University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 168

Mixed Equilibrium in a Pure Location Game: The Case of n ≥ 4 Firms

Christian Ewerhart

July 2014

Mixed Equilibrium in a Pure Location Game: The Case of $n \geq 4$ Firms*

Christian Ewerhart**

This version: July 18, 2014

Abstract. The Hotelling game of pure location allows interpretations in spatial competition, political theory, and professional forecasting. In this paper, the doubly symmetric mixed-strategy equilibrium for $n \geq 4$ firms is characterized as the solution of a well-behaved boundary value problem. The analysis suggests that, in contrast to the cases $n = 3$ and $n \to \infty$, the equilibrium for a finite number of $n \geq 4$ firms tends to overrepresent locations at the periphery of its support interval. Moreover, in the class of examples considered, an increase in the number of firms universally leads to a wider range of location choices and to a more dispersed distribution of individual locations. The results are used to comment on the potential benefit of competition in forecasting markets.

*) The problem studied below was kindly suggested to me by Marco Ottaviani. For useful discussions, I am grateful to Pavlo Blavatskyy and Jean Gabszewicz.

**) University of Zurich, Department of Economics, Schönberggasse 1, 8001 Zürich, Switzerland. Phone +41-44-6343733. Fax +41-44-6344978. Email: christian.ewerhart@econ.uzh.ch.

1. Introduction

The present paper deals with what is known as the Hotelling (1929) model of pure location, in which each of a given finite number of firms simultaneously and independently chooses a location on the unit interval so as to maximize its expected market share.[1] While traditional applications related to spatial competition and political theory remain important, the framework has more recently been recognized as capturing also strategic aspects of the competition between professional forecasters (Laster et al., 1999; Ottaviani and Sørensen, 2006; Marinovic et al., 2013).

The game-theoretic analysis of the pure location game was initially concerned mainly with equilibria in pure strategies (Lerner and Singer, 1937; Eaton and Lipsey, 1975).[2] However, for $n = 3$, there is no pure-strategy equilibrium. Moreover, in cases where the pure-strategy equilibrium exists, the equilibrium typically vanishes if the density function associated with the underlying distribution of consumer preferences is either strictly convex or strictly concave (Osborne and Pitchik, 1986). Finally, pure-strategy equilibria may sometimes be harder to coordinate upon (see, e.g., Xefteris, 2014).[3] It should, therefore, not come as a surprise that attention has also been devoted to the analysis of mixed-strategy equilibria.

Of particular interest has been the so-called doubly symmetric equilib-

[1] The pure location model is a simplified variant of Hotelling's original set-up (cf. Chamberlin, 1938, Appendix C). For an introduction to the literature on spatial competition, see Gabszewicz and Thisse (1992).

[2] See also Graitson (1982), Denzau et al. (1985), and Cox (1987).

[3] As an illustration, consider the location model with $n = 5$ firms. In the pure-strategy equilibrium, two firms locate at the first sextile, two others at the fifth sextile, and one firm at the market center. Thus, the market share of the central firm is twice as large as that of its competitors, making coordination on the pure-strategy equilibrium potentially difficult.

rium in which each firm uses the same mixed strategy and in which, in addition, the distribution of individual choices that represents the mixed equilibrium strategy is symmetric with respect to the midpoint of the location interval. Shaked (1982) showed that the doubly symmetric mixed-strategy equilibrium with $n = 3$ firms is unique, and uniform on the interval $[\frac{1}{4}, \frac{3}{4}]$. For general $n \geq 3$, Osborne and Pitchik (1986) proved that there exists an atomless doubly symmetric mixed-strategy equilibrium, where the support is necessarily an interval if consumer preferences are distributed uniformly. Moreover, as the number of firms n goes to infinity, any convergent sequence of twice continuously differentiable equilibrium distributions must ultimately approach the underlying distribution of customer preferences. Despite these general insights, however, a more qualitative description of the mixed-strategy equilibrium for $n \geq 4$ firms remained elusive.[4]

The contribution of this paper is a re-formulation of the equilibrium condition for the location game with $n \geq 4$ firms in terms of a well-behaved boundary value problem. Based on the resulting characterization of the equilibrium distribution, a numerical solution is obtained for small values of n by studying trajectories that depart from the midpoint of the location interval. It turns out that, in all cases considered, the doubly symmetric equilibrium involves a tendency to overrepresent locations at the periphery of its support interval. Moreover, an increase in the number

[4] Osborne and Pitchik (1986, p. 227) write: "Even if C is uniform this is a difficult problem — (2) is a nonlinear integral equation, about which little in general is known." Also the brute-force approach via discretization of the strategy space has remained ineffective. See, e.g., Huck et al. (2002) for the case of $n = 4$ firms.

of firms universally leads to a wider range of locations that are used in equilibrium, and to a more dispersed distribution of individual choices. Based upon those findings, we comment on the potential impact of policy measures that mitigate competition among professional forecasters.

The remainder of the paper is structured as follows. Section 2 reviews the location game. Section 3 discusses the first-order condition. The equilibrium is characterized and discussed in Section 4. Section 5 concludes. Two appendices provide details on the numerical procedure as well as technical proofs, respectively.

2. Review of the location game

This section introduces the set-up, and reviews some well-known results regarding the doubly symmetric mixed-strategy equilibrium of the location game. The exposition will follow the literature on professional forecasting cited above. However, alternative interpretations in spatial competition and political theory will be immediate.

A public authority wishes to collect information about the value of a macroeconomic indicator $\xi \in \mathbb{R}$, which is ex-ante distributed according to an uninformative uniform prior.[5] A finite number $n \geq 3$ of professional forecasters is assumed to have access to privileged information about ξ. Specifically, nature draws a value $\xi_0 \in \mathbb{R}$, which is observable information to any forecaster but not to the public authority, such that the true state of the world ξ lies somewhere in the interval $[\xi_0, \xi_0 + 1]$. Ex-post, i.e., after the public authority has taken any decisions on the basis of the solicited

[5]The assumption of an improper prior simplifies the chosen interpretation, and may be dropped without loss in alternative interpretations.

estimates, ξ becomes publicly known. To elicit the forecasters' private information, the public authority organizes a contest in which the forecaster whose estimate turns out to be closest to the true state of the world receives a prize (where the value of the prize is normalized to unity). Submitting an estimate is costless, yet any forecaster may submit at most one estimate.

As set forth more generally in Osborne and Pitchik (1986), the expected payoff of forecaster 1 when it chooses the estimate $\xi_0 + z$ and each of the competitors $2, ..., n$ randomizes according to a distribution F is given as

$$\begin{aligned}
\Pi(z) \quad = \quad & (n-1) \int_z^1 f(y)(1-F(y))^{n-2} \frac{z+y}{2} dx \\
& + \sum_{k=1}^{n-2} \binom{n-1}{k} k(n-k-1) \\
& \cdot \int_0^z \int_z^1 f(x) f(y) F(x)^{k-1} (1-F(y))^{n-k-2} \frac{y-x}{2} dy dx \\
& + (n-1) \int_0^z f(x) F(x)^{n-2} (1 - \frac{z+x}{2}) dx,
\end{aligned} \tag{1}$$

where $f = F'$ denotes the density of the equilibrium distribution.[6] The right-hand side of equation (1) obviously reflects the variety of possible scenarios for the representative forecaster: Ending up below all $n-1$ competing estimates; then, for $k = 1, ..., n-2$, having a total of k competing estimates below and $n-k-1$ competing estimates above; or, finally, ending up above all other estimates.

For a mixed-strategy equilibrium to be *doubly symmetric*, it is required that (i) all forecasters use the same mixed strategy F, and (ii) the strategy F is unchanged when reflected at the midpoint of the location interval, i.e.,

[6] Only distributions allowing a density will be considered in this paper. Moreover, for convenience, all functions depending on a location will be treated as functions on the unit interval, i.e., as functions of z rather than of $\xi_0 + z$.

$F(1-z) = 1 - F(z)$ for all $z \in [0,1]$. The following result summarizes what is known about the doubly symmetric mixed-strategy equilibrium of the location game in the uniform case.

Theorem 1. *For $n \geq 3$, there exists a doubly symmetric mixed-strategy equilibrium $F = F_n$, where the distribution F has support $[\alpha, 1-\alpha]$ for some $\alpha = \alpha_n \in [0, \frac{1}{2})$. For $n = 3$, the equilibrium is unique, and such that individual estimates are distributed uniformly over the interval $[\xi_0 + \frac{1}{4}, \xi_0 + \frac{3}{4}]$. Moreover, if F_n is twice continuously differentiable and converges uniformly (including in terms of its first and second derivatives) to some twice continuously differentiable F_∞, then F_∞ induces a uniform distribution of estimates on $[\xi_0, \xi_0 + 1]$.*

Proof. See Osborne and Pitchik (1986, Prop. 3 and 4). The case $n = 3$ is treated in Shaked (1982). □

3. Discussion of the first-order condition

For any given number of competitors $k \geq 1$, consider the function

$$G_k(z) = \int_\alpha^z F(x)^k dx. \tag{2}$$

As will become clear from the proof of the lemma below, $G_k(z)$ corresponds to the average distance between z and the highest of k lower estimates.[7] Using this notation, marginal expected payoffs of the representative forecaster may be written in a relatively compact way.

Lemma 1. *On the support of F, forecaster 1's marginal expected payoffs are*

[7]Similarly, provided F is symmetric, $G_k(1-z)$ corresponds to the average distance between z and the lowest of k higher estimates.

given as

$$\Pi'(z) = -\varphi(z) + \varphi(1-z) + f(z)\left\{\psi(z) - \psi(1-z)\right\}, \tag{3}$$

where $\varphi(z) = F(z)^{n-1}/2$ *and*

$$\begin{aligned}\psi(z) \quad &= \quad \frac{1}{2}\sum_{k=1}^{n-2}\binom{n-1}{k} kF(z)^{k-1}G_{n-k-1}(1-z) \\ &\quad +(n-1)F(z)^{n-2}(1-z).\end{aligned} \tag{4}$$

Proof. See Appendix B. □

Condition (3) captures two pairs of mirror-image effects resulting from a marginal increase in forecaster 1's estimate. First, there is a marginal cost $\varphi(z)$, due to a reduced probability of winning the contest in the scenario in which forecaster 1's estimate is the highest, and a mirror-image marginal benefit $\varphi(1-z)$, due to an increased probability of winning in the scenario in which forecaster 1's estimate is the lowest. Second, there is a marginal benefit, represented by $\psi(z)$ and measured in units of the density, due to an increased probability that the estimates of any given set of competitors end up below forecaster 1's estimate, and a mirror-image cost represented by $\psi(1-z)$, due to a reduced probability that the estimates of any complementary set of competitors end up above forecaster 1's estimate. The doubly symmetric mixed-strategy equilibrium just balances these two pairs of effects at any point of the support interval.

Setting marginal payoffs to zero, one finds the key equation

$$f(z) = \frac{\varphi(z) - \varphi(1-z)}{\psi(z) - \psi(1-z)}. \tag{5}$$

An obvious obstacle to interpreting equation (5) as a differential equation in the usual meaning of the term is that the functions φ and ψ are evaluated at both z and $1-z$. This problem is addressed by a functional equation that is stated in the following lemma.

Lemma 2. *The functions $G_1, G_2, ...$ satisfy the functional equation*

$$G_k(1-z) = C_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} G_k(z) \tag{6}$$

for any integer $k \geq 1$, with constants

$$C_k = \tfrac{1}{2} + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} G_m(\tfrac{1}{2}) + \left\{1 + (-1)^k\right\} G_k(\tfrac{1}{2}). \tag{7}$$

Proof. See Appendix B. □

4. Equilibrium characterization

After these preparations, the equilibrium distribution can be characterized as the solution of a boundary value problem with a relatively simple structure.

Theorem 2. *Let $n \geq 3$. Then there exists a function $\Phi_n : \mathbb{R}^{2n-2} \to \mathbb{R} \cup \{+\infty, -\infty\}$ such that any doubly symmetric mixed-strategy equilibrium $F = F_n$ of the location game with n firms corresponds to the first element of a tuple*

$$(\widetilde{F}, \widetilde{G}_1, ..., \widetilde{G}_{n-2}, \widetilde{C}_1, ..., \widetilde{C}_{n-2}, \widetilde{\alpha}), \tag{8}$$

composed of functions $\widetilde{F}, \widetilde{G}_1, ..., \widetilde{G}_{n-2} : [\widetilde{\alpha}, 1-\widetilde{\alpha}] \to \mathbb{R}$ and constants $\widetilde{C}_1, ...,$ $\widetilde{C}_{n-2} \in \mathbb{R}$, $\widetilde{\alpha} \in [0, \frac{1}{2})$, such that (8) satisfies the system of ordinary first-order *differential equations*

$$\widetilde{F}'(z) = \Phi_n(\widetilde{F}(z), \widetilde{G}_1(z), ..., \widetilde{G}_{n-2}(z), \widetilde{C}_1, ..., \widetilde{C}_{n-2}, z), \quad (9)$$

$$\widetilde{G}_k'(z) = \widetilde{F}(z)^k \qquad (k = 1, ..., n-2), \quad (10)$$

as well as the boundary conditions $\widetilde{F}(\widetilde{\alpha}) = \widetilde{G}_1(\widetilde{\alpha}) = ... = \widetilde{G}_{n-2}(\widetilde{\alpha}) = 0$, $\widetilde{F}(\frac{1}{2}) = \frac{1}{2}$, *and*

$$\widetilde{C}_k = \frac{1}{2} + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} \widetilde{G}_m(\tfrac{1}{2}) + \left\{1 + (-1)^k\right\} \widetilde{G}_k(\tfrac{1}{2}) \quad (11)$$

for $k = 1, ..., n-2$. *Conversely, if the first component* $\widetilde{F}$ *of a solution of the boundary value problem stated above is restricted to be monotone increasing and symmetric with respect to a reflection at* $z = \frac{1}{2}$, *then* $\widetilde{F}$ *represents a doubly symmetric mixed-strategy equilibrium of the location game.*

Proof. See Appendix B. □

The proof of Theorem 2 is constructive. Specifically, the function Φ_n used in the characterization simply corresponds to the right-hand side of equation (5).

In the case $n = 3$, one can check that the two-dimensional system (9-10) reduces to the differential equation

$$\widetilde{F}'(z) = \frac{2\widetilde{F}(z) - 1}{4\widetilde{F}(z) - 6z + 1}, \quad (12)$$

with boundary conditions $\widetilde{F}(\widetilde{\alpha}) = 0$ and $\widetilde{F}(\frac{1}{2}) = \frac{1}{2}$.[8] As shown by Shaked (1982), equation (12) has precisely one solution satisfying $\widetilde{F}(\frac{1}{2}) = \frac{1}{2}$. Thus,

[8]More generally, it can be seen as a consequence of Lemma 2 that, for n odd, the function Φ_n defined in the proof of Theorem 2 does not depend on $\widehat{G}_{n-2}$. Thus, for n odd, the dimension of the system (9-10) reduces to $n-2$. For n even, however, this simplification is not possible.

the unique solution of the boundary value problem is $\widetilde{F}(z) = 2z - \frac{1}{2}$, with $\widetilde{\alpha} = \frac{1}{4}$.

In cases where $n \geq 4$, the differential equation (9) becomes more involved, so that an explicit solution is not readily available. In particular, there is no obvious substitution that would simplify the equation.[9] We also checked that, in general, there is no distribution with a quadratic density function that solves (9). However, the characterization paves the way to a numerical computation of the equilibrium distribution.[10]

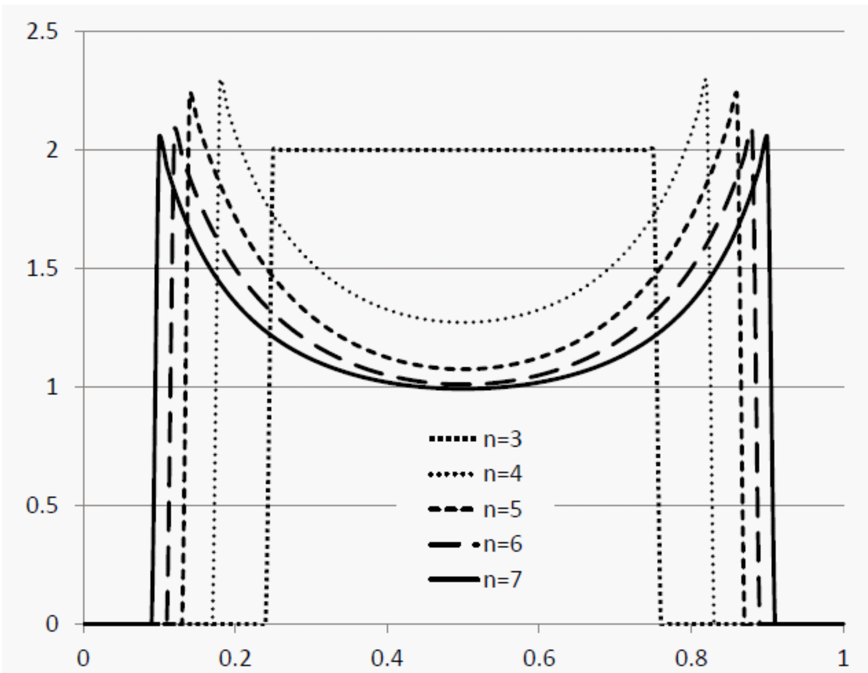

Figure 1. The density of the doubly symmetric mixed-strategy equilibrium for $n = 3, ..., 7$ firms.

[9]E.g., in the case $n = 4$, an application of Shaked's (1982) substitution $h(z) = (\widetilde{F}(z) - \frac{1}{2})/(z - \frac{1}{2})$ does not lead to a substantial simplification of the three-dimensional system (9-10).

[10]A description of the numerical approach is provided in Appendix A.

Figure 1 shows the doubly symmetric mixed-strategy equilibrium for $n \in \{3, 4, 5, 6, 7\}$. As can be seen, the numerical density $f(z)$ is strictly M-shaped in all calculated examples where $n \geq 4$. This finding is somewhat puzzling because it implies that the equilibrium for a finite number of forecasters $n \geq 4$ differs qualitatively from the equilibria in the well-understood cases $n = 3$ and $n \to \infty$.[11]

A second observation from Figure 1 is that, as n increases, the estimates submitted by any individual forecaster cover a larger support, and become more dispersed (in the sense of a mean-preserving spread). The widening of the support is intuitive, however, because an increase in the population density with unchanged α would probably reduce expected payoffs more substantially in the interior of the support than at the boundary.

5. Concluding remarks

In this paper, we characterized the doubly symmetric mixed-strategy equilibrium in the Hotelling game of pure location for $n \geq 4$ firms and subsequently used the characterization to compute the equilibrium for small values of n. It turned out that, in the cases considered, increasing the number of professional forecasters lowers the quality of the individual forecasts even though the costs of providing an accurate estimate were assumed to be negligible.

We conclude that competition is not universally beneficial in forecast-

[11]Huck et al. (2002) hypothesize that the probability of getting "squeezed" between two competitors should be relatively small to make locations at the center as attractive as locations at the periphery. However, that intuition does not really explain our findings because the same intuition should apply likewise in the cases $n = 3$ and $n \to \infty$, where the equilibrium is, however, not markedly M-shaped.

ing markets, which might explain why, occasionally, competition at the level of individual estimates is explicitly avoided. This is the case, for example, for the Joint Economic Forecast prepared twice yearly since 1950 by leading economic research institutes on behalf of the German Ministry of Economic Affairs.[12]

Appendix A: Details on the numerical approach

An effective way to approximate the equilibrium is a "shooting method" that works with trajectories starting at the midpoint of the location interval.[13] For intuition, note that the starting point of the trajectory at $z = \frac{1}{2}$ is an $(n-1)$-dimensional vector

$$X_0 = (F(\tfrac{1}{2}), G_1(\tfrac{1}{2}), ..., G_{n-2}(\tfrac{1}{2})), \tag{13}$$

whose first component is fixed through the symmetry condition $F(\frac{1}{2}) = \frac{1}{2}$, whereas the remaining components $G_1(\frac{1}{2}), ..., G_{n-2}(\frac{1}{2})$ are initially unknown. Any given approximation for X_0 may then be improved by adapting the values $G_1(\frac{1}{2}), ..., G_{n-2}(\frac{1}{2})$ until the corresponding trajectory satisfies the boundary conditions at the boundary of the support interval with sufficient accuracy.

The details of the approximation are described below. The unknown components of the vector X_0 were initialized with the corresponding values for the uniform distribution, i.e., with

$$G_k(\tfrac{1}{2}) = \int_0^{1/2} z^k dz = \tfrac{1}{k+1}(\tfrac{1}{2})^{k+1} \tag{14}$$

[12] For a general approach to mitigating the inefficiencies caused by strategic information transmission, see Ewerhart and Schmitz (2000).

[13] The alternative computation of trajectories from the boundary of the equilibrium support proved to be numerically instable.

for $k = 1, ..., n-2$. The iteration repeated the following steps. First, the gradient of the trajectory at the midpoint of the location interval was computed using the relationship[14]

$$f(\tfrac{1}{2}) = \frac{1 + 2^{n-3}}{(n-2)\left\{1 + \sum_{k=1}^{n-3} \binom{n-3}{k} 2^k G_k(\tfrac{1}{2})\right\}}. \tag{15}$$

Next, the trajectory itself was computed on the basis of a discrete variant of system (9-10), where the grid step was $\varepsilon = 10^{-4}$. Finally, α was determined to be the left-most grid point z at which the first component of the trajectory exceeded unity. The multivariate approximation was executed by a solver plug-in of a standard spreadsheet software, where we used $\sum_{k=1}^{n-2} (G_k(\alpha))^2 < 10^{-9}$ as a stopping condition.

Appendix B: Proofs

Proof of Lemma 1. Differentiation of equation (1) yields

$$\begin{aligned}
\Pi'(z) \;=\; & \frac{n-1}{2}\int_z^1 f(y)(1-F(y))^{n-2}dy - (n-1)f(z)(1-F(z))^{n-2}z \\
& + f(z)\sum_{k=1}^{n-2}\binom{n-1}{k}k(n-k-1) \\
& \cdot\left\{F(z)^{k-1}\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy\right. \\
& \left. -(1-F(z))^{n-k-2}\int_0^z f(x)F(x)^{k-1}\frac{z-x}{2}dx\right\} \\
& + (n-1)f(z)F(z)^{n-2}(1-z) - \frac{n-1}{2}\int_0^z f(x)F(x)^{n-2}dx.
\end{aligned} \tag{16}$$

We will now rewrite the two integrals in the interior of the curly brackets.

[14] A proof of this equation can be found in Appendix B.

First, applying integration by parts, one can check that

$$\int_0^z f(x)F(x)^{k-1}\frac{z-x}{2}dx = \left.\frac{F(x)^k}{k}\frac{z-x}{2}\right|_{x=0}^{x=z} + \frac{1}{2}\int_0^z \frac{F(x)^k}{k}dx \quad (17)$$

$$= \frac{G_k(z)}{2k}, \quad (18)$$

where we have used that $F(x) = 0$ for $x \in [0, \alpha]$. Second, applying the substitution $x = 1 - y$, and noting the symmetry property $1 - F(1-x) = F(x)$, one obtains

$$\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy \quad (19)$$

$$= \int_0^{1-z} f(x)F(x)^{n-k-2}\frac{1-z-x}{2}dx.$$

Hence, equations (17-18), with z and k replaced by $1-z$ and $n-k-1$, respectively, imply

$$\int_z^1 f(y)(1-F(y))^{n-k-2}\frac{y-z}{2}dy = \frac{G_{n-k-1}(1-z)}{2(n-k-1)}. \quad (20)$$

Next, the terms obtained for the integrals via (17-18) and (20) are plugged into equation (16). Using also the obvious relationships

$$\frac{n-1}{2}\int_z^1 f(y)(1-F(y))^{n-2}dy = \frac{1}{2}(1-F(z))^{n-1}, \quad (21)$$

$$\frac{n-1}{2}\int_0^z f(x)F(x)^{n-2}dx = \frac{1}{2}F(z)^{n-1}, \quad (22)$$

one arrives at

$$\begin{aligned}\Pi'(z) = {} & \frac{1}{2}(1-F(z))^{n-1} - (n-1)f(z)(1-F(z))^{n-2}z \quad (23)\\ & +\frac{f(z)}{2}\sum_{k=1}^{n-2}\binom{n-1}{k}\\ & \cdot\left\{kF(z)^{k-1}G_{n-k-1}(1-z) - (n-k-1)(1-F(z))^{n-k-2}G_k(z)\right\}\\ & +(n-1)f(z)F(z)^{n-2}(1-z) - \frac{1}{2}F(z)^{n-1}.\end{aligned}$$

A simple re-ordering of terms, mapping index k to $n-k-1$ and vice versa, shows finally that

$$\begin{aligned} & \sum_{k=1}^{n-2} \binom{n-1}{k} (n-k-1)(1-F(z))^{n-k-2} G_k(z) \\ = & \sum_{k=1}^{n-2} \binom{n-1}{k} k(1-F(z))^{k-1} G_{n-k-1}(z). \end{aligned} \tag{24}$$

Using now (24) to rewrite (23), and exploiting the symmetry of F once more, the lemma follows. □

Proof of Lemma 2. By definition, $G_k(1-z) = \int_\alpha^{1-z} F(x)^k dx$. Splitting the integral and subsequently exploiting symmetry, one finds

$$G_k(1-z) = \int_\alpha^{1-\alpha} F(x)^k dx - \int_{1-z}^{1-\alpha} F(y)^k dy \tag{25}$$

$$= G_k(1-\alpha) - \int_\alpha^z F(1-x)^k dx \tag{26}$$

$$= G_k(1-\alpha) - \int_\alpha^z (1-F(x))^k dx. \tag{27}$$

Thus,

$$G_k(1-z) = G_k(1-\alpha) - z + \alpha - \sum_{m=1}^{k} (-1)^m \binom{k}{m} G_m(z), \tag{28}$$

for any $z \in [\alpha, 1-\alpha]$. Evaluating equation (28) at $z = \frac{1}{2}$ yields

$$G_k(1-\alpha) = \tfrac{1}{2} - \alpha + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} G_m(\tfrac{1}{2}) + \left\{1 + (-1)^k\right\} G_k(\tfrac{1}{2}). \tag{29}$$

Plugging this back into (28) proves the claim. □

Proof of Theorem 2. *(Necessity)* To construct Φ_n, one first writes differential equation (5) in explicit form, i.e., using the definitions of $\varphi(z)$ and

$\psi(z)$ provided in Lemma 1. This yields

$$F'(z) = \left\{ \frac{F(z)^{n-1} - F(1-z)^{n-1}}{2} \right\} \tag{30}$$

$$\cdot \left\{ \begin{array}{l} (n-1)(1-z)F(z)^{n-2} - (n-1)zF(1-z)^{n-2} + \frac{1}{2}\sum_{k=1}^{n-2}\binom{n-1}{k}k \\ \cdot \left\{ F(z)^{k-1}G_{n-k-1}(1-z) - F(1-z)^{k-1}G_{n-k-1}(z) \right\} \end{array} \right\}^{-1}.$$

Re-ordering the terms of the sum by mapping index k to $n-k-1$, and subsequently using the relationship $\binom{n-1}{k}(n-k-1) = (n-1)\binom{n-2}{k}$, one obtains

$$F'(z) = \left\{ \frac{F(z)^{n-1} - F(1-z)^{n-1}}{n-1} \right\} \tag{31}$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)F(z)^{n-2} - 2zF(1-z)^{n-2} + \sum_{k=1}^{n-2}\binom{n-2}{k} \\ \cdot \left\{ F(z)^{n-k-2}G_k(1-z) - F(1-z)^{n-k-2}G_k(z) \right\} \end{array} \right\}^{-1}.$$

Replacing all occurrences of $F(1-z)$ by $1-F(z)$, and similarly, all occurrences of $G_1(1-z), \ldots, G_{n-2}(1-z)$ by the corresponding expressions in Lemma 2, we arrive at

$$F'(z) = \left\{ \frac{F(z)^{n-1} - (1-F(z))^{n-1}}{n-1} \right\} \tag{32}$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)F(z)^{n-2} - 2z(1-F(z))^{n-2} + \sum_{k=1}^{n-2}\binom{n-2}{k} \\ \cdot \left\{ F(z)^{n-k-2}\left\{ C_k - z - \sum_{m=1}^{k}(-1)^m\binom{k}{m}G_k(z) \right\} \right. \\ \qquad \left. - (1-F(z))^{n-k-2}G_k(z) \right\} \end{array} \right\}^{-1}.$$

In analogy with (32), define the function $\Phi_n : \mathbb{R}^{2n-2} \to \mathbb{R} \cup \{+\infty, -\infty\}$ by

$$\Phi_n(\widehat{F}, \widehat{G}_1, ..., \widehat{G}_{n-2}, \widehat{C}_1, ..., \widehat{C}_{n-2}, z) = \left\{ \frac{\widehat{F}^{n-1} - (1-\widehat{F})^{n-1}}{n-1} \right\} \tag{33}$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)\widehat{F}^{n-2} - 2z(1-\widehat{F})^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{ \widehat{F}^{n-k-2} \left\{ \widehat{C}_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} \widehat{G}_m \right\} - (1-\widehat{F})^{n-k-2} \widehat{G}_k \right\} \end{array} \right\}^{-1}.$$

Then, by construction, F is the first component of a solution of the boundary value problem stated in Theorem 2, thereby proving the first part of the theorem. *(Sufficiency)* Suppose that $\widetilde{F}$ is monotone increasing, and that $\widetilde{F}$ is symmetric in the sense that $\widetilde{F}(1-z) = 1 - \widetilde{F}(z)$ for any $z \in [\widetilde{\alpha}, 1-\widetilde{\alpha}]$. Then, from $\widetilde{G}_k(\widetilde{\alpha}) = 0$ and $\widetilde{G}_k'(z) = \widetilde{F}(z)^k$, it follows that $\widetilde{G}_k(z) = \int_{\alpha}^{z} \widetilde{F}(x)^k dx$. From the symmetry of $\widetilde{F}$, one may derive just as in the proof of Lemma 2 that

$$\widetilde{G}_k(1-z) = \widetilde{C}_k - z - \sum_{m=1}^{k} (-1)^m \binom{k}{m} \widetilde{G}_k(z), \tag{34}$$

for any integer $k \geq 1$, where

$$\widetilde{C}_k = \tfrac{1}{2} + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} \widetilde{G}_m(\tfrac{1}{2}) + \left\{ 1 + (-1)^k \right\} \widetilde{G}_k(\tfrac{1}{2}). \tag{35}$$

By assumption, equation (32) holds with $F, G_1, ..., G_{n-2}, C_1, ..., C_{n-2}$ replaced by $\widetilde{F}, \widetilde{G}_1, ..., \widetilde{G}_{n-2}, \widetilde{C}_1, ..., \widetilde{C}_{n-2}$. Using the symmetry of $\widetilde{F}$ and the functional equations (34) for $k = 1, ..., n-2$, one arrives at

$$\widetilde{F}'(z) = \left\{ \frac{\widetilde{F}(z)^{n-1} - \widetilde{F}(1-z)^{n-1}}{n-1} \right\} \tag{36}$$

$$\cdot \left\{ \begin{array}{l} 2(1-z)\widetilde{F}(z)^{n-2} - 2z\widetilde{F}(1-z)^{n-2} + \sum_{k=1}^{n-2} \binom{n-2}{k} \\ \cdot \left\{ \widetilde{F}(z)^{n-k-2} \widetilde{G}_k(1-z) - \widetilde{F}(1-z)^{n-k-2} \widetilde{G}_k(z) \right\} \end{array} \right\}^{-1}.$$

Hence, invoking Lemma 1, $\widetilde{F}$ solves the first-order condition, and expected payoffs are constant on the interval $[\widetilde{\alpha}, 1-\widetilde{\alpha}]$. Moreover, by the nature of expected payoffs in the location game, any location $z < \widetilde{\alpha}$ yields strictly lower expected payoffs than $\widetilde{\alpha}$, and similarly, any location $z > 1-\widetilde{\alpha}$ yields strictly lower expected payoffs than $1-\widetilde{\alpha}$. Thus, $\widetilde{F}$ really corresponds to a doubly symmetric mixed-strategy equilibrium. □

Proof of Equation (15). A straightforward application of the rule of L'Hôpital to the differential equation (5) shows that

$$f(\tfrac{1}{2}) = \frac{\varphi'(\frac{1}{2})}{\psi'(\frac{1}{2})}. \tag{37}$$

Noting that $F(\frac{1}{2}) = \frac{1}{2}$, one readily verifies that

$$\varphi'(\tfrac{1}{2}) = \frac{n-1}{2^{n-1}} f(\tfrac{1}{2}). \tag{38}$$

Moreover, using $G'_{n-k-1}(\frac{1}{2}) = F^{n-k-1}(\frac{1}{2}) = (\frac{1}{2})^{n-k-1}$, one can check that

$$\begin{aligned}\psi'(\tfrac{1}{2}) &= f(\tfrac{1}{2}) \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k(k-1)}{2^{k-1}} G_{n-k-1}(\tfrac{1}{2}) - \sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k}{2^{n-1}} \\ &\quad - \frac{n-1}{2^{n-2}} + f(\tfrac{1}{2}) \frac{(n-1)(n-2)}{2^{n-2}}. \end{aligned} \tag{39}$$

Exploiting the identities

$$\begin{aligned} &\sum_{k=1}^{n-2} \binom{n-1}{k} \frac{k(k-1)}{2^{k-1}} G_{n-k-1}(\tfrac{1}{2}) \\ &= \frac{1}{2^{n-2}} \sum_{k=1}^{n-2} \binom{n-1}{k} k(k-1) 2^{n-k-1} G_{n-k-1}(\tfrac{1}{2}) && (40) \\ &= \frac{1}{2^{n-2}} \sum_{k=1}^{n-2} \binom{n-1}{k} (n-k-1)(n-k-2) 2^{k} G_{k}(\tfrac{1}{2}) && (41) \\ &= \frac{(n-1)(n-2)}{2^{n-2}} \sum_{k=1}^{n-3} \binom{n-3}{k} 2^{k} G_{k}(\tfrac{1}{2}) && (42) \end{aligned}$$

and

$$\sum_{k=1}^{n-2} \binom{n-1}{k} k = (n-1) \sum_{k=1}^{n-2} \binom{n-2}{k-1} = (n-1)(2^{n-2}-1), \tag{43}$$

it follows that

$$\begin{aligned} \psi'(\tfrac{1}{2}) &= f(\tfrac{1}{2}) \frac{(n-1)(n-2)}{2^{n-2}} \left\{ 1 + \sum_{k=1}^{n-3} \binom{n-3}{k} 2^k G_k(\tfrac{1}{2}) \right\} \\ &\quad - \frac{(n-1)(1+2^{n-2})}{2^{n-1}}. \end{aligned} \tag{44}$$

Plugging now (38) and (44) into (37), and subsequently solving for $f(\frac{1}{2})$, one arrives at equation (15). □

References

Chamberlin, E. (1938), *The Theory of Monopolistic Competition*, 3rd ed., Harvard University Press.

Cox, G.W. (1987), Electoral equilibrium under alternative voting institutions, *American Journal of Political Science* 31(1), 82-108.

Denzau, A., Kats, A., Slutsky, S. (1985), Multi-agent equilibria with market share and ranking objectives, *Social Choice and Welfare* 2, 95-117.

Eaton, B.C., Lipsey, R.G. (1975), The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition, *Review of Economic Studies* 42(1), 27-49.

Ewerhart, C., Schmitz, P.W. (2000), "Yes men," integrity, and the optimal design of incentive contracts, *Journal of Economic Behavior and Organization* 43, 115-125.

Gabszewicz, J.J., Thisse, J.-F. (1992), "Location," pp. 281-304 in: *Handbook of Game Theory with Economic Applications I*, by Aumann, R.J., Hart, S. (eds.), Elsevier.

Graitson, D. (1982), Spatial competition a la Hotelling: A selective survey, *Journal of Industrial Economics* 31(1/2), 13-25.

Hotelling, H. (1929), Stability in competition, *Economic Journal* 39(153), 41-57.

Huck, S., Müller, W., Vriend, N.J. (2002), The East End, the West End, and King's Cross: On clustering in the four-player Hotelling game, *Economic Inquiry* 40(2), 231-240.

Laster, D., Bennett, P., Geoum, I.S. (1999), Rational bias in macroeconomic forecasts, *Quarterly Journal of Economics* 114(1), 293-318.

Lerner, A.P., Singer, H.W. (1937), Some notes on duopoly and spatial competition, *Journal of Political Economy* 45(2), 145-186.

Marinovic, I., Ottaviani, M., Sørensen, P.N. (2013), "Forecasters' objectives and strategies," ch. 12 in: *Handbook of Economic Forecasting*, Volume 2B, by Elliott, G., Timmermann, A. (eds.), North Holland.

Osborne, M.J., Pitchik, C. (1986), The nature of equilibrium in a location model, *International Economic Review* 27(1), 223-237.

Ottaviani, M., Sørensen, P.N. (2006), The strategy of professional forecasting, *Journal of Financial Economics* 81(2), 441-466.

Shaked, A. (1982), Existence and computation of mixed strategy equilibrium for 3-firms location problem, *Journal of Industrial Economics* 31, 93-97.

Xefteris, D. (2014), Mixed equilibria in runoff elections, *Games and Economic Behavior*, forthcoming.