Some Results in Asymptotic Analysis and Nonlocal Problems

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Contents

Abstract

The major part of this thesis deals with an eigenvalue problem with mixed boundary type conditions, considered on cylindrical domain that tends to infinity in one direction.

The first chapter is a general introduction to eigenvalue problems with mixed (Dirichlet-Neumann) type boundary conditions and similar problems that are closely related to the work of this thesis.

In the second chapter we consider the following problem on the cylinder, $\Omega_\ell = (-\ell, \ell) \times \omega\,(\subset \mathbb{R}^n)$

$$\left.\begin{array}{ll} -\mathrm{div}(A(X_2)\nabla u_\ell) = \lambda_\ell^k u_\ell & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \gamma_\ell, \\ A\nabla u_\ell \cdot \nu = 0 & \text{on } \Gamma_\ell, \end{array}\right\} \qquad (0.0.1)$$

where λ_ℓ^k denotes the k-th eigenvalue, $\omega \subset \mathbb{R}^{n-1}$, $\gamma_\ell = (-\ell, \ell) \times \partial\omega$, $\Gamma_\ell = \{-\ell, \ell\} \times \omega$ and ν is the outer normal to $\partial\Omega_\ell$. The asymptotic behavior of λ_ℓ^1 and λ_ℓ^2 are studied, as $\ell \to \infty$. We identify the correct limiting problem which shows in general, that the limiting behavior of λ_ℓ^k is very different from the one for the Dirichlet boundary conditions. Convergence results for the eigenfunctions are also addressed. The case for full Dirichlet boundary conditions were studied in **[19]**.

In the third chapter, problems in variational settings are considered, set on cylindrical domain. Asymptotic behavior of solutions, which are minimizer of some convex energy functional is addressed, when the cylinder tends to infinity. Convergence results for the appropriate energy is also considered.

In the last chapter the following nonlocal problem is considered

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (0.0.2)$$

where $\mathcal{A}$ is the nonlocal function defined on $\Omega \times L^p(\Omega)$, $p \geq 1$, with values in $\mathbb{R}$. Such equations are used to model various problems, that arise in Physics and Population Biology. We mainly consider existence results for such kind of problems. This is a generalization of the work **[13]**.

Zusamenfassung

Der größte Teil dieser Arbeit behandelt ein Eigenwertproblem mit gemischten Rand bedingungen auf einem zylinderförmigem Wertebereich, welcher in eine Richtung gegen unendlich strebt.

Das erste Kapitel beinhaltet eine generelle Einführung in Eigenwertprobleme mit gemischten Rand bedingungen und ähnliche Probleme mit enger Verwandtschaft zu der hier präsentierten Arbeit. Im zweiten Kapitel wird folgendes Problem auf dem Zylinder $\Omega_\ell = (-\ell, \ell) \times \omega \, (\subset \mathbb{R}^n)$ behandelt:

$$\left.\begin{array}{ll} -\mathrm{div}(A(X_2)\nabla u_\ell) = \lambda_\ell^k u_\ell & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{auf } \gamma_\ell, \\ A\nabla u_\ell \cdot \nu = 0 & \text{auf } \Gamma_\ell, \end{array}\right\} \tag{0.0.3}$$

wobei λ_ℓ^k den k-ten Eigenwert bezeichnet und $\omega \subset \mathbb{R}^{n-1}$, $\gamma_\ell = (-\ell, \ell) \times \partial\omega$, $\Gamma_\ell = \{-\ell, \ell\} \times \omega$, und ν die äußere Normale zu $\partial\Omega_\ell$ ist. Das asymptotische Verhalten von λ_ℓ^1 und λ_ℓ^2 für $\ell \to \infty$ wird untersucht. Wir identifizieren das korrekte Grenzwertproblem welches im Allgemeinen zeigt, dass das Grenzwertproblem der Eigenwerte sich stark von dem der Dirichlet Randbedingungen unterscheidet. Konvergenzresultate für die Eigenfunktion werden auch betrachtet. Der Fall der vollen Dirichlet Randbedingungen wurden in **[19]** studiert.

Im dritten Kapitel werden Probleme mit zylinderförmigem Wertebereich in verschiedenen Ausführungen behandelt. Das asymptotisches Verhalten der Lösungen welche gewisse Energiefunktionale minimierien, werden besprochen, falls der Zylinder gegen Unendlich strebt. Auch werden Konvergenzresultate für die entprechende Energie betrachtet.

Das letzte Kapitel behandelt das folgende nicht-lokale Problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{auf } \partial\Omega, \end{cases} \tag{0.0.4}$$

wobei $\mathcal{A}$ the nichtlokale Funktion auf $\Omega \times L^p(\Omega)$ mit Werten in $\mathbb{R}$ bezeichnet.

Solche Gleichungen werden zur Modellierung verschiedener Probleme, welche in der Physik und der Populationsbiologie entstehen, verwendet. Wir betrachten hauptsächlich Existenzresultate für solche Probleme. Dies ist eine Verallgemeinerung der Arbeit in **[13]**.

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Chapter 1

Introduction

Eigenvalues in general are real valued solutions of an equation of the type

$$A(u) = \lambda u$$

where A is an operator from the space to itself and λ is called an eigenvalue of A. In linear algebra an eigenvector v of a matrix A is a vector in $\mathbb{R}^n$, such that the direction of v remains unchanged (possibly reversed), but the length may get scaled, under the action of A on v. This scaling factor is called an eigenvalue of the matrix A. Eigenvalue problems are one of the classical problems that are studied in Mathematics, Quantum mechanics, Ecology and in various other fields of science. In Physics, the vibration of a string is modeled by an eigenvalue problem. In linear theory, the eigenvalues or the spectrum of the operator

$$- \triangle + V(\cdot)$$

are mainly related to the study of standing waves for Schrödinger equation.

1.1 Differences between Dirichlet and Neumann Eigenvalue Problems

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Consider the following two eigenvalue problems:

$$\left.\begin{array}{ll} - \triangle u_k = \lambda^k(\Omega) u_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega \end{array}\right\} \tag{1.1.1}$$

and

$$\left.\begin{array}{ll} - \triangle v_k = \sigma^k(\Omega) v_k & \text{in } \Omega, \\ \frac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{1.1.2}$$

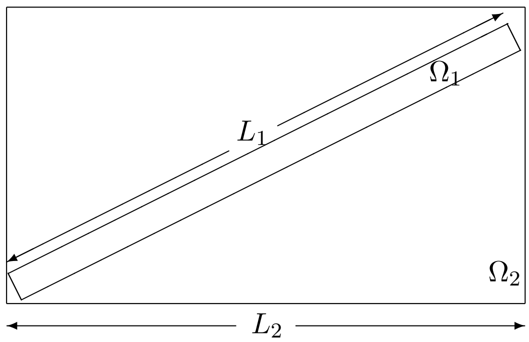
In the above equation ν denotes the outer normal to $\partial\Omega$. The first problem is called Dirichlet Eigenvalue problem and the second one is known as Neumann Eigenvalue problem. Here $\lambda^k(\Omega)$ and $\sigma^k(\Omega)$ denotes the k-th eigenvalues for the problem (1.1.1) and (1.1.2) respectively. From many mathematical point of views, the above mentioned two problems are very different in general. Consider the following two examples.

1. "Domain Monotonicity" property

Suppose Ω_1, Ω_2 be two open bounded subsets of $\mathbb{R}^n$ such that $\Omega_1 \subset \Omega_2$. For the Dirichlet case there is a natural embedding of $H_0^1(\Omega_1) \hookrightarrow H_0^1(\Omega_2)$ by extending the $H_0^1(\Omega_1)$ functions by 0 outside Ω_1. The min-max principle [**see,[32]**] trivially implies that

$$\sigma^k(\Omega_2) \leq \sigma^k(\Omega_1).$$

Similar result is not true for $\lambda^k(\Omega)$. The following example [**see, [38]**] with rectangles will explain the situation.



Figure

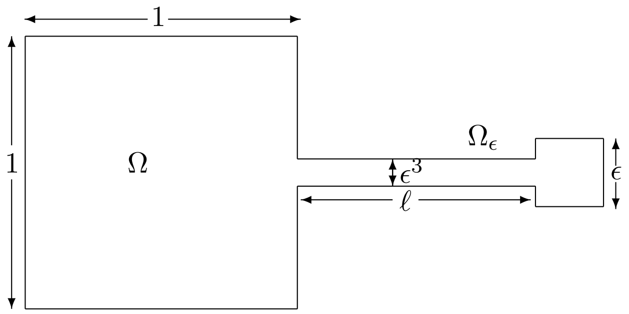
In the above diagram $L_1 > L_2$. Here $\Omega_1 \subset \Omega_2$, but one gets $\lambda^2(\Omega_1) = \frac{\pi^2}{L_1^2} < \lambda^2(\Omega_2) = \frac{\pi^2}{L_2^2}$. For more examples of this kind we refer to [**42**].

2. "Domain Continuity" property

Let B be a fixed compact subset of $\mathbb{R}^n$. Let $\{\Omega_\epsilon\}_{\epsilon \geq 0} \subset B$ be uniformly Lipschitz domains (it means that the boundary of Ω_ϵ is locally the graph of a Lipschitz continuous functions and the Lipschitz constant can be chosen uniformly) and $\Omega \subset B$. Then the following theorem [**see, [38]**] shows continuity of Dirichlet eigenvalues with respect to the domain.

Theorem 1.1.1. *If Ω_ϵ converges to Ω, in the sense of Hausdorff distance, then for all fixed k, it holds that $\sigma^k(\Omega_\epsilon) \to \sigma^k(\Omega)$.*

Such kind of convergence result is no more true for the Neumann case. One of the main reason for such kind of difficulty, is that small perturbation of the boundary can reduce the energy in such an amount that can pollute the spectrum. To explain such a non convergence result, let us recall the classical example, due to Courant-Hilbert [**see, [28]**].



Figure

In the above diagram Ω is the unit square and Ω_ϵ is obtained by joining a small square of side ϵ through a thin rectangular channel of width ϵ^3 and fixed length ℓ. Here it can be shown that Ω_ϵ converges to Ω as $\epsilon \to 0$, in the sense of Hausdorff distance but $\lambda^2(\Omega_\epsilon)$ doesn't converges to $\lambda^2(\Omega)$. The following choice of test function will work.

Define $\phi_\epsilon : \Omega_\epsilon \to \mathbb{R}$ as

$$\phi_\epsilon(x,y) = \begin{cases} c_1 := \epsilon^2 + \frac{\ell\epsilon^3}{2} & \text{on the unit square } (-1,0)\times(-1,0) \\ c_2 := -1 - \frac{\ell\epsilon^3}{2} & \text{on the small square} \\ \frac{c_2 - c_1}{\ell}x + c_1 & \text{on the channel } (x \in [0,\ell]). \end{cases}$$

Then $\int_{\Omega_\epsilon} \phi_\epsilon = 0$ and

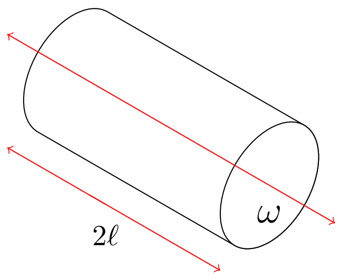
$$\lambda^2(\Omega_\epsilon) \leq \frac{\int_{\Omega_\epsilon} |\nabla\phi_\epsilon|^2}{\int_{\Omega_\epsilon} \phi_\epsilon^2} \leq \frac{(c_2 - c_1)^2\epsilon^3}{\ell(c_1^2 + c_2^2\epsilon^3)}$$

holds. Hence $\lambda^2(\Omega_\epsilon) \to 0$, but $\lambda^2(\Omega) = \pi^2$.

1.2 Theory of $\ell \to \infty$

Suppose $p < n$, define $\Omega_\ell := \ell\omega_1 \times \omega$, where ω_1 and ω are open, bounded subsets of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively. As $\ell \to \infty$, the domain Ω_ℓ tends to infinity in the first p directions.

Diagram of Ω_ℓ in dimension 3, with circular cross section is drawn below:



Figure

Study of Elliptic or Parabolic problems set on such domains Ω_ℓ is shortly abbreviated as "Problems of $\ell \to \infty$".

Suppose the points in ω are denoted by X_2. Consider the following problems:

$$\left.\begin{array}{ll} -\triangle u_\ell = f(X_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell \end{array}\right\} \tag{1.2.1}$$

and

$$\left.\begin{array}{ll} -\triangle_{X_2} u_\infty = f(X_2) & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega \end{array}\right\} \tag{1.2.2}$$

where $f \in L^2(\omega)$. The following asymptotic behavior of u_ℓ is obtained in **[26]**.

Theorem 1.2.1. [Chipot-Yeressian] *For some constants $C, \alpha > 0$, it holds that*

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla u_\ell - u_\infty|^2 \leq Ce^{-\alpha\ell}$$

where u_∞ is as in (1.2.2).

We observe that u_ℓ converges to the solution of the problem (1.2.2), which is a similar problem set on the cross section ω of the cylinder Ω_ℓ. This shows how the problem (1.2.2) is related to the problem (1.2.1), as $\ell \to \infty$.

Now we introduce the general theory of $\ell \to \infty$ for eigenvalue problems. Let us start with a very simple situation. Suppose $\omega_1 = \omega := (-1, 1)$, then it is well known **[see,[19]]** that $\sigma^k(\Omega_\ell)$ has the following representation:

$$\sigma^k(\Omega_\ell) = \left(\frac{\pi}{2}\right)^2 + \left(\frac{k\pi}{2\ell}\right)^2. \tag{1.2.3}$$

Note that $\sigma^1((-1,1)) = \left(\frac{\pi}{2}\right)^2$. Therefore as $\ell \to \infty$, one has

$$\sigma^k(\Omega_\ell) \to \left(\frac{\pi}{2}\right)^2 = \sigma^1(-1,1).$$

That is the sequence $\left\{\sigma^k(\Omega_\ell)\right\}_{\ell \geq 1}$ converges to the first eigenvalue of the *Dirichlet eigenvalue problem* on the cross section.

Consider the following two eigenvalue problems:

$$\left.\begin{array}{ll} -\text{div}(A(X_2)\nabla v_\ell) = \sigma^k(\Omega_\ell)v_\ell & \text{in } \Omega_\ell, \\ v_\ell = 0 & \text{on } \partial\Omega_\ell \end{array}\right\} \tag{1.2.4}$$

and the problem set on the cross section ω, as

$$\left.\begin{array}{ll} -\text{div}(A_{22}(X_2)\nabla u) = \sigma^1(\omega)u & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega \end{array}\right\} \tag{1.2.5}$$

where

$$A = A(X_2) := \begin{pmatrix} A_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix} \tag{1.2.6}$$

is a symmetric, uniformly positive definite matrix. The above problem in the general setting was studied in **[19]** and the following result was obtained.

Theorem 1.2.2. [Chipot-Rougirel] *It holds that*

$$\sigma^1(\omega) \leq \sigma^k(\Omega_\ell) \leq \sigma^1(\omega) + \frac{C}{\ell^2}, \quad \textit{for all } k. \tag{1.2.7}$$

where C is a constant independent of ℓ.

Therefore in the Dirichlet case, the asymptotic behavior of $\left\{\sigma^k(\Omega_\ell)\right\}_{\ell\geq 1}$ is nice, as it converges to the first eigenvalue of the *eigenvalue problem* (1.2.5) set on the cross section ω.

The main aim of this thesis is to consider an eigenvalue problem with mixed boundary type (Dirichlet-Neumann) data set on $\Omega_\ell := (-\ell,\ell)\times\omega$. More precisely consider the following problem:

$$\left.\begin{array}{ll} -\mathrm{div}(A(X_2)\nabla u_\ell) = \lambda^k(\Omega_\ell)u_\ell & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \gamma_\ell, \\ A\nabla u_\ell\cdot\nu = 0 & \text{on } \Gamma_\ell, \end{array}\right\} \tag{1.2.8}$$

where $\gamma_\ell := (-\ell,\ell)\times\partial\omega$, $\Gamma_\ell := \{-\ell,\ell\}\times\omega$ and ν is the outward normal vector to $\partial\Omega_\ell$. The goal of this thesis is to analyze the asymptotic behavior of eigenvalues and eigenfunctions of the above problem. Note that the corresponding problem defined on the cross section ω for both the problems (1.2.8) and (1.2.4) are the same, since we assumed $u_\ell = 0$ on γ_ℓ. We will see that the asymptotic behavior of $\lambda^1(\Omega_\ell)$ and $\lambda^2(\Omega_\ell)$ as $\ell\to\infty$ is very different from the Dirichlet case, as the limiting behavior of eigenvalues in this case is independent of the problem (1.2.5) defined on the cross section. In particular we will show [**see, Theorem (2.3.2)**], **Theorem (2.6.1)**] that for $k = 1, 2$

$$\limsup_{\ell\to\infty}\lambda^k(\Omega_\ell) < \sigma^1(\omega)$$

under appropriate assumptions on the matrix A. Therefore $\sigma^1(\Omega_\ell)$ and $\lambda^1(\Omega_\ell)$ cannot converge to the same limit, which again points out the difference between Dirichlet eigenvalue problems and Neumann eigenvalue problems. This gap phenomenon is explained by the appearance of boundary effects near the side boundary $\mathit{\Gamma}_\ell$ of Ω_ℓ. In order to describe the asymptotic behavior of $\lambda^1(\Omega_\ell)$ as $\ell\to\infty$, we will first need to study the behavior of $\lambda^1(\Omega_\ell)$ as $\ell\to 0$.

Elliptic problems set on domains which tends to 0 in some directions, are generally known as "Dimension Reduction" problems and are addressed in [**1**], [**2**], [**6**] and [**35**]. The work of this thesis establishes a relation between the theory of "Dimension reduction" and the theory for "$\ell\to\infty$".

This kind of issues, namely the approximation of the solution of the problem set on cylindrical domains by the solution of the problem on section, is addressed in [**10**], [**17**], [**19**], [**20**], [**21**], [**22**], [**23**], [**24**] for some differential equations, variational inequalities or systems.

The second part of this thesis deals with nonlocal problems. Suppose Ω is an open, bounded subset of $\mathbb{R}^n$. Consider the following problem:

$$\begin{cases} -\operatorname{div}\left(\mathcal{A}(x,u)\nabla u\right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.9}$$

where $\mathcal{A}$ is a nonlocal function defined on $\Omega \times L^p(\Omega)$, $p \geq 1$, with values in $\mathbb{R}$. f is nonlinear function having "loop" structure. The main motivation to study this problem comes from **[13]**. Related problems in a local framework were well studied in **[4], [29], [33]** and **[34]**. Problems of such kind in nonlocal setting were considered in **[13], [17], [20], [21], [22], [23]** and **[32]**. Similar issues were also studied in the frame work of asymptotic behavior of parabolic equations **[7], and [15]**.

Chapter 2

Asymptotic Behavior of Eigenmodes as $\ell \to \infty$

Let ω be a bounded open set in $\mathbb{R}^{n-1}$. For every $\ell > 0$ set $\Omega_\ell = (-\ell, \ell) \times \omega$ and write each $x \in \Omega_\ell$ as $x = (x_1, X_2)$ with $X_2 = (x_2, \ldots, x_n)$. We assume that the matrices

$$A(X_2) = \begin{pmatrix} a_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

are uniformly elliptic and uniformly bounded on ω (precise assumptions will be made in Section 2.1). The limiting behavior, when ℓ goes to infinity, of the eigenvalues and eigenfunctions of the elliptic operator $-\text{div}(A(X_2)\nabla u)$ on Ω_ℓ with zero Dirichlet boundary conditions, was studied by Chipot and Rougirel in [**19**]. We shall recall below one of their main results that was the principal motivation for the current chapter. Let μ^k and σ_ℓ^k denote, respectively, the k-th eigenvalues for the problems

$$\begin{cases} -\text{div}(A_{22}(X_2)\nabla u) = \mu u \quad \text{in } \omega, \\ \quad u = 0 \quad \text{on } \partial\omega, \end{cases} \tag{2.0.1}$$

and

$$\begin{cases} -\text{div}(A(X_2)\nabla u) = \sigma u \quad \text{in } \Omega_\ell, \\ \quad u = 0 \quad \text{on } \partial\Omega_\ell. \end{cases} \tag{2.0.2}$$

The following relation between problem (2.0.2) (for large ℓ) and problem (2.0.1) was established in [**19**].

Theorem 2.0.3. [Chipot-Rougirel]

$$\mu^1 \le \sigma_\ell^1 \le \mu^1 + \frac{C}{\ell^2}\,, \tag{2.0.3}$$

where C is a constant independent of ℓ.

The main goal of the present article is to study the analogous problem for *mixed* boundary conditions, at least for $k = 1$. Let us write $\partial\Omega_\ell = \Gamma_\ell \cup \gamma_\ell$ where

$$\Gamma_\ell = \{-\ell, \ell\} \times \omega \text{ and } \gamma_\ell = (-\ell, \ell) \times \partial\omega, \tag{2.0.4}$$

and denote by λ_ℓ^k the k-th eigenvalue for the mixed Neumann-Dirichlet problem

$$\begin{cases} -\mathrm{div}(A(X_2)\nabla u) = \sigma u \quad \text{in } \Omega_\ell, \\ u = 0 \quad \text{on } \gamma_\ell, \\ (A(X_2)\nabla u) \cdot \nu = 0 \quad \text{on } \Gamma_\ell. \end{cases} \tag{2.0.5}$$

One of our main results establishes that

$$\lim_{\ell\to\infty} \lambda_\ell^1$$

exists, but in general it is strictly smaller than μ^1. This "gap phenomenon" is explained by the appearance of boundary effects near Γ_ℓ. To gain better understanding of these effects we are led to consider first the limit

$$\lim_{\ell\to 0} \lambda_\ell^1.$$

Asymptotic behavior of elliptic problems set on domains shrinking to zero in some directions are generally known as "Dimension Reduction" problems and are addressed in [**1, 6, 35**] and in a setting particularly suitable for us, in [**2**]. Our work establishes a somewhat surprising connection between the theory of dimension reduction (i.e., "$\ell \to 0$") and the theory for "$\ell \to \infty$".

In order to have a more precise description of the boundary effects and to characterize the value of the limit $\lim_{\ell\to\infty} \lambda_\ell^1$, we introduce eigenvalue problems on the two semi-infinite cylinders

$$\Omega_\infty^+ = (0, \infty) \times \partial\omega \quad \text{and} \quad \Omega_\infty^- = (-\infty, 0) \times \partial\omega,$$

with mixed boundary conditions. Let $\nu_\infty^\pm$ denote the first eigenvalue for the operator

$$-\mathrm{div}(A(X_2)\nabla u) \quad \text{on } \Omega_\infty^\pm$$

with zero boundary condition on the lateral part of the boundary $\partial\Omega_\infty^\pm$. One might be tempted to expect that the equality

$$\nu_\infty^+ = \nu_\infty^-$$

always hold because of "symmetry considerations". However, as we shall see in Section 2.5, this equality is false in general. We denote by W_1 the positive normalized eigenfunction corresponding to μ^1.

Main Theorem. *We have* $\lim_{\ell\to\infty} \lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-)$ *. If* $A_{12} \cdot \nabla W_1 \neq 0$ *a.e. on* ω, *then* $\lim_{\ell\to\infty} \lambda_\ell^1 < \mu^1$. *Otherwise,* $\lambda_\ell^1 = \mu^1$, $\forall \ell$.

Many problems of the type "$\ell \to \infty$" were studied in the past. Besides the eigenvalue problem already mentioned [**19**], these include elliptic and parabolic equations, variational inequalities and systems, see [**10, 17, 20, 21, 22, 23, 24**]. In all these problems it is found that the limit is characterized by the solution of the corresponding problem on the section ω. We emphasize that the limiting behavior in our problem is very different.

The chapter is organized as follows. In Section 2.1 we give the main definitions and notation needed in the subsequent sections. In Section 2.2 we illustrate the gap phenomenon in a simple model case where $\omega = (-1, 1)$ and A is a 2×2 matrix with constant coefficients, namely,

$$A := A_\delta = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}.$$

In Section 2.3 we prove the gap phenomenon for the general case. In Section 2.4 we prove that the limit $\lim_{\ell\to\infty} \lambda_\ell^1$ exists, and identify its value using the eigenvalue problems on the semi-infinite cylinders Ω_∞^+ and Ω_∞^-. In Section 2.5 we investigate further the problem on a semi-infinite cylinder and use it to give a more precise description of the first eigenfunction u_ℓ for large ℓ. In Section 2.6 we study the limiting behavior of the second eigenvalues (λ_ℓ^2). Under some symmetry assumptions on the matrix A, we will show that λ_ℓ^1 and λ_ℓ^2 has the same limit. In the last section we allow the cylinder to go to infinity in several directions. Also we remark the asymptotic behavior of the eigenvalues for full Neumann problem

2.1 Preliminaries

For each $\ell > 0$ consider $\Omega_\ell = (-\ell, \ell) \times \omega$ with ω a bounded domain in $\mathbb{R}^{n-1}$ as in the Introduction. The lateral part of $\partial\Omega_\ell$ and the remaining part of the cylinder (i.e., the two ends) will be denoted by γ_ℓ and Γ_ℓ, respectively. Let us denote by $H^1(\Omega_\ell)$ and $H_0^1(\Omega_\ell)$ the usual spaces of functions defined by

$$H^1(\Omega_\ell) = \left\{ v \in L^2(\Omega_\ell) | \ \frac{\partial v}{\partial x_i} \in L^2(\Omega_\ell), i = 1, 2, \ldots, n \right\},$$

and

$$H_0^1(\Omega_\ell) = \left\{ v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \ \partial\Omega_\ell \right\},$$

or in a more precise way, $H_0^1(\Omega_\ell)$ is the closure of $C_c^\infty(\Omega_\ell)$ in $H^1(\Omega_\ell)$. The space $H_0^1(\Omega_\ell)$ is equipped with the norm

$$\|\nabla v\|_{2,\Omega_\ell}^2 = \int_{\Omega_\ell} |\nabla v|^2. \tag{2.1.1}$$

A suitable space for our problem is

$$V(\Omega_\ell) = \left\{ v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \gamma_\ell \right\},$$

where the boundary condition should be interpreted in the sense of traces. Thanks to the Poincaré inequality, $V(\Omega_\ell)$ becomes an Hilbert space when equipped with the norm (2.1.1). For later use we define the sets

$$\Omega_\ell^+ = [0,\ell) \times \omega \quad \text{and} \quad \Omega_\ell^- = (-\ell, 0) \times \omega, \tag{2.1.2}$$

We decompose Γ_ℓ (see (2.0.4)) into two parts as $\Gamma_\ell = \Gamma_\ell^+ \cup \Gamma_\ell^-$, where

$$\Gamma_\ell^+ = \{\ell\} \times \omega \quad \text{and} \quad \Gamma_\ell^- = \{-\ell\} \times \omega\,. \tag{2.1.3}$$

Similarly, for the lateral part of $\partial\Omega_\ell$ we define,

$$\gamma_\ell^+ = (0,\ell) \times \partial\omega \quad \text{and} \quad \gamma_\ell^- = (-\ell, 0) \times \partial\omega\,. \tag{2.1.4}$$

We shall be concerned with the operator $-\mathrm{div}(A(X_2)\nabla u)$ where, for each $X_2 \in \omega$,

$$A(X_2) = \begin{pmatrix} a_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

is a symmetric $n \times n$ matrix, $a_{11} \in \mathbb{R}$, A_{12} is a $1 \times (n-1)$ matrix and A_{22} is a $(n-1) \times (n-1)$ matrix. The components of $A(X_2)$ are assumed to be bounded measurable functions on ω and we assume the following bound

$$\|A(X_2)\| \le C_A\,, \quad a.e.\ X_2 \in \omega, \tag{2.1.5}$$

for the Euclidean operator norm. We also assume that $A(X_2)$ is uniformly elliptic and denote by λ_A the largest positive number for which the following inequality holds,

$$A(X_2)\xi \cdot \xi \ge \lambda_A |\xi|^2\,, \quad \forall \xi \in \mathbb{R}^n,\ a.e.\ X_2 \in \omega. \tag{2.1.6}$$

The weak formulation of the eigenvalue problem (2.0.1) is to find $u \in H_0^1(\omega) \setminus \{0\}$ and $\mu \in \mathbb{R}$ such that

$$\int_\omega (A_{22}\nabla u) \cdot \nabla v \, dX_2 = \mu \int_\omega uv \, dX_2\,, \quad \forall v \in H_0^1(\omega)\,. \tag{2.1.7}$$

Denote by μ^1 the first eigenvalue of the problem (2.1.7) with the corresponding normalized eigenfunction W_1, i.e.,

$$\int_\omega |W_1|^2 = 1.$$

It is well known that μ^1 has a variational characterization by the Rayleigh quotient:

$$\mu^1 = \inf\left\{ \int_\omega (A_{22}(X_2)\nabla u) \cdot \nabla u \,\Big|\, u \in H_0^1(\omega) \text{ s.t. } \int_\omega u^2 = 1 \right\}$$
$$= \inf_{u \in H_0^1(\omega)\setminus\{0\}} \frac{\int_\omega (A_{22}(X_2)\nabla u) \cdot \nabla u}{\int_\omega u^2}. \tag{2.1.8}$$

Moreover, W_1 is simple and has constant sign in Ω **(see [32])**. The choice of positive sign leaves us with a unique W_1.

Similarly, the eigenvalue problem (2.0.5) has the following weak formulation: find $u \in V(\Omega_\ell) \setminus \{0\}$ and a real number λ such that

$$\int_{\Omega_\ell} A\nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_\ell} uv \, dx \,, \quad \forall v \in V(\Omega_\ell). \tag{2.1.9}$$

It is well known, see **[8]**, that the first eigenvalue λ_ℓ^1 for (2.1.9) is associated with a variational characterization,

$$\lambda_\ell^1 = \inf \left\{ \int_{\Omega_\ell} A\nabla u \cdot \nabla u \, : \, u \in V(\Omega_\ell), \, \int_{\Omega_\ell} u^2 = 1 \right\}$$
$$= \inf_{u \in V(\Omega_\ell) \setminus \{0\}} \frac{\int_{\Omega_\ell} A(X_2)\nabla u \cdot \nabla u}{\int_{\Omega_\ell} u^2}. \tag{2.1.10}$$

It is also true, and can be proved in the same way as it is done for the corresponding Dirichlet problem, that λ_ℓ^1 is simple and the corresponding eigenfunction u_ℓ has constant sign in Ω_ℓ, that we should fix as the positive sign in the sequel. For some of our results we shall need to impose a certain symmetry condition on ω and A.

Definition 2.1.1. *We shall say that property (S) holds if ω is symmetric w.r.t. the origin (i.e., $-\omega = \omega$) and $A(-X_2) = A(X_2)$.*

From the uniqueness of u_ℓ we deduce easily the following symmetry result.

Proposition 2.1.1. *If property (S) holds then $u_\ell(x_1, X_2) = u_\ell(-x_1, -X_2)$.*

Proof. Clearly $v_\ell(x_1, X_2) := u_\ell(-x_1, -X_2)$ is a positive normalized eigenfunction for λ_ℓ^1, so it must be equal to u_ℓ. □

2.2 The Gap Phenomenon in a Model Problem

In this section we treat a two dimensional model problem in order to illustrate the main ideas behind the analysis of the general case in the next sections. Throughout this section $\omega = (-1, 1)$, $\Omega_\ell = (-\ell, \ell) \times (-1, 1)$, and the matrix A is a constant matrix depending on the parameter $\delta \in [0, 1)$, namely,

$$A = A_\delta = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}. \tag{2.2.1}$$

Clearly A_δ satisfies all the assumptions made on A in Section 2.1. Since the eigenvalues of A_δ are $1 \pm \delta$, $\lambda_A = 1 - \delta$ (see (2.1.6)). In this section we shall denote a point in $\mathbb{R}^2$ by $x = (x_1, x_2)$. The problem (2.1.7) has the following simple form

$$\begin{cases} -W_1'' = \mu^1 W_1 \text{ in } (-1, 1)\,, \\ W_1(-1) = W_1(1) = 0\,. \end{cases}$$

where μ^1 denotes the first eigenvalue and W_1 is the corresponding positive normalized eigenfunction. Therefore,

$$\mu^1 = (\frac{\pi}{2})^2 \text{ and } W_1(t) = \cos(\frac{\pi}{2}t).$$

Proposition 2.2.1. *For $\delta = 0$ we have $\lambda_\ell^1 = \mu^1$ for all $\ell > 0$. For $\delta \in (0,1)$ we have*

$$(1-\delta^2)\mu^1 < \lambda_\ell^1 < \mu^1, \forall \ell > 0. \tag{2.2.2}$$

Proof. (i) Since

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the corresponding operator is just $-\Delta$, and the function $v(x_1,x_2) = W_1(x_2)$ is clearly a positive eigenfunction in (2.0.5) with $\sigma = \mu^1$, for all $\ell > 0$. It follows that $\lambda_\ell^1 = \mu^1$ as claimed.

(ii) Assume now that $\delta \in (0,1)$. Using the function

$$v(x_1,x_2) = W_1(x_2)$$

in the Rayleigh quotient (2.1.10) yields the inequality

$$\lambda_\ell^1 \le \mu^1 . \tag{2.2.3}$$

We claim that the inequality in (2.2.3) is strict as stated in (2.2.2). Indeed, an equality would imply that the function v (as defined above) is a positive eigenfunction in (2.0.5) for

$$\sigma = \lambda_\ell^1 = \mu^1,$$

and in particular, it satisfies the Neumann boundary condition

$$0 = (A_\delta \nabla v)\cdot \nu = \frac{\partial v}{\partial x_1} + \delta \frac{\partial v}{\partial x_2} = \delta \frac{\partial v}{\partial x_2} \quad \text{on } \Gamma_\ell^+ = \{\ell\} \times (-1,1) .$$

But this clearly contradicts the fact that

$$(W_1)'(x_2) \neq 0 \quad \text{for } x_2 \in (-1,1) \setminus \{0\}.$$

To prove the inequality of the left in (2.2.2) we first notice the elementary inequality

$$(A_\delta \vec{\xi}).\vec{\xi} \ge (1-\delta^2)|\xi_2|^2, \ \forall \vec{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^2 . \tag{2.2.4}$$

Indeed, (2.2.4) follows from the identity

$$(A_\delta \vec{\xi}) \cdot \vec{\xi} = \xi_1^2 + 2\delta\xi_1\xi_2 + \xi_2^2 = (1-\delta^2)\xi_2^2 + (\xi_1 + \delta\xi_2)^2 . \tag{2.2.5}$$

By (2.2.4) and(2.1.8) we get

$$\lambda_\ell^1 = \int_{\Omega_\ell} (A_\delta \nabla u_\ell) \cdot \nabla u_\ell \geq (1-\delta^2) \int_{\Omega_\ell} |\frac{\partial u_\ell}{\partial x_2}|^2 \geq (1-\delta^2)\mu^1 \int_{\Omega_\ell} |u_\ell|^2 = (1-\delta^2)\mu^1. \tag{2.2.6}$$

To conclude, we show that the inequality

$$\lambda_\ell^1 \geq (1-\delta^2)\mu^1$$

is strict. Indeed, equality would imply equalities in all the inequalities in (2.2.6), implying in particular that $u_\ell(x_1, x_2) = W_1(x_2)$ in Ω_ℓ. It would then follow that $\lambda_\ell^1 = \mu^1$. Contradiction. □

From now on we shall assume that $\delta \in (0,1)$ (the first part of Proposition 2.2.1 settles completely the case $\delta = 0$). Our main result in this section establishes the following estimate about the behavior of λ_ℓ^1 as ℓ goes to infinity.

Theorem 2.2.1. $\limsup_{\ell\to\infty} \lambda_\ell^1 < \mu^1$, *for every* $\delta \in (0,1)$.

In the next section, when dealing with the general case, we shall actually see that the limit

$$\lim_{\ell\to\infty} \lambda_\ell^1$$

exists. As mentioned in the Introduction, an important ingredient in the proof of Theorem 2.2.1 is a study of the asymptotic behavior of λ_ℓ^1 as $\ell \to 0$ (a dimension reduction problem).

Theorem 2.2.2. *We have* $\lim_{\ell\to 0} \lambda_\ell^1 = (1-\delta^2)\mu^1$.

Proof. It suffices to consider $\ell < 1$. Fix any $\alpha \in (0,1)$ and let ρ_ℓ be the piecewise-linear function defined by

$$\rho_\ell(t) = \begin{cases} \frac{t+1}{\ell^\alpha} & t \in [-1, -1+\ell^\alpha), \\ 1 & t \in [-1+\ell^\alpha, 1-\ell^\alpha], \\ \frac{1-t}{\ell^\alpha} & t \in (1-\ell^\alpha, 1]. \end{cases}$$

Consider the following test function

$$v_\ell(x_1,\ x_2) = W_1(x_2) - \delta x_1 W_1'(x_2)\rho_\ell(x_2)\,. \tag{2.2.7}$$

Then clearly $v_\ell \in V(\Omega_\ell)$ is a valid test function. From (2.1.10), we have

$$\lambda_\ell^1 \leq \frac{\int_{\Omega_\ell} A_\delta \nabla v_\ell \cdot \nabla v_\ell}{\int_{\Omega_\ell} v_\ell^2} = \frac{\int_{\Omega_\ell} |\frac{\partial v_\ell}{\partial x_1}|^2 + \int_{\Omega_\ell} |\frac{\partial v_\ell}{\partial x_2}|^2 + 2\delta \int_{\Omega_\ell} \frac{\partial v_\ell}{\partial x_1}\frac{\partial v_\ell}{\partial x_2}}{\int_{\Omega_\ell} v_\ell^2} = \frac{I_1 + I_2 + I_3}{I}\,. \tag{2.2.8}$$

We consider each of the terms I_1, I_2, I_3 and I separately. First,

$$I_1 = \delta^2 \int_{\Omega_\ell} \rho_\ell^2 |W_1'(x_2)|^2 \, dx = 2\ell\delta^2 \int_{-1}^{1} \rho_\ell^2 |W_1'(x_2)|^2 \, dx_2 \,. \tag{2.2.9}$$

Next, calculating for I_2,

$$\begin{aligned} I_2 &= \int_{\Omega_\ell} \Big[W_1'(x_2) - \delta x_1 \{\rho_\ell W_1''(x_2) + W_1'(x_2)\rho_\ell'(x_2)\} \Big]^2 \\ &= \int_{\Omega_\ell} |W_1'|^2 - 2\delta \int_{\Omega_\ell} x_1 W_1'(x_2) \big\{ \rho_\ell W_1''(x_2) + W_1'(x_2)\rho_\ell'(x_2) \big\} \\ &\quad + \delta^2 \int_{\Omega_\ell} x_1^2 |\rho_\ell W_1''(x_2) + W_1'(x_2)\rho_\ell'(x_2)|^2. \end{aligned}$$

The integral in the middle vanishes since $\int_{-\ell}^{\ell} x_1 = 0$. Hence, using $|\rho_\ell'| \le \frac{1}{\ell^\alpha}$ and (2.1.7) we get

$$I_2 = 2\ell\mu^1 + \frac{2\delta^2\ell^3}{3} \int_{-1}^{1} |\rho_\ell W_1'' + W_1'\rho_\ell'|^2 \le 2\ell\mu^1 + \frac{2\delta^2\ell^3}{3}(C_1 + C_2\ell^{-2\alpha}), \tag{2.2.10}$$

where C_1, C_2 are two constants independent of ℓ. Next, for I_3 we find,

$$\begin{aligned} I_3 &= 2\delta \int_{\Omega_\ell} -\delta W_1'\rho_\ell \left[W_1' - x_1\delta \left\{ W_1'\rho_\ell' + \rho_\ell W_1'' \right\} \right] \\ &= -4\ell\delta^2 \int_{-1}^{1} \rho_\ell |W_1'|^2 + 2\delta^2 \int_{\Omega_\ell} x_1 W_1' \rho_\ell \left\{ W_1'\rho_\ell' + \rho_\ell W_1'' \right\} = -4\ell\delta^2 \int_{-1}^{1} \rho_\ell |W_1'|^2. \end{aligned} \tag{2.2.11}$$

Finally we compute the term I.

$$\begin{aligned} I = \int_{\Omega_\ell} \left(W_1 - \delta x_1 W_1' \rho_\ell \right)^2 &= \int_{\Omega_\ell} W_1^2 + \delta^2 \int_{\Omega_\ell} x_1^2 \rho_\ell^2 |W_1'|^2 \\ &= 2\ell + \frac{2\ell^3\delta^2}{3} \int_{-1}^{1} \rho_\ell^2 |W_1'|^2 \ge 2\ell. \end{aligned} \tag{2.2.12}$$

Plugging (2.2.9)–(2.2.12) in (2.2.8) yields

$$\lambda_\ell^1 \le \delta^2 \int_{-1}^{1} \rho_\ell^2 |W_1'|^2 + \mu^1 - 2\delta^2 \int_{-1}^{1} \rho_\ell |W_1'|^2 + \varepsilon(\ell), \tag{2.2.13}$$

where $\varepsilon(\ell) \to 0$ as $\ell \to 0$. Since $\rho_\ell \to 1$ pointwise, passing to the limit $\ell \to 0$ and using dominated convergence for the RHS of (2.2.13) gives

$$\limsup_{\ell \to 0} \lambda_\ell^1 \le (1 - \delta^2)\mu^1 \,. \tag{2.2.14}$$

Combining (2.2.14) with (2.2.2) we obtain the result of the theorem. □

Now we turn to the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. Let ℓ_0 and η be two positive constants whose values will be determined later. For $\ell > \ell_0 + \eta$ define ϕ_ℓ by

$$\phi_\ell = \begin{cases} v_{\ell_0}(x_1 - \ell + \ell_0, x_2) & \text{on } (\ell - \ell_0, \ell) \times (-1,1)\,, \\ \frac{(x_1 - (\ell - \ell_0 - \eta))W_1(x_2)}{\eta} & \text{on } (\ell - \ell_0 - \eta, \ell - \ell_0) \times (-1,1)\,, \\ 0 & \text{on } \Omega_{\ell - \ell_0 - \eta}\,, \\ \frac{(-x_1 - (\ell - \ell_0 - \eta))W_1(x_2)}{\eta} & \text{on } (\ell_0 - \ell, -\ell + \ell_0 + \eta) \times (-1,1)\,, \\ v_{\ell_0}(x_1 + \ell - \ell_0, x_2) & \text{on } (-\ell, \ell_0 - \ell) \times (-1,1)\,, \end{cases}$$

where v_{ℓ_0} is given by (2.2.7). We have

$$\begin{aligned} \int_{\Omega_\ell} \phi_\ell^2 &= \int_{\Omega_\ell \setminus \Omega_{\ell-\ell_0}} \phi_\ell^2 + \int_{\Omega_{\ell-\ell_0}} \phi_\ell^2 \\ &= \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + 2\left(\int_{\ell-\ell_0-\eta}^{\ell-\ell_0} \frac{(x_1 - \ell + \ell_0 + \eta)^2}{\eta^2} dx_1\right)\left(\int_{-1}^{1} W_1^2\right) \\ &= \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta\,, \end{aligned} \tag{2.2.15}$$

where we used the fact that ϕ_ℓ is an even function in x_1 on $\Omega_\ell \setminus \Omega_{\ell-\ell_0}$. Also,

$$\int_{\Omega_\ell} A_\delta \nabla\phi_\ell \cdot \nabla\phi_\ell = \int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0} + \int_{\Omega_{\ell-\ell_0}} A_\delta \nabla\phi_\ell \cdot \nabla\phi_\ell\,. \tag{2.2.16}$$

Setting $\mathcal{D} = \Omega_{\ell-\ell_0} \setminus \Omega_{\ell-\ell_0-\eta}$ and using the fact that ϕ_ℓ is even in $\mathcal{D}$ while $\frac{\partial\phi_\ell}{\partial x_1}$ is odd on $\mathcal{D}$ we get

$$\begin{aligned} \int_{\Omega_{\ell-\ell_0}} A_\delta \nabla\phi_\ell \cdot \nabla\phi_\ell &= \frac{1}{\eta^2}\int_{\mathcal{D}} W_1^2 + 2\delta \int_{\mathcal{D}} \frac{\partial\phi_\ell}{\partial x_1}\frac{\partial\phi_\ell}{\partial x_2} \\ &\quad + \frac{2}{\eta^2}\int_{(\ell-\ell_0-\eta,\ell-\ell_0)\times(-1,1)} |W_1'|^2 (x_1 - \ell + \ell_0 + \eta)^2 \\ &= \frac{2}{\eta}\int_{-1}^{1} W_1^2 + \frac{2\eta}{3}\int_{-1}^{1} |W_1'|^2 = \frac{2}{\eta} + \frac{2\eta\mu^1}{3}\,. \end{aligned} \tag{2.2.17}$$

From (2.2.15)–(2.2.17) we obtain

$$\lambda_\ell^1 \le \frac{\int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0} + \frac{2}{\eta} + \frac{2\eta\mu^1}{3}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta}\,. \tag{2.2.18}$$

Noting that Theorem 2.2.2 implies that

$$\frac{\int_{\Omega_{\ell_0}} A_\delta \nabla v_{\ell_0} \cdot \nabla v_{\ell_0}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2} = (1 - \delta^2)\mu^1 + \varepsilon(\ell_0)\,,$$

we obtain from (2.2.18) that

$$\begin{aligned} \lambda_\ell^1 - \mu^1 &\le \frac{\left\{(1-\delta^2)\mu^1 + \varepsilon(\ell_0)\right\} \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{\eta} + \frac{2\eta\mu^1}{3}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta} - \mu^1 \\ &= \frac{(\varepsilon(\ell_0) - \delta^2\mu^1) \int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{\eta}}{\int_{\Omega_{\ell_0}} v_{\ell_0}^2 + \frac{2}{3}\eta}\,. \end{aligned} \tag{2.2.19}$$

Choosing ℓ_0 small enough such that

$$\varepsilon(\ell_0) - \delta^2\mu^1 < 0,$$

and then taking η sufficiently large, makes the RHS of (1.2.6) equal a negative number, say $-\delta_0$. Hence

$$\lambda_\ell^1 \le \mu^1 - \delta_0 \ \text{ for } \ell > \ell_0 + \eta,$$

and the result follows. □

2.3 The Gap Phenomenon in the General Case.

In this section we extend the results from Section 2.2 to a more general framework. We shall use the notation from Section 2.1 and study the limit $\lim_{\ell\to\infty} \lambda_\ell^1$ for λ_ℓ^1 given by (2.1.10). As in Section 2.2 our strategy is to study first the limit as ℓ goes to 0.

Theorem 2.3.1. *We have* $\lim_{\ell\to 0} \lambda_\ell^1 = \Lambda^1$ *where*

$$\Lambda^1 = \inf\left\{ \int_\omega A_{22}(X_2)\nabla u \cdot \nabla u - \frac{|A_{12}(X_2)\cdot\nabla u|^2}{a_{11}(X_2)} \,:\, u \in H_0^1(\omega),\ \int_\omega u^2 = 1 \right\}. \tag{2.3.1}$$

Proof. The reason why we find Λ^1 as the limiting value will be clarified by the following simple observation. Let

$$B = \begin{pmatrix} b_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}$$

be a positive definite $n \times n$ matrix and represent any vector $\vec{z}$ in $\mathbb{R}^n$ as $\vec{z} = (z_1, Z_2)$ with $Z_2 \in \mathbb{R}^{n-1}$. Then, elementary calculus shows that for any fixed $Z_2 \in \mathbb{R}^{n-1}$ we have

$$\min_{z_1\in\mathbb{R}} (B\vec{z})\cdot\vec{z} = (B_{22}Z_2)\cdot Z_2 - \frac{|B_{12}Z_2|^2}{b_{11}}\,. \tag{2.3.2}$$

Furthermore, the minimum in (2.3.2) is attained for

$$z_1 = -\frac{B_{12}Z_2}{b_{11}}\,. \tag{2.3.3}$$

Applying (2.3.2) with $B = A(X_2)$ we obtain, for any $\ell > 0$,

$$\begin{aligned}\int_{\Omega_\ell} (A(X_2)\nabla u_\ell) \cdot \nabla u_\ell &\geq \int_{\Omega_\ell} (A_{22}(X_2)\nabla_{X_2} u_\ell) \cdot \nabla_{X_2} u_\ell - \frac{|A_{12}(X_2)\nabla_{X_2} u_\ell|^2}{a_{11}(X_2)} \\ &\geq \Lambda^1 \int_{\Omega_\ell} u_\ell^2 \,.\end{aligned} \tag{2.3.4}$$

By (2.3.4) the lower-bound

$$\liminf_{\ell \to 0} \lambda_\ell^1 \geq \Lambda^1 \,, \tag{2.3.5}$$

is clear. We note that from the above it follows in particular that

$$\Lambda^1 \geq \lambda_A$$

(see (2.1.6)) and the infimum in (2.3.1) is actually a minimum, which is realized by a positive function $w_1 \in H_0^1(\omega)$ satisfying

$$\int_\omega w_1^2 = 1.$$

In order to complete the proof of Theorem 2.3.1 we need to establish the upper-bound part. A natural generalization of the construction used in the proof of Theorem 2.2.2 would be to use

$$v_\ell(x) = w_1(X_2) - \frac{(A_{12}(X_2) \cdot \nabla w_1)\, x_1 \rho_\ell(X_2)}{a_{11}(X_2)} \,, \tag{2.3.6}$$

where ρ_ℓ is an appropriate cut-off function. However, since the coefficients of the matrix $A(X_2)$ are only assumed to be L^∞-functions, the function on the RHS of (2.3.6) does not necessarily belong to H^1. To overcome this difficulty, we use an approximation argument, motivated by **[2, Ch. 14]**. We apply standard mollification to define a family of $\mathbb{R}^{n-1}$-valued functions $\{G_\varepsilon\}_{\varepsilon>0} \subset C_c^\infty(\omega, \mathbb{R}^{n-1})$ satisfying

$$\lim_{\varepsilon \to 0} G_\varepsilon(X_2) = \frac{A_{12}(X_2) \cdot \nabla w_1}{a_{11}(X_2)} \quad \text{in } L^2(\omega) \text{ and a.e..} \tag{2.3.7}$$

We then define

$$v_\ell^\varepsilon(x_1, X_2) = w_1(X_2) - G_\varepsilon(X_2) x_1 \,. \tag{2.3.8}$$

First notice that

$$\int_{\Omega_\ell} |v_\ell^\varepsilon|^2 = \int_{-\ell}^{\ell} \int_\omega w_1^2 - 2x_1 w_1 G_\varepsilon + (x_1 G_\varepsilon)^2 \geq 2\ell \int_\omega w_1^2 = 2\ell, \tag{2.3.9}$$

since $\int_{-\ell}^{\ell} x_1 \, dx_1 = 0$. Now

$$\begin{aligned}\int_{\Omega_\ell} A\nabla v_\ell^\varepsilon . \nabla v_\ell^\varepsilon &= \int_{\Omega_\ell} a_{11} \left(\frac{\partial v_\ell^\varepsilon}{\partial x_1}\right)^2 + 2(A_{12}.\nabla_{X_2} v_\ell^\varepsilon)\frac{\partial v_\ell^\varepsilon}{\partial x_1} + (A_{22}\nabla_{X_2} v_\ell^\varepsilon).\nabla_{X_2} v_\ell^\varepsilon \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)\,.\end{aligned}$$

For the first integral we have

$$I_1(\varepsilon) = \int_{\Omega_\ell} a_{11} G_\varepsilon^2 = 2\ell \int_\omega a_{11} G_\varepsilon^2 \,. \tag{2.3.10}$$

For the second integral,

$$I_2(\varepsilon) = 2\int_{-\ell}^{\ell}\int_\omega A_{12}.\Big\{\nabla_{X_2} w_1 - x_1 \nabla_{X_2} G_\varepsilon(X_2)\Big\}\Big\{-G_\varepsilon(X_2)\Big\}\,.$$

Since the integral of the term containing x_1 vanishes, we get

$$I_2(\varepsilon) = -4\ell \int_\omega (A_{12}.\nabla w_1) G_\varepsilon \,. \tag{2.3.11}$$

For the last integral we have (after dropping the term with the vanishing integral),

$$\begin{aligned} I_3(\varepsilon) = \int_{-\ell}^{\ell}\int_\omega & (A_{22}\nabla_{X_2} w_1).\nabla_{X_2} w_1 + (A_{22}\nabla_{X_2} G_\varepsilon).\nabla_{X_2} G_\varepsilon x_1^2 \\ &= 2\ell \Big\{\int_\omega (A_{22}\nabla_{X_2} w_1).\nabla w_1 + \frac{\ell^2}{3}\int_\omega (A_{22}\nabla_{X_2} G_\varepsilon).\nabla G_\varepsilon\Big\}\,. \end{aligned} \tag{2.3.12}$$

By (2.3.9)–(2.3.12) we deduce that

$$\begin{aligned} \limsup_{\ell\to 0} \lambda_\ell^1 \le \limsup_{\ell \to 0} \frac{\int_{\Omega_\ell} A\nabla v_\ell^\varepsilon.\nabla v_\ell^\varepsilon}{\int_{\Omega_\ell} |v_\ell^\varepsilon|^2} \le \\ \int_\omega a_{11}\,(G_\varepsilon)^2 - 2\int_\omega (A_{12}.\nabla w_1).G_\varepsilon + \int_\omega \big(A_{22}\nabla w_1\big).\nabla w_1 \,. \end{aligned} \tag{2.3.13}$$

Passing to the limit $\varepsilon \to 0$ in (2.3.13), using (2.3.7), gives

$$\limsup_{\ell\to 0} \lambda_\ell^1 \le \int_\omega (A_{22}\nabla w_1).\nabla w_1 - \frac{|A_{12}\nabla w_1|^2}{a_{11}} = \Lambda^1 \,,$$

which together with (2.3.5) yields the result. □

Remark 2.3.1. *Replacing* (2.3.8) *by*

$$\tilde{v}_\ell^\varepsilon(x_1, X_2) = W_1(X_2) - G_\varepsilon(X_2) x_1 \,, \tag{2.3.14}$$

with w_1 *replaced by* W_1 *in* (2.3.7) *and carrying out the same computation as in the last part of the proof of Theorem 2.3.1 yields*

$$\inf_{\varepsilon>0} \lim_{\ell\to 0} \frac{\int_{\Omega_\ell} (A\nabla \tilde{v}_\ell^\varepsilon).\tilde{\nabla} v_\ell^\varepsilon}{\int_{\Omega_\ell} |\tilde{v}_\ell^\varepsilon|^2} = \int_\omega (A_{22}\nabla W_1).\nabla W_1 - \frac{|A_{12}\nabla W_1|^2}{a_{11}}\,. \tag{2.3.15}$$

Our next theorem provides an analog of Theorem 2.2.1 to the general case.

Theorem 2.3.2. *We have*

$$\limsup_{\ell\to\infty} \lambda_\ell^1 < \mu^1, \tag{2.3.16}$$

provided the following condition holds,

$$A_{12} \cdot \nabla W_1 \not\equiv 0 \text{ a.e. on } \omega. \tag{2.3.17}$$

In case (2.3.17) *does not hold we have* $\lambda_\ell^1 = \mu^1$ *for all* $\ell > 0$.

Remark 2.3.2. *It is easy to construct examples where condition* (2.3.17) *doesn't hold. Take for example for* ω *the unit disc in* $\mathbb{R}^2$. *For*

$$A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the eigenfunction W_1 *is radially symmetric. We use polar coordinates on* ω *and represent each* X_2 *as*

$$X_2 = r(\cos\theta, \sin\theta).$$

Taking $a_{11} = 1$ *and* $A_{1,2}(X_2) = t(-\sin\theta, \cos\theta)$ *for* $|t|$ *small enough (in order for the uniform ellipticity condition* (2.1.6) *to hold for the* 3 *by* 3 *matrix* A*) yields an example for which* (2.3.17) *doesn't hold.*

Proof. (i) Assume first that (2.3.17) holds. Then,

$$\Lambda_1 < \mu^1. \tag{2.3.18}$$

Indeed, this follows from

$$\Lambda_1 \le \int_\omega A_{22}(X_2)\nabla W_1.\nabla W_1 - \frac{|A_{12}(X_2)\nabla W_1|^2}{a_{11}(X_2)} < \int_\omega A_{22}(X_2)\nabla W_1.\nabla W_1 = \mu^1 .$$

By the proof of Theorem 2.3.1 there exist positive values of ℓ_0 and ε_0 such that $\tilde v_{\ell_0}^{\varepsilon_0}$ defined by (2.3.14) satisfies

$$\int_{\Omega_{\ell_0}} A\nabla \tilde v_{\ell_0}^{\varepsilon_0} \cdot \nabla \tilde v_{\ell_0}^{\varepsilon_0} < \mu^1 \int_{\Omega_{\ell_0}} |\tilde v_{\ell_0}^{\varepsilon_0}|^2 . \tag{2.3.19}$$

Notice that

$$\tilde v_{\ell_0}^{\varepsilon_0}(0, X_2) = W_1(X_2).$$

Let $\eta > 0$ be a parameter whose value will be determined later. For $\ell > \ell_0 + \eta$ define ϕ_ℓ as follows,

$$\phi_\ell = \begin{cases} \tilde v_{\ell_0}^{\varepsilon_0}(x_1 - \ell + \ell_0, X_2) & \text{on } (\ell - \ell_0, \ell) \times \omega\,, \\ \frac{(x_1 - (\ell - \ell_0 - \eta))W_1(X_2)}{\eta} & \text{on } (\ell - \ell_0 - \eta, \ell - \ell_0) \times \omega\,, \\ 0 & \text{on } \Omega_{\ell - \ell_0 - \eta}\,, \\ \frac{(-x_1 - (\ell - \ell_0 - \eta))W_1(X_2)}{\eta} & \text{on } (\ell_0 - \ell, -(\ell - \ell_0 - \eta)) \times \omega\,, \\ \tilde v_{\ell_0}^{\varepsilon_0}(x_1 + \ell - \ell_0, X_2) & \text{on } (-\ell, \ell_0 - \ell) \times \omega\,. \end{cases} \tag{2.3.20}$$

Since

$$\int_{\Omega_\ell \setminus \Omega_{\ell-\ell_0}} \phi_\ell^2 = \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 \,,$$

and

$$\int_{\Omega_{\ell-\ell_0}} \phi_\ell^2 = 2 \left(\int_{\ell-\ell_0-\eta}^{\ell-\ell_0} \frac{(x_1 - \ell + \ell_0 + \eta)^2}{\eta^2} dx_1 \right) \left(\int_\omega W_1^2 \, dX_2 \right) = \frac{2}{3}\eta \,,$$

we have

$$\int_{\Omega_\ell} \phi_\ell^2 = \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta \,. \tag{2.3.21}$$

Similarly

$$\int_{\Omega_\ell} A\nabla\phi_\ell \cdot \nabla\phi_\ell = \int_{\Omega_{\ell_0}} A\nabla\tilde{v}_{\ell_0} \cdot \nabla\tilde{v}_{\ell_0} + \int_{\Omega_{\ell-\ell_0}} A\nabla\phi_\ell \cdot \nabla\phi_\ell \,, . \tag{2.3.22}$$

Setting $D = \Omega_{\ell-\ell_0} \setminus \Omega_{\ell-\ell_0-\eta}$ and $D^+ = (\ell-\ell_0-\eta, \ell-\ell_0) \times \omega$, the last integral above can be written as

$$\int_{\Omega_{\ell-\ell_0}} A\nabla\phi_\ell \cdot \nabla\phi_\ell = \frac{1}{\eta^2} \int_D a_{11} W_1^2 + 2 \int_D A_{12} \cdot \nabla_{X_2}\phi_\ell \frac{\partial\phi_\ell}{\partial x_1} + \frac{2}{\eta^2} \int_{D^+} (x_1 - \ell + \ell_0 + \eta)^2 A_{22} \nabla W_1 \cdot \nabla W_1.$$

The second integral vanishes since its integrand is an odd function of x_1 on D. Therefore,

$$\int_{\Omega_{\ell-\ell_0}} A\nabla\phi_\ell \cdot \nabla\phi_\ell = \frac{2}{\eta} \int_\omega a_{11} W_1^2 + \frac{2\eta}{3} \int_\omega A_{22} \nabla W_1 \cdot \nabla W_1 = \frac{2}{\eta} \int_\omega a_{11} W_1^2 + \frac{2\eta\mu^1}{3} \,. \tag{2.3.23}$$

Combining (2.3.21), (2.3.22) and (2.3.23) we obtain

$$\lambda_\ell^1 \le \frac{\int_{\Omega_\ell} A\nabla\phi_\ell \cdot \nabla\phi_\ell}{\int_{\Omega_\ell} \phi_\ell^2} \le \frac{\int_{\Omega_{\ell_0}} A\nabla\tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla\tilde{v}_{\ell_0}^{\varepsilon_0} + \frac{2}{\eta}\int_\omega a_{11} W_1^2 + \frac{2}{3}\eta\mu^1}{\int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta} \,.$$

Therefore,

$$\lambda_\ell^1 - \mu^1 \le \frac{\int_{\Omega_{\ell_0}} A\nabla\tilde{v}_{\ell_0}^{\varepsilon_0} \cdot \nabla\tilde{v}_{\ell_0}^{\varepsilon_0} - \mu^1 \int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{\eta}\int_\omega a_{11} W_1^2}{\int_{\Omega_{\ell_0}} |\tilde{v}_{\ell_0}^{\varepsilon_0}|^2 + \frac{2}{3}\eta} . \tag{2.3.24}$$

By (2.3.19) it is clear that we can fix a large enough value for η such that the RHS of (2.3.24) is negative, and the result for case (i) follows.

(ii) By (2.3.4) we have $\Lambda_1 \le \lambda_\ell^1$ for all $\ell > 0$. On the other hand, using $u(x) = W_1(X_2)$ as a test function in (2.1.10) gives $\lambda_\ell^1 \le \mu^1$. Thus we have,

$$\Lambda_1 \le \lambda_\ell^1 \le \mu^1 \,, \quad \forall \ell > 0 \,. \tag{2.3.25}$$

In view of (2.3.25), the result for the case where (2.3.17) doesn't hold would follow once we show that in this case

$$\Lambda_1 = \mu^1.$$

The Euler-Lagrange equation for an eigenfunction v of the quadratic form in (2.3.1), with eigenvalue λ is

$$\begin{cases} -\mathrm{div}(A_{22}\nabla v) + \mathrm{div}((A_{12} \cdot \nabla v)A_{12}) = \lambda v \quad \text{in } \omega\,, \\ \quad v = 0 \quad \text{on } \partial\omega\,. \end{cases} \tag{2.3.26}$$

Of course $v = w_1$ satisfies (2.3.26) with $\lambda = \Lambda_1$. But since we assume that (2.3.17) doesn't hold, $v = W_1$ is also a solution of (2.3.26) with $\lambda = \mu^1$. However, only the first eigenvalue of the problem (2.3.26) can have a positive eigenfunction, so we must have $\Lambda_1 = \mu^1$ as claimed. □

2.4 Characterization of the Limit $\lim_{\ell\to\infty} \lambda_\ell^1$

In this section we study the asymptotic behavior of u_ℓ in L^2 and H^1 norms. We shall see that when (2.3.17) holds, the eigenfunctions decay to zero in the bulk of the cylinder and concentration occurs near the bases of the cylinder. We denote by $[x]$ the integer part of x.

Theorem 2.4.1. *Assume* (2.3.17) *holds. Then, there exist $\alpha \in (0,1)$ and a positive constants c such that for $\ell > \ell_0$ we have, for every $0 < r \le \ell - 1$,*

$$\int_{\Omega_r} u_\ell^2 \le \alpha^{[\ell - r]}\,, \tag{2.4.1}$$

and

$$\int_{\Omega_r} |\nabla u_\ell|^2 \le c\alpha^{[\ell - r]}\,. \tag{2.4.2}$$

Proof. Let ℓ and ℓ' satisfy $0 < \ell' \le \ell - 1$. Define $\rho_{\ell'} = \rho_{\ell'}(x_1)$ by

$$\rho_{\ell'}(x_1) = \begin{cases} 1 & |x_1| \le \ell'\,, \\ \ell' + 1 - |x_1| & |x_1| \in (\ell', \ell'+1)\,, \\ 0 & |x_1| \ge \ell' + 1\,. \end{cases} \tag{2.4.3}$$

Using $v = \rho_{\ell'}^2 u_\ell \in V(\Omega_\ell)$ in (2.1.9), we get

$$\int_{\Omega_\ell} (A\nabla u_\ell) \cdot \nabla(\rho_{\ell'}^2 u_\ell) = \lambda_\ell^1 \int_{\Omega_\ell} \rho_{\ell'}^2 u_\ell^2\,,$$

i.e.,

$$\int_{\Omega_\ell} \big(A\nabla(\rho_{\ell'} u_\ell)\big) \cdot \nabla(\rho_{\ell'} u_\ell) - \int_{\Omega_\ell} u_\ell^2 (A\nabla\rho_{\ell'}) \cdot \nabla\rho_{\ell'} = \lambda_\ell^1 \int_{\Omega_\ell} \rho_{\ell'}^2 u_\ell^2\,. \tag{2.4.4}$$

Since $\rho_{\ell'} u_\ell \in H_0^1(\Omega_\ell)$, by the Rayleigh quotient characterization of σ_ℓ^1 (see (2.0.2)) we have

$$\sigma_\ell^1 \int_{\Omega_\ell} u_\ell^2 \rho_{\ell'}^2 \le \int_{\Omega_\ell} A\nabla(\rho_{\ell'} u_\ell) \cdot \nabla(\rho_{\ell'} u_\ell)\,. \tag{2.4.5}$$

Combining (2.4.4)–(2.4.5) with (2.1.5) we get

$$\begin{aligned}(\sigma_\ell^1 - \lambda_\ell^1) \int_{\Omega_\ell} u_\ell^2 \rho_{\ell'}^2 \le \int_{\Omega_\ell} u_\ell^2 (A\nabla\rho_{\ell'}) \cdot \nabla\rho_{\ell'} &= \int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} u_\ell^2 (A\nabla\rho_{\ell'}) \cdot \nabla\rho_{\ell'} \\ &\le C_A \int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} u_\ell^2\,.\end{aligned} \tag{2.4.6}$$

By (2.0.3) and (2.3.16) there exists $\beta > 0$ such that for $\ell > \ell_0$ we have $\sigma_\ell^1 - \lambda_\ell^1 \ge \beta$. Therefore, from (2.4.6) we deduce that

$$(C_A + \beta) \int_{\Omega_{\ell'}} u_\ell^2 \le C_A \int_{\Omega_{\ell'+1}} u_\ell^2\,.$$

This leads to

$$\int_{\Omega_{\ell'}} u_\ell^2 \le \alpha \int_{\Omega_{\ell'+1}} u_\ell^2\,, \tag{2.4.7}$$

with $\alpha = \frac{C_A}{C_A+\beta} < 1$. Applying (2.4.7) successively for $\ell' = r, r+1, \ldots, r + [\ell - r] - 1$ yields

$$\int_{\Omega_r} u_\ell^2 \le \alpha^{[\ell-r]} \int_{\Omega_\ell} u_\ell^2 = \alpha^{[\ell-r]}\,. \tag{2.4.8}$$

To prove (2.4.2), we fix $r \in (0, \ell-2)$ and then use (2.4.4), with $\ell' = r$, combined with (2.1.6) and (2.2.3), to obtain

$$\begin{aligned}\lambda_A \int_{\Omega_r} |\nabla u_\ell|^2 &\le \int_{\Omega_\ell} A\nabla(\rho_r u_\ell) \cdot \nabla(\rho_r u_\ell) \\ &= \int_{\Omega_\ell} u_\ell^2 (A\nabla\rho_r) \cdot \nabla\rho_r + \lambda_\ell^1 \int_{\Omega_\ell} \rho_r^2 u_\ell^2 \le (C_A + \mu^1) \int_{\Omega_{r+1}} u_\ell^2\,.\end{aligned} \tag{2.4.9}$$

Finally, (2.4.2) follows from (2.4.8)–(2.4.9) for $r \le \ell - 2$. Choosing a step size of $\frac{1}{2}$ in the first part of the proof will allow $r \le \ell - 1$. □

The decay of the eigenfunction in the bulk immediately implies concentration near the two ends of the cylinder.

Corollary 2.4.1. *If* (2.3.17) *holds then for every* $r \in (0, \ell - 1]$ *we have*

$$\int_{\Omega_\ell\setminus\Omega_r} u_\ell^2 \ge 1 - \alpha^{[\ell-r]} \quad \textit{and} \quad \int_{\Omega_\ell\setminus\Omega_{\ell-1}} A\nabla u_\ell \cdot \nabla u_\ell \ge \lambda_\ell^1 - c_1 \alpha^{[\ell-r]}\,. \tag{2.4.10}$$

To have a more precise description of the asymptotic behavior of λ_ℓ^1 we introduce two variational problems on semi-infinite cylinders. Set

$$\Omega_\infty^+ = (0,\infty) \times \omega \quad \text{and} \quad \Omega_\infty^- = (-\infty,0) \times \omega\,,$$

and denote the corresponding lateral parts of the boundary by

$$\gamma_\infty^+ = (0,\infty) \times \partial\omega \quad \text{and} \quad \gamma_\infty^- = (-\infty,0) \times \partial\omega\,.$$

Define the spaces

$$V(\Omega_\infty^\pm) := \{u \in H^1(\Omega_\infty^\pm) \,:\, u = 0 \text{ on } \gamma_\infty^\pm\}\,,$$

and set

$$\nu_\infty^\pm = \inf_{0 \neq u \in V(\Omega_\infty^\pm)} \frac{\int_{\Omega_\infty^\pm} A\nabla u \cdot \nabla u}{\int_{\Omega_\infty^\pm} u^2}\,. \tag{2.4.11}$$

Remark 2.4.1. *In case property (S) holds (see Definition 2.1.1) we clearly have*

$$\nu_\infty^+ = \nu_\infty^-$$

as we can use the transformation

$$v(x_1, X_2) \mapsto v(-x_1, -X_2)$$

to pass from a function in $V(\Omega_\infty^+)$ to a function in $V(\Omega_\infty^-)$ (and vice versa) that has the same Rayleigh quotient. In general we can only assert that $\nu_\infty^- = \tilde{\nu}_\infty^+$ where $\tilde{\nu}_\infty^+$ is defined as in (2.4.11)*, but with A being replaced by $\tilde{A}$, given by*

$$\tilde{A}(X_2) = \begin{pmatrix} a_{11}(X_2) & -A_{12}(X_2) \\ -A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}\,.$$

This is easily seen by applying the transformation $v(x_1, X_2) \mapsto v(-x_1, X_2)$.

The next lemma gives the possible range of values for $\nu_\infty^\pm$.

Lemma 2.4.1. *We have*

$$0 < \nu_\infty^\pm \leq \mu^1\,. \tag{2.4.12}$$

Proof. By Remark 2.4.1 it is enough to consider ν_∞^+. The fact that $\nu_\infty^+ > 0$ follows from the Poincaré inequality. In order to show that $\nu_\infty^+ \leq \mu^1$ we set for each $\varepsilon > 0$,

$$v_\varepsilon(x) = e^{-\varepsilon x_1} W_1(X_2)\,.$$

Clearly $v_\varepsilon \in V(\Omega_\infty^+)$ and a direct computation gives

$$\begin{aligned} \int_{\Omega_\infty^+} A\nabla v_\varepsilon \cdot \nabla v_\varepsilon &= \int_{\Omega_\infty^+} e^{-2\varepsilon x_1}\Big(a_{11}\varepsilon^2 W_1^2 - 2\varepsilon(A_{12}\cdot\nabla W_1)W_1 + A_{22}\nabla W_1 \cdot \nabla W_1\Big) \\ &= \left(\int_0^\infty e^{-2\varepsilon x_1}\right)\Big(\mu^1 + \varepsilon^2 \int_\omega a_{11} W_1^2 - 2\varepsilon \int_\omega (A_{12}\cdot\nabla W_1)W_1\Big)\,, \end{aligned} \tag{2.4.13}$$

and

$$\int_{\Omega_\infty^+} v_\varepsilon^2 = \int_0^\infty e^{-2\varepsilon x_1} \left(= \frac{1}{2\varepsilon}\right). \tag{2.4.14}$$

By (2.4.13)–(2.4.14) we obtain

$$\frac{\int_{\Omega_\infty^+} A\nabla v_\varepsilon \cdot \nabla v_\varepsilon}{\int_{\Omega_\infty^+} v_\varepsilon^2} = \mu^1 - 2\varepsilon \int_\omega (A_{12} \cdot \nabla W_1) W_1 + \varepsilon^2 \int_\omega a_{11} W_1^2 \,,$$

so by sending ε to 0 we deduce that $\nu_\infty^+ \le \mu^1$. □

It is easy to identify $\nu_\infty^\pm$ with the limits, as $\ell \to \infty$, of certain minimization problems on $\Omega_\ell^\pm$. This is the content of the next lemma (see (2.1.3) and (2.1.4) for the definitions of $\gamma_\ell^\pm$ and $\Gamma_\ell^\pm$).

Lemma 2.4.2. *We have* $\nu_\infty^\pm = \lim_{\ell\to\infty} \tilde\lambda_\ell^{1,\pm}$, *where*

$$\tilde\lambda_\ell^{1,\pm} = \inf\{\int_{\Omega_\ell^\pm} A\nabla u \cdot \nabla u \,:\, u \in H^1(\Omega_\ell^\pm),\ \int_{\Omega_\ell^\pm} u^2 = 1, u = 0 \ \text{on}\ \gamma_\ell^\pm \cup \Gamma_\ell^\pm\}\,. \tag{2.4.15}$$

Remark 2.4.2. *It is a standard fact that the infimum in* (2.4.15) *is actually attained. The unique positive normalized minimizer will be denoted by* $\tilde u_\ell^\pm$.

Proof. We present the proof for $\tilde\lambda_\ell^{1,+}$ as the proof for $\tilde\lambda_\ell^{1,-}$is completely analogous. Note first that the limit $\lim_{\ell\to\infty} \tilde\lambda_\ell^{1,+}$ exists since the function

$$\ell \to \tilde\lambda_\ell^{1,+}$$

is non increasing. Indeed, if $\ell_1 < \ell_2$ then any admissible function in (2.4.15) for $\tilde\lambda_{\ell_1}^{1,+}$ can be extended to an admissible function for $\tilde\lambda_{\ell_2}^{1,+}$ by setting it to zero on $\Omega_{\ell_2}^+ \setminus \Omega_{\ell_1}^+$. A similar argument shows that

$$\tilde\lambda_\ell^{1,+} \ge \nu_\infty^+, \ \ \forall \ell > 0.$$

On the other hand, the density of the space

$$V_s(\Omega_\infty^+) = \{u \in C^\infty(\Omega_\infty^+) \cap V(\Omega_\infty^+) \,:\, \exists M = M(u) > 0 \text{ s.t. } u = 0 \text{ on } (M,\infty)\times\omega\}\,, \tag{2.4.16}$$

in $V(\Omega_\infty^+)$ implies that for each $u \in V(\Omega_\infty^+) \setminus \{0\}$ and any $\varepsilon > 0$ we can find an ℓ_ε and $v_\varepsilon \in V_s(\Omega_\infty^+)$ with $\text{supp}(v_\varepsilon) \subset \Omega_{\ell_\varepsilon}^+$ such that

$$\left| \frac{\int_{\Omega_\infty^+} (A\nabla v_\varepsilon).\nabla v_\varepsilon}{\int_{\Omega_\infty^+} v_\varepsilon^2} - \frac{\int_{\Omega_\infty^+} (A\nabla u)\cdot \nabla u}{\int_{\Omega_\infty^+} u^2} \right| \le \varepsilon\,,$$

and (2.4.15) follows (for $\tilde\lambda_\ell^{1,+}$). □

Our next result complements the result of Theorem 2.3.2 in two ways: by showing that the limit $\lim_{\ell\to\infty} \lambda_\ell^1$ exists and by identifying its value.

Theorem 2.4.2. *We have*

$$\lim_{\ell\to\infty} \lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-)\,. \tag{2.4.17}$$

Proof. (i) We shall first show that

$$\limsup_{\ell\to\infty} \lambda_\ell^1 \le \min(\nu_\infty^+, \nu_\infty^-)\,. \tag{2.4.18}$$

We may assume w.l.o.g. that

$$\nu_\infty^+ = \min(\nu_\infty^+, \nu_\infty^-).$$

Given $\varepsilon > 0$ we may find by Lemma 2.4.2 an $\ell_\varepsilon > 1/\varepsilon$ such that

$$\tilde\lambda_{\ell_\varepsilon}^{1,+} \le \nu_\infty^+ + \varepsilon.$$

Since $\lambda_{\ell/2}^1 \le \tilde\lambda_\ell^{1,+}$ by the definitions (2.1.10) and (2.4.15), we easily deduce (2.4.18).

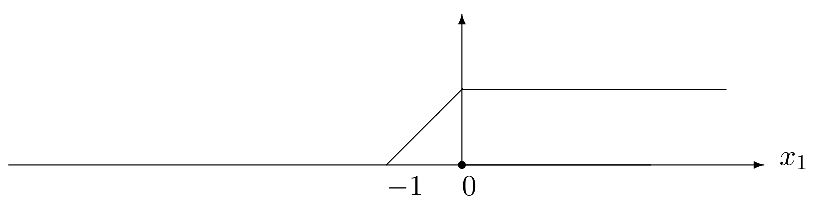
(ii) We now treat the case where (2.3.17) holds. Let u_ℓ denote the positive normalized minimizer in (2.1.10). Define

$$v_\ell(x) = \rho(x_1)u_\ell(x)$$

where ρ is given by

$$\rho(x_1) = \begin{cases} 0 & x_1 \le -1\,, \\ 1 + x_1 & x_1 \in (-1,0)\,, \\ 1 & x_1 \ge 0\,. \end{cases} \tag{2.4.19}$$

The graph of ρ is given below.



Figure

By (2.1.5) and (2.4.19) we have

$$\int_{(-1,\ell)\times\omega} (A\nabla v_\ell)\cdot\nabla v_\ell \le \int_{\Omega_\ell^+} (A\nabla u_\ell)\cdot\nabla u_\ell + C_A \int_{(-1,0)\times\omega} |\nabla(\rho u_\ell)|^2\,. \tag{2.4.20}$$

Define $w_{\ell+1}(x_1, X_2) = v_\ell(x_1 + \ell, X_2)$ on $\Omega^-_{\ell+1}$ and notice that it is an admissible function for the infimum defining $\tilde{\lambda}^{1,-}_{\ell+1}$ (see (2.4.15)). By (2.4.20) and (2.4.1)–(2.4.2) we obtain, for some positive constant C,

$$\int_{\Omega^-_{\ell+1}} (A\nabla w_{\ell+1}) \cdot \nabla w_{\ell+1} \le \int_{\Omega^+_\ell} (A\nabla u_\ell) \cdot \nabla u_\ell + C\alpha^\ell \,. \tag{2.4.21}$$

Denote

$$N^\pm_\ell = \int_{\Omega^\pm_\ell} (A\nabla u_\ell) \cdot \nabla u_\ell \quad \text{and} \quad D^\pm_\ell = \int_{\Omega^\pm_\ell} |u_\ell|^2 \,,$$

so that in particular we have

$$N^+_\ell + N^-_\ell = \lambda^1_\ell \quad \text{and} \quad D^+_\ell + D^-_\ell = 1 \,. \tag{2.4.22}$$

By (2.4.21) and an analogous construction on $\Omega^+_{\ell+1}$ we have

$$\tilde{\lambda}^{1,-}_{\ell+1} \le \frac{N^+_\ell + C\alpha^\ell}{D^+_\ell} \quad \text{and} \quad \tilde{\lambda}^{1,+}_{\ell+1} \le \frac{N^-_\ell + C\alpha^\ell}{D^-_\ell} \,. \tag{2.4.23}$$

From (2.4.23) and (2.4.22) it follows that

$$\min\{\tilde{\lambda}^{1,-}_{\ell+1}, \tilde{\lambda}^{1,+}_{\ell+1}\} \le D^+_\ell \tilde{\lambda}^{1,-}_{\ell+1} + D^-_\ell \tilde{\lambda}^{1,+}_{\ell+1} \le \lambda^1_\ell + C\alpha^\ell \,. \tag{2.4.24}$$

Passing to the limit $\ell \to \infty$ in (2.4.24) and using Lemma 2.4.2 yields

$$\min(\nu^+_\infty, \nu^-_\infty) \le \liminf_{\ell\to\infty} \lambda^1_\ell \,, \tag{2.4.25}$$

which combined with (2.4.18) clearly implies (2.4.17) (in the case that (2.3.17) holds).

(iii) Finally, we turn to the case where (2.3.17) doesn't hold. In this case we know already from Theorem 2.3.2 that $\lambda^1_\ell = \mu^1$ for all ℓ. The proof of (2.4.17) will be clearly completed if we show that

$$\nu^+_\infty = \nu^-_\infty = \mu^1.$$

We shall only show that $\nu^+_\infty = \mu^1$ as the argument for ν^-_∞ is identical. By Lemma 2.4.1 we have

$$\nu^+_\infty \le \mu^1.$$

For the reverse inequality we notice that in our case, for any $u \in V(\Omega^+_\infty)$ we have,

$$\int_{\Omega^+_\infty} A\nabla u \cdot \nabla u = \int_{\Omega^+_\infty} a_{11} u^2_{x_1} + (A_{22}\nabla_{X_2} u) \cdot \nabla_{X_2} u \ge \mu^1 \int_{\Omega^+_\infty} u^2 \,,$$

implying that $\nu^+_\infty \ge \mu^1$. □

The argument used in the above proof can be used to derive an additional information that will be useful in the next section.

Proposition 2.4.1. *If* $\nu_\infty^+ < \nu_\infty^-$ *then*

$$\lim_{\ell\to\infty} \int_{\Omega_\ell^+} |\nabla u_\ell|^2 + |u_\ell|^2 = 0.$$

Proof. We use the same notation as in the proof of Theorem 2.4.2. Passing to the limit $\ell \to \infty$ in (2.4.24), using Lemma 2.4.2 and (2.4.17) yields

$$\Big(\limsup_{\ell\to\infty} D_\ell^+\Big)\nu_\infty^- + \Big(1 - \limsup_{\ell\to\infty} D_\ell^+)\nu_\infty^+ \le \lim_{\ell\to\infty} \lambda_\ell^1 = \nu_\infty^+ \,,$$

so necessarily

$$\limsup_{\ell\to\infty} D_\ell^+ = 0.$$

Next, by (2.4.23) we have for ℓ large,

$$\frac{N_\ell^+}{D_\ell^-} + \tilde\lambda_{\ell+1}^{1,+} - C\alpha^\ell \le \frac{N_\ell^+ + N_\ell^-}{D_\ell^-} \le \frac{N_\ell^+ + N_\ell^-}{D_\ell^+ + D_\ell^-} = \lambda_\ell^1 \,. \tag{2.4.26}$$

Since in our case,

$$\lim_{\ell\to\infty} \lambda_\ell^1 = \lim_{\ell\to\infty} \tilde\lambda_{\ell+1}^{1,+} = \nu_\infty^+,$$

and we know already that $\lim_{\ell\to\infty} D_\ell^- = 1$, we deduce from (2.4.26) that

$$\lim_{\ell\to\infty} N_\ell^+ = 0.$$

This completes the proof of the proposition. □

2.5 The Problem on a Semi-Infinite Cylinder

In this section we further investigate the minimization problem (2.4.11). By Remark 2.4.1 it is enough to consider ν_∞^+. There are two main questions we are interested in. First, we want to identify the conditions under which the infimum in (2.4.11) is attained. Second, we would like to know when the inequality $\nu_\infty^+ < \mu^1$ hold. The next proposition shows that the two questions are closely related to each other.

Proposition 2.5.1. *If*

$$\nu_\infty^+ < \mu^1 \,, \tag{2.5.1}$$

then ν_∞^+ *is attained. The minimizer* $\tilde u^+$ *is unique up to multiplication by a constant, has constant sign and satisfies*

$$\begin{cases} -\mathrm{div}(A(X_2)\nabla \tilde u^+) = \tilde u^+ & \text{in } \Omega_\infty^+, \\ \tilde u^+ = 0 \quad \text{on } \gamma_\infty^+, \end{cases} \tag{2.5.2}$$

Proof. This existence of a minimizer will be achieved by taking the limit $\ell \to \infty$ of the minimizers $\{\tilde{u}_\ell^+\}$ in (2.4.15) (see Remark 2.4.2). Since $\{\tilde{u}_\ell^+\}$ is bounded in $H^1(\Omega_\infty^+)$, a subsequence $\{\tilde{u}_{\ell_k}\}$ converges weakly to some limit $\tilde{u}^+ \in H^1(\Omega_\infty^+)$. Take any $\varphi \in V_s(\Omega_\infty^+)$. Since

$$\nu_\infty^+ = \lim_{k\to\infty} \tilde{\lambda}_{\ell_k}^{1,+}$$

by Lemma 2.4.2, we can pass to the limit in the following equality, that holds for $\ell_k > M(\varphi)$ (see (2.4.16)),

$$\int_{\Omega_\infty^+} A\nabla \tilde{u}_{\ell_k} \cdot \nabla\varphi = \tilde{\lambda}_{\ell_k}^{1,+} \int_{\Omega_\infty^+} \tilde{u}_{\ell_k}\varphi \,,$$

and obtain that

$$\int_{\Omega_\infty^+} A\nabla \tilde{u}^+ \cdot \nabla\varphi = \nu_\infty^+ \int_{\Omega_\infty^+} \tilde{u}^+\varphi \,. \tag{2.5.3}$$

Since (2.5.3) is valid for any $\varphi \in V_s(\Omega_\infty^+)$, and by density also for any $\varphi \in V(\Omega_\infty^+)$, we obtain that $\tilde{u}^+$ is a solution of (2.5.2). To conclude that it is a minimizer realizing ν_∞^+ in (2.4.11) we only need to prove that it is *nontrivial*, i.e., that $\tilde{u}^+ \not\equiv 0$. Actually, we are going to show that

$$\int_{\Omega_\infty^+} (\tilde{u}^+)^2 = 1 \quad \text{and} \quad \tilde{u} > 0.$$

For that matter we will prove decay estimates for $\tilde{u}_\ell$ for large x_1, implying concentration near $x_1 = 0$, using the same technique as the one used in the proof of Theorem 2.4.1.

Let ℓ and ℓ' satisfy $0 < \ell' \le \ell - 1$. Define $\tilde{\rho}_{\ell'} = \tilde{\rho}_{\ell'}(x_1)$ by

$$\tilde{\rho}_{\ell'}(x_1) = \begin{cases} 0 & x_1 \le \ell' \,, \\ x_1 - \ell' & x_1 \in (\ell', \ell'+1) \,, \\ 1 & x_1 \ge \ell' + 1 \,. \end{cases}$$

By the Euler-Lagrange equation satisfied by $\tilde{u}_\ell^+$ we have

$$\int_{\Omega_\ell^+} (A\nabla \tilde{u}_\ell^+) \cdot \nabla(\tilde{\rho}_{\ell'}^2 \tilde{u}_\ell^+) = \tilde{\lambda}_\ell^{1,+} \int_{\Omega_\ell^+} \tilde{\rho}_{\ell'}^2 |\tilde{u}_\ell^+|^2 \,.$$

Repeating the argument used to derive (2.4.6) we obtain

$$\begin{aligned}(\sigma_{\ell/2}^1 - \tilde{\lambda}_\ell^{1,+}) \int_{\Omega_\ell^+ \setminus \Omega_{\ell'+1}} |\tilde{u}_\ell^+|^2 \le (\sigma_{\ell/2}^1 - \tilde{\lambda}_\ell^{1,+}) \int_{\Omega_\ell^+} |\tilde{u}_\ell^+|^2 \tilde{\rho}_{\ell'}^2 \le \int_{\Omega_\ell^+} |\tilde{u}_\ell^+|^2 (A\nabla\tilde{\rho}_{\ell'}) \cdot \nabla\tilde{\rho}_{\ell'} \\ = \int_{\Omega_{\ell'+1}^+ \setminus \Omega_{\ell'}} |\tilde{u}_\ell^+|^2 (A\nabla\tilde{\rho}_{\ell'}) \cdot \nabla\tilde{\rho}_{\ell'} \le C_A \int_{\Omega_{\ell'+1}^+ \setminus \Omega_{\ell'}} |\tilde{u}_\ell^+|^2 \,.\end{aligned} \tag{2.5.4}$$

Using (2.0.3) together with (2.5.1) and Lemma 2.4.2 we deduce that there exist $\tilde{\ell}_0 > 0$ and $\tilde{\beta} > 0$ such that for $\ell > \tilde{\ell}_0$ we have

$$\sigma^1_{\ell/2} - \tilde{\lambda}^{1,+}_\ell \geq \tilde{\beta}.$$

Therefore, we deduce from (2.5.4) that

$$\int_{\Omega^+_\ell \setminus \Omega_{\ell'+1}} \tilde{(u^+_\ell)}^2 \leq \tilde{\alpha} \int_{\Omega^+_\ell \setminus \Omega_{\ell'}} |\tilde{u}^+_\ell|^2 \quad \text{with} \quad \tilde{\alpha} := \frac{C_A}{\tilde{\beta} + C_A} \,. \tag{2.5.5}$$

Fix any $r > 1$. Applying (2.5.5) successively for $\ell' = r-1, r-2, \ldots, r-[r]$ yields

$$\int_{\Omega^+_\ell \setminus \Omega_r} |\tilde{u}^+_\ell|^2 \leq \tilde{\alpha}^{[r]} \int_{\Omega^+_\ell} |\tilde{u}^+_\ell|^2 = \tilde{\alpha}^{[r]}, \ \forall \ell > r \,.$$

In other words,

$$\int_{\Omega^+_r} |\tilde{u}^+_\ell|^2 \geq 1 - \tilde{\alpha}^{[r]} \,. \tag{2.5.6}$$

Since $\tilde{u}_{\ell_k} \to \tilde{u}^+$ strongly in $L^2(\Omega^+_r)$, we deduce from (2.5.6) that

$$\int_{\Omega^+_r} (\tilde{u}^+)^2 \geq 1 - \tilde{\alpha}^{[r]} \,. \tag{2.5.7}$$

This already implies that $\tilde{u}^+$ is a nontrivial non negative solution to (2.5.2) and therefore, a minimizer in (2.4.11). Applying (2.5.7) with arbitrary large r, we get that

$$\int_{\Omega^+_\infty} (\tilde{u}^+)^2 = 1.$$

The uniqueness of the minimizer follows by a standard argument, using the fact that any minimizer must have a constant sign. □

The next result provides a sufficient condition for (2.5.1) to hold and another one for it to fail.

Theorem 2.5.1. (i) *Assume that* (2.3.17) *is satisfied. If the following condition holds,*

$$\int_\omega (A_{12} \cdot \nabla W_1) W_1 \geq 0 \,, \tag{2.5.8}$$

then (2.5.1) *holds.*

(ii) *If*

$$A_{12} \cdot \nabla W_1 \leq 0 \ \textit{a.e. in}\ \omega \tag{2.5.9}$$

then $\nu^+_\infty = \mu^1$. Moreover, in this case there is no minimizer realizing ν^+_∞.

Proof. (i) Assume that (2.5.8) is satisfied. A similar computation to the one done in the proof of Theorem 2.3.1 (see also Remark 2.3.1) shows that $\{\tilde{v}_\ell^\varepsilon\}$ given by (2.3.14), satisfy not only (2.3.15), but also

$$\inf_{\varepsilon>0} \lim_{\ell\to 0} \frac{\int_{\Omega_\ell^-} (A\nabla\tilde{v}_\ell^\varepsilon)\cdot\nabla\tilde{v}_\ell^\varepsilon}{\int_{\Omega_\ell^-} |\tilde{v}_\ell^\varepsilon|^2} = \int_\omega (A_{22}\nabla W_1)\cdot\nabla W_1 - \frac{|A_{12}\nabla W_1|^2}{a_{11}}\,.$$

Hence we can fix values of ℓ_1 and ε_1 such that the following analog of (2.3.19) holds,

$$-\gamma_1 := \int_{\Omega_{\ell_1}^-} (A\nabla\tilde{v}_{\ell_1}^{\varepsilon_1})\cdot\nabla\tilde{v}_{\ell_1}^{\varepsilon_1} - \mu^1 \int_{\Omega_{\ell_1}^-} |\tilde{v}_{\ell_1}^{\varepsilon_1}|^2 < 0\,. \tag{2.5.10}$$

For each $\alpha > 0$ we define a test function in $V_\infty(\Omega_\infty^+)$ by

$$z_\alpha(x_1, X_2) = \begin{cases} v_{\ell_1}^{\varepsilon_1}(x_1-\ell_1, X_2) & x_1 \in [0,\ell_1)\,, \\ W_1(X_2)e^{-\alpha(x_1-\ell_1)} & x_1 \in [\ell_1,\infty)\,. \end{cases}$$

Above we used the fact that $v_{\ell_1}^{\varepsilon_1}(0,X_2) = W_1(X_2)$. We have,

$$\int_{\Omega_\infty^+} |z_\alpha|^2 = \int_{\Omega_{\ell_1}^-} |v_{\ell_1}^{\varepsilon_1}|^2 + \left(\int_0^\infty e^{-2\alpha x_1}\right)\int_\omega W_1^2 = \int_{\Omega_{\ell_1}^-} |v_{\ell_1}^{\varepsilon_1}|^2 + \frac{1}{2\alpha}\,,$$

and

$$\int_{\Omega_\infty^+} (A\nabla z_\alpha)\cdot\nabla z_\alpha = \int_{\Omega_{\ell_1}^-} (A\nabla v_{\ell_1}^{\varepsilon_1})\cdot\nabla v_{\ell_1}^{\varepsilon_1} + \frac{1}{2\alpha}\left(\alpha^2\int_\omega a_{11}W_1^2 - 2\alpha\int_\omega (A_{12}\cdot\nabla W_1)W_1 + \int_\omega A_{22}\nabla W_1\cdot\nabla W_1\right)$$

Therefore, using (2.5.10) we get

$$\nu_\infty^+ - \mu^1 \le \frac{\int_{\Omega_\infty^+} A\nabla z_\alpha\cdot\nabla z_\alpha}{\int_{\Omega_\infty^+} |z_\alpha|^2} - \mu^1 < \frac{\frac{\alpha}{2}\int_\omega a_{11}W_1^2 - \int_\omega (A_{12}\cdot\nabla W_1)W_1 - \gamma_1}{\int_{\Omega_{\ell_1}^-} |v_{\ell_1}^{\varepsilon_1}|^2 + \frac{1}{2\alpha}}\,. \tag{2.5.11}$$

Since $\gamma_1 > 0$ and

$$\int_\omega (A_{12}\cdot\nabla W_1)W_1 \ge 0$$

by (2.5.8), it is clear that we can choose α small enough to ensure that the RHS of (2.5.11) is negative, completing the proof of (2.5.1).

(ii) We notice that not only $V_s(\Omega_+^\infty)$ is dense in $V(\Omega_+^\infty)$ (see (2.4.16)), but its subspace

$$V_s^0(\Omega_\infty^+) = \left\{u \in V_s(\Omega_\infty^+) \,:\, \exists\delta = \delta(u) > 0 \text{ s.t. } u(x) = 0 \text{ for } \operatorname{dist}(x,\gamma_\infty^+) \le \delta\right\},$$

is dense as well. By elliptic regularity and the strong maximum principle we know that W_1 is continuous and positive in ω **(see [36, Chapter 8])**. We shall use the following version of Picone identity,

$$(A\nabla u)\cdot\nabla u-(A\nabla v)\cdot\nabla\big(\frac{u^2}{v}\big)=A\big(\nabla u-\frac{u}{v}\nabla v\big)\cdot\big(\nabla u-\frac{u}{v}\nabla v\big)\geq 0\,. \tag{2.5.12}$$

Using (2.5.12) with any $u\in V_s^0(\Omega_\infty^+)$ and $v=W_1$, integrating and applying the generalized Green formula yields

$$\begin{aligned}0&\leq\int_{\Omega_\infty^+}A\big(\nabla u-\frac{u}{W_1}\nabla W_1\big)\cdot\big(\nabla u-\frac{u}{W_1}\nabla W_1\big)=\int_{\Omega_\infty^+}A\nabla u\cdot\nabla u-A\nabla W_1\cdot\nabla\big(\frac{u^2}{W_1}\big)\\&=\int_{\Omega_\infty^+}A\nabla u\cdot\nabla u+\int_{\Omega_\infty^+}\operatorname{div}(A\nabla W_1)\big(\frac{u^2}{W_1}\big)-\int_{\{0\}\times\omega}(A\nabla W_1\cdot\nu)\big(\frac{u^2}{W_1}\big)\\&=\int_{\Omega_\infty^+}A\nabla u\cdot\nabla u-\mu^1u^2+\int_\omega\Big(A_{12}\cdot\nabla W_1\Big)\frac{u^2(0,X_2)}{W_1(X_2)}\,.\end{aligned} \tag{2.5.13}$$

By (2.5.13) and (2.5.9) we deduce that

$$0\leq\int_{\Omega_\infty^+}A\big(\nabla u-\frac{u}{W_1}\nabla W_1\big)\cdot\big(\nabla u-\frac{u}{W_1}\nabla W_1\big)\leq\int_{\Omega_\infty^+}A\nabla u\cdot\nabla u-\mu^1u^2\,. \tag{2.5.14}$$

By the density of $V_s^0(\Omega_\infty^+)$ in $V(\Omega_\infty^+)$ it follows that (2.5.14) holds for every $u\in V(\Omega_\infty^+)$, i.e.,

$$\nu_\infty^+\geq\mu^1.$$

Finally, applying (2.4.12) we conclude that $\nu_\infty^+=\mu^1$.

To conclude, assume by negation that ν_∞^+ is realized by a minimizer u. Then, by (2.5.14) we get that

$$\nabla\big(\frac{u}{W_1}\big)=0\ \ a.e.,$$

implying that

$$u=cW_1$$

for some constant $c\neq0$. But this is clearly a contradiction since $W_1\notin V(\Omega_\infty^+)$. □

Remark 2.5.1. *An immediate consequence of Theorem 2.5.1 and Remark 2.4.1 is that if* (2.3.17) *holds and*

$$\int_\omega(A_{12}\cdot\nabla W_1)W_1=0,$$

then we have both

$$\nu_\infty^+<\mu^1\quad\textit{and}\ \ \nu_\infty^-<\mu^1.$$

A special case is when property (S) holds. Another direct consequence is that whenever (2.3.17) *holds we have*

$$\min(\nu_\infty^+,\nu_\infty^-)<\mu^1.$$

However, this fact follows already from our previous results, by combining Theorem 2.3.2 and Theorem 2.4.2.

Our last result provides a description of the asymptotic profile of the eigenfunctions $\{u_\ell\}$ near the ends of the cylinder. We denote by $\tilde{u}^\pm$ the unique positive renormalized minimizer for $\nu_\infty^\pm$, when it exists. For each $\ell > 0$ we define:

$$\begin{aligned} \tilde{u}_\ell^+(x_1, X_2) &= u_\ell(x_1 - \ell, X_2) \quad \text{on} \quad \Omega_\ell^+\,, \\ \tilde{u}_\ell^-(x_1, X_2) &= u_\ell(x_1 + \ell, X_2) \quad \text{on} \quad \Omega_\ell^-\,. \end{aligned} \tag{2.5.15}$$

The next theorem describes two possible scenarios that may occur: concentration near one of the ends of the cylinder, or concentration near both ends.

Theorem 2.5.2. (i) *If $\nu_\infty^+ < \nu_\infty^-$ then, for every $r > 0$,*

$$\tilde{u}_\ell^+ \to \tilde{u}^+ \ \textit{in}\ H^1(\Omega_r^+) \quad \textit{and} \quad \tilde{u}_\ell^- \to 0 \ \textit{in}\ H^1(\Omega_r^-)\,. \tag{2.5.16}$$

(ii) *If both* (2.5.2) *and property (S) hold then we have*

$$\tilde{u}^+(x_1, X_2) = \tilde{u}^-(-x_1, -X_2)$$

and for every $r > 0$,

$$\tilde{u}_\ell^+ \to \tilde{u}^+ \ \textit{in}\ H^1(\Omega_r^+) \quad \textit{and} \quad \tilde{u}_\ell^- \to \tilde{u}^- \ \textit{in}\ H^1(\Omega_r^-)\,. \tag{2.5.17}$$

Proof. (i) The convergence of $\{\tilde{u}_\ell^-\}$ to 0 in $H^1(\Omega_r^-)$ for all $r > 0$ is clear from Proposition 2.4.1, so we only need to prove the result for $\{\tilde{u}_\ell^+\}$. Since $\{\tilde{u}_\ell^+\}$ is bounded in $H^1(\Omega_\ell^+)$, given any sequence $\ell_k \to \infty$, we can apply a diagonal argument to $\{\tilde{u}_{\ell_k}^+\}$ to extract a subsequence, still denoted by $\{\ell_k\}$, such that $\tilde{u}_{\ell_k}^+$ converges weakly in $H^1(\Omega_r^+)$ and strongly in $L^2(\Omega_r^+)$ to some function $v^+ \in H^1(\Omega_\infty^+)$, for every $r > 0$. By (2.4.1) and Proposition 2.4.1 we have

$$\int_{\Omega_r^+} |\tilde{u}_\ell^+|^2 = \int_{\Omega_\ell^- \setminus \Omega_{\ell-r}} |u_\ell|^2 = 1 - \int_{\Omega_r^-} |u_\ell|^2 - \int_{\Omega_\ell^+} |u_\ell|^2 \geq 1 - \alpha^{[r]} + o(1)\,, \tag{2.5.18}$$

where $o(1)$ stands for a quantity that tends to 0 when $\ell \to \infty$. Passing to the limit in (2.5.18) with $\ell = \ell_k$, yields,

$$\int_{\Omega_r^+} |v^+|^2 \geq 1 - \alpha^{[r]}\,, \tag{2.5.19}$$

and since r is arbitrary, we get that

$$\int_{\Omega_\infty^+} |v^+|^2 = 1.$$

In addition, we clearly have

$$\nu_\infty^+ = \lim_{k\to\infty} \lambda_\ell^1 \geq \limsup_{k\to\infty} \int_{\Omega_r^+} (A\nabla \tilde{u}_{\ell_k}^+) \cdot \nabla \tilde{u}_{\ell_k}^+ \geq \int_{\Omega_r^+} (A\nabla v^+) \cdot \nabla v^+\,. \tag{2.5.20}$$

From (2.5.20) we deduce that

$$\int_{\Omega_\infty^+} (A\nabla v^+) \cdot \nabla v^+ = \nu_\infty^+,$$

i.e., v^+ is a nonnegative normalized minimizer, realizing ν_∞^+ in (2.4.11). Therefore, it must coincide with $\tilde{u}^+$. Finally, defining on $(0, \infty)$ the function

$$f(r) = \limsup_{k\to\infty} \int_{\Omega_r^+} (A\nabla \tilde{u}_{\ell_k}^+) \cdot \nabla \tilde{u}_{\ell_k}^+ - \int_{\Omega_r^+} (A\nabla v^+) \cdot \nabla v^+ ,$$

we see that on the one hand it is a nonnegative and nondecreasing function, while on the other hand

$$\lim_{r\to\infty} f(r) = 0.$$

Hence $f(r) \equiv 0$, implying the strong convergence

$$\tilde{u}_{\ell_k} \to \tilde{u}^+ \quad \text{in } H^1(\Omega_r^+)$$

for all $r > 0$. The uniqueness of the possible limit implies the the same convergence holds for the whole family $\{\tilde{u}_\ell\}$.

(ii) In this case we have the symmetry relation

$$u_\ell(x_1, X_2) = u_\ell(-x_1, -X_2)$$

by Proposition 2.1.1, and the same argument as in (i) gives the result. □

2.6 Characterization of the Limit $\lim_{\ell\to\infty} \lambda_\ell^2$

Now we study the asymptotic behavior of the second eigenvalues λ_ℓ^2.

Theorem 2.6.1. *Under the assumptions that property* (S) *holds and* (2.3.17) *we have*

$$\lim_{\ell\to\infty} \lambda_\ell^2 = \nu_\infty^+.$$

Proof. Note from Remark 2.4.1 that $\nu_\infty^+ = \nu_\infty^-$. Define h_ℓ^- and h_ℓ^+ on Ω_ℓ as

$$h_\ell^- = \begin{cases} \tilde{u}_\ell^+(x_1 + \ell, X_2) & \text{on } \Omega_\ell^-, \\ 0 & \text{on } \Omega_\ell^+ \end{cases}$$

and

$$h_\ell^+ = \begin{cases} \tilde{u}_\ell^+(\ell - x_1, -X_2) & \text{on } \Omega_\ell^+, \\ 0 & \text{on } \Omega_\ell^-, \end{cases}$$

where $\tilde{u}_\ell^-, \tilde{u}_\ell^+$ is as in Remark 2.4.2. Now define $\mathcal{H}_\ell = \alpha_\ell h_\ell^- + \beta_\ell h_\ell^+$, where $\alpha_\ell,\ \beta_\ell$ are chosen such that both of them does not vanish together and

$$\int_{\Omega_{2\ell}} u_\ell \mathcal{H}_\ell = 0.$$

Such a choice of α_ℓ and β_ℓ is possible since we have two unknown variables and one equation. From the Rayleigh quotient characterization of λ_ℓ^2, we get

$$\lambda_\ell^2 \le \frac{\int_{\Omega_\ell} A\nabla\mathcal{H}_\ell \cdot \nabla\mathcal{H}_\ell}{\int_{\Omega_\ell} \mathcal{H}_\ell^2}$$
$$= \frac{\alpha_\ell^2 \int_{\Omega_\ell^-} A\nabla h_\ell^- \cdot \nabla h_\ell^- + \beta_\ell^2 \int_{\Omega_\ell^+} A\nabla h_\ell^+ \cdot \nabla h_\ell^+ + 2\alpha_\ell\beta_\ell \int_{\Omega_\ell} A\nabla h_\ell^+ \cdot \nabla h_\ell^-}{\alpha_\ell^2 \int_{\Omega_\ell^-} (h_\ell^-)^2 + \beta_\ell^2 \int_{\Omega_\ell^+} (h_\ell^+)^2 + 2\alpha_\ell\beta_\ell \int_{\Omega_\ell} h_\ell^+ h_\ell^-}. \quad (2.6.1)$$

Since the function h_ℓ^+ and h_ℓ^- have disjoint supports, the terms

$$\int_{\Omega_{2\ell}} A\nabla h_\ell^+ \cdot \nabla h_\ell^- \quad \text{and} \quad \int_{\Omega_\ell} h_\ell^+ h_\ell^-$$

vanishes. Using the assumption that the property (S) holds, it is easy to see that

$$\int_{\Omega_\ell^-} A\nabla h_\ell^- \cdot \nabla h_\ell^- = \tilde{\lambda}_\ell^{1,+} \text{ and } \int_{\Omega_\ell^+} A\nabla h_\ell^+ \cdot \nabla h_\ell^+ = \tilde{\lambda}_\ell^{1,+}.$$

Also

$$\int_{\Omega_\ell^+} (h_\ell^+)^2 = \int_{\Omega_\ell^-} (h_\ell^-)^2 = 1.$$

We have

$$\lambda_\ell^2 \le \frac{\alpha_\ell^2 \tilde{\lambda}_\ell^{1,+} + \beta_\ell^2 \tilde{\lambda}_\ell^{1,+}}{\alpha_\ell^2 + \beta_\ell^2} = \tilde{\lambda}_\ell^{1,+}.$$

Hence we have

$$\lambda_\ell^1 < \lambda_\ell^2 \le \tilde{\lambda}_\ell^{1,+}.$$

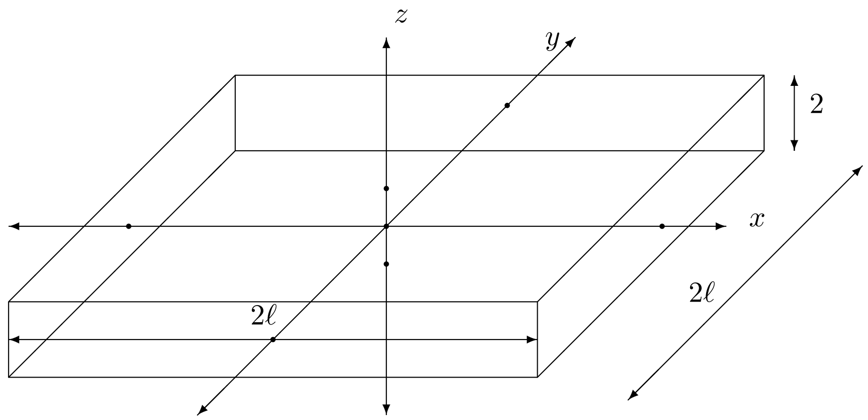
The theorem then follows from Lemma 2.4.2. □

2.7 $\ell \to \infty$ in Several Directions

In the previous section we considered the case of a cylinder which goes to infinity in one direction. In this section we consider the case when the cylinder can tend to infinity in more directions. For this section we change the notation of Ω_ℓ. Let us now set

$$\Omega_\ell = (-\ell, \ell)^p \times \omega,$$

where $1 \le p < n$ and ω is a bounded subset of $\mathbb{R}^{n-p}$ containing the origin. The figure of $\Omega_\ell = (-\ell, \ell)^2 \times (-1, 1)$ is shown below.



Figure

The points in Ω_ℓ are denoted as,

$$X := (X_1, X_2)$$

where $X_1 = (x_1, x_2, \ldots, x_p)$ and $X_2 = (x_{p+1}, \ldots, x_n)$. Let A be a $n \times n$ symmetric, positive definite matrix

$$A := A(X_2) = \begin{pmatrix} A_{11}(X_2) & A_{12}(X_2) \\ A_{12}^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

where $A_{11},\ A_{12}$ and A_{22} are $p \times p, p \times (n-p)$ and $(n-p) \times (n-p)$ matrices respectively. Let C_i denotes the i-th row of the matrix A_{12}, that is

$$C_i = (a_{i(p+1)}, a_{i(p+2)}, \ldots, a_{in})$$

and B_i denotes the matrix

$$B_i := B_i(X_2) = \begin{pmatrix} a_{ii}(X_2) & C_i(X_2) \\ C_i^t(X_2) & A_{22}(X_2) \end{pmatrix}$$

for $1 \leq i \leq p$.

Lemma 2.7.1. *The matrix B_i is uniformly positive definite.*

Proof. Let $\xi := (\xi_1, \xi_2, \ldots, \xi_{n-p+1}) \in \mathbb{R}^{n-p+1}$. Without any loss of generality let us assume that $i = 1$. We have to show that for some $C > 0$,

$$B_1 \xi \cdot \xi \geq C|\xi|^2.$$

Take $\eta := (\xi_1, 0, 0, \ldots, 0, \xi_2, \ldots, \xi_{n-p+1}) \in \mathbb{R}^n$. Easy computation shows that

$$B_1(X_2)\xi \cdot \xi = A(X_2)\eta \cdot \eta, \quad \forall X_2 \in \omega.$$

Since A is uniformly positive definite and $|\xi| = |\eta|$, the lemma follows. Note that by $|\xi|$ and $|\eta|$ we mean the euclidean norms in $\mathbb{R}^{p+1}$ and $\mathbb{R}^n$ respectively. $\square$

Let us consider the problem

$$\left.\begin{array}{ll} -\mathrm{div}(A(X_2)\nabla u_\ell) = \lambda_\ell^1 u_\ell & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } (-\ell,\ell)^p \times \partial\omega, \\ A\nabla u_\ell \cdot \nu = 0 & \text{on } \partial(-\ell,\ell)^p \times \omega, \\ \int_{\Omega_\ell} u_\ell^2 = 1. & \end{array}\right\} \tag{2.7.1}$$

where λ_ℓ^1 denotes the first eigenvalue.

Theorem 2.7.1. *If for some $1 \le i \le p$,*

$$C_i \cdot \nabla_{X_2} W_1 \neq 0 \quad a.e.\ x \in \omega,$$

then

$$\limsup_{\ell\to\infty} \lambda_\ell^1 < \mu^1,$$

where μ^1 and W_1 is as in Section (2.1).

Proof. Let $\Omega_{i,\ell}$ denotes only the i-th direction of Ω_ℓ, that is

$$\Omega_{i,\ell} = \left\{ X := (X_1, X_2) \in \Omega_\ell \;\middle|\; x_j = 0 \text{ if } j \neq i, \text{ for } 1 \le j \le p \right\}.$$

In other word $\Omega_{i,\ell} = (-\ell,\ell) \times \omega$. Let us denote points in $\Omega_{i,\ell}$ as $X = (x_i, X_2)$ where $X_2 \in \omega$. Since the matrix B_i satisfies the criterion of Theorem (2.3.2), we can find $\phi_\ell(x_1, X_2) \in V(\Omega_{i,\ell})$ such that

$$\limsup_{\ell\to\infty} \frac{\int_{\Omega_{i,\ell}} B_i \nabla_{i,X_2}\phi_\ell \cdot \nabla_{i,X_2}\phi_\ell}{\int_{\Omega_{i,\ell}} \phi_\ell^2} < \mu^1.$$

Here ∇_{i,X_2} denotes the gradient in (x_i, X_2) direction.

Define $v_\ell(X_1, X_2) := \phi_\ell(x_i, X_2)$. It is easy to check that $v_\ell \in V(\Omega_\ell)$,

$$(2\ell)^{p-1} \int_{\Omega_{i,\ell}} B_i \nabla_{i,X_2}\phi_\ell \cdot \nabla_{i,X_2}\phi_\ell = \int_{\Omega_\ell} A\nabla v_\ell \cdot \nabla v_\ell$$

and

$$(2\ell)^{p-1} \int_{\Omega_{i,\ell}} \phi_\ell^2 = \int_{\Omega_\ell} v_\ell^2.$$

Hence we have

$$\limsup_{\ell\to\infty} \lambda_\ell^1 \le \frac{\int_{\Omega_\ell} A\nabla v_\ell \cdot \nabla v_\ell}{\int_{\Omega_\ell} v_\ell^2} < \mu^1.$$

This finishes the proof of the theorem. □

Remarks for the full Neumann Problem

Let A, Ω_ℓ and ν be as in Section (2.1). Consider the following eigenvalue problem with full Neumann boundary conditions.

$$\left.\begin{array}{ll} -\mathrm{div}\left(A(X_2)\nabla w_\ell^k\right) = \mu_\ell^k w_\ell^k & \text{in } \Omega_\ell, \\ A\nabla w_\ell^k \cdot \nu = 0 & \text{on } \partial\Omega_\ell, \\ \int_{\Omega_\ell} (w_\ell^k)^2 = 1. & \end{array}\right\} \tag{2.7.2}$$

In the above equation μ_ℓ^k denotes the k-th eigenvalue and w_ℓ^k is the corresponding eigenfunction. Note that the corresponding problem on the cross section for the above problem, is not the problem (2.1.8). In this case the correct problem on the cross section is the following one.

$$\left.\begin{array}{ll} -\mathrm{div}\left(A_{22}(X_2)\nabla w^k\right) = \mu^k w^k & \text{in } \omega, \\ A_{22}\nabla_{X_2} w^k \cdot \gamma = 0 & \text{on } \partial\omega, \\ \int_{\omega} (w^k)^2 = 1, & \end{array}\right\} \tag{2.7.3}$$

where γ denotes the outer normal to $\partial\omega$. With obvious notation, μ^k and w^k denote the k-th eigenvalue and the eigenfunction respectively. The asymptotic behavior of the first eigenvalue (μ_ℓ^1) for this case turns out to be uninteresting, since $w_\ell^1 = \frac{1}{2\sqrt{\ell}}$ and therefore $\mu_\ell^k = 0$ holds for all ℓ and similarly $\mu^1 = 0$. Therefore, trivially we have $\mu_\ell^1 \to \mu^1$. Next theorem will provide us with information about the behavior of the sequence $\{\mu_\ell^2\}_{\ell \geq 1}$.

Theorem 2.7.2.

$$\lim_{\ell\to\infty} \mu_\ell^2 = 0.$$

Proof. Consider the following 1 dimensional Neumann eigenvalue problem

$$\left.\begin{array}{l} -\frac{\partial^2 \phi_\ell}{\partial x_1^2} = s_\ell^2 \phi_\ell \quad \text{on } (-\ell, \ell), \\ \frac{\partial \phi_\ell}{\partial x_1}(\ell) = \frac{\partial \phi_\ell}{\partial x_1}(-\ell) = 0. \end{array}\right\} \tag{2.7.4}$$

In the above problem s_ℓ^2 denotes the second eigenvalue and ϕ_ℓ denotes the second eigenfunction. It is well known **[see, [38]]** that $s_\ell^2 = \frac{\pi^2}{4\ell^2}$. Define $v_\ell(x_1, X_2) = \phi_\ell(x_1)$. Clearly $v_\ell \in H^1(\Omega_\ell)$. Since $\int_{-\ell}^{\ell} \phi_\ell = 0$, it follows that $\int_{\Omega_\ell} v_\ell = 0$. Therefore we have

$$\begin{aligned} \mu_\ell^2 \leq \frac{\int_{\Omega_\ell} A\nabla v_\ell \cdot \nabla v_\ell}{\int_{\Omega_\ell} v_\ell^2} = \frac{\int_{\Omega_\ell} a_{11} \left|\frac{\partial \phi_\ell}{\partial x_1}\right|^2}{\int_{\Omega_\ell} \phi_\ell^2} \leq ||a_{11}||_\infty \frac{\int_{-\ell}^{\ell} \left|\frac{\partial \phi_\ell}{\partial x_1}\right|^2}{\int_{-\ell}^{\ell} \phi_\ell^2} &= ||a_{11}||_\infty s_\ell^2 \\ &= ||a_{11}||_\infty \frac{\pi^2}{4\ell^2} \to 0 \end{aligned}$$

as $\ell \to \infty$. This finishes the proof of the theorem. $\square$

Chapter 3

$\ell \to \infty$ for Variational Problems

Suppose $\omega := (-1, 1)$, in this chapter we will consider the following minimization problem

$$J_\ell(u_\ell) = \inf_{u \in H_0^1(\Omega_\ell)} J_\ell(u)$$

where $\Omega_\ell := \ell\omega \times \omega$. Points in Ω_ℓ are simply denoted by the pair (x_1, x_2).

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying the following conditions:

$$F \text{ is convex,} \tag{3.0.1}$$

there exists $\alpha, \beta > 0$ such that $\forall \xi \in \mathbb{R}^2$,

$$\alpha|\xi|^2 \leq F(\xi) \leq \beta|\xi|^2. \tag{3.0.2}$$

Define the functional $J_\ell : H_0^1(\Omega_\ell) \to \mathbb{R}$ as

$$J_\ell(u) := \int_{\Omega_\ell} F(\nabla u) - \int_{\Omega_\ell} f(x_2)u,$$

where $f \in L^2(\omega)$. Let us now define the functional $J_\infty : H_0^1(\omega) \to \mathbb{R}$ on the cross section ω of the cylinder Ω_ℓ as

$$J_\infty(u) := \int_\omega F\left(0, \frac{\partial u}{\partial x_2}\right) - \int_\omega fu.$$

It is well known **[see,[31]]** that there exists $u_\ell \in H_0^1(\Omega_\ell)$ and $u_\infty \in H_0^1(\omega)$ such that

$$J_\ell(u_\ell) = \inf_{u \in H_0^1(\Omega_\ell)} J_\ell(u) \quad \text{and} \quad J_\infty(u_\infty) = \inf_{u \in H_0^1(\omega)} J_\infty(u). \tag{3.0.3}$$

If F is assumed to be defined as $F(x, y) := \frac{1}{2}(x^2 + y^2)$, then u_ℓ and u_∞ satisfies the following equations respectively,

$$\left.\begin{array}{ll} -\triangle u_\ell = f(x_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell \end{array}\right\} \tag{3.0.4}$$

and

$$\left.\begin{array}{ll} -\frac{\partial^2 u_\infty}{\partial x_2^2} = f(x_2) & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega. \end{array}\right\} \tag{3.0.5}$$

It is clear from Theorem (1.2.1) that in this case u_ℓ converges to u_∞, in H_0^1 norm, in the middle of the cylinder at an exponential rate.

In this chapter we will consider the asymptotic behavior of u_ℓ as $\ell \to \infty$ for the problem (3.0.3). In particular we will show under some smoothness assumptions on F [**see, Theorem (3.1.1)**], that u_ℓ converges to u_∞ at least as fast as exponential, in the middle of the cylinder. In Section (3.2), we consider an issue of convergence of an appropriate energy functional, for general cylindrical domains. At the end of this section we will make some remark on the problem

$$J_\ell(u_\ell) = \inf_{u \in V(\Omega_\ell)} J_\ell(u)$$

where $V(\Omega_\ell) := \left\{u \in H^1(\Omega) \mid u = 0 \text{ on } \ell\omega \times \partial\omega\right\}$.

3.1 The Case of Smooth Convexity

In this section we will work with some extra smoothness assumptions on F. We assume that

$$(1)\ \nabla F \text{ is Lipschitz continuous,} \tag{3.1.1}$$

$$(2)\ \nabla F(X) - \nabla F(Y).(X - Y) \geq \eta |X - Y|^2, \quad \forall X, Y \in \mathbb{R}^2, \tag{3.1.2}$$

where η is a positive number.

Lemma 3.1.1. *Under the assumption* (3.0.1) *and* (3.0.2) *on* F*,*

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 \leq C\ell \int_\omega f^2 \tag{3.1.3}$$

where C *is some positive constant independent of* ℓ*.*

Proof. Using $u = 0$ as a test function in (3.0.3) we have $J_\ell(u_\ell) \leq J_\ell(0)$. Using $\xi = 0$ in (3.0.2) we get $F(0) = 0$. Hence $J_\ell(u_\ell) \leq 0$. This implies that

$$\int_{\Omega_\ell} F(\nabla u_\ell) \leq \int_{\Omega_\ell} f u_\ell.$$

Using Hölder's inequality, we get

$$\int_{\Omega_\ell} F(\nabla u_\ell) \leq \left(\int_{\Omega_\ell} u_\ell^2\right)^{\frac{1}{2}} \left(\int_{\Omega_\ell} f^2\right)^{\frac{1}{2}}.$$

Applying (3.0.2) and Poincaré inequality we get

$$\alpha \int_{\Omega_\ell} |\nabla u_\ell|^2 \leq \lambda_1^{-\frac{1}{2}} \left(\int_{\Omega_\ell} |\nabla u_\ell|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_\ell} f^2\right)^{\frac{1}{2}}.$$

where λ_1 satisfies

$$\lambda_1 \le \inf_{u \in H_0^1(\Omega_\ell)} \frac{\int_{\Omega_\ell} |\nabla u|^2}{\int_{\Omega_\ell} u^2}, \qquad \forall \ell.$$

It is clear from (1.2.3) that we can choose $\lambda_1 = \frac{\pi^2}{4}$. Therefore

$$\alpha^2 \int_{\Omega_\ell} |\nabla u_\ell|^2 \le \lambda_1^{-1} \int_{\Omega_\ell} f^2 = \frac{2\ell}{\lambda_1} \int_\omega f^2.$$

This completes the proof of the lemma. □

Theorem 3.1.1. *Under the assumptions* (3.0.1), (3.0.2), (3.1.1) *and* (3.1.2), *it holds that*

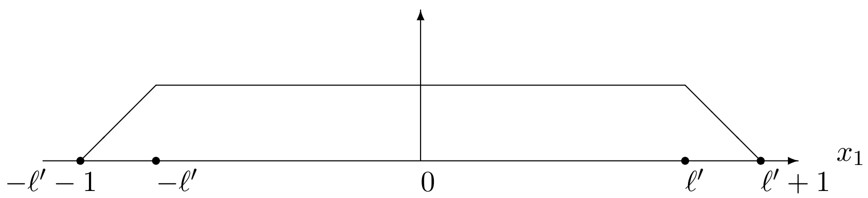
$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \le C\ell e^{-C'\ell} \tag{3.1.4}$$

where C, C' *are some positive constants independent of* ℓ.

Proof. Let $\ell' \le \ell - 1$ and $\rho_{\ell'} : \mathbb{R} \to \mathbb{R}$ be the Lipschitz continuous function defined as

$$\rho_{\ell'}(x_1) = \begin{cases} 1 & \text{on } [-\ell', \ell'] \\ 1 - x_1 + \ell' & \text{on } (\ell', \ell' + 1) \\ 1 + x_1 + \ell' & \text{on } (-\ell' - 1, -\ell') \\ 0 & \text{otherwise.} \end{cases}$$

The graph of $\rho_{\ell'}$ is given below.



Figure

For any $\lambda \in (0,1)$, define $v_{\ell'} := u_\ell - \lambda\rho_{\ell'}(u_\ell - u_\infty)$ and $w := u_\infty + \lambda\rho_{\ell'}(u_\ell(x_1,\cdot) - u_\infty)$. Then clearly $v_{\ell'} \in H_0^1(\Omega_\ell)$ and $w \in H_0^1(\omega)$. Using $v_{\ell'}$ and w as test functions in (3.0.3), we get

$$\begin{aligned} &\int_{\Omega_\ell} F(\nabla u_\ell) - \int_{\Omega_\ell} f(x_2) u_\ell \\ &\quad \le \int_{\Omega_\ell} F(\nabla u_\ell - \lambda \nabla\{\rho_{\ell'}(u_\ell - u_\infty)\}) - \int_{\Omega_\ell} f(x_2)\{u_\ell - \lambda\rho_{\ell'}(u_\ell - u_\infty)\} \end{aligned} \tag{3.1.5}$$

and

$$\int_\omega F\left(0, \frac{\partial u_\infty}{\partial x_2}\right) - \int_\omega f(x_2)u_\infty \le \int_\omega F\left(0, \frac{\partial}{\partial x_2}\{u_\infty + \lambda\rho_{\ell'}(u_\ell - u_\infty)\}\right) - \int_\omega f\left\{u_\infty + \lambda\rho_{\ell'}(u_\ell - u_\infty)\right\}. \quad (3.1.6)$$

Integrating the last inequality in x_1 direction from $-\ell$ to ℓ, we get

$$\int_{\Omega_\ell} F\left(0, \frac{\partial u_\infty}{dx_2}\right) - \int_{\Omega_\ell} f(x_2)u_\infty \le \int_{\Omega_\ell} F\left(0, \frac{\partial}{\partial x_2}\{u_\infty + \lambda\rho_{\ell'}(u_\ell - u_\infty)\}\right) - \int_{\Omega_\ell} f\left\{u_\infty + \lambda\rho_{\ell'}(u_\ell - u_\infty)\right\}. \quad (3.1.7)$$

Summing (3.1.5) and (3.1.7), we have

$$\int_{\Omega_\ell} F(\nabla u_\ell - \lambda\nabla\{\rho_{\ell'}(u_\ell - u_\infty)\}) - \int_{\Omega_\ell} F(\nabla u_\ell) + \int_{\Omega_\ell} F\left(0, \frac{\partial}{\partial x_2}\{u_\infty + \lambda\rho_{\ell'}(u_\ell - u_\infty)\}\right) - \int_{\Omega_\ell} F\left(0, \frac{\partial u_\infty}{\partial x_2}\right) \ge 0. \quad (3.1.8)$$

Using the formula

$$F(Y) - F(X) = \int_0^1 \frac{d}{dt}F(tY + (1-t)X)dt = \int_0^1 \nabla F(X + t(X - Y)) \cdot Y - Xdt,$$

for all $X, Y \in \mathbb{R}^2$, we obtain from the last equation that

$$\lambda\int_{\Omega_{\ell'+1}}\int_0^1 \nabla F\left(\nabla u_\ell - t\lambda\rho_{\ell'}\nabla(u_\ell - u_\infty)\right) \cdot \nabla - \rho_{\ell'}(u_\ell - u_\infty) + \lambda\int_{\Omega_{\ell'+1}}\int_0^1 \frac{\partial}{\partial x_2}F\left(0, \frac{\partial}{\partial x_2}\{u_\infty + t\lambda\rho_{\ell'}(u_\ell - u_\infty)\}\right)\frac{\partial}{\partial x_2}\left\{\rho_{\ell'}(u_\ell - u_\infty)\right\} \ge 0. \quad (3.1.9)$$

Dividing the above equation by λ and letting $\lambda \to 0$ we get

$$\int_{\Omega_{\ell'+1}} \nabla F\left(\nabla u_\ell\right) \cdot \nabla\rho_{\ell'}(u_\ell - u_\infty) - \int_{\Omega_{\ell'+1}} \frac{\partial}{\partial x_2}F\left(0, \frac{\partial u_\infty}{\partial x_2}\right)\frac{\partial}{\partial x_2}\rho_{\ell'}(u_\ell - u_\infty) \le 0. \quad (3.1.10)$$

Now since u_∞ is independent of x_1 one has $(0, \frac{\partial u_\infty}{\partial x_2}) = \nabla u_\infty$ and

$$\int_{\Omega_{\ell'+1}} \frac{\partial}{\partial x_1}F\left(0, \frac{\partial u_\infty}{\partial x_2}\right)\frac{\partial}{\partial x_1}\rho_{\ell'}(u_\ell - u_\infty) = 0.$$

This allows us to write (3.1.10) as

$$\int_{\Omega_{\ell'+1}} \nabla F(\nabla u_\ell) \cdot \nabla\rho_{\ell'}(u_\ell - u_\infty) - \nabla F(\nabla u_\infty) \cdot \nabla\rho_{\ell'}(u_\ell - u_\infty) \le 0.$$

This leads to

$$\int_{\Omega_{\ell'+1}} \rho_{\ell'} \left\{\nabla F(\nabla u_\ell) - \nabla F(\nabla u_\infty)\right\} \cdot \nabla(u_\ell - u_\infty) \leq \int_{\Omega_{\ell'+1}} (u_\ell - u_\infty) \left\{\nabla F(\nabla u_\ell) - \nabla F(\nabla u_\infty) \cdot \nabla \rho_{\ell'}\right\}.$$

Using (3.1.2) and the fact that $\rho_{\ell'} = 1$ on $\Omega_{\ell'}$, we deduce that

$$\eta \int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2 \leq \int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |\nabla F(\nabla u_\ell) - \nabla F(\nabla u_\infty)||u_\ell - u_\infty|. \tag{3.1.11}$$

Applying Hölder's inequality and (3.1.1) we get

$$\begin{aligned}\eta \int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2 &\leq L \int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)||u_\ell - u_\infty| \\ &\leq L \left(\int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |u_\ell - u_\infty|^2\right)^{\frac{1}{2}},\end{aligned}$$

where L denotes the Lipschitz constant of ∇F. Now from Poincaré inequality we deduce that

$$\int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2 \leq C \int_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2,$$

which can also be written as

$$\int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2 \leq \frac{C}{C+1} \int_{\Omega_{\ell'+1}} |\nabla(u_\ell - u_\infty)|^2, \tag{3.1.12}$$

where $C = \frac{L\lambda_1^{-\frac{1}{2}}}{\eta}$. Let $[x]$ denotes the greatest integer less than or equal to x. Now choosing $\ell' = [\frac{\ell'}{2} + 1], [\frac{\ell'}{2} + 1] + 1, \ \ldots, [\ell]$ and iterating the formula (3.1.12) we get

$$\int_{\Omega_{[\frac{\ell'}{2}+1]}} |\nabla(u_\ell - u_\infty)|^2 \leq \left(\frac{C}{C+1}\right)^{[\frac{\ell}{2}-1]} \int_{\Omega_{[\ell]}} |\nabla(u_\ell - u_\infty)|^2.$$

Now since $[\frac{\ell'}{2} + 1] \geq \frac{\ell}{2}$, $[\ell] \leq \ell$ and $[\frac{\ell}{2} - 1] \geq \frac{\ell}{2} - 2$ we have

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \leq \left(\frac{C}{C+1}\right)^{\frac{\ell}{2}-2} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 = C_1 e^{-C_2\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2,$$

where $C_1 = \left(\frac{C+1}{C}\right)^2$ and $C_2 = -\frac{1}{2}\log(\frac{C}{C+1})$. Note that C_2 is strictly positive since $\log(\frac{C}{C+1}) < 0$. Finally using the inequality $(a-b)^2 \leq 2(a^2 + b^2)$ and Lemma (3.1.1), we have

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \le 2C_1 e^{-C_2\ell} \int_{\Omega_\ell} \left(|\nabla u_\ell|^2 + |\nabla u_\infty|^2\right)$$
$$\le 2C_1 \left\{ C \int_\omega f^2 + 2 \int_\omega |\frac{\partial u_\infty}{\partial x_2}|^2 \right\} \ell e^{-C_2\ell}. \quad (3.1.13)$$

This finishes the proof of the theorem.

Remark 3.1.1. *Suppose $\alpha \in (0,1)$, then one can replace $\frac{\ell}{2}$ with $\alpha\ell$ in the statement of the last theorem.*

□

3.2 Convergence of Energy

We consider here a function $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$F \text{ is continuous, convex} \quad (3.2.1)$$

$$\lambda|\xi|^2 \le F(\xi) \le \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \quad (3.2.2)$$

We set $\Omega_\ell := \ell\omega_1 \times \omega_2$ where ω_1 and ω_2 are bounded open sets of $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$ respectively, ω_1 star shaped with respect to 0. One uses the notation $x = (X_1, X_2), X_1 \in \mathbb{R}^p, X_2 \in \mathbb{R}^{n-p}$.

Define

$$J_\ell(u) = \int_{\Omega_\ell} F(\nabla u) - f(x_2)u \; dx, \quad \forall u \in H_0^1(\Omega_\ell)$$

$$J_\infty(v) = \int_{\omega_2} F(0, \nabla_{X_2} v) - f(X_2)u \; dX_2, \quad \forall v \in H_0^1(\omega_2).$$

Lemma 3.2.1. *If u_ℓ, u_∞ are minimizers of J_ℓ, J_∞ on $H_0^1(\Omega_\ell), H_0^1(\omega)$ one has*

$$\limsup_{\ell\to\infty} \frac{J_\ell(u_\ell)}{|\ell\omega_1|} \le J_\infty(u_\infty). \quad (3.2.3)$$

Proof. For $u \in H_0^1(\Omega_\ell)$ one defines $\tilde{u} \in H_0^1(\Omega_1)$ as

$$\tilde{u}(X_1, X_2) = u(\ell X_1, X_2).$$

One has by a change of variable for $u \in H_0^1(\Omega_\ell)$,

$$J_\ell(u) = \int_{\Omega_\ell} F(\nabla u(\xi)) - fu(\xi) \; d\xi$$
$$= \int_{\Omega_1} \{F(\nabla_{X_1} u(\ell X_1, X_2), \nabla_{X_2} u(\ell X_1, X_2)) - f(X_2)u(\ell X_1, X_2)\} \ell^p \; dX_1 dX_2$$
$$= \int_{\Omega_1} \left\{ F\left(\frac{1}{\ell}\nabla_{X_1}\tilde{u}(\ell X_1, X_2), \nabla_{X_2}\tilde{u}(\ell X_1, X_2)\right) - f(X_2)\tilde{u}(\ell X_1, X_2) \right\} \ell^p \; dX_1 dX_2.$$

For $\eta > 0$ we consider the function

$$\rho_\eta = 1 \wedge \frac{\operatorname{dist}(X_1, \partial\omega_1)}{\eta} \in H_0^1(\omega_1).$$

Choosing u such $\tilde{u} = u_\infty \rho_\eta$ we get

$$J_\ell(u_\ell) \le J_\ell(u) \le \int_{\Omega_1} \left\{ F\left(\frac{1}{\ell}\nabla_{X_1} u_\infty \rho_\eta, \rho_\eta \nabla_{X_2} u_\infty\right) - f(X_2) u_\infty \rho_\eta \right\} \ell^p \, dX_1 dX_2.$$

and thus

$$\frac{J_\ell(u_\ell)}{|\ell\omega_1|} \le \frac{1}{|\omega_1|} \int_{\Omega_1} \left\{ F\left(\frac{1}{\ell}\nabla_{X_1} u_\infty \rho_\eta, \rho_\eta \nabla_{X_2} u_\infty\right) - f(X_2) u_\infty \rho_\eta \right\} \ell^p \, dX_1 dX_2.$$

One has clearly

$$|\rho_\eta| \le \frac{1}{\eta}.$$

Choosing $\eta = \eta(\ell) \to 0$ such that $\ell\eta(\ell) \to \infty$ we get by the Lebesgue theorem

$$\limsup_{\ell \to \infty} \frac{J_\ell(u_\ell)}{|\ell\omega_1|} \le \frac{1}{|\omega_1|} \int_{\omega_1} F(0, \nabla_{X_2} u_\infty) - f u_\infty \, dX_1 dX_2 = J_\infty(u_\infty).$$

□

Lemma 3.2.2.

$$J_\infty(u_\infty) \le \frac{J_\ell(u_\ell)}{|\ell\omega_1|}. \tag{3.2.4}$$

Proof. Set

$$v_\ell = \fint_{\ell\omega_1} u_\ell(X_1, X_2) dX_1 = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} u_\ell(X_1, X_2) dX_1.$$

It is easy to see that $v_\ell \in H_0^1(\omega_2)$. Thus

$$J_\infty(u_\infty) \le J_\infty(v_\ell) = \int_{\omega_2} F\left(0, \nabla_{X_2} v_\ell(X_2)\right) - f(X_2) v_\ell(X_2) \, dX_2.$$

By the divergence theorem one has

$$0 = \fint_{\ell\omega_1} \nabla_{X_1} u_\ell(X_1, X_2) \, dX_1$$

and by differentiation under the integral

$$\nabla_{X_2} v_\ell(X_2) = \fint_{\ell\omega_1} \nabla_{X_2} u_\ell(X_1, X_2) \, dX_1.$$

Therefore we have by Jensen's inequality

$$
\begin{aligned}
&J_\infty(u_\infty) \\
&\quad \le \int_{\omega_2} \left\{ F\left(⨍_{\ell\omega_1} \nabla_{X_1} u_\ell(X_1, X_2)\, dX_1, ⨍_{\ell\omega_1} \nabla_{X_2} u_\ell(X_1, X_2) \right) dX_1 \right\} dX_2 \\
&\qquad\qquad - \int_{\omega_2} f(X_2) \left\{ ⨍_{\ell\omega_1} u_\ell(X_1, X_2) dX_1 \right\} dX_2 \\
&\qquad\qquad\qquad \le \int_{\omega_2} ⨍_{\ell\omega_1} \{ F(\nabla u_\ell) - f(X_2) u_\ell \}\, dX_1 dX_2 = \frac{J_\ell(u_\ell)}{|\ell\omega_1|}.
\end{aligned}
$$

This finishes the proof of the lemma. □

Mixed Boundary Conditions

In this section we consider minimization of J_ℓ over a closed subspace $V(\Omega_\ell)$ of $H^1(\Omega_\ell)$, which is defined as

$$V(\Omega_\ell) := \left\{ u \in H^1(\Omega_\ell) \;\middle|\; u = 0 \text{ on } \ell\omega \times \partial\omega \right\}.$$

In other words we are considering free boundary conditions or Neumann boundary conditions at the side boundary of the cylinder.

We will consider the following minimization problem

$$J_\ell(v_\ell) = \inf_{u \in V(\Omega_\ell)} J_\ell(u). \tag{3.2.5}$$

We mention here that the energy J_∞ remains same as considered in (3.0.3), since the functions in $V(\Omega_\ell)$ vanishes on $\ell\omega \times \partial\omega$. Existence of v_ℓ is again well known **[see,[31]]**.

Lemma 3.2.3. *We have*

$$\frac{J_\ell(v_\ell)}{2\ell} \le J_\infty(u_\infty).$$

Proof. First of all we notice that $u_\infty(x_2) \in V(\Omega_\ell)$. Using $u = u_\infty$ as test function in (3.2.5), we have,

$$\frac{J_\ell(v_\ell)}{2\ell} \le J_\infty(u_\infty), \quad \forall \ell. \tag{3.2.6}$$

□

Lemma 3.2.4. *If F is assumed to be a radial function then under the assumptions* (3.0.1) *and* (3.0.2)*,*

$$\frac{J_\ell(v_\ell)}{2\ell} = J_\infty(u_\infty), \quad \forall \ell.$$

Proof. First we claim that for any fixed $y_0 \in \mathbb{R}$,

$$F(0, y_0) = \inf_{x \in \mathbb{R}} F(x, y_0).$$

Since F is radial, we have

$$F(x, y_0) = F(-x, y_0), \quad \forall x \in \mathbb{R}. \tag{3.2.7}$$

If possible, let there exists $X_0 \in \mathbb{R}$ such that

$$F(X_0, y_0) = \inf_{x \in \mathbb{R}} F(x, y_0).$$

Therefore from convexity of F and (3.2.7), this implies that $F(z, y_0) = F(X_0, y_0)$ for all $z \in [-X_0, X_0]$. Since $0 \in [-X_0, X_0]$ the claim follows.

From the above claim it implies that

$$J_\ell(v_\ell) = \int_{\Omega_\ell} F(\nabla v_\ell) - \int_{\Omega_\ell} f(x_2) v_\ell \geq \int_{-\ell}^{\ell} \left(\int_{-1}^{1} \left\{ F\left(0, \frac{\partial v_\ell}{\partial x_2}\right) - f(x_2) v_\ell \right\} \right).$$

Finally using (3.0.3), we get

$$J_\ell(v_\ell) \geq \int_{-\ell}^{\ell} J_\infty(u_\infty) = 2\ell J_\infty(u_\infty). \tag{3.2.8}$$

The result then follows after combining the last equation and (3.2.6). □

Remark 3.2.1. *Lemma 3.2.4 shows that u_∞ is the minimizer of J_ℓ. If $F(X) = A_\delta X \cdot X$ where*

$$A_\delta := \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} \tag{3.2.9}$$

it could not be so since by the Euler equation one would have

$$\int_{\Omega_\ell} A_\delta \nabla u_\infty \cdot \nabla v = 0, \quad \forall v \in V(\Omega_\ell),$$

$$\Rightarrow \int_{\Omega_\ell} \delta \frac{\partial u_\infty}{\partial x_2} \frac{\partial v}{\partial x_1} = 0, \quad \forall v \in V(\Omega_\ell)$$

which is not possible.

Now we show for the above particular choice of $F(X) = A_\delta X \cdot X$, that

$$\frac{J_\ell(v_\ell)}{2\ell} \to J_\infty(u_\infty)$$

from below.

Lemma 3.2.5. *There exists* $C_1, C_2 > 0$ *such that*

$$\int_{\Omega_\ell} |\nabla v_\ell|^2 \leq C_1 \ell \quad \textit{and} \int_{\Omega_\ell} v_\ell^2 \leq C_2 \ell. \tag{3.2.10}$$

where v_ℓ *is as in (1.2.1).*

Proof. It is enough to proof the first inequality, as the second inequality follows by the application of Poincaré inequality and the first inequality.

Since $J_\ell(v_\ell) \leq J_\ell(0)$, we have

$$\int_{\Omega_\ell} F(\nabla v_\ell) \leq \int_{\Omega_\ell} f v_\ell.$$

From (3.0.1) and application of Hölder inequality it implies that

$$\alpha \int_{\Omega_\ell} |\nabla v_\ell|^2 \leq \left(\int_{\Omega_\ell} f^2(x_2) \right)^{\frac{1}{2}} \left(\int_{\Omega_\ell} v_\ell^2 \right)^{\frac{1}{2}}.$$

Application of Poincaré inequality again gives that

$$\alpha \left(\int_{\Omega_\ell} |\nabla v_\ell|^2 \right)^{\frac{1}{2}} \leq C \left(\int_{\Omega_\ell} f^2(x_2) \right)^{\frac{1}{2}}.$$

The required result then follows after noting that

$$\int_{\Omega_\ell} f^2(x_2) = 2\ell \int_{-1}^{1} f^2(x_2).$$

□

Theorem 3.2.1.

$$J_\infty(u_\infty) - \frac{C}{\ell^{\frac{1}{3}}} \leq \frac{J_\ell(v_\ell)}{2\ell} < J_\infty(u_\infty)$$

for some positive constant C*, independent of* ℓ*.*

Proof. Suppose $\gamma \in (0,1)$. Define $\rho_\ell : [-\ell, \ell] \to \mathbb{R}$ a Lipschitz continuous function such that $\rho_\ell = 1$ on $[-\ell + \ell^\gamma, \ell - \ell^\gamma]$, $\rho_\ell(-\ell) = \rho_\ell(\ell) = 0$ and $|\frac{\partial \rho_\ell}{\partial x_1}| \leq \frac{C}{\ell^\gamma}$ for some $C > 0$.

Define $Z_\ell := \rho_\ell v_\ell \in H_0^1(\Omega_\ell)$. From Lemma (3.2.4) we know

$$2\ell J_\infty(u_\infty) \leq J_\ell(v_\ell \rho_\ell). \tag{3.2.11}$$

Let us calculate the term $J_\ell(v_\ell \rho_\ell)$ explicitly after setting $\mathcal{D}_\ell := \Omega_\ell \setminus \Omega_{\ell-\ell^\gamma}$. By definition

$$
\begin{aligned}
J_\ell(\rho_\ell v_\ell) &= \int_{\Omega_\ell} F(\nabla \rho_\ell v_\ell) - f(x_2)\rho_\ell v_\ell \\
&= \int_{\Omega_{\ell-\ell^\gamma}} F(\nabla v_\ell) - f(x_2) v_\ell + \int_{\mathcal{D}_\ell} F(\nabla \rho_\ell v_\ell) - f(x_2)\rho_\ell v_\ell \\
&= \int_{\Omega_\ell} F(\nabla v_\ell) - f(x_2) v_\ell + \int_{\mathcal{D}_\ell} \{F(\nabla \rho_\ell v_\ell) - F(\nabla v_\ell)\} + \int_{\mathcal{D}_\ell} \{f(x_2) v_\ell - f(x_2)\rho_\ell v_\ell\} \\
&= J_\ell(v_\ell) + \int_{\mathcal{D}_\ell} \{F(\nabla \rho_\ell v_\ell) - F(\nabla v_\ell)\} + \int_{\mathcal{D}_\ell} \{f(x_2) v_\ell - f(x_2)\rho_\ell v_\ell\} \\
&:= J_\ell(v_\ell) + I_1(\ell) + I_2(\ell). \qquad (3.2.12)
\end{aligned}
$$

Let us estimate the term $I_1(\ell)$ using explicitly the definition of F.

$$
\begin{aligned}
I_1(\ell) &= \int_{\mathcal{D}_\ell} A_\delta \nabla \rho_\ell v_\ell \cdot \nabla \rho_\ell v_\ell - \int_{\mathcal{D}_\ell} A_\delta \nabla v_\ell \cdot \nabla v_\ell \\
&= \int_{\mathcal{D}_\ell} (\rho_\ell^2 - 1) A_\delta \nabla v_\ell \cdot \nabla v_\ell + 2\int_{\mathcal{D}_\ell} \rho_\ell v_\ell A_\delta \nabla \rho_\ell \cdot \nabla v_\ell \\
&\quad + \int_{\mathcal{D}_\ell} v_\ell^2 A_\delta \nabla \rho_\ell \cdot \nabla \rho_\ell .
\end{aligned}
$$

Since $\rho_\ell \le 1$ and $A_\delta \nabla v_\ell \cdot \nabla v_\ell \ge 0$, we have

$$
I_1(\ell) \le 2\int_{\mathcal{D}_\ell} \rho_\ell v_\ell A_\delta \nabla \rho_\ell \cdot \nabla v_\ell + \int_{\mathcal{D}_\ell} v_\ell^2 A_\delta \nabla \rho_\ell \cdot \nabla \rho_\ell . \qquad (3.2.13)
$$

Hence for some positive constant C, we have

$$
I_1(\ell) \le C\int_{\mathcal{D}_\ell} v_\ell |\nabla v_\ell||\nabla \rho_\ell| + C\int_{\mathcal{D}_\ell} v_\ell^2 |\nabla \rho_\ell|^2 .
$$

Using $|\nabla \rho_\ell| \le \frac{C}{\ell^\gamma}$ and Hölder's inequality we have

$$
\begin{aligned}
I_1(\ell) &\le \frac{C}{\ell^\gamma}\int_{\mathcal{D}_\ell} |v_\ell||\nabla v_\ell| + \frac{C}{\ell^{2\gamma}}\int_{\mathcal{D}_\ell} v_\ell^2 \\
&\le \frac{2C}{\ell^\gamma}\left(\int_{\mathcal{D}_\ell} |\nabla v_\ell|^2\right)^{\frac{1}{2}} \left(\int_{\mathcal{D}_\ell} v_\ell^2\right)^{\frac{1}{2}} + \frac{C}{\ell^{2\gamma}}\int_{\mathcal{D}_\ell} v_\ell^2 . \qquad (3.2.14)
\end{aligned}
$$

From Lemma (3.2.5) it comes that

$$
I_1(\ell) \le \frac{C}{\ell^{\gamma-1}} + \frac{C}{\ell^{2\gamma-1}} . \qquad (3.2.15)
$$

Let us estimate the term $I_2(\ell)$ as follows:

$$|I_2(\ell)| \le \int_{\mathcal{D}_\ell} |1-\rho_\ell||f||v_\ell| \le 2\int_{\mathcal{D}_\ell} |f||v_\ell| \le 2\left(\int_{\mathcal{D}_\ell} |v_\ell|^2\right)^{\frac{1}{2}} \left(\int_{\mathcal{D}_\ell} |f|^2\right)^{\frac{1}{2}}$$
$$\le 2(2\ell^{\frac{\gamma}{2}})\left(\int_{-1}^{1} |f|^2\right)^{\frac{1}{2}} \left(\int_{\mathcal{D}_\ell} v_\ell^2\right)^{\frac{1}{2}} \le 4\ell^{\frac{\gamma+1}{2}} \left(\int_{-1}^{1} |f|^2\right)^{\frac{1}{2}}. \quad (3.2.16)$$

Therefore combining (3.2.12) together with (3.2.15) and (3.2.16), we get for some constant $C > 0$,

$$\frac{J_\ell(v_\ell)}{2\ell} \ge J_\infty(u_\infty) - C\left\{\frac{1}{\ell^\gamma} + \frac{1}{\ell^{2\gamma}} + \frac{1}{\ell^{\frac{1-\gamma}{2}}}\right\}.$$

The best rate of convergence is obtained after choosing $\gamma = \frac{1}{3}$, which gives the left hand side of the claim. The inequality $\frac{J_\ell(v_\ell)}{2\ell} < J_\infty(u_\infty)$ follows from the previous Remark. This finishes the proof of the theorem. □

Note that the last theorem also provides us the rate of convergence as well, which is $\frac{1}{\ell^{\frac{1}{3}}}$.

Chapter 4

Nonlocal problems

Let Ω be a bounded, open subset of $\mathbb{R}^d$. We denote by $\mathcal{A}$ a functional from $\Omega \times L^p(\Omega)$, $p \geq 1$, with values in $\mathbb{R}$. We suppose that

$$x \mapsto \mathcal{A}(x, u) \text{ is measurable } \forall u \in L^p(\Omega) \tag{4.0.1}$$

and that the mapping

$$u \mapsto \mathcal{A}(x, u) \text{ is continuous from } L^p(\Omega) \text{ into } \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{4.0.2}$$

We make in addition the following ellipticity assumption, namely we assume that for some positive constants $a_0,\ a_\infty$ one has

$$0 < a_0 \leq \mathcal{A}(x, u) \leq a_\infty \ \text{ a.e. } x \in \Omega,\ \forall u \in L^p(\Omega). \tag{4.0.3}$$

We are interested in finding solutions of the following problem

$$\begin{cases} -\mathcal{A}(x, u) \triangle u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.0.4}$$

Here $\partial\Omega$ denotes the boundary of Ω, f some function which will be described later and λ is a positive parameter. Such a problem has been studied in **[13]** under some assumptions on f. The key argument in **[13]** is the Schauder fixed point theorem applied on a convex set involving the first eigenfunction of the Dirichlet problem in Ω. Here we replace the first eigenfunction by the minimizer of a functional related to the problem. This allows to relax some of the assumptions of **[13]** and to discover new solutions. Problems of this type in local frame work were considered in [4], [30]-[40].

The chapter is divided as follows. The next section is devoted to our main result, which mainly shows if f has n-loops, then the problem (4.0.4) admits atleast n non trivial solution. The next section is devoted to an nonlocal Eigenvalue problem. In the last section we deal with a nonlocal problem in divergence form.

4.1 The main existence results

Let us denote by f a Lipschitz continuous function from $[0, \infty)$ into itself (i.e. $f \geq 0$) and suppose that there exists two non negative numbers $0 \leq \theta < \theta'$ such that

$$f(\theta) = f(\theta') = 0, \quad f > 0 \text{ on } (\theta, \theta'). \tag{4.1.1}$$

We will denote by $\tilde{f}$ the function defined as

$$\tilde{f}(u) = \begin{cases} f(u) & \text{for } u \in (\theta, \theta'), \\ 0 & \text{for } u \notin (\theta, \theta'). \end{cases} \tag{4.1.2}$$

Then we have:

Lemma 4.1.1. *For λ sufficiently large the problem*

$$\begin{cases} - \triangle u = \lambda \tilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.3}$$

admits a weak solution ψ such that

$$0 \leq \psi \leq \theta', \ |\psi|_\infty > \theta. \tag{4.1.4}$$

(Here $|\psi|_\infty$ denotes the usual $L^\infty(\Omega)$- norm of ψ.)

Proof. Let us set

$$F(v) = \int_0^v \tilde{f}(s)ds \tag{4.1.5}$$

$$J[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(u)dx, \quad \forall u \in H_0^1(\Omega). \tag{4.1.6}$$

Claim 1: J admits a global minimizer ψ on $H_0^1(\Omega)$.
Indeed, due to $f \geq 0$ and (4.1.2) one has

$$F(v) \leq F(\theta') \quad \forall v \in \mathbb{R}. \tag{4.1.7}$$

It follows that

$$J[v] \geq -\lambda|\Omega|F(\theta') \quad \forall v \in H_0^1(\Omega) \tag{4.1.8}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, i.e. J is bounded from below. The usual direct method of calculus of variations allows then to conclude to the existence of a $\psi \in H_0^1(\Omega)$ minimizing J on $H_0^1(\Omega)$. Note at this point that ψ might not be unique. In what follows ψ is such a minimizer.
Claim 2: $|\psi|_\infty > \theta$ for λ large enough.
Suppose on the contrary that $|\psi|_\infty \leq \theta$. Then clearly

$$F(\psi) = 0$$

and

$$J[\psi] \geq 0. \tag{4.1.9}$$

Consider a function approximating the constant function equal to θ', for instance

$$w_\delta(x) := \theta' \left(1 \wedge \frac{\mathrm{dist}(x, \partial\Omega)}{\delta}\right) \tag{4.1.10}$$

($\delta > 0, \mathrm{dist}(x, \partial\Omega)$ is the euclidean distance from x to $\partial\Omega$, $\wedge$ denotes the minimum of two numbers).

It is clear that $w_\delta \in H_0^1(\Omega)$. Moreover if $\Omega_\delta = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) < \delta\}$ one has $w_\delta = \theta'$ on $\Omega \setminus \Omega_\delta$. Let us fix δ such that $|\Omega_\delta| < |\Omega|$ that is

$$F(\theta')|\Omega_\delta| - F(\theta')|\Omega| < 0. \tag{4.1.11}$$

Then we have

$$\begin{aligned} J[w_\delta] &= \frac{1}{2}\int_\Omega |\nabla w_\delta|^2 dx - \lambda \int_\Omega F(w_\delta) dx \\ &= \frac{1}{2}\int_\Omega |\nabla w_\delta|^2 dx - \lambda \int_{\Omega\setminus\Omega_\delta} F(\theta') dx - \lambda \int_{\Omega_\delta} F(w_\delta) dx \\ &\leq \frac{1}{2}\int_\Omega |\nabla w_\delta|^2 dx - \lambda \int_{\Omega\setminus\Omega_\delta} F(\theta') dx \\ &= \frac{1}{2}\int_\Omega |\nabla w_\delta|^2 dx + \lambda\{F(\theta')|\Omega_\delta| - F(\theta')|\Omega|\} \\ &< 0 \end{aligned} \tag{4.1.12}$$

for λ large enough (see (4.1.11)). This contradicts (4.1.9) and completes the proof of the claim.

Claim 3: $0 \leq \psi \leq \theta'$**.**

Note first that (4.1.3) is the Euler equation of the minimizing problem of J on $H_0^1(\Omega)$. Thus ψ is a weak solution to (4.1.3) and thus is a non negative. Let us suppose that $\psi > \theta'$ on a set of positive measure. Set

$$v := \psi \wedge \theta' \in H_0^1(\Omega)$$

where as above $\wedge$ denotes the minimum of two numbers. Then one has

$$J[v] = \frac{1}{2}\int_{\{\psi\leq\theta'\}} |\nabla\psi|^2 dx - \lambda \int_{\{\psi\leq\theta'\}} F(\psi) dx - \lambda \int_{\{\psi>\theta'\}} F(\theta') dx$$

where $\{\psi \leq \theta'\} = \{x \in \omega \mid \psi(x) \leq \theta'\}$ and $\{\psi > \theta'\}$ is defined in a similar way.

It follows

$$J[v] = \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx - \lambda \int_\Omega F(\psi) dx - \frac{1}{2}\int_{\{\psi>\theta'\}} |\nabla\psi|^2 dx + \lambda \int_{\{\psi>\theta'\}} \{F(\psi) - F(\theta')\}\, dx.$$

Due to (4.1.2), (4.1.5) the last integral above vanishes and one gets

$$J[\psi] \le J[v] \le J[\psi] - \frac{1}{2}\int_{\{\psi \ge \theta'\}} |\nabla\psi|^2 dx.$$

This implies that

$$\int_{\{\psi \ge \theta'\}} |\nabla\psi|^2 dx = 0$$

and thus $\psi \wedge \theta' = \psi$. This completes the proof of the lemma. □

Remark 4.1.1. *Due to the strong maximum principle one has in fact*

$$\psi > 0 \quad in\ \Omega.$$

We can now establish our main result:

Theorem 4.1.1. *Under the assumptions* (4.0.1)-(4.0.3), (4.1.1)*, for λ' sufficiently large there exists a weak solution u to*

$$\begin{cases} -\triangle u = \lambda' \frac{f(u)}{\mathcal{A}(x,u)} & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4.1.13}$$

satisfying

$$0 < u \le \theta', \quad |u|_\infty > \theta. \tag{4.1.14}$$

Proof. Let us denote by ψ a function satisfying (4.1.3), (4.1.4). For any $w \in L^2(\Omega)$ one has

$$-\triangle\psi \;=\; \lambda\tilde{f}(\psi) \;=\; \lambda\mathcal{A}(x,w)\frac{\tilde{f}(\psi)}{\mathcal{A}(x,w)} \;\le\; \lambda a_\infty \frac{\tilde{f}(\psi)}{\mathcal{A}(x,w)} \;\le\; \frac{\lambda' f(\psi)}{\mathcal{A}(x,w)} \tag{4.1.15}$$

where we have set $\lambda' = \lambda a_\infty$.

Consider the function

$$g(t) = \lambda' f(t) + \mu t.$$

If L denotes the Lipschitz constant of f. For $t > t' \ge 0$ one has

$$g(t) - g(t') = \mu(t-t') + \lambda'\{f(t) - f(t')\} \ge \mu(t-t') - L\lambda'(t-t')$$

and thus for $\mu > L\lambda'$ the function g is increasing. We fix μ in such a way that g is increasing and set

$$\mathcal{K} := \left\{ w \in L^2(\Omega) \mid \psi \le w \le \theta' \text{ a.e. in } \Omega \right\}. \tag{4.1.16}$$

It is clear that $\mathcal{K}$ is a closed convex subset of $L^2(\Omega)$. For $w \in \mathcal{K}$ let us denote by $u = T(w)$ the unique weak solution to

$$\begin{cases} -\triangle u + \frac{\mu u}{\mathcal{A}(x,w)} = \frac{g(w)}{\mathcal{A}(x,w)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.17}$$

It is clear that T is a map from $\mathcal{K}$ into $H_0^1(\Omega)$. Moreover a fixed point for T in $\mathcal{K}$ is clearly a solution to (4.1.13), (4.1.14). Let us first prove:

1. T maps $\mathcal{K}$ to itself.

Indeed let $w \in \mathcal{K}$. One has by the monotonicity of g and (4.1.15)

$$- \triangle u + \frac{\mu u}{\mathcal{A}(x,w)} = \frac{g(w)}{\mathcal{A}(x,w)} \geq \frac{g(\psi)}{\mathcal{A}(x,w)} \geq - \triangle \psi + \frac{\mu \psi}{\mathcal{A}(x,w)}$$

and

$$- \triangle u + \frac{\mu u}{\mathcal{A}(x,w)} \leq \frac{g(w)}{\mathcal{A}(x,w)} \leq \frac{g(\theta')}{\mathcal{A}(x,w)} \leq - \triangle \theta' + \frac{\mu \theta'}{\mathcal{A}(x,w)}.$$

Since $\psi \leq u \leq \theta'$ on $\partial\Omega$, it follows from the weak maximum principle

$$\psi \leq u \leq \theta' \quad a.e \text{ in } \Omega,$$

that is $u \in \mathcal{K}$.

2. $T : \mathcal{K} \to \mathcal{K}$ is compact.

For $w \in \mathcal{K}$ if u is a solution to (4.1.17) one has clearly

$$\left|\frac{g(w)}{\mathcal{A}(x,w)}\right| \leq \frac{g(\theta')}{a_0}$$

and u remains in a fixed ball of $H_0^1(\Omega)$. Due to the compactness of the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ it is then enough to show that T is continuous from $\mathcal{K}$ into itself. Thus let $w_n \in \mathcal{K}$ with

$$w_n \to w \quad \text{in } L^2(\Omega).$$

We are going to show that

$$u_n := T(w_n) \to T(w) := u \quad \text{in } H_0^1(\Omega).$$

Indeed, due to the definition of u_n and u (see, (4.1.17)) one has

$$- \triangle (u - u_n) + \frac{\mu u}{\mathcal{A}(x,w)} - \frac{\mu u_n}{\mathcal{A}(x,w_n)} = \frac{g(w)}{\mathcal{A}(x,w)} - \frac{g(w_n)}{\mathcal{A}(x,w_n)}.$$

This can be written as

$$\begin{aligned}
- \triangle(u - u_n) + \frac{\mu(u-u_n)}{\mathcal{A}(x,w)} &= \mu u_n \left\{ \frac{1}{\mathcal{A}(x,w_n)} - \frac{1}{\mathcal{A}(x,w)} \right\} \\
&\quad + \left\{ \frac{g(w)}{\mathcal{A}(x,w)} - \frac{g(w_n)}{\mathcal{A}(x,w_n)} \right\} \\
&= (\mu u_n - g(w)) \left\{ \frac{1}{\mathcal{A}(x,w_n)} - \frac{1}{\mathcal{A}(x,w)} \right\} + \frac{g(w) - g(w_n)}{\mathcal{A}(x,w_n)} \\
&= \frac{\mu u_n - g(w)}{\mathcal{A}(x,w_n)\mathcal{A}(x,w)} \{\mathcal{A}(x,w_n) - \mathcal{A}(x,w)\} + \frac{g(w) - g(w_n)}{\mathcal{A}(x,w_n)} = f_n \quad (4.1.18)
\end{aligned}$$

One notices that u_n, w are uniformly bounded, $\mathcal{A}$ is bounded from below, g is Lipschitz continuous. Thus for some constant C one has

$$|f_n| \le C|\mathcal{A}(x,w) - \mathcal{A}(x,w_n)| + C|w - w_n|.$$

Since $w_n \to w$ in $L^2(\Omega)$, up to a subsequence we have

$$w_n \to w \quad a.e. \text{ in } \Omega$$

and thus by the Lebesgue theorem (recall the definition of $\mathcal{K}$)

$$w_n \to w \quad \text{in } L^p(\Omega),\ \forall p \ge 1.$$

It follows from (4.0.2) that

$$\mathcal{A}(x,w_n) \to \mathcal{A}(x,w) \quad a.e. \text{ in } \Omega$$

and by Lebesgue's theorem again

$$\mathcal{A}(x,w_n) \to \mathcal{A}(x,w) \quad \text{in } L^2(\Omega).$$

Thus the only possible limit of f_n in $L^2(\Omega)$ is 0 i.e.

$$f_n \to 0 \quad \text{in } L^2(\Omega).$$

This shows (see (4.1.18)) that $u_n \to u$ in $H_0^1(\Omega)$. This completes the proof of the theorem. □

As a corollary consider a Lipschitz continuous function f, non negative and having n bumps- see the figure below:

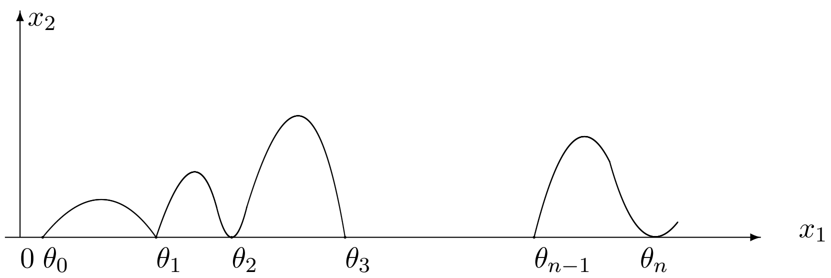


Figure 1

Theorem 4.1.2. *Under the assumptions* (4.0.1)-(4.0.3) *and if* f *is Lipschitz continuous function which graph is depicted in the figure above, then for* λ *large enough the problem*

$$\begin{cases} -\mathcal{A}(x,u) \triangle u = \lambda f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{4.1.19}$$

admits at least n *non trivial solutions.*

Proof. It is enough to apply repeatedly Theorem (4.1.1). □

Remark 4.1.2. *If in addition* f *admits* n *negative bumps for* $x < 0$ *then the problem* (4.1.19) *posses at least* $2n$ *non trivial solutions. It is indeed enough to apply the theorem above with the function* $-f(-x)$.

Now we delve in to finding suitable $\mathcal{A}$ to fulfill our assumptions. Let $\mathcal{B}(x,u)$ denote a Carathéodory function, that is $\mathcal{B}$ is defined from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ such that

$$(1)\ x \mapsto \mathcal{B}(x,u) \quad \text{is measurable} \quad \forall u \in \mathbb{R},$$
$$(2)\ u \mapsto \mathcal{B}(x,u) \quad \text{is Lipschitz continuous} \quad \text{a.e. } \forall x \in \Omega,$$

with the Lipschitz constant independent of x, and satisfying for some positive constants

$$0 < a_0 \leq \mathcal{B}(x,u) \leq a_\infty \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R}.$$

At first we look at the population distribution model. Let

$$\mathcal{A}(x,u) = \mathcal{B}(x, \int_\Omega u). \tag{4.1.20}$$

If u denotes the density of population, then the total population is denoted by $\int_\Omega u$. Replacing $\int_\Omega u$ by $\int_{\Omega'} u$ in the last equation, we can also look at the total population of a sub region Ω' of Ω. Then it is quite obvious that $\mathcal{A}(x,u)$ defined by (4.1.20) satisfies our assumptions.

Another important class of nonlocal operator that suits our criterion is as follows. If Ω is a domain of $A-type$, that is for fixed $0 < r < diam(\Omega)$, there exists a constant $A > 0$ such that $|\Omega(x,r)| \geq Ar^d$ where $\Omega(x,r) = \Omega \cap B(x,r)$. If a is a Lipschitz continuous function then the nonlocal operator defined by

$$\mathcal{A}(x,u) = a\left(\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} u(y)dy\right)$$

also satisfies our criterion.

4.2 An Eigenvalue Problem

In this section we are interested in finding nontrivial solution of the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}(x,u)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 = 1. \end{cases} \tag{4.2.1}$$

In this section we restrict ourself for the case $p = 2$.

Let λ_1 and u_1 denotes the first eigenvalue and first eigenfunction of the problem

$$\begin{cases} -\triangle u_1 = \lambda_1 u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \\ u_1 > 0,\ \int_\Omega u_1^2 = 1. \end{cases} \tag{4.2.2}$$

The main result of this section is the following.

Theorem 4.2.1. *Under the assumptions* (4.0.1)-(4.0.3)*, the problem*

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}(x,u)\nabla u) = \lambda u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \\ \int_\Omega u^2 = 1, \end{cases} \tag{4.2.3}$$

admits a nontrivial solution for some $\lambda = \lambda^*$. Further $\lambda^* \in [a_0\lambda_1, a_\infty\lambda_1]$, where λ_1 is as in (4.2.2)*.*

Proof. Fix $w \in L^2(\Omega)$ and consider the following problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}(x,w)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 = 1,\ u > 0 \quad \text{a.e. } x \in \Omega. \end{cases} \tag{4.2.4}$$

The above problem is an eigenvalue problem for an elliptic operator in divergence form. From the standard results of elliptic theory, we can conclude that there exists unique $\lambda = \lambda_w^1$ and $u = u_w$ that solves (4.2.4), where λ_w^1 denotes the first eigenvalue and u_w is the corresponding first eigenfunction. It is also well known that λ_w^1 has the following characterization,

$$\lambda_w^1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \mathcal{A}(x,w)|\nabla u|^2}{\int_\Omega u^2} \tag{4.2.5}$$

and $u_w > 0$ a.e. $x \in \Omega$.

1. For all $w \in L^2(\Omega)$, we have $\lambda_w^1 \in [a_0\lambda_1, a_\infty\lambda_1]$.

From (4.0.3) we have

$$a_0 \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \leq \frac{\int_\Omega \mathcal{A}(x,w)|\nabla u|^2}{\int_\Omega u^2} \leq a_\infty \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

The claim then follows by taking infimum over the set $H_0^1(\Omega) \setminus \{0\}$.

Define the set

$$\mathcal{K} = \left\{ u \in L^2(\Omega) \middle| \ \int_\Omega |\nabla u|^2 \leq \frac{a_\infty \lambda_1}{a_0} \right\}.$$

The set $\mathcal{K}$ is a compact, convex subset of $L^2(\Omega)$. Define the map $T : \mathcal{K} \to L^2(\Omega)$ as

$$T(w) = u_w,$$

where u_w solves (4.2.4). Clearly any fixed point of T is a solution of the problem (4.2.3).

2. T **maps** $\mathcal{K}$ **to** $\mathcal{K}$.

Fix $w \in \mathcal{K}$. From (4.2.4) and (4.2.5) we have

$$\lambda_w^1 = \int_\Omega \mathcal{A}(x,w)|\nabla u_w|^2.$$

Now using the last claim and (4.0.3), we get $u_w \in \mathcal{K}$.

3. $T : \mathcal{K} \to \mathcal{K}$ **is continuous.**

Let $\{w_k\}_k \subset \mathcal{K}$ be such that

$$w_k \to w \quad \text{in } L^2(\Omega). \tag{4.2.6}$$

Since $T(w_k) \in \mathcal{K}$, the sequence $\{T(w_k)\}_k$ is bounded in $H_0^1(\Omega)$. Hence there exists a function $p \in H_0^1(\Omega)$ such that up to a subsequence $\{w_{k_m}\}_m$ of $\{w_k\}_k$, we can have

$$\begin{aligned} T(w_{k_m}) &\to p && \text{in } L^2(\Omega), \\ T(w_{k_m}) &\rightharpoonup p && \text{in } H_0^1(\Omega), \\ T(w_{k_m}) &\to p && \text{a.e. } x \in \Omega. \end{aligned} \tag{4.2.7}$$

Since $T(w_{k_m}) > 0$ a.e. $x \in \Omega$, it follows from convergence above that

$$p \geq 0 \quad \text{a.e. } x \in \Omega$$

and $\int_\Omega p^2 = 1$. Since $\lambda^1_{w_{k_m}} \in [a_0\lambda_1, a_\infty\lambda_1]$, there exists a further subsequence $\left\{k_{m_j}\right\}_j$ of $\{k_m\}_m$, such that

$$\lambda^1_{w_{k_{m_j}}} \to \lambda^*_w$$

where $\lambda^*_w \in [a_0\lambda_1, a_\infty\lambda_1]$. The Euler-Lagrange equation satisfied by $T(w_{k_{m_j}})$ is given by

$$\int_\Omega \mathcal{A}(x, w_{k_{m_j}}) \nabla T(w_{k_{m_j}}) \cdot \nabla v = \lambda^1_{w_{k_{m_j}}} \int_\Omega T(w_{k_{m_j}}) v, \quad \forall v \in H_0^1(\Omega). \tag{4.2.8}$$

Consider the left hand side of (4.2.8),

$$\int_\Omega \mathcal{A}(x, w_{k_{m_j}})\nabla T(w_{k_{m_j}}) \cdot \nabla v$$
$$= \int_\Omega \{\mathcal{A}(x, w_{k_{m_j}}) - \mathcal{A}(x, w)\}\nabla T(w_{k_{m_j}}) \cdot \nabla v + \int_\Omega \mathcal{A}(x, w)\nabla T(w_{k_{m_j}}) \cdot \nabla v$$
$$:= I_1^j + I_2^j.$$

We first estimate the term I_1^j.

$$|I_1^j| \le \int_\Omega |\mathcal{A}(x, w_{k_{m_j}}) - \mathcal{A}(x, w)||\nabla T(w_{k_{m_j}})||\nabla v|$$
$$\le \left(\int_\Omega |\mathcal{A}(x, w_{k_{m_j}}) - \mathcal{A}(x, w)|^2|\nabla v|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla T(w_{k_{m_j}})|^2\right)^{\frac{1}{2}}.$$

Now using $\int_\Omega |\nabla T(w_{k_{m_j}})|^2 \le \frac{a_\infty \lambda_1}{a_0}$ we get

$$|I_1^j| \le \left(\frac{a_\infty \lambda_1}{a_0}\right)^{\frac{1}{2}} \left(\int_\Omega |\mathcal{A}(x, w_{k_{m_j}}) - \mathcal{A}(x, w)|^2|\nabla v|^2\right)^{\frac{1}{2}}. \tag{4.2.9}$$

From (4.0.2) and (4.2.6) we have

$$\mathcal{A}(x, w_{k_{m_j}}) \to \mathcal{A}(x, w) \quad \text{a.e. } x \in \Omega$$

and

$$|\mathcal{A}(x, w_{k_{m_j}}) - \mathcal{A}(x, w)|^2|\nabla v|^2 \le 4a_\infty^2|\nabla v|^2, \quad \forall v \in H_0^1(\Omega).$$

Now since $4a_\infty^2|\nabla v|^2 \in L^1(\Omega)$, we can pass trough the limit in (4.2.9) using dominated convergence theorem to get

$$I_1^j \to 0.$$

From (4.2.7),

$$I_2^j \to \int_\Omega \mathcal{A}(x, w)\nabla p \cdot \nabla v.$$

Therefore

$$\int_\Omega \mathcal{A}(x, w_{k_{m_j}})\nabla T(w_{k_{m_j}}) \cdot \nabla v \to \int_\Omega \mathcal{A}(x, w)\nabla p \cdot \nabla v.$$

From (4.2.7) it also follows that

$$\int_\Omega T(w_{k_{m_j}})v \to \int_\Omega pv.$$

Therefore we have

$$\int_\Omega \mathcal{A}(x, w)\nabla p \cdot \nabla v = \lambda_w^* \int_\Omega pv, \quad \forall v \in H_0^1(\Omega).$$

It is known **see,[32]** that the first eigenfunction of the problem (4.2.4) is its only solution that does not change its sign. Since p is non negative and nontrivial, it has to be the first eigenfunction and λ_w^* has to be the first eigenvalue (λ_w^1). Therefore

$$\int_\Omega \mathcal{A}(x,w)\nabla p \cdot \nabla v = \lambda_w^1 \int_\Omega pv, \quad \forall v \in H_0^1(\Omega).$$

Hence $T(w) = p$ holds. Since the possible limit is unique, we have

$$T(w_k) \to T(w) \ \text{ in } L^2(\Omega).$$

This proves continuity of the map T.

4. Schauder fixed point theorem.

The map $T : \mathcal{K} \to \mathcal{K}$ is continuous where $\mathcal{K}$ is compact and convex subset of $L^2(\Omega)$. Therefore from Schauder fixed point theorem the map T has a fixed point, that is $T(z) = z$ for some $z \in \mathcal{K}$.

Non triviality of z follows since

$$\int_\Omega |T(w)|^2 = 1, \ \forall w \in \mathcal{K}.$$

It is also clear from the definition of T that $\lambda^* = \lambda_z^1$ and hence $\lambda^* \in [a_0\lambda_1, a_\infty\lambda_1]$. This finishes the proof of the theorem. □

4.3 The General Case

Let $\theta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function satisfying

$$\begin{aligned} &f'(0) > 0, \\ &f(t) > 0 \ \ \forall t \in (0,\theta) \quad \text{and} \quad f(t) = 0 \ \text{ otherwise}, \\ &t \mapsto f(t)/t \ \text{ is strictly decreasing in } [0,\theta]. \end{aligned} \tag{4.3.1}$$

In this section we study existence of nontrivial solution for the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.2}$$

The condition $t \mapsto \frac{f(t)}{t}$ is strictly decreasing is generally assumed to get existence of unique solution for semilinear problems [3]. Let $g : [0,\theta] \to \mathbb{R}$ be any strictly decreasing function such that $g(\theta) = 0$. Define $f(t) := tg(t)$. It can be easily checked that such a f satisfies the conditions in (4.3.1).

A solution of (1.2.8) is understood in weak sense, i.e. a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \mathcal{A}(x,u)\nabla u \cdot \nabla\phi = \lambda \int_\Omega f(u)\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{4.3.3}$$

In this section we further assume that the operator

$$\tilde{T} : L^p(\Omega) \to L^\infty(\Omega)$$

defined by,

$$\tilde{T}(u)(x) = \mathcal{A}(x, u) \quad \text{is continuous.} \tag{4.3.4}$$

Notice that the above assumption clearly implies (4.0.2).

Define for all $u \in L^p(\Omega)$,

$$\mathcal{A}_n(x, u) = \mathcal{A}(x, u) * \psi_{\frac{1}{n}},$$

where $\mathcal{A}(x, u)$ is extended by a_0 outside Ω, $\psi_{\frac{1}{n}}$ is the standard mollifier and " $*$ " denotes the operation of mollification. From the definition of the operation of mollification

$$\mathcal{A}_n(x, u) := \int_{B(0,\frac{1}{n})} \mathcal{A}(x - y, u)\psi_{\frac{1}{n}}(y)dy = \int_\Omega \mathcal{A}(y, u)\psi_{\frac{1}{n}}(x - y)dy.$$

For the sake of completeness let us recall the definition and some well known properties of standard mollifier $\psi_{\frac{1}{n}}$ that will be used later. Define $\psi \in C^\infty(\mathbb{R}^d)$ by

$$\psi(x) := \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

the constant C is chosen such that $\int_{\mathbb{R}^d} \psi = 1$. For each positive integer n, set

$$\psi_{\frac{1}{n}}(x) = n^d \psi(nx).$$

The function $\psi_{\frac{1}{n}} \in C^\infty(\Omega)$ and satisfies $\int_{\mathbb{R}^d} \psi_{\frac{1}{n}} = 1$, with support$(\psi_{\frac{1}{n}}) \subset B(0, \frac{1}{n})$.

Lemma 4.3.1. *For each $u \in L^p(\Omega)$ it holds $\forall n$,*

$$a_0 \leq \mathcal{A}_n(x, u) \leq a_\infty \quad a.e.\ x \in \Omega. \tag{4.3.5}$$

Proof. By definition of $\mathcal{A}_n$,

$$\mathcal{A}_n(x, u) = \int_{B(0,\frac{1}{n})} \mathcal{A}(x - y, u)\psi_{\frac{1}{n}}(y)dy \leq a_\infty \int_{B(0,\frac{1}{n})} \psi_{\frac{1}{n}}(y)dy = a_\infty.$$

As $\mathcal{A}$ is extended by a_0 outside Ω, the other inequality also holds similarly. □

From the definition of $\mathcal{A}_n(x, u)$ it is clear that $\mathcal{A}_n(x, u) \in C^\infty(\Omega)$.

Lemma 4.3.2. *For each fixed n and $x \in \Omega$, the mapping $u \to \mathcal{A}_n(x, u)$ is continuous from $L^p(\Omega)$ to $\mathbb{R}$.*

Proof. Let $w_m \to w$ in $L^p(\Omega)$, then for fixed x and n,

$$|\mathcal{A}_n(x,w_m)-\mathcal{A}_n(x,w)| \le \int_\Omega |\mathcal{A}(y,w_m)-\mathcal{A}(y,w)|\psi_{\frac{1}{n}}(x-y)dy$$
$$\le |\Omega|||\psi_{\frac{1}{n}}||_\infty \; ||\mathcal{A}(x,w_m)-\mathcal{A}(x,w)||_\infty.$$

The lemma then follows from (4.3.4). □

Consider the problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}_n(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.6}$$

First we will prove existence of nontrivial solution for the above problem and then pass through the limit as $n \to \infty$, to get existence results for the problem (4.3.3).

For fixed $w \in L^2(\Omega)$, consider the problem

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}_n(x,w)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.7}$$

A solution of (4.3.7) is understood in weak sense, i.e. a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathcal{A}_n(x,w)\nabla u\cdot\nabla\phi = \lambda\int_\Omega f(u)\phi, \quad \forall \phi \in H_0^1(\Omega).$$

For fixed $w \in L^2(\Omega)$, consider the energy functional $J_w^n : H_0^1(\Omega) \to \mathbb{R}$ associated to (4.3.7), given by

$$J_w^n[u] = \frac{1}{2}\int_\Omega \mathcal{A}_n(x,w)|\nabla u|^2 - \lambda\int_\Omega F(u),$$

where $F(t) = \int_0^t f(s)ds$. Put

$$m_w^n = \inf_{u\in H_0^1(\Omega)} J_w^n[u]. \tag{4.3.8}$$

From the standard results of calculus of variation, we know that m_w^n is attained by some function $u_w^n \in H_0^1(\Omega)$, that is

$$m_w^n = J_w^n(u_w^n) \tag{4.3.9}$$

and the same function solves (4.3.7) weakly. The function u_w^n may not be unique.

Fix an $\epsilon > 0$ small enough such that $f'(0)-\epsilon > 0$. Such a choice of ϵ is possible since it is assumed that $f'(0) > 0$. Again using $f'(0) > 0$ and $u_1 \in L^\infty(\Omega)$ it is possible to find small $t_\epsilon > 0$ such that

$$f(t_\epsilon u_1) \ge (f'(0)-\epsilon)t_\epsilon u_1, \tag{4.3.10}$$

where u_1 is as in (4.2.2).

Lemma 4.3.3. *If $\delta > 0$ be a fixed small positive number and $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$. Then $\forall n$ and $\forall w \in L^2(\Omega)$,*

$$-\lambda F(\theta)|\Omega| \leq m_w^n \leq -\frac{\delta t_\epsilon^2}{2} \tag{4.3.11}$$

where ϵ, t_ϵ is as in (4.3.10) and $|\Omega|$ denotes the n dimensional Lebesgue measure of the set Ω.

Proof. For fixed $u \in H_0^1(\Omega)$, we have from (4.3.1) and the definition of F,

$$F(u) \leq F(\theta).$$

Since $\mathcal{A}_n > 0$, we have

$$J_w^n[u] \geq -\lambda F(\theta)|\Omega|.$$

Now the left hand side inequality in (4.3.11) follows since u is arbitrary in the above inequality. For the other side of the inequality, first we estimate the term $F(t_\epsilon u_1)$ by using (4.3.1) and (4.3.10).

$$F(t_\epsilon u_1) = \int_0^{t_\epsilon u_1} f(s)ds = \int_0^{t_\epsilon u_1} \frac{f(s)}{s} s ds \geq \frac{f(t_\epsilon u_1)}{t_\epsilon u_1} \int_0^{t_\epsilon u_1} s ds$$

$$\geq \frac{f(t_\epsilon u_1) t_\epsilon u_1}{2} \geq (f'(0) - \epsilon)\frac{t_\epsilon^2 u_1^2}{2}. \tag{4.3.12}$$

Using $u = t_\epsilon u_1$ in (4.3.8) along with (4.3.12), we get

$$m_w^n \leq J_w^n[t_\epsilon u_1]$$

$$\leq \frac{a_\infty t_\epsilon^2}{2} \int_\Omega |\nabla u_1|^2 - \lambda \int_\Omega F(t_\epsilon u_1) \leq \frac{\lambda_1 a_\infty t_\epsilon^2}{2} - \frac{\lambda t_\epsilon^2}{2}(f'(0) - \epsilon).$$

Now if we choose $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$, we have

$$m_w^n \leq -\frac{\delta t_\epsilon^2}{2}.$$

This finishes the proof of the lemma. □

Remark 4.3.1. *Last lemma says us that for $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$, u_w^n is nontrivial, since $m_w^n < 0$.*

Lemma 4.3.4. *If $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$ holds then*

$$0 < u_w^n \leq \theta, \quad \text{a.e. } x \in \Omega,$$

where u_w^n is as in (4.3.9).

Proof. Since $f \geq 0$ and $\lambda > 0$ we have

$$\operatorname{div}(\mathcal{A}_n(x,w)\nabla u_w^n) \leq 0.$$

Hence from Strong maximum principle we have, either $u_w^n > 0$ a.e. $x \in \Omega$ or $u \equiv 0$. From the last remark, we know that u_w^n is nontrivial. Hence we can conclude that $u_w^n > 0$ a.e. $x \in \Omega$.
For the other side of the inequality, let us assume that $u_w^n > \theta$ on a set of positive measure in Ω. Define $v \in H_0^1(\Omega)$ by

$$v = u_w^n \wedge \theta$$

where $a \wedge b = \min\{a, b\}$. Clearly $0 \leq v \leq \theta$ a.e. $x \in \Omega$. Moreover

$$\begin{aligned} J_w^n[v] = \frac{1}{2}&\left\{\int_{\{u_w^n \leq \theta\}} \mathcal{A}_n(x,w)|\nabla u_w^n|^2 + \int_{\{u_w^n \geq \theta\}} \mathcal{A}_n(x,w)|\nabla \theta|^2\right\} \\ &- \lambda\left\{\int_{\{u_w^n \leq \theta\}} F(u_w^n) + \int_{\{u_w^n > \theta\}} F(\theta)\right\} \\ = \frac{1}{2}&\int_\Omega \mathcal{A}_n(x,w)|\nabla u_w^n|^2 - \lambda\int_\Omega F(u_w^n) - \frac{1}{2}\int_{\{u_w^n > \theta\}} \mathcal{A}_n(x,w)|\nabla u_w^n|^2 \\ &+ \lambda\int_{\{u_w^n > \theta\}} \{F(u_w^n) - F(\theta)\}. \end{aligned}$$

Since $F(t) = F(\theta)$ for all $t \geq \theta$, we have

$$J_w^n[v] \leq J_w^n[u_w^n] - \frac{1}{2}\int_{\{u_w^n > \theta\}} \mathcal{A}_n(x,w)|\nabla u_w^n|^2 < J_w^n[u_w^n],$$

which contradicts (4.3.8). □

Remark 4.3.2. *One should note that if u is any nontrivial solution of* (4.3.7) *then $u > 0$ a.e. $x \in \Omega$ holds from maximum principle.*

It is to be noted that if u is any solution of (4.3.7), then $u \in L^\infty(\Omega)$. This follows from the fact that f is Lipschitz continuous, hence bounded in $[0,\ \theta]$. Since $f \in L^\infty(\mathbb{R})$, $f(u) \in L^p(\Omega)$, for all $p \in \mathbb{R}$. Now since $\mathcal{A}_n$ is smooth, we have by well known regularity results of elliptic theory that $u \in W^{2,\,p}(\Omega)$. In particular when $p > d$, by Morey's theorem $u \in C^{1,\,1-\frac{d}{p}}(\overline{\Omega})$ which implies $u \in L^\infty(\Omega)$.

Next lemma is a well known result **[5]**, but we present a sketch of the proof.

Lemma 4.3.5. *Let u_1 and u_2 be two distinct non trivial solutions of* (4.3.7)*, then $\frac{u_1}{u_2}$ and $\frac{u_2}{u_1}$ are in $L^\infty(\Omega)$.*

Proof. We have for $i = 1, 2$, that $f(u_i) \in L^p(\Omega)$, $\forall p \geq 1$. Also since $\mathcal{A}_n$ is smooth, we have $u_i \in W^{2,p}(\Omega)$ which implies $\nabla u_i \in W^{1,\,p}(\Omega) \subset C^{1,1-\frac{d}{p}}(\overline{\Omega})$ for $p > d$. Thus both u_i, ∇u_i are continuous.

Interior estimate:

Let K be a relatively compact subset of Ω. Since $u_i \in L^\infty(\Omega)$ are continuous and positive, there exists $C_K > 0$ which depends on n and K, such that

$$\frac{u_1}{u_2},\ \frac{u_2}{u_1} \leq C_K \quad \text{in } \overline{K}.$$

Boundary estimate:

Let $x_0 \in \partial\Omega$. By Hopf's Strong maximum principle $\frac{\partial u_i}{\partial \nu}(x_0) < 0$ for $i = 1, 2$. Here ν denotes the outer normal vector to Ω at x_0. For $t > 0$,

$$\frac{u_1(x_0 - t\nu)}{u_2(x_0 - t\nu)} = \frac{\frac{u_1(x_0)-u_1(x_0-t\nu)}{t}}{\frac{u_2(x_0)-u_2(x_0-t\nu)}{t}}.$$

Using $\frac{\partial u_i}{\partial \nu}(x_0) < 0$,

$$\lim_{t\to 0} \frac{u_1(x_0 - t\nu)}{u_2(x_0 - t\nu)} = \frac{\frac{\partial u_1}{\partial \nu}(x_0)}{\frac{\partial u_2}{\partial \nu}(x_0)} \ (> 0). \tag{4.3.13}$$

The mapping defined by

$$x_0 \to \frac{\frac{\partial u_1}{\partial \nu}(x_0)}{\frac{\partial u_2}{\partial \nu}(x_0)}$$

from the compact boundary $\partial\Omega$, is continuous and strictly positive. Hence there exists $a > 0$ such that

$$\frac{\frac{\partial u_1}{\partial \nu}(x_0)}{\frac{\partial u_2}{\partial \nu}(x_0)} > a, \quad \forall x_0 \in \partial\Omega. \tag{4.3.14}$$

The boundary estimate then follows from (4.3.13) and (4.3.14).

□

Lemma 4.3.6. *There exists at most one nontrivial solution to* (4.3.7).

Proof. Let $u_1, u_2 \in H_0^1(\Omega)$ be two nontrivial solutions of (4.3.7). Fix $\epsilon > 0$. Using $\phi_1 = (u_1^2 - u_2^2)/(u_1 + \epsilon) \in H_0^1(\Omega)$ in the Euler-Lagrange equation of u_1, we get

$$\int_\Omega \mathcal{A}_n(x, w)\nabla u_1 \cdot \nabla \phi_1 = \lambda \int_\Omega f(u_1)\phi_1\,. \tag{4.3.15}$$

Similarly, using $\phi_2 = (u_1^2 - u_2^2)/(u_2 + \epsilon) \in H_0^1(\Omega)$, in the Euler-Lagrange equation of u_2, we obtain

$$\int_\Omega \mathcal{A}_n(x, w)\nabla u_2 \cdot \nabla \phi_2 = \lambda \int_\Omega f(u_2)\phi_2\,. \tag{4.3.16}$$

Explicit calculations in (4.3.15) gives

$$\int_\Omega \mathcal{A}_n(x,w)\nabla u_1 \cdot \left\{ \frac{(u_1+\epsilon)(2u_1\nabla u_1 - 2u_2\nabla u_2) - (u_1^2-u_2^2)\nabla u_1}{(u_1+\epsilon)^2} \right\} = \lambda \int_\Omega f(u_1)\frac{u_1^2-u_2^2}{u_1+\epsilon}. \quad (4.3.17)$$

Similarly, from (4.3.16) it follows

$$\int_\Omega \mathcal{A}_n(x,w)\nabla u_2 \cdot \left\{ \frac{(u_2+\epsilon)(2u_1\nabla u_1 - 2u_2\nabla u_2) - (u_1^2-u_2^2)\nabla u_2}{(u_2+\epsilon)^2} \right\} = \lambda \int_\Omega f(u_2)\frac{u_1^2-u_2^2}{u_2+\epsilon}. \quad (4.3.18)$$

Substarcting the right hand side of (4.3.18) from the right hand side of (4.3.17), we get

$$\begin{aligned}
&\int_\Omega \mathcal{A}_n(x,w)\nabla u_1 \cdot \left\{ \frac{(u_1+\epsilon)(2u_1\nabla u_1 - 2u_2\nabla u_2) - (u_1^2-u_2^2)\nabla u_1}{(u_1+\epsilon)^2} \right\} \\
&\qquad - \int_\Omega \mathcal{A}_n(x,w)\nabla u_2 \cdot \left\{ \frac{(u_2+\epsilon)(2u_1\nabla u_1 - 2u_2\nabla u_2) - (u_1^2-u_2^2)\nabla u_2}{(u_2+\epsilon)^2} \right\} \\
&= \int_\Omega \mathcal{A}_n(x,w)|\nabla u_1|^2 \left\{ \frac{u_1^2+u_2^2+2\epsilon u_1}{(u_1+\epsilon)^2} \right\} + \int_\Omega \mathcal{A}_n(x,w)|\nabla u_2|^2 \left\{ \frac{u_1^2+u_2^2+2\epsilon u_1}{(u_2+\epsilon)^2} \right\} \\
&\qquad - 2\int_\Omega \mathcal{A}_n(x,w)\nabla u_1\cdot\nabla u_2 \left\{ \frac{u_1}{u_2+\epsilon} + \frac{u_2}{u_1+\epsilon} \right\} \\
&= \int_\Omega \mathcal{A}_n(x,w)|\nabla u_1|^2 \left\{ 1 + \frac{u_2^2}{(u_1+\epsilon)^2} - \frac{\epsilon^2}{(u_1+\epsilon)^2} \right\} \\
&\qquad + \int_\Omega \mathcal{A}_n(x,w)|\nabla u_2|^2 \left\{ 1 + \frac{u_1^2}{(u_2+\epsilon)^2} - \frac{\epsilon^2}{(u_2+\epsilon)^2} \right\} \\
&\qquad - 2\int_\Omega \mathcal{A}_n(x,w)\nabla u_1\cdot\nabla u_2 \left\{ \frac{u_1}{u_2+\epsilon} + \frac{u_2}{u_1+\epsilon} \right\} \\
&= \int_\Omega \mathcal{A}_n(x,w) \left\{ \left|\nabla u_1 - \frac{u_1}{u_2+\epsilon}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1+\epsilon}\nabla u_1\right|^2 \right\} \\
&\qquad - \epsilon^2 \int_\Omega \mathcal{A}_n(x,w) \left\{ \frac{|\nabla u_1|^2}{(u_1+\epsilon)^2} + \frac{|\nabla u_2|^2}{(u_2+\epsilon)^2} \right\} \quad (4.3.19)
\end{aligned}$$

Subtracting (4.3.18) from (4.3.17), we get

$$\int_\Omega \mathcal{A}_n(x,w)\left\{\left|\nabla u_1 - \frac{u_1}{u_2+\epsilon}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1+\epsilon}\nabla u_1\right|^2\right\} - \epsilon^2\int_\Omega \mathcal{A}_n(x,w)\left\{\frac{|\nabla u_1|^2}{(u_1+\epsilon)^2}+\frac{|\nabla u_2|^2}{(u_2+\epsilon)^2}\right\} = \lambda\int_\Omega\left\{\frac{f(u_1)}{u_1+\epsilon}-\frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2). \quad (4.3.20)$$

Let us denote by $\mathcal{L}_\epsilon$ and $\mathcal{R}_\epsilon$ the left and the right hand side of (4.3.20) respectively. We want to show that,

$$\lim_{\epsilon\to 0}\mathcal{L}_\epsilon = \lim_{\epsilon\to 0}\mathcal{R}_\epsilon = 0.$$

First note that

$$\mathcal{L}_\epsilon \geq -\int_\Omega \mathcal{A}_n(x,w)g_\epsilon(x), \quad (4.3.21)$$

where

$$g_\epsilon(x) = \epsilon^2\left\{\frac{|\nabla u_1|^2}{(u_1+\epsilon)^2}+\frac{|\nabla u_2|^2}{(u_2+\epsilon)^2}\right\}.$$

Clearly $g_\epsilon \geq 0$ in Ω and $g_\epsilon \to 0$ point wise. For each fixed $x\in\Omega$ as $u_1(x),\ u_2(x) > 0$, we have

$$g_\epsilon(x) \leq |\nabla u_1(x)|^2 + |\nabla u_2(x)|^2.$$

Since $|\nabla u_1(x)|^2+|\nabla u_2(x)|^2 \in L^1(\Omega)$, we can apply dominated convergence theorem to get

$$\lim_{\epsilon\to 0}\int_\Omega \mathcal{A}_n(x,\ w)g_\epsilon(x) = 0.$$

Hence from (4.3.21),

$$\liminf_{\epsilon\to 0}\mathcal{L}_\epsilon \geq 0.$$

Set $\mathcal{R}_\epsilon = \lambda I_\epsilon + \lambda J_\epsilon$, where

$$I_\epsilon = \int_{\{u_1>u_2\}}\left\{\frac{f(u_1)}{u_1+\epsilon}-\frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2)$$

and

$$J_\epsilon = \int_{\{u_1\leq u_2\}}\left\{\frac{f(u_1)}{u_1+\epsilon}-\frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2).$$

Using (4.3.1) we estimate I_ϵ from above,

$$\begin{aligned} I_\epsilon = \int_{\{u_1>u_2\}} &\left\{\frac{f(u_1)}{u_1+\epsilon} - \frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2) \\ &\leq \epsilon \int_{\{u_1>u_2\}} \frac{f(u_2)}{u_2}\left\{\frac{(u_1-u_2)(u_1^2-u_2^2)}{(u_1+\epsilon)(u_2+\epsilon)}\right\} \\ &\leq \epsilon L \int_{\{u_1>u_2\}} \frac{u_1^3+u_2^3}{(u_1+\epsilon)(u_2+\epsilon)} \\ &\leq \epsilon L \int_{\{u_1>u_2\}} \frac{u_1^2}{u_2} + \epsilon L \int_{\{u_1>u_2\}} \frac{u_1^2}{u_1} \\ &\leq \epsilon L|\Omega|(||\frac{u_1}{u_2}||_\infty ||u_1||_\infty + ||\frac{u_2}{u_1}||_\infty ||u_2||_\infty). \end{aligned}$$

Since the right hand side goes to 0 as $\epsilon \to 0$. We have

$$\limsup_{\epsilon\to 0} I_\epsilon \leq 0.$$

Using a similar argument for J_ϵ, it can be shown that

$$\limsup_{\epsilon\to 0} J_\epsilon \leq 0.$$

Combining the last two inequality, we have

$$\limsup_{\epsilon\to 0} \mathcal{R}_\epsilon \leq 0$$

and hence

$$\lim_{\epsilon\to 0} \mathcal{L}_\epsilon = \lim_{\epsilon\to 0} \mathcal{R}_\epsilon = 0.$$

This means that

$$\lim_{\epsilon\to 0}\int_\Omega \left\{\frac{f(u_1)}{u_1+\epsilon} - \frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2) = 0. \tag{4.3.22}$$

Let us now consider the sequence $h_\epsilon(x) = \left\{\frac{f(u_1)}{u_1+\epsilon} - \frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2-u_2^2)$. For any fixed $x \in \Omega$ one has

$$h_\epsilon(x) \to \left\{\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right\}(u_1^2-u_2^2), \quad \text{as} \quad \epsilon \to 0. \tag{4.3.23}$$

Using $f(0) = 0$ and the Lipschitz continuity of f, we get

$$\begin{aligned} |h_\epsilon(x)| \leq \left\{\frac{f(u_1)}{u_1+\epsilon} + \frac{f(u_2)}{u_2+\epsilon}\right\}(u_1^2+u_2^2) &\leq \left\{\frac{f(u_1)}{u_1} + \frac{f(u_2)}{u_2}\right\}(u_1^2+u_2^2) \\ &\leq 2L(||u_1||_\infty^2 + ||u_2||_\infty^2), \end{aligned}$$

where L denotes the Lipschitz constant of f. Hence from dominated convergence theorem, we obtain

$$\int_\Omega \left\{ \frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right\} (u_1^2 - u_2^2) = 0$$

which is possible if and only if $u_1 = u_2$ *a.e.* $x \in \Omega$. This concludes the proof of the lemma. □

Remark 4.3.3. *From the last theorem we know that* (4.3.7) *has an unique nontrivial solution. Also from Lemma (4.3.3) we have* u_w^n *is a nontrivial solution of* (4.3.7), *for* $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$. *Hence* u_w^n *is the only nontrivial solution of the problem* (4.3.7).

Theorem 4.3.1. *For* $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$ *the problem* (4.3.6) *admits a positive solution.*

Proof. Define the set

$$\mathcal{K} = \left\{ u \in L^2(\Omega) \;\middle|\; 0 \leq u \leq \theta \; a.e. \; x \in \Omega \right\}.$$

Clearly $\mathcal{K}$ is a closed convex subset of $L^2(\Omega)$. Fix $w \in \mathcal{K}$. Define the map $T : \mathcal{K} \to L^2(\Omega)$ as

$$T(w) = u_w^n$$

where u_w^n is as in the last remark.

1. 0 **does not belong to** $T(\mathcal{K})$**.**

The claim follows from the definition of T.

2. T **maps** $\mathcal{K}$ **to** $\mathcal{K}$**.**

This claim is a consequence of Lemma (4.3.4).

3. Continuity of T**.**

Let $\{w_k\}_k \subset \mathcal{K}$ be such that

$$w_k \to w \quad \text{in } L^2(\Omega). \tag{4.3.24}$$

The Euler-Lagrange equation associated to (4.3.7) satisfied by w_k is given by

$$\int_\Omega \mathcal{A}_n(x, w_k) \nabla T(w_k) \cdot \nabla v = \lambda \int_\Omega f(T(w_k)) v, \quad \forall v \in H_0^1(\Omega). \tag{4.3.25}$$

Taking $v = T(w_k)$ in (4.3.25), we obtain using Hölder and Poincaré inequality

$$\left(\int_\Omega |\nabla T(w_k)|^2 \right)^{\frac{1}{2}} \leq \frac{\lambda ||f||_\infty}{a_0} \sqrt{\frac{|\Omega|}{\lambda_1}}$$

where λ_1 is as in (4.2.2). Thus the sequence $\{T(w_k)\}_k$ is bounded in $H_0^1(\Omega)$, hence there exists a function $p \in H_0^1(\Omega)$ such that up to a subsequence $\{w_{k_m}\}_m$ of $\{w_k\}_k$, we have

$$\begin{aligned} T(w_{k_m}) &\to p \quad && \text{in } L^2(\Omega), \\ T(w_{k_m}) &\rightharpoonup p \quad && \text{in } H_0^1(\Omega), \\ \nabla T(w_{k_m}) &\rightharpoonup \nabla p \quad && \text{in } L^2(\Omega). \end{aligned} \tag{4.3.26}$$

First we show that p is nontrivial. From Lemma (4.3.3) we have

$$\frac{1}{2}\int_\Omega \mathcal{A}_n(x, w_{m_k})|\nabla T(w_{m_k})|^2 - \lambda \int_\Omega F(T(w_{m_k})) \leq -\frac{t_\epsilon^2 \delta}{2}.$$

Using $\mathcal{A}_n \geq a_0$, we have

$$\frac{a_0}{2}\int_\Omega |\nabla T(w_{m_k})|^2 - \lambda \int_\Omega F(T(w_{m_k})) \leq -\frac{t_\epsilon^2 \delta}{2}.$$

Using the lower semi continuity for the weak convergence of H_0^1 norm and the continuity of F, we have

$$\frac{a_0}{2}\int_\Omega |\nabla p|^2 - \lambda \int_\Omega F(p) \leq -\frac{t_\epsilon^2 \delta}{2} < 0.$$

This proves that p cannot be trivial. Now considering the left hand side of (4.3.25), we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}_n(x, w_{k_m})\nabla T(w_{k_m}) \cdot \nabla v \\
&\quad = \int_\Omega \{\mathcal{A}_n(x, w_{k_m}) - \mathcal{A}_n(x, w)\}\nabla T(w_{k_m}) \cdot \nabla v + \int_\Omega \mathcal{A}_n(x, w)\nabla T(w_{k_m}) \cdot \nabla v \\
&\qquad := I_1^m + I_2^m.
\end{aligned}$$

We first estimate the term I_1^m.

$$\begin{aligned}
|I_1^m| &\leq \int_\Omega |\mathcal{A}_n(x, w_{k_m}) - \mathcal{A}_n(x, w)||\nabla T(w_{k_m})||\nabla v| \\
&\leq \left(\int_\Omega |\mathcal{A}_n(x, w_{k_m}) - \mathcal{A}_n(x, w)|^2 |\nabla v|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla T(w_{k_m})|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Now using $\int_\Omega |\nabla T(w_{k_m})|^2 \leq C^2$, where $C = \frac{\lambda ||f||_\infty}{a_0}\sqrt{\frac{|\Omega|}{\lambda_1}}$, we get

$$|I_1^m| \leq C\left(\int_\Omega |\mathcal{A}_n(x, w_{k_m}) - \mathcal{A}_n(x, w)|^2 |\nabla v|^2\right)^{\frac{1}{2}}. \tag{4.3.27}$$

From (4.3.24) we have

$$w_n \to w \text{ a.e. } x \in \Omega.$$

Since $w_n \leq \theta$ forall n, this implies from Lebesgue theorem that

$$w_n \to w \text{ in } L^p(\Omega),\ \forall p.$$

Therefore from (4.0.2) we have

$$\mathcal{A}_n(x, w_{k_m}) \to \mathcal{A}_n(x, w) \quad \text{a.e. } x \in \Omega.$$

Also

$$|\mathcal{A}_n(x, w_{k_m}) - \mathcal{A}_n(x, w)|^2 |\nabla v|^2 \leq 4a_\infty^2 |\nabla v|^2, \quad \forall v \in H_0^1(\Omega).$$

Now since $4a_\infty^2 |\nabla v|^2 \in L^1(\Omega)$, we can pass trough the limit in (4.3.27) using dominated convergence theorem to get

$$I_1^m \to 0.$$

Also by (4.3.26),

$$I_2^m \to \int_\Omega \mathcal{A}_n(x, w) \nabla p \cdot \nabla v.$$

Therefore

$$\int_\Omega \mathcal{A}_n(x, w_{k_m}) \nabla T(w_{k_m}) \cdot \nabla v \to \int_\Omega \mathcal{A}_n(x, w) \nabla p \cdot \nabla v.$$

Using Lipschitz continuity of f and (4.3.25), we have

$$\int_\Omega f(T(w_{k_m}))v \to \int_\Omega f(p)v.$$

Therefore we have

$$\int_\Omega \mathcal{A}_n(x, w) \nabla p \cdot \nabla v = \lambda \int_\Omega f(p)v, \qquad \forall v \in H_0^1(\Omega).$$

Hence from the above equation we get $T(w) = p$ and since the possible limit is unique we have,

$$T(w_k) \to T(w), \quad \text{in } L^2(\Omega).$$

This completes the proof of continuity of T.

4. Compactness of T

Let $w_n \to w$ in $L^2(\Omega)$. We want to show that

$$T(w_k) \to T(w) \ \text{ in } H_0^1(\Omega).$$

Compactness of the mapping T then follows from the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$. The Euler-Lagrange equation satisfied by $T(w_k)$ is

$$\int_\Omega \mathcal{A}_n(x, w_k) \nabla T(w_k) \cdot \nabla v = \lambda \int_\Omega f(T(w_k))v, \quad \forall v \in H_0^1(\Omega). \tag{4.3.28}$$

That is

$$\int_\Omega \mathcal{A}_n(x, w_k) \nabla (T(w_k) - T(w)) \cdot \nabla v + \int_\Omega (\mathcal{A}_n(x, w_k) - \mathcal{A}_n(x, w)) \nabla T(w) \cdot \nabla v$$
$$= \lambda \int_\Omega f(T(w_k))v - \int_\Omega \mathcal{A}_n(x, w) \nabla T(w) \cdot \nabla v = \lambda \int_\Omega \{f(T(w_k)) - f(T(w))\} v.$$

Using $v = T(w_k) - T(w)$, (4.0.3) and Lipschitz continuity of f, we have

$$a_0 \int_\Omega |\nabla (T(w_k) - T(w))|^2 \leq \lambda L \int_\Omega |T(w_k) - T(w)|^2 + \int_\Omega |\mathcal{A}_n(x, w_k) - \mathcal{A}_n(x, w)||\nabla (T(w_k) - T(w))||\nabla T(w)|. \quad (4.3.29)$$

Application of Young's inequality gives

$$a_0 \int_\Omega |\nabla (T(w_k) - T(w))|^2 \leq \lambda L \int_\Omega |T(w_k) - T(w)|^2 + \frac{a_0}{2} \int_\Omega |\nabla (T(w_k) - T(w))|^2 + \frac{2}{a_0} \int_\Omega |\mathcal{A}_n(x, w_k) - \mathcal{A}_n(x, w)|^2 |\nabla T(w)|^2.$$

This implies that

$$\frac{a_0}{2} \int_\Omega |\nabla (T(w_k) - T(w))|^2 \leq \lambda L \int_\Omega |T(w_k) - T(w)|^2 + \frac{2}{a_0} \int_\Omega |\mathcal{A}_n(x, w_k) - \mathcal{A}_n(x, w)|^2 |\nabla T(w)|^2.$$

The first integral on the RHS of the above inequality tends to 0 from the last part and the second integral converges to 0, following a similar argument, that shows the convergence of the I_m^1 in (4.3.27).

Schauder fixed point theorem.

The map $T : \mathcal{K} \to \mathcal{K}$ is compact and $\mathcal{K}$ is closed, convex set in $L^2(\Omega)$. By Schauder fixed point theorem T has a fixed point. Since the function 0 doesn't belongs to $T(\mathcal{K})$, the above obtained fixed point is nontrivial. This finishes the proof of the theorem. □

Let u_n denotes the nontrivial solution obtained for the problem (4.3.6) for large λ. In the above theorem it should be noted that the choice of λ doesn't depends on n. Now the goal is to pass through the limit in (4.3.6) and obtain a nontrivial solution for the problem (1.2.8).

Theorem 4.3.2. *If* $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \epsilon}$, *problem* (1.2.8) *admits a positive solution under the assumptions* (4.0.1)*,* (4.0.3) *and* (4.3.4)*.*

Proof. First of all it is clear that

$$\mathcal{A}_n(x, u) \to \mathcal{A}(x, u)$$

for each fixed $x \in \Omega$ and $u \in L^2(\Omega)$. This follows from the property of mollification. The equation satisfied by u_n is written as, for fixed $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \mathcal{A}_n(x, u_n) \nabla u_n \cdot \nabla \phi = \lambda \int_\Omega f(u_n) \phi. \quad (4.3.30)$$

Using $\phi = u_n$ in (4.3.30), we get

$$\int_\Omega \mathcal{A}_n(x,u_n)|\nabla u_n|^2 = \lambda \int_\Omega f(u_n)u_n.$$

Since $\mathcal{A}_n \geq a_0$, we have

$$a_0 \int_\Omega |\nabla u_n|^2 \leq \lambda ||f||_\infty \int_\Omega u_n \leq \lambda ||f||_\infty |\Omega|^{\frac{1}{2}} \left(\int_\Omega |u_n|^2 \right)^{\frac{1}{2}}.$$

Now using Poincaré's inequality, we get

$$\left(\int_\Omega |\nabla u_n|^2 \right)^{\frac{1}{2}} \leq \frac{\lambda ||f||_\infty |\Omega|^{\frac{1}{2}}}{a_0\sqrt{\lambda_1}}.$$

Thus for a subsequence, which we again denote by $\{n\}$, there exist $u_0 \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega)$$

and strongly in $L^2(\Omega)$. The theorem will be proved if we show that $u_0 \in L^p(\Omega),\ \forall p \geq 1$, nontrivial and for fixed ϕ, the following holds

$$\int_\Omega \mathcal{A}(x,u_0)\nabla u_0 \cdot \nabla\phi = \lambda \int_\Omega f(u_0)\phi.$$

For all n, one has from Lemma (4.3.4) that

$$0 < u_n \leq \theta, \text{ a.e. } x \in \Omega.$$

This implies that $0 \leq u_0 \leq \theta$ a.e. $x \in \Omega$ from almost every where convergence of u_n to u_0. Hence $u_0 \in L^p(\Omega),\ \forall p \geq 1$.

Now let us start from the left hand side of (4.3.30).

$$\begin{aligned}\int_\Omega \mathcal{A}_n(x,u_n)\nabla u_n \cdot \nabla\phi &= \int_\Omega \{\mathcal{A}_n(x,u_n) - \mathcal{A}_n(x,u_0)\}\nabla u_n \cdot \nabla\phi \\ &+ \int_\Omega \{\mathcal{A}_n(x,u_0) - \mathcal{A}(x,u_0)\}\nabla u_n \cdot \nabla\phi + \int_\Omega \mathcal{A}(x,u_0)\nabla u_n \cdot \nabla\phi \\ &:= I_n^1 + I_n^2 + I_n^3.\end{aligned}$$

Clearly from the weak convergence of u_n to u_0, we have

$$I_n^3 \to \int_\Omega \mathcal{A}(x,u_0)\nabla u_0 \cdot \nabla\phi.$$

We claim that both I_n^1 and I_n^2 converges to 0. First we will estimate the term I_n^1.

$$\begin{aligned}|I_n^1| &\leq \int_\Omega |\mathcal{A}_n(x,u_n) - \mathcal{A}_n(x,u_0)||\nabla u_n||\nabla\phi| \\ &\leq \int_\Omega \left(\int_{B(0,\frac{1}{n})} \left|\mathcal{A}(x-y,u_n) - \mathcal{A}(x-y,u_0)\right|\psi_{\frac{1}{n}}\,dy \right) |\nabla u_n|\ |\nabla\phi|.\end{aligned}$$

Using $|\mathcal{A}(x-y,u_n)-\mathcal{A}(x-y,u_0)| \leq ||\mathcal{A}(x,u_n)-\mathcal{A}(x,u_0)||_\infty$ and $\int_{B(0,\frac{1}{n})}\psi_{\frac{1}{n}}=1$, we get

$$\begin{aligned}|I_n^1| &\leq ||\mathcal{A}(x,u_n)-\mathcal{A}(x,u_0)||_\infty \int_\Omega |\nabla u_n||\nabla\phi| \\ &\leq ||\mathcal{A}(x,u_n)-\mathcal{A}(x,u_0)||_\infty \; ||\nabla u_n||_{L^2}||\nabla\phi||_{L^2} \\ &\leq C||\nabla\phi||_{L^2}||\mathcal{A}(x,u_n)-\mathcal{A}(x,u_0)||_\infty,\end{aligned}$$

where $C=\frac{\lambda||f||_\infty|\Omega|^{\frac{1}{2}}}{a_0\sqrt{\lambda_1}}$.

Now as $u_n\to u_0$ in $L^p(\Omega)$, this implies from (4.3.4) that

$$||\mathcal{A}(x,u_n)-\mathcal{A}(x,u_0)||_\infty \to 0$$

and hence

$$I_n^1\to 0.$$

Let us now estimate the term I_n^2.

$$\begin{aligned}|I_n^2| &\leq \int_\Omega |\mathcal{A}_n(x,u_0)-\mathcal{A}(x,u_0)||\nabla\phi||\nabla u_n| \\ &\leq ||\left\{\mathcal{A}_n(x,u_0)-\mathcal{A}(x,u_0)\right\}|\nabla\phi|||_{L^2}||\nabla u_n||_{L^2} \\ &\leq C||\left\{\mathcal{A}_n(x,u_0)-\mathcal{A}(x,u_0)\right\}|\nabla\phi|||_{L^2}.\end{aligned}$$

As mentioned above in the beginning of the proof, we have

$$\mathcal{A}_n(x,u_0)\to\mathcal{A}(x,u_0) \quad \text{a.e. } x\in\Omega.$$

Also

$$|\mathcal{A}_n(x,u_0)-\mathcal{A}(x,u_0)|^2|\nabla\phi|^2 \leq 4a_\infty^2|\nabla\phi|^2$$

where $|\nabla\phi|^2\in L^1(\Omega)$. Therefore by dominated convergence theorem, we have

$$||\left\{\mathcal{A}_n(x,u_0)-\mathcal{A}(x,u_0)\right\}|\nabla\phi|||_{L^2}\to 0.$$

Thus we have proved that

$$\int_\Omega \mathcal{A}_n(x,u_n)\nabla u_n\cdot\nabla\phi \to \int_\Omega \mathcal{A}(x,u_0)\nabla u_0\cdot\nabla\phi.$$

The right hand side of (4.3.30) can be written as

$$\int_\Omega f(u_n)\phi=\int_\Omega\{f(u_n)-f(u_0)\}\phi+\int_\Omega f(u_0)\phi.$$

Now as $n\to\infty$, we have

$$|\int_\Omega\left\{f(u_n)-f(u_0)\right\}\phi| \leq L\int_\Omega |u_n-u_0||\phi| \leq L||u_n-u_0||_{L^2}||\phi||_{L^2}\to 0.$$

Hence we have

$$\int_\Omega f(u_n)\phi \to \int_\Omega f(u_0)\phi.$$

The proof will be completed once we show u_0 is not identically equals to 0. For proving that we use the weak lower semi continuity of the H_0^1 norm, the Lipschitz continuity of F and the energy estimates done in Lemma 4.3.3. We have

$$a_0\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega F(u_n) \le \int_\Omega \mathcal{A}_n(x,u_n)|\nabla u_n|^2 - \lambda\int_\Omega F(u_n) \le -\frac{t_\epsilon^2\delta}{2}.$$

Again since $u_n \rightharpoonup u_0$, we have

$$a_0\int_\Omega |\nabla u_0|^2 - \lambda\int_\Omega F(u_0) \le \liminf_{n\to\infty} a_0\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega F(u_n).$$

Combining the last two equations we get

$$a_0\int_\Omega |\nabla u_0|^2 - \lambda\int_\Omega F(u_0) \le -\frac{t_\epsilon^2\delta}{2} < 0$$

which is impossible if u_0 identically vanishes. In particular, if u_0 is not trivial then it has to be strictly positive in Ω. This again follows from the maximum principle. □

Since the choice of $\epsilon,\ \delta > 0$ is kept arbitrary, we have the following theorem.

Theorem 4.3.3. *Under the assumptions* (4.0.1), (4.0.3) *and* (4.3.4), *if* $\lambda > \frac{a_\infty\lambda_1}{f'(0)}$, *problem* (1.2.8) *admits a positive solution.*

A Liouville type theorem.

To end this section we present a Liouville type result. Non locality in this case will be expressed as sum of two monotone operators. For $p > 1$, denote by

$$\triangle_p u := -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

the $p-$ Laplace operator.

Theorem 4.3.4. *Suppose* $p \ge 2$ *and* $n \le 2$ *(i.e.* 1 *or* 2*), then any bounded solution of*

$$-\triangle_p u - \triangle u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

is constant, where $\mathcal{D}'(\mathbb{R}^n)$ *denotes the set of distributions in* $\mathbb{R}^n$.

Proof. We suppose that u is a weak solution to

$$\int_{\mathbb{R}^n} \nabla u\cdot\nabla\phi + \int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u\cdot\nabla\phi = 0 \quad \forall\phi\in\mathcal{D}(\mathbb{R}^n) \tag{4.3.31}$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of C^∞ functions with compact support. Suppose B_r denotes the ball of radius r centered at 0. Let ρ be a radial function satisfying $0 \leq \rho \leq 1$, $\rho = 1$ on $B_{\frac{1}{2}}$, $\rho = 0$ outside B_1 and $|\nabla \rho| \leq C$, for some constant $C > 0$.

We set $\rho_r := \rho(\frac{x}{r})$ and consider in (4.3.31), $\phi = \rho_r^p u$. Setting $D_r := B_r \setminus B_{\frac{r}{2}}$ we get

$$\int_{B_r} \nabla u \cdot \nabla \left\{\rho_r^p u\right\} + |\nabla u|^{p-2} \nabla u \cdot \nabla \left\{\rho_r^p u\right\} = 0$$

that is

$$\int_{B_r} \rho_r^p \left\{|\nabla u|^2 + |\nabla u|^p\right\} = -p \int_{D_r} \rho_r^{p-1} u \nabla u \cdot \nabla \rho_r + p \int_{D_r} \rho_r^{p-1} u |\nabla u|^{p-2} \nabla u \cdot \nabla \rho_r.$$

Now if u is bounded , that is $|u| \leq M$ for some positive constant M, we have

$$\int_{B_r} \rho_r^p \left\{|\nabla u|^2 + |\nabla u|^p\right\} \leq pM \int_{D_r} \rho_r^{\frac{p}{2}} |\nabla u| |\nabla \rho_r| \rho_r^{\frac{p}{2}-1} + pM \int_{D_r} \rho_r^{p-1} |\nabla u|^{p-1} |\nabla \rho_r|. \quad (4.3.32)$$

Using the fact that $p \geq 1$ and the assumptions made on ρ_r, we get

$$\int_{B_r} \rho_r^p \left\{|\nabla u|^2 + |\nabla u|^p\right\} \leq pM \int_{D_r} \rho_r^{\frac{p}{2}} |\nabla u| |\nabla \rho_r| + pM \int_{D_r} \rho_r^{p-1} |\nabla u|^{p-1} |\nabla \rho_r|. \quad (4.3.33)$$

Using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we derive

$$\int_{B_r} \rho_r^p \left\{|\nabla u|^2 + |\nabla u|^p\right\} \leq \left(\int_{D_r} \rho_r^p |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{D_r} |\nabla \rho_r|^2\right)^{\frac{1}{2}} + \left(\int_{D_r} \rho_r^p |\nabla u|^p\right)^{\frac{1}{q}} \left(\int_{D_r} |\nabla \rho_r|^p\right)^{\frac{1}{p}} \quad (4.3.34)$$

Since $n \leq 2$, we get for r large enough and for some constant $C, C_1, C_2 > 0$,

$$\int_{D_r} |\nabla \rho_r|^2 \leq C \int_{D_r} \frac{1}{r^2} \leq C_1$$

and

$$\left(\int_{D_r} |\nabla \rho_r|^p\right)^{\frac{1}{p}} \leq C \left(\int_{D_r} \frac{1}{r^p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-1} \leq C_2.$$

Define

$$X_r := \int_{B_r} \rho_r^p \left\{|\nabla u|^2 + |\nabla u|^p\right\}.$$

Therefore from (4.3.34), for some constant $K > 0$, we have

$$X_r \leq K\left(\int_{D_r} \rho_r^p |\nabla u|^2\right)^{\frac{1}{2}} + K\left(\int_{D_r} \rho_r^p |\nabla u|^p\right)^{\frac{1}{q}} \tag{4.3.35}$$

That is

$$X_r \leq K X_r^{\frac{1}{2}} + K X_r^{\frac{1}{q}}$$

and X_r is bounded independently of r. Since

$$\int_{B_{\frac{r}{2}}} |\nabla u|^2, \quad \int_{B_{\frac{r}{2}}} |\nabla u|^p \leq X_r$$

these two quantities are bounded independently of r and therefore have a limit as $r \to \infty$. Since $\rho_r = 1$ on $B_{\frac{r}{2}}$ and $|\rho_r| \leq 1$, we get from (4.3.34) that

$$\int_{B_{\frac{r}{2}}} |\nabla u|^2 + |\nabla u|^p \leq K\left(\int_{B_r} |\nabla u|^2 - \int_{B_{\frac{r}{2}}} |\nabla u|^2\right)^{\frac{1}{2}} + K\left(\int_{B_r} |\nabla u|^p - \int_{B_{\frac{r}{2}}} |\nabla u|^p\right)^{\frac{1}{q}} \to 0$$

as $r \to \infty$. This implies that u is constant. $\square$

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