), \gamma_{o,\bar{\eta}}(D + \delta_0)) \leq \delta_0.$$

On the other hand, both the points $\gamma_\eta(D + \delta_0)$ and $\gamma_{\bar{\eta}}(D + \delta_0)$ are contained in the horoballs C_1 and C_2 respectively, at distance at least δ_0 to the boundaries of the respective horoballs. Therefore, if the horoballs C_1 and C_2 are disjoint, then

$$d(\gamma_{o,\eta}(D + \delta), \gamma_{o,\bar{\eta}}(D + \delta_0)) \geq 2\delta_0,$$

which is a contradiction. Hence, $C_1 = C_2$ and $\{\eta\} = \{\bar{\eta}\}$.
Case 2: By definition, $l_2 = T + 2\delta_0$, and hence,

$$d(\gamma_{o,\eta}(D+T+2\delta_0), \gamma_{o,\bar{\eta}}(D+T+2d_0)) \leq \delta_0.$$

Let o_j be the projection of o on the closed convex set C_j with $d(o, o_j) = d(o, C_j)$, $j = 1, 2$. For any point x on the ray $\gamma_{o,\eta}$ at distance $d(o,x) > D + \delta_0$ and y on $[o, o_j]$, we have $d(x,y) \geq d(o,x) - d(y,o) > \delta_0$. Since Z is a Gromov δ_0-hyperbolic space, we see that x cannot belong the δ_0-neighborhood of the segment $[o, o_1]$ and must therefore be contained in $\mathcal{N}_{\delta_0}([o_1, \eta])$. Since C_1 is convex, $o_1 \in C$ and $\eta \in \partial_\infty C$, we have $[o_1, \eta] \subset C_1$. Thus, we showed

$$\gamma_{o,\eta}([D+\delta_0, D+T+2\delta_0]) \subset \mathcal{N}_{\delta_0}([o_1,\eta]) \subset \mathcal{N}_{\delta_0}(C_1),$$

and the analogous results is true for $\gamma_{o,\bar{\eta}}$. Therefore, by convexity of the distance function and by (3.3.1), we have

$$\gamma_{o,\eta}([D+\delta_0, D+T+2\delta_0]) \subset \mathcal{N}_{2\delta_0}(C_2),$$

and hence

$$\gamma_{o,\eta}([D+\delta_0, D+T+2\delta_0]) \subset \mathcal{N}_{\delta_0}(C_1) \cap \mathcal{N}_{2\delta_0}(C_2).$$

In particular, since $\mathcal{C}_2$ is $(2\delta_0, T)$-embedded and

$$\operatorname{diam}(\mathcal{N}_{2\delta_0}(C_1) \cap \mathcal{N}_{2\delta_0}(C_2)) \geq L(\gamma_{o,\eta}([D+\delta_0, D+T+2\delta_0])) = T + \delta_0 > T,$$

we must have $C_1 = C_2$ and $\eta, \bar{\eta} \in C_1$.

Now, for *Case 3.,* we switch to the hyperbolic space. For a subset $M \subset S^n$ and $0 \leq a \leq \bar{a}$, consider the truncated cone of M with respect to o,

$$M(a, \bar{a}) \equiv \{\gamma_{o,\xi}(t) \in \mathbb{H}^{n+1} : \xi \in M, a \leq t \leq \bar{a}\}.$$

Fix $b > 0$, a ball $B = B_{d_o}(\eta, 2e^{-t})$ (with $t \geq b$) and note that a point ξ_m^∞ with $t - b < d(o, x_m) \leq t$ lies in B if and only if $x_m \in B(t-b, t)$. It therefore suffices to estimate the number of $x_m \in B(t-b,t)$ which we denote by $G(\eta, t, b)$.

Moreover, we claim that $B(t-b,t)$ is contained in the $(\delta_0 + 2\log(2))$-neighborhood of the geodesic segment $\gamma_{o,\eta}((t-b,t])$. To see this, note that for any point $\xi \in B$, we have $(\xi, \eta)_0 \geq -\log(d_o(\xi,\eta)) \geq t - \log(2)$ and hence, by (2.3.5) $d(\gamma_{o,\xi}(s), \gamma_{o,\eta}(s)) \leq \delta_0$, for all $s \leq t - \log(2)$. For $t - \log(2) \leq s \leq t$ we have

$$\begin{aligned} d(\gamma_{o,\xi}(s), \gamma_{o,\eta}(s)) &\leq d(\gamma_{o,\xi}(s), \gamma_{o,\xi}(t-\log(2))) + \delta_0 + d(\gamma_{o,\eta}(s), \gamma_{o,\eta}(t-\log(2))) \\ &\leq \delta_0 + 2\log(2), \end{aligned}$$

concluding the claim.

Clearly, since $\mathbb{H}^{n+1}$ is of constant sectional curvature, there exists a universal constant $C > 0$ such that the hyperbolic volume of $\mathcal{N}_{\delta_0+2\log(2)}(\gamma_{o,\eta}((t-b,t]))$ is bounded by $C \cdot b$. Finally, since moreover $\mathcal{C}_3$ is τ_0-separated for some $\tau_0 > 0$, it also follows that there exists a constant $\bar{c} = \bar{c}(\tau_0) > 0$ such that the (hyperbolic) volume of every ball $B(x_m, \tau_0/2)$ is at least $\bar{c}$. Thus, we conclude that $G(\eta, r, b) \leq C/\bar{c} \cdot b$, finishing the proof. □

Proof of Theorem 2.30. For Case 1. and Case 2., if every set C_m is a geodesic line, assume that X is β-diffuse. For the second case we need to remark that for distinct points η, $\bar{\eta} \in \partial_\infty C$, for a geodesic line $C \in \mathcal{C}_2$, then by (2.3.6) we have

$$d_o(\eta, \bar{\eta}) = e^{-(\eta,\bar{\eta})_o} \geq e^{-d(o,[\eta,\bar{\eta}])} = e^{-d(o,C)}.$$

Hence, using this remark and Proposition 2.29, we see that the special case (2.2.25) is satisfied for $c_i = e^{-l_i}$ and $\sigma = 1$. Moreover, $(\bar{X}, d_o)$ is compact so that $(\bar{\Omega}, B_1)$ is $\log(3)$-contracting. Thus, Proposition 2.12 implies that (Ω, B_1) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}_i$, where $\bar{b}_* = l_i + \log(3) + \log(2) + \beta$. In addition, $\mathbf{Bad}_X^{B_1}(\mathcal{F}_i)$ is absolute-winning (in the sense of McMullen) in the respective cases.

For Case 2. it follows from Proposition 2.29 that, given a ball $B = B_{d_o}(\xi, e^{-(t+l_*+\log(2))})$, $\xi \in X$, $t > t_0$, then for every size $s_m \leq t$ we have that $B \cap R_m$ is either empty or equals the set $B \cap \partial_\infty C_j$ for some set $C_j \in \mathcal{C}_2$. We showed that $\mathcal{F}$ is locally contained in $\mathcal{C}_2^\infty$ for $n_* = 1$. Since (Ω, B_1) is b_*-diffuse with respect to $\mathcal{C}_2^\infty$, Theorem 2.11 shows that (Ω, ψ) is strongly $\bar{b}_*$-diffuse with respect to $\mathcal{F}_2$ where $\bar{b}_* = l_2 + \log(2) + \log(3) + b_*$, and, moreover, that $\mathbf{Bad}_X^{B_1}(\mathcal{F}_1)$ is absolute B_1-winning with respect to $\mathcal{C}_2^\infty$. Finally, the same is true for Case 1., and Theorem 2.6 concludes the first two cases.

For Case 3., using Theorem 2.7 and again Theorem 2.6, it suffices to show that (Ω, B_1) is (b_*, n_*, L_*)-diffuse with respect to $\mathcal{F}$ for every $n_* \in \mathbb{N}$ and $L_* \geq 0$. In fact, assume that $X = \operatorname{supp}(\mu)$ for a locally finite Borel measure on S^n which satisfies a power law with respect to τ. Given a ball $B = B(\xi, e^{-t})$, $\xi \in X$, $t > t_0$, consider the set of boundary projections $\xi_m^\infty \in \mathcal{C}_3^\infty$ with $d(o, x_m) \in (t-b, t]$ and $\xi_m^\infty \cap 2B = B(\xi, 2e^{-t})$. This is precisely the set $2B \cap R^3(t,b)$ and Proposition 2.29 implies that $|2B \cap R^3(t,b)|$ is bounded by $C \cdot b$. Moreover, a ball $B(\xi_m^\infty, e^{-(t+s)})$ with $\xi_m^\infty \notin 2B$ cannot intersect B. Hence, we have

$$\begin{aligned} \mu(B \cap \mathcal{N}_{e^{-(t+s)}}(R(t,b))) &\leq \cup_{\xi_m^\infty \in 2B \cap R(t,b)} \mu(B_{d_o}(\xi_m^\infty, e^{-(t+s)})) \\ &\leq Cb \cdot c_2 e^{-\tau(t+s)} \\ &\leq \tfrac{Cc_2}{c_1} b \cdot e^{-\tau s} \mu(B) \equiv f(b,s)\mu(B), \end{aligned}$$

and we showed that (Ω, B_1, μ) is f-decaying with respect to the family $\mathcal{F}_3$. Clearly, there exists $b_* = b_*(n_*, L_*, \tau, \tau_0) > 2\log(3)$ sufficiently large such that the function f satisfies $f(n_*(b + L_*) + \log(3), b - 2\log(3)) \leq c_0 < 1$ for all $b > b_*$. Since $R^3(t,b)$ is a discrete set for all $t, b > 0$, (Ω, B_1) is $\log(3)$-separating with respect to $\mathcal{F}_3$. Thus, Proposition 2.16 concludes that (Ω, B_1) is $(\bar{b}_*, n_*, L_*)$-diffuse with respect to $\mathcal{F}$, where $\bar{b}_* = b_* + \log(3)$.

Since (3.2.32) and (MSG1-2) are satisfied, we have $\dim(\mathbf{Bad}_X^{B_1}(\mathcal{F}_3) \cap U) = d_\mu(U) = \tau$ from Theorem 2.17, for $U \subset X$ open. This finishes the proof. □

We will make use of the following results.

Lemma 2.42 ([45], Lemma 2.1)**.** *Let $x, y \in Z$ and for $z \in Z \cup \partial_\infty Z$ let $\gamma = [x,z]$. Then, for all $t \in [0, d(x,z)]$,*

$$d(\gamma(t), [y,z]) \leq \tfrac{1}{2} e^{d(x,y)-t}.$$

If $\varepsilon > 0$ and α is a geodesic segment, let $\mathcal{N}_\varepsilon(\alpha)$ be the closed ε-neighborhood of α which is itself convex. As a consequence of Lemma 2.42, we prove that a ray which penetrates in the D-neighborhood of a geodesic segment for a sufficiently long time must also penetrate in its ε-neighborhood.

Lemma 2.43. *Let $D \geq \varepsilon > 0$. Let γ and α be two geodesics in X such that $d(\gamma(-L), \alpha) \leq D$ and $d(\gamma(L), \alpha) \leq D$, where $L \geq 2(D - \log(\varepsilon))$. Then there exists a constant $c = c(D, \varepsilon) \leq D - \log(\varepsilon)$ such that $\gamma([-L + c, L - c]) \subset \mathcal{N}_\varepsilon(\alpha)$.*

Proof. First, consider the case when γ and α do not intersect. Let p and $q \in \alpha$ be the closest points of $a = \gamma(L)$ and $b = \gamma(-L)$ respectively at distance at most D on α. We subdivide the quadrilateral (a, b, p, q) in two geodesic triangles (a, b, p) and (b, p, q) with a connecting geodesic $\tilde{\gamma} = [b, p]$. Note that $\tilde{L} \equiv d(b, p) \geq 2L - D$. For $t \in [0, \tilde{L}]$, we let $b_t \in \gamma$ and $q_t \in \alpha$ be the closest points of $\tilde{\gamma}(t)$ on γ and α respectively. Let $t_0 = D - \log(\varepsilon)$. From Lemma 2.42 we have $d(\tilde{\gamma}(t_0), q_{t_0}) \leq e^{-t_0} e^D / 2 = \varepsilon/2$, as well as, since $L \geq 2(D - \log(\varepsilon))$,

$$d(\tilde{\gamma}(t_0), b_{t_0}) \leq \tfrac{1}{2} e^{-(\tilde{L} - t_0)} e^D \leq \tfrac{1}{2} e^{-2L + D + \log(\varepsilon)} \leq \tfrac{\varepsilon}{2}.$$

Thus, $d(b_{t_0}, \alpha) \leq \varepsilon$. Note that $d(\gamma(L), b_{t_0}) \leq t_0$ by properties of the closest point map. In the same way, we define a_{t_0} for the two geodesic triangles (a, b, q) and (a, p, q). Similarly, we obtain that also $d(a_{t_0}, \alpha) \leq \varepsilon$ with $d(\gamma(-L), a_{t_0}) \leq t_0$. Therefore, we see by convexity of the distance function that $\gamma([-L + t_0, L - t_0]) \subset [a_{t_0}, b_{t_0}] \subset \mathcal{N}_\varepsilon(\alpha)$.

The case when γ and α intersect follows from the same arguments (and is simpler). □

Lemma 2.44 ([45], Lemma 2.9)**.** *Let C_0 be a horoball in Z and $o \in Z - C_0$. Then, for two geodesic rays starting in o and entering in C_0 at x and $\bar{x}$ respectively, we have*

$$d(x, \bar{x}) \leq 2 \log(1 + \sqrt{2}) \equiv c_0.$$

If $\tau \geq 0$ and $C_0 = \beta^{-1}((-\infty, 0])$ is a (closed) horoball with respect to the Busemann function β, let $C_0[\tau] \equiv \beta^{-1}((-\infty, -\tau]) = \{x \in C_0 : d(x, \partial C_0) \geq \tau\} \subset C_0$ denote the horoball shrinked by the factor τ. Let $o \in X - C_0$ and assume that for $\xi \in \partial_\infty X$ the ray $\gamma_{o,\xi}$ enters in C_0. Define the *shrinking parameter* of ξ by $s(\xi) = \sup\{\tau \in [0, \infty] : \gamma_{o,\xi} \cap C_0[\tau] \neq \emptyset\}$. Then the ray $\gamma_{o,\xi}$ penetrates the horoball C_0 for a long time if and only if it enters deeply into C_0, that is, its shrinking parameter is large.

Lemma 2.45. *Let $o \in Z - C_0$. Assume that for $\xi \in \partial_\infty Z$ the ray $\gamma_{o,\xi}$ enters in C_0 at time $t \geq 0$ and leaves at time $t + p$, $0 < p < \infty$. Let $s \geq 0$ be the shrinking parameter of ξ. Then*

$$2s - c_0 \leq p \leq 2s + 2c_0.$$

Proof. Let C_0 be based at the point $\eta \in \partial_\infty Z$, $\eta \neq \xi$, and let $d_o = d(o, C_0) \geq 0$ such that $\gamma_{o,\eta}(d_o) \in \partial C_0$. Note that the function $s \mapsto \beta \circ \gamma_{o,\xi}(s)$ is continuous and convex. Hence, there exists a point $\xi_s \equiv \gamma_{o,\xi}(t + p_1)$ on $\partial C_0[s] = \beta^{-1}(-s)$. By Lemma 2.44, we have $d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d_o)) \leq c_0$ as well as $d(\xi_s, \gamma_{o,\eta}(d_o + s)) \leq c_0$. Note that for all $\tau \geq 0$, $\gamma_{o,\eta}(d_o + \tau)$ is the closest point of o to $\partial C_0[\tau]$. Hence, $d_o \leq t \leq d_o + c_0$ as well as

$$d_o + s \leq t + p_1 \leq d_o + c_0 + s.$$

Starting with the point $\tilde{o} = \gamma_{o,\xi}(t + p) \in \partial C_0$ with $d_{\tilde{o}} = d(\tilde{o}, C_0) = 0$, we obtain in the same way by Lemma 2.44 that $s \leq p_2 \equiv p - p_1 \leq s + c_0$. Thus,

$$\begin{aligned} 2s - c_0 \leq 2s + d_o - t &\leq p_1 + p_2 = p \\ &\leq d_o - t + 2s + 2c_0 \leq 2s + 2c_0, \end{aligned}$$

which finishes the proof. □

Finally, we are able to prove Lemma 2.32.

Proof of Lemma 2.32. For the first case, given a horoball C based at η and a point $\xi \in \partial_\infty Z$, assume that $\gamma_{o,\xi}([t, t+l]) \subset C$. We may assume that t is the entering and $t+l$ the exiting time. Then from Lemma 2.45, $l \leq 2s + 2c_0$, where s denotes the shrinking parameter of ξ in C. Moreover, if x is the closest point of o on $[\xi, \eta]$, we claim that $d(o,x) \geq d(o, C[s]) - 2\delta_0$. Assuming the claim, we have

$$d(o,x) \geq d(o, C[s]) - 2\delta_0 = d(o,C) + s \geq d(o,C) + l/2 - c_0 - 2\delta_0,$$

and it follows from (2.3.6) that

$$d_o(\xi,\eta) \quad = \quad e^{-(\xi,\eta)_o} \leq e^{-d(o,x)+\kappa} \leq e^{-l/2+(\kappa+c_0+2\delta_0)} e^{-d(o,C)}.$$

For the claim, assume that $d(o,x) < d(o, C[s]) - 2\delta_0$. Consider the geodesic triangle given by (o, x, ξ). For any point y on the ray $[o, \xi]$ at distance $d(o,y) > d(o,x) + \delta_0$ and z on $[o,x]$, we have $d(y,z) \geq d(o,y) - d(z,o) > \delta_0$. Since Z is a Gromov δ_0-hyperbolic space, we see that y cannot belong the δ_0-neighborhood of the segment $[o,x]$ and must therefore be contained in $\mathcal{N}_{\delta_0}([x,\xi])$. Thus, let x_s be a point on $\partial C[s] \cap [o,\xi]$ and note that $d(o,x_s) \geq d(o,C[s]) \geq d(o,x) + 2\delta_0$. From the above, we find a point y on $[x,\xi]$ which is δ_0-close to x_s and hence, $y \in C[s-\delta_0]$. However, by convexity of the horoball $C[s-\delta_0]$ and since $\eta = \partial_\infty C[s-\delta_0]$, we have

$$[x,\eta] \subset [y,\eta] \subset C[s-\delta_0].$$

This shows $d(o,x) = d(o,[x,\eta]) \geq d(o, C[s-\delta_0]) = d(o,C[s]) - \delta_0$; a contradiction implying the claim.

Conversely, let $d_o(\xi,\eta) \leq c(l)e^{-d(o,C)}$ with $c(l) \leq \bar{c}e^{-l/2}$ where $\bar{c} > 0$ is sufficiently large. Set $t := D(o,C) + \delta_0$ and

$$t + \bar{l} \equiv -\log(d_o(\xi, C)) = (\xi,\eta)_o \geq \log(\bar{c}) + d(o,C) + l/2.$$

For $l > l_0 = l_0(\bar{c})$ sufficiently large, we have $t + \bar{l} > t$. Since $t + \bar{l} = (\xi,\eta)_o$, we have from (2.3.5) that $d(\gamma_{o,\xi}(t+\bar{l}), \gamma_{o,\eta}(t+\bar{l})) \leq \delta_0$, and by convexity also, $d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(t)) \leq \delta_0$. Thus, we obtain that

$$\gamma_{o,\xi}([t, t+\bar{l}]) \subset \mathcal{N}_{\delta_0}(\gamma_{o,\eta}([d(o,C)+\delta_0, \infty))) \subset C.$$

The second case follows with similar arguments using Lemma 2.43. The proof can be found in [46], Lemma 4.1.

For the third case, from Lemma 3.1 in [24], there exist positive (universal) constants c_1, c_2, c_3 such that for all $x_m \in \mathbb{H}^{n+1}$, with $d(o,c) \geq c_2$ (which we may assume if t_0 is sufficiently large), for all $0 < R \leq c_3$ and $R \leq d(o,x_m)$ we have

$$B_{d_o}(\xi_m^\infty, Re^{-d(o,x_m)}) \subset \mathcal{S}_o(B(x_m,R)) \subset B_{d_o}(\xi_m^\infty, c_1 Re^{-d(o,x_m)}),$$

where $\mathcal{S}_o(B(x_m,R))$ denotes the shadow at infinity of the metric ball $B(x_m,R)$ which is disjoint to $\{o\}$. ∎

Chapter 3

Jarník-type Inequalities

Large parts of this chapter are published in [61].

Abstract of Chapter 3. It is well known due to Jarník [29] that the set $\mathbf{Bad}^1_{\mathbb{R}}$ of badly approximable numbers is of Hausdorff-dimension one. If $\mathbf{Bad}^1_{\mathbb{R}}(c)$ denotes the subset of $x \in \mathbf{Bad}^1_{\mathbb{R}}$ for which the approximation constant $c(x) \geq c$, then Jarník was in fact more precise and gave nontrivial lower and upper bounds of the Hausdorff-dimension of $\mathbf{Bad}^1_{\mathbb{R}}(c)$ in terms of the parameter $c > 0$. Our aim is to determine simple conditions on a framework which allow to extend 'Jarník's inequality' to further examples; among the applications, we discuss the set $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$ of badly approximable vectors in $\mathbb{R}^n$ with weights $\bar{r}$ and the set of geodesics in the hyperbolic space $\mathbb{H}^n$ which avoid a suitable collection of convex sets.

Outline of Chapter 3. In the introduction, we begin with a motivation (Subsection 3.1.1) and the statement of the main results in their simplest settings (Subsection 3.1.2).

In Section 3.2, we introduce the framework and conditions which lead to an abstract formalism for the lower and upper bound on the Hausdorff dimension of a set of badly approximable points with respect to a given lower bound on the approximation constant (see Subsections 3.2.2 and 3.2.3 respectively). We can distinguish between 'separation conditions' and 'measure conditions', which both concern the parameter space as well as the structure and distribution of the resonant sets. General criteria are deduced when the required conditions are satisfied (Subsection 3.2.4).

In Section 3.3, we apply the abstract formalism to the set of badly approximable vectors with weights (Subsection 3.3.1), to the set of words in the Bernoulli shift which avoid a periodic word (Subsection 3.3.2), to the set of geodesics in a geometrically finite hyperbolic manifold which are bounded with respect to a suitable collection convex sets (Subsection 3.28) and to the set of orbits of toral endomorphisms which avoid separated sets of $\mathbb{R}^n$ (Subsection 3.3.4).

3.1 Introduction and Main Results

3.1.1 Introduction

An irrational number $x \in \mathbb{R}$ is called *badly approximable* if there exists a positive constant $c = c(x) > 0$, called *approximation constant*,[1] such that

$$|x - \frac{p}{q}| \geq \frac{c}{q^2} \tag{3.1.1}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The set $\mathbf{Bad}^1_{\mathbb{R}}$ of badly approximable numbers is a Lebesgue null-set, yet it is well known due to Jarník [29] that $\mathbf{Bad}^1_{\mathbb{R}}$ is of Hausdorff-dimension one. Note that a positive irrational number $x \in \mathbb{R}$ is badly approximable if and only if the entries $a_n \in \mathbb{N}$ of the continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ of x are bounded by some integer $N \in \mathbb{N}$. Moreover, a small bound N on the entries corresponds to a larger approximation constant $c(x)$ in (3.1.1). In fact, if p_n/q_n are the approximates given by the continued fraction expansion of x, then

$$\frac{1}{(a_{n+1}+2)q_n^2} < |x - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2}.$$

Moreover, if $|x - p/q| < 1/(2q^2)$, then $p/q = p_n/q_n$ for a suitable n. Using this correspondence, Jarník was more precise and gave nontrivial lower and upper estimates on the Hausdorff-dimension of the set of badly approximable numbers with an approximation constant bounded below.

Theorem 3.1 ([29], Satz 4)**.** *: If M_N denotes the set of irrational numbers for which the entries of the continued fraction expansion are bounded by N, where $N > 8$, then*

$$1 - \frac{4}{N\log(2)} \leq dim(M_N) \leq 1 - \frac{1}{8N\log(N)}. \tag{3.1.2}$$

Here and in the following, 'dim' stands for the Hausdorff-dimension.

In particular, inequality (3.1.2), which we call *Jarník's inequality*, implies Jarník's theorem on full Hausdorff-dimension of $\mathbf{Bad}^1_{\mathbb{R}}$.

There is a further correspondence between Diophantine approximation and hyperbolic geometry. Let $\mathbb{H}^2/SL(2,\mathbb{Z})$ be the modular surface, which is a hyperbolic orbifold with a cusp; for details, we refer to Section 3.3. Let H_0 be the maximal standard cusp neighborhood and denote by $H_t \subset H_0$ the standard cusp neighborhood at *height* t with $d(H_t, H_0) = t$. The set of complete 'cuspidal' geodesics γ with $\gamma(0) \in \partial H_0$, $\gamma(-t) \in H_t$ (hence starting from the cusp) can be identified with the set $[0,1)$ via the endpoint $\tilde{\gamma}(\infty) \in [0,1)$ of a suitable lift $\tilde{\gamma}$ of γ, starting from ∞. We say that γ is *bounded* with height $t = t(\gamma)$ if $\gamma|_{\mathbb{R}^+}$ does not enter H_t. Again, γ is bounded if and only if $x = \tilde{\gamma}(\infty) \in [0,1) \setminus \mathbb{Q}$ is a badly approximable number and a small height $t(\gamma)$ corresponds to a large approximation constant $c(x)$. Hence, Jarník's inequality (3.1.2) also shows that the Hausdorff-dimension of the cuspidal geodesics in the modular surface with a sufficiently large given upper bound on the height can be bounded below and above nontrivially.

While Kristensen, Thorn and Velani [35] extended Jarník's Theorem on full Hausdorff-dimension to a more general setting, our intention is to determine simple conditions on a framework which

[1] In fact, the approximation constant should be given in terms of the supremum of all the constants c satisfying (3.1.1). However, this plays no role in the following.

enable to extend Jarník's inequality to further examples. We remark that implicitly in the proof of [35], a lower bound on the Hausdorff-dimension of a given set of badly approximable points with a lower bound on the approximation constant can be determined. However, the bound is neither stated explicitly, nor is it effective. In particular, they only use the trivial upper bound, which is the dimension of the space.

3.1.2 Main results

Among the applications in Section 3.3, we now present two of the main results in their simplest settings. For $n \geq 1$, let $\mathbf{Bad}^n_{\mathbb{R}^n}$ be the set of points $\bar{x} \in \mathbb{R}^n$ for which there exists a positive constant $c(\bar{x}) > 0$ such that the distances (say in the maximum-norm) from $\bar{x}$ to all rational vectors satisfy

$$\|\bar{x} - \frac{\bar{p}}{q}\| \geq \frac{c(\bar{x})}{q^{1+1/n}},$$

for every $q \in \mathbb{N}$ and $\bar{p} \in \mathbb{Z}^n$. The set $\mathbf{Bad}^1_{\mathbb{R}^1}$ is the classical set of badly approximable numbers and $\mathbf{Bad}^n_{\mathbb{R}^n}$ is called the set of badly approximable vectors. For $c > 0$, let moreover $\mathbf{Bad}^n_{\mathbb{R}^n}(c)$ be the subset of $\bar{x} \in \mathbf{Bad}^n_{\mathbb{R}^n}$ with approximation constant $c(\bar{x}) \geq c$.

Theorem 3.2. *There exist positive constants k_l, k_u, $\bar{k}_l$, $\bar{k}_u > 0$ and $t_0 > 0$ such that for all $t > t_0$ we have*

$$n - k_l \frac{|\log(1 - \bar{k}_l \; e^{-t/2})|}{t} \leq \dim(\mathbf{Bad}^n_{\mathbb{R}^n}(e^{-t})) \leq n - k_u \frac{|\log(1 - \bar{k}_u \; e^{-(n+1)t})|}{t}.$$

In particular, $\dim(\mathbf{Bad}^n_{\mathbb{R}^n}) = n$. Using the Taylor expansion, we recover for $n = 1$ and large $c = \log(N)$ an inequality which is similar to Jarník's inequality (3.1.2).

We will in fact prove a similar result for the set $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$ and for intersections with suitable 'diffuse' sets. It is worth pointing out that a positive lower bound on the Hausdorff-dimension, is a lower bound for the Hurwitz-constant of the spectrum of approximation constants. Very little seems to be known about the Hurwitz-constant (of $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}$).

Now let $M = \mathbb{H}^{n+1}/\Gamma$ be a $(n+1)$-dimensional finite volume hyperbolic manifold with exactly one cusp. As above, let H_0 be a standard cusp neighborhood and let $H_t \subset H_0$ be the standard cusp neighborhood at height t. Fix a base point $o \in M - H_0$ in the compact part of M and let SM_o be the n-dimensional unit tangent space of M at o. Identity a vector $v \in SM_o$ with the unique geodesic ray γ_v starting at o such that $\dot{\gamma}_v(0) = v$. For a constant $t_0 > 0$, we define for $t > t_0$ the set of rays γ_v which avoid H_t, i.e. stay in the compact part H_t^C, by

$$\mathbf{Bad}_{M,H_0,o}(t) \equiv \{v \in SM_o : \gamma_v(s) \notin H_t \text{ for all } s \geq 0\}.$$

Theorem 3.3. *There exist positive geometric[2] constants k_l, k_u, $\bar{k}_l$, $\bar{k}_u > 0$, depending on Γ, and a height t_0 such that for all $t > t_0$ we have*

$$n - k_l \frac{|\log(1 - \bar{k}_l \; e^{-nt/2})|}{t} \leq \dim(\mathbf{Bad}_{M,H_0,o}(t)) \leq n - k_u \frac{|\log(1 - \bar{k}_u \; e^{-2nt})|}{t}.$$

[2] By 'geometric' we mean that the constants depend on geometric quantities such as the diameter of $M - H_0$ as well as further universal constants depending on the group Γ and on the hyperbolic space.

In particular, the set of 'bounded' rays is of full Hausdorff-dimension.

We will prove a similar result even in a geometrically finite setting, which yields information about the distribution of the horoballs (lifts of H_0) in $\mathbb{H}^{n+1}$ as well as of the orbit of the parabolic fixed points (base points of the lifts of H_0) in the limit set of Γ.
Note that the Hurwitz-constant is given in terms of the infimum of the heights of closed geodesics in M (see [25]).

3.1.3 Further remarks

The property of full Hausdorff-dimension of a set of badly approximable points (with respect to a suitable setting of Diophantine approximation) has been established for various examples specifically and, as mentioned above, by [35] in an abstract fashion. With respect to the examples we consider in Section 3.3, we point out Patterson [47], for the case of Diophantine approximation in Fuchsian groups, and, Pollington, Velani [48], for the set $\mathbf{Bad}_{\mathbb{R}^2}^{(r_1,r_2)}$. Again, a lower bound on the Hausdorff-dimension of the set of badly approximable points with a lower bound on the approximation constants can be determined from the proofs for these specific examples.

Moreover, Schmidt [50] showed that $\mathbf{Bad}_{\mathbb{R}}^1$ is actually a Schmidt-winning set. Schmidt's game also applies to further examples from number theory and dynamical systems (see for instance [13]) when there is a suitable set of badly approximable points. Since winning sets of Schmidt's game (and modifications of it) enjoy a remarkable rigidity, only recently several modifications, adopted to the specific setting of the considered examples, have been introduced.[3] In particular, the property of full Hausdorff-dimension is, at least in a reasonably nice setting, a 'byproduct' of a winning set. We want to refer, for instance, to the work of Kleinbock, Weiss [34] in which a technique is given that determines a lower bound for the Hausdorff-dimension of the set $S_{\alpha,\beta}$ in terms of α, β, where α, β are parameters of the Schmidt game. However, we remark that, although we will use this technique for our purpose, the set $S_{\alpha,\beta}$ in general contains badly approximable elements with arbitrarily small approximation constants.

3.2 The Geometry of Parameter Spaces and the Abstract Formalism

The idea of the formalism and the required conditions are simple, yet hidden below technicalities. We therefor want to explain it for the basic example $\mathbf{Bad}_{\mathbb{R}}^1$, the set of badly approximable numbers (see Subsection 3.3.1). For $r > 0$, let $R(r) \equiv \{p/q \in \mathbb{Q} : \frac{1}{q^2} \geq r\}$. Fix a sufficiently large parameter $c > 0$. For the lower bound, we start with any closed metric ball $B_1 = B(x, 1)$. Now, given a closed metric ball $B = B_{1i_2\ldots i_k}$ of radius $r = e^{-2kc}$ at the k.th step, we consider the 'relevant set' $A_k^l = \bigcup_{p/q \in R(r \cdot l_*)} B(p/q, e^{-2c}r)$. The constant $l_* = 3$ guarantees that at most one of the balls $B(p/q, e^{-2c}r)$ with $p/q \in R(r \cdot l_*)$ can intersect B. Hence, with respect to the Lebesgue measure μ, the following condition is satisfied

$$\mu(B \cap \bigcup_{p/q \in R(r \cdot l_*)} B(p/q, e^{-2c}r)) \leq e^{-2c}\mu(B) \equiv \tau_c \cdot \mu(B). \tag{3.2.1}$$

[3] Further details can be found in the probably incomplete list [1, 2, 3, 12, 13, 14, 17, 19, 20, 32, 34, 28, 40, 42, 11, 10, 9, 50, 55, 60].

Up to further separation constants, we can find disjoint balls $B_{1i_2\dots i_k i_{k+1}}$ of radius $e^{-2c}r$ contained in B and in the complement of A^l_k. The number of these balls can be estimated from below in terms of τ_c. Thus, step by step, we construct a treelike collection of 'sub-covers' of the set $\mathbf{Bad}^1_{\mathbb{R}}(e^{-2\tilde{c}})$ with $\tilde{c}$ related to c. This will yield a lower bound on the Hausdorff-dimension of $\mathbf{Bad}^1_{\mathbb{R}}(e^{-2\tilde{c}})$ in terms of τ_c.

For the upper bound, given again a closed metric ball $B = B_{1i_2\dots i_k}$ of radius $r_k = u_*^k e^{-4kc}$ at the k.th step, we consider the 'relevant set' $A^u_k = \bigcup_{p/q\in R(r_k\cdot u^c)} B(p/q, \frac{e^{-2c}}{q^2})$. The parameter $u^c = u_* e^{-2c}$ guarantees that either B is contained in a set $B(p/q, \frac{e^{-2c}}{q^2})$ with $p/q \in R(r_{k-1}\cdot u^c)$ or that there exists a point $p/q \in R(r_k \cdot u^c)$ with $B(p/q, e^{-4c}r_k) \subset B$. Hence, the following condition is satisfied

$$\mu(B \cap \bigcup_{p/q\in R(r_k\cdot u^c)} B(p/q, \tfrac{e^{-2c}}{q^2})) \geq e^{-4c}\mu(B) \equiv \tau^c \cdot \mu(B). \tag{3.2.2}$$

Again, up to further separation constants, we can find closed balls $B_{1i_2\dots i_k i_{k+1}}$ of radius $u_* e^{-4c} r_k$ covering the complement of A^u_k in B, for which the number can be estimated from above in terms of τ^c. Thus, step by step, we construct a treelike collection of covers of the set $\mathbf{Bad}^1_{\mathbb{R}}(e^{-2c}) \cap B_1$. This will yield an upper bound on the Hausdorff-dimension of $\mathbf{Bad}^1_{\mathbb{R}}(e^{-2c})$ in terms of τ^c.

For our abstract formalism, we will in fact assume the conditions (3.2.1) and (3.2.2) as well as separation conditions and construct treelike collections of sub-covers and covers respectively as above.

Remark 3.4. Our setting and formalism is similar to the *local ubiquity* setup of Beresnevich, Dickinson and Velani [4]. In particular, our main conditions (3.2.1) and (3.2.2) (as well as (3.2.13) and (3.2.23) respectively) are similar to their *intersection conditions*. However, their formalism served the purpose of determining the Hausdorff-dimension of the complementary set, that is the set of well-approximable points and of 'limsup sets' in general.

3.2.1 The general framework.

We first introduce the setting of this section that bases on the notion of [34] and was adopted in Chapter 2. However, some of the following terminology differs from these works.

Let $(\bar{X}, d)$ be a proper metric space. Fix $t_* \in \mathbb{R} \cup \{-\infty\}$ and define the parameter space $\bar{\Omega} \equiv \bar{X} \times (t_*, \infty)$, the set of *formal balls* in $\bar{X}$. Let $\mathcal{C}(\bar{X})$ be the set of nonempty compact subsets of $\bar{X}$. Assume that there exists a function

$$\bar{\psi} : \bar{\Omega} \to \mathcal{C}(\bar{X})$$

which is *monotonic*, that is, for all $(x,t) \in \bar{\Omega}$ and $s \geq 0$ we have

$$\bar{\psi}(x, t+s) \subset \bar{\psi}(x,t). \tag{3.2.3}$$

For a subset $Y \subset \bar{X}$ and $t > t_*$, we call $(Y,t) \equiv \{(y,t) : y \in Y\}$ *formal neighborhood*, and define $\mathcal{P} = \mathcal{P}(\bar{X}) \times (t_*, \infty)$ to be the set of formal neighborhoods. Define the $\bar{\psi}$*-neighborhood* of $(Y,t) \in \mathcal{P}$ by

$$\bar{\psi}(Y,t) \equiv \bigcup_{y\in Y} \bar{\psi}(y,t).$$

Note that by monotonicity (3.2.3), $\bar{\psi}(Y, t+s) \subset \bar{\psi}(Y,t)$ for all $s \geq 0$.

For instance, since $\bar{X}$ is proper, set $t_* = -\infty$ and for $x \in \bar{X}$, $r > 0$, let $B(x,r) \equiv \{y \in \bar{X} : d(x,y) \leq r\} \in \mathcal{C}(\bar{X})$. For $\sigma > 0$, the *standard function* $\bar{\psi}_\sigma \equiv B_\sigma$ is given by the monotonic function

$$\bar{\psi}_\sigma(x,t) \equiv B(x, e^{-\sigma t}). \tag{3.2.4}$$

In many applications, we are interested in badly approximable points of a closed subset X of $\bar{X}$ which is, with the induced metric, a complete metric space. However, we do not require the resonant sets to be contained in X but in $\bar{X}$. Therefore, let also $\Omega = X \times (t_*, \infty) \subset \bar{\Omega}$. The monotonic function $\bar{\psi}$ induces the monotonic function $\psi : \Omega \to \mathcal{C}(X)$, defined by

$$\psi(\omega) \equiv \bar{\psi}(\omega) \cap X, \quad \omega \in \Omega.$$

The family of resonant sets

Now, let Λ be a countable index set and $\{R_\lambda \subset \bar{X} : \lambda \in \Lambda\}$ be a family of *resonant sets* in $\bar{X}$, where we assign a *size* $s_\lambda \geq s_*$ to every R_λ with $t_* < s_* \in \mathbb{R}$. We consider the contractions of the $(\bar{\psi}, s_\lambda)$-neighborhoods of R_λ,

$$f_\lambda(s) \equiv \bar{\psi}(R_\lambda, s_\lambda + s) \subset \bar{\psi}(R_\lambda, s_\lambda), \quad s \geq 0.$$

Denote this family by

$$\mathcal{F} = (\Lambda, R_\lambda, s_\lambda).$$

Assume that the family $\mathcal{F}$ satisfies the following conditions.

(N) The resonant sets $\{R_\lambda\}$ are *nested* with respect to their sizes, that is, for $\lambda, \beta \in \Lambda$ we have

$$s_\lambda \leq s_\beta \implies R_\lambda \subset R_\beta.$$

(D) The sizes $\{s_\lambda\}$ are *discrete*, that is, for all $t > t_*$ we have

$$|\{\lambda \in \Lambda : s_\lambda \leq t\}| < \infty.$$

We then define the set of *badly approximable points* with respect to $\mathcal{F}$ by

$$\mathbf{Bad}_X^{\bar{\psi}}(\mathcal{F}) = \{x \in X : \exists\, c = c(x) < \infty \text{ such that } x \notin \bigcup_{\lambda \in \Lambda} f_\lambda(c)\},$$

or simply by $\mathbf{Bad}(\mathcal{F})$ if there is no confusion about the parameter spaces under consideration. The constant $c(x) \equiv \sup\{c \in \mathbb{R} : x \in \bigcup_{\lambda \in \Lambda} f_\lambda(c)\}$ is called the *approximation constant* of $x \in \mathbf{Bad}(\mathcal{F})$. In the following, we are interested in the subset

$$\mathbf{Bad}(\mathcal{F}, c) \equiv \{x \in \mathbf{Bad}(\mathcal{F}) : c(x) \leq c\}.$$

Note moreover that the resonant sets can be ordered with respect their sizes by (N). We will therefore assume in the following that $\Lambda = \mathbb{N}$, $s_n \leq s_m$ for $n \leq m$. For a parameter $t \geq s_1$, we define the *relevant resonant set* with respect to the parameter t by

$$R(t) \equiv \bigcup_{s_n \leq t} R_n = R_{n_t},$$

where $n_t \in \mathbb{N}$ is the largest integer such $s_n \leq t$ (see (N) and (D)), and we call s_{n_t} the *relevant size*.

Rigiditiy assumptions

The following requirements will be standing assumptions in Section 2. For $c > 0$ assume there exist constants d_c, $d^c \geq 0$ such that, for $(y,t) \in \bar{\Omega}$,

$$\begin{aligned} x \in \bar{\psi}(y, t + d_c) &\implies \bar{\psi}(x, t + c) \subset \bar{\psi}(y, t) \\ x \in \bar{\psi}(y, t) &\implies \bar{\psi}(x, t + c) \subset \bar{\psi}(y, t - d^c). \end{aligned} \tag{3.2.5}$$

Moreover, require that (Ω, ψ) is *d_*-separating with respect to $\mathcal{F}$*, that is, there exists a constant $d_* \geq 0$ such that for all resonant sets $Y = R_n \subset \bar{X}$, or points $Y = y \in X$, $t > t_*$ and for all $x \in X$, $\bar{\psi}$ satisfies

$$x \notin \bar{\psi}(Y, t) \implies \psi(x, t + d_*) \cap \bar{\psi}(Y, t + d_*) = \emptyset. \tag{3.2.6}$$

Assume in addition that for every Borel set $Y \subset \bar{X}$ as above also the $\bar{\psi}$-neighborhood $\bar{\psi}(Y,t)$ is a Borel set. Let μ be a locally finite Borel measure on $\bar{X}$ which is positive on ψ-balls, that is, for all $\omega \in \Omega$ we have

$$\mu(\psi(\omega)) > 0. \tag{3.2.7}$$

Finally, we require that for all $\omega = (x,t) \in \Omega$, the diameter of $\psi(\omega)$ is bounded by

$$\operatorname{diam}(\psi(x,t)) \leq c_\sigma e^{-\sigma t}, \tag{3.2.8}$$

where $c_\sigma, \sigma > 0$.

Further considerations.

Denote by $O(x,r) \equiv \{y \in \bar{X} : d(x,y) < r\}$ the open metric ball around $x \in \bar{X}$. Let μ be a locally finite Borel measure on $\bar{X}$. The *lower pointwise dimension* of μ at $x \in \operatorname{supp}(\mu)$ is defined by

$$d_\mu(x) \equiv \liminf_{r \to 0} \frac{\log(\mu(O(x,r))}{\log r}.$$

If μ satisfies a *power law*, that is, there exist constants δ, c_1, c_2 and $R > 0$ such that for every $0 < r < R$ and $x \in \operatorname{supp}(\mu)$ we have

$$c_1 r^\delta \leq \mu(O(x,r)) \leq c_2 r^\delta,$$

then we have $d_\mu(x) = \delta$.

We say that (Ω, ψ, μ) satisfies a *power law with respect to the parameters* (τ, c_1, c_2), where $\tau > 0$, $c_2 \geq c_1 > 0$, if $\operatorname{supp}(\mu) = X$ and

$$c_1 e^{-\tau t} \leq \mu(\psi(x,t)) \leq c_2 e^{-\tau t} \tag{3.2.9}$$

for all formal balls $(x,t) \in \Omega$.

Note that, depending on the considered function ψ, the exponent τ from (3.2.9) may differ from δ. For $x \in X$, define

$$\Delta_{\mu,\psi}(x) \equiv \limsup_{t \to \infty} \frac{1}{t} \log\big(\frac{\mu(O(x, 2c_\sigma e^{-\sigma t}))}{\mu(\psi(x,t))}\big),$$

if the limit exists. We remark that by (3.2.8) we have $\psi(x,t) \subset O(x, 2c_\sigma e^{-\sigma t})$ for all $t > t_*$; hence $\Delta_{\mu,\psi}(x) \geq 0$. The following lemma is readily checked.

Lemma 3.5. *Let* (Ω, ψ, μ) *satisfy a power law. If the limit exists, we have*

$$\tau = \sigma d_\mu(x) + \Delta_{\mu,\psi}(x).$$

Finally, let (Ω, ψ, μ) satisfy a power law with respect to the parameters (τ, c_1, c_2). For later purpose, we state that the following inequalities are satisfied. For $c > 0$, a constant $u^c \geq 0$ and $\omega = (x, t - d^c) \in \Omega$, we have for all $y \in \psi(\omega)$

$$\begin{aligned} \mu(\psi(y, t+c)) &\geq \tfrac{c_1}{c_2} e^{-\tau c} \mu(\psi(x,t)) \equiv k_c \ \mu(x,t), \\ \mu(\psi(x, t+d_c)) &\geq \tfrac{c_1}{c_2} e^{\tau(c-d_*-d_c)} \mu(\psi(y, t+c-d_*)) \equiv \bar{k}_c^{-1} \ \mu(\psi(y, t+c-d_*)), \\ \mu(\psi(y, t+(c+u^c)+d_*)) &\geq \tfrac{c_1}{c_2} e^{-\tau(c+u^c+d_*+d^c)} \mu(\psi(x, t-d^c)) \equiv k^c \ \mu(\psi(x, t-d^c)). \end{aligned} \tag{3.2.10}$$

Moreover, if $\bar{\psi}$ is given by the standard function $B_\sigma(x,t) \equiv B(x, e^{-\sigma t})$, then for $c > 0$ we have

$$d_* \leq \log(2)/\sigma, \quad d_c \leq -\log(1 - e^{-\sigma c})/\sigma, \quad d^c \leq \log(1 - e^{-\sigma c})/\sigma.$$

3.2.2 The lower bound.

We fix a constant $c > 0$ and let $l_c \geq 0$. For $k \geq 1$, define $t_k \equiv s_1 + kc + l_c$ and

$$L_k(c) = L_k^{\bar{\psi}}(c) \equiv \bigcap_{i=1}^{k} \bar{\psi}(R(t_i - l_c), t_i + c)^C. \tag{3.2.11}$$

Assume that there exist positive constants $\bar{k}_c, k_c > 0$ such that (Ω, ψ, μ) satisfies, for all formal balls $\omega = (x, t_k) \in \Omega$ with $x \in L_{k-1}(c - d_*)$ and $y \in \psi(\omega) \cap L_k(c - d_*)$,

$$k_c \ \mu(\psi(x, t_k)) \leq \mu(\psi(y, t_{k+1})) \leq \mu(\psi(y, t_{k+1} - d_*)) \leq \bar{k}_c \ \mu(\psi(x, t_k + d_c)). \tag{3.2.12}$$

The concept of (absolutely) decaying measures was introduced in [31] and we adopted it to our setting in Chapter 2. (Ω, ψ, μ) is called τ_c*-decaying with respect to* $\mathcal{F}$ *and the parameters* (c, l_c) if all formal balls $\omega = (x, t_k + d_c) \in \Omega$ with $x \in L_{k-1}(c - d_*)$ we have

$$\mu(\psi(\omega) \cap \bar{\psi}(R(t_k - l_c), t_k + c - d_*)) \leq \tau_c \cdot \mu(\psi(\omega)), \tag{3.2.13}$$

where $\tau_c < 1$.[4]

Remark 3.6. For $c \geq d_*$, the condition that $x \in L_{k-1}(c - d_*)$ implies that $\psi(x, t_k)$ is disjoint to $\bar{\psi}(R(t_{k-1} - l_c), t_{k-1} + c) \supset \bar{\psi}(R(t_{k-1} - l_c), t_k + c - d_*)$ by (3.2.6). Hence it would actually suffice to consider the set $R(t_k - l_c, c) \equiv R(t_k - l_c) - R(t_{k-1} - l_c)$ in (3.2.13). Note that also the proof of Lemma 3.9 will work if we only consider the sets $R(t_k - l_c, c)$.

In order to determine the lower bound of $\dim(\mathbf{Bad}(\mathcal{F}, 2c + l_c))$, we first construct a strongly treelike family of sets such that its limit set, A_∞, is a subset of $\mathbf{Bad}(\mathcal{F}, 2c + l_c)$. Using the method of [34, 33] (which is a generalization of the ones of [39, 56]), based on the 'Mass Distribution Principle', we derive a lower bound of $\dim(A_\infty)$.

Let $\omega_1 = (x_1, s_1 + l_c) \in \Omega$ be a formal ball and note that $L_0(c)$ is $\bar{X}$.

[4] In fact, we should call this condition 'absolutely τ_c-decaying' rather than τ_c-decaying according to [31]. For the sake of simplicity we omit the term 'absolutely'.

Lemma 3.7. *Given a formal ball* $\omega_{i_1\ldots i_k} = (x_{i_1\ldots i_n}, t_k) \in \Omega$ *with* $x_{i_1\ldots i_k} \in L_{k-1}(c-d_*)$ *there exist formal balls* $\omega_{i_1\ldots i_n i_{k+1}} = (x_{i_1\ldots i_k i_{k+1}}, t_{k+1}) \in \Omega$ *satisfying*

$$\psi(\omega_{i_1\ldots i_k i_{k+1}}) \subset \psi(\omega_{i_1\ldots i_k}) - \bar{\psi}(R(t_k - l_c), t_k + c) \tag{3.2.14}$$

where $x_{i_1\ldots i_k i_{k+1}} \in L_k(c-d_*)$*, such that* $\psi(\omega_{i_1\ldots i_k i_{k+1}})$ *are disjoint, and moreover,*

$$\mu\Big(\bigcup_{i_{k+1}} \psi(\omega_{i_1\ldots i_k i_{k+1}})\Big) \geq \frac{(1-\tau_c)k_c}{2\bar{k}_c}\mu(\psi(\omega_{i_1\ldots i_k})) \tag{3.2.15}$$

Proof. Given the formal ball $\omega_{i_1\ldots i_k} = (x_{i_1\ldots i_k}, t_k) \in \Omega$ where $x_{i_1\ldots i_k} \in L_{k-1}(c-d_*)$, assume that we have $m \geq 0$ formal balls $\omega_{i_1\ldots i_k i_{k+1}} = (x_{i_1\ldots i_k i_{k+1}}, t_{k+1}) \in \Omega$, for which (3.3.6) is satisfied and such that $\psi(\omega_{i_1\ldots i_k i_{k+1}})$ are disjoint for $i_{k+1} = 1, \ldots, m$.

We apply (3.2.13) on the formal ball $\omega_0 \equiv (x_{i_1\ldots i_k}, t_k + d_c) \in \Omega$ and use (3.2.12) so that we obtain

$$\begin{aligned} & \mu\Big(\psi(\omega_0) - \psi(R(t_k - l_c), t_k + c - d_*)) - \bigcup_{i_{k+1}=1}^{m} \psi(x_{i_1\ldots i_k i_{k+1}}, t_{k+1} - d_*)\Big)\Big) \\ = \; & \mu(\psi(\omega_0)) - \mu\Big(\psi(\omega_0) \cap \Big(\psi(R(t_k - l_c), t_k + c - d_*) \cup \bigcup_{i_{k+1}=1}^{m} \psi(x_{i_1\ldots i_n i_{k+1}}, t_{k+1} - d_*)\Big)\Big) \\ \geq \; & (1 - \tau_c - m \cdot \bar{k}_c)\mu(\psi(\omega_0)). \end{aligned}$$

As long as $m < (1-\tau_c)\bar{k}_c^{-1}$, by (3.2.7) there exists a point

$$x' \in \psi(\omega_0) - \psi(R(t_k - l_c), t_k + c - d_*) - \bigcup_{i_{k+1}=1}^{m} \psi(x_{i_1\ldots i_k i_{k+1}}, t_{k+1} - d_*).$$

Define $\omega_{i_1\ldots i_k(m+1)} \equiv (x', t_k + c) \in \Omega$. By (3.2.6) we know that $\psi(\omega_{i_1\ldots i_k(m+1)})$ is disjoint from both, $\cup_{i_{k+1}=1}^{m}\psi(\omega_{i_1\ldots i_k i_{k+1}})$ as well as $\bar{\psi}(R(t_k - l_c), t_k + c)$. Moreover, by (3.2.5) we have that $\psi(\omega_{i_1\ldots i_k(m+1)}) \subset \psi(\omega_{i_1\ldots i_k})$. In particular, by construction, $x' \in L_k(c-d_*) \cap L_{k-1}(c)$. Iterating this argument until

$$(m+1) \geq \frac{1-\tau_c}{\bar{k}_c}$$

we see by (3.2.12) that

$$\begin{aligned} \mu\Big(\bigcup_{i_{k+1}=1}^{m} \psi(\omega_{i_1\ldots i_k i_{k+1}})\Big) & \geq m \cdot k_c \cdot \mu(\psi(\omega_{i_1\ldots i_k})) \\ & \geq \frac{m+1}{2}k_c \cdot \mu(\psi(\omega_{i_1\ldots i_k})) \\ & \geq \frac{(1-\tau_c)k_c}{2\bar{k}_c} \cdot \mu(\psi(\omega_{i_1\ldots i_k})), \end{aligned} \tag{3.2.16}$$

which shows the claim. □

We now construct a strongly treelike family $\mathcal{A}$ of subsets of $X \cap \psi(\omega_1)$ relative to μ as follows. Let $\mathcal{A}_1 = \{\psi(\omega_1)\}$. Given the subfamily $\mathcal{A}_k$ at the k.th step and a set $\psi(\omega_{i_1\ldots i_k}) \in \mathcal{A}_k$, Lemma

3.7 implies the existence of sets $\psi(\omega_{i_1\dots i_k i_{k+1}})$, which are disjoint subsets of $\psi(\omega_{i_1\dots i_k})$, disjoint to $\bar{\psi}(R(t_k-l_c),t_k+c)$ and satisfy (3.2.15). We therefore define

$$\mathcal{A}_{k+1}=\cup_{i_1\dots i_k}\{\psi(\omega_{i_1\dots i_k i_{k+1}})\}.$$

If $\mathcal{A}$ (a countable family of compact subsets of X) denotes the union of the subcollections $\mathcal{A}_k$, $k\in\mathbb{N}$, the following properties are satisfied with respect to μ:

(TL0) $\mu(A)>0$ for all $A\in\mathcal{A}$,

(TL1) for all $k\in\mathbb{N}$, for all $A,B\in\mathcal{A}_n$, either $A=B$ or $\mu(A\cap B)=0$,

(TL2) for all $k\in\mathbb{N}_{\geq 2}$, for all $B\in\mathcal{A}_k$, there exists $A\in\mathcal{A}_{k-1}$ such that $B\subset A$,

(TL3) for all $k\in\mathbb{N}$, for all $A\in\mathcal{A}_k$, there exists $B\in\mathcal{A}_{k+1}$ such that $B\subset A$.

We can therefore define $\cup\mathcal{A}_k=\cup_{A\in\mathcal{A}_k}A$ and obtain a decreasing sequence of nonempty compact subsets $X\supset\cup\mathcal{A}_1\supset\cup\mathcal{A}_2\supset\cup\mathcal{A}_3\supset\dots$. Since X is complete, the limit set

$$A_\infty\equiv\bigcap_{k\in\mathbb{N}}\cup\mathcal{A}_k$$

is nonempty. Define moreover the k.th stage diameter $d_k(\mathcal{A})\equiv\max_{A\in\mathcal{A}_k}\operatorname{diam}(A)$, which by (3.2.8) satisfies $d_k(\mathcal{A})\leq c_\sigma e^{-\sigma t_k}$, and hence

(STL) $\lim_{k\to\infty}d_k(\mathcal{A})=0$.

Finally, by (3.2.15), we obtain a lower bound for the k.th stage 'density of children'

$$\Delta_k(\mathcal{A})\equiv\min_{B\in\mathcal{A}_k}\frac{\mu(\cup A_{k+1}\cap B)}{\mu(B)}\geq\frac{(1-\tau_c)k_c}{2\bar{k}_c}\tag{3.2.17}$$

of $\mathcal{A}$. This gives a lower bound on the Hausdorff-dimension of A_∞.

Lemma 3.8. *If $\mathcal{A}$ as above satisfies (TL0-3) and (STL), then*

$$\dim(A_\infty)\geq\inf_{x_0\in A_\infty}d_\mu(x_0)-\limsup_{k\to\infty}\frac{\sum_{i=1}^k\log(\Delta_i(\mathcal{A}))}{\log(d_k(\mathcal{A}))}.$$

Proof. In [33], Lemma 2.5 (which is stated for $\bar{X}=\mathbb{R}^n$ but also true for general complete metric spaces, see [34]) a measure ν is constructed for which its support equals A_∞. Moreover, ν satisfies for every $x\in A_\infty$ that

$$d_\nu(x)\geq d_\mu(x)-\limsup_{k\to\infty}\frac{\sum_{i=1}^k\log(\Delta_i(\mathcal{A}))}{\log(d_k(\mathcal{A}))}.$$

For every open set $U\subset\bar{X}$ with $\nu(U)>0$, let

$$d_\nu(U)\equiv\inf_{x\in U\cap\operatorname{supp}(\mu)}d_\nu(x),$$

which is known to be a lower bound for the Hausdorff-dimension of $\operatorname{supp}(\nu)\cap U=A_\infty\cap U$ (see [18], Proposition 4.9 (a)). Setting $U=\bar{X}$ shows the claim. □

Using Lemma 3.8, (3.2.8) and (3.2.17), we obtain

$$\begin{aligned}\dim(A_\infty) &\geq \inf_{x_0\in A_\infty} d_\mu(x_0) - \limsup_{k\to\infty} \frac{k\log((1-\tau_c)k_c(2\bar{k}_c)^{-1})}{\log(c_\sigma e^{-\sigma t_k})} \\ &\geq \inf_{x_0\in A_\infty} d_\mu(x_0) - \limsup_{k\to\infty} \frac{k\big(\log(1-\tau_c) - \log(2\bar{k}_c k_c^{-1})\big)}{-\sigma k c} \\ &= \inf_{x_0\in A_\infty} d_\mu(x_0) - \frac{\log(2\bar{k}_c k_c^{-1}) + |\log(1-\tau_c)|}{\sigma c}.\end{aligned} \tag{3.2.18}$$

We establish our lower bound by showing the following Lemma.

Lemma 3.9. *$A_\infty \subset \psi(\omega_1) \cap \mathbf{Bad}(\mathcal{F}, 2c + l_c)$; hence, $\dim(\mathbf{Bad}(\mathcal{F}, 2c + l_c)) \geq \dim(A_\infty)$.*

Proof. Let $x_0 \in A_\infty$. Let $\{i_1 \dots i_k\}_{k\in\mathbb{N}}$ be a sequence such that $x_0 \in \cap_{k\in\mathbb{N}}\psi(\omega_{i_1\dots i_k})$. For the above case, since for every $k \in \mathbb{N}$ the sets $\psi(\omega_{i_1\dots i_k})$ of the construction of $\mathcal{A}$ are in particular disjoint, the sequence $\{i_1 \dots i_k\}_{k\in\mathbb{N}}$ is in fact unique - this might not be true for the standard case in the following.

Assume that $x_0 \in \bar{\psi}(R_m, s_m)$ for some $m \in \mathbb{N}$ (if no such m exists, then the claim already follows). Let $k \in \mathbb{N}$ such that $s_m + l_c \in [t_k, t_{k+1})$. By construction, $x_0 \in \psi(\omega_{i_1\dots i_{k+2}})$ which is disjoint to $\bar{\psi}(R(t_{k+1} - l_c), t_{k+1} + c)$ by (3.3.6). Since $t_k - l_c \leq s_m < t_{k+1} - l_c$ we have $R_m \subset R(t_{k+1} - l_c)$ and

$$\begin{aligned} x_0 &\notin \bar{\psi}(R(t_{k+1} - l_c), t_{k+1} + c) \\ &= \bar{\psi}(R(t_{k+1} - l_c), t_k - l_c + 2c + l_c) \supset \bar{\psi}(R_m, s_m + (2c + l_c)),\end{aligned}$$

by monotonicity of $\bar{\psi}$. This shows that $x_0 \in \mathbf{Bad}(\mathcal{F}, 2c + l_c)$. □

The Standard Case $X = \mathbb{R}^n$

Let $X = \bar{X} = \mathbb{R}^n$ and μ be the Lebesgue measure. For $\sigma > 0$, let $\psi(x, t) = B_\sigma(x, t) \equiv B(x, e^{-\sigma t})$. Then μ satisfies a power law. However, even in this case, our given bound (3.2.18) might not be sharp, because the constants k_c and $\bar{k}_c$ respectively depend sensitively on the separation constants d_* and d_c respectively.

For this standard case we now want to sharpen the lower bound. We only need to modify the above arguments by shifting the separation constants into τ_c. Consider therefore the monotonic function on $\Omega = \mathbb{R}^n \times \mathbb{R}^+$ given by

$$Q_\sigma(x, t) \equiv B(x_1, e^{-\sigma t}) \times \cdots \times B(x_n, e^{-\sigma t}),$$

which denotes the n-dimensional cube of edge length $2e^{-\sigma t}$ with center $x = (x_1, \dots, x_n)$. Note that for all formal balls $(x, t) \in \Omega$ we have

$$Q_\sigma(x, t + \sqrt{n}/\sigma) \subset B_\sigma(x, t) \subset Q_\sigma(x, t).$$

Thus, $\mathbf{Bad}^{Q_\sigma}_{\mathbb{R}^n}(\mathcal{F}, c) \subset \mathbf{Bad}^{B_\sigma}_{\mathbb{R}^n}(\mathcal{F}, c) \subset \mathbf{Bad}^{Q_\sigma}_{\mathbb{R}^n}(\mathcal{F}, c + \sqrt{n}/\sigma)$. Moreover, the Lebesgue measure satisfies $\mu(Q_\sigma(x, t)) = 2^n e^{-n\sigma t}$ so that (Ω, Q_σ, μ) satisfies a power law. Also, if (Ω, B_σ) is d_*-separating with respect to $\mathcal{F}$, let $\bar{d}_* \equiv d_* + \sqrt{n}/\sigma$, and we see that (Ω, Q_σ) is at least $\bar{d}_*$-separating with respect to $\mathcal{F}$.

The crucial point is that, given any cube $Q_\sigma(x,t) \subset \mathbb{R}^n$ and $c = \log(m)/\sigma$ for some $m \in \mathbb{N}$, we can find a partition into m^n cubes $Q_i = Q_i(x,t) \equiv Q_\sigma(x_i, t+c)$ satisfying

$$\begin{aligned} \mu(Q_i \cap Q_j) &= 0 \quad \text{for } i \neq j, \text{ and} \\ \bigcup_i Q_i &= Q_\sigma(x,t). \end{aligned} \tag{3.2.19}$$

Now let $c = \log(m)/\sigma \geq \bar{d}_* + \log(2)/\sigma$ for some integer $m \in \mathbb{N}$, $\bar{l}_c \geq 0$ and modify (3.2.13) such that for all formal balls $\omega = (x, t_k) \in \Omega$ with $x \in L_k^{Q_\sigma}(c)$ we have

$$\mu(Q_\sigma(\omega) \cap Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \bar{d}_* - \log(2)/\sigma)) \leq \bar{\tau}_c \cdot \mu(Q_\sigma(\omega)), \tag{3.2.20}$$

where $\bar{\tau}_c < 1$. Note that for all $(x,t) \in \Omega$, $y \in \mathbb{R}^n$ and $s \geq 0$ we already have

$$\mu(Q_\sigma(y, t+s)) = e^{-n\sigma s}\mu(Q_\sigma(x,t))$$

and in particular (3.2.12).

We modify the arguments from Lemma 3.7 where we replace the choices $\psi(\omega_{i_1 \ldots i_k})$ by cubes $Q_{i_1 \ldots i_k}$ in order to construct a strongly treelike family $\mathcal{A}$ with a limit set contained in $\mathbf{Bad}_{\mathbb{R}^n}^{Q_\sigma}(\mathcal{F}, 2c + \bar{l}_c) \cap Q_{i_1}$.

In fact, if $Q = Q_{i_1 \ldots i_k} = Q_\sigma(x_{i_1 \ldots i_k}, t_k)$ is a given cube, let $Q_{i_1 \ldots i_k i_{k+1}} = Q_\sigma(x_{i_1 \ldots i_k i_{k+1}}, t_k + c)$ be precisely the cubes of the partition of Q as above, which intersect

$$Q \cap Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \bar{d}_* - \log(2)/\sigma)^C,$$

and hence cover $Q \cap Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \bar{d}_* - \log(2)/\sigma)^C$. From (3.2.20), we obtain

$$\mu\Big(\bigcup_{i_{k+1}} Q_{i_1 \ldots i_k i_{k+1}}\Big) \geq (1 - \bar{\tau}_c)\mu(Q_{i_1 \ldots i_k}),$$

which improves (3.2.15). Moreover, let $\bar{Q} = Q_{i_1 \ldots i_k i_{k+1}}$ be such a cube intersecting $Q \cap Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \bar{d}_* - \log(2)/\sigma)^C$ in a point y. Then $\bar{Q} \subset Q_\sigma(y, t_{k+1} - \log(2)/\sigma)$, and, since $y \notin Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \bar{d}_* - \log(2)/\sigma)$, the supset is disjoint to

$$Q_\sigma(R(t_k - \bar{l}_c), t_k + c - \log(2)/\sigma) \supset Q_\sigma(R(t_k - \bar{l}_c), t_k + c).$$

Hence, every cube chosen as above is contained in $L_k^{Q_\sigma}(c)$ which shows (3.3.6) for the setting of cubes. Using the results of the strongly treelike construction as well as (3.2.18) and Lemma 3.9, we obtain our lower bound in the standard case

$$\dim(\mathbf{Bad}_{\mathbb{R}^n}^{Q_\sigma}(\mathcal{F}, 2c + \bar{l}_c) \cap Q_1) \;\geq\; n - \frac{|\log(1 - \bar{\tau}_c)|}{\sigma c}. \tag{3.2.21}$$

Remark 3.10. The improvement relies on the partition (3.2.19) of cubes. This is no longer possible in general, not even for subsets of the Euclidean space. Note also that the restriction to $c = \log(m)/\sigma$ will not be a severe restriction in the applications, since, for sufficiently large $c > 0$ we can choose a $\bar{c} = \log(m)/\sigma$ with $\bar{c} \leq c$ and obtain a lower bound with respect to $\bar{c}$. The defect can again be shifted to a multiplicative constant in $\bar{\tau}_c$.

In special cases, when (3.2.13) is satisfied with respect to B_σ the parameters (c, l_c) and even for all $x \in L_k^{Q_\sigma}(c)$, we can in fact already estimate $\bar{\tau}_c$ using a sufficiently small τ_c - however, a more precise bound can be determined in the particular examples.

Lemma 3.11. *If μ is τ_c-decaying with respect to $\mathcal{F}$, B_σ and the parameters (c, l_c), let $\bar{l}_c = l_c + a$ where $a \equiv 2\sqrt{n}/\sigma + \log(2)/\sigma + d_c$. Then for all $x \in L_{k-1}^{B_\sigma}(c - d_*)$ and t_k we have (3.2.20) with*

$$\bar{\tau}_c \leq e^{n\sigma(a-d_c)}\tau_c.$$

Proof. Recall that for all formal balls $\omega = (x, t_k + d_c) \in \Omega$ with $x \in L_{k-1}^{B_\sigma}(c)$ we have

$$\mu(B_\sigma(\omega) \cap B_\sigma(R(t_k - l_c), t_k + c - d_*)) \leq \tau_c \cdot \mu(B_\sigma(\omega)),$$

where $\tau_c < 1$. Define $\bar{l}_c = l_c + a$ as well as $t_k = s_1 + kc + l_c$ and $\tilde{t}_k = s_1 + kc + \bar{l}_c$. Then note that $R(\tilde{t}_k - \bar{l}_c) = R(s_1 + kc) = R(t_k - l_c)$ and

$$\begin{aligned}\tilde{t}_k + c - \bar{d}_* - \log(2)/\sigma &= (t_k + a) + c - d_* - \sqrt{n}/\sigma - \log(2)/\sigma \\ &\geq t_k + c - d_* + \sqrt{n}/\sigma.\end{aligned}$$

as well as $\tilde{t}_k \geq t_k + d_c + \sqrt{n}/\sigma$. Hence, $Q_\sigma(x, \tilde{t}_k) \subset B_\sigma(x, t_k + d_c)$ and

$$Q_\sigma(R(\tilde{t}_k - \bar{l}_c), \tilde{t}_k + c - \bar{d}_* - \log(2)/\sigma) \subset B_\sigma(R(t_k - l_c), t_k + c - d_*).$$

For $x \in L_{k-1}^{B_\sigma}(c - d_*)$ and $\omega = (x, \tilde{t}_k)$ this shows

$$\begin{aligned}\mu(Q_\sigma(\omega) \cap Q_\sigma(R(\tilde{t}_k - \bar{l}_c), \tilde{t}_k + c - \bar{d}_* - \log(2)/\sigma)) &\leq \tau_c \cdot \mu(B_\sigma(x, t_k + d_c)) \\ &\leq \tau_c \cdot \mu(Q_\sigma(x, t_k + d_c)) \\ &= e^{n\sigma(a-d_c)}\tau_c \cdot \mu(Q_\sigma(\omega)),\end{aligned}$$

proving the claim. □

3.2.3 The upper bound

For a given parameter $c > 0$ we let $u^c \geq 0$ and define $t_k = s_1 + (k-1)(c + u^c)$ as well as $\bar{t}_k = t_k - u^c$ for $k \geq 1$. Moreover, we consider the sequence of the sets

$$\begin{aligned}U_k(c) = U_k^{\bar{\psi}}(c) &\equiv \bigcap_{s_n \leq t_k} \bar{\psi}(R_n, s_n + c)^C \\ &\supset \bigcap_{n \in \mathbb{N}} \bar{\psi}(R_n, s_n + c)^C \\ &= \Big(\bigcup_{n \in \mathbb{N}} \bar{\psi}(R_n, s_n + c)\Big)^C = \mathbf{Bad}(\mathcal{F}, c).\end{aligned}$$

At this step, we require that (Ω, ψ, μ) satisfies for every $k \in \mathbb{N}$ and for all formal balls $(x, \bar{t}_k - d^c) \in \Omega$ with $x \in U_{k-1}(c)$ and $y \in \psi(x, \bar{t}_k) \cap U_k(c)$, that

$$\mu(\psi(y, \bar{t}_{k+1} + d_*)) \geq k^c \mu(\psi(x, \bar{t}_k - d^c)), \tag{3.2.22}$$

where k^c is a positive constant. In addition, consider a further condition on μ, which is the counterpart of the notion of decaying measures. (Ω, ψ, μ) is called *τ^c-Dirichlet with respect to the* *family $\mathcal{F}$ and the parameters* (c, u^c) if for any formal ball $\omega = (x, \bar{t}_k - d^c) \in \Omega$ with $x \in U_{k-1}(c)$ we have

$$\mu(\psi(\omega) \cap \bigcup_{s_n \leq t_k} \bar{\psi}(R_n, s_n + c + d_*)) \geq \tau^c \cdot \mu(\psi(\omega)), \tag{3.2.23}$$

where $\tau^c \geq 0$. Note that we called this condition 'Dirichlet' since (3.2.23) will follow from Dirichlet-type results in the applications.

Remark 3.12. It would actually suffice to require (3.2.22) and (3.2.23) for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$ and modify the arguments below. For clarity of the proof, we however let $k_0 = 1$.

Assume moreover that $X - \bar{\psi}(R_1, s_1 + c)$ can be covered by countably many ψ-balls $X_{i_1} \equiv \psi(x_{i_1}, s_1 - u_c)$ where $x_{i_1} \notin \bar{\psi}(R_1, s_1 + c)$. Using the arguments given below, this is for instance the case if X has finite μ-measure and (Ω, ψ, μ) satisfies a power law with respect to (Ω, ψ). Note that, by the countable stability of the Hausdorff-dimension - that is,

$$\dim(\mathbf{Bad}(\mathcal{F}, c)) \leq \dim(\cup_i \mathbf{Bad}(\mathcal{F}, c) \cap X_{i_1}) \leq \sup_{i_1} \dim(\mathbf{Bad}(\mathcal{F}, c) \cap X_{i_1}) -$$

it suffices to estimate the dimension of each $\mathbf{Bad}(\mathcal{F}; c) \cap X_{i_1}$. In order to determine the upper bound of $\dim(\mathbf{Bad}(\mathcal{F}, c) \cap X_{i_1})$, we construct a suitable covering of $\mathbf{Bad}(\mathcal{F}, c) \cap X_{i_1}$ with uniform bounds on the diameters converging to zero.

We start with X_{i_1}. Suppose that the we are already given $\omega_{i_1 \dots i_k} = (x_{i_1 \dots i_k}, \bar{t}_k) \in \Omega$ with $x_{i_1 \dots i_k} \in U_{k-1}(c)$ and let $U_{i_1 \dots i_k} \equiv U_k(c) \cap \psi(\omega_{i_1 \dots i_k})$. If possible, let $x_{i_1 \dots i_k 1}, \dots, x_{i_1 \dots i_k m} \in U_{i_1 \dots i_k}$ be chosen such that $\psi(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_{k+1} + d_*)$ are disjoint. Note that if there exists $x' \in U_{i_1 \dots i_k}$ such that $x' \notin \cup \psi(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_{k+1})$, then $\psi(x', \bar{t}_{k+1} + d_*)$ is disjoint to $\cup \psi(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_{k+1} + d_*)$ by (3.2.6) and we set $\omega_{i_1 \dots i_k (m+1)} \equiv (x', \bar{t}_{k+1}) \in \Omega$. Therefore, we obtain a covering of $U_{i_1 \dots i_k}$ by the ψ-balls $\psi(\omega_{i_1 \dots i_k i_{k+1}})$ which is finite (see (3.2.24)), bounded by a number N_k.

In fact, by (3.2.6) and since

$$\bar{t}_{k+1} + d_* = t_{k+1} - u^c + d_* = t_k + c + d_* \geq s_n + c + d_*,$$

for all $s_n \leq t_k$, the ψ-balls $\psi(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_{k+1} + d_*)$ are disjoint to $\bar{\psi}(R_n, s_n + c + d_*)$, $s_n \leq t_k$. Moreover, they are contained in $\psi(x_{i_1 \dots i_k}, \bar{t}_k - d^c)$ by (3.2.5). Hence, (3.2.22) and (3.2.23) applied to the formal ball $\omega_0 = (x_{i_1 \dots i_k}, \bar{t}_k - d^c)$ imply

$$\begin{aligned} \mu(\psi(\omega_0)) &\geq \mu(\psi(\omega_0) \cap \bigcup_{s_n \leq t_k} \bar{\psi}(R_n, t_k + c + d_*)) + \sum_{i_{k+1}=1}^{N_k} \mu(\psi(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_{k+1} + d_*)) \\ &\geq (\tau^c + N_k \cdot k^c) \mu(\psi(\omega_0)). \end{aligned}$$

Using (3.2.7), this shows that the above collection of ψ-balls $\psi(\omega_{i_1 \dots i_k i_{k+1}})$, $i_{k+1} = 1, \dots, N_k$, must be finite where N_k is bounded by

$$N_k \leq \frac{(1 - \tau^c)}{k^c}. \tag{3.2.24}$$

For every $k \in \mathbb{N}$ we thus constructed a finite cover of

$$\mathbf{Bad}(\mathcal{F}, c) \cap X_{i_1} \subset U_k(c) \cap X_{i_1} \subset \bigcup_{i_1 \dots i_k} U_{i_1 \dots i_k} \subset \bigcup_{i_1 \dots i_k i_{k+1}} \psi(\omega_{i_1 \dots i_k i_{k+1}}),$$

where the indices run over all $i_2 \dots i_{k+1}$ from the above construction. The sets of the covering are of diameter at most $c_\sigma e^{-\sigma \bar{t}_{k+1}}$ (by (3.2.8)) and, using (3.2.24), the number of this covering is bounded by

$$\bar{N}_k \equiv \prod_{i=1}^{k} N_i \leq \frac{(1-\tau^c)^k}{(k^c)^k}.$$

Finally, it is readily checked (or seen from [18], Proposition 4.1) that

$$\begin{aligned}
\dim(\mathbf{Bad}(\mathcal{F}, c) \cap X_i) &\leq \liminf_{k\to\infty} \frac{\log(\bar{N}_k)}{-\log(c_\sigma e^{-\sigma \bar{t}_{k+1}})} \\
&\leq \liminf_{k\to\infty} \frac{k\big(\log(1-\tau^c) - \log(k^c)\big)}{\sigma \bar{t}_{k+1}} \\
&\leq \frac{\log(1-\tau^c) - \log(k^c)}{\sigma(c+u^c)}.
\end{aligned} \tag{3.2.25}$$

This gives our first formula for the upper bound.

In the case that $\bar{\psi}$ is not the standard function B_σ, the constructed covering might not be the best suitable for an optimal upper bound and we want to consider further conditions. Let μ satisfy a power law with respect to both parameter spaces (Ω, ψ) and (Ω, B_1) and the parameters (τ, c_1, c_2) and $(\delta, c_{1,\delta}, c_{2,\delta})$ respectively. Assume moreover that there exists $\bar{\sigma} \geq \sigma$ and $d^* \geq 0$ such that $B_{\bar{\sigma}}(y,t) = B(y, e^{-\bar{\sigma} t}) \cap X \subset \psi(x, t-d^*)$, whenever $y \in \psi(x,t)$.

Then, given one of the $\bar{N}_{k-1}$ formal balls $\omega_{i_1 \dots i_k}$ constructed above, we can cover $\psi(\omega_{i_1\dots i_k})$ by $Z(\omega_{i_1\dots i_k})$ metric balls $B_j(\omega_{i_1\dots i_k}) \equiv B(z_j(\omega_{i_1\dots i_k}), e^{-\bar{\sigma}\bar{t}_k})$. Moreover, using the same arguments as above we can assume that $z_j(\omega_{i_1\dots i_k}) \in \psi(\omega_{i_1\dots i_k})$ for $j = 1, \dots, Z(\omega_{i_1\dots i_k})$ and that the balls $B(z_j(\omega_{i_1\dots i_k}), \frac{1}{2}e^{-\bar{\sigma}\bar{t}_k})$ are disjoint and contained in $\psi(x_{i_1\dots i_k}, \bar{t}_k - d^*)$. Hence, we obtain

$$\begin{aligned}
c_2 e^{-\tau(\bar{t}_k - d^*)} &\geq \mu(\psi(x_{i_1\dots i_k}, \bar{t}_k - d^*)) \\
&\geq \mu(\cup_{j=1}^{Z(\omega_{i_1\dots i_k})} B(z_j(\omega_{i_1\dots i_k}), \tfrac{1}{2}e^{-\bar{\sigma}\bar{t}_k})) \geq Z(\omega_{i_1\dots i_k}) \cdot c_{1,\delta} \tfrac{1}{2^\delta} e^{-\delta\bar{\sigma}\bar{t}_k},
\end{aligned}$$

and the number $Z(\omega_{i_1\dots i_k})$ can be bounded by

$$Z_k \equiv \tfrac{2^\delta c_2 e^{\tau d^*}}{c_{1,\delta}} e^{(\delta\bar{\sigma} - \tau)\bar{t}_k}.$$

Thus, this gives a covering of $\mathbf{Bad}(\mathcal{F}, c) \cap X_i$ by $\bar{N}_k \cdot Z_k$ metric balls of diameter $2e^{-\bar{\sigma}\bar{t}_k}$. Replacing $\bar{N}_k$ by $\bar{N}_k \cdot Z_k$ in (3.2.25), we obtain a new upper bound

$$\begin{aligned}
\dim(\mathbf{Bad}(\mathcal{F}, c) \cap X_i) &\leq \delta - \frac{\tau}{\bar{\sigma}} + \frac{\log(1-\tau^c) - \log(k^c)}{\bar{\sigma}(c+u^c)} \\
&\leq \delta - \frac{|\log(1-\tau^c)| + \log(\frac{c_1}{c_2}) - \tau(d_* + d^c)}{\bar{\sigma}(c+u^c)},
\end{aligned} \tag{3.2.26}$$

where we used (3.2.10) in the last inequality.

The Standard Case $X = \mathbb{R}^n$

Let again $X = \bar{X} = \mathbb{R}^n$, $\psi = B_\sigma$ and μ be the Lebesgue measure. Note that, even in this case, our given bounds (3.2.25) and (3.2.26) respectively might not be sharp, that is, the bounds might in fact be bigger than the actual dimension of $\mathbb{R}^n$ because of the strictly positive separation constant d_* and d^c.

For this standard case we now want to sharpen the upper bound. We again only need to modify the above arguments by shifting the separation constant into τ^c.

For $c > 0$, let $\bar{u}^c \geq 0$ such that

$$c + \bar{u}^c = \log(m)/\sigma$$

for some $m \in \mathbb{N}$, and modify (3.2.23) such that for any formal ball $\omega = (x, \bar{t}_k) \in \Omega$ with $x \in U_{k-1}^{Q_\sigma}(c + \bar{d}_*) \equiv \bigcap_{s_n \leq t_{k-1} - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c + \bar{d}_*)^C$ we have

$$\mu(Q_\sigma(\omega) \cap \bigcup_{s_n \leq t_k - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c + \bar{d}_*)) \geq \bar{\tau}^c \cdot \mu(Q_\sigma(\omega)), \tag{3.2.27}$$

where $\bar{\tau}^c$ is a positive constant; here, t_k and $\bar{t}_k$ are with respect to $\bar{u}_c$. Note that we already have $\mu(Q_\sigma(x, \bar{t}_{k+1})) = e^{-n\sigma(c+\bar{u}_c)}\mu(Q_\sigma(x, \bar{t}_k))$ and hence (3.2.22).

We modify our construction (as before (3.2.24)) where we replace the choices $\psi(\omega_{i_1 \dots i_k})$ by cubes $Q_{i_1 \dots i_k}$ in order to obtain a covering of $\mathbf{Bad}_{\mathbb{R}^n}^{Q_\sigma}(\mathcal{F}, c) \cap Q_{i_1}$ by the cubes $Q_{i_1 \dots i_k}$. In fact, if $Q = Q_{i_1 \dots i_k} = Q_\sigma(x_{i_1 \dots i_k}, \bar{t}_k)$ is a given cube, let $Q_{i_1 \dots i_k i_{k+1}} = Q_\sigma(x_{i_1 \dots i_k i_{k+1}}, \bar{t}_k + (c + \bar{u}_c))$ be precisely the cubes of the partition of Q as in (3.2.19), which intersect

$$U_{i_1 \dots i_k}^{Q_\sigma}(c) \equiv Q \cap \bigcap_{s_n \leq t_k - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c)^C,$$

and hence cover $U_{i_1 \dots i_k}^{Q_\sigma}(c)$. Let $\bar{Q} = Q_{i_1 \dots i_k i_{k+1}}$ be such a cube intersecting $U_{i_1 \dots i_k}^{Q_\sigma}(c)$ in a point y. Then $\bar{Q} \subset Q_\sigma(y, \bar{t}_{k+1} - \log(2)/\sigma)$. Moreover, for every n with $s_n \leq t_k - \bar{d}_* - \log(2)/\sigma$, we have

$$\bar{t}_k + (c + \bar{u}_c) - \log(2)/\sigma = t_k + c - \log(2)/\sigma \geq s_n + c + \bar{d}_*$$

so that $Q_\sigma(y, \bar{t}_{k+1} - \log(2)/\sigma) \subset Q_\sigma(y, s_n + c + \bar{d}_*)$ where the supset is disjoint to $Q_\sigma(R_n, s_n + c + \bar{d}_*)$. Hence, every cube chosen as above is contained in $U_{i_1 \dots i_k}^{Q_\sigma}(c + \bar{d}_*)$. Using the above arguments with (3.2.27) shows

$$N_k \leq (1 - \bar{\tau}^c)e^{n\sigma(c+\bar{u}_c)},$$

which improves (3.2.24). As in (3.2.25), we obtain our upper bound in the standard case

$$\begin{aligned} \dim(\mathbf{Bad}_{\mathbb{R}^n}^{Q_\sigma}(\mathcal{F}, c) \cap Q_{i_1}) &\leq \frac{\log(1 - \bar{\tau}^c) - \log(e^{-n\sigma(c+u_c)})}{\sigma(c + \bar{u}^c)} \\ &= n - \frac{|\log(1 - \bar{\tau}^c)|}{\sigma(c + \bar{u}^c)}. \end{aligned} \tag{3.2.28}$$

Remark 3.13. Again, it suffices to require (3.2.27) for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$ and start with a covering of $X = \mathbb{R}^n$ by cubes $Q_\sigma(x_{i_1}, \bar{t}_{k_0})$.

In special situations, when (3.2.13) is satisfied with respect to B_σ and the parameters $(\tilde{c}, u^{\tilde{c}})$ (as below) and is even satisfied for all times t, we can in fact already estimate $\bar{\tau}^c$ using $\tau^{\tilde{c}}$. More precisely, we have the following, where we remark that (3.2.29) corresponds to (3.2.23) with $t = \bar{t}_k - d^{\tilde{c}}$ and the parameters $(\tilde{c}, u^{\tilde{c}})$ (where $\bar{t}_k$ is with respect to the parameters $(c, \bar{u}^c)$ in the following).

Lemma 3.14. *For $c > 0$, let $\tilde{c} = c + a$ and $\bar{u}^c \geq u^{\tilde{c}} + a$, where $a = d^c + \bar{d}_* + \log(2)/\sigma + \sqrt{n}/\sigma$. Then, $U_{k-1}^{Q_\sigma}(c + \bar{d}_*) \subset \bigcap_{s_n \leq \bar{t}_k - \tilde{c} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c})^C$. Moreover, assume that for all $\omega = (x,t) \in \Omega$ with $x \in \bigcap_{s_n \leq t - \tilde{c} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c})^C$ we have*

$$\mu(B_\sigma(\omega) \cap \bigcup_{s_n \leq t + u^{\tilde{c}} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c} + d_*)) \geq \tau^{\tilde{c}} \cdot \mu(B_\sigma(\omega)), \tag{3.2.29}$$

where $\tau^{\tilde{c}}$ is a positive constant. Then, (3.2.27) is satisfied for ω with

$$\bar{\tau}^c \geq e^{-n\sqrt{n}} \tau^{\tilde{c}}.$$

Proof. Note that $\bar{t}_k - \tilde{c} + d^{\tilde{c}} = t_{k-1} - a + d^{\tilde{c}} \leq t_{k-1} - \bar{d}_* - \log(2)/\sigma$, since $a \geq d^{\tilde{c}} + \bar{d}_* + \log(2)/\sigma$ (as $d^{\tilde{c}} \leq d^c$). Using that $Q_\sigma(y, s + \sqrt{n}/\sigma) \subset B_\sigma(y,s)$ and $a \geq \bar{d}_* + \sqrt{n}/\sigma$, we see that

$$\begin{aligned} U_{k-1}^{Q_\sigma}(c + \bar{d}_*) &= \bigcap_{s_n \leq t_{k-1} - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c + \bar{d}_*)^C \\ &\subset \bigcap_{s_n \leq \bar{t}_k - \tilde{c} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c})^C, \end{aligned}$$

which shows the first claim.

Similarly, since $t + u^{\tilde{c}} + d^{\tilde{c}} \leq t + \bar{u}^c - \bar{d}_* - \log(2)/\sigma$ and $\tilde{c} + d_* \geq c + \bar{d}_*$, we have

$$B_\sigma(x,t) \cap \bigcup_{s_n \leq t + u^{\tilde{c}} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c} + d_*) \subset Q_\sigma(x,t) \cap \bigcup_{s_n \leq t + \bar{u}^c - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c + \bar{d}_*).$$

Hence, by (3.2.29), we obtain

$$\begin{aligned} \mu(Q_\sigma(x,t) \cap \bigcup_{s_n \leq t + \bar{u}^c - \bar{d}_* - \log(2)/\sigma} Q_\sigma(R_n, s_n + c + \bar{d}_*)) &\geq \tau^{\tilde{c}} \cdot \mu(B_\sigma(x,t)) \\ &\geq \tau^{\tilde{c}} \cdot \mu(Q_\sigma(x, t + \sqrt{n}/\sigma)) \\ &= \tau^{\tilde{c}} e^{-n\sqrt{n}} \cdot \mu(Q_\sigma(x,t)), \end{aligned}$$

proving the lemma if we set $t = \bar{t}_k$. □

3.2.4 Dirichlet and absolutely decaying measures

Let $\mathcal{S} \equiv \{S \subset \bar{X}\}$ be a given collection of nonempty Borel sets. For instance, consider $\mathcal{S}$ to be the collection of metric spheres $S(x,t) \equiv \{y \in \bar{X} : d(x,y) = e^{-t}\}$ in $\bar{X}$, or the set of hyperplanes in the Euclidean space $\mathbb{R}^n$. Assume moreover, that $\bar{\psi}(S,t)$ is a Borel-set for all $t > t_*$ and $S \in \mathcal{S}$.

For the lower bound, given a locally finite Borel measure μ on X, (Ω, ψ, μ) is said to be *absolutely (c_δ, δ)-decaying with respect to $\mathcal{S}$* if for all $(x,t) \in \Omega$ and for all $S \in \mathcal{S}$ and $s \geq 0$ we have

$$\mu(\psi(x,t) \cap \bar{\psi}(S, t+s)) \leq c_\delta e^{-\delta s} \mu(\psi(x,t)). \tag{3.2.30}$$

Moreover, we say that a nested discrete family $\mathcal{F}$ is *locally contained in $\mathcal{S}$* (with respect to $(\bar{\Omega}, \bar{\psi})$) if there exists $l_* \geq 0$ and a number $n_* \in \mathbb{N}$ such that for all $(x,t) \in \Omega$ we have

$$\bar{\psi}(x, t + l_*) \cap R(t) \subset \bigcup_{i=1}^{n_*} S_i \tag{3.2.31}$$

is contained in at most n_* sets S_i of $\mathcal{S}$.

We say that $(\Omega, \bar{\psi})$ is d_**-separating* if for all formal balls $(x,t) \in \Omega$ and for any set M disjoint to $\bar{\psi}(x,t)$, we have

$$\bar{\psi}(x, t + d_*) \cap \bar{\psi}(M, t + d_*) = \emptyset. \tag{3.2.32}$$

Clearly, the standard function B_σ is $\log(3)/\sigma$-separating in a proper metric space $\bar{X}$.

Proposition 3.15. *Let (Ω, ψ, μ) be d_*-separating and let $\mathcal{F}$ be locally contained in $\mathcal{S}$. Then, if $(\Omega, \bar{\psi})$ is absolutely (c_δ, δ)-decaying with respect to $\mathcal{S}$, it is τ_c-decaying with respect to $\mathcal{F}$ and the parameters $(c, l_* + d_*)$, where*

$$\tau_c = n_* c_\delta e^{-\delta(c-2d_*)},$$

for all $c \geq 2d_*$ such that $\tau_c < 1$.

Proof. Fix $c \geq 2d_*$. Given $\omega = (x, t + l_* + d_* + d_c) \in \Omega$ and l_*, $n_* \in \mathbb{N}$ as well as $S_1, \ldots, S_{n_*}$ from the definition of (3.2.31), we claim that

$$\psi(\omega) \cap \bar{\psi}(R(t), t + l_* + d_c + c - d_*) \subset \psi(\omega) \cap \bigcup_{i=1}^{n_*} \psi(S_i, t + l_* + c - d_*).$$

In fact, let M be the set $R(t) - \cup S_i$ which is disjoint to $\bar{\psi}(x, t + l_*)$ by (3.2.31). By monotonicity of $\bar{\psi}$, we have

$$\bar{\psi}(x, t + l_* + d_c + d_*) \subset \bar{\psi}(x, t + l_* + d_*)$$

which, by (3.2.32), is disjoint to

$$\bar{\psi}(M, t + l_* + d_*) \supset \bar{\psi}(M, t + l_* + c - d_*),$$

for $c \geq 2d_*$ again by monotonicity of $\bar{\psi}$. This shows the above claim.

Set $l_c = l_* + d_*$ so that $\omega = (x, t + l_c + d_c) \in \Omega$. Finally, (3.2.30) implies

$$\begin{aligned} \mu(\psi(\omega) \cap \bigcup_{i=1}^{n_*} \psi(S_i, t + l_* + d_c + c - d_*)) &= \mu(\psi(\omega) \cap \bigcup_{i=1}^{n_*} \psi(S_i, t + l_c + d_c + (c - 2d_*))) \\ &\leq n_* c_\delta e^{-\delta(c-2d_*)} \mu(\psi(\omega)), \end{aligned}$$

which shows that μ is τ_c-decaying with respect to $\mathcal{F}$ and the parameters $(c, l_* + d_*)$. □

As a special case, let $\bar{\psi} = \bar{B}_\sigma$ be the standard function and $\bar{X}$ be a proper metric space. Recall that $d_* \leq \log(3)/\sigma$, and assume that for all distinct points x, $y \in R_n$ we have

$$d(x, y) > \bar{c} \cdot e^{-\sigma s_n}, \tag{3.2.33}$$

for some constant $\bar{c} > 0$.

Lemma 3.16. *Let (Ω, ψ, μ) satisfy a power law with respect to the parameters (τ, c_1, c_2). If (3.2.33) is satisfied, then μ is τ_c-decaying with respect to $\mathcal{F}$, where $\tau_c = \frac{c_2}{c_1} e^{\tau(c-2d_*)}$, for all $c \geq 2d_*$ and $l_c = -\log(\bar{c})/\sigma + d_* + \log(2)$.*

Proof. Let $l_* = -\log(\bar{c})/\sigma + \log(2)$. Given a formal ball $(x, t + l_*) \in \Omega$, at most one point $y \in R(t)$ can lie in $B(x, e^{-\sigma(t+l_*)})$. In fact, for distinct y and $y' \in R_{n_t}$ (where $n_t \in \mathbb{N}$ was the largest integer such that $s_n \leq t$), (3.2.33) implies

$$d(y, y') > e^{-\sigma(s_n + \log(\bar{c})/\sigma)} \geq 2e^{-\sigma(t+l_*)}.$$

Hence, $\mathcal{F}$ is locally contained in the set $\mathcal{S} \equiv \{y \in R_n : n \in \mathbb{N}\}$ with $n_* = 1$. Since μ satisfies the power law, it is $(\frac{c_2}{c_1}, \tau)$-decaying with respect to $\mathcal{S}$ and $\bar{B}_\sigma$. The proof follows from Proposition 3.15. □

Analogously, for the upper bound and a possibly different collection of Borel sets $\mathcal{S}$, for a locally finite Borel measure μ on X, (Ω, ψ, μ) is called (c_δ, δ)-*Dirichlet with respect to* $\mathcal{S}$ if for all $\omega = (x, t) \in \Omega$, for all $S \in \mathcal{S}$ such that $S \cap \bar{\psi}(\omega) \neq \emptyset$ and $s \geq 0$ we have

$$\mu(\psi(\omega) \cap \bar{\psi}(S, t + s)) \geq c_\delta e^{-\delta s} \mu(\psi(\omega)). \tag{3.2.34}$$

We say that the family $\mathcal{F}$ *locally contains* $\mathcal{S}$ (with respect to (Ω, ψ)) if there exists $u_* \geq 0$ such that for all formal balls $\omega = (x, t - u_*) \in \Omega$ there exists $S \in \mathcal{S}$ with

$$\bar{\psi}(\omega) \cap R(t) \supset \bar{\psi}(\omega) \cap S. \tag{3.2.35}$$

Proposition 3.17. *If $\mathcal{F}$ locally contains $\mathcal{S}$ and (Ω, ψ, μ) is (c_δ, δ)-Dirichlet with respect to $\mathcal{S}$, then (Ω, ψ, μ) is τ^c-decaying with respect to $\mathcal{F}$ and the parameters (c, u_*), where $\tau^c \geq c_\delta e^{-\delta(c+d_*)}$.*

In the special case when $\mathcal{F}$ locally contains $\mathcal{S}$, where $\mathcal{S}$ consists of subsets of X, and (Ω, ψ, μ) satisfies a power law with respect to the parameters (τ, c_1, c_2), we have that (Ω, ψ, μ) is τ^c-decaying with respect to $\mathcal{F}$ and the parameters (c, u_*), where

$$\tau^c \geq \tfrac{c_1}{c_2} e^{-\tau(c+d_*+u_*+d^c)}.$$

Proof. The first statement is readily checked. For the second one, let $\omega = (x, t - u_*) \in \Omega$ and $S \in \mathcal{S}$ such that $S \cap \bar{\psi}(\omega) \subset R_{n_t} \cap \bar{\psi}(\omega)$. Let $y \in S \cap \bar{\psi}(\omega)$. By monotonicity of $\bar{\psi}$ and (3.2.5), $\bar{\psi}(y, t + c + d_*) \subset \bar{\psi}(y, t - u_* + c) \subset \bar{\psi}(x, t - u_* - d^c)$. Hence, for $\omega_0 = (x, t - u_* - d^c) \in \Omega$ we see that

$$\begin{aligned} \mu(\psi(\omega_0) \cap \bigcup_{s_n \leq t} \bar{\psi}(R_n, s_n + c + d_*)) &\geq \mu(\psi(\omega_0) \cap \bar{\psi}(y, t + c + d_*)) \\ &\geq \mu(\psi(y, t + c + d_*)) \geq \tfrac{c_1}{c_2} e^{-\tau(c+d_*+u_*+d^c)} \mu(\psi(\omega_0)), \end{aligned}$$

which shows the second claim. □

3.3 Applications

We want to determine the upper and lower bounds on the Hausdorff-dimension of $\mathbf{Bad}(\mathcal{F}, c)$ of several examples by checking the conditions of the abstract formalism.

3.3.1 $\mathbf{Bad}^{\bar r}_{\mathbb{R}^n}$

For $n \geq 1$, let $\bar r \in \mathbb{R}^n$ with $r^1, \ldots, r^n \geq 0$ such that $\sum r^i = 1$. Recall that $\mathbf{Bad}^{\bar r}_{\mathbb{R}^n}$ is the set of points $\bar x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a positive constant $c(\bar x) > 0$ such that

$$\max_{i=1,\ldots,n} |qx_i - p_i|^{1/r^i} \geq c(\bar x)/q,$$

for every $q \in \mathbb{N}$ and $\bar p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$.

As in [34, 60], we let let $\bar\Omega = \mathbb{R}^n \times \mathbb{R}$ and define $\bar\psi_{\bar r} : \bar\Omega \to \mathcal{C}(\mathbb{R}^n)$ to be monotonic function given by the rectangle determined by $\bar r$, that is, the product of metric balls

$$\bar\psi_{\bar r}(\bar x, t) \equiv B(x_1, e^{-(1+r^1)t}) \times \cdots \times B(x_n, e^{-(1+r^n)t}).$$

Denote by $r^+ \equiv \max\{r^i\}$ and by $r^- \equiv \min\{r^i\}$ which we assume to be non-zero. Clearly, we have $\mathrm{diam}(\bar\psi_{\bar r}(x,t)) \leq 2e^{-(1+r^-)t}$, hence $\sigma = 1 + r^-$. Moreover, for $c > 0$ it is readily checked that

$$d^c = \tfrac{\log(1+e^{-(1+r^-)c})}{1+r^-} \leq \tfrac{\log(2)}{1+r^-}, \qquad d_c = -\tfrac{\log(1-e^{-(1+r^+)c})}{1+r^+}.$$

In the following, let $\mathcal{S}$ be the set of affine hyperplanes in $\mathbb{R}^n$. Note that, if μ denotes the Lebesgue-measure on $\mathbb{R}^n$, it follows from [31], Lemma 9.1 (recall also Lemma 2.20), that $(\Omega, \psi_{\bar r}, \mu)$ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$ for $\delta = 1 + \min\{r^1, \ldots, r^n\} = 1 + r^-$ and some $c_\delta > 0$. Moreover, $(\Omega, \psi_{\bar r}, \mu)$ satisfies a power law with respect the exponent $n+1$; in fact, for all $(x,t) \in \Omega$ we have $\mu(\bar\psi_{\bar r}(x,t)) = 2^n e^{-(n+1)t}$.

For $c > 0$, define

$$\begin{aligned} l_* &= \tfrac{\log(n!)}{n+1} + \tfrac{n}{n+1}\log(2) + \tfrac{\log(3)}{1+r^-}, \\ u_c &= \tfrac{1}{r^-}(c + (1+r^+)\log(2)) + \log(n+1) + 2\log(2) \equiv \frac{c}{r^-} + u_*, \\ d_* &= \tfrac{\log(3)}{1+r^-}. \end{aligned}$$

Theorem 3.18. *Let $X \subset \mathbb{R}^n$ be the support of a locally finite Borel measure μ on $\mathbb{R}^n$ such that $(\Omega, \psi_{\bar r}, \mu)$ satisfies a power law with respect to the parameters (τ, c_1, c_2).*

For the lower bound, assume that $(\Omega, \psi_{\bar r}, \mu)$ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$. Then, for $c > \log(c_\delta)/\delta + 2d_*$, we have

$$\dim(\boldsymbol{Bad}^{\bar r}_{\mathbb{R}^n}(\tfrac{1}{2}e^{-(1+r^+)(2c+l_*)}) \cap X) \geq d_\mu(X) - \frac{\log(2)+2\log(\frac{c_2}{c_1})+\tau(d_*+d_c)+|\log(1-c_\delta e^{2\delta d_*}e^{-\delta c})|}{(1+r^-)c}.$$

For the upper bound, assume that (Ω, B_1, μ) satisfies also a power law with respect to the exponent δ and with $X = \mathbb{R}^n$. Then, for $c > 0$, we have

$$\dim(\boldsymbol{Bad}^{\bar r}_{\mathbb{R}^n}(e^{-(1+\frac{1}{r^+})c})) \quad \leq \quad \delta - \frac{\log(\frac{c_1}{c_2})-\tau(d_*+d^c)+|\log(1-\frac{c_1}{c_2}e^{-\tau(d_*+d^c)}.e^{-\tau(c+u_c)})|}{(1+r^+)(c+u^c)}.$$

The theorem will be sharpened for the standard case when μ is the Lebesgue measure.

Proof. For $k \in \Lambda \equiv \mathbb{N}$ we define the set of rational vectors

$$R_k \equiv \{\bar p/q : \bar p \in \mathbb{Z}^n, 0 < q \leq k\}$$

as resonant set and define its size by $s_k \equiv \log(k+1)$. The family $\mathcal{F} = (\mathbb{N}, R_k, s_k)$ is nested and discrete. Moreover, since $R(t)$ is a discrete set for all $t \geq s_1$ and $\psi_{\bar{r}}$ is a product of metric balls, it is readily checked that $(\Omega, \psi_{\bar{r}})$ is $\frac{\log(2)}{1+r^-}$-separating with respect to $\mathcal{F}$, and that $(\Omega, \bar{\psi}_{\bar{r}})$ is d_*-separating.

For the lower bound, choose any $\bar{l}_* > \log(n!)/(n+1) + n/(n+1)\log(2)$. Note that for a formal ball $\omega = (\bar{x}, t + \bar{l}_*)$ such that $t \leq s_k$ the sidelights ρ_i of the box $\psi_{\bar{r}}(\omega)$ satisfy

$$\begin{aligned} \rho_1 \cdots \rho_n &= 2e^{-(1+r^1)(t+l_*)} \cdots 2e^{-(1+r^n)(t+l_*)} \\ &< 2^n e^{-(1+n)s_k - \log(n!) - n\log(2)} \leq \tfrac{1}{n!(k+1)^{n+1}}. \end{aligned}$$

We now use the following version of the 'Simplex Lemma' due to Davenport and Schmidt where the version of this lemma can be found in [35], Lemma 4 or in Lemma 2.21.

Lemma 3.19. *Let $D \subset \mathbb{R}^n$ be a box of side lengths $\rho_1, \ldots, \rho_n$ such that $\rho_1 \ldots \rho_n < 1/(n!(k+1)^{n+1})$. Then there exists an affine hyperplane L such that $R_k \cap D \subset L$.*

This shows that $\mathcal{F}$ is locally contained in the collection of affine hyperplanes $\mathcal{S}$ with $n_* = 1$. Since $(\Omega, \psi_{\bar{r}}, \mu)$ is absolutely (δ, c_δ)-decaying with respect to $\mathcal{S}$, if follows from Proposition 3.15 that $(\Omega, \psi_{\bar{r}}, \mu)$ is τ_c-decaying with respect to $\mathcal{F}$ for all $c > 2d_*$ where $l_c \equiv l_* = \bar{l}_* + d_*$ and $\tau_c = c_\delta e^{-\delta(c-2d_*)}$. Note that $c > 0$, such that $\tau_c < 1$, is given when $c > \log(c_\delta)/\delta + 2d_*$.

Finally, if $\bar{x} \in \mathbf{Bad}_X^{\bar{\psi}_{\bar{r}}}(\mathcal{F}, c)$, then for every $\bar{p}/q$, where $\bar{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, and $q \in \mathbb{N}$, $\bar{x} \notin \bar{\psi}_{\bar{r}}(R_q, s_q + c) \supset \bar{\psi}_{\bar{r}}(\bar{p}/q, s_q + c)$. Hence, for some $i \in \{1, \ldots, n\}$, we have

$$|x_i - p_i/q| \geq e^{-(1+r^i)(s_q+c)} \geq \frac{e^{-(1+r^+)c}}{(q+1)^{1+r^i}} \geq \frac{e^{-(1+r^+)c}}{2q^{1+r^i}},$$

which shows that $\mathbf{Bad}_X^{\psi_{\bar{r}}}(\mathcal{F}; 2c + l_*) \subset \mathbf{Bad}_{\mathbb{R}^n}^{\bar{r}}(\frac{1}{2}e^{-(1+r^+)(2c+l_*)}) \cap X$. Applying (3.2.18), the formula for the lower bound, together with (3.2.10) gives that $\dim(\mathbf{Bad}_{\mathbb{R}^n}^{\bar{r}}(\frac{1}{2}e^{-(1+r^+)(2c+l_*)}) \cap X)$ is bounded below by

$$\begin{aligned} & d_\mu(X) - \frac{\log(2\frac{c_2}{c_1}e^{-\tau(c-d_*-d_c)}) - \log(1 - c_\delta e^{-\delta(c-2d_*)}) - \log(\frac{c_1}{c_2}e^{-\tau c})}{\sigma c}. \\ = \; & d_\mu(X) - \frac{\log(2) + 2\log(\frac{c_2}{c_1}) + \tau(d_* + d_c) + |\log(1 - c_\delta e^{2\delta d_*} e^{-\delta c})|}{(1+r^-)c}. \end{aligned}$$

For the upper bound, note that using the pigeon-hole lemma as for the classical Dirichlet Theorem, the following Lemma can be shown.

Lemma 3.20. *Let $x \in \mathbb{R}^n$. For every $N \in \mathbb{N}$ there exists a vector $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $1 \leq q \leq (n+1)N$ such that, for $i = 1, \ldots, n$, we have*

$$|x_i - \frac{p_i}{q}| \leq \frac{1}{qN^{r^i}}.$$

Define $u_c = c/r^- + u_*$ as given above and let $\bar{x} \in X$. Given t_k, let $N \in \mathbb{N}$ be the maximal integer such that $t_k \geq \log((n+1)N) + \log(2)$; hence $t_k \leq \log((n+1)(N+1)) + \log(2)$. Let $\bar{p}$, q

as in the above lemma and note that $s_q \leq \log(q) + \log(2) \leq t_k$. In the case when $\log(q) \leq t_{k-1} = t_k - (c + u^c)$, we have

$$\begin{aligned} |x_i - \frac{p_i}{q}| &\leq \frac{1}{qN^{r^i}} \\ &\leq \frac{2^{r^i}}{q(N+1)^{r^i}} \\ &\leq e^{-\log(q) - r^i(t_k - \log(n+1) - 2\log(2))} \\ &\leq e^{-(1+r^i)\log(q) - r^i(c+u_c - \log(n+1) - 2\log(2))} \\ &\leq e^{-(1+r^i)(\log(q+1)+c)} = e^{-(1+r^i)(s_q+c)}, \end{aligned}$$

for every $i = 1, \dots, n$, since

$$\begin{aligned} r^i(u_c - \log(n+1) - 2\log(2)) &= \tfrac{r^i}{r^-}(c + (1+r^+)\log(2)) \\ &\geq c + (1+r^i)(\log(q+1) - \log(q)). \end{aligned}$$

This shows

$$\bar{x} \in \bar{\psi}_{\bar{r}}(\bar{p}/q, s_q + c) \subset \bigcup_{s_q \leq t_{k-1}} \bar{\psi}_{\bar{r}}(R_q, s_q + c).$$

Hence, we may assume $\log(q) > t_k - (c + u^c)$ and obtain that for every $i = 1, \dots, n$,

$$\begin{aligned} |x_i - \frac{p_i}{q}| &\leq e^{-\log(q) - r^i t_k} \\ &\leq e^{-(1+r^i)t_k + (c+u_c))} \leq e^{-(1+r^i)(t_k - u_c)}, \end{aligned}$$

since $r^i u^c \geq c$. This yields $\bar{p}/q \in \bar{\psi}_{\bar{r}}(\bar{x}, t_k - u^c)$. Since by assumption $X = \mathbb{R}^n$, replacing u_* by u^c in the proof of Proposition 3.17 shows that $(\Omega, \psi_{\bar{r}}, \mu)$ is τ^c-Dirichlet with respect to $\mathcal{F}$ and the parameters (c, u_c), where $\tau^c = \frac{c_1}{c_2} e^{-\tau(d_* + d^c)} \cdot e^{-\tau(c+u^c)}$.

Finally, let $\bar{x} \in \mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}(e^{-(1+\frac{1}{r^+})c})$, where $c > 0$. Thus, for every $\bar{p}/q$ with $\bar{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ there exists $i \in \{1, \dots, n\}$ such that

$$|qx_i - p_i|^{1/r^i} > e^{-(1+\frac{1}{r^+})c}/q.$$

Equivalently, we have

$$|x_i - p_i/q| > (e^{-(1+\frac{1}{r^+})c})^{r^i} q^{-(1+r^i)} \quad \geq \quad e^{-(1+r^i)(s_q+c)},$$

which shows that $\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}(e^{-(1+\frac{1}{r^+})c}) \cap X \subset \mathbf{Bad}^{\bar{\psi}_{\bar{r}}}_X(\mathcal{F}; c)$. Applying (3.2.26) with $\bar{\sigma} = 1 + r^+$ and $d^* = 2\log(2)$, yields the upper bound

$$\begin{aligned} \dim(\mathbf{Bad}^{\bar{r}}_{\mathbb{R}^n}(e^{-(1+\frac{1}{r^+})c}) \cap X) &\leq \delta - \frac{|\log(1-\tau^c)| + \log(\frac{c_1}{c_2}) - \tau(d_* - d^c)}{(1+r^+)(c+u^c)} \\ &\leq \delta - \frac{|\log(1 - \frac{c_1}{c_2} e^{-\tau(d_*+d^c)} \cdot e^{-\tau(c+u_c)})| + \log(\frac{c_1}{c_2}) - \tau(d_* + d^c)}{(1+r^+)(c+u^c)}. \end{aligned}$$

This finishes the proof. □

The Standard Case. Let $X = \mathbb{R}^n$ and $\sigma = 1 + 1/n$. For the lower bound, we let $c = \log(m)/\sigma > \bar{d}_* + \log(2)/\sigma$ be sufficiently large for some $m \in \mathbb{N}$ (such that $\bar{\tau}_c < 1$ below). Note that we nowhere used the condition that $x \in L_k^{B_{1+1/n}}(c)$ which hence becomes obsolete in this setting. Thus, Lemma 3.11 shows that (3.2.20) is satisfied for the parameters $\bar{l}_c = \bar{l}_* + d_* + a$, where $a \equiv 2\sqrt{n} + \log(2) + d_c$, and $\bar{\tau}_c \leq e^{n\sigma(a-d_c)}\tau_c \equiv \bar{k}_l e^{-(1+1/n)c}$. Thus, (3.2.21) yields the lower bound

$$\begin{aligned} \dim(\mathbf{Bad}^n_{\mathbb{R}^n}(\tfrac{1}{2}e^{-(1+1/n)(2c+\bar{l}_c)})) &\geq \dim(\mathbf{Bad}^{Q_{1+1/n}}_{\mathbb{R}^n}(\mathcal{F}, 2c+\bar{l}_c)) \\ &\geq n - \frac{|\log(1-\bar{k}_l\, e^{-(1+1/n)c})|}{(1+1/n)c}. \end{aligned}$$

Up to modifying $\bar{k}_l$ to a suitable constant depending on $c_0 > 0$ sufficiently large, the lower bound also follows for general $c > c_0$.

For the upper bound, for $c > 0$ we let $\tilde{c} = c + a$, where $a = d^c + \bar{d}_* + \log(2)/\sigma + \sqrt{n}/\sigma$ and and $\bar{u}^c \geq u^{\tilde{c}} + a$ such that $c + \bar{u}^c = \log(m)/\sigma$ (with m minimal). Recall that from Lemma 3.14, $U^{Q_\sigma}_{k-1}(c+\bar{d}_*) \subset \bigcap_{s_n \leq \bar{t}_k - \tilde{c} + d^{\tilde{c}}} B_\sigma(R_n, s_n + \tilde{c})^C$. Moreover, we remark that in the above arguments for determining τ^c, it was in fact nowhere necessary to require $t = t_k$ and we showed that, if $(x, \bar{t}_k) \in \Omega$ with $x \in U^{Q_\sigma}_{k-1}(c+\bar{d}_*)$ then (3.2.29) is satisfied. Hence, Lemma 3.14 implies (3.2.27) with respect to $\bar{\tau}^c \geq e^{-n\sqrt{n}}\tau^{\tilde{c}} \equiv \bar{k}_u e^{-(n+1)^2 c}$. Finally, since m above was chosen minimal there exists a constant $k_u \geq 0$ (independent on c) such that $\tilde{c} + u^{\tilde{c}} \leq (n+1)c + k_u$, and (3.2.28) shows

$$\begin{aligned} \dim(\mathbf{Bad}^n_{\mathbb{R}^n}(e^{-(1+n)c})) &\leq \dim(\mathbf{Bad}^{Q_{1+1/n}}_{\mathbb{R}^n}(\mathcal{F}, c)) \\ &\leq n - \frac{|\log(1-\bar{k}_u e^{-(n+1)^2 c})|}{(1+1/n)(n+1)(c+k_u)}. \end{aligned}$$

This proves Theorem 3.2.

Remark 3.21. Let $n = 2$ and $X = \mathbb{R}^2$. If $\frac{1+r_1}{1+r_2} \in \mathbb{Q}$, then there exist parameters $c > 0$ such that $c = \frac{\log(m_1)}{1+r_1} = \frac{\log(m_2)}{1+r_2}$ with $m_1, m_2 \in \mathbb{N}$. For these parameters a partition of the rectangles $\psi_{\bar{r}}(x,t)$ as in (3.2.19) is possible, which in turn, following the arguments of the formalism, allows more precise bounds as in the standard case.

3.3.2 The Bernoulli shift Σ^+

For $n \geq 1$, let $\Sigma^+ = \{1, \ldots, n\}^{\mathbb{N}}$ be the set of one-sided sequences in symbols from $\{1, \ldots, n\}$. Let T denote the shift and let d^+ be the metric given by $d^+(w, \bar{w}) \equiv e^{-\min\{i \geq 1 : w(i) \neq \bar{w}(i)\}}$ for $w \neq \bar{w}$ and $d(w, w) \equiv 0$.

Fix a periodic word $\bar{w} \in \Sigma^+$ of period $p \in \mathbb{N}$. For $c \in \mathbb{N}$, consider the set

$$S_{\bar{w}}(c) = \{w \in \Sigma^+ : T^k w \notin B(\bar{w}, e^{-(c+1)}) \text{ for all } k \in \mathbb{N}\}.$$

Theorem 3.22. *For every $c \in \mathbb{N}$ we have*

$$\dim(S_{\bar{w}}(c)) \leq \log(n) - \frac{|\log(1-n^{-c})|}{c},$$

as well as

$$\dim(S_{\bar{w}}(2c+p+1)) \geq \log(n) - \frac{|\log(1-n^{-c})|}{c}.$$

Remark 3.23. Note that the Morse-Thue sequence w in $\{0,1\}^{\mathbb{N}}$ is a particular example of a word in $S_{\bar{w}}(2p)$ for any periodic word $\bar{w}$ or period p. In fact, w does not contain any subword of the form WWa where a is the first letter of the subword W; for details and more general words in $S_{\bar{w}}$, we refer to Section 4.3.

Proof. For $k \in \mathbb{N}$ and $w_k \in \{1, .., n\}^k$, let $\bar{w}_k \in \Sigma^+$ denote the word $\bar{w}_k = w_k \bar{w}$. Let $\Lambda \equiv \mathbb{N}_0$ and consider the resonant sets

$$R_0 = \{\bar{w}\}, \qquad R_k = \{\bar{w}_l \in \Sigma^+ : w_l \in \{1, .., n\}^l, l \leq k\}) \cup R_0, \text{ for } k \in \mathbb{N}$$

which we give the size $s_k = k + 1$. Then, the family $\mathcal{F} = (\mathbb{N}_0, R_k, s_k)$ is nested and discrete.

Note that we let $\Omega = \Sigma^+ \times \mathbb{N}$ and consider the standard function ψ_1. Hence, we have $d_l(c) = d_u(c) = d_* = 0$. Moreover, we have $\mathbf{Bad}(\mathcal{F}, c) = S_{\bar{w}}(c)$. In fact, $d^+(T^{k-1}w, \bar{w}) \leq e^{-(c+1)}$ if and only if $w(k) \ldots w(k+c) = \bar{w}(1) \ldots \bar{w}(c)$. Thus, for $w_k = w(1) \ldots w(k)$ and $\bar{w}_k = w_k \bar{w}$ we have $d^+(w, \bar{w}_k) \leq e^{-(k+c+1)}$ if and only if $w \in B(\bar{w}_k, e^{-(s_k+c)}) \subset \psi_1(R_k, s_k + c)$.

For the lower bound, let $\bar{w}_m$ and $\tilde{w}_m \in R_m$ be distinct. By definition of $\bar{w}_m$ and $\tilde{w}_m$ there exists $i \in \{1, \ldots, m+p\}$ such that $\bar{w}_m(i) \neq \tilde{w}_m(i)$; hence

$$d^+(\bar{w}_m, \tilde{w}_m) \geq e^{-(p+m+1)} = e^{-p} e^{-s_m}$$

and we are given the special case (3.2.33) with $\bar{c} = e^{-p}$. Moreover, for the probability measure $\mu = \{1/n, \ldots, 1/n\}^{\mathbb{N}}$, (Ω, B_1, μ) satisfies

$$\mu(B(w, e^{-(t+1)})) = n^{-t} = n e^{-\log(n)(t+1)},$$

and hence a $(\log(n), n, n)$-power law. From Lemma 3.16 we see that (Ω, B_1, μ) is $(\log(n), 1)$-decaying with respect to $\mathcal{F}$ and $l_* = p + 1$. Applying (3.2.18), we obtain

$$\dim(S_{\bar{w}}(2c + p + 1)) \geq \log(n) - \frac{\log(2) + |\log(1 - n^{-c})|}{c}.$$

Finally, note that, checking the arguments in (3.2.16) (and (3.2.15) respectively), we can see that the constant 'log(2)' can be omitted. (In fact, we even have a partition as in (3.2.19)).

For the upper bound, let $(w, s_k) = (w, k+1) \in \Omega$. If $w_k \equiv w(1) \ldots w(k)$, let $\bar{w}_k \equiv w_k \bar{w} \in R_k$ which lies in $B(w, e^{-s_k})$; hence, $R_k \cap \psi_1(w, s_k) \neq \emptyset$. Thus, Lemma 3.17 shows that (Ω, B_1, μ) is $(\log(n), 1)$-Dirichlet with respect to $\mathcal{F}$ for $u_* = 0$. Hence, (3.2.25) yields

$$\dim(S_{\bar{w}}(c)) \leq \log(n) - \frac{|\log(1 - n^{-c})|}{c},$$

finishing the proof. □

3.3.3 The geodesic flow in $\mathbb{H}^{n+1}$

Although the following setting is even suitable for proper geodesic CAT(-1) metric spaces, we restrict to the real hyperbolic space $\mathbb{H}^{n+1}$. The reason is that, given a geometrically finite Kleinian group Γ, there exists a nice measure satisfying the *Global Measure Formula* (see Theorem 3.31). We start by introducing the setting and a model of Diophantine approximation developed by Hersonsky, Paulin and Parkkonen in [25, 26, 46], which allows a dynamical interpretation of badly approximable elements.

In the following, $\mathbb{H}^{n+1}$ denotes the $(n+1)$-dimensional real hyperbolic ball-model. For $o \in \mathbb{H}^{n+1}$, we define the *visual metric* $d_o : S^n \times S^n \to [0,\infty)$ at o by $d_o(\xi,\xi) \equiv 0$ and

$$d_o(\xi,\eta) \equiv e^{-(\xi,\eta)_o},$$

for $\xi \neq \eta$, where $(\cdot,\cdot)_o$ denotes the Gromov-product at o. Note that if $o = 0$ is the center of the ball $\mathbb{H}^{n+1}$ then the visual distance d_0 is bi-Lipschitz equivalent to the angle metric on the unit sphere S^n. The boundary $S^n = \partial_\infty \mathbb{H}^{n+1}$ is a compact metric space with respect to d_o and we will consider all metric balls to be with respect to d_o in the following.

Let Γ be a discrete subgroup of the isometry group $I(\mathbb{H}^{n+1})$ of $\mathbb{H}^{n+1}$. The *limit set* $\Lambda\Gamma$ of Γ is given by the set $\overline{\Gamma.o} \cap S^n$, which is the set of all accumulation points of subsequences from $\Gamma.o \equiv \{\varphi(o) : \varphi \in \Gamma\}$. Recall that a subgroup $\Gamma_0 \subset \Gamma$ is called *convex cocompact* if $\Lambda\Gamma_0$ contains at least two points and the action of Γ_0 on the convex hull $\mathcal{C}\Gamma_0$ has compact quotient. We call Γ_0 *bounded parabolic* if Γ_0 is the maximal subgroup of Γ stabilizing a parabolic fixed point $\xi_0 \in \Lambda\Gamma$ and Γ_0 acts cocompactly on $\Lambda\Gamma - \{\xi_0\}$. Moreover, we call Γ_0 *almost malnormal* if $\varphi.\Lambda\Gamma_0$[5] $\cap \Lambda\Gamma_0 = \emptyset$ for every $\varphi \in \Gamma - \Gamma_0$.

Let Γ be a geometrically finite group where we refer to [49] for the following. Recall that for the convex hull $\mathcal{C}\Gamma$ of $\Lambda\Gamma$, the subset $\mathcal{C}\Gamma \cap \mathbb{H}^{n+1}$ of $\mathbb{H}^{n+1}$ is closed, convex and Γ-invariant. The convex core $\mathcal{C}M \subset M$ of $M = \mathbb{H}^{n+1}/\Gamma$ is the convex closed connected set

$$\mathcal{C}M \equiv (\mathcal{C}\Gamma \cap \mathbb{H}^{n+1})/\Gamma = K \cup \bigcup_i V_i,$$

which can be decomposed into a compact set K, and, unless Γ is convex cocompact, finitely many open disjoint sets V_i corresponding to the conjugacy classes of maximal parabolic subgroups of Γ which are bounded parabolic and almost malnormal. Moreover, if π denotes the projection to $M = \mathbb{H}^n/\Gamma$ we may assume that each $V_i = \pi(C_i) \cap \mathcal{C}M$ is the projection of a horoball C_i in $\mathcal{C}M$, where the collection $\varphi(C_i)$, $\varphi \in \Gamma -$ $\mathrm{Stab}_\Gamma(C_i)$, is disjoint.

The setting.

Let Γ be a geometrically finite group without elliptic elements as above and $\Gamma_i \subset \Gamma$, $i = 1,2$, be an almost malnormal subgroup in Γ of infinite index. We treat the following two 'disjoint' cases simultaneously.

1. There is precisely one conjugacy class of a maximal parabolic subgroup Γ_1 of Γ. Let m be the rank of Γ_1 and let C_1 be a horoball based at the parabolic fixed point ξ_0 of Γ_1 as above.

2. Let Γ be convex-cocompact such that $\Lambda\Gamma \subset S^n$ is not contained in a finite union of spheres of S^n of codimension at least 1. Let Γ_2 be a convex-cocompact subgroup and $C_2 = \mathcal{C}\Gamma_2$ be the convex hull of Γ_2 which is a hyperbolic subspace (that is, C_2 is totally geodesic and isometric[6] to the hyperbolic space $\mathbb{H}^m$).

Remark 3.24. The requirements that there is only one parabolic subgroup in *Case 1.* or that Γ itself is convex-cocompact in *Case 2.* will be necessary in the Global Measure Formula. In fact, we need to control the 'depth of geodesic rays in the cuspidal end' which would not be possible in *Case 2.* if $\mathcal{C}M$ was not compact.

[5] Note that an isometry φ of $\mathbb{H}^{n+1}$ extends to a homeomorphism of S^n. We denote the image of a set $S \subset S^n$ under φ by $\varphi.S$.

[6] With respect to the induced metric on C_2.

Note that, since Γ_i is almost malnormal, we have $\Gamma_i = \mathrm{Stab}_\Gamma(C_i)$ so that Γ_i is determined by C_i. In addition, C_i is (ε, T)-*embedded*, that is, for every $\varepsilon > 0$ there exists $T = T(\varepsilon) \geq 0$ such that for all $\varphi \in \Gamma - \Gamma_i$ we have that $\mathrm{diam}(\mathcal{N}_\varepsilon(C_i) \cap \varphi(\mathcal{N}_\varepsilon(C_i)) \leq T$; see [46]. In the first case, we therefore assume, after shrinking C_1, that the images $\varphi(C_1)$, $[\varphi] \in \Gamma/\Gamma_1$, form a disjoint collection of horoballs. For the second case, we let $\varepsilon = \delta_0$ and $T_0 = T(2\delta_0)$ where δ_0 is the constant such that $\mathbb{H}^{n+1}$ is a tripod-δ_0-hyperbolic space.

Example 3.25. Clearly, if $M = \mathbb{H}^{n+1}/\Gamma$ is a finite volume hyperbolic manifold with exactly one cusp, then *Case 1.* is satisfied with $m = n$. If Γ is even cocompact, then every closed geodesic α in M determines a subgroup Γ_2 as in *Case 2.* and C_2 (a lift of α) is one-dimensional. Moreover, T can be estimated in terms of the length of α and the length of a systole of M.

A model of Diophantine approximation and the main result.

Given Γ, Γ_i, $i = 1, 2$, as above, we fix a base point $o \in \mathbb{H}^{n+1}$ such that $\pi(o) \in K$. For technical reasons, we also fix a sufficiently large constant $t_0 \geq 0$. For the respective cases, $i = 1, 2$, denote the quadruple of data by

$$\mathcal{D}_i = (\Gamma, C_i, o, t_0).$$

For $r = [\varphi] \in \Gamma/\Gamma_i$ we define

$$D_i(r) = d(o, \varphi C_i)$$

which does not depend on the choice of the representative φ of r. Note that the set $\{D_i(r) : r \in \Gamma/\Gamma_i\}$ is discrete and unbounded (see [46, 60]); that is, for every $D \geq 0$ there are only finitely many elements $r \in \Gamma/\Gamma_i$ such that $D_i(r) \leq D$ and there exists an $r \in \Gamma/\Gamma_i$ with $D_i(r) > D$.

Now, for $i = 1, 2$ and for $\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i$ define the *approximation constant*

$$c_i(\xi) = \inf_{r=[\varphi]\in\Gamma/\Gamma_i:\ D(r)>t_0} e^{D_i(r)} d_o(\xi, \varphi.\Lambda\Gamma_i),$$

If $c_i(\xi) = 0$ then ξ is called *well approximable*, otherwise it is called *badly approximable* (with respect to $\mathcal{D}_i$). Define the set of badly approximable limit points by

$$\mathbf{Bad}(\mathcal{D}_i) = \{\xi \in \Lambda\Gamma - \Gamma.\Lambda\Gamma_i : c_i(\xi) > 0\} \subset \Lambda\Gamma,$$

and $\mathbf{Bad}(\mathcal{D}_i, e^{-c})$ the subset of elements for which $c_i(\xi) \geq e^{-c}$.

Theorem 3.26. *Let δ be the Hausdorff-dimension of $\Lambda\Gamma$ and $\tau > 0$ be the exponent of Theorem 3.32 below. There exists $c_0 > 0$ and geometric constants $k_l, \bar{k}_l, k_u, \bar{k}_u, \tilde{k}_u > 0$, determined in the following, such that for all $c > c_0$ we have*

$$\delta - \frac{k_l + |\log(1 - \bar{k}_l\ e^{-(2\delta - m)c/2})|}{c/2 - (\delta_0 + \log(2))} \leq \dim(\mathbf{Bad}(\mathcal{D}_1, e^{-c})) \leq (2\delta - m) - \frac{|\log(1 - \bar{k}_u\ e^{-(3\delta - m)c})| - \tilde{k}_u}{2c + k_u},$$

as well as

$$\delta - \frac{k_l + |\log(1 - \bar{k}_l\ e^{-\tau c/2})|}{c/2 - (T_0 + \delta_0 + 2\log(3))} \leq \dim(\mathbf{Bad}(\mathcal{D}_2, e^{-c})) \leq \delta - \frac{|\log(1 - \bar{k}_u\ e^{-\delta c})| - \tilde{k}_u}{c + k_u}.$$

Remark 3.27. It is well known (see [43]) that $2\delta \geq m$. In fact, it follows from the lower and upper bound that $\delta \geq m$ in our case. Therefore, the upper bound is only suitable for $c > 0$ such that the right hand side is smaller than the trivial bound δ. For the second case, note that if C_2 is an axis, we can choose $\tau = \delta$. We moreover expect that τ is dependent on the dimension of C_2 (and of course on δ).

In the special case, when Γ is of the *first kind*, that is $\Lambda\Gamma = S^n$ (for instance if Γ is a lattice), we can improve the above theorem to the following.

Theorem 3.28. *Let again $\tau > 0$ be the exponent of Theorem 3.32 below. If in addition $\Lambda\Gamma = S^n$, then there exists $c_0 > 0$ and geometric constants $k_l, \bar{k}_l, k_u, \bar{k}_u$*[7] *$> 0$, such that for all $c > c_0$ we have*

$$n - \frac{|\log(1 - \bar{k}_l\ e^{-nc/2})|}{c/2 - k_l} \leq \dim(\mathbf{Bad}(\mathcal{D}_1, e^{-c})) \leq n - \frac{|\log(1 - \bar{k}_u\ e^{-2nc})|}{2c + k_u},$$

as well as

$$n - \frac{|\log(1 - \bar{k}_l\ e^{-\tau c/2})|}{c/2 - k_l} \leq \dim(\mathbf{Bad}(\mathcal{D}_2, e^{-c})) \leq n - \frac{|\log(1 - \bar{k}_u\ e^{-nc})|}{c + k_u}.$$

The above theorem and the following dynamical interpretation of the set $\mathbf{Bad}(\mathcal{D}_i, e^{-c})$ yield the proof of Theorem 3.3. Note that this is Lemma 2.32 which is in the context of CAT(-1)-spaces.

Lemma 3.29. *There exist positive constants c_0, $\kappa_0 > 0$ (we may assume $\kappa_0 \geq 1$) and $t_0 \geq 0$ such that, if C_1 is a horoball based at $\partial_\infty C = \eta \in \partial_\infty \mathbb{H}^n$ or C_2 is a hyperbolic subspace with $d(o, C_i) \geq t_0$, then for all $\xi \in \Lambda\Gamma$ and $c > c_0$ we have*

1. $\gamma_{o,\xi}([t, t + c]) \subset C_1$,
2. $\gamma_{o,\xi}([t, t + c]) \subset \mathcal{N}_{\delta_0}(C_2)$,

for some $t \geq d(o, C_i)$, if and only if

1. $d_o(\xi, \eta) \leq \kappa_0\ e^{-c/2} \cdot e^{-d(o,C_1)}$,
2. $d_o(\xi, \partial_\infty C_2) \leq \kappa_0\ e^{-c} \cdot e^{-d(o,C_2)}$.

A measure on $\Lambda\Gamma$.

Let $o = 0$ be the center so that the visual distance d_o is bi-Lipschitz equivalent to the angle metric on the unit sphere S^n. Hence, if Γ is of the first kind, then the Lebesgue measure on S^n satisfies a power law with respect to the visual metric d_o and the exponent n. More generally, recall that the *critical exponent* of a discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ is given by

$$\delta(\Gamma) \equiv \inf \{ s > 0 : \sum_{\varphi \in \Gamma} e^{-sd(x,\varphi(x))} < \infty \},$$

for any $x \in \mathbb{H}^{n+1}$. If Γ is non-elementary and discrete then the Hausdorff-dimension of the conical limit set of $\Lambda\Gamma$ equals $\delta(\Gamma)$ and if Γ is moreover geometrically finite, then $\dim(\Lambda\Gamma) = \delta(\Gamma)$ (see [6]).

[7] The constants may differ from the ones in the proof.

Moreover, associated to Γ, there is a canonical measure, the *Patterson-Sullivan* measure μ_Γ, which is a $\delta(\Gamma)$-conformal probability measure supported on $\Lambda\Gamma$. For a precise definition we refer to [43]. There are various results concerning the Patterson-Sullivan measure. Here, we will make use of the following.

Let Γ be a geometrically finite Kleinian group as in *Cases 1.* and *2.* above. Let moreover D_0 be the diameter of the compact part K of the convex core $\mathcal{C}M$ of M.

For a limit point $\xi \in \Lambda\Gamma$, we let $\gamma_{o,\xi}$ be the unique geodesic ray starting in o and asymptotic to ξ. In *Case 1.* define the *depth* $D_t(\xi)$ of the point $\gamma_{o,\xi}(t)$ in the collection of horoballs $\{\varphi(C_1)\}_{\varphi\in\Gamma}$, where $D_t(\xi) \equiv 0$ if $\gamma_{o,\xi}(t)$ does not belong to $\cup_{\varphi\in\Gamma}\varphi(C_1)$, and $D_t(\xi) \equiv d(\gamma_{o,\xi}(t), \partial\varphi(C_1))$ otherwise; in *Case 2.* we simply set $D_t(\xi) = 0$ for all $t > 0$.

We need the following Lemma.

Lemma 3.30. *We have*

$$D_t(\xi) \leq d(\gamma_{o,\xi}(t), \Gamma.o) \leq D_t(\xi) + 4\log(1+\sqrt{2}) + D_0.$$

Proof. By the arguments given below, the proof is obvious if Γ is convex-cocompact (and hence the set V is empty) and we may assume that we are given *Case 1.* Recall that the convex core $\mathcal{C}M = (\mathcal{C}\Gamma \cap \mathbb{H}^{n+1})/\Gamma$ consists of (the disjoint union of) the compact set K and the set V which we may assume to be the projection of $C_1 \cap \mathcal{C}\Gamma$. Since $\mathcal{C}\Gamma$ is convex and $o \in \mathcal{C}\Gamma$, for every limit point $\xi \in \mathcal{C}\Gamma$ the ray $\gamma_{o,\xi}(\mathbb{R}^+)$ is contained in $\mathcal{C}\Gamma$ and hence covered by lifts of K and of V. Since $\pi(o) \in K$, if $\gamma_{o,\xi}(t) \in \mathcal{C}\Gamma - \cup_\varphi\varphi(C_1)$ for some $t > 0$, then $d(\gamma_{o,\xi}(t), \Gamma.o) \leq D_0$.

Hence, fix $t > 0$ such that $\gamma_{o,\xi}(t) \in \varphi(C_1) \equiv C$ for some $\varphi \in \Gamma$, where we let $\eta \equiv \varphi(\xi_0)$. If we let t_0 be the entering time of $\gamma_{o,\xi}$ in C, that is, $\gamma_{o,\xi}(t_0) \in \partial C$, then clearly by the above remark and since $\gamma_{o,\xi}(t_0)$ belongs to some lift of K, we have

$$d(\gamma_{o,\xi}(t), \Gamma.o) \leq d(\gamma_{o,\xi}(t), \gamma_{o,\xi}(t_0)) + D_0 \equiv \bar{d} + D_0.$$

Moreover, let $\tilde{C}$ be the horoball based at η (and contained in C) such that $\gamma_{o,\xi}(t) \in \partial\tilde{C}$ and note that $\gamma_{o,\eta}(d(o,C) + D_t(\xi)) \in \partial\tilde{C}$. It then follows from [45], Lemma 2.9, that both $d(\gamma_{o,\xi}(t_0), \gamma_{o,\eta}(d(o,C)))$ and $d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d(o,C) + D_t(\xi)))$ are bounded above by the constant $2\log(1+\sqrt{2})$. This shows

$$\begin{aligned}
\bar{d} &= d(\gamma_{o,\xi}(t), \gamma_{o,\xi}(t_0)) \\
&\leq d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(d(o,C) + D_t(\xi))) + d(\gamma_{o,\eta}(d(o,C) + D_t(\xi)), \gamma_{o,\xi}(t_0)) \\
&\leq 2\log(1+\sqrt{2}) + (D_t(\xi) + d(\gamma_{o,\eta}(d(o,C)), \gamma_{o,\xi}(t_0))) \\
&\leq D_t(\xi) + 4\log(1+\sqrt{2}).
\end{aligned}$$

Finally, since $o \notin C$ (used in the first inequality) we have

$$\begin{aligned}
D_t(\xi) &\leq d(\gamma_{o,\xi}(t), \Gamma.o) \\
&\leq \bar{d} + D_0 \\
&\leq D_t(\xi) + 4\log(1+\sqrt{2}) + D_0,
\end{aligned}$$

proving the claim. □

In the following, let $\mu = \mu_o$ be the Patterson-Sullivan measure given at the base point o. By the above lemma, we can reformulate the *Global Measure Formula* due to [52], Theorem 2, to the following.

Theorem 3.31. *There exist positive constants $c_1, c_2 > 0$ and $t_0 > 0$ such that for all $\xi \in \Lambda\Gamma$ and for all $t > t_0$, we have that*

$$c_1 e^{-\delta t} \cdot e^{-(\delta-m)D_t(\xi)} \leq \mu(B_{d_o}(\xi, e^{-t})) \leq c_2 e^{-\delta t} \cdot e^{-(\delta-m)D_t(\xi)}.$$

In particular, if Γ is convex-cocompact, then μ satisfies a power law with respect to δ.[8]

For the second case, let again $o = 0$ and note that, since Γ_2 is almost malnormal in Γ, C_2 can be of dimension at most $m \leq n$. Moreover, since C_2 is an m-dimensional hyperbolic subspace, the boundary $\partial_\infty C_2 = \Lambda\Gamma_2 \subset \Lambda\Gamma$ of C_2 is an $(m-1)$-dimensional sphere (with respect to d_0). Hence, every image $\varphi.\Lambda\Gamma_2$, $\varphi \in \Gamma$, is contained in the set $\mathcal{H}(\Gamma) \equiv \{S \cap \Lambda\Gamma : S \text{ is a sphere in } S^n \text{ of codimension at least } 1\}$. A finite Borel measure μ on S^n is called $\mathcal{H}(\Gamma)$-*friendly*, if μ is Federer and $(\Lambda\Gamma \times (t_0, \infty), B_1, \mu)$ is absolutely (τ, c_τ)-decaying with respect to $\mathcal{H}(\Gamma)$.

Theorem 3.32 ([53], Theorem 2)**.** *For every non-elementary convex-cocompact discrete group $\Gamma \subset I(\mathbb{H}^{n+1})$ (without elliptic elements), such that $\Lambda\Gamma$ is not contained in a finite union of elements of $\mathcal{H}(\Gamma)$, the Patterson-Sullivan measure μ at o is $\mathcal{H}(\Gamma)$-friendly.*

Note that if we consider only 0-dimensional spheres, we can clearly choose $\tau = \delta$.

The resonant sets.

Let $\bar{\Omega} = \Omega = \Lambda\Gamma \times (t_0, \infty)$, where t_0 is sufficiently large as in Theorem 3.31 and Theorem 3.32 above (as well as Lemma 3.29 and 3.37 below). We are given the discrete set of sizes $\{D_i([\varphi]) : [\varphi] \in \Gamma/\Gamma_i, D_i([\varphi]) > t_0\}$ which we relabel to $\{s^i_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^+$ and reorder such that $s^i_m \leq s^i_k$ for $m \leq k$. For $m \in \Lambda_i \equiv \mathbb{N}$ let

$$\begin{aligned} R^i_m &\equiv \{\xi \in \varphi.\Lambda\Gamma_i : [\varphi] \in \Gamma/\Gamma_i \text{ such that } t_0 < D_i([\varphi]) \leq s^i_m\} \\ &= \{\xi \in \varphi.\Lambda\Gamma_i : [\varphi] \in \Gamma/\Gamma_i \text{ such that } e^{-t_0} > e^{-D_i([\varphi])} \geq e^{-s^i_m}\}. \end{aligned}$$

Since Γ is discrete, for every metric ball $B = B(\xi, e^{-t})$, $(\xi, t) \in \Omega$, only finitely many sets $\varphi.\Lambda\Gamma_i$ with $D_i([\varphi]) \leq t$ can intersect B and it is readily checked that (Ω, B_1) is d_*-contracting with respect to $\mathcal{F}_i$ where $d_* = \log(2)$. Moreover, since $\Lambda\Gamma$ is compact, (Ω, B_1) is $\log(3)$-separating. Also, $d^c \leq \log(2)$ for all $c > 0$ and $d_c \leq \log(2)$ for all $c \geq \log(2)$.

For $\mathcal{F}_i \equiv (\mathbb{N}, R^i_m, s^i_m)$, since $\Lambda\Gamma_i \subset S^n$ is closed (hence compact), we remark that

$$\mathbf{Bad}(\mathcal{D}_i, e^{-c}) = \mathbf{Bad}^{B_1}_{\Lambda\Gamma}(\mathcal{F}_i, c).$$

The lower bound

For the lower bound, recall that the following is shown in Propositon 2.29, using that C_i is $(2\delta_0, T_0)$-embedded: For two different cosets $[\bar{\varphi}]$, $[\varphi] \in \Gamma/\Gamma_i$ let $\eta \in \varphi.\Lambda\Gamma_i$ and $\bar{\eta} \in \bar{\varphi}.\Lambda\Gamma_i$. Then

$$d_o(\eta, \bar{\eta}) \geq e^{-c_i} e^{-\max\{D_i([\varphi]), D_i([\bar{\varphi}])\}}, \tag{3.3.1}$$

where

$$c_1 \equiv \delta_0, \qquad c_2 \equiv T(2\delta_0) + 2\delta_0,$$

[8] The same is true if δ equals m and in particular if Γ is of the first kind in which case μ is equivalent to the Lebesgue measure on S^n.

and δ_0 is the hyperbolicity constant of $\mathbb{H}^{n+1}$ (and i stands for the respective case).

For **Case 2.** we obtain that, for $l_* = c_2 + \log(3)$, for any formal ball $(\xi, t) \in \Omega$ we have

$$B(\xi, e^{-(t+\bar{l}_*)}) \cap R(t) = B(\xi, e^{-(t+\bar{l}_*)}) \cap S,$$

where S is either empty or $S = \varphi.\Lambda\Gamma_2 \in \mathcal{S}$ for some $[\varphi] \in \Gamma/\Gamma_2$. Thus, (3.2.31) is satisfied with $n_* = 1$. Proposition 3.15 and Theorem 3.32 show that (Ω, B_1, μ) is τ_c-decaying with respect to $\mathcal{F}_2$, where $\tau_c = c_\tau e^{-\tau(c-2\log(3))}$, for all $c \geq 2\log(3)$ and the parameters (c, l_c), $l_c = T_0 + \delta_0 + 2\log(3)$. We let $c_0 \geq 2\log(3)$ such that for all $c \geq c_0$ we have $\tau_c < 1$. Recall that (Ω, B_1, μ) satisfies a power law with respect to the parameters (δ, c_1, c_2). Thus, remarking that $\sigma = 1$ and using (3.2.10), (3.2.18) establishes the lower bound

$$\begin{aligned}
\dim(\mathbf{Bad}(\mathcal{D}_2, e^{-(2c+l_c)})) &\geq \delta - \frac{\log(2\bar{k}_c k_c^{-1}) - \log(1-\tau_c)}{c} \\
&\geq \delta - \frac{\log(2) + 2\delta(\log(\frac{c_2}{c_1}) + \log(3)) + |\log(1 - c_\tau e^{2\tau\log(3)} \cdot e^{-\tau c})|}{c}.
\end{aligned}$$

For **Case 1.** we have that (3.2.33) is satisfied for $l_* = \delta_0 + \log(2)$ by (3.3.1). Using the Global Measure Formula, we can determine the required constants.

Proposition 3.33. *For the parameters c, $l_c = \delta_0 + \log(2)$ and $d_c \leq \log(2)$ we have*

$$\begin{aligned}
k_c &\geq \tfrac{c_1}{c_2} e^{-\delta\delta_0} e^{-(2\delta-m)c} \equiv \bar{c}_1 e^{-(2\delta-m)c}, \\
\bar{k}_c &\leq \tfrac{c_2}{c_1} e^{2\delta(d_c+d_*)} e^{-m(d_c+d_*)} e^{-mc} \equiv \bar{c}_2 e^{-mc} \leq \bar{c}_2 e^{(2\delta-m)c} \\
\tau_c &\leq \tfrac{c_2}{c_1} e^{2\delta(d_*+d_c+\delta_0)} e^{-m(\delta_0+d_c-d_*+\delta_0)} e^{-(2\delta-m)c} \equiv \bar{c}_3 e^{-(2\delta-m)c},
\end{aligned}$$

in (3.2.12) and (3.2.13).

Proof. For any $\eta \in B(\xi, e^{-t})$ with t sufficiently large, since $e^{-(\xi,\eta)_o} = d_o(\xi, \eta) \leq e^{-t}$ and $\mathbb{H}^{n+1}$ is a δ_0-tripod-hyperbolic space, we have $d(\gamma_{o,\xi}(t), \gamma_{o,\eta}(t)) < \delta_0$. Hence $|D_t(\xi) - D_t(\eta)| \leq \delta_0$. Moreover, we have $|D_h(\eta) - D_s(\eta)| \leq |h - s|$ for all h, s. This shows that for $\eta \in B(\xi, e^{-t})$ and $s, h \geq 0$,

$$|D_{t+s}(\xi) - D_{t+h}(\eta)| \leq \delta_0 + s + h. \tag{3.3.2}$$

Recall that $t_k = s_1^1 + kc + l_c$ and let $(\xi, t_k) \in \Omega$ be a given formal ball. From the above (3.3.1), we know that $B(\xi, e^{-t_k}) \cap R(t_k - l_c)$ contains at most one point, say $\eta = \varphi.\Lambda\Gamma_1$. By (3.3.2), $D_{t_k+d_c}(\xi)$ and $D_{t_k+c-d_*}(\eta)$ can differ by at most $c + \delta_0 + d_c - d_*$. Moreover, since $D_1([\varphi]) \leq t_k - l_c$, we have for the depth of η that

$$D_{t_k+c-d_*}(\eta) = t_k + c - d_* - D([\varphi]) \geq c + l_c - d_*.$$

Assuming that $c + l_c \geq c + \delta_0 + d_c$ (which is the case for $c \geq \log(2)$), we have $D_{t_k+c-d_*}(\eta) \geq D_{t_k+d_c}(\xi) + c + \delta_0 + d_c - d_*$. Using the Global Measure Formula, we obtain

$$\begin{aligned}
\mu(B(\xi, e^{-(t_k+d_c)})) &\geq c_1 e^{-\delta(t_k+d_c)} \cdot e^{-(\delta-m)D_{t_k+d_c}(\xi)} \\
&\geq c_2 e^{-\delta(t_k+c-d_*)} \cdot \tfrac{c_1}{c_2} e^{\delta(c-d_*-d_c)} e^{-(\delta-m)(D_{t_k+c-d_*}(\eta)-(c+\delta_0+d_c-d_*))} \\
&\geq c_2 e^{-\delta(t_k+c-d_*)} e^{-(\delta-m)D_{t_k+c-d_*}(\eta)} \cdot \tfrac{c_1}{c_2} e^{2\delta(c-d_*-d_c)} e^{-m(c+\delta_0+d_c-d_*)} \\
&\geq \mu(B(\eta, e^{-(t_k+c-d_*)})) \cdot \tfrac{c_1}{c_2} e^{-2\delta(d_*+d_c)} e^{m(\delta_0+d_c-d_*)} e^{(2\delta-m)c} \\
&\geq \mu(B(\xi, e^{-(t_k+d_c)}) \cap B(\eta, e^{-(t_k+c-d_*)})) \cdot \tau_c^{-1}.
\end{aligned}$$

As above, using (3.3.2) for $\eta \in B(\xi, e^{-t_k})$ and the Global Measure Formula, we obtain

$$\begin{aligned}\mu(B(\eta, e^{-t_{k+1}})) &\geq c_1 e^{-\delta t_{k+1}} \cdot e^{-(\delta-m)D_{t_{k+1}}(\xi)} \\ &\geq \mu(B(\xi, e^{-t_k})) \cdot \tfrac{c_1}{c_2} e^{-\delta c} e^{-(\delta-m)(c+\delta_0)} \\ &\equiv \mu(B(\xi, e^{-t_k})) \cdot k_c,\end{aligned}$$

as well as

$$\begin{aligned}\mu(B(\xi, e^{-(t_k+d_c)})) &\geq c_1 e^{-\delta(t_k+d_c)} \cdot e^{-(\delta-m)D_{t_k+d_c}(\xi)} \\ &\geq \mu(B(\eta, e^{-(t_{k+1}-d_*)}) \cdot \tfrac{c_1}{c_2} e^{\delta(c-d_c-d_*)} e^{-(\delta-m)(c+d_c+d_*+\delta_0)} \\ &\geq \mu(B(\eta, e^{-(t_{k+1}-d_*)}) \cdot \tfrac{c_1}{c_2} e^{-2\delta(d_c+d_*+\delta_0)} e^{m(d_c+d_*+\delta_0)} e^{mc} \\ &\equiv \mu(B(\eta, e^{-(t_{k+1}-d_*)}) \cdot \bar{k}_c^{-1}.\end{aligned}$$

This finishes the proof. □

Assuming that $c > c_0$, where c_0 is as in Lemma 3.29 and such that $\tau_{c_0} < 1$, the following Lemma will finish determining the parameters for the lower bound.

Lemma 3.34. *For any $\xi \in \boldsymbol{Bad}(\mathcal{F}, 2c + l_c)$ we have $d_\mu(\xi) \geq \delta$.*

Proof. If $\xi \in \mathbf{Bad}(\mathcal{F}, 2c + l_c)$, then $d_o(\xi, \varphi.\Lambda\Gamma_1) > e^{-(D_1([\varphi])+2c+l_c)}$ for every $[\varphi] \in \Gamma/\Gamma_1$ with $D_1([\varphi]) > t_0$. Hence, Lemma 3.29 states that the length of $\gamma_{o,\xi}(\mathbb{R}^+) \cap \varphi(C_1)$ is bounded by $2(2c + l_c + 2\log(\kappa_0))$ for every $[\varphi] \in \Gamma/\Gamma_1$. In particular, the distance from $\gamma_{o,\xi}(t)$ to $\partial\varphi(C_1)$ is less than $2c + l_c + 2\log(\kappa_0)$ for all $t > t_0$ and we see that $0 \leq D_t(\xi) \leq 2c + l_c + 2\log(\kappa_0)$. The Global Measure Formula yields that $c_1 e^{-\delta t} C^{-1} \leq \mu(B(\xi, e^{-t})) \leq c_2 e^{-\delta t} C$ for all $t > t_0$, for some $C = C(c) > 0$. In particular, $d_\mu(\xi) \geq \delta$. □

Finally, using Proposition 3.33, (3.2.18) gives the lower bound

$$\begin{aligned}\dim(\mathbf{Bad}(\mathcal{D}_1, e^{-(2c+l_c)})) &\geq \delta - \frac{\log(2\bar{k}_c k_c^{-1}) - \log(1-\tau_c)}{c} \\ &\geq \delta - \frac{\log(2\bar{c}_2\bar{c}_1^{-1}) + |\log(1 - \bar{c}_3 e^{-(2\delta-m)c})|}{c}.\end{aligned}$$

The Standard Case. Let $\Lambda\Gamma = S^n$. Note that for any formal ball (ξ, t_0), $\xi \in S^n$ we can take an isometry from the hyperbolic ball to the upper half space model (again denoted by $\mathbb{H}^{n+1}$) which maps o to $(0, \ldots, 0, 1) \in \mathbb{H}^{n+1}$ and ξ to $0 \in \mathbb{R}^n \subset \partial_\infty\mathbb{H}^{n+1}$. If $t_0 > 0$ is sufficiently large then $B(0, e^{-t_0})$ (with respect to the visual distance) is contained in the Euclidean unit ball $B \subset \mathbb{R}^n$ and we remark that the visual metric d_o restricted to B is bi-Lipschitz equivalent to the Euclidean metric on B; let $c_B \geq 1$ be the bi-Lipschitz constant.

We let $c = \log(m) > c_0$ for some $m \in \mathbb{N}$ sufficiently large (such that $\bar{\tau}_c^i < 1$ below). Up to modifying l_c^i and τ_c^i to $\tilde{l}_c^i = l_c^i + \log(c_B)$ and $\tilde{\tau}_c^i = c_B^n \tau_c^i$ respectively, we may use the same arguments as above and assume for any point $\xi \in B$ that (3.2.13) is satisfied with respect to the Lebesgue measure and the function B_1 (which is with respect to the Euclidean metric). Note also that we nowhere used the condition that $\xi \in L_k^{B_1}(c)$ so that the condition becomes obsolete in this setting. Hence, Lemma 3.11 shows that (3.2.20) is satisfied for the parameters $\bar{l}_c^i = \tilde{l}_c^i + a$, with $a \equiv 2\sqrt{n}+\log(2)+d_c$, and $\bar{\tau}_c^i \leq e^{n(a-d_c)}\tilde{\tau}_c^i$, where i stands for the respective cases. Recalling that $\tilde{\tau}_c^1 = \bar{c}_1\ e^{-nc}$ and $\tilde{\tau}_c^2 = \bar{c}_2\ e^{-\tau c}$, (3.2.21) yields the lower bound

$$\begin{aligned}\dim(\mathbf{Bad}(\mathcal{D}_i, e^{-(2c+\bar{l}_c^i)})) &\geq \dim(\mathbf{Bad}_{\mathbb{R}^n}^{B_1}(\mathcal{F}_i, 2c+\bar{l}_c)\cap B)\\ &\geq \dim(\mathbf{Bad}_{\mathbb{R}^n}^{Q_1}(\mathcal{F}_i, 2c+\bar{l}_c)\cap B) \geq n - \frac{|\log(1-\bar{\tau}_c^i)|}{c}.\end{aligned}$$

Again, up to modifying $\bar{\tau}_c^i$ to $\bar{\tau}_c^1 \equiv \bar{k}_l^1 e^{-nc}$ and $\bar{\tau}_c^1 \equiv \bar{k}_l^i e^{-\tau c}$ for suitable constants $\bar{k}_l^i > 0$ (depending only on c_0), gives the result for sufficiently large general $c \geq c_0$.

The upper bound

We again distinguish between the cases and start with **Case 2.** by showing a Dirichlet-type Lemma. Recall that D_0 denotes the diameter of the compact set K covering the convex core $\mathcal{C}M$.

Lemma 3.35. *There exists a constant $\kappa_1 \geq 0$ such that for all $\xi \in \Lambda\Gamma$ and $t > t_0$, where $t_0 > 2D_0$, there exists $[\varphi] \in \Gamma/\Gamma_2$ with $D_2([\varphi]) \leq t$ such that*

$$d_o(\xi, \varphi.\Lambda\Gamma_2) < e^{2D_0+\kappa_1}e^{-t}.$$

Proof. Let $\tilde{K}$ be a lift of K such that $o \in \tilde{K}$. The geodesic ray $\gamma_{o,\xi}$ is contained in $\mathcal{C}\Gamma$, which is covered by images $\varphi(\tilde{K})$, $\varphi \in \Gamma$. Hence, let $\varphi \in \Gamma$ such that $\gamma_{o,\xi}(t-D_0) \in \varphi(\tilde{K})$. Since $C_2 \subset \mathcal{C}\Gamma$, some image of C_2 under Γ, say C_2 itself, intersects $\tilde{K}$. Thus, $\varphi(C_2)$ intersects $\varphi(\tilde{K})$, and we see that

$$D_2([\varphi]) = d(o, \varphi(C_2)) \leq d(o, \gamma_{o,\xi}(t-D_0)) + d(\gamma_{o,\xi}(t-D_0)), \varphi(C_2)) \leq t.$$

Moreover, there exists a geodesic line α contained in $\varphi(C_2)$ at distance at most D_0 to $\gamma_{o,\xi}(t-D_0)$. Let H be the hyperbolic half-space such that $\gamma_{o,\xi}(t-2D_0) \in \partial H$, H orthogonal to $\gamma_{o,\xi}$ and $\xi \in \partial_\infty H$. Hence, one of the endpoints of α (which belongs to $\varphi.\Lambda\Gamma_2$) must lie in the boundary $\partial_\infty H$ of H. Remarking that $\partial_\infty H$ is a subset of $B(\xi, e^{-(d(o,H)-\kappa_1)})$ for some universal constant $\kappa_1 > 0$, yields the claim. □

Setting $u_* = 2D_0 + \kappa_1$, we see that $\mathcal{F}$ locally contains $\mathcal{S}$, which denotes the set of points of $\Lambda\Gamma$. Moreover, since Γ is convex-cocompact, (Ω, B_1, μ) satisfies a power law with respect to the parameters (δ, c_1, c_2), Proposition 3.17 shows that (Ω, B_1, μ) is τ^c-Dirichlet with respect to $\mathcal{F}$ and the parameters (c, u_*), where $\tau^c = \frac{c_1}{c_2}e^{-\delta(d_*+d^c+u_*)}\cdot e^{-\delta c}$.

Using (3.2.10), (3.2.25) implies the upper bound

$$\begin{aligned}\dim(\mathbf{Bad}(\mathcal{D}_2, e^{-c})) &\leq \frac{\log(1-\frac{c_1}{c_2}e^{-\delta(d_*+d^c+u_*)}e^{-\delta c}) - \log(\frac{c_1}{c_2}e^{-\delta(c+u_*+d_*+d^c)})}{c+u_*}\\ &= \delta - \frac{|\log(1-\frac{c_1}{c_2}e^{-\delta(d_*+d^c+u_*)}\cdot e^{-\delta c})| + \log(\frac{c_1}{c_2}) - \delta(d_*+d^c)}{c+u_*}.\end{aligned}$$

We are left with **Case 1.** We start again with the following Dirichlet-type Lemma that follows from [52], Theorem 1, which we reformulated in a version best suitable for us.

Lemma 3.36. *There exists a $t_0 \geq 0$ and a constant $\kappa_1 > 0$ (we may assume $\kappa_1 \geq 1$) such that for any $\xi \in \Lambda\Gamma$, for any $t > t_0$ there exists $[\varphi] \in \Gamma/\Gamma_1$ with $D_1([\varphi]) \leq t$, such that*

$$d_o(\xi, \varphi.\Lambda\Gamma_1) \leq \kappa_1\ e^{-t/2}\ e^{-D_1([\varphi])/2}. \tag{3.3.3}$$

Fix $c > 0$ and let $u^c \equiv c + 2\log(\kappa_1)$. Recall that $t_k = s_1^1 + (k-1)(c+u^c)$ and $\bar{t}_k = t_k - u^c$. We need the following refinement of the above lemma.

Lemma 3.37. *For $\xi \in \Lambda\Gamma$ with $\xi \in U_{k-1}(c)$ and $t_k > t_0$, there exists $[\varphi] \in \Gamma/\Gamma_1$ with $t_{k-1} < D_1([\varphi]) \leq t_k$ such that $\varphi.\Lambda\Gamma_1 \in B(\xi, e^{-\bar{t}_k})$.*

Proof. Let $\xi \in \Lambda\Gamma$ and $t_k > t_0$. There exists $[\varphi] \in \Gamma/\Gamma_1$ with $D([\varphi]) \leq t_k$ such that (3.3.3) is satisfied. If $D_1([\varphi]) \leq t_{k-1} = t_k - (c+u^c)$, then

$$\begin{aligned} d_o(\xi, \varphi.\Lambda\Gamma_1) &\leq \kappa_1 e^{-t_k/2} e^{-D_1([\varphi])/2} \\ &\leq \kappa_1 e^{-(D_1([\varphi])+1/2(c+u^c))} \\ &= e^{-(D_1([\varphi])+1/2(c+c+2\log(\kappa_1)))} \leq e^{-(D_1([\varphi])+c)}. \end{aligned}$$

Thus, we see that

$$\xi \in B(\varphi.\Lambda\Gamma_1, e^{-(D_1([\varphi])+c)}) \subset \bigcup_{s_n \leq t_{k-1}} \psi_s(R_n, s_n + c) = U_{k-1}(c)^C,$$

and we may assume that $t_{k-1} < D_1([\varphi]) \leq t_k$. In this case, we have

$$\begin{aligned} d_o(\xi, \varphi.\Lambda\Gamma_1) &\leq \kappa_1 e^{-t_k/2} e^{-D_1([\varphi])/2} \\ &< \kappa_1 e^{-t_k + 1/2(c+u^c))} \\ &= e^{-(t_k - 1/2(c+u^c+2\log(\kappa_1)))} \\ &\leq e^{-(t_k - u^c)} = e^{-\bar{t}_k} \end{aligned}$$

and hence, $\varphi.\Lambda\Gamma_1 \in B(\xi, e^{-\bar{t}_k})$ which finishes the proof. $\square$

Combining the Global Measure Formula and the above lemma yields the parameters.

Proposition 3.38. *For the parameters c, $u^c \equiv c + 2\log(\kappa_1)$ and $d^c \equiv \log(2)$ (independent of c) we have*

$$\begin{aligned} k^c &\geq \tfrac{c_1}{c_2} e^{-(\delta-m)(2d_* + 2d^c + \delta_0)} e^{-(2\delta - m)(c+u^c)} \equiv \bar{c}_1 e^{-(2\delta-m)(c+u^c)} \\ \tau^c &\geq \tfrac{c_1}{c_2} e^{-\delta(2c + u_c + 2d_* + d^c) + m(c+d_*)} \equiv \bar{c}_2 e^{-(3\delta - m)c} \end{aligned}$$

in (3.2.22) and (3.2.23).

Proof. Let $(\xi, \bar{t}_k - d^c) \in \Omega$ be a given a formal ball and $\eta \in B(\xi, e^{-\bar{t}_k}) \subset B(\xi, e^{-(\bar{t}_k - d^c)})$. Using (3.3.2) we obtain

$$D_{\bar{t}_{k+1}+d_*}(\eta) \leq D_{\bar{t}_k - d^c}(\xi) + c + u^c + d_* + d^c + \delta_0.$$

The Global Measure Formula shows that

$$\begin{aligned} \mu(B(\eta, e^{-(\bar{t}_{k+1}+d_*)})) &\geq c_1 e^{-\delta(\bar{t}_{k+1}+d_*)} \cdot e^{-(\delta-m)D_{\bar{t}_{k+1}+d_*}(\eta)} \\ &\geq c_1 e^{-\delta(\bar{t}_k + c + u_c + d_*)} e^{-(\delta-m)D_{\bar{t}_k - d^c}(\xi)} \cdot e^{-(\delta-m)(c+u^c+d_*+d^c+\delta_0)} \\ &\geq \mu(B(\xi, e^{-(\bar{t}_k - d^c)}) \cdot \tfrac{c_1}{c_2} e^{-(\delta-m)(2d_*+2d^c+\delta_0)} e^{-(2\delta-m)(c+u^c)} \\ &\geq \mu(B(\xi, e^{-(\bar{t}_k - d^c)}) \cdot k^c. \end{aligned}$$

Similarly, let $(\xi, \bar{t}_k - d^c) \in \Omega$ be a given a formal ball such that $\xi \in U_{k-1}(c)$. By Lemma 3.37, there exists $[\varphi] \in \Gamma/\Gamma_1$ with $t_{k-1} < D_1([\varphi]) \leq t_k$ and $\eta \in B(\xi, e^{-\bar{t}_k})$, where $\eta \equiv \varphi.\Lambda\Gamma_1$. Moreover, since

$$D_1([\varphi]) + c + d_* > t_{k-1} + c + d_* \geq \bar{t}_k + d_*,$$

the ball $B(\eta, e^{-(D_1([\varphi])+c+d_*)}) \subset B(\eta, e^{-(\bar{t}_k+d_*)})$ which in turn is contained in $B(\xi, e^{-(\bar{t}_k-d^c)})$ (since $d^c = \log(2)$). Finally, we have $D_{D_1([\varphi])+c+d_*}(\eta) = c + d_*$ and $D_1([\varphi]) \leq t_k = \bar{t}_k + u^c$, the Global Measure Formula shows

$$\begin{aligned} \mu(B(\xi, e^{-(\bar{t}_k-d^c)}) \cap B(\eta, e^{-(D_1([\varphi])+c+d_*)})) &\geq \mu(B(\eta, e^{-(D_1([\varphi])+c+d_*)}) \\ &\geq c_1 e^{-\delta(D_1([\varphi])+c+d_*)} \cdot e^{-(\delta-m)(c+d_*)} \\ &\geq c_2 e^{-\delta(\bar{t}_k-d^c)} \cdot \tfrac{c_1}{c_2} e^{-\delta(c+u_c+d_*+d^c)-(\delta-m)(c+d_*)} \\ &\geq \mu(B(\xi, e^{-(\bar{t}_k-d^c)}) \cdot \tfrac{c_1}{c_2} e^{-\delta(2c+u_c+2d_*+d^c)+m(c+d_*)} \\ &\equiv \tau^c \cdot \mu(B(\xi, e^{-(\bar{t}_k-d^c)}). \end{aligned}$$

This finishes the proof. □

Using Proposition 3.38, (3.2.25) gives the upper bound

$$\begin{aligned} \dim(\mathbf{Bad}(\mathcal{D}_1, e^{-c})) &\leq \frac{-\log(k^c) + \log(1-\tau^c)}{c + u^c} \qquad (3.3.4) \\ &\leq \frac{-\log(\bar{c}_1) + (2\delta - m)(c+u^c) + \log(1 - \bar{c}_2 e^{-(3\delta-m)c})}{c+u^c} \\ &\leq (2\delta - m) - \frac{|\log(1-\bar{c}_2 e^{-(3\delta-m)c})| + \log(\bar{c}_1)}{2c + 2\log(k_1)}. \end{aligned}$$

The Standard Case. Let again $\Lambda\Gamma = S^n$ and, for $c > 0$, let $\tilde{c} = c + a$, where $a = d^c + \bar{d}_* + \log(2) + \sqrt{n}$ and and $\bar{u}_i^c \geq u_i^{\tilde{c}} + a$ such that $c + \bar{u}_i^c = \log(m_i)$ (with m_i minimal). Let $k_0 = 1$ and note that $\bar{t}_{k_0} \geq t_0$. Moreover, let $\xi_j \in S^n$ be finitely many points such that $B_j = B(\xi_j, \bar{t}_1)$ cover S^n. As for the lower bound, for each ξ_j we can take again an isometry to the upper half space model which maps o to $(0, \dots, 0, 1) \in \mathbb{H}^{n+1}$ and ξ to $0 \in \mathbb{R}^n \subset \partial_\infty \mathbb{H}^{n+1}$ as well as B_j to a subset contained in the Euclidean unit ball B. Up to modifying a to $\tilde{a} = a + \log(c_B)$, we may even assume that the cube $Q = Q_1(0, \bar{t}_1) \supset B_j$ is contained in B.

Recall that from Lemma 3.14, $U_{k-1}^{Q_1}(c + \bar{d}_*) \subset \bigcap_{s_n \leq \bar{t}_k - \tilde{c} + d^{\tilde{c}}} B_1(R_n, s_n + \tilde{c})^C$. Moreover, we remark that in the above arguments for determining τ^c, it was in fact nowhere necessary to require $t = t_k$ and we showed that, if $(\xi, \bar{t}_k) \in \Omega$ with $\xi \in U_{k-1}^{Q_1}(c + \bar{d}_*)$, $k \geq k_0$ and $\xi \in B$, then (3.2.29) is satisfied with respect to the Lebesgue measure and the function ψ_1 (with respect to the visual metric d_o). Again, up to adding the constant $\log(c_B)$ to $\bar{u}_i^c$ as well as $\bar{\tau}_i^c = c_B^{-n} \tilde{\tau}_i^c$, we see that $(\xi, \bar{t}_k)$ satisfies (3.2.29) with respect to the Lebesgue measure and the function B_1 (with respect to the Euclidean metric). Hence, Lemma 3.14 implies (3.2.27) for all $k \geq k_0$ with respect to $\bar{\tau}_i^c \geq c_B^{-n} e^{-n\sqrt{n}} \tau_i^{\tilde{c}} \equiv \bar{k}_u^i e^{-n_i c}$. Finally, since m_i above was chosen minimal there exists a constant $k_u^i \geq 0$ (independent on c) such that $\tilde{c} + u_i^{\tilde{c}} \leq n_i c + k_u^i$ with $n_1 = 2$ and $n_2 = 1$. Thus, (3.2.28) and (the remark after (3.2.28)) show

$$\begin{aligned} \dim(\mathbf{Bad}(\mathcal{D}_i, e^{-c+\sqrt{n}}) \cap B_j) &\leq \dim(\mathbf{Bad}_{\mathbb{R}^n}^{Q_1}(\mathcal{F}, c) \cap Q) \\ &\leq n - \frac{|\log(1 - \bar{k}_u^i \, e^{-n_i c})|}{n_i c + k_u^i}. \end{aligned}$$

3.3.4 Toral Endomorphisms

For the motivation of the following result, we refer to Broderick, Fishman, Kleinbock [10] and references therein. For $n \in \mathbb{N}$, let $\mathcal{M} = (M_k)$ be a sequence of real matrices $M_k \in GL(n, \mathbb{R})$, with $t_k = \|M_k\|_{op}$ (the operator norm), and $\mathcal{Z} = (Z_k)$ be a sequence of τ_k-separated[9] subsets of $\mathbb{R}^n$. Define

$$E_{\mathcal{M},\mathcal{Z}} \equiv \{x \in \mathbb{R}^n : \exists\, c = c(x) > 0 \text{ such that } d(M_k x, Z_k) \geq c \cdot \tau_k \text{ for all } k \in \mathbb{N}\},$$

where d is the Euclidean distance. For $c > 0$, let $E_{\mathcal{M},\mathcal{Z}}(c)$ be the elements $x \in E_{\mathcal{M},\mathcal{Z}}$ with $c(x) \geq c$. We assume that, independently of $t \in \mathbb{R}^+$, for all $c > 0$ we have

$$|\{k \in \mathbb{N} : \log(t_k/\tau_k) \in (t - c, t]\}| \leq f(c), \tag{3.3.5}$$

for some function $f : \mathbb{R}^+ \to \mathbb{R}^+$. The sequence $\mathcal{M}$ is *lacunary*, if $\inf_{k\in\mathbb{N}} \frac{t_{k+1}}{t_k} \equiv \lambda > 1$, and the sequence $\mathcal{Z}$ is *uniformly discrete*, if there exists $\tau_0 > 0$ such that every set Z_k is τ_0-separated. Note that if $\mathcal{M}$ is lacunary and $\mathcal{Z}$ is uniformly discrete, then (3.3.5) holds and f is in fact bounded by $f(c) \leq c/\log(\lambda)$.

Let again $\mathcal{S}$ denote the set of affine hyperplanes in $\mathbb{R}^n$ and recall that the Lebesgue measure is absolutely $(1, c_0)$-decaying with respect to $\mathcal{S}$ and the function $\psi = B_1$. Using similar arguments for the proof as [10, 60], we want to show the following lower bounds.

Theorem 3.39. *Let $X \subset \mathbb{R}^n$ be the support of an absolutely (τ, c_τ)-decaying measure μ (with respect to $\mathcal{S}$ and $\psi = B_1$) which also satisfies a power law with respect to the exponent δ. Let $\mathcal{M}$ and $\mathcal{Z}$ be as above satisfying (3.3.5) with $f(c) \leq e^{\bar{\tau}c}$, where $0 < \bar{\tau} < \tau$. Then, there exists $c_0 > 0$ such that for all $c > c_0$ we have*

$$dim(E_{\mathcal{M},\mathcal{Z}}(e^{-(2c+\log(12))}) \cap X) \quad \geq \quad \delta - \frac{\log(2\frac{c_1^2}{c_2^2}) + \delta(\log(2) + d_c) + |\log(1 - c_\tau 2^\tau f(c) e^{-\tau c}|}{c}.$$

If μ denotes the Lebesgue measure, then there exist constants $k_l, \bar{k}_l > 0$ and $c_0 > 0$, such that for all $c > c_0$ we have

$$dim(E_{\mathcal{M},\mathcal{Z}}(e^{-c})) \quad \geq \quad n - \frac{|\log(1 - \bar{k}_l\, f(c/2)e^{-c/2})|}{c/2 - k_l}.$$

Proof. Let $v_k \in \mathbb{R}^n$ be the unit vector such that $\|M_k v_k\| = t_k$ and if $V_k \equiv \{M_k v_k\}^\perp$ is the subspace orthogonal to $M_k v_k$, let $W_k \equiv M_k^{-1}(V_k)$. Then, for $k \in \mathbb{N}$ and $z \in Z_k$ we define the subsets

$$Y_k(z) \equiv (M_k^{-1}(z) + W_k) \cap M_k^{-1}(B(z, \tau_k/4)).$$

Set $s_k \equiv \log(\tau_k/t_k)$, which we reorder such that $s_k \leq s_{k+1}$, so that we obtain a discrete set of sizes. For $k \in \Lambda \equiv \mathbb{N}$ let the resonant set R_k be given by

$$\begin{aligned} R_k &\equiv \{x \in Y_l(z_l) : z_l \in Z_l \text{ and } \log(t_l/\tau_l) \leq s_k\} \\ &= \{x \in Y_l(z_l) : z_l \in Z_l \text{ and } \frac{\tau_l}{t_l} \geq \frac{\tau_k}{t_k}\}, \end{aligned}$$

[9] That is, for every $y_1, y_2 \in Z_k$ we have $d(y_1, y_2) \geq \tau_k > 0$.

which gives a nested and discrete family $\mathcal{F} = \{\mathbb{N}, R_k, s_k\}$.

Note that for all $x \in \mathbb{R}^n$ we have $\|x\| \geq \|M_k x\|/t_k$. Hence, for distinct points $z_1, z_2 \in Z_k$, $Y_k(z_1)$ and $Y_k(z_2)$ are subsets of parallel affine hyperplanes and we have

$$\begin{aligned} \|Y_k(z_1) - Y_k(z_2)\| &\geq \|M_k^{-1}(B(y_1, \tau_k/4)) - M_k^{-1}(B(y_2, \tau_k, /4))\| \\ &\geq \frac{\tau_k - 2\tau_k/4}{t_k} = \frac{\tau_k}{2t_k} \geq \tfrac{1}{2}e^{-s_k}, \end{aligned} \tag{3.3.6}$$

since Z_k is τ_k-separated. Let $l_c = \log(4) + \log(3)$. Given a closed ball $B = B(x, 2e^{-(t+l_c)}) \subset \mathbb{R}^n$ with $x \in X$, for every $k \in \mathbb{N}$ with $s_k \leq t$, it follows from (3.3.6) that at most one of the sets $Y_k(y)$, $y \in Z_k$, can intersect B. Moreover, for $c > 0$, the number of $k \in \mathbb{N}$ with $s_k \in (t-c, t]$ is bounded by $f(c)$ by (3.3.5). Recall that $R(t, c) \equiv R(t) - R(t-c)$. Thus, there exist at most $N = \lfloor f(c) \rfloor$ affine hyperplanes $L_1, \ldots, L_N \in \mathcal{S}$ such that

$$B(x, 2e^{-(t+l_c)}) \cap R(t, c) \subset B(x, 2e^{-(t+l_c)}) \cap \bigcup_{i=1}^{N} L_i.$$

Since (Ω, B_1, μ) is absolutely (τ, c_τ)-decaying with respect to $\mathcal{S}$, for $c \geq \log(3) + \log(2)$ and $B = B(x, e^{-(t+l_c+d_c)})$ we have

$$\begin{aligned} \mu(B \cap \mathcal{N}_{e^{-(t+l_c+d_c+c-\log(2))}}(R(t, c))) &\leq \sum_{i=1}^{N} \mu(B \cap \mathcal{N}_{e^{-(t+l_c+d_c+c-\log(2))}}(L_i)) \\ &\leq f(c) \cdot c_\tau e^{-\tau(c-\log(2))} \mu(B) \equiv \tau_c\, \mu(B). \end{aligned}$$

Note that, since $f(c) \leq e^{\bar{\tau}c}$ with $\bar{\tau} < \tau$, for all $c > c_0 = \log(c_\tau 2^\tau)/(\tau - \bar{\tau})$ we have $\tau_c < 1$. Using the remark after (3.2.23), we in fact showed that (Ω, B_1, μ) is τ_c-decaying with respect to $\mathcal{F}$ and the parameters (c, l_c). Moreover, $\psi = B_1$ is $\log(2)$-separating with respect to the sets $R(t, c)$.

Finally, let $x \in \mathbf{Bad}_X^{B_1}(\mathcal{F}, c)$, that is, for every $k \in \mathbb{N}$ and $y \in Z_k$ we have $d(x, Y_k(y)) \geq e^{-(s_k+c)} \geq e^{-c}\tau_k/t_k$. Assume that $M_k x \in B(y, \tau_k/4)$. Then, $x \in \mathcal{N}_{e^{-c}\tau_k/t_k}(M_k^{-1}(y) + W_k)^C \cap M_k^{-1}(B(y, \tau_k/4))$ and we can write the vector $v = x - M_k^{-1}(y)$ as $v = w + \tilde{c}\tau_k/t_k v_k$ with $w \in W_k$ and $\tilde{c} \geq e^{-c}$. Hence, since $M_k W_k = V_k$ is orthogonal to $M_k v_k$,

$$d(M_k x, y) = \|M_k v\| = \|M_k w + \tilde{c}\, \tfrac{\tau_k}{t_k} M_k v_k\| \geq \tilde{c}\, \tfrac{\tau_k}{t_k} \|M_k v_k\| \geq e^{-c}\tau_k,$$

and we showed that $\mathbf{Bad}_X^{B_1}(\mathcal{F}, c) \subset E_{\mathcal{M},\mathcal{Z}}(e^{-c}) \cap X$.

Thus, using (3.2.10), our formula for the lower bound (3.2.18) yields for $c > c_0$ that

$$\dim(E_{\mathcal{M},\mathcal{Z}}(e^{-(2c+l_c)}) \cap X) \geq \delta - \frac{\log(2\frac{c_1^2}{c_2^2}) + \delta(\log(2) + d_c) + |\log(1 - c_\tau 2^\tau f(c) e^{-\tau c}|}{c}.$$

For the second part, when μ denotes the Lebesgue measure and $\psi = Q_1$, we let $c = \log(m) > \bar{c}_0 \geq c_0 + \log(2) + \sqrt{n}$ be sufficiently large such that $\bar{\tau}_c < 1$ (as below). The above arguments apply analogously for the function $\psi = Q_1$ with possibly different parameters $\bar{l}_c = l_c + c_1$ and $\bar{\tau}_c = c_2\tau_c \equiv \bar{k}_l f(s) e^{-\tau c}$, where $\tau = 1$, for some constants $c_1, c_2, \bar{k}_l > 0$. Hence, (3.2.21) shows

$$\dim(E_{\mathcal{M},\mathcal{Z}}(e^{-(2c+\bar{l}_c)}) \geq n - \frac{|\log(1 - \bar{k}_l\, f(c) e^{-c})|}{c},$$

since $\mathbf{Bad}_{\mathbb{R}^n}^{Q_1}(2c + \bar{l}_c) \subset \mathbf{Bad}_{\mathbb{R}^n}^{B_1}(2c + \bar{l}_c)$. This finishes the proof. □

For a nontrivial upper bound, we restrict to the following example. Let $Z = Z_k$ for all $k \in \mathbb{N}$ where Z is a τ_0-spanning set[10] of $\mathbb{R}^2$. Let $\mathcal{M} = (M_k)$ with $M_k = M^k$, where $M \in GL(2,\mathbb{R})$ is a real diagonizable matrix with eigenvalues $\lambda \geq \beta > 1$.[11] In fact, for simplicity, let $M = \mathrm{diag}(\lambda, \beta)$[12] , where λ and β are integers, and let $Z = \mathbb{Z}^2$. For these assumptions, there exist constants $c_0 > 0$ and $k_u, \bar{k}_u > 0$ such that for $c > c_0$ we have

$$\dim(E_{\mathcal{M},\mathcal{Z}}(e^{-c}) \cap [0,1]^2) \quad \leq \quad 2 - \frac{\log(\beta)}{\log(\lambda)} \frac{|\log(1 - \bar{k}_u \; e^{-(1+\log(\lambda)/\log(\beta))c})|}{c + k_u}. \tag{3.3.7}$$

Sketch of the proof of (3.3.7). We let $\sigma = \log(\beta)$ and $\bar{\sigma} = \log(\lambda)$. On $\Omega = \mathbb{R}^2 \times \mathbb{R}^+$, define the monotonic function $\psi = \psi_{(\lambda,\beta)}$ by the rectangle centered at $x \in \mathbb{R}^2$,

$$\psi(x,t) = x + B(0, e^{-\bar{\sigma}t}) \times B(0, e^{-\sigma t}).$$

Since $\lambda,\ \beta \in \mathbb{N}_{\geq 2}$, there exist parameters $\bar{c} > 0$[13] such that $\lambda^{\bar{c}} = p$, $\beta^{\bar{c}} = q$ with $p, q \in \mathbb{N}_{\geq 2}$, and we can partition $\psi(x,t)$ into pq rectangles $\psi(x_i, t + \bar{c})$ as in (3.2.19). Note that we have $\mu(\psi(x_i, t + \bar{c})) = e^{-(\bar{\sigma}+\sigma)\bar{c}} \mu(\psi(x,t))$, where μ denotes the Lebesgue measure.

Fix a parameter $c > 0$ sufficiently large and let $\bar{c} \geq c/\sigma + \log(6)/\sigma + 1$ be the minimal parameter such that a partition as above is possible. Let $Q_0 = Q_1(x_0, 0) = [0,1]^2$. Now, assume we are given a rectangle $Q = Q_{i_0 \ldots i_k} = \psi(x_{i_0 \ldots i_k}, k\bar{c})$. Let $\tilde{k} \in \mathbb{N}$ be the minimal integer such that $\tilde{k} \geq t_k + \log(2)/\sigma$. Thus, we have

$$R \equiv M^{\tilde{k}} Q = M^{\tilde{k}} x_{i_0 \ldots i_k} + B(0, \lambda^{\tilde{k}} e^{-\bar{\sigma} t_k}) \times B(0, \beta^{\tilde{k}} e^{-\sigma t_k})$$

which is a rectangle of edge lengths in $[2, 2^{\bar{\sigma}/\sigma}\lambda] \times [2, 2\beta]$ since $\tilde{k}$ was chosen minimal. In particular, there exists an integer point $z \in \mathbb{Z}^2$ such that $Q_1(z, c)$ is contained in R. Hence,

$$\begin{aligned} Q = M^{-\tilde{k}} R \quad &\supset \quad M^{-\tilde{k}} Q_1(z, c) \\ &= \quad M^{-\tilde{k}} z + B(0, e^{-\bar{\sigma}\tilde{k} - c}) \times B(0, e^{-\sigma \tilde{k} - c}) \supset \psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma). \end{aligned}$$

This in particular shows that $Q \cap E_{\mathcal{M},\mathcal{Z}}(e^{-c}) \subset Q \cap \psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma)^C$ and it suffices to cover the sup set. Again since $\tilde{k}$ is minimal, $t_k + c/\sigma + \log(2)/\sigma + 1 \geq \tilde{k} + c/\sigma$, so that a rectangle $\psi(x, t_k + \bar{c}) \subset Q$ that intersects $Q \cap \psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma))^C$ does not intersect $\psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma + \log(3)/\sigma)$. Moreover, we have

$$\mu(Q \cap \psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma + \log(3)/\sigma) \geq \bar{k}_u \; e^{-(\bar{\sigma}+\sigma)c/\sigma} \mu(Q) \equiv \tau^{\bar{c}} \mu(Q),$$

for some constant $\bar{k}_u > 0$. Thus, if $Q_{i_0 \ldots i_k i_{k+1}} = \psi(x_{i_0 \ldots i_k i_{k+1}}, t_k + \bar{c})$ are the rectangles from the partition of Q, then we can bound the number of rectangles $Q_{i_0 \ldots i_k i_{k+1}}$ not intersecting $\psi(M^{-\tilde{k}} z, \tilde{k} + c/\sigma + \log(3)/\sigma)$ by $pq(1 - \tau^{\bar{c}}) = e^{(\bar{\sigma}+\sigma)\bar{c}}(1 - \tau^{\bar{c}})$. In particular, we can bound the number of rectangles of the covering constructed this way at stage k by

$$N_k \leq \left(e^{(\bar{\sigma}+\sigma)\bar{c}}(1 - \tau^{\bar{c}})\right)^k.$$

[10] That is, for any $x \in \mathbb{R}^2$, there exists $z \in Z$ such that $d(x, z) < \tau$.

[11] Note that for $\beta = 1$ and c sufficiently large, there might even exist M-invariant strips of $\mathbb{R}^2$, consisting of badly approximable elements x with $c(x) \geq e^{-c}$.

[12] If $M = DAD^{-1}$ for $D \in GL(2,\mathbb{R})$, then consider $\tilde{\psi}(x,t) = x + D\psi(0,t)$, for ψ as below, in the following.

[13] This is in particular true for every integer $\bar{c} \in \mathbb{N}$.

Thus we obtain a covering of $E_{\mathcal{M},\mathcal{Z}}(e^{-c})$ by N_k rectangles of diameter at most $2e^{-\sigma t_{k+1}}$. The argument used to obtain (3.2.26) actually shows that we can cover each rectangle $Q_{i_0\dots i_k i_{k+1}}$ with Z_{k+1} cubes $Q_{\bar{\sigma}}(y_{i_1\dots i_k i_{k+1}}, t_{k+1})$, hence of diameter at most $2e^{-\bar{\sigma}t_{k+1}}$, where $Z_{k+1} \leq c_2 e^{(2\bar{\sigma}-(\sigma+\bar{\sigma}))t_{k+1}}$ for some constant $c_2 > 0$. Finally, as in (3.2.26), we obtain

$$\begin{aligned}
\dim(E_{\mathcal{M},\mathcal{Z}}(e^{-c}) \cap Q_0) &\leq \liminf_{k\to\infty} \frac{\log(Z_{k+1}N_k)}{-\log(2e^{-\bar{\sigma}t_{k+1}})} \\
&\leq 2 - \frac{\bar{\sigma}-\sigma}{\bar{\sigma}} + \frac{\log(1-\tau^{\bar{c}}) - (\bar{\sigma}+\sigma)\bar{c}}{\bar{\sigma}\bar{c}} \\
&\leq 2 - \frac{|\log(1-\bar{k}_u\, e^{-(\bar{\sigma}+\sigma)c/\sigma})|}{\bar{\sigma}\bar{c}} \\
&\leq 2 - \tfrac{\log(\beta)}{\log(\lambda)} \frac{|\log(1-\bar{k}_u\, e^{-(\bar{\sigma}+\sigma)c/\sigma})|}{c+k_u},
\end{aligned}$$

where we used that $\bar{c}$ is chosen minimally, so that $\bar{c} \leq c/\sigma + \log(6)/\sigma + 2 \equiv c/\sigma + k_u/\sigma$. This finishes the proof. □

Chapter 4

Quantitative Recurrence and F-Aperiodicity

This chapter results from a joint work with Viktor Schroeder. Large parts of this chapter are published in [51].

Abstract of Chapter 4. We introduce a quantitative condition on orbits of dynamical systems which measures their aperiodicity. We show the existence of sequences in the Bernoulli-shift and geodesics on closed hyperbolic manifolds which are as aperiodic as possible with respect to this condition. Finally, we discuss consequences of the existence of these special orbits.

Organization of Chapter 4. In Section 4.1 we state our main results in the simplest setting. In Section 4.2 we introduce the measure of aperiodcitiy for general dynamical systems and deduce immediate properties. Then, in Sections 4.3 and 4.4 respectively, we examine two examples and state the main results, namely of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold. These will be proven in Section 4.5. Finally, in the Appendix 4.6, we relate the existence of φ-aperiodic geodesics to Diophantine approximation in negatively curved spaces, hence with the Chapters 2 and 3, as well as with the topological and volume entropy.

4.1 Main Results.

In this section we state our main results in the case of sequences in a finite alphabet and of geodesics in hyperbolic manifolds. Denote by $\mathbb{N}_0$ the natural numbers including 0 and let $\mathbb{N} = \mathbb{N} \setminus \{0\}$. Given a finite set $\mathcal{A}$ with $k \geq 2$ elements, let $\Sigma = \mathcal{A}^{\mathbb{Z}}$ be the set of biinfinite sequences in the alphabet $\mathcal{A}$, which we call *words*. With $[w(i) \dots w(i+l)]$ denote the subword of $w \in \Sigma$ starting at *time* $i \in \mathbb{Z}$ and of *length* $l \in \mathbb{N}_0$. For a word $w \in \Sigma$ define the *recurrence time* $R^i_w : \mathbb{N}_0 \to \mathbb{N} \cup \{\infty\}$ at time $i \in \mathbb{Z}$ by

$$R^i_w(l) = \min\{s \geq 1 : [w(i+s) \dots w(i+s+l)] = [w(i) \dots w(i+l)]\},$$

(i.e. the first instant when the sub word $[w(i) \dots w(i+l)]$ of w is seen again), and by

$$R_w(l) \equiv \min\{R^i_w(l) : i \in \mathbb{Z}\}.$$

For a periodic word $w \in \Sigma$ with period $p \in \mathbb{N}$, i.e. $w(i) = w(i+p)$ for all $i \in \mathbb{Z}$, we have $R_w(l) \leq p$ for all $l \in \mathbb{N}_0$. Thus, if R_w is unbounded, then w is aperiodic and we view the growth rate of R_w as a measure for the aperiodicity of the word w. Note that R_w is nondecreasing and by a trivial counting argument we have $R_w(l) \leq k^{l+1}$ for every word w, in particular

$$\limsup_{l\to\infty} \frac{1}{l} \log R_w(l) \leq \log(k).$$

One of our main results is the existence of words w such that the growth rate is as near as possible to this bound.

Theorem 4.1. *Let $\varphi : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing function such that*

$$\limsup_{l\to\infty} \frac{1}{l} \log(\varphi(l)) \leq \delta \log(k) \tag{4.1.1}$$

for some $0 < \delta < 1$. Then there exist $l_0 = l_0(\varphi, k, \delta) \in \mathbb{N}_0$ and a word $w \in \Sigma$ such that, for every $l_0 \leq l \in \mathbb{N}_0$, we have $R_w(l) \geq \varphi(l)$.

Now let M be a closed n-dimensional hyperbolic manifold, where $n \geq 2$. Let $i_M > 0$ denote the injectivity radius of M and let d be the Riemannian distance function on M. For a unit speed geodesic $\gamma : \mathbb{R} \to M$ we define the *recurrence time* $R^{t_0}_\gamma : [0,\infty) \to [i_M/2, \infty]$ at time $t_0 \in \mathbb{R}$ by

$$R^{t_0}_\gamma(l) = \inf\{s > i_M/2 : d(\gamma(t_0+t), \gamma(t_0+s+t)) < \frac{i_M}{2} \text{ for all } 0 \leq t \leq l\}.$$

and

$$R_\gamma(l) \equiv \inf\{R^{t_0}_\gamma(l) : t_0 \in \mathbb{R}\}.$$

If γ is a periodic geodesic, then R_γ is bounded and again one can view the growth rate of R_γ as a measure for the aperiodicity of γ.

Theorem 4.2. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that*

$$\limsup_{l\to\infty} \frac{1}{l} \log(\varphi(l)) \leq \delta(n-1) \tag{4.1.2}$$

for some $0 < \delta < 1$. If $i_M > 2\log(2)$ then there exist $l_0 = l_0(\varphi, \delta, n, i_M) \geq 0$ and a unit speed geodesic $\gamma : \mathbb{R} \to M$ such that for all $l \geq l_0$, we have $R_\gamma(l) \geq \varphi(l)$.

The theorems will be shown in greater generality.

Remark 4.3. The bounds $\log(k)$ and $n-1$ equal the topological entropies of the respective dynamical systems and are both optimal. Moreover, we believe that the assumption on the injectivity radius in Theorem 4.2 is not necessary. A version of this theorem is also true if M is of strictly negative curvature. However, for the sake of clarity we restrict to these assumptions. We also remark that a version of Theorem 4.2 was shown in [37] for a special case in dimension 2.

4.2 F-Aperiodic Points.

Let (X, d) be a compact metric space and let $T : X \to X$ be a given continuous transformation. For $n \in \mathbb{N}_0$ let T^n be the n-times composition of T (where $T^0 = id_X$) and for a point $x \in X$ let $T^n x$ be the point in the orbit $\mathcal{T}(x) \equiv \{T^n x\}_{n\in\mathbb{N}_0}$ of x at *time* n. Let moreover μ be a finite Borel-measure on the Borel-σ-algebra $\mathcal{B}$ of (X, d) such that T is measure-preserving; see [59].

A point $x \in X$ is called *periodic* (with respect to T) if there exists an integer $p \in \mathbb{N}$, called a *period* of x, such that $T^p x = x$. Denote by $\mathcal{P}_T$ the T-invariant set of T-periodic points of X. A point is called *aperiodic*, if it is not periodic.

A point $x \in X$ is *recurrent* with respect to T, if for any $\varepsilon > 0$ there exists $s = s(x, \varepsilon) \in \mathbb{N}$ such that $d(T^s x, x) < \varepsilon$. Periodic points are obviously recurrent. The set $\mathcal{R}_T$ of recurrent points is nonempty (see [21]) and T-invariant. However $s(T^i x, \varepsilon)$ can differ from $s(x, \varepsilon)$ in general, unless T is an isometry on its orbit $\mathcal{T}(x)$; that is, $d(T^{i+s}x, T^i x) = d(T^s x, x)$ for all i and $s \in \mathbb{N}_0$. We recall that by the Poincaré-recurrence theorem, μ-almost every point is recurrent.

In this paper we give a quantitative version of recurrence and aperiodicity. Given a point $x \in X$ and a *time* $i \in \mathbb{N}_0$, we ask for a lower bound on the *shift* s such that $T^{i+s}x$ is allowed to be ε-close to $T^i x$:

Definition 4.4. *For a non-increasing function $F : (0, \infty) \to [0, \infty)$ a point $x \in X$ is called* F-aperiodic *at* time $i \in \mathbb{N}_0$ *if for every $\varepsilon > 0$, whenever*

$$d(T^i x, T^{i+s}x) < \varepsilon$$

for some $s \in \mathbb{N}$, then $s > F(\varepsilon)$. If x is F-aperiodic at every time $i \in \mathbb{N}_0$ then it is called F-aperiodic.

We emphasize that although we called the condition "F-aperiodic", a periodic point x is F-aperiodic for a suitable bounded function F. However, if the function F is unbounded, an F-aperiodic point must be aperiodic. Moreover, if x is not recurrent, then it is easy to find an unbounded function F such that x is F-aperiodic at least at time 0.

Let $F : (0, \infty) \to [0, \infty)$ be a given non-increasing function. Clearly, if a non-increasing function $\bar{F}$ satisfies $\bar{F}(s) \le F(s)$ for all $s \in (0, \infty)$ then an F-aperiodic point is also $\bar{F}$-aperiodic. On the other hand, using the upper box dimension $\dim_B(X)$ for metric spaces, we obtain an upper bound on the growth rate (as ε tends to 0) of functions F such that an F-aperiodic point might exist. For $\varepsilon > 0$ let $N(X, \varepsilon)$ denote the largest number of disjoint metric balls of radius ε. Then the upper box dimension ([18]) is given by

$$\dim_B(X) = \limsup_{\varepsilon\to 0} \frac{\log(N(X, \varepsilon))}{-\log(\varepsilon)}.$$

Lemma 4.5. *Let x be an F-aperiodic point. Then*

$$\limsup_{\varepsilon \to 0} \frac{\log(F(\varepsilon))}{\log(2/\varepsilon)} \le \dim_B(X).$$

Proof. Let $\varepsilon > 0$. If $B(T^{s_1}x, \varepsilon/2) \cap B(T^{s_2}x, \varepsilon/2) \neq \emptyset$ for some $0 \le s_1 < s_2 \le F(\varepsilon)$, we have $d(T^{s_1}x, T^{s_2}x) < \varepsilon_0$ which is impossible since $s_2 - s_1 \le F(\varepsilon_0)$. Therefore the metric balls $B(T^s x, \varepsilon/2)$ must be disjoint for $s \le F(\varepsilon)$. Hence we have $F(\varepsilon) \le N(X, \varepsilon/2)$. □

Moreover, since F is independent of the time $i \in \mathbb{N}_0$, the set $\mathcal{F}_T \subset X$ of F-aperiodic points is T-invariant. In the case when $(X, \mathcal{B}, \mu, T)$ is ergodic, $\mathcal{F}_T$ is either of full or of zero μ-measure. When $\mathcal{P}_T$ is nonempty, this question is related to the distribution of periodic orbits. In fact, let $x_0 \in \mathcal{P}_T$ be of minimal period p_0 and assume that $F(\varepsilon) \ge p_0$ for some $\varepsilon_{p_0} > 0$. In the case when F is continuous, we may choose $\varepsilon_{p_0} \equiv \sup\{\varepsilon > 0 : F(\varepsilon) \ge p_0\}$. Define the *critical neighborhood* of x_0 with respect to F and p_0 by

$$\mathcal{N}_{x_0} \equiv B(x_0, \varepsilon_{p_0}/2) \cap T^{-p_0}(B(x_0, \varepsilon_{p_0}/2)). \tag{4.2.1}$$

Whenever $x \in \mathcal{N}_{x_0}$ we have by the triangle inequality that $d(x, T^{p_0}x) < \varepsilon_{p_0}$, but $p_0 \le F(\varepsilon_{p_0})$. Thus, no point in $\mathcal{N}_{x_0}$ can be F-aperiodic and we see that the orbit of an F-aperiodic point must avoid the critical neighborhoods of periodic points. If in addition $\mu(\mathcal{N}_{x_0}) > 0$ then the set of F-aperiodic points cannot be of full and must therefore be of zero μ-measure. Thus, we showed the following criterion.

Lemma 4.6. *Assume $\mathcal{P}_T \neq \emptyset$ and let x_0 be a periodic point of period p_0 and $F(\varepsilon) \ge p_0$ for some $\varepsilon > 0$. If μ is ergodic and positive on $\mathcal{N}_{x_0}$ then the set $\mathcal{F}_T$ has μ-measure 0.*

In particular, this result is interesting for the *systolic point* $x_0 \in \mathcal{P}_T$ of *systolic period* $p_0 \in \mathbb{N}$, that is, x_0 has minimal period p_0 and for every periodic point in X of period p we have $p \ge p_0$.

Lemma 4.7. *F-aperiodicity is a closed condition.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of F-aperiodic points in X converging to $x \in X$. Let i and $s \in \mathbb{N}$ be fixed. For $\varepsilon > 0$ such that $d(T^i x, T^{i+s}x) < \varepsilon$ let $d \equiv \frac{1}{2}(\varepsilon - d(T^i x, T^{i+s}x))$. Since T is continuous, there exists $N = N(i, s, d) \in \mathbb{N}_0$ such that for all $n \ge N$ we have $d(T^i x, T^i x_n) < d$ and $d(T^{i+s}x, T^{i+s}x_n) < d$. From the triangle inequality we obtain

$$d(T^i x_n, T^{i+s}x_n) \le d(T^i x_n, T^i x) + d(T^i x, T^{i+s}x) + d(T^{i+s}x, T^{i+s}x_n) < \varepsilon$$

for $n \ge N$, so that $s > F(\varepsilon)$ since x_n is F-aperiodic. Hence, x is also F-aperiodic. □

Finally, note that if T acts as an isometry on the orbit $\mathcal{T}(x)$ of a point $x \in X$, then x is F-aperiodic as soon as it is F-aperiodic at a given time. For instance, we consider the rotation on the circle as a motivating example:

Example 4.8. Let $\mathbb{Z}$ act on $\mathbb{R}$ by translations and let $X = \mathbb{R}/\mathbb{Z}$ be the compact quotient space with the induced metric d obtained from the Euclidean metric. Given an irrational number $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$, we let $T = T_\alpha : X \to X$ be the automorphism induced by the translation $\tilde{T} : \mathbb{R} \to \mathbb{R}$, $\tilde{T}(x) \equiv x + \alpha$. For $c > 0$ we let $F_c : (0, \infty) \to [0, \infty)$, $F_c(t) = ct^{-1}$. In fact, since $\dim_B(X) = 1$, -1 is the optimal exponent due to Lemma 4.5. The point $[0]$ is F_c-aperiodic if and only if every point $[x]$ is F_c-aperiodic and hence $\mathcal{F}_T$ is either empty or X itself. Moreover, since T is an isometry, $[0]$ is F_c-aperiodic as soon as it is F_c-aperiodic at time 0. The question for which c and α there exist F_c-aperiodic points can be answered by classical Diophantine approximation; see for instance [5] for the following well-known results: Let μ be the Lebesgue measure on $\mathbb{R}$. For μ-almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have $c_0(\alpha) = 0$, where

$$c_0(\alpha) = \inf\{c > 0 : \text{there exist infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } |\alpha - \frac{p}{q}| < \frac{c}{q^2}\}.$$

However, there exists a set of Hausdorff-dimension one such that $c_0(\alpha)$ is positive. Such an α is called badly approximable. The supremum $\sup_{\alpha \in \mathbb{R} \setminus \mathbb{Q}} c_0(\alpha)$ of this set, called the Hurwitz-constant, is equal to $1/\sqrt{5}$ and attained at the golden ratio.

First, let α such that $c_0(\alpha) = 0$. Then for $c > 0$ we have for infinitely many $p \in \mathbb{Z}$, $q \in \mathbb{N}$,

$$|\tilde{T}^q 0 - p| = |q\alpha - p| = q|\alpha - \frac{p}{q}| < cq^{-1}, \tag{4.2.2}$$

hence $q \leq F_c(q^{-1})$ and we see that $[0]$ is not F_c-aperiodic for any $c > 0$. Thus, $\mathcal{F}_T$ is empty. In particular, this shows that for $c > 1/\sqrt{5}$ the set $\mathcal{F}_T$ is empty for every $T = T_\alpha$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational. However, for α a badly approximable number we have $c_0(\alpha) > 0$ and for $c < c_0(\alpha)$ there are only finitely many p, q as in (4.2.2). Hence we can choose some $0 < \bar{c} \leq c_0(\alpha)$ such that $[0]$ is $F_{\bar{c}}$-aperiodic and therefore $\mathcal{F}_T = X$.
If we conversely assume that $[0]$ is F_c-aperiodic, then whenever $|\tilde{T}^q 0 - p| < \varepsilon$ for some $\varepsilon > 0$ we have $q > F_c(\varepsilon) = c/\varepsilon > \frac{c}{q|\alpha - p/q|}$. Thus,

$$|\alpha - \frac{p}{q}| > \frac{c}{q^2}$$

for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and α is necessarily a badly approximable number.

In the following we are concerned with the examples of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold where the question of existence of F-aperiodic points is more delicate.

Remark 4.9. A somewhat orthogonal problem has been studied by many authors. For instance, [7] showed that the rate of recurrence can be quantified in the case when X has finite Hausdorff-dimension. More precisely, assume that the α-dimensional Hausdorff-measure H_α is σ-finite for some $\alpha > 0$, then, given a T-invariant probability measure μ, for μ-almost every point $x \in X$ there exists a finite constant $c(x) \geq 0$ such that

$$\liminf_{n \to \infty} n^{1/\alpha} d(x, T^n(x)) \leq c(x).$$

Assume that there exists a point $x \in X$ which is F-aperiodic at time 0 for the function $F(\varepsilon) = c \cdot \varepsilon^{-\alpha}$ for some $c > 0$ (compare with Lemma 4.5). Then it is not hard to show that for every $n > 0$,

$$n^{1/\alpha} d(x, T^n x) \geq c^{1/\alpha}.$$

The main point in our paper is that we study the recurrence for every point of the orbit and not only for the initial one.

4.3 Sequences.

Let $\mathcal{A}$ be a finite set of $k \geq 2$ elements which we call *alphabet*. Let $\Sigma^+ = \{w : \mathbb{N} \to \mathcal{A}\}$ and $\Sigma = \{w : \mathbb{Z} \to \mathcal{A}\}$ be the set two-sided sequences in symbols from $\mathcal{A}$. The elements of Σ are called *words*. Given words w and $\bar{w}$ in Σ we let $a(w, \bar{w}) = \max\{i \geq 0 : w(i) = \bar{w}(i) \text{ for } |j| \leq i\}$ for $w \neq \bar{w}$ and define $\bar{d}(w, \bar{w}) \equiv 2^{-a(w,\bar{w})}$, and $\bar{d}(w, w) \equiv 0$ otherwise. Let T denote the shift operator acting on Σ, with $T(w) = \bar{w}$ where $\bar{w}(i) = w(i+1)$. Then, $(\Sigma, \bar{d})$ is a compact metric space such that T is a homeomorphism. Moreover, let $\mathcal{B}$ denote the product σ-algebra of the power set $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ which equals the Borel-σ-algebra of $(\Sigma, \bar{d})$. Let (the probability measure) $\mu = \prod_{\mathbb{Z}} \mu_{\mathcal{A}}$ be the infinite product measure of $\mathcal{B}$ where $\mu_{\mathcal{A}}$ is a probability measure on $(\mathcal{A}, \mathcal{P}(\mathcal{A}))$. Then the *Bernoulli-shift* $(\Sigma, \mathcal{B}, \mu, T)$ is ergodic. For details we refer to [59].

Note that by definition of $\bar{d}$, two words are close if and only if the length of their subwords around position 0 on which they agree is large. In particular, if $w \in \mathcal{R}_T$ then, by recurrence applied to the word $T^i w$, for every length $l \in \mathbb{N}_0$ we can find an $s = s(i, l) \in \mathbb{N}$ such that $[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$. In the case of sequences it is suitable to reformulate F-aperiodicity as follows (see Proposition 4.11).

Definition 4.10. *For a non-decreasing function $\varphi : \mathbb{N}_0 \to [0, \infty)$ a word $w \in \Sigma$ is called φ-*aperiodic *at* time $i \in \mathbb{Z}$, *if for every* length $l \in \mathbb{N}_0$, *whenever*

$$[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)] \tag{4.3.1}$$

for some shift $s \in \mathbb{N}$, *then $s > \varphi(l)$. If w is φ-aperiodic at every time $i \in \mathbb{Z}$ it is called φ-aperiodic.*

A φ-aperiodic word $w \in \Sigma$ is F-aperiodic for the following function F.

Proposition 4.11. *A φ-aperiodic word $w \in \Sigma$ is F-aperiodic for $F(\varepsilon) = \varphi(-2\lceil\log_2(\varepsilon)\rceil)$. Conversely, an F-aperiodic word w is φ-aperiodic for $\varphi(l) = F(2^{-(l-3)/2})$.*

Proof. Let $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. For every $l \in \mathbb{N}_0$ such that $\bar{d}(T^i w, T^{i+s} w) \leq 2^{-l}$ we have

$$[w(i-l) \dots w(i+l)] = [w(i-l+s) \dots w(i+s+l)].$$

Thus, for $2^{-l} < \varepsilon \leq 2^{-(l-1)}$,

$$s > \varphi(2l) = \varphi(-2\lceil\log_2(\varepsilon)\rceil) = F(\varepsilon).$$

Since $F(\bar{\varepsilon}) \leq F(\varepsilon)$ for $\bar{\varepsilon} \geq \varepsilon$, the first implication follows.

Conversely, if w is F-aperiodic, assume that

$$[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$$

for $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ and let $\bar{l} = l/2$ if l is even and $\bar{l} = (l-1)/2$ if l is odd. Hence, $\bar{d}(T^{i+\bar{l}} w, T^{i+\bar{l}+s} w) \leq 2^{-\bar{l}}$ and for every $2^{-\bar{l}} < \varepsilon \leq 2^{-(\bar{l}-1)}$ we have

$$s > F(\varepsilon) \geq F(2^{-(\bar{l}-1)}) \geq F(2^{-(l-3)/2}) = \varphi(l).$$

This finishes the proof. □

If a φ-aperiodic word contains a periodic subword of infinite length then the function φ is bounded, whereas if a word is φ-aperiodic for an unbounded function, the word must be aperiodic. We want to give some examples in order to make the definition more familiar, among them the prominent Morse-Thue-sequence:

Example 4.12. First, let $a, b \in \mathcal{A}$. One checks that the (non-recurrent) words $w_1 = \ldots bbbaaa \ldots$ and $w_2 = ..abaabaaabaaaab \ldots$ are φ-aperiodic only for a function φ such that $1 = s > \varphi(l)$ for all $l \in \mathbb{N}_0$. Both, the orbits of w_1 and w_2, come closer and closer to the periodic word $\ldots aaa \ldots$ with respect to the metric $\bar{d}$. This is not the case for φ-aperiodic words when φ is unbounded; see Proposition 4.17.

Consider the Morse-Thue recurrent sequence $w \in \{0,1\}^{\mathbb{Z}}$ which is determined as follows: Let $a_0 = 0$, $b_0 = 1$. Then for $n \in \mathbb{N}_0$, let $a_{n+1} = a_n b_n$ and $b_{n+1} = b_n a_n$ be finite words of length $2^{n+1}-1$. Then w is defined such that it satisfies $[w(0) \ldots w(2^n-2)] = a_n$ and $[w(-n)] = [w(n-1)]$ for every $n \in \mathbb{N}$. In particular, w contains the sub words $a_{n+2} = a_n b_n b_n a_n$. Hence for every length $l = 2^n - 1$, w contains subwords of the form WW where W has length l. A function φ such that w is φ-aperiodic must therefore be bounded by $\varphi(2^n - 1) \leq 2^n - 1$ for every $n \in \mathbb{N}$. On the other hand there are no sub words of the form WWa where a is the first letter of a sub word W (see [41]). In other words, w is overlap-free (which means that there are no sub words of the form $aWaWa$ for a finite sub word W and a letter a), from which follows that there are even no sub words of the form $wWwWw$ for w and W finite subwords. Hence we may choose $\varphi(l) \geq l$. We conclude that w is at least φ-aperiodic for the function $\varphi(l) = l$, $l \in \mathbb{N}_0$.

The example shows that the set of φ-aperiodic words $\mathcal{F}_T = \mathcal{F}_T(\varphi)$ is nonempty for the unbounded function $\varphi(l) = l$ and moreover, the Morse-Thue sequence gives an explicit example of such a word. However, let $a \in \mathcal{A}$ such that $\mu_{\mathcal{A}}(\{a\}) > 0$ and let $w = \ldots aaa \ldots$ be a periodic word which is of systolic period 1. Moreover, μ is positive on the critical neighborhood of w and hence by Lemma 4.6, $\mathcal{F}_T$ is of zero μ-measure unless φ is strictly bounded by 1.

Our main result for sequences is the following. It will be proved in Section 4.5.

Theorem 4.13. *Let $\varphi : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing unbounded function such that there exists $c \in (1, k)$ satisfying*

$$k - \lfloor \varphi(0) \rfloor - \sum_{l=1}^{\infty} \frac{\lfloor \varphi(l) \rfloor - \lfloor \varphi(l-1) \rfloor}{c^l} \geq c, \tag{4.3.2}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Then there exists a φ-aperiodic word in Σ.

Remark 4.14. The condition is satisfied for the following set of parameters:
(1) $k \geq 4$, then $\varphi(l) = l$ satisfies (4.3.2) for $c = 2$,
(2) $k \geq 5$, then $\varphi(l) = 2^l$ satisfies (4.3.2) for $c = 3$,
(3) $k \geq 2$, $0 < \delta < 1$ and $k^\delta < c < k$, then there exists $l_0 = l_0(k, \delta, c) \in \mathbb{N}_0$ such that

$$\varphi(l) = \begin{cases} 0, & \text{for } l \leq l_0 \\ k^{\delta l}, & \text{for } l > l_0 \end{cases} \tag{4.3.3}$$

satisfies (4.3.2).

Note that if a word w is φ-aperiodic then $R_w(l) > \varphi(l)$ for every $l \in \mathbb{N}_0$ where R_w is the recurrence time introduced in Section 4.1. Theorem 4.1 is hence a corollary of Theorem 4.13.

Proof of Theorem 4.1. By condition (4.1.1), for every $\varepsilon_0 > 0$ there exists $l_1 = l_1(\varepsilon_0) \in \mathbb{N}$ such that for all $l \geq l_1$,

$$\frac{1}{l}\log(\varphi(l)) \leq \delta \log(k)(1+\varepsilon_0).$$

Since $\delta < 1$ we let $\varepsilon_0 > 0$ such that $\tilde{\delta} = (1+\varepsilon_0)\delta < 1$. Then, $\varphi(l) \leq k^{\tilde{\delta} l}$ for $l \geq l_1$. If we take $c \equiv (k - k^{\tilde{\delta}})/2$ then by (4.3.3) there exists $l_2 = l_2(k, \tilde{\delta})$ such that condition (4.3.2) is satisfied for the function $\bar{\varphi}(l) \equiv k^{\tilde{\delta} l}$ for $l > l_2$ and $\bar{\varphi}(l) = 0$ for $l \leq l_2$, $l \in \mathbb{N}_0$. Theorem 4.13 implies the existence of a $\bar{\varphi}$-aperiodic word $w \in \Sigma$. Thus, setting $l_0 \equiv \max\{l_1, l_2\} + 1$, we have that $\bar{\varphi}(l) \geq \varphi(l)$ for all $l \geq l_0$ and the claim follows. □

Remark 4.15. The critical function φ for which φ-aperiodic words cannot exist is the function $\varphi(l) = k^{l+1}$. The critical exponent $\log(k)$ equals the topological entropy of the system $(\Sigma, \bar{d}, T)$ (see [59]) and is optimal. To see that there exists no $w \in \Sigma$ which is φ-aperiodic for a function φ such that $\varphi(l) \geq k^{l+1} - 1$ for some $l \in \mathbb{N}_0$, fix a subword $[w(1)\dots w(1+l)]$ of any $w \in \Sigma$. Inductively one shows that at each step $1 \leq s \leq \varphi(l)$ one has at most $k^{l+1} - s$ possibilities to choose a sub word $[w(1+s)\dots w(1+s+l)]$ such that w stays φ-aperiodic. Then, at step $s = k^{l+1}$, there is no choice left such that w is φ-aperiodic.

Remark 4.16. Let $\Sigma^+(m) = \{w : \{1,\dots,m\} \to \mathcal{A}\}$ be the set of words of length m in $\mathcal{A}$ and $\mathcal{W}^g(m) \subset \Sigma^+(m)$ be the set of *good* words of length m which satisfy (4.3.1) for all $i, s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i + s + l \leq m$. If φ satisfies (4.3.2) with respect to the parameter $c > 1$ we will see in the proof of Theorem 4.13 (see Lemma 4.34) that the good words $\mathcal{W}^g(m)$ increase in m by the factor c. Thus, $|\mathcal{W}^g(m)| \geq c^m$ which is a lower bound on the asymptotic growth of $|\mathcal{W}^g(m)|$, where $|\cdot|$ denotes its cardinality.

We may reformulate the critical neighborhood of a periodic point given in (4.2.1) to the setting of φ-aperiodicity. Moreover, since $\mathcal{P}_T$ is dense in Σ we can also give a sufficient condition on φ-aperiodicity in terms of periodic words. Therefore, for a non-decreasing unbounded function $\varphi : \mathbb{N}_0 \to [0,\infty)$, we define a discrete form of a right-inverse for φ by $\ell : \mathbb{N} \to \mathbb{N}_0$,

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\}, \tag{4.3.4}$$

which is also non-decreasing and unbounded.

A word is φ-aperiodic if and only if its orbit avoids all periodic words $\bar{w}$ with respect to a distance depending on φ and the period of $\bar{w}$; more precisely we have the following.

Proposition 4.17. *Let $\varphi : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing unbounded function. If $w \in \Sigma$ is φ-aperiodic, then for every periodic word $\bar{w} \in \Sigma$ of period s and for all $i \in \mathbb{Z}$ we have*

$$\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s))/2}.$$

Conversely, if $\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s)-1)/2}$ for every periodic word $\bar{w}$ of period s and all $i \in \mathbb{Z}$, then w is φ-aperiodic.

Proof. If w is φ-aperiodic, w is aperiodic and there exists $m \in \mathbb{N}_0$ such that $\bar{d}(T^i w, \bar{w}) = 2^{-m}$ where we assume $2m \geq s$ (otherwise the first statement follows). Hence, $[w(i-m)\dots w(i+m)] = [\bar{w}(-m)\dots \bar{w}(m)]$ and we see that

$$[w(i-m)\dots w(i-m+s+(2m-s))] = [w(i-m+s)\dots w(i+m)].$$

Thus, $s > \varphi(2m-s)$ and $m < (s+\ell(s))/2$ from (4.5.1).

Conversely, assume that $[w(i)\dots w(i+l)] = [w(i+s)\dots w(i+s+l)]$ for $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ and let $\bar{l} = (s+l)/2$ if $s+l$ even and $\bar{l} = (s+l-1)/2$ if $s+l$ is odd. Moreover, let $\bar{w}$ be the periodic word of period s such that $[\bar{w}(i)\dots\bar{w}(i+s-1)] = [w(i)\dots w(i+s-1)]$. Thus,

$$2^{-\bar{l}} \geq d(T^{i+\bar{l}}w, T^{i+\bar{l}}\bar{w}) > 2^{-(s+\ell(s)-1)/2}$$

and we see that

$$s+\ell(s)-1 > 2\bar{l} \geq s+l-1.$$

Hence, $l < \ell(s)$ and from (4.5.1) we have $s > \varphi(l)$. □

Remark 4.18. Consider the overlap-free recurrence time $\tilde{R}^0_w : \mathbb{N}_0 \to \mathbb{N}$ of the initial sub word,

$$\tilde{R}^0_w(l) = \min\{s > l : [w(s)\dots w(s+l)] = [w(0)\dots w(l)]\}.$$

Clearly, $R_w(l) \leq R^0_w(l) \leq \tilde{R}^0_w(l)$ for $l \in \mathbb{N}_0$. Then it follows from [44] that, since the Bernoulli-shift is ergodic, for μ-almost all $w \in \Sigma$ the limit

$$\lim_{l\to\infty} \frac{\log \tilde{R}^0_w(l)}{l}$$

exists and equals the measure-entropy $h_\mu(T)$.

4.4 Geodesic flow on hyperbolic manifolds

Let M be a closed n-dimensional hyperbolic manifold, that is a compact connected Riemannian manifold without boundary of constant negative curvature -1, where $n \geq 2$. We denote by d the distance function on M and by $i_M > 0$ the injectivity radius.

Let SM be the unit tangent bundle of M and d^S the Sasaki-distance function on SM. For $v \in SM$ let $\gamma_v : \mathbb{R} \to M$ be the unit speed geodesic such that $\gamma_v'(0) = v$. The geodesic flow $\phi^t : SM \to SM$, $t \in \mathbb{R}$, acts on the compact metric space (SM, d^S) by diffeomorphisms, where $\phi^t v = \gamma_v'(t)$. For details and background we refer to [16].

A vector $v \in SM$ is *periodic*, if there exists a $t > 0$ such that $\phi^t v = v$ and v is *recurrent* if for every $\varepsilon > 0$ there exists $s > 0$ such that $d^S(\phi^s v, v) < \varepsilon$. Denote by $\mathcal{P}_\phi$ and $\mathcal{R}_\phi$ the flow-invariant sets of periodic respectively of recurrent vectors. Thus if $v \in \mathcal{R}_\phi$ then for a given $t \in \mathbb{R}$, $\varepsilon > 0$, there exists $s = s(t,\varepsilon)$ such that $d^S(\phi^{t+s}v, \phi^t v) < \varepsilon$.

We now adjust the definitions of F-aperiodic and φ-aperiodic points to the setting of the geodesic flow.

Definition 4.19. *Let $F : (0,\infty) \to [0,\infty)$ be a non-increasing function and $s_0 > 0$ be a constant, called the* minimal shift. *A vector $v \in SM$ is called F*-aperiodic *(with minimal shift s_0) at $t_0 \in \mathbb{R}$ if for every $\varepsilon > 0$, whenever*

$$d^S(\phi^{t_0}v, \phi^{t_0+s}v) < \varepsilon$$

for some shift $s > s_0$, then $s > F(\varepsilon)$. If v is F-aperiodic at every time t_0 then v is called F-aperiodic *(with minimal shift s_0).*

Note that in contrast to the discrete setting in Section 2 (where $s \in \mathbb{N}$, i.e. $s \geq 1$) we now have to specify the additional parameter s_0, since $d^S(\phi^{t_0}v, \phi^{t_0+s}v) = s$ for s small enough.

We also have to generalize the notion of φ-aperiodicity. All geodesics will be assumed to be unit speed. Note that as in the case of the Bernoulli-shift, two vectors in the Sasaki-distance are very close if and only if the trajectories of the corresponding geodesics are close (in the Riemannian distance) to each other for a long time. Thus we may reformulate φ-aperiodicity in terms of the *fellow traveller length*.

Herefore we introduce a second parameter, the *distance constant* $\varepsilon_0 > 0$.

Definition 4.20. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function, let $0 < \varepsilon_0 < i_M$ and $s_0 \geq \varepsilon_0$. A geodesic $\gamma : \mathbb{R} \to M$ is called φ-*aperiodic *at* time $t_0 \in \mathbb{R}$ *if for every* length $l > \varepsilon_0$, *whenever*

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0 \qquad \text{for all } 0 \leq t \leq l$$

for some shift $s > s_0$, *then* $s > \varphi(l)$. *If γ is φ-aperiodic at every time t_0, it is called φ-*aperiodic *(with parameters (s_0, ε_0)).*

The geodesic flow on compact hyperbolic manifolds is ergodic with respect to the Liouville measure μ (on the Borel-σ-algebra of SM). A systole of M has length $2i_M$ which equals the systolic period. For a non-decreasing function φ let $\mathcal{F}_\phi$ be the set of φ-aperiodic geodesics (with respect to (s_0, ε_0)), which is invariant under the geodesic flow ϕ^t. Since μ is positive on open sets, one can show as in Lemma 4.6, that the set $\mathcal{F}_\phi$ is of zero μ-measure if and only if φ is not bounded by either s_0 or $2i_M - \varepsilon_0$.

The main result of this section is the following, which will be proved in the Section 4.5.

Theorem 4.21. *Assume that $i_M > \log(2)$ and let $\varepsilon_0 > 0$ such that $\log(2) + \varepsilon_0 < i_M$. Let*

$$\varphi_\delta(l) = e^{\delta(n-1)l},$$

where $0 < \delta < 1$. Then there exists a minimal length $l_0 = l_0(\delta, i_M, n, \varepsilon_0)$ and a geodesic $\gamma : \mathbb{R} \to M$ which satisfies for every $t_0 \in \mathbb{R}$ and all $l \geq l_0$, whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t) < \varepsilon_0 \qquad \text{for all } 0 \leq t \leq l \tag{4.4.1}$$

for some shift $s > \varepsilon_0$, then $s > \varphi_\delta(l)$.

Remark 4.22. The critical function φ such that φ-aperiodic geodesics might exist is the function $c \cdot \varphi(s)_{n-1} = c \cdot e^{(n-1)s}$, $c \geq 0$, where the critical exponent $n - 1$ equals the topological entropy of (SM, ϕ^t) and is optimal.

In fact, although the box dimension of the manifold SM is $2n-1$, we will prove in Subsection 4.6.1 that there exist a $\bar{c} = \bar{c}(\mathrm{diam}(M), \varepsilon_0) > 0$ such that no $c \cdot \varphi_{n-1}$-aperiodic geodesic can exist when $c > \bar{c}$.

Note that for $\varepsilon_0 = i_M/2$, if a geodesic $\gamma : \mathbb{R} \to M$ satisfies (4.4.1), then $R_\gamma(l) \geq \varphi_\delta(l)$ for all $l \geq l_0$, where R_γ is the recurrence time introduced in Section 4.1. Theorem 4.2 is hence a corollary of Theorem 4.21.

Proof of Theorem 4.2. By (4.1.2), there exists for every $\tau > 0$ some $l_1 = l_1(\tau) \geq 0$ such that for all $l \geq l_1$ we have

$$\varphi(l) \leq e^{(1+\tau)(n-1)\delta l}.$$

Since $\delta < 1$ we let $\tau_0 > 0$ such that $\bar{\delta} \equiv (1+\tau_0)\delta < 1$. From Theorem 4.21 for $\varepsilon_0 = i_M/2$, there exists an $l_2 = l_2(\bar{\delta}, i_M, n)$ and a geodesic geodesic $\gamma : \mathbb{R} \to M$ such that for every $t_0 \in \mathbb{R}$ and $l \geq l_2$, whenever

$$d(\gamma(t_0+t), \gamma(t_0+s+t)) < \frac{i_M}{2} \quad \text{for all } \; 0 \leq t \leq l,$$

for some shift $s > i_M/2$, then $s > e^{\bar{\delta}(n-1)l}$. If we set $l_0 \equiv \max\{l_1, l_2\}$ then $s > e^{\bar{\delta}(n-1)l} \geq \varphi(l)$ whenever $l \geq l_0$ and the proof is finished. □

In order to prove Theorem 4.21 we discretize our geodesics. Therefore we need a third parameter, the *discretization constant* $r_0 > 0$. To a geodesic $\gamma : \mathbb{R} \to M$ we consider the *discrete* geodesic

$$\bar{\gamma} : \mathbb{Z} \to M, \qquad \bar{\gamma}(i) \equiv \gamma(i \cdot r_0).$$

Definition 4.23. (Discrete Definition) *Let $\bar{\varphi} : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing function and let the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ be given where $\bar{s}_0 \in \mathbb{N}_0$, $0 < \bar{\varepsilon}_0 < i_M$ and $0 < r_0 < \bar{\varepsilon}_0$. A discrete geodesic $\bar{\gamma} : \mathbb{Z} \to M$ is called* $\bar{\varphi}$-aperiodic *at time $i \in \mathbb{Z}$ if for $l \in \mathbb{N}$, whenever*

$$d(\bar{\gamma}(i+j), \bar{\gamma}(i+s+j)) < \bar{\varepsilon}_0 \qquad \textit{for all } j \in \{0, \ldots, l\} \tag{4.4.2}$$

for some shift $s > \bar{s}_0$, then $s > \bar{\varphi}(l)$. $\bar{\gamma}$ is called $\bar{\varphi}$-aperiodic *(with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$) if it is $\bar{\varphi}$-aperiodic at every time $i \in \mathbb{Z}$.*

Note that, given a $\bar{\varphi}$-aperiodic geodesic $\bar{\gamma} : \mathbb{Z} \to M$ (with the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$), the corresponding geodesic $\gamma : \mathbb{R} \to M$ is continuously φ-aperiodic in the following way.

Lemma 4.24. *For a non-decreasing function $\bar{\varphi} : [0, \infty) \to [0, \infty)$ and the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ let $\bar{\gamma} : \mathbb{Z} \to M$ be a $\bar{\varphi}|_{\mathbb{N}_0}$-aperiodic geodesic. For $r_0 \leq l \in \mathbb{R}$, define*

$$\varphi(l) \equiv r_0 \cdot \bar{\varphi}(\frac{l - r_0}{r_0}) - r_0.$$

Then γ is φ-aperiodic with respect to the minimal shift $s_0 = (\bar{s}_0 + 1)r_0$ and the distance constant $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 > 0$.

Conversely, if $\gamma : \mathbb{R} \to M$ is φ-aperiodic with parameters (s_0, ε_0) then for $r_0 < \varepsilon_0$, let

$$\bar{\varphi}(l) \equiv \varphi(l \cdot r_0)/r_0.$$

Then $\bar{\gamma} : \mathbb{Z} \to M$ is $\bar{\varphi}$-aperiodic with parameters $(\lceil s_0/r_0 \rceil, \varepsilon_0, r_0)$.

Proof. For $t_0 \in \mathbb{R}$, $L \geq r_0$ and $s > (\bar{s}_0 + 1)r_0$ assume that $d(\gamma(t_0+t), \gamma(t_0+s+t)) < \varepsilon_0$ for all $0 \leq t \leq L$. If we set $i \equiv \lceil \frac{t_0}{r_0} \rceil$ and $i + \bar{s} \equiv \lceil \frac{t_0+s}{r_0} \rceil$ whereas $l \equiv \lfloor \frac{L}{r_0} \rfloor$, we have i, $l \geq 1$ and $\bar{s} > \bar{s}_0$. Then, since $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 < i_M$ and the distance function is locally convex, one checks by the triangle inequality that $d(\bar{\gamma}(i), \bar{\gamma}(i+\bar{s})) < \bar{\varepsilon}_0$ and $d(\bar{\gamma}(i+l), \bar{\gamma}(i+\bar{s}+l)) < \bar{\varepsilon}_0$. In particular, $d(\bar{\gamma}(i+j), \bar{\gamma}(i+\bar{s}+j)) < \bar{\varepsilon}_0$ for all $0 \leq j \leq l$. Thus, $\bar{s} > \bar{\varphi}(l)$ so that

$$s \geq (\bar{s} - 1)r_0 > (\bar{\varphi}(l) - 1)r_0 \geq (\bar{\varphi}(\frac{L}{r_0} - 1) - 1)r_0 = \varphi(L)$$

since $(l+1)r_0 \geq L$. This finishes the first part of the Lemma. The second part follows analogously. □

In terms of Lemma 4.24 we are left with stating the existence theorem for discrete $\bar{\varphi}$-aperiodic geodesics. Recall that for an unbounded function $\bar{\varphi}$ we defined its discrete right-inverse $\bar{\ell} : \mathbb{N} \to \mathbb{N}_0$ in (4.3.4) which is also non-decreasing and unbounded.

Theorem 4.25. *Let $\bar{\varphi} : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing, unbounded function. Assume that $\log(2) < r_0 < \bar{\varepsilon}_0 < i_M$ and $\bar{s}_0 \in \mathbb{N}_0$ such that for all $l \geq \bar{s}_0$,*

$$\lfloor \bar{\varphi}(l) \rfloor > l, \qquad \textit{and} \qquad \bar{\ell}(\bar{s}_0) \geq 1, \tag{4.4.3}$$

and moreover, that there exists a constant $c \in (1, 2^{n-1})$ such that

$$2^{n-1} - \bar{c} \cdot \sum_{l=\bar{\ell}(\bar{s}_0)}^{\infty} \frac{\lfloor \bar{\varphi}(l) \rfloor - \lfloor \bar{\varphi}(l-1) \rfloor}{c^l} \geq c, \tag{4.4.4}$$

where $\bar{c}$ is an explicit constant depending only on n and i_M. Then there exist a $\bar{\varphi}$-aperiodic geodesic $\gamma : \mathbb{Z} \to M$ with the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$.

Remark 4.26. Since $\bar{\ell}$ is unbounded, condition (4.4.4) depends again essentially on the convergence of the sum in (4.4.4). For instance, let $\delta \in (0,1)$ and define $\bar{\varphi}(l) = 2^{\delta(n-1)l}$ and let $c \in (2^{\delta(n-1)}, 2^{n-1})$. Then, since $\bar{\ell}(s) = \lceil \frac{1}{\delta(n-1)\log(2)} \log(s) \rceil$ for $s \geq 0$, there exists a minimal shift $\bar{s}_0 = \bar{s}_0(n, \delta, \bar{c}, c)$ such that (4.4.3) and (4.4.4) are satisfied.

The constant $\bar{c}$ of condition (4.4.4) can in fact be sharpened to be also dependent on $\bar{s}_0$, in which case it is strictly decreasing in $\bar{s}_0$. It will be explicitly defined in the proof of claim 4.40. We may give a rough upper bound of $\bar{c}$ which is independent of $\bar{s}_0$ by

$$\bar{c} \leq \lceil (3\cosh(i_M)\sqrt{n+1})^{n-1} \rceil \lceil \frac{\int_0^{5i_M + 4\log(\sqrt{n+1}/2)} \sinh(t)^{n-1} dt}{\int_0^{i_M/2} \sinh(t)^{n-1} dt} \rceil. \tag{4.4.5}$$

The lower bound $\log(2)$ on the injectivity radius is necessary for the proof. However we believe that the result should be valid without this bound. Moreover, a version of Theorem 4.25 remains true for M a closed n-dimensional Riemannian manifold of negative sectional curvature.

Remark 4.27. For a closed geodesic $\alpha : \mathbb{R} \to M$, let $\mathcal{N}_{\varepsilon_0}(\alpha)$ be the (closed) $\varepsilon_0/2$-neighborhood of α in M, where $\varepsilon_0 > 0$ sufficiently small. When a geodesic $\gamma : \mathbb{R} \to M$ enters $\mathcal{N}_{\varepsilon_0}(\alpha)$ at time t_0 let $\mathfrak{p}_\alpha(\gamma, t_0)$ be the *penetration length* of γ in α at time t_0, that is, the maximal length $L \in [0, \infty]$ of an interval I, $t_0 \in I$, such that $\gamma(t) \in \mathcal{N}_{\varepsilon_0}(\alpha)$ for all $t \in I$. Set $\mathfrak{p}_\alpha(\gamma, t_0) = 0$ if $\gamma(t_0) \notin \mathcal{N}_{\varepsilon_0}(\alpha)$. Then by [26], for μ-almost every $v \in SM$ the limit

$$\limsup_{t \to \infty} \frac{\mathfrak{p}_\alpha(\gamma_v, t)}{\log(t)} \tag{4.4.6}$$

exists and equals $1/(n-1)$.

Moreover, the penetration length reflects the *depth* in which γ enters the neighborhood $\mathcal{N}_{\varepsilon_0}(\alpha)$. The study of depths or penetration lengths in an adequate convex set of negatively curved manifolds, such as the ε-neighborhood of totally geodesic embedded submanifold or the cusp-neighborhood of a finite-volume hyperbolic manifold, leads to the theory of diophantine approximation in negatively curved manifolds; see for instance [22, 25, 26, 46, 45, 47, 54, 57] to give only a short and incomplete list. In general, a sequence of depths or penetration lengths and times of γ in these convex sets reflects "how well γ is approximated", where γ is called *badly approximable* if any such sequence is bounded; see [25, 26, 46].

Now, let γ be a φ-aperiodic geodesic (φ unbounded) with respect to the parameters s_0 and ε_0 and let α be **any** closed geodesic in M. Then, it can be seen that the penetration lengths of γ in $\mathcal{N}_{\varepsilon_0/8}(\alpha)$ are bounded by a constant depending only on φ, ε_0 and the length of α (and s_0 respectively). Therefore, the notion of φ-aperiodicty is linked to bad approximation; recall also Example 4.8. In particular, the limit of (4.4.6) equals 0 for γ. For more details, see Subsection 4.6.2.

4.5 Proofs

Let $\varphi : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing unbounded function. Recall the definition of the function $\ell : \mathbb{N} \to \mathbb{N}_0$ given by

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\},$$

see (4.3.4). The following properties hold: ℓ is non-decrasing and for s and $l \in \mathbb{N}_0$, we have

$$\begin{aligned} \varphi(\ell(s)) &\geq s, \\ l < \ell(s) \iff \varphi(l) &< s, \\ l \geq \ell(s) \iff \varphi(l) &\geq s. \end{aligned} \tag{4.5.1}$$

Proof. For the first property, clearly $\varphi(\min\{j : \varphi(j) \geq s\}) \geq s$. Let $l < \ell(s)$ and assume $s \leq \varphi(l)$. Then $\ell(s) = \min\{j : \varphi(j) \geq s\} \leq l$; a contradiction. If $s > \varphi(l)$ then $\ell(s) = \min\{j : \varphi(j) \geq s\} > l$ and if $\varphi(l) \geq s$ then $\ell(j) = \min\{j : \varphi(j) \geq s\} \leq l$. Also, if $l \geq \ell(s)$ then $\varphi(l) \geq \varphi(\ell(s)) \geq s$. □

4.5.1 Proof of Theorem 4.13.

Recall that $\Sigma^+(m) = \{w : \{1, \ldots, m\} \to \mathcal{A}\}$ is the set of words of length $m - 1$. We consider $\Sigma^+(m)$ to be a subset of $\Sigma^+ = \mathcal{A}^{\mathbb{N}}$ (for example, by extending an element $w \in \Sigma^+(m)$ to an element $\bar{w} \in \Sigma^+$ by setting $\bar{w}(i) = a$ for all $i > m$, where $a \in \mathcal{A}$ is fixed).

Definition 4.28. *Let $m \in \mathbb{N}$. $w \in \Sigma^+(m)$ is called* φ-aperiodic *if for all $i, s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i + s + l \leq m$ whenever*

$$[w(i) \ldots w(i + l)] = [w(i + s) \ldots (w(i + s + l)]$$

we have $s > \varphi(l)$.

Let $l_0 \equiv \min\{j \in \mathbb{N}_0 \cup \{-1\} : \varphi(j + 1) \neq 0\}$ and note that $\ell(s) > l_0$ for all $s \in \mathbb{N}$. For $m \in \mathbb{N}$, define the *admissible set* by

$$A(m) \equiv \{(i, s) \in \mathbb{N} \times \mathbb{N} : i + s + \ell(s) = m\},$$

if $m \geq m_0 \equiv 2 + \ell(1) > 2 + l_0$ and let $A(m)$ be empty for $m < m_0$. Then, for $(i, s) \in A(m)$ where $m \geq m_0$, we define the sets

$$C_{is} \equiv \{w \in \Sigma^+(m) : [w(i) \ldots w(i + \ell(s))] \neq [w(i + s) \ldots w(i + s + \ell(s))]\},$$

called *conditions*.

Remark 4.29. Note that $s > \varphi(\ell(s)-1)$ for $\ell(s) > 0$ but $s \le \varphi(\ell(s))$. Therefore $\ell(s)$ determines the critical length of a given shift s with respect to φ.

For $w \in \Sigma^+(m)$ and $1 \le n \le m$ let $w|_n \equiv [w(1)\dots w(n)] \in \Sigma^+(n)$. This leads to the reformulation of φ-aperiodic words:

Lemma 4.30. *For $m < m_0$ every word $w \in \Sigma^+(m)$ is φ-aperiodic. For $m \ge m_0$, a word $w \in \Sigma^+(m)$ is φ-aperiodic if and only if for all $n \le m$ and all $(i,s) \in A(n)$ we have $w|_n \in C_{is}$.*

Proof. First, let $m < m_0$. Then for every i,$s \in \mathbb{N}$, $l \in \mathbb{N}_0$ such that $i+s+l \le m < 2+\ell(1)$ we have in particular $l < \ell(1)$. Equivalently, $\varphi(l) < 1$ so that $s > \varphi(l)$ and every word $[w(1)\dots w(m)]$ follows to be φ-aperiodic.

Now let $m \ge m_0$. Let w be φ-aperiodic and assume $w|_n \notin C_{is}$ for some i and s in $\mathbb{N}$ such that $i + s + \ell(j) = n \le m$. Then

$$[w(i)\dots w(i+\ell(s))] = [w(i+s)\dots w(i+s+\ell(s))]$$

and by (4.3.1), we have $s > \varphi(\ell(s))$; a contradiction to $\varphi(\ell(s)) \ge s$.

Conversely, assume that w is not φ-aperiodic. Then there are i, $s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i+s+l \le m$ and

$$[w(i)\dots w(i+l)] = [w(i+s)\dots w(i+s+l)]$$

with $s \le \varphi(l)$. This implies that $\ell(s) \le l$ and in particular

$$[w(i)\dots w(i+\ell(s))] = [w(i+s)\dots w(i+s+\ell(s))].$$

Hence, it follows that $w|_n \notin C_{is}$ since $i+s+\ell(s) = n \le m$ so that $(i,s) \in A(n)$. □

Note that by the same arguments as in the previous proof, a word $w \in \Sigma^+$ is φ-aperiodic if and only if for all $n \ge m_0$ and all $(i,s) \in A(n)$ we have $w|_n \in C_{is}$.

For $m \in \mathbb{N}$ such that $m \ge m_0$ the set of good words of length m is therefore given by

$$\mathcal{W}^g(m) = \{w \in \Sigma^+(m) : w|_n \in C_{is} \text{ for all } (i,s) \in A(n) \text{ where } n \le m\},$$

and by $\mathcal{W}^g(m) = \Sigma^+(m)$ otherwise. Let

$$\mathcal{C}_m = \{C_{is} : (i,s) \in A(m)\}$$

be the set of conditions at place m which is empty if and only if $m < m_0$. Clearly, if $w \in \mathcal{W}^g(m)$ then $w|_n \in \mathcal{W}^g(n)$ for $n \le m$.

Lemma 4.31. *For $m \in \mathbb{N}$,*

$$|\mathcal{W}^g(m+1)| \ge k \cdot |\mathcal{W}^g(m)| - \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{W}^g(i+s-1)|$$

Proof. If $m+1 < m_0$ then $\mathcal{C}_{m+1}$ is empty and the claim follows. Hence let $m+1 \ge m_0$. Set $L = \{w \in \Sigma^+(m+1) : w|_m \in \mathcal{W}^g(m)\}$. Then

$$\mathcal{W}^g(m+1) = L \cap (\bigcap_{C_{is}\in\mathcal{C}_{m+1}} C_{is}) = L \setminus (\bigcup_{C_{is}\in\mathcal{C}_{m+1}} (L \cap C_{is}^C)),$$

where C_{is}^C denotes the complement of C_{is}. Fix some condition $C_{is} \in \mathcal{C}_{m+1}$. Since $|L| = k\cdot|\mathcal{W}^g(m)|$ the Lemma follows from the following claim. □

Claim 4.32. $|L \cap C_{is}^C| \le |\mathcal{W}^g(i+s-1)|$.

Proof. If $Q \equiv \{w|_{i+s-1} \in \Sigma^+(i+s-1) : w \in L\}$ then clearly $|Q| \le |\mathcal{W}^g(i+s-1)|$. Decompose L into $L = \cup_{q\in Q} L_q$ where $L_q = \{w \in L : w|_{i+s-1} = q\}$. By definition, different elements in L_q have different subwords $[w(i+s)\dots w(m+1)]$ and moreover

$$L \cap C_{is}^C = \{w \in L : [w(i)\dots w(i+\ell(s)] = [w(i+s)\dots w(m+1)]\}.$$

Hence, if $s > \ell(s)$ then an element w of L_q, which is also in C_{is}^C, is uniquely determined by q, that means, w is of the form $w|_{i+s-1} = q$ and

$$[w(i+s)\dots w(m+1)] = [q(i)\dots q(i+\ell(s))].$$

If $s \le \ell(s)$ then one inductively checks that a word w in $L_q \cap C_{is}^C$ is of the form $w|_{i+s-1} = q$,

$$\begin{aligned} [w(i+js)\dots w(i+(j+1)s-1)] &= [w(i+(j-1)j)\dots w(i+js-1)] \\ &= \dots \\ &= [w(i)\dots w(i+s-1)] = [q(i)\dots q(i+s-1)] \end{aligned}$$

for $1 \le j \le j_0$ where j_0 is the maximal j such that $i+(j+1)s-1 \le m+1$, and

$$[w(i+(j_0+1)j)\dots w(m+1)] = [q(i)\dots q(m+1-(i+(j_0+1)s))],$$

if $i+(j_0+1)s < m+1$. Again, w is uniquely determined by q. Hence in both cases, $|L_q \cap C_{is}^C| \le 1$ and therefore

$$|L \cap C_{is}^C| \le |Q| \le |\mathcal{W}^g(i+s-1)|$$

which proves the claim. □

The above Lemma yields the following crucial estimate:

Lemma 4.33. *For $m \in \mathbb{N}$,*

$$|\mathcal{W}^g(m+1)| \ge (k - \lfloor\varphi(0)\rfloor)|\mathcal{W}^g(m)| - \sum_{j=1}^{m} (\lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor)|\mathcal{W}^g(m-j)|. \tag{4.5.2}$$

Proof. For $0 \le j \le m$ let

$$H_j = \{C_{is} \in \mathcal{C}_{m+1} : i+s-1 = m-j\}, \tag{4.5.3}$$

possibly empty. If $C_{is} \in H_j$ then $i+s+\ell(s) = m+1$ and $i+s-1 = m-j$; hence $\ell(s) = j$. Therefore, $|H_j| \le |\{s : \ell(s) = j\}|$. We have $\ell(s) \le j$ if and only if $s \le \varphi(j)$ and thus

$$|\{s : \ell(s) \le j\}| = |\{s : s \le \varphi(j)\}| = \lfloor\varphi(j)\rfloor.$$

For $j \ge 1$ this implies that

$$\begin{aligned} |H_j| &\le |\{s : \ell(s) = j\}| \\ &= |\{s : \ell(s) \le j\} \setminus \{s : \ell(s) \le j-1\}| \\ &= \lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor. \end{aligned}$$

Moreover,

$$|\{s : \ell(s) = 0\}| = |\{s \in \mathbb{N}_0 : \varphi(0) \ge s\}| = \lfloor\varphi(0)\rfloor.$$

Lemma 4.31 concludes the proof. □

Finally we show the existence of a φ-aperiodic word in Σ^+.

Lemma 4.34. *If condition (4.3.2) is satisfied, then $|\mathcal{W}^g(m)| \geq c^m$. In particular, there exists a φ-aperiodic word in Σ^+.*

Proof. For $m+1 < m_0$ we have that $|\mathcal{W}^g(m+1)| = k^{m+1} \geq c^{m+1}$. For $m+1 \geq m_0$ assume that $|\mathcal{W}^g(n)| \geq c \cdot |\mathcal{W}^g(n-1)|$ for all $n \leq m$. Then, by the previous Lemma,

$$\begin{aligned}
|\mathcal{W}^g(m+1)| &\geq (k - \lfloor\varphi(0)\rfloor)|\mathcal{W}^g(m)| - \textstyle\sum_{j=1}^m(\lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor)|\mathcal{W}^g(m-j)| \\
&\geq (k - \lfloor\varphi(0)\rfloor)|\mathcal{W}^g(m)| - \textstyle\sum_{j=1}^m \frac{\lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor}{c^j}|\mathcal{W}^g(m)| \\
&\geq \Big(k - \lfloor\varphi(0)\rfloor - \textstyle\sum_{j=1}^\infty \frac{\lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor}{c^j}\Big)|\mathcal{W}^g(m)| \geq c \cdot |\mathcal{W}^g(m)|,
\end{aligned} \tag{4.5.4}$$

where we used condition (4.3.2) in the last inequality. Now Lemma 4.30 implies the existence of a φ-aperiodic word in Σ^+. □

Given a φ-aperiodic word $w \in \Sigma^+$ and a letter $a \in \mathcal{A}$, extend w to a word $\ldots aaaw \equiv \bar{w} \in \Sigma$ (in the obvious way). Consider the sequence $\{T^n\bar{w}\}_{n\in\mathbb{N}}$ in the compact space Σ and let w_0 be an accumulation point. Note that from the definition of the metric $\bar{d}$, a sequence w^n in Σ converges to a word $w_0 \in \Sigma$ if and only if for every $l \in \mathbb{N}_0$ there exists $N \in \mathbb{N}$ such that $[w^n(-l)\ldots w^n(l)] = [w_0(-l)\ldots w_0(l)]$ for every $n \geq N$. It therefore follows that φ-aperiodicity is a closed condition (as showed similarly in Lemma 4.7). Since every $T^n\bar{w}$ is φ-aperiodic starting at time $-(n-1)$, w_0 is a φ-aperiodic word in Σ. This proves Theorem 4.13.

4.5.2 Proof of Theorem 4.25.

Recall that M is a closed hyperbolic manifold of dimension $n \geq 2$ and we have $\log(2) < r_0 < \bar{\varepsilon}_0 < i_M$. Moreover $\bar{\varphi} : \mathbb{N}_0 \to [0, \infty)$ is a non-decreasing unbounded function for which conditions (4.4.3) and (4.4.4) are satisfied with respect to the given minimal shift $\bar{s}_0 \in \mathbb{N}_0$.

A reference for the following is given by [16, 58]. Let $\mathbb{H}^n$ be the n-dimensional hyperbolic upper half-space model where d denotes the hyperbolic distance function on $\mathbb{H}^n$. Let Γ be the discrete, torsion-free subgroup of the isometry group of $\mathbb{H}^n$ identified with the fundamental group $\pi_1(M)$ of M acting cocompactly on $\mathbb{H}^n$ such that the manifold $\Gamma\backslash\mathbb{H}^n$ with the induced smooth and metric structure is isometric to M. Let $\pi : \mathbb{H}^n \to \Gamma\backslash\mathbb{H}^n \cong M$ be the projection map. Assume all geodesic segments, rays or lines to be parametrized by arc length and identify their images with their point sets in $\mathbb{H}^n$. Let $\partial_\infty\mathbb{H}^n$ be the set of equivalence classes of asymptotic rays in $\mathbb{H}^n$ which we identify with the set $\mathbb{R}^{n-1} \cup \{\infty\}$, where $\bar{\mathbb{H}}^n - \{\infty\} = \mathbb{H}^n \cup \mathbb{R}^{n-1}$ is equipped with the induced Euclidean topology. If γ is a ray in $\mathbb{H}^n$ we will simply write $\gamma(\infty)$ for the corresponding point in $\partial_\infty\mathbb{H}^n$. For any two points p and q in $\bar{\mathbb{H}}^n$ denote by $[p, q]$ the geodesic segment, ray or line in $\mathbb{H}^n$ - depending on if $p, q \in \mathbb{H}^n$, $p \in \mathbb{H}^n$ and $q \in \partial_\infty\mathbb{H}^n$, or $p, q \in \partial_\infty\mathbb{H}^n$ respectively - connecting p and q.

For $t \in \mathbb{R}$ let $H_t \equiv \mathbb{R}^{n-1} \times \{e^{-t}\} \subset \mathbb{H}^n$. This equals the horosphere based at ∞ through the point $\gamma(t)$ of the unit speed geodesic $\gamma(t) = (0, e^{-t})$. Let h_t be the induced length metric on H_t with respect to d . The geometry of horospheres in the hyperbolic space is well-known; see for instance [23] for the following facts. (H_t, h_t) is a complete and flat metric space, isometric to the $(n-1)$-dimensional Euclidean space. If $\gamma_i : \mathbb{R} \to \mathbb{H}^n$ with $\gamma_i(0) \in H_0$, $i = 1, 2$, are two geodesic lines in $\mathbb{H}^n$ with $\gamma_1(-\infty) = \gamma_2(-\infty) = \infty$ and $\gamma_1(0)$, $\gamma_2(0)$ in the same horosphere, let $\mu(t) \equiv h_t(\gamma_1(t), \gamma_2(t))$. Then, for $t \geq 0$,

$$\mu(t) = e^t \mu(0). \tag{4.5.5}$$

Moreover, for two points p, q in the same horosphere H_t we have

$$h_t(p, q) = 2\sinh(d(p, q)/2). \tag{4.5.6}$$

Now let $\tau > 0$ such that the discretization constant satisfies $r_0 = \log 2 + \tau$. Let $R > 0$ be a fixed length, say $R = 1$. Define Q to be an isometric copy of a closed $(n-1)$-dimensional cube $[-R/2, R/2]^{n-1}$ of edge lengths R in the Euclidean space $\mathbb{E}^{n-1}$ and contained in the horosphere H_0. Starting with the cube Q as a reference, we inductively shed shadows in the horospheres H_{mr_0}, $m \in \mathbb{N}$, as follows:

Definition 4.35. *Given two disjoint sets S and S' in $\bar{\mathbb{H}}^n$, the set $\mathcal{S}(S; S') \equiv \{q \in S' : S \cap [\infty, q] \neq \emptyset\}$ is called the* shadow of S in S' *(with respect to ∞).*

By (4.5.5), the shadow $\mathcal{S}(Q; H_{r_0})$ of Q is an isometric copy of a closed $(n-1)$-dimensional cube of edge lengths $e^{r_0}R = (2 + e^{\tau})R$, contained in H_{r_0}. Hence, there exist 2^{n-1} disjoint isometric copies Q_j, $j \in \{1, \ldots, 2^{n-1}\}$, of Q in $\mathcal{S}(Q; H_{r_0})$; see Figure 4.5.2.

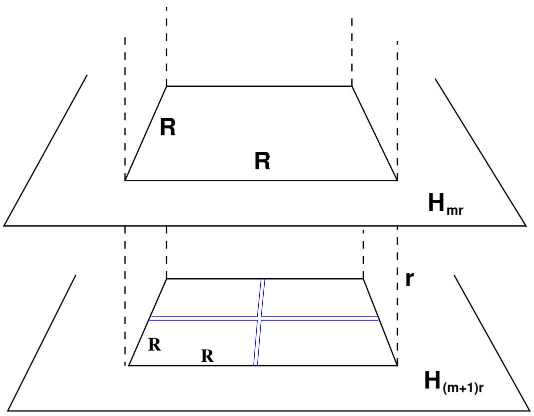

Figure 4.5.2: Shadowing cubes in horospheres ($n = 3$).

For $m \geq 1$, let the closed disjoint cubes $Q_{i_1 \ldots i_m}$ in H_{mr_0} be already defined. Fix a *cube* $Q_{i_1 \ldots i_m}$, then, as above, the shadow

$$\mathcal{S}(Q_{i_1 \ldots i_m}; H_{(m+1)r_0}) \subset H_{(m+1)r_0}$$

contains 2^{n-1} disjoint isometric copies $Q_{i_1 \ldots i_m j}$ of Q, $j \in \{1, \ldots, 2^{n-1}\}$. Hence, for an alphabet $\mathcal{A} = \{1, \ldots, 2^{n-1}\}$, we associate a finite word $[w(1) \ldots w(m+1)] \in \Sigma^+(m+1)$ to the cube $Q_{i_1 \ldots i_{m+1}}$ in $H_{(m+1)r_0}$ where $w(n) = i_n$ for all $n \in \{1, \ldots, m+1\}$. In particular, we obtain a bijection of finite words $\Sigma^+(m)$ of length m with the set of cubes

$$\mathcal{Q}(m) \equiv \{Q_{i_1 \ldots i_m} \subset H_{mr_0} : i_n \in \{1, \ldots, 2^{n-1}\} \text{ for } 1 \leq n \leq m\}.$$

We denote the closed cubes $Q_{i_1 \ldots i_m}$ obtained in this way by $q(1) \ldots q(m)$ where $q(n) \in \{1, \ldots, 2^{n-1}\}$ for $n \in \{1, \ldots, m\}$. Every sequence of cubes $\{q(1)q(2) \ldots q(m)\}_{m \in \mathbb{N}}$, successively shadowed from the previous ones, determines a unique point

$$\eta \equiv \bigcap_{m \in \mathbb{N}} \mathcal{S}(q(1) \ldots q(m); \mathbb{R}^{n-1}) \in \mathbb{R}^{n-1}, \tag{4.5.7}$$

since $\mathcal{S}(q(1)\ldots q(m);\mathbb{R}^{n-1})$, $m\in\mathbb{N}$, is a sequence of closed nested subsets of $\mathbb{R}^{n-1}$ with diameters converging to 0. Define $\eta\equiv q(1)q(2)\ldots$ in $\mathbb{R}^{n-1}$. By construction, the geodesic line $[\infty,\eta]$ runs through every cube $q(1)\ldots q(m)$, $m\in\mathbb{N}$, of the particular sequence. Hence, we obtain a bijection of infinite sequences $q(1)q(2)\ldots$ of cubes and words $w=:[w(1)w(2)\ldots]$ in Σ^+.

Notation. Given a cube $q(1)\ldots q(m)$ in $\mathcal{Q}(m)$ and an integer $n\le m$, let $q(1)\ldots q(m)|_n\in\mathcal{Q}(n)$ be the unique cube such that $q(1)\ldots q(m)$ lies in the shadow of $q(1)\ldots q(m)|_n$. Moreover, for $\xi\in\mathbb{R}^n$ we denote the geodesic subsegment $[i,j](\xi)$ by

$$[i,j](\xi)\equiv[\infty,\xi]|_{[ir_0,jr_0]}\colon[ir_0,jr_0]\to\mathbb{H}^n,$$

where we assume that $[\infty,\xi](0)\in H_0$ and that $i,j\in\mathbb{N}_0$ with $i\le j$, which connects the horospheres H_{ir_0} to H_{jr_0} and is orthogonal to both. If $i=j$, then we write $[i](\xi)\equiv[i,i](\xi)$ which is the orthogonal projection of ξ on the horosphere H_{ir_0}.

We again define the *admissible set*

$$A(m)\equiv\{(i,s)\in\mathbb{N}\times\mathbb{N}: i+s+\bar{\ell}(s)=m, s>\bar{s}_0\},$$

if $m\ge m_0\equiv 2+\bar{s}_0+\bar{\ell}(\bar{s}_0+1)$ and set $A(m)$ to be empty for $m<m_0$.

Definition 4.36. *Let $\psi\in\Gamma$ be an isometry and let $i,\ s\in\mathbb{N}$, $l\in\mathbb{N}_0$. If $\xi\in\mathbb{R}^{n-1}$ such that $d(\psi([i](\xi)),[i+s](\xi))<\bar{\varepsilon}_0$ and also $d(\psi([i+l](\xi)),[i+s+l](\xi))<\bar{\varepsilon}_0$ we write*

$$\psi([i,i+l](\xi))\sim_{\bar{\varepsilon}_0}[i+s,i+s+l](\xi).$$

In particular, by convexity of the distance function, we have for all $j\in\{0,\ldots,l\}$,

$$d(\psi([i,i+j](\xi)),[i+s,i+s+j](\xi))<\bar{\varepsilon}_0. \tag{4.5.8}$$

We are now able to translate the proof of Theorem 4.13 for the existence of φ-aperiodic words into the existence of φ-aperiodic geodesics by counting good cubes:

Definition 4.37. *Let $m\in\mathbb{N}$. A cube $q(1)\ldots q(m)$ in $\mathcal{Q}(m)$ is called* good *if for every $\xi\in\mathcal{S}(q(1)\ldots q(m);\mathbb{R}^{n-1})$, every $\psi\in\Gamma$ and every $i\in\mathbb{N}$, $l\in\mathbb{N}_0$, whenever*

$$\psi([i,i+l](\xi))\sim_{\bar{\varepsilon}_0}[i+s,i+s+l](\xi) \tag{4.5.9}$$

for some shift $s>\bar{s}_0$ such that $i+s+l\le m$, then $s>\bar{\varphi}(l)$. Otherwise $q(1)\ldots q(m)$ is called bad.

If the cube $q(1)\ldots q(m)$ is good, then, since $\bar{\varepsilon}_0<i_M$, for every $x\in q(1)\ldots q(m)$ the projection of the geodesic segment $[\infty,x]|_{[r_0,mr_0]}$ into M is $\bar{\varphi}$-aperiodic, up to length mr_0, with respect to condition (4.4.2) (see the proof Lemma 4.38 (2)).

Analogously to the proof of Theorem 4.13, for $(i,s)\in A(m)$ and $m\ge m_0$, define

$$\begin{aligned}C_{is}\equiv\{q(1)\ldots q(m)\in\mathcal{Q}(m):\ &\text{for all } \xi\in\mathcal{S}(q(1)\ldots q(m);\mathbb{R}^{n-1})\text{ and }\psi\in\Gamma,\\ &\psi([i,i+\bar{\ell}(s)](\xi))\not\sim_{\bar{\varepsilon}_0}[i+s,m](\xi)\}\end{aligned}$$

and let $\mathcal{C}_m$ be the set of all C_{ij} for $(i,j)\in A(m)$. Note that $\mathcal{C}_m$ is empty if $m<m_0$.

With respect to these definitions, the relationship between Definitions 4.23 and 4.37 respectively and the sets C_{is} is given by the following Lemma:

Lemma 4.38. (1) *For $m < m_0$ every cube $q(1)\dots q(m) \in \mathcal{Q}(m)$ is good. For $m \geq m_0$, the cube $q(1)\dots q(m) \in \mathcal{Q}(m)$ is good if $q(1)\dots q(m)|_n \in C_{is}$ for all $n \leq m$ and $(i,s) \in A(n)$.*

(2) *Let $q(1)q(2)\dots$ be an infinite sequence of cubes and let $\eta \in \mathbb{R}^{n-1}$ be the unique corresponding limit point. The discrete geodesic $\overline{\pi \circ [r_0,\infty)(\eta)}$ in M is $\bar{\varphi}$-aperiodic at every time $i \in \mathbb{N}$ if for all $m \in \mathbb{N}$ and $(i,s) \in A(m)$ the cube $q(1)\dots q(m)$ in $\mathcal{Q}(m)$ of the sequence $q(1)q(2)\dots$ belongs to C_{is}.*

Proof. For (1), let first $m < m_0$. Let i, $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ such that $s > \bar{s}_0$ and $i+s+l \leq m < 2+\bar{s}_0+\bar{\ell}(\bar{s}_0+1)$. In particular, $l < \bar{\ell}(\bar{s}_0+1)$ so that $\varphi(l) < \bar{s}_0+1 \leq s$ and every cube $q(1)\dots q(m)$ follows to be good.

Now let $m \geq m_0$. Assume by absurd that $q(1)\dots q(m)$ is not good and let $\xi \in \mathcal{S}(q(1)\dots q(m);\mathbb{R}^{n-1})$ and $\psi \in \Gamma$ such that for some $i \in \mathbb{N}$, $l \in \mathbb{N}_0$, we have

$$\psi([i,i+l](\xi)) \sim_{\bar{\varepsilon}_0} [i+s,i+s+l](\xi),$$

where $s > \bar{s}_0$ with $i+s+l \leq m$ and $s \leq \bar{\varphi}(l)$. Hence, $\bar{\ell}(s) \leq l$ and for $n \equiv i+s+\bar{\ell}(s)$ we have in particular by (4.5.8),

$$\psi([i,i+\bar{\ell}(s)](\xi)) \sim_{\bar{\varepsilon}_0} [i+s,n](\xi).$$

Hence, we see that $q(1)\dots q(m)|_n \notin C_{is}$ where $(i,s) \in A(n)$ for $n \leq m$; a contradiction.

For (2), assume that $\bar{\gamma} \equiv \overline{\pi \circ [r_0,\infty)(\eta)}$ is not $\bar{\varphi}$-aperiodic at time $i \in \mathbb{N}$. Then there must be a shift $s \in \mathbb{N}$ with $s > \bar{s}_0$, and $l \in \mathbb{N}_0$ such that

$$d(\bar{\gamma}(i+j),\bar{\gamma}(i+s+j)) < \bar{\varepsilon}_0 \quad \text{for all } j \in \{0,\dots,l\},$$

where $s \leq \bar{\varphi}(l)$. Since $\bar{\varepsilon}_0 < i_M$ and the distance function is convex, we also have $d(\gamma((i+t)r_0),\gamma((i+s+t)r_0) < \bar{\varepsilon}_0$ for all $0 \leq t \leq l$ for the corresponding extended geodesic $\gamma : \mathbb{R} \to M$. By discreteness of Γ, there exist finitely many isometries $\psi_1,\dots,\psi_q \in \Gamma$ and a subdivision of the interval $[ir_0,(i+l)r_0]$ into $[l_0r_0,l_1r_0],[l_1r_0,l_2r_0],\dots,[l_{q-1}r_0,l_qr_0]$ where $l_0 = i$ and $l_q = i+l$ and $l_j \in \mathbb{R}$, such that (with analogous notation as above)

$$\psi_{j+1}([l_j,l_{j+1}](\eta)) \sim_{\bar{\varepsilon}_0} [s+l_j,s+l_{j+1}](\eta), \quad j = 0,\dots,q-1.$$

We thus have $d(\psi_{j+1}([l_{j+1}](\eta)),[s+l_{j+1}](\eta)) < \bar{\varepsilon}_0$ and $d(\psi_{j+2}([l_{j+1}](\eta)),[s+l_{j+1}](\eta)) < \bar{\varepsilon}_0$. Since $\bar{\varepsilon}_0 < i_M$ and every orbit of Γ is $2i_M$-separated (that is, for ψ, $\bar{\psi} \in \Gamma$ we have $d(\psi x,\bar{\psi}x) \geq 2i_M$ for any $x \in \mathbb{H}^n$) it follows from the triangle inequality that $\psi_{j+1}([l_{j+1}](\eta)) = \psi_{j+2}([l_{j+1}](\eta))$; hence $\psi_{j+1} = \psi_{j+2}$ for all $j = 0,\dots,q-2$ since Γ acts freely. Therefore, we have an isometry $\psi \in \Gamma$ such that

$$\psi([i,i+l](\eta)) \sim_{\bar{\varepsilon}_0} [i+s,i+s+l](\eta)$$

where $s \leq \bar{\varphi}(l)$. The proof is now finished analogously to the case of (1). □

In view of Lemma 4.38, let for $m \geq m_0$,

$$\mathcal{Q}^g(m) = \{q(1)\dots q(m) \in \mathcal{Q}(m) : q(1)\dots q(m)|_n \in C_{is} \text{ for all } (i,s) \in A(n), n \leq m\},$$

and $Q^g(m) = \mathcal{Q}(m)$ for $m < m_0$, which is a subset of all good cubes at step m.

Lemma 4.39. *Assume that condition (4.4.3) is satisfied. Then, for $m \in \mathbb{N}$,*

$$|\mathcal{Q}^g(m+1)| \geq k|\mathcal{Q}^g(m)| - \bar{c} \cdot \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{Q}^g(i+s-1)|, \tag{4.5.10}$$

where $\bar{c}$ is a constant depending only on n, i_M and $\bar{s}_0$, and is strictly decreasing in $\bar{s}_0$.

Proof. If $m+1 < m_0$ then $\mathcal{C}_{m+1}$ is empty and the claim follows. Hence assume $m+1 \geq m_0$. Let

$$L = \{q(1)\dots q(m+1) \in \mathcal{Q}(m+1) : q(1)\dots q(m+1)|_m \in \mathcal{Q}^g(m)\}$$

and note that $|L| = k|Q^g(m)|$. Then

$$\mathcal{Q}^g(m+1) = L \cap (\bigcap_{C_{is} \in \mathcal{C}_{m+1}} C_{is}) = L \setminus (\bigcup_{C_{is} \in \mathcal{C}_{m+1}} (L \cap C_{is}^C)),$$

where C_{is}^C is the complement of C_{is}. Fix some $C = C_{is} \in \mathcal{C}_{m+1}$. Define

$$Q = \{q(1)\dots q(m+1)|_{i+s-1} \in \mathcal{Q}(i+s-1) : q(1)\dots q(m+1) \in L\},$$

One checks that $|Q| \leq |\mathcal{Q}^g(i+s-1)|$. Let $L = \cup_{q \in Q} L_q$ where

$$L_q = \{q(1)\dots q(m+1) \in L : q(1)\dots q(m)|_{i+s-1} = q\}.$$

It remains to show that each $L_q \cap C^C$ contains at most $\bar{c}$ cubes; in this case,

$$|L \cap C^C| \leq \bar{c} \cdot |Q| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|.$$

The following claim concludes the proof. □

Claim 4.40. $|L_q \cap C^C| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|$.

For the proof of the claim note that if (4.4.4) is satisfied, then for all $l \geq \bar{s}_0$,

$$\lfloor \bar{\varphi}(l) \rfloor > l,$$

which implies that for all $s > \bar{s}_0$,

$$\bar{\ell}(s) < s. \tag{4.5.11}$$

To see this, assume $\bar{\ell}(s) \geq s$ for some $s > \bar{s}_0$. Then, by definition of $\bar{\ell}$, $\bar{\varphi}(j) < s$ for all $s > j \in \mathbb{N}_0$. In particular, for $\bar{s}_0 < s$ we have $\bar{\varphi}(\bar{s}_0) \geq \lfloor \bar{\varphi}(\bar{s}_0) \rfloor$; a contradiction to $\lfloor \bar{\varphi}(\bar{s}_0) \rfloor > \bar{s}_0$.

Proof of the Claim 4.40. L_q consists of cubes of the form $q \cdot q(i+s)\dots q(m+1) \in \mathcal{Q}(m+1)$. Hence, consider the point set W of all geodesic segments $[i, i+\bar{\ell}(s)](\xi)$ where $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$; see Figure 4.5.2. Since $s > \bar{s}_0$ we have $\bar{\ell}(s) < s$ by (4.5.11), and therefore $s - 1 - \bar{\ell}(s) \geq 0$. Moreover, by definition, the cube q in $H_{(i+s-1)r_0}$ has h-edge lengths R. Thus from (4.5.5), the subset $H_{i+\bar{\ell}(s)} \cap W$ is isometric to an Euclidean cube with h-edge length

$$e^{-(i+s-1)r_0 + (i+\bar{\ell}(s))r_0} R = e^{-(s-1-\bar{\ell}(s))r_0} R \leq R.$$

Since an Euclidean cube in $\mathbb{E}^{n-1}$ of edge length L has diameter at most $\sqrt{n-1}L$, we obtain from (4.5.6) that the d-diameter of $H_{i+\bar{\ell}(s)} \cap W$ is bounded above by

$$2\text{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2). \tag{4.5.12}$$

In the same way, the h-edge length of $H_{ir_0} \cap W$ is given by

$$e^{-(s-1)r_0}R. \tag{4.5.13}$$

Now, by definition, for every $q \cdot q(i+s)\dots q(m+1) \in L_q \cap C^C$ there exists $\psi \in \Gamma$ such that $\psi([i, i+\bar{\ell}(s)](\xi)) \sim_{\bar{\varepsilon}_0} [i+s, m+1](\xi)$ for some $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$. In particular, $x \equiv [m+1](\xi)$ must belong to the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. Thus, we want to estimate the maximal number of cubes in $\mathcal{Q}(m+1)$ which intersect with the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. Let therefore also $y \in H_{(m+1)r_0}$ belong to the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. By the triangle inequality and by (4.5.12), we have

$$d(x, y) \leq 2\bar{\varepsilon}_0 + 2\text{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2).$$

Therefore, again from (4.5.6), the h-diameter of the intersection of the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ with $H_{(m+1)r_0}$ is bounded above by

$$\bar{r}_1(s) \equiv 2\sinh(\bar{\varepsilon}_0 + \text{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2)).$$

On the other hand, the cubes $q \cdot q(i+s)\dots q(m+1) \in \mathcal{Q}(m+1)$ are disjoint and have Euclidean volume R^{n-1}. Therefore, we set

$$\bar{c}_1(s) \equiv \lceil \frac{(\bar{r}_1(s) + \sqrt{n-1}R)^{n-1}}{R^{n-1}} \rceil.$$

Hence, the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ can intersect at most $\bar{c}_1(s)$ qubes in $\mathcal{Q}(m+1)$. Since $q(1)\dots q(m)$ is good for every $q(1)\dots q(m+1) \in L_q$, we conclude that, with respect to ψ, at most $\bar{c}_1(s)$ cubes can become bad in $L_q \cap C^C$.

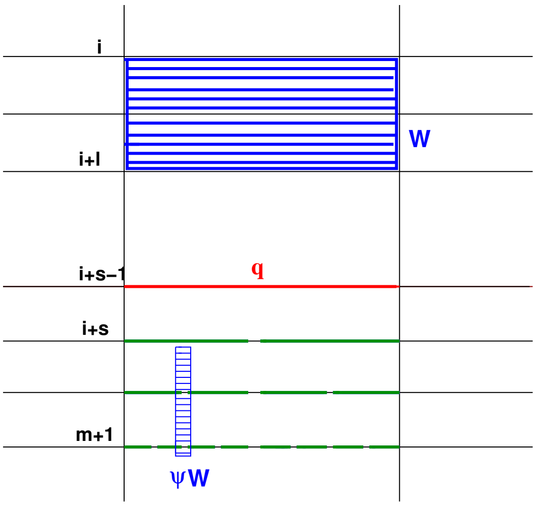

Figure 4.5.2: Cubes which become bad by the isometry ψ ($n = 2$).

Now, let $\bar{y}$ be the center of $W \cap H_{ir_0}$, which is isometric to a cube in the Euclidean space of edge length $e^{-(s-1)r_0}R$ by (4.5.13) and contained in the cube $q|_i$. From (4.5.6), $W \cap H_{ir_0}$ must be contained in the hyperbolic ball $B_d(\bar{y}, \bar{r}_2(s))$, where

$$\bar{r}_2(s) = 2\text{arcsinh}(e^{-(s-1)r_0}\sqrt{n-1}R/4).$$

Note that if there is some point $p \in W \cap H_{ir_0}$ and some $\psi \in \Gamma$ such that $d(\psi p, \bar{q}) < \bar{\varepsilon}_0$, where $\bar{q} \equiv \mathcal{S}(q, H_{(i+s)r_0})$, then $d(\psi\bar{y}, \bar{q}) < \bar{\varepsilon}_0 + \bar{r}_2(s)$. In particular, for every cube $q \cdot q(i+s) \dots q(m+1) \in L_q \cap C^C$ there exists such an isometry ψ. But since the orbit $\Gamma\bar{y}$ is $2i_M$-separated, the open metric balls $B(\psi\bar{y}, i_M)$, $\psi \in \Gamma$, are disjoint and there can only be finitely many, say $\bar{c}_2(j)$, intersecting the $\max\{\bar{\varepsilon}_0 + \bar{r}_2(s) - i_M, 0\}$-neighborhood of $\bar{q}$. In fact, from (4.5.5) and (4.5.6), the h-diameter of $\bar{q}$ is bounded above by $e^{r_0}\sqrt{n-1}R$ and $\bar{q}$ must be contained in a hyperbolic ball of radius $2\text{arcsinh}(e^{r_0}\sqrt{n-1}R/4)$. Therefore, $\bar{c}_2(s)$ is bounded above by

$$\left\lceil \frac{\text{vol}(B(2\text{arcsinh}(e^{r_0}\sqrt{n-1}R/4) + 2\text{arcsinh}(e^{-(s-1)r_0}\sqrt{n-1}R/4) + \bar{\varepsilon}_0))}{\text{vol}(B(i_M/2))} \right\rceil.$$

Since both, $\bar{c}_1(s)$ and $\bar{c}_2(s)$ are non-increasing in s, we conclude the claim by setting $\bar{c} \equiv \bar{c}_1(\bar{s}_0+1)\bar{c}_2(\bar{s}_0+1)$. □

Analogously to the proof of Lemma 4.33, the previous Lemma yields the following.

Lemma 4.41. *Assume that condition (4.5.11) is satisfied. Then, for $m \in \mathbb{N}$,*

$$\begin{aligned} |\mathcal{Q}^g(m+1)| \quad &\geq \quad (k - \mathbf{1}_{\{\bar{\ell}(\bar{s}_0+1)=0\}}\bar{c}\lfloor\bar{\varphi}(0)\rfloor)|\mathcal{Q}^g(m)| \\ &- \quad \bar{c} \cdot \textstyle\sum_{j=\max(\bar{\ell}(\bar{s}_0+1),1)}^{m}(\lfloor\bar{\varphi}(j)\rfloor - \lfloor\bar{\varphi}(j-1)\rfloor)|\mathcal{Q}^g(m-j)|. \end{aligned}$$

Proof. Recall the definition of the set $H_j = \{C_{is} \in \mathcal{C}_{m+1} : i+s-1 = m-j\}$ in (4.5.3). Since $\bar{\ell}$ is non-decreasing we have

$$j = m+1-(i+s) = \bar{\ell}(s) \geq \bar{\ell}(\bar{s}_0+1)$$

if $s > \bar{s}_0$. □

Finally, if moreover condition (4.4.4) is satisfied, then the same inductive proof as in Lemma 4.34 shows that the number of good cubes in $Q^g(m+1)$ increases in $m+1$ by the factor $c > 1$; see (4.5.4). Lemma 4.38.(2) then shows the existence of a $\bar{\varphi}$-aperiodic geodesic $\bar{\gamma} : \mathbb{N} \to M$. Thus, we have shown the following.

Lemma 4.42. *Assume that conditions (4.4.3) and (4.4.4) are satisfied. Then, for $m \in \mathbb{N}$, $|\mathcal{Q}^g(m)| \geq c^m$. In particular, there exists a $\bar{\varphi}$-aperiodic geodesic $\bar{\gamma} : \mathbb{N} \to M$ with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$.*

Now, let $\bar{\gamma} : \mathbb{N} \to M$ be a $\bar{\varphi}$-aperiodic geodesic (with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ and let $\gamma : \mathbb{R} \to M$ be the corresponding extended geodesic. Consider the sequence $v^n \equiv \phi^n\gamma'(r_0)$, $n \in \mathbb{N}$, in the compact space SM and let γ_0 be an accumulation point. The space of unit speed geodesics (identified with SM) is endowed with the topology of uniform convergence on bounded sets. Therefore note that a sequence v^n converges to v in SM if and only if for every $l \geq 0$ and every $\tau > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $d(\gamma_{v^n}(t), \gamma_v(t)) < \tau$ for every $t \in [-l, l]$. Therefore $\bar{\varphi}$-aperiodicity can be shown to be a closed condition (similarly as in Lemma 4.7). Since $\bar{\gamma}_{v^n}$ is $\bar{\varphi}$-aperiodic beginning at $t_n \geq -(n-1)$ (with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$), it follows that $\bar{\gamma}_0 : \mathbb{Z} \to M$ is $\bar{\varphi}$-aperiodic. This completes the proof of Theorem 4.25.

4.5.3 Proof of Theorem 4.21.

For $\delta \in (0,1)$ choose $\bar{\delta} \in [\delta, 1)$ such that for $r_0 = \log(3-\bar{\delta})$ we have $\log(3-\bar{\delta}) + \varepsilon_0 < i_M$. Note that $\tilde{\delta} = \bar{\delta}\log(2)/\log(3-\bar{\delta}) \to 1$ as $\bar{\delta} \to 1$ and assume therefore that $\tilde{\delta} > \delta$. For $l \geq 0$ let $\bar{\psi}(l) = 2^{\bar{\delta}(n-1)l}$ so that its right inverse $\lceil \frac{1}{\bar{\delta}(n-1)\log(2)} \log(s) \rceil$ is an unbounded function. Then, for $c = \frac{1}{2}(2^{n-1} + 2^{\bar{\delta}(n-1)})$, we have that for sufficiently large $\bar{s}_0 = \bar{s}_0(\bar{\delta}, n, i_M, \varepsilon_0) \in \mathbb{N}_0$ the conditions (4.4.3) and (4.4.4) are satisfied. Thus, from Theorem 4.25 there exists a discrete geodesic $\bar{\gamma} : \mathbb{Z} \to M$ which is $\bar{\psi}$-aperiodic with respect to $(\bar{s}_0, r_0 + \varepsilon_0, r_0)$. From Lemma 4.24 we obtain that $\gamma : \mathbb{R} \to M$ is continuously ψ-aperiodic with parameters $s_0 = (\bar{s}_0 + 1)r_0$ and ε_0, where for $l \geq r_0$,

$$\begin{aligned}
\psi(l) &= \log(3-\bar{\delta}) \cdot \bar{\psi}(\tfrac{l}{\log(3-\bar{\delta})} - 1) - \log(3-\bar{\delta}) \\
&= \tfrac{\log(3-\bar{\delta})}{2^{\bar{\delta}(n-1)}} e^{\frac{\bar{\delta}\log(2)}{\log(3-\bar{\delta})}(n-1)l} - \log(3-\bar{\delta}) \\
&= \left(\tfrac{\log(3-\bar{\delta})}{2^{\bar{\delta}(n-1)}} - \tfrac{\log(3-\bar{\delta})}{e^{\tilde{\delta}(n-1)l}}\right) e^{\tilde{\delta}(n-1)l} \\
&\equiv c(\tilde{\delta}, l) \cdot e^{\tilde{\delta}(n-1)l} = c(\tilde{\delta}, l)\varphi_{\tilde{\delta}}(l).
\end{aligned}$$

Note that $c(\tilde{\delta}, l)$ is increasing in l and we restrict ψ to the interval $[l_1, \infty)$ for some $l_1 > \log(3-\bar{\delta})$ such that $c(\tilde{\delta}, l_1) > 0$.

We now translate the minimal shift s_0 into a sufficiently large minimal length $l_0 \geq l_1$. Assume that for some t_0 and shift $s > \varepsilon_0$ we have

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0$$

for all $0 \leq t \leq l$ where $l \geq l_0$.

First, we assume that $s \leq s_0$. The closing lemma implies the existence of a closed geodesic nearby; in fact, the following Lemma can be show using standard arguments in hyperbolic geometry (for the proof, see Subsection 4.6.4).

Lemma 4.43. *In this setting, there exists a closed geodesic α of period $p \leq s + \varepsilon_0$ and a constant $s' = s'(s_0, \varepsilon_0, i_M)$ such that (up to parametrization of α),*

$$d(\alpha(t), \gamma(t_0 + s' + t)) < \varepsilon_0/2 \quad \textit{for all } 0 \leq t \leq s + l - 2s'.$$

Let $N = \lceil s_0/p \rceil \in \mathbb{N}$ be the smallest integer such that $Np > s_0$. Then

$$\begin{aligned}
&d(\gamma(t_0 + s' + t), \gamma(t_0 + s' + Np + t)) \\
&\leq d(\gamma(t_0 + s' + t), \alpha(t)) + d(\gamma(t_0 + s' + Np + t), \alpha(t)) < \varepsilon_0
\end{aligned}$$

for all $0 \leq t \leq s + l - 2s' - Np$. Thus,

$$Np > c(\tilde{\delta}, l_1)\varphi_{\tilde{\delta}}(l - 2s' - Np) = \frac{c(\tilde{\delta}, l_1)}{e^{\tilde{\delta}(n-1)(2s'+Np)}} \varphi_{\tilde{\delta}}(l).$$

Since $Np \leq cs$ and hence $2s' + Np \leq 2s' + cs_0$, for some $c = c(s_0, \varepsilon_0, i_M)$, we can find a positive constant $c_0 = c_0(\tilde{\delta}, i_M, n, \varepsilon_0)$ such that $s > c_0\varphi_{\tilde{\delta}}(l)$.

In the case when $s > s_0$, we have

$$s > c(\tilde{\delta}, l_1)\varphi_{\tilde{\delta}}(l) \geq \; c_0\varphi_{\tilde{\delta}}(l).$$

Finally, since $\delta < \tilde{\delta}$, we restrict if necessary to $\tilde{l}_0 \geq l_0$ such that $c_0\varphi_{\tilde{\delta}}(l) \geq \varphi_\delta(l)$ for all $l \geq \tilde{l}_0$. This finishes the proof of the theorem.

4.6 Appendix

In this section we want to treat several questions and proofs which were not included in [51]. Unless stated otherwise, we consider the setting of the proof of Theorem 4.25 in the following.

4.6.1 Proof of the Remark after Theorem 4.21.

Assume that there exist a φ-aperiodic geodesic with respect to $\varepsilon_0 > 0$ for the function $\varphi(l) = c_0 \cdot e^{(n-1)l}$ for some large constant $c_0 > 0$. Let $\gamma : \mathbb{R} \to \mathbb{H}^n$ be a lift of this geodesic and let $p_0 = \gamma(0)$. Let D_{p_0} be a Dirichlet fundamental domain of Γ and let $R = \text{diam}(D_{p_0})$ such that $B \equiv B_R(p_0) \supset D_{p_0}$. Now, fix $l > R$ sufficiently large with respect to R, say $l = e^{100R}$, and let $A = B_{l+3R}(p_0) - B_{l-3R}(p_0)$ be an annulus around p_0. Finally, let $\mathcal{S}_B$ and $\mathcal{S}_A$ be maximal $\varepsilon_0/4$-separated sets of B and A respectively. In particular, $\cup_{x\in\mathcal{S}_B} B_{\varepsilon_0/2}(x) \supset B$ and $\cup_{x\in\mathcal{S}_A} B_{\varepsilon_0/2}(x) \supset A$.

Then, for every $i \in \mathbb{N}$, for $t_i = 2Ri$, we can find an isometry $\psi_i \in \Gamma$ such that the endpoints

$$e_-^i \equiv \psi_i(\gamma(t_i)) \in B, \quad e_+^i \equiv \psi_i(\gamma(t_i + l)) \in A,$$

and moreover $\psi_i \neq \psi_j$ for $i \neq j$. In particular, there exists a pair $(x, y) \in \mathcal{S}_B \times \mathcal{S}_A$ such that $e_-^i \in B_{\varepsilon_0/2}(x)$ and $e_+^i \in B_{\varepsilon_0/2}(y)$. Assume that for $i \neq j$ also $e_-^j \in B_{\varepsilon_0/2}(x)$ and $e_+^j \in B_{\varepsilon_0/2}(y)$. Hence, $d(e_-^i, e_-^j) < \varepsilon_0$ and $d(e_+^i, e_+^j) < \varepsilon_0$, and since $\psi_i \neq \psi_j$, φ-aperiodicity of γ implies that $|t_i - t_j| > \varphi(l)$. This shows that, if we let

$$N = \max\{i \in \mathbb{N} : 2Ri \leq \varphi(l)\},$$

then for every pair $(x, y) \in \mathcal{S}_B \times \mathcal{S}_A$ there exists at most one pair $(e_-^i, e_+^i) \in B_{\varepsilon_0/2}(x) \times B_{\varepsilon_0/2}(y)$ when $i \leq N$. This shows that $N \leq |\mathcal{S}_B||\mathcal{S}_A|$.

It thus remains to estimate the number of the separating sets $\mathcal{S}_B$ and $\mathcal{S}_A$ respectively. First, note that the annulus A is contained in the $3R$-neighborhood of the metric sphere $S_l(p_0)$. The volume of A can therefore be bounded above by $ce^{(n-1)l}$ where $c = c(R)$. Hence, since the balls $B_{\varepsilon_0/8}(x)$, $x \in \mathcal{S}_A$, are disjoint,

$$|\mathcal{S}_A| \leq \frac{c}{\text{vol}(B(\varepsilon_0/8))} e^{(n-1)l} \equiv \bar{c}_1 \; e^{(n-1)l},$$

where $\bar{c}_1 = \bar{c}_1(R, \varepsilon_0, n-1)$ and $\text{vol}(B(\varepsilon_0/8))$ is the volume of any metric ball in $\mathbb{H}^n$ of radius $\varepsilon_0/8$. In the same way, the number $|\mathcal{S}_B|$ can be bounded by a constant $\bar{c}_2 = \bar{c}_2(R, \varepsilon_0, n-1)$. Finally, we have by definition of N that $2RN \geq \varphi(l)$ and hence

$$\tfrac{1}{4R} \; \varphi(l) \leq N \leq |\mathcal{S}_B||\mathcal{S}_A| \leq \bar{c}_1\bar{c}_2 \; e^{(n-1)l} \equiv \bar{c} \; e^{(n-1)l},$$

which proves the claim.

Remark 4.44. For details of the following we refer to Manning [36]. Note that in the hyperbolic space $\mathbb{H}^n$, a metric ball of radius $\varepsilon_0/4$ with respect to the Bowen metric d_l (defined by a suitable metric on $S\mathbb{H}^n$ inducing the topology) can be characterized by $B(x, \varepsilon_0/4) \times B(y, \varepsilon_0/4)$ for a pair $(x, y) \in \mathcal{S}_B \times \mathcal{S}_A$. We therefore see in the proof a relation between the concepts of topological entropy, volume entropy and φ-aperiodicity.

In fact, note that from the above arguments, it is readily checked that if M denotes a compact Riemannian manifold of strictly negative curvature, then

$$\lambda \equiv \limsup_{l\to\infty} \frac{\log(\mathrm{vol}(B(p_0,l)))}{l} \geq \limsup_{l\to\infty} \frac{\log(\varphi(l))}{l},$$

where λ denotes the *volume entropy* of the universal covering of M and the right hand side the exponential growth rate of φ. Since λ is known to be a lower bound for the *topological entropy* $h(\phi^t)$ of the geodesic flow on SM (see [36]), we see that

$$h(\phi^t) \geq \lambda \geq \limsup_{l\to\infty} \frac{\log(\varphi(l))}{l},$$

and the existence of a φ-aperiodic geodesic implies, for a suitable function φ, a positive topological entropy.

4.6.2 φ-aperiodic geodesics and Diophantine approximation in $\mathbb{H}^n$

In this subsection, we want to use the model of Hersonsky, Paulin and Parkkonen to relate the concept of φ-aperiodic geodesics to Diophantine approximation in negatively curved spaces.

Note that every isometry $\psi_0 \in \Gamma$ (which we assume is primitive) in the cocompact discrete group Γ is of hyperbolic type and determines an axis C_0 as well as an almost malnormal subgroup $\Gamma_0 \equiv \langle\psi_0\rangle \subset \Gamma$ of infinite index. Denote by $|\psi_0|$ the translation length of ψ_0, that is, the translation along the axis C_0. Given a base point $o \in \mathbb{H}^n$, a time $t_0 > 0$, we thus obtain the setting of Subsubsection 2.3.6 and let

$$\mathcal{D}_{[\psi_0]} \equiv (\Gamma, C_0, o, t_0).$$

Denote by $K_{\mathcal{D}_{[\psi_0]}}$ the Hurwitz-constant of the spectrum $\{c_{[\psi_0]}(\xi) : \xi \in \mathbf{Bad}(\mathcal{D}_{[\psi_0]})\}$. Recall that we have

$$\begin{aligned}\mathbf{Bad}(\mathcal{D}_{[\psi_0]}) &= S_{[\psi_0]}\\ &\equiv \{\xi \in S^{n-1} : \exists\, l = l(\xi) < \infty \text{ such that } L(\gamma_{o,\xi}(\mathbb{R}^+) \cap \mathcal{N}_{2\delta_0}(\varphi C_0)) \leq l \text{ for all } [\varphi] \in \Gamma/\Gamma_0\},\end{aligned}$$

and, as corollary from Theorem 2.30, that $S_{[\psi_0]}$ is absolute winning.

Corollary 4.45. *The intersection $\bigcap S_{[\psi_0]}$ over every (conjugacy class of a) primitive hyperbolic isometry $\psi_0 \in \Gamma$ is an absolute winning set (in the sense of McMullen); hence of Hausdorff-dimension $dim(\Lambda\Gamma) = n-1$.*

Using Jarník's inequality for $\mathcal{D}_{[\psi_0]}$, we can estimate the Hausdorff-dimension with respect to a given lower bound on the approximation constant $c_{[\psi_0]}(\xi)$. However, the following questions remains open:

Question 4.46. Given a sequence $c_{[\psi_0]} > 0$, $[\psi_0]$ a conjugacy class of a primitive hyperbolic isometry $\psi_0 \in \Gamma$, what is the Hausdorff-dimension of the set of elements $\xi \in \bigcap S_{[\psi_0]}$, such that $c_{[\psi_0]}(\xi) \geq c_{[\psi_0]}$ for every conjugacy class of a primitive hyperbolic isometry $\psi_0 \in \Gamma$?

In the following, we want to show that $\xi \in \bigcap S_{[\psi_0]}$ if ξ is the endpoint of a lift of a φ-aperiodic geodesic and compute the approximation constants $c_{[\psi_0]}(\xi)$.

In fact, we first give another definition of φ-aperiodic geodesics which is adopted to the interpretation in terms of Diophantine approximation.

Definition 4.47. *Let $\varepsilon_0 < i_M/2$ and $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function.*

For a shift $s > \varepsilon_0$ *and a* minimal length $l_s \geq 0$, *a geodesic ray $\gamma : \mathbb{R}^+ \to X$ is called* (φ, s)-aperiodic *if the following property is satisfied for all times $t_0 \in \mathbb{R}^+$: whenever*

$$d(\gamma(t_0 + s + t), \psi(\gamma(t_0 + t)) \leq \varepsilon_0 \qquad \forall t \in [0, l] \tag{4.6.1}$$

for some $\psi \in \Gamma$ and length $l \geq l_s$, then $s \geq \varphi(l)$.

Clearly, the restriction of the lift γ of a φ-aperiodic geodesic in $M = \mathbb{H}^n/\Gamma$ to the ray $\gamma|_{\mathbb{R}^+}$ is (φ, s)-aperiodic for every shift $s > \varepsilon_0$ with $l_s = l_0$.

To keep things simple, assume that φ is strictly increasing in the following. Then the penetration of a φ-aperiodic ray into $\varepsilon_0/8$-neighborhoods of axes of Γ is bounded above depending only on the period of the axis, φ, and ε_0; more precisely we have the following.

Lemma 4.48. *Let α be the axis of (a not necessarily primitive isometry) $\psi \in \Gamma$ of translation length $|\psi| = s$. If $\gamma : \mathbb{R}^+ \to \mathbb{H}^n$ is a (φ, s)-aperiodic ray with minimal length l_s, such that $\gamma([t_0, t_0 + l]) \subset \mathcal{N}_{\varepsilon_0/8}(\alpha)$ for some $t_0 > 0$ and $l > 0$, then*

$$l \leq s + \max\{\varphi^{-1}(s), l_s\} + \varepsilon_0. \tag{4.6.2}$$

Proof. Assume that $l > l_s + s + \varepsilon_0$ (otherwise the claim follows). Let $\alpha(\bar{t}_0)$ be the closest point projection of $\gamma(t_0)$ onto the convex subspace α. It follows from simple arguments that

$$d(\alpha(\bar{t}_0 + t), \gamma(t_0 + t)) < \varepsilon_0/2$$

for at least all $t \in [0, l - \varepsilon_0]$. Hence, we have

$$\begin{aligned} & d(\gamma(t_0 + s + t), \psi\gamma(t_0 + t)) \\ & \leq d(\gamma(t_0 + s + t), \alpha(\bar{t}_0 + s + t)) + d(\alpha(\bar{t}_0 + s + t), \psi\gamma(t_0 + t)) \\ & = d(\gamma(t_0 + s + t), \alpha(\bar{t}_0 + s + t)) + d(\alpha(\bar{t}_0 + t), \gamma(t_0 + t)) < \varepsilon_0 \end{aligned}$$

for at least all $t \in [0, l - s - \varepsilon_0]$. Since γ is (φ, s)-aperiodic and $l - s - \varepsilon_0 > l_s$, we have $s \geq \varphi(l - s - \varepsilon_0)$. This shows that $l \leq s + \varphi^{-1}(s) + \varepsilon_0$. □

Hence, in our above setting, let ψ_0 be of minimal translation length $|\psi_0| = p_0$ and γ be a lift of a φ-aperiodic geodesic in M with minimal length l_0. Lemma 4.48 implies that the penetration lengths $l_{[\psi]}$ of γ into the images $\psi(\mathcal{N}_{\varepsilon_0/8}(C_0))$, $[\psi] \in \Gamma/\Gamma_0$, are bounded by

$$l_{[\psi]} \leq p_0 + \max\{\varphi^{-1}(p_0), l_0\} + \varepsilon_0 \equiv p_0 + \varphi_{l_0}^{-1}(p_0) + \varepsilon_0.$$

Note that, since Γ is cocompact, we may assume that the lift γ is at distance at most $d(o, \gamma) \leq \operatorname{diam}(\mathbb{H}^n/\Gamma) \equiv D$. Hence, up to changing to a base point $\tilde{o} \in \gamma(\mathbb{R})$ with $d(o, \tilde{o}) \leq D$ and remarking that visual metrics d_o and $d_{\tilde{o}}$ are Bi-Lipschitz equivalent with respect to a constant depending on $d(o, \tilde{o}) \leq D$, Lemma 3.29 shows that for $\xi = \gamma(\infty) \in \Lambda\Gamma = \partial_\infty \mathbb{H}^n$,

$$d_o(\xi, \psi.\Lambda\Gamma_0) = d_o(\xi, \psi.\partial_\infty C_0) \geq \bar{c}e^{-(p_0 + \varphi_{l_0}^{-1}(p_0) + \varepsilon_0)}e^{-D([\psi])},$$

for every $[\psi] \in \Gamma/\Gamma_0$, where $\bar{c} > 0$ is a universal constant. Hence, we have

$$c_{[\psi_0]}(\xi) \geq \bar{c} \cdot \exp(-(p_0 + \varphi_{l_0}^{-1}(p_0) + \varepsilon_0)). \tag{4.6.3}$$

This already shows that $\xi = \gamma(\infty)$ determines a limit point as in Question 1. Moreover, we obtain the following relation for the Hurwitz constant $K_{\mathcal{D}_{[\psi_0]}} \equiv \sup\{c_{[\psi_0]}(x) : \xi \in S^{n-1}\}$.

Corollary 4.49. *Let $\psi_0 \in \Gamma$ be a primitive hyperbolic isometry with $|\psi_0| = p_0$. Then*

$$K_{\mathcal{D}_{[\psi_0]}} \geq \bar{c} \cdot \sup\{\exp(-(p_0 + \varphi_{l_0}^{-1}(p_0) + \varepsilon_0))\},$$

where the supremum is taken over all $(\varphi, \varepsilon_0, l_0)$-aperiodic geodesics in $M = \mathbb{H}^n/\Gamma$.

4.6.3 An estimate on the Hausdorff-dimension of φ-aperiodic geodesics.

In view of (4.6.3), we consider the following special case regarding Question 1.

Recall the construction for the proof of Theorem 4.25, where we are given the subsets of good cubes $\mathcal{Q}^g(m)$ in the constructed collections of shadowed cubes $\mathcal{Q}(m)$, $m \in \mathbb{N}$. Every nested sequence of good cubes determined a unique limit point η (see (4.5.7)) and Lemma 4.38 (2) showed that the projection of the discrete geodesic $\gamma_{\infty,\eta}|_{\mathbb{N}}\colon \mathbb{N} \to \mathbb{H}^n$ to M is $\bar{\varphi}$-aperiodic with the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ of Theorem 4.25. Denote the set of all these limit points by $A_\infty \subset \mathbb{R}^{n-1}$.

Proposition 4.50. *Given the parameters $1 < c < 2^{n-1}$ and $r_0 > \log(2)$ from Theorem 4.25, we have*

$$dim(A_\infty) \geq \frac{\log(c)}{r_0}.$$

Sketch of the proof. Given a cube $Q = Q_{i_1 \dots i_m} \in \mathcal{Q}(m)$, we let $\bar{Q} \equiv \mathcal{S}(Q, \mathbb{R}^{n-1})$ be its shadow in $\mathbb{R}^{n-1} \subset \partial_\infty \mathbb{H}^n$ with respect to the point ∞. For $m \in \mathbb{N}$, define $\mathcal{A}_m \equiv \cup_{Q \in \mathcal{Q}^g(m)}\{\bar{Q}\}$ and by construction of the collections $\mathcal{Q}(m)$, $m \in \mathbb{N}$, the conditions (TL0-3) are readily checked (see Subsection 3.2.18). Hence, we obtain a treelike family $\mathcal{A}$ with respect to the Lebesgue measure μ on $\mathbb{R}^{n-1}$ and the limit set of $\mathcal{A}$ agrees with A_∞ defined above. Moreover, using (4.5.5), we know that every cube $\bar{Q} \in \mathcal{A}_m$ is a Euclidean cube in $\mathbb{R}^{n-1}$ of edge-length e^{-mr_0}; thus its diameter is bounded by $\sqrt{n-1}e^{-mr_0}$ and we have (STL). Also, from Lemma 4.41, we obtain that

$$\Delta_m(\mathcal{A}) = \min_{B \in \mathcal{A}_m} \frac{\mu(\cup \mathcal{A}_{m+1} \cap B)}{\mu(B)} \geq ce^{-(n-1)r_0}.$$

Applying Lemma 3.8 finishes the proof. □

4.6.4 Proof of Lemma 4.43

Let (X, d) be a proper geodesic CAT(-1)-space. For a hyperbolic isometry ψ, denote by A_ψ the axis of ψ. We need the following Lemma, which gives estimates for the displacement function $d_\psi(x) \equiv d(x, \psi(x))$.

Lemma 4.51. *For $\psi \in \Gamma$ hyperbolic with $|\psi| \geq 4\delta_0$ (with δ_0 the constant such that X is a Gromov-hyperbolic space), and $x \in X$, we have*

$$\max\{2d(x, A_\psi), |\psi|\} - 4\delta_0 \leq d_\psi(x) \leq |\psi| + 2d(x, A_\psi),$$

Proof. Note that if $pr : X \to A_\psi$ denotes the closest point projection on the convex closed set A_ψ, then $pr(\psi(x)) = \psi(pr(x))$. Hence, $d(x, A_\psi) = d(\psi(x), A_\psi)$ and $d(pr(x), pr(\psi(x)) = |\psi|$. Therefore, the upper bound follows easily.

Let $m \in [pr(x), pr(\psi(x))]$ such that $d(m, pr(x)) = |\psi|/2$. Note that if m is δ_0-close to $[x, pr(x)]$, then $|\psi|/2 = d(pr(x), m) < \delta_0$. Hence, assume there is a point $\bar{m} \in [x, pr(\psi(x))]$ which is δ_0-close to m. If $\bar{m}$ is in turn δ_0-close to $[x, \psi x]$, say to the point $\bar{x}$, then let $d_1 = d(x, \bar{x})$ and $d_2 = d(\bar{x}, \psi(x))$. Considering the triangle $(x, pr(x), m)$ with $\angle_{pr(x)}(x, m) \geq \pi/2$, we have

$$\begin{aligned} d_1 &\geq d(x,m) - 2\delta_0 \\ &\geq \max\{d(x, pr(x)), d(pr(x), m)\} - 2\delta_0 = \max\{d(x, A_\psi), |\psi|/2\} - 2\delta_0. \end{aligned}$$

The same lower bound holds for d_2 which shows the claim in this case.

If there exists no such point $\bar{x}$, then $\bar{m}$ is δ_0-close to a point $\bar{y}$ in $[pr(\psi(x)), \psi(x)]$ and, since $\angle_{pr(\psi(x))}(\bar{y}, m) \geq \pi/4$, we have $|\psi|/2 = d(m, pr(\psi(x)) \leq d(m, \bar{y}) < 2\delta_0$. □

We now want to prove a version of the 'closing lemma' in CAT(-1)-spaces, where we assume that $i_\Gamma \equiv \inf_{\psi \in \Gamma, x \in X}\{d_\psi(x)\} > 0$.

Lemma 4.52. *For $s > 0$, let $l \geq l_s = 2c - s$, where $c = c(s, \varepsilon, i_\Gamma)$ is increasing in the shift s. Let $\gamma : [0, s + l] \to X$ be a geodesic segment such that,*

$$d(\gamma(s+t), \psi(\gamma(t)) \leq \varepsilon \qquad \textit{for all } t \in [0, l],$$

where $\psi \in \Gamma$ is of hyperbolic type. Then we have $s - 2\varepsilon \leq |\psi| \leq s + \varepsilon$ and

$$\gamma([c, s + l - c]) \subset \mathcal{N}_{\varepsilon/8}(A_\psi). \tag{4.6.4}$$

Proof. Set $x = \gamma(0)$ and $y = \gamma(l)$. Let $n \in \mathbb{N}$ be the minimal integer such that $|\psi^n| = n|\psi| \geq 4\delta_0$. Note that since $|\psi| \geq i_\Gamma > 0$ we have $n \leq \lceil 4\delta_0 / i_\Gamma \rceil$. By Lemma 4.51,

$$\begin{aligned} n(s+\varepsilon) &\geq n(d(\gamma(0), \gamma(s)) + d(\gamma(s), \psi(\gamma(0))) \\ &\geq n d_\psi(x) \geq d_{\psi^n}(x) \\ &\geq \max\{2d(x, A_\psi), n|\psi|\} - 4\delta_0 \geq 2d(x, A_\psi) - 4\delta_0. \end{aligned}$$

Hence, $d(x, A_\psi) \leq D = D(s, \varepsilon, i_M)$ and analogously, $d(y, A_\psi) \leq D$. Moreover,

$$d(\gamma(s+l), A_\psi) \leq d(\gamma(s+l), \psi(\gamma(l)) + d(\psi(\gamma(l)), A_\psi) \leq D + \varepsilon.$$

Hence, by Lemma 2.43, there exists a $c = c(D + \varepsilon, \varepsilon/8)$ such that (4.6.4) holds. Finally, since

$$s - \varepsilon \leq d_\psi(\gamma(c)) \leq |\psi| + 2d(\gamma(c), A_\psi) \leq |\psi| + \varepsilon,$$

and since $|\psi| \leq d_\psi(x) \leq s + \varepsilon$, we have $s - 2\varepsilon \leq |\psi| \leq s + \varepsilon$. □

Finally, in the case of our setting of Lemma 4.43 we have $i_M = i_\Gamma/2 > 0$ and the shifts are bounded, $\varepsilon_0 < s \leq s_0$, so that we can find $l_0, c_0 > 0$ sufficiently large such that Lemma 4.52 holds for all $s \leq s_0$. Moreover, we have for some t_0 and shift $s > \varepsilon_0$ that

$$d(\gamma(t_0 + t), \psi\gamma(t_0 + s + t)) < \varepsilon_0$$

for all $0 \leq t \leq l$ where $l \geq l_0$. Lemma 4.52 shows that $\gamma([t_0 + c_0, t_0 + s + l - c_0]) \subset \mathcal{N}_{\varepsilon_0/8}(A_\psi)$. Up to increasing c_0 to s' as well as l_0, we obtain with simple arguments that

$$d(\gamma(t_0 + s' + t), A_\psi(t)) < \varepsilon_0/2$$

(up to the parametrization of A_ψ) for all $0 \leq t \leq s + l - 2s'$, which finishes the proof.

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