Translated Poisson Approximation to Equilibrium Distributions of Markov Population Processes

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Zusammenfassung

Wir untersuchen in dieser Arbeit dichteabhängige markovsche Populationsprozesse, die häufig in der Modellierung der Entwicklung von Populationen verwendet werden. Wir konzentrieren uns auf das Gleichgewichtsverhalten solcher Prozesse und beweisen zuerst ein Existenztheorem für die Gleichgewichtsverteilung der Populationsgrösse. Danach zeigen wir wie die im allgemeinen etwas kompliziertere Gleichgewichtsverteilung durch eine einfachere Verteilung, nämlich eine verschobene Poissonverteilung, approximiert werden kann. Letztere kann durch zwei Parametern beschrieben werden, einen zur Bestimmung der Position und einen zur Skalierung. Ferner machen wir Aussagen über die Approximationsfehler indem wir Schranken für die Distanz der Verteilungen in totaler Variation und für die Distanz zwischen ihren Punktwahrscheinlichkeiten angeben.

Abstract

In this thesis, we study density-dependent Markov population processes, which are frequently used to model the way in which populations evolve. We focus on their behavior at equilibrium, and prove first an existence theorem for the equilibrium distribution of the population size. We then show how to approximate this, in general complicated, equilibrium distribution by a simpler translated Poisson distribution, which is characterized by the values of just two parameters, that can be used to fix location and scale. We establish error bounds for our approximations for the total-variation distance between the two distributions, and for the distance between their point probabilities.

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CHAPTER 1

Introduction

Overview and motivation

It is often of practical importance to be able to approximate the unknown distribution of a random quantity of interest by a common, well-known distribution. For this reason, distributional approximations have been the subject of mathematical research for a long time. A classical example is that of approximating the distribution of a sum of independent identically distributed random variables. Such sums frequently appear in statistical estimation and test problems, as do sums for which the conditions of independence and identical distribution are both, to some extent, relaxed. An extensive literature already exists on the resulting laws of large numbers and central limit theorems and, in the case of sums of Bernoulli distributed variables, the so-called law of small numbers, where the approximating distribution is a Poisson distribution. These are all limit theorems, that is to say, limits as the number n of random variables that are summed goes to infinity.

Rather then letting n go to infinity, one could also choose to compute bounds on the distance between the unknown distribution and the approximating distribution, for fixed n; that is, one could choose to quantify the error. This is a much harder problem. A classical example of such an error bound is the Berry-Esseen bound for asymptotic normality in the central limit theorem, which was first proved in 1941, 40 years later than Lyapounov's corresponding central limit theorem.

While classical proofs of this kind of result make use of generating functions, Stein's method for normal approximation and the Stein-Chen method for Poisson approximation have in recent years proved really successful in yielding precise error bounds. The big advantage of Stein's method over transform methods is that it adapts well to settings in which dependence is an important feature. A good example is Bolthausen's (1984) bound for the error in the combinatorial central limit theorem, the exact analogue of the Berry-Esseen bound, obtained using Stein's method. Bolthausen's result came 40 years after Wald and Wofowitz set the problem of approximating the distribution of statistics for non-parametric tests based on permutations of the observations, and 33 years after Hoeffding's proof of the combinatorial central limit theorem using moment methods. There had been many earlier attempts to establish such a bound by classical methods, but none were fully successful.

There are, of course, many interesting random quantities that are not sums of independent or identically distributed random variables, but whose unknown distributions we would nevertheless like to be able to say more about. Because of their importance in modeling many real life phenomena, stochastic processes and in particular their finite-dimensional and also equilibrium distributions have for example been intensively studied. In this thesis, we are concerned with the equilibrium distributions of a particular class of stochastic processes.

Population processes, as their name suggests, are used to model the behavior of populations and the interactions between their individuals, with a wide range of applications in the social and natural sciences.

For our purposes, a *population process* is a continuous-time, piecewise constant stochastic process taking values in $\mathbb{Z}_+^d$, for some $d \geq 1$, which jumps after random waiting times. That is to say, a population process is a d-dimensional jump process taking values on the non-negative integers. In this thesis, we concentrate on Markov population processes. The Markov property implies that the random waiting times must be exponentially distributed, and that jump rates distributions are determined by the current state alone. Note that, in this thesis, the processes that we consider may actually take values on the whole set of integers, and not just the positive integers. A simple example of a Markov population process is the linear-death with immigration process, with transition rates given by

$$i \to i+1 \quad \text{at rate} \quad \mu > 0,$$

$$i \to i-1 \quad \text{at rate} \quad \nu i \geq 0,$$

where $i \in \mathbb{Z}_+$ denotes the state of the process, interpreted as the number of individuals in the population. The transition rates have the following meaning. The waiting time in state i is exponentially distributed with mean $1/(\mu + \nu i)$, $\mu + \nu i$ being the total rate of leaving state i, and the first jump is of size $+1$ with probability $\mu/(\mu + \nu i)$. We call the positive constant μ the immigration rate, and ν the per capita death rate.

A *density dependent* population process is an element of a sequence of population processes, which model the same underlying process in ever larger systems. For instance, we can use density dependent processes to model the behavior of similar populations living inside different regions, whose sizes vary from small to very large. For a fixed n, where n stands for a population size typical for a given region, the density dependent process that we shall denote by $X_n(t)$, with $t > 0$, has rates that depend on the ratio between the number of individuals in the population at a given time and n.

We now give an example of a density-dependent population process, that is also meant to introduce the reader to some of the terminology used in this work. This is the so called logistic growth model due to Verhulst, in 1838, which was advanced as an alternative to the Malthusian exponential growth model. The idea behind the Malthusian model dates back to 1798, and the model itself is the simplest possible, being described by the differential equation:

$$\dot{x} = mx, \tag{1.0.1}$$

where $x(t)$ is the population density at time t, and $m > 0$ is the per capita growth rate, see Bailey (1967) equation (1.1). This model is purely deterministic. A (stochastic)-population process corresponding to this dynamic model is the linear birth-death process

$$\begin{aligned} i &\to i+1 \quad \text{at rate} \quad \mu i, \\ i &\to i-1 \quad \text{at rate} \quad \nu i, \end{aligned}$$

where μ and ν are both positive constants such that $\mu - \nu > 0$. We call μ the per capita birth rate and ν the per capita death rate. Since the "probability" of jumping 1 up, instantly, from state i is $\mu i dt$, and jumping 1 down, instantly, from state i is $\nu i dt$, the infinitesimal expected change in the value of this process, that we denote by N_t, is given by

$$\frac{1}{dt}\mathbb{E}\{N_{t+dt} - N_t \mid N_t = i\} = \mu i - \nu i = (\mu - \nu)i.$$

If we write $m := \mu - \nu$ and $x(t) = \mathbb{E}N_t$, then the above equation becomes essentially equivalent to the differential equation (1.0.1). The solution for (1.0.1) is $x(t) = ae^{mt}$, with a the initial number of individuals in the population, and we have that $m > 0$, at least for the model Malthus suggested. Malthus' intuition was that populations tend to grow exponentially (in geometric progression), while resources such as food only grow in arithmetic progression. He had foreseen a resulting drastic decrease in the size of the human population, caused by famine. Even though his theories proved to be far too pessimistic, they helped establish demography as a science, and they strongly influenced Darwin's theory of evolution. Worth mentioning at this point is the very interesting, though less mathematical discussion on the dynamics of the human population today, and the future tightness of space and food resources on our planet, to be found in Cohen (2005). Malthus' critics appreciated the logistic growth model introduced by Verhulst. He also included in his model, as an additional element, "some kind of retardation factor which would increase as the population grew", see Bailey (1967) equation (1.3):

$$\dot{x} = mx - rx^2, \tag{1.0.2}$$

where $m > 0$, the per capita growth rate is now $m - rx$, and r is called "retardation constant". Actually, the correct interpretation of the form of the Verhulst model was given by Pearl and Reed (1920). Again, this model is deterministic. A random process corresponding to the logistic model of Verhulst was given by Feller in 1939 and is a density-dependent population process, that we denote here by X_n, which jumps as follows:

$$\begin{aligned} i &\to i+1 \quad \text{at rate} \quad \mu i\Big(1 - \frac{i}{n}\Big), \\ i &\to i-1 \quad \text{at rate} \quad \nu i, \end{aligned}$$

with μ and ν as before, and where $i \in \mathbb{N}$ denotes again the number of individuals in the population, and i/n represents the probability that any newborn dies right after birth. The infinitesimal expected change in the value of the process $X_n(t)$ is given by

$$\frac{1}{dt}\mathbb{E}\{X_n(t+dt) - X_n(t) \mid X_n(t) = i\} = \mu i\Big(1 - \frac{i}{n}\Big) - \nu i = (\mu - \nu)i - \frac{\mu i^2}{n}.$$

If we write $x_n := X_n/n$ for the density of the population inside an area of "maximum size" n, when $x := i/n$ would become equal to 1 so that there would be no more surviving newborns, then the above equation becomes

$$\frac{1}{dt}\mathbb{E}\{x_n(t+dt) - x_n(t) \mid x_n(t) = x\} = (\mu - \nu)x - \mu x^2,$$

which, for $m := \mu - \nu$, $r := \mu$ and $x(t) = \mathbb{E}x_n(t)$ is essentially equivalent to (1.0.2). The per capita growth rate of X_n is then $n(\mu - \nu - \mu x)$, and the function $nF(x) := n(\mu - \nu)x - n\mu x^2$ is the average growth of the population size X_n from the state $i = nx$. Any solution x of the equation $F(x) = 0$ is a state of equilibrium for the solution $x(\cdot)$ of (1.0.2), since then $\dot{x} = 0$. Hence, the logistic growth model has two deterministic equilibrium points: 0 and $n(\mu - \nu)/\mu$. The constant $n(\mu - \nu)/\mu$ is also called "carrying capacity" for $x(\cdot)$. In case $\mu - \nu > 0$, 0 is an unstable equilibrium point, since, once the density is very close to 0, there is the tendency for exponential increase in the population size. In exchange, $n(\mu - \nu)/\mu$ is stable because, once $x(\cdot)$ reaches in some neighborhood of this equilibrium point, then it tends to exponentially decrease to $n(\mu - \nu)/\mu$, as t becomes larger and larger. In case $\mu - \nu < 0$, then it is 0, the absorbing state, which is stable. Indeed, since there is no immigration, the state 0 can become absorbing, and this with probability one. Therefore, only when $n(\mu - \nu)/\mu$ is the stable equilibrium point can the process X_n have an equilibrium distribution, which is not a true equilibrium distribution, but is conditional on the fact that absorbtion did not happen yet. We call such a limiting conditional distribution a *quasi-stationary* distribution.

By the true *equilibrium distribution (or stochastic equilibrium)* of a time homogeneous Markov process X we understand a vector of probabilities $\Pi := \Pi(i)$ with $\sum_i \Pi(i) = 1$, having the property that

$$\Pi = \Pi \cdot \mathbb{P}_t, \quad \text{for any} \quad t \geq 0,$$

where $\mathbb{P}_t$ denotes the matrix of transition probabilities of the process X over a time interval of length t. Thus, in this case, it is the distribution of X that remains constant over time. Note that, for the simple death with immigration process mentioned before, the growth rate $F(i) = \mu - \nu i$ has an unique deterministic equilibrium at μ/ν, where $\nu > 0$. It is known that the equilibrium distribution of the linear- (or simple-) death with immigration process is the Poisson distribution $\text{Po}(\mu/\nu)$, and the mean of the Poisson equilibrium distribution coincides with the unique deterministic equilibrium point μ/ν.

Important results on the relationship between deterministic and stochastic models based on jump Markov processes are due to Kurtz (1970,1971,1981) and Ethier and Kurtz (1986). Kurtz (1970) gives a limit theorem on the convergence of a sequence of d-dimensional density-dependent pure jump Markov processes $X_n(t)/n$, with $t > 0$, to the solution $\xi(t)$ of the system of ordinary differential equations

$$\frac{d}{dt}\xi(t) = F(\xi(t)),$$

where F is the average growth rate function of the process X_n/n, and is defined, just as in the case of the logistic model above, as the infinitesimal expected change in the value of the process when in a given state. For our population processes, this average growth rate is defined in (2.0.1). Kurtz's result is a law of large numbers which states that, if $\lim_{n\to\infty} X_n(0)/n = \xi(0)$, then

$$\lim_{n\to\infty} \mathbb{P}\Big\{ \sup_{0\le s\le t} \Big| \frac{X_n(s)}{n} - \xi(s) \Big| > \delta \Big\} = 0,$$

for every $\delta > 0$, and any $t \ge 0$. Kurtz (1971) gives, under slightly stronger assumptions than above, a functional central limit theorem saying that, on finite time intervals, $\sqrt{n}\big(X_n(t)/n - \xi(t)\big)$ converges weakly to a diffusion $V(t)$ with drift $DF(\xi(t))$, where DF denotes the first derivative of the function F, and with infinitesimal variance the variance function of the process $X_n/\sqrt{n}$, evaluated at $\xi(t)$. For our population processes, the variance function is given in (2.0.2). Further references on density dependent processes include Norman (1972), a discrete-time version of Kurtz (1971) giving a central limit theorem for the normalized equilibrium distribution of population processes used to modeling slow learning, Alm (1978) and Barbour (1980). Alm's (1978) Berry-Esseen result gives the bound $K(t)/\sqrt{n}$ on the Kolmogorov distance between the distribution of the process $\sqrt{n}\big(X_n(t)/n - \xi(t)\big)$ and the distribution of the diffusion process $V(t)$ from Kurtz (1971), for any fixed $t > 0$, where $K(t)$ is a constant that depends on the time t, but not on n, and the *Kolmogorov distance* between two probability measures P and Q on $\mathbb{R}$ is defined as:

$$d_K\{P, Q\} := \sup_{x\in\mathbb{R}} |P\{(-\infty, x]\} - Q\{(-\infty, x]\}|.$$

The results of Barbour (1980) will be presented in more detail in what follows.

In this work, we focus on the behavior of one-dimensional density-dependent Markov population processes at equilibrium. Our starting point is the paper of Barbour (1980), in which conditions are given for the existence of an equilibrium distribution concentrated close to the deterministic equilibrium. In this paper, a bound of order $O(1/\sqrt{n})$ on the Kolmogorov distance between the equilibrium distribution and a suitable normal distribution is also given.

His results can be formulated as follows. Let z_n, $n \ge 1$, be a family of irreducible continuous-time density dependent, pure jump Markov processes on the lattice $n^{-1}\mathbb{Z}$, with transition rates

$$z \to z + j/n \quad \text{at rate} \quad n\lambda_j(z), \quad j \in J, \quad z \in E_n,$$

where J is a finite set of integers, $E_n := E \cap n^{-1}\mathbb{Z}$ for some subset E of the real line, and, for any $j \in J$, $\lambda_j(\cdot)$ are real-valued functions defined on E.

Let $F(z) := \sum_{j\in J} j\lambda_j(z)$ denote the average growth rate from the state z, for any $z \in E$. Assume c is the unique constant solution to the differential equation $\dot{x} = F(x)$, and let $\eta(t; 0, z)$ denote the flow from z for this differential equation, for any $t \ge 0$. Let $\sigma^2(z) := \sum_{j\in J} j^2\lambda_j(z)$ denote the variation of the process from state z, for any $z \in E$.

The existence theorem for the equilibrium distribution is as follows.

Theorem 1.1 (Barbour 1980, Theorem 2.1). *Let F be twice continuously differentiable over E, and suppose that $F'(c) < 0$. Define $V(z) = \int_0^\infty \{\eta(t; 0, z) - c\}^2 dt$. Then, if $\sup_{z\in E} \sum_{j\in J} \lambda_j(z) < \infty$ and if $V''(z) \le K\{(z - c)^2 + 1\}$, there exists an* *equilibrium distribution* π_n *for* z_n *whenever* n *is sufficiently large. Furthermore, there exist positive constants* K_1 *and* K_2 *such that*

$$\mathbb{E}_{\pi_n}\{|z_n - c| \cdot \mathbb{1}(|z_n - c| \geq 1)\} \leq K_1/n$$
$$\mathbb{E}_{\pi_n}\{|z_n - c|^2 \cdot \mathbb{1}(|z_n - c| \leq 1)\} \leq K_2/n.$$

In Remark 1, Barbour indicates how the theorem still holds for processes for which $\sup_{z\in E}\sum_{j\in J}\lambda_j(z)$ is possibly unbounded.

The normal approximation theorem states the following:

THEOREM 1.2 (Barbour 1980, Theorem 2.3). *Write* $\lambda := \frac{\sigma^2(c)}{-2F'(c)}$. *Under the hypotheses of Theorem 1.1, and if also*

$$|\sigma^2(z) - \sigma^2(c)| \leq K_3|z - c|,$$

for some positive constant K_3, *then there exists a constant* $C > 0$ *so that*

$$\sup_{y\in\mathbb{R}} |\mathbb{P}_{\pi_n}\{\sqrt{n/\lambda}(z_n - c) \leq y\} - \Phi(y)| \leq Cn^{-\frac{1}{2}},$$

where Φ *denotes the distribution function of the standard Normal distribution.*

Results and open questions

Here, we are able to sharpen the results of Barbour (1980) in three ways. First, we relax the conditions under which an equilibrium distribution can be shown to exist. Secondly, we prove approximation not just with respect to the Kolmogorov distance, but in the stronger total-variation distance, where the *distance in total variation* between two probability measures P and Q on $\mathbb{R}$ is defined as:

$$d_{TV}\{P, Q\} := \sup_{A\subset\mathbb{R}} |P\{A\} - Q\{A\}|.$$

Since we take the supremum over any subset of $\mathbb{R}$, and not just over the sets $(-\infty, x]$, with $x \in \mathbb{R}$, it is clear why the total-variation distance is stronger than the Kolmogorov distance. Finally, we are also able to prove a local limit approximation for point probabilities, again with error bounds.

The processes in Barbour (1980) take values in $n^{-1}\mathbb{Z}$, whereas the approximating distribution for the equilibrium distribution defined above is a normal distribution, which is absolutely continuous with respect to the Lebesgue measure. If finer results are desired, we need to choose a discrete approximating distribution, such as the Poisson. In such a case, we would need to approximate the unknown equilibrium distribution which, in view of the results of Theorems 1.1 and 1.2, has mean nc and variance $n\lambda = \frac{n\sigma^2(c)}{-2F'(c)}$, by a Poisson distribution which has only one parameter to fit, mean and variance being equal. To overcome this problem, we translate both distributions close to 0 and then approximate the translated equilibrium distribution by the translated Poisson distribution with parameter $n\lambda$.

We work under assumptions which are generally weaker than those of Theorems 1.1 and 1.2. We do not assume J to be finite, replacing this requirement by moment conditions on the λ_j's. We are also able to relax the condition

$$\sup_{z\in E}\sum_{j\in J}\lambda_j(z) < \infty,$$

which is actually not even satisfied by the simple immigration-death process. Our first result is an existence theorem, which is the subject of the third chapter.

THEOREM 1.3 (**Existence theorem for** Π_n, Theorem 3.1). *Under suitable assumptions, for n fixed large enough, there exists an equilibrium distribution π_n for the process z_n. Moreover, for a certain $\delta > 0$ and for some $\alpha \in (0, 1]$, there exist positive constants $K_1(\delta, \alpha)$ and $K_2(\delta, \alpha)$ so that:*

$$\mathbb{E}_{\pi_n}\{|z_n - c| \cdot \mathbb{1}(|z_n - c| \geq \delta)\} \leq K_1(\delta, \alpha)/n$$
$$\mathbb{E}_{\pi_n}\{(z_n - c)^2 \cdot \mathbb{1}(|z_n - c| < \delta)\} \leq K_2(\delta, \alpha)/n.$$

In order to prove our translated Poisson approximation theorems, we need the following auxiliary result, which is of independent interest, proved in the fourth chapter.

THEOREM 1.4 (**TV-distance between** Π_n **and** $\Pi_n * \delta_1$, Theorem 4.1). *Under suitable assumptions, there exists a constant $K > 0$ so that*

$$d_{TV}\{\Pi_n, \Pi_n * \delta_1\} \leq Kn^{-1/2},$$

where Π_n denotes the equilibrium distribution of the process $Z_n := nz_n$, and $\Pi_n * \delta_1$ denotes the distribution Π_n, translated by 1.

The approximation of Π_n is then proved in the following two theorems. First, we prove a translated Poisson approximation theorem in the fifth chapter.

THEOREM 1.5 (**Translated Poisson approximation to** Π_n, Theorem 5.4). *Under suitable assumptions, there exists a constant $C > 0$ so that*

$$d_{TV}\Big\{(\Pi_n - \lfloor nc \rfloor), \widehat{\mathrm{Po}}(n\lambda)\Big\} \leq Cn^{-\alpha/2},$$

where $\alpha \in (0, 1]$ is determined by the assumptions on the λ_j functions.

A local limit approximation bounding the distance between the point probabilities of the equilibrium distribution and those of the translated Poisson distribution functions, and is proved in chapter six.

THEOREM 1.6 (**Local limit theorem for** Π_n, Theorem 6.1). *Under suitable assumptions, there exists a constant $C > 0$ so that*

$$\sup_{k \in \mathbb{Z}} \Big|(\Pi_n - \lfloor nc \rfloor)(k) - \widehat{\mathrm{Po}}(n\lambda)(k)\Big| \leq Cn^{-\frac{\alpha}{2} - \frac{1}{4}},$$

where $\alpha \in (0, 1]$ is as above.

The second chapter contains the necessary prerequisites. We establish our general framework and list our main assumptions. We then introduce the infinitesimal generator of a Markov process, and give a brief account of the Stein-Chen method for Poisson and translated Poisson approximation. The last chapter, chapter seven, is an appendix containing the proof of a lemma from chapter two.

Note that, in this project, we only consider one-dimensional processes. While one-dimensional population processes can be used to model population growth and its possible extinction, phenomena such as the spread of epidemics, the competition between species and chemical reaction processes need to be modeled using two or higher-dimensional processes, since there are typically several different populations (such as infectious and susceptible individuals, or prey and predators, or chemical reactants) that interact with one another, see Kurtz (1981). Our results are only useful for describing the behavior of single populations under equilibrium.

However, it is to be hoped that our approach provides useful tools for further research on stable equilibria, in the more complicated two or higher-dimensional setting, where approximation in total-variation has as yet not been satisfactorily studied.

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CHAPTER 2

Preliminaries

We start by defining our density dependent sequence of Markov processes. Let $Z_n(t)$, with $t \in [0, \infty)$ and $n \in \mathbb{N}$, be a sequence of irreducible continuous-time pure jump Markov processes, defined on some probability space $(\Omega, \mathcal{K}, \mathbb{P})$ and taking values in $\mathbb{Z}$, so that

$$i \longrightarrow i + j$$

at rate

$$n\lambda_j\Big(\frac{i}{n}\Big),$$

for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z} \setminus \{0\}$, and where λ_j, $j \in \mathbb{Z} \setminus \{0\}$, are prescribed functions on $\mathbb{R}$; set

$$z_n(t) := n^{-1} Z_n(t), \quad \text{for all } t \in [0, \infty).$$

Define the average growth rate of the process z_n at $z \in \frac{1}{n}\mathbb{Z}$ as

$$F(z) := \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{j}{n} \cdot n\lambda_j(z) = \sum_{j \in \mathbb{Z} \setminus \{0\}} j \cdot \lambda_j(z) \tag{2.0.1}$$

and its variation function by

$$\frac{1}{n}\sigma^2(z) := \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{j^2}{n^2} \cdot n\lambda_j(z), \tag{2.0.2}$$

so that

$$\sigma^2(z) = \sum_{j \in \mathbb{Z} \setminus \{0\}} j^2 \cdot \lambda_j(z).$$

For the intensity with which the process leaves the state z, no matter the size of the jump it makes, we write

$$n\lambda(z) := \sum_{j \in \mathbb{Z} \setminus \{0\}} n\lambda_j(z).$$

Example 1. *An immigration birth death process with birth occurring in groups, equivalent to a Markov branching process with immigration, has the following jump-rates:*

$$\lambda_{-1}(z) := dz, \quad \lambda_1(z) := a + b_1 z \quad \text{and} \quad \lambda_j(z) := b_j z, \ j \geq 2,$$

while $\lambda_j(z) := 0, \ j < -1$. *Then,*

$$F(z) = a + z\Big(\sum_{j \geq 1} j b_j - d\Big), \quad \sigma^2(z) = a + z\Big(\sum_{j \geq 1} j^2 b_j + d\Big)$$

and $\lambda(z) = a + z(\sum_{j \geq 1} b_j + d)$.

For the proofs that follow we shall need certain assumptions on the functions λ_j to be satisfied.

Assumption 1: We assume that there exists an unique c satisfying $F(c) = 0$, and that, for any $\delta > 0$, $\mu_\delta := \inf_{|z-c|\geq\delta} |F(z)| > 0$.

Assumption 2: The transition rates λ_j are of class $C^2(\mathbb{R})$, for each $j \in \mathbb{Z} \setminus \{0\}$. We also assume that there exists $\lambda^0 > 0$ and $c_1 > 0$ such that

$$2\lambda^0 \leq \lambda_1(z) \leq c_1(1 + |z - c|), \quad \text{for all} \quad z \in \mathbb{R},$$

and suitable $\varepsilon, \delta > 0$ such that, for each $j \in \mathbb{Z} \setminus \{0, 1\}$, either

$$\inf_{|z-c|\leq\delta} \lambda_j(z) \geq \varepsilon\lambda_j(c) \quad \text{or} \quad \lambda_j(z) = 0, \quad \text{on } |z - c| \leq \delta.$$

Also, for each $j \in \mathbb{Z} \setminus \{0, 1\}$, there exist $c_j > 0$, such that

$$\lambda_j(z) \leq c_j(1 + |z - c|), \quad \text{for all} \quad z \in \mathbb{R}. \tag{2.0.3}$$

Assumption 3: Assume that, for some $\alpha \in (0, 1]$,

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2+\alpha} c_j < \infty.$$

Assumptions 2 and 3 imply that the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z)$, $\sum_{j\in\mathbb{Z}\setminus\{0\}} j\lambda_j(z)$ and $\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2\lambda_j(z)$ are uniformly convergent on bounded sets and that their sums, λ, F and σ^2 respectively, are continuous on bounded sets.

Assumption 4: We write $\|\lambda'_j\|_\delta := \sup_{|z-c|\leq\delta} |\lambda'_j(z)|$, for the δ in **Assumption 2** and for any jump-size j, and assume that

$$\sup_{j\in\mathbb{Z}\setminus\{0\}} \frac{\|\lambda'_j\|_\delta}{|j|\lambda_j(c)} < \infty.$$

This assumption implies, in view of the **Assumptions 1 to 3**, that the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda'_j(z)$ and $\sum_{j\in\mathbb{Z}\setminus\{0\}} j\lambda'_j(z)$ are uniformly convergent on $|z - c| \leq \delta$, their sums are λ' and F' respectively, and λ and F are of class C^1 on $|z - c| \leq \delta$. We further assume that $F'(c) < 0$.

Assumption 4s: Assume that

$$\sup_{j\in\mathbb{Z}\setminus\{0\}} \frac{\|\lambda'_j\|_\delta}{\lambda_j(c)} < \infty.$$

This assumption further implies that the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2\lambda'_j(z)$ is uniformly convergent on $|z - c| \leq \delta$, its sum is $(\sigma^2)'$ and σ^2 is also of class C^1 on $|z - c| \leq \delta$.

Assumption 5: We write $\|\lambda''_j\|_\delta := \sup_{|z-c|\leq\delta} |\lambda''_j(z)|$, for the δ in **Assumption 2** and for any jump-size j, and assume that

$$\sup_{j\in\mathbb{Z}\setminus\{0\}} \frac{\|\lambda''_j\|_\delta}{j^2\lambda_j(c)} < \infty.$$

This assumption implies, in view of the **Assumptions 1 to 3**, that the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda''_j(z)$ is uniformly convergent on $|z - c| \leq \delta$, its sum is λ'' and λ is now of class C^2 on $|z - c| \leq \delta$.

Assumption 5s: Assume that

$$\sup_{j\in\mathbb{Z}\setminus\{0\}} \frac{\|\lambda_j''\|_\delta}{|j|\lambda_j(c)} < \infty.$$

This assumption further implies that the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} j\lambda_j''(z)$ is uniformly convergent on $|z-c|\le\delta$, its sum is F'' and F is now of class C^2 on $|z-c|\le\delta$.

In the example, **Assumption 1** is satisfied if $d > \sum_{j\ge1} jb_j$, with $c = a/(d - \sum_{j\ge1} jb_j)$. **Assumption 2** is satisfied with $\lambda^0 = a/2$ and $c_1 = \max\{b_1, a + b_1 c\}$, and, for instance, for $\delta = c/2$ and $\varepsilon = 1/2$, for $c_j = b_j \max\{1, c\}$, with $j \ge 2$, and $c_{-1} = d\max\{1, c\}$. For any $\delta > 0$, we have $\mu_\delta = \delta(d - \sum_{j\ge1} jb_j)$. **Assumption 3** is satisfied if $\sum_{j\ge1} |j|^{2+\alpha} b_j < \infty$, in which case the other assumptions follow immediately.

The first section introduces the infinitesimal generator of a Markov process and its nice property, especially under equilibrium, known as Dynkin's formula, as given by Hamza and Klebaner (1995).

The second section gives a brief account of the Stein-Chen method for Poisson approximation, together with a couple of useful inequalities. This technique, along with the mentioned inequalities, will then be adapted for the translated Poisson approximation.

1. The generator of a Markov process and Dynkin's formula

The infinitesimal generator $\mathcal{A}$ of a Markov process X_t is the generator of a semigroup of linear operators $\{T(t),\ t \ge 0\}$ on a Banach space L, where

$$T(t)h(x) := \mathbb{E}\{h(X_t) \mid X_0 = x\},$$

for any $h \in L$. The semigroup conditions are indeed fulfilled: we have that $T(0) = I$ and $T(s+t) = T(s)T(t)$, for any $s, t \ge 0$.

The infinitesimal generator of a semigroup of linear operators $\{T(t)\}$ on L is the linear operator $\mathcal{A}$ defined by

$$\mathcal{A}h := \lim_{t\to0} \frac{1}{t}\{T(t)h - h\}.$$

The domain of $\mathcal{A}$ is the subspace of all $h \in L$ for which this limit exists.

Acting on a function $h\colon \mathbb{Z} \to \mathbb{R}$ in its domain, the infinitesimal generator $\mathcal{A}_n$ of the Markov process Z_n has the form

$$(\mathcal{A}_n h)(i) := \sum_{j\in\mathbb{Z}\setminus\{0\}} n\lambda_j\Big(\frac{i}{n}\Big)\big[h(i+j) - h(i)\big],$$

for any $i \in \mathbb{Z}$.

The next lemma gives an alternative expression for $\mathcal{A}_n$, more suitable to our purposes.

LEMMA 2.1. *Under* **Assumptions 1 to 3**, *for any function* $h\colon \mathbb{Z} \to \mathbb{R}$ *whose first difference* $g(i) := \triangle h(i) := h(i+1) - h(i)$ *is a bounded function on* $\mathbb{Z}$, *we have*

$$\begin{aligned}
(\mathcal{A}_n h)(i) \quad &= \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) \bigtriangledown g(i) + nF\Big(\frac{i}{n}\Big)g(i)\\
&-\frac{n}{2}F\Big(\frac{i}{n}\Big) \bigtriangledown g(i)\\
&+\sum_{j=2}^{\sqrt{n}}\Big[\sum_{k=2}^{j}\binom{k}{2} \bigtriangledown^2 g(i+j-k+1)\Big]n\lambda_j\Big(\frac{i}{n}\Big)\\
&+\sum_{j>\sqrt{n}}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown g(i+j-k) - \bigtriangledown g(i)\Big)\Big]n\lambda_j\Big(\frac{i}{n}\Big)\\
&-\sum_{j=2}^{\sqrt{n}}\Big[\sum_{k=2}^{j}\binom{k}{2} \bigtriangledown^2 g(i-j+k)\Big]n\lambda_{-j}\Big(\frac{i}{n}\Big)\\
&-\sum_{j>\sqrt{n}}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown g(i) - \bigtriangledown g(i-j+k)\Big)\Big]n\lambda_{-j}\Big(\frac{i}{n}\Big), \quad i \in \mathbb{Z},
\end{aligned}$$

where we write $\bigtriangledown g(i) := g(i) - g(i-1)$.

PROOF. A proof is given in the **Appendix**. □

PROPOSITION 2.2 (Application of Hamza and Klebaner (1995), Corollary 2.1). *Assume that*

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} |j| n\lambda_j\Big(\frac{i}{n}\Big) = O(|i|), \quad |i| \to \infty,$$

then the process Z_n *is regular a.s. and for all* t, $\mathbb{P}\{|Z_n(t)| < \infty\} = 1$.

PROOF. Indication: Note that, in our notation, the equivalent of $\lambda(z)$ in Hamza and Klebaner (1995) is $n\lambda(\frac{i}{n})$, and of $|m(z)|$ is $\sum_{j\in\mathbb{Z}\setminus\{0\}} |j| \cdot \frac{\lambda_j(\frac{i}{n})}{\lambda(\frac{i}{n})}$, since $\pi(z, dx)$ in the Hamza and Klebaner (1995) notation is, in our notation, the ration between $n\lambda_j(\frac{i}{n})$ and $n\lambda(\frac{i}{n})$. □

THEOREM 2.3 (Application of Hamza and Klebaner (1995), Theorem 3.2). *Assume that the process* Z_n *is regular and integer-valued. Let* h *be an unbounded function such that there exists* $c^0 > 0$ *so that*

$$(|\mathcal{A}_n|h)(i) := \sum_{j\in\mathbb{Z}\setminus\{0\}} n\lambda_j\Big(\frac{i}{n}\Big)|h(i+j) - h(i)| \le nc^0(1 \vee |h(i)|), \quad |i| \to \infty.$$

If $h(Z_n(0))$ *is integrable, then, for any* $t \ge 0$, *so is* $h(Z_n(t))$; *moreover*

$$h(Z_n(t)) - h(Z_n(0)) - \int_0^t (\mathcal{A}_n h)(Z_n(s)))ds$$

is a martingale, and Dynkin's formula holds:

$$\mathbb{E}[h(Z_n(t)) - h(Z_n(0))] = \int_0^t \mathbb{E}(\mathcal{A}_n h)(Z_n(s))ds.$$

Hence, if Z_n is in equilibrium, then

$$\mathbb{E}(\mathcal{A}_n h)(Z_n) = 0. \tag{2.1.1}$$

PROOF. Indication: Again, $\lambda(z) \cdot \pi(z, dx)$ in Hamza and Klebaner (1995) is the equivalent, in our notation, of $n\lambda_j(\frac{i}{n})$, and we have used the notation h for the function f. □

2. The Stein-Chen method for translated Poisson approximation

First developed by Chen (1975), Poisson approximation using Stein's method is based on the following argument, as described in Barbour (2001).

For any $\lambda > 0$ and any subset $A \subset \mathbb{Z}_+$, it is possible to find a function $g_{\lambda,A} \colon \mathbb{Z}_+ \to \mathbb{R}$ such that

$$\mathbb{1}_A(i) - \mathrm{Po}(\lambda)\{A\} = \lambda\, g_{\lambda,A}(i+1) - i\, g_{\lambda,A}(i), \quad i \geq 0. \tag{2.2.1}$$

The function $g_{\lambda,A}$ is called *the solution to the Stein Equation* (2.2.1) and can be recursively determined on $\mathbb{Z}_+ \setminus \{0\}$; the value of $g_{\lambda,A}(0)$ is irrelevant, and taken here to be 0. Moreover, the function $g_{\lambda,A}$ is bounded on $\mathbb{Z}_+$, and by Barbour and Eagleson (1983) it holds that

$$\begin{aligned} \sup_{A\subset\mathbb{Z}_+} \sup_{i\geq 1} |g_{\lambda,A}(i)| &\leq \min\left\{1, \frac{1.4}{\sqrt{\lambda}}\right\} \\ \sup_{A\subset\mathbb{Z}_+} \sup_{i\geq 1} |\bigtriangledown g_{\lambda,A}(i)| &\leq \frac{1-e^{-\lambda}}{\lambda}\ . \end{aligned} \tag{2.2.2}$$

Now if we write $l := i - \lfloor\lambda\rfloor$ and $B := A - \lfloor\lambda\rfloor = \{i - \lfloor\lambda\rfloor,\ i \in A\}$, and define $\tilde{g}_{\lambda,B}(l) := g_{\lambda,B+\lfloor\lambda\rfloor}(l + \lfloor\lambda\rfloor) = g_{\lambda,A}(i)$, then (2.2.1) becomes:

$$\mathbb{1}_B(l) - \widehat{\mathrm{Po}}(\lambda)\{B\} = \lambda\ \triangle\, \tilde{g}_{\lambda,B}(l) - l\tilde{g}_{\lambda,B}(l) + \langle\lambda\rangle\tilde{g}_{\lambda,B}(l) \tag{2.2.3}$$

for any $l \in \mathbb{Z}$, $l \geq -\lfloor\lambda\rfloor$, where the notation $\langle\lambda\rangle := \lambda - \lfloor\lambda\rfloor$ stands for the fractional part of λ, and $\widehat{\mathrm{Po}}(\lambda) := \mathrm{Po}(\lambda) - \lfloor\lambda\rfloor$. Then (2.2.2) imply that:

$$\begin{aligned} \sup_{B\subset\mathbb{Z}_+-\lfloor\lambda\rfloor} \sup_{l\geq 1-\lfloor\lambda\rfloor} |\tilde{g}_{\lambda,B}(l)| &\leq \min\left\{1, \frac{1.4}{\sqrt{\lambda}}\right\} \\ \sup_{B\subset\mathbb{Z}_+-\lfloor\lambda\rfloor} \sup_{l\geq 1-\lfloor\lambda\rfloor} |\bigtriangledown \tilde{g}_{\lambda,B}(l)| &\leq \frac{1-e^{-\lambda}}{\lambda}\ . \end{aligned} \tag{2.2.4}$$

Remember that, since $g_{\lambda,A}(0) = 0$, then also $\tilde{g}_{\lambda,B}(-\lfloor\lambda\rfloor) = 0$.

CHAPTER 3

Existence theorem for the equilibrium distribution Π_n

THEOREM 3.1. *Under* **Assumptions 1 to 4**, *for n fixed large enough, there exists an equilibrium distribution π_n for the process z_n. Moreover, the following inequalities hold:*

$$\begin{aligned} \mathbb{E}_{\pi_n}\{|z_n - c| \cdot \mathbb{1}(|z_n - c| \geq \delta)\} &= O\Big(\frac{1}{n}\Big) \\ \mathbb{E}_{\pi_n}\{(z_n - c)^2 \cdot \mathbb{1}(|z_n - c| < \delta)\} &= O\Big(\frac{1}{n}\Big), \end{aligned} \tag{3.0.1}$$

for the δ in **Assumption 2**.

PROOF. The proof is inspired by Barbour (1980, Proof of Theorem 2.1). We look for a suitable function, which in a neighborhood of c is a Lyapounov function and otherwise has the nice property that it fulfils the conditions of Theorem 2.3, so that we can use Dynkin's formula. The inequality that we obtain proves both the existence of the limiting distribution π_n and the first inequality in (3.0.1). To prove the second inequality, we use a similar procedure, but with a different function.

Since F is of class C^1 on $|z-c| \leq \delta$, a solution to the equation $\dot{x} = F(x)$ exists locally and is unique, for every initial point z in $|z-c| \leq \delta$.

Consider the twice continuously differentiable function $V \colon \mathbb{R} \to \mathbb{R}_+$ defined by $V(z) := |z-c|^{2+\alpha}$, for the α in **Assumption 3**. Since $(z-c)F(z) < 0$, we have that $\operatorname{sign}\{z-c\}F(z) = -|F(z)|$ for all $z \neq c$. Also, since $V(c) = 0$ and $V(z) > 0$ for any $z \neq c$, and given that

$$F(z)V'(z) = -|F(z)|(2+\alpha)|z-c|^{1+\alpha} < 0, \quad \text{for any } z \neq c,$$

while $F(c)V'(c) = 0$, we conclude that V is a good candidate for a Lyapounov function, whose existence guarantees the asymptotic stability of the constant solution c of the equation $\dot{x} = F(x)$.

A useful remark for the following computations is that $(x+y)^\alpha \leq x^\alpha + y^\alpha$, for any positive x and y, and any $\alpha \in (0,1]$. We let δ_z denote the point mass at z.

LEMMA 3.2. *Under the assumptions of Theorem 3.1, the function $h(i) := V\big(\frac{i}{n}\big) = \big|\frac{i}{n} - c\big|^{2+\alpha}$, with $i \in \mathbb{Z}$, fulfils the conditions of Theorem 2.3 with respect to the initial distribution δ_l, for any $l \in \mathbb{Z}$.*

PROOF. Let $\mathcal{A}_n$ denote the infinitesimal generator of the process z_n. Then, for any $i \in \mathbb{Z}$, writing $z = i/n$ and for some $\theta_j(z) \in (0,1)$, $j \in \mathbb{Z} \setminus \{0\}$, we have

$$(|\mathcal{A}_n|\, h)(i) \quad := \sum_{j \in \mathbb{Z}\setminus\{0\}} n\lambda_j\Big(\frac{i}{n}\Big)\Big|V\Big(\frac{i+j}{n}\Big) - V\Big(\frac{i}{n}\Big)\Big|$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z)\Big|jV'(z) + \frac{j^2}{2n}V''\Big(z+\theta_j(z)\frac{j}{n}\Big)\Big| \\
&\le (2+\alpha)|z-c|^{1+\alpha} \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|\ c_j(1+|z-c|) \\
&+\frac{(2+\alpha)(1+\alpha)|z-c|^{\alpha}}{2n} \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j(1+|z-c|) \\
&+\frac{(2+\alpha)(1+\alpha)}{2n^{1+\alpha}} \sum_{j\in\mathbb{Z}\setminus\{0\}} \theta_j^{\alpha}(z)|j|^{2+\alpha} c_j(1+|z-c|).
\end{aligned} \tag{3.0.2}
$$

For $|z-c|<\delta$, the right hand side is, under **Assumption 3**, uniformly bounded by C_{1n}, where

$$
C_{1n} := (1+\delta)^{2+\alpha}(2+\alpha)\Big\{\sum_j |j|c_j + \frac{(1+\alpha)}{2n}\sum_j j^2 c_j + \frac{(1+\alpha)}{2n^{1+\alpha}}\sum_j |j|^{2+\alpha}c_j\Big\}.
$$

For $|z-c|\ge\delta$, **Assumption 3** implies that

$$
(|\mathcal{A}_n|\,V)(z) \le C_{1n}|z-c|^{2+\alpha} = C_{1n}V(z).
$$

□

The above lemma allows us to apply Dynkin's formula to the function V. In view of (3.0.2), on $|z-c|<\delta$, the quantity

$$
(\mathcal{A}_n\,V)(z) := \sum_{j\in\mathbb{Z}\setminus\{0\}} n\lambda_j(z)\Big[V\Big(z+\frac{j}{n}\Big)-V(z)\Big]
$$

has the property that

$$
\begin{aligned}
(\mathcal{A}_n\,V)(z) &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z)\Big[jV'(z) + \frac{j^2}{2n}V''\Big(z+\theta_j(z)\frac{j}{n}\Big)\Big] \\
&\le -|F(z)|(2+\alpha)|z-c|^{1+\alpha} + \frac{C_2}{n} \le \frac{C_2}{n},
\end{aligned} \tag{3.0.3}
$$

for some $C_2>0$. On $|z-c|\ge\delta$ and under **Assumptions 1 and 3**, the generator has the property that

$$
\begin{aligned}
(\mathcal{A}_n\,V)(z) \quad &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z)\Big[jV'(z) + \frac{j^2}{2n}V''\Big(z+\theta_j(z)\frac{j}{n}\Big)\Big] \\
&= -|F(z)|(2+\alpha)|z-c|^{1+\alpha}\ \Big[1- \\
&-\frac{(1+\alpha)}{2n|F(z)|\cdot|z-c|} \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j(1+|z-c|) \\
&-\frac{(1+\alpha)}{2n^{1+\alpha}|F(z)|\cdot|z-c|^{1+\alpha}} \sum_{j\in\mathbb{Z}\setminus\{0\}} \theta_j^{\alpha}(z)|j|^{2+\alpha} c_j(1+|z-c|)\Big],
\end{aligned}
$$

$$\leq -\frac{\mu_\delta(2+\alpha)}{2}|z-c|^{1+\alpha} \leq -\mu_\delta |z-c|^{1+\alpha}, \tag{3.0.4}$$

as long as n is large enough that

$$\frac{4(1+\alpha)}{2n\mu_\delta} \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2+\alpha} c_j < \frac{1}{2}.$$

Dynkin's formula implies, for such n, that

$$\begin{aligned} 0 &\leq \mathbb{E}_{\delta_z} V(z_n(t)) = V(z) + \int_0^t \mathbb{E}_{\delta_z}(\mathcal{A}_n V)(z_n(s))ds \\ &\leq V(z) + \int_0^t \frac{C_2}{n}\mathbb{P}_{\delta_z}(|z_n(s)-c| < \delta)ds \\ &\quad -\mu_\delta \int_0^t \mathbb{E}_{\delta_z}\{|z_n(s)-c|^{1+\alpha}\cdot \mathbb{1}(|z_n(s)-c|\geq\delta)\}\,ds, \end{aligned}$$

for any $t>0$ and $z\in \frac{1}{n}\mathbb{Z}$. It now follows, for any $y\geq\delta$, that

$$\begin{aligned} &\frac{\mu_\delta\, y^{1+\alpha}}{t}\int_0^t \mathbb{P}_{\delta_z}(|z_n(s)-c|\geq y)ds \\ &\leq \frac{\mu_\delta}{t}\int_0^t \mathbb{E}_{\delta_z}\{|z_n(s)-c|^{1+\alpha}\cdot \mathbb{1}(|z_n(s)-c|\geq y)\}\,ds \\ &\leq \frac{1}{t}V(z) + \frac{C_2}{nt}\int_0^t \mathbb{P}_{\delta_z}(|z_n(s)-c|<\delta)ds, \end{aligned} \tag{3.0.5}$$

and by letting $t\to\infty$, we obtain that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}_{\delta_z}(|z_n(s)-c|\geq y)ds \leq \frac{C_2}{n\mu_\delta\, y^{1+\alpha}}.$$

By the general theory of Markov processes, see for instance Ethier and Kurtz (1986) Theorem 9.3, Chapter 4, this is equivalent to the fact that a limiting equilibrium distribution π_n for z_n exists, and satisfies

$$\mathbb{P}_{\pi_n}(|z_n-c|\geq y) \leq \frac{C_2}{n\mu_\delta\, y^{1+\alpha}},\ \text{ for any } y\geq\delta. \tag{3.0.6}$$

Furthermore, $|z_n-c|$ is a positive random variable, so we may now write

$$\begin{aligned} \mathbb{E}_{\pi_n}\{|z_n-c|\cdot\mathbb{1}(|z_n-c|\geq\delta)\} &= \int_\delta^\infty \mathbb{P}_{\pi_n}(|z_n-c|\geq y)dy \\ &\leq \int_\delta^\infty \frac{C_2}{n\mu_\delta\, y^{1+\alpha}}\,dy = O\Big(\frac{1}{n}\Big), \end{aligned}$$

proving the first inequality in (3.0.1).

To prove the second inequality in (3.0.1), we define a function $\tilde{V}\colon\mathbb{R}\to\mathbb{R}$, which has four properties important to us: it is of class $C^2(\mathbb{R})$, it has uniformly bounded first and second derivatives on $\mathbb{R}$, it fulfils the conditions of Theorem 2.3 and $F(z)\tilde{V}'(z) = -(z-c)^2$ on $|z-c|\leq\delta$.

In view of the latter property, we begin by letting $v\colon [c-\delta, c+\delta] \to \mathbb{R}_+$ be a function defined as follows:

$$v(z) := \int_c^z \frac{-(x-c)^2}{F(x)} dx,$$

with $v(c) = 0$. Note that v is well defined, since $F'(x) < 0$ on a small enough neighborhood of c, and that $v(z) > 0$, for any $z \neq c$.

Remark 1. *Note that, under* **Assumptions 1 and 4**,

$$v'(z) = -\frac{(z-c)^2}{F(z)} \quad \text{and} \quad v''(z) = \frac{(z-c)^2 F'(z) - 2(z-c)F(z)}{F^2(z)},$$

exist and are continuous on $|z-c| \leq \delta$, since $|F(z)| > 0$ for $z \neq c$, $F(z) \sim F'(c)(z-c)$ for $z \to c$, and F' is continuous; in particular, we have that

$$v'(c) = \lim_{z\to c} v'(z) = 0 \quad \text{and} \quad v''(c) = \lim_{z \to c} v''(z) = -\frac{1}{F'(c)} > 0.$$

Define the function $\tilde{V}(z)$ as follows:

$$\tilde{V}(z) := \begin{cases} v(c-\delta) - \delta + |z-c| & +[v'(c-\delta)+1]\sin(z-c+\delta) \\ & +\frac{1}{2}v''(c-\delta)\sin^2(z-c+\delta), \qquad \text{if } z < c-\delta \\ v(z), & \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |z-c| \leq \delta \\ v(c+\delta) - \delta + |z-c| & +[v'(c+\delta)-1]\sin(z-c-\delta) \\ & +\frac{1}{2}v''(c+\delta)\sin^2(z-c-\delta), \qquad \text{if } z > c+\delta. \end{cases}$$

Remark 2. *Note that the function $\tilde{V}$ is of class $C^2(\mathbb{R})$, and that*

$$|\tilde{V}'(z)| \leq C_3 \quad \text{and} \quad |\tilde{V}''(z)| \leq C_3,$$

for any $z \in \mathbb{R}$, where

$$C_3 := \max\{2+|v'(c-\delta)|+3|v''(c-\delta)|, \sup_{|z-c|\leq\delta}(|v'(z)|+|v''(z)|), 2+|v'(c+\delta)|+3|v''(c+\delta)|\}.$$

LEMMA 3.3. *Under the assumptions of Theorem 3.1, and in view of the above remark, the function $\tilde{h}(i) := \tilde{V}\left(\frac{i}{n}\right)$, $i \in \mathbb{Z}$, fulfils the conditions of Theorem 2.3 with respect to the initial distribution π_n.*

PROOF. For any $i \in \mathbb{Z}$, write $z = i/n$. By definition, $\tilde{V}(z) \sim |z-c|$, for $|z| \to \infty$, and hence $\mathbb{E}_{\pi_n}|\tilde{V}(z_n)| < \infty$, by the first inequality in (3.0.1). Then, for some $\tilde{\theta}_j(z) \in (0,1)$, $j \in \mathbb{Z}\setminus\{0\}$, we have under **Assumptions 1, 2 and 3** that

$$\begin{aligned} (|\mathcal{A}_n|\,\tilde{h})(i) \quad &:= \sum_{j\in\mathbb{Z}\setminus\{0\}} n\lambda_j\Big(\frac{i}{n}\Big)\left|\tilde{V}\Big(\frac{i+j}{n}\Big) - \tilde{V}\Big(\frac{i}{n}\Big)\right| \\ &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z)\left|j\tilde{V}'(z) + \frac{j^2}{2n}\tilde{V}''\Big(z+\tilde{\theta}_j(z)\frac{j}{n}\Big)\right| \\ &\leq C_3\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}}|j|c_j + \frac{1}{2n}\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\Big)(1+|z-c|) \\ &= O(1+|\tilde{V}(z)|) = O(1 \vee |\tilde{h}(i)|). \end{aligned}$$

□

We now apply Dynkin's formula to $\tilde V$, and obtain from

$$0 = \mathbb{E}_{\pi_n}\{(\mathcal{A}_n \tilde V)(z_n)\} = \mathbb{E}_{\pi_n}\Big\{F(z_n)\tilde V'(z_n) + \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z_n)\frac{j^2}{2n}\tilde V''\Big(z_n + \tilde\theta_j(z_n)\frac{j}{n}\Big)\Big\}$$

that

$$\begin{aligned}&\mathbb{E}_{\pi_n}\{-F(z_n)\tilde V'(z_n)\cdot \mathbb{1}(|z_n-c|<\delta)\}\\ &= \mathbb{E}_{\pi_n}\Big\{F(z_n)\tilde V'(z_n)\cdot \mathbb{1}(|z_n-c|\ge\delta) + \sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z_n)\frac{j^2}{2n}\tilde V''\Big(z_n + \tilde\theta_j(z_n)\frac{j}{n}\Big)\Big\}\end{aligned}$$

from where it follows that

$$\begin{aligned}&\mathbb{E}_{\pi_n}\{(z_n-c)^2\cdot \mathbb{1}(|z_n-c|<\delta)\}\\ &\le \mathbb{E}_{\pi_n}\{|F(z_n)\tilde V'(z_n)|\cdot \mathbb{1}(|z_n-c|\ge\delta)\}\\ &+ \mathbb{E}_{\pi_n}\Big\{\sum_{j\in\mathbb{Z}\setminus\{0\}} \lambda_j(z_n)\frac{j^2}{2n}\Big|\tilde V''\Big(z_n + \tilde\theta_j(z_n)\frac{j}{n}\Big)\Big|\Big\}\\ &\le C_3 \sum_{j\in\mathbb{Z}\setminus\{0\}} \Big(2|j| + \frac{j^2}{n}\Big)c_j\ \mathbb{E}_{\pi_n}\{|z_n-c|\cdot \mathbb{1}(|z_n-c|\ge\delta)\} + \frac{C_3}{2n}\sup_{|z-c|<\delta}\sigma^2(z).\end{aligned}$$

Using the first inequality in (3.0.1), we finally obtain that there exists a constant $C_4 > 0$ so that

$$\mathbb{E}_{\pi_n}\{(z_n-c)^2\cdot \mathbb{1}(|z_n-c|<\delta)\} \le \frac{C_4}{n}, \tag{3.0.7}$$

proving the second inequality in (3.0.1).

□

COROLLARY 3.4. *Under* **Assumptions 1 to 4**, *if $z_n \sim \pi_n$ is in equilibrium, then*

$$\mathbb{E}\{|z_n-c|\} = O\Big(\frac{1}{\sqrt{n}}\Big).$$

PROOF. Using truncation and Hölder's inequality, we write as follows:

$$\begin{aligned}&\mathbb{E}\{|z_n-c|\}\\ &= \mathbb{E}\{|z_n-c|\cdot \mathbb{1}(|z_n-c|\ge\delta)\} + \mathbb{E}\{|z_n-c|\cdot \mathbb{1}(|z_n-c|<\delta)\}\\ &\le \mathbb{E}\{|z_n-c|\cdot \mathbb{1}(|z_n-c|\ge\delta)\} + \sqrt{\mathbb{E}\{(z_n-c)^2\cdot \mathbb{1}(|z_n-c|<\delta)\}}.\end{aligned}$$

With the estimates (3.0.1) we now deduce that

$$\mathbb{E}\{|z_n-c|\} \le O\Big(\frac{1}{n}\Big) + O\Big(\frac{1}{\sqrt{n}}\Big) = O\Big(\frac{1}{\sqrt{n}}\Big),$$

which is the desired bound. □

COROLLARY 3.5. *There exists, for any fixed $a > 0$, a constant $C_\delta > 0$ so that*

$$\mathbb{P}_{\pi_n}(|z_n-c|\ge a) \le \frac{C_\delta}{n(a\wedge\delta)^2}.$$

PROOF. If $a \geq \delta$, then by (3.0.6) it follows that

$$\mathbb{P}_{\pi_n}(|z_n - c| \geq a) \leq \frac{C_2}{n\mu_\delta\, a^{1+\alpha}} \leq \frac{C_2}{n\mu_\delta\, \delta^{1+\alpha}} \leq \frac{C_2\delta^{1-\alpha}}{n\mu_\delta\, \delta^2}.$$

If $a < \delta$, then by Cauchy's inequality and in view of (3.0.6) and (3.0.7), we have that

$$\begin{aligned}
\mathbb{P}_{\pi_n}(|z_n - c| \geq a) &= \mathbb{P}_{\pi_n}(|z_n - c| \cdot \mathbb{1}(|z_n - c| < \delta) \geq a) + \mathbb{P}_{\pi_n}(|z_n - c| \geq \delta) \\
&\leq \frac{\mathbb{E}_{\pi_n}\{|z_n - c|^2 \cdot \mathbb{1}(|z_n - c| < \delta)\}}{a^2} + \frac{C_2}{n\mu_\delta\, \delta^{1+\alpha}} \\
&\leq \frac{C_4}{na^2} + \frac{C_2\delta^{1-\alpha}}{n\mu_\delta\, \delta^2} \leq \frac{C_4}{na^2} + \frac{C_2\delta^{1-\alpha}}{n\mu_\delta\, a^2}.
\end{aligned}$$

Now choose $C_\delta := C_4 + \frac{C_2\delta^{1-\alpha}}{\mu_\delta}$, and we have obtained the desired result.

□

The existence of an equilibrium distribution π_n for the process z_n implies the existence of an equilibrium distribution Π_n for the process $Z_n = nz_n$. One writes

$$\Pi_n\{i\} := \pi_n\{i/n\}, \quad \text{for all} \;\; i \in \mathbb{Z}.$$

CHAPTER 4

The distance between Π_n and its unit translation

If the equilibrium distribution Π_n, suitably translated, is indeed $O(1/\sqrt{n})$ close to a Poisson distribution with parameter $n\rho$, say, then its unit translate is correspondingly close to $\mathrm{Po}(n\rho)$ translated by 1. Then, since the total-variation distance between $\mathrm{Po}(n\rho)$ and $\mathrm{Po}(n\rho)+1$ is of order $O(1/\sqrt{n})$, the same has to be true of the total-variation distance between Π_n and its unit translate. However, as illustrated in Barbour and Xia (1999) and Barbour and Čekanavičius (2002), it is extremely useful to be able to establish this latter fact in advance, in order to prove the translated Poisson approximation theorem, using Stein's method, in the same way that proving a concentration inequality is a useful prerequisite for deriving Berry-Esseen approximations in the central limit theorem, see Chen and Shao (2003). In this chapter, we establish such a bound.

THEOREM 4.1. *Under* **Assumptions 1 to 3, 4s and 5**, *there exists a constant* $K > 0$ *so that*

$$d_{TV}\{\Pi_n, \Pi_n * \delta_1\} \leq Kn^{-1/2},$$

where $\Pi_n * \delta_1$ *denotes the equilibrium distribution* Π_n *of* Z_n, *translated with 1.*

PROOF. The proof of Theorem 4.1 requires a number of steps.

Our first problem is that we have little direct information about Π_n, other than that it is the equilibrium distribution of the Markov process with generator $\mathcal{A}_n$. This we can exploit more easily if we can translate our problem into one involving transition probabilities instead, which we do at **Step 1**.

We now need to compare $\mathcal{L}(Z_n(U) \mid Z_n(0) = z)$ with $\mathcal{L}(Z_n(U) + 1 \mid Z_n(0) = z)$ for any given and fixed U, where Z_n is the Markov process with generator $\mathcal{A}_n$. A good way of doing this is to find a random variable N embedded in $Z_n(U)$ for which $\mathcal{L}(N)$ and $\mathcal{L}(N + 1)$ are close enough, and to exploit this. Here, we use a Poisson process with jumps of size $+1$ at rate $n\lambda^0$ to provide our N. We thus need to split Z_n into a sum of two processes, one of which is this Poisson process. The construction of the appropriate bivariate process (X_n, N_n) is done at **Step 2**.

Using the newly defined processes, we are able to bound $d_{TV}\{\Pi_n, \Pi_n * \delta_1\}$ by the sum of two terms, see **Step 3**. The first one of these we can further bound by the total-variation distance between a Poisson distribution and its unit translation. This distance, by Barbour, Holst and Janson (1992, Theorem 1.C), can be bounded by $O(1/\sqrt{n})$, see **Step 4**.

The last step, **Step 5**, is concerned with finding a bound of the same size on the second term in the sum from **Step 3**. Here, the essence of the argument is to show that the conditional distribution of the paths of X_n on $[0, U]$, given the path of the Poisson process $\{N_n(t),\ 0 \leq t \leq U\}$, changes only little in total variation if an extra jump is added to N_n : that is to say, $\mathcal{L}(X_n(U) \mid N_n(t) = n(t), 0 \le t \le U; X_n(0) = z)$ and $\mathcal{L}(X_n(U) \mid N_n(t) = n(t) + \mathbb{1}_{[s^*,\infty)}(t), 0 \le t \le U; X_n(0) = z)$, where s^* denotes the time of the extra jump, are $O(1/\sqrt{n})$ close. The detailed argument requires several sub-steps, that we shall explain in due time.

Step 1: Fix a time $U > 0$. Since Π_n is the equilibrium distribution of Z_n, we have that $\Pi_n = \Pi_n \cdot P_n(U)$, where $P_n(t)$ denotes the transition matrix for the chain Z_n over a time interval of length t. It thus follows that

$$\begin{aligned} &d_{TV}\{\Pi_n, \Pi_n * \delta_1\} \\ &\le \sum_{z\in\mathbb{Z}} \Pi_n(z) d_{TV}\{\mathcal{L}(Z_n(U) \mid Z_n(0) = z), \mathcal{L}(Z_n(U) + 1 \mid Z_n(0) = z)\}, \end{aligned} \tag{4.0.1}$$

so that we have reduced the problem to one concerning the conditional distribution of $Z_n(U)$, given $Z_n(0)$.

Step 2: Construct a 2-dimensional Markov process $(N_n(t), X_n(t))_{t\ge 0}$ on the probability space $(\Omega, \mathcal{K}, \mathbb{P})$, having as the first component a Poisson process with rate $n\lambda^0$. This process starts at $(N_n(0), X_n(0)) = (0, z)$ with probability $\Pi_n(z)$, and its transition rates are, under **Assumption 2**, well defined through

$$\begin{aligned} &(l,x) \to (l+1,x) && \text{at rate } n\lambda^0 \\ &(l,x) \to (l,x+1) && \text{at rate } n[\lambda_1\left(\tfrac{l+x}{n}\right) - \lambda^0] \\ &(l,x) \to (l,x+j) && \text{at rate } n\lambda_j\left(\tfrac{l+x}{n}\right), \text{ for any } j \in \mathbb{Z}, j \ne 0, 1. \end{aligned}$$

Note that N_n and X_n never jump simultaneously, a.s., and that the process $N_n + X_n$ has the same transition probabilities as Z_n.

Step 3: We are now able to split the right hand side of (4.0.1) into a sum of two terms that will both prove (the second one with considerable difficulty) to have very small expectations with respect to Π_n.

Note that $X_n(0)$ and $Z_n(0)$ have the same distribution, Π_n. We may write

$$\begin{aligned} &\mathbb{P}(Z_n(U) = k \mid Z_n(0) = z) \\ &= \sum_{l\ge 0} \mathbb{P}(N_n(U) = l)\mathbb{P}(X_n(U) = k - l \mid N_n(U) = l, X_n(0) = z) \end{aligned} \tag{4.0.2}$$

and

$$\begin{aligned} &\mathbb{P}(Z_n(U) = k - 1 \mid Z_n(0) = z) \\ &= \sum_{l\ge 1} \mathbb{P}(N_n(U) = l - 1)\mathbb{P}(X_n(U) = k - l \mid N_n(U) = l - 1, X_n(0) = z). \end{aligned} \tag{4.0.3}$$

Since

$$\begin{aligned} &d_{TV}\{\mathcal{L}(Z_n(U) \mid Z_n(0) = z), \mathcal{L}(Z_n(U) + 1 \mid Z_n(0) = z)\} \\ &= \frac{1}{2} \sum_{k\in\mathbb{Z}} |\mathbb{P}(Z_n(U) = k \mid Z_n(0) = z) - \mathbb{P}(Z_n(U) = k - 1 \mid Z_n(0) = z)|, \end{aligned}$$

it follows from (4.0.2) − (4.0.3), by adding and subtracting the term

$$\sum_{l\geq 1}\mathbb{P}(N_n(U)=l-1)\mathbb{P}(X_n(U)=k-l\mid N_n(U)=l,X_n(0)=z),$$

that

$$\begin{aligned} &d_{TV}\{\mathcal{L}(Z_n(U)\mid Z_n(0)=z),\mathcal{L}(Z_n(U)+1\mid Z_n(0)=z)\}\\ &\quad\leq\frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 0}|\mathbb{P}(N_n(U)=l)-\mathbb{P}(N_n(U)=l-1)|f^{U*}_{l,z}(k-l)\\ &\quad+\frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 1}\mathbb{P}(N_n(U)=l-1)|f^{U*}_{l,z}(k-l)-f^{U}_{l,z}(k-l)|, \end{aligned} \tag{4.0.4}$$

where

$$\begin{aligned} f^{U*}_{l,z}(k-l)&:=\mathbb{P}(X_n(U)=k-l\mid N_n(U)=l,X_n(0)=z)\\ f^{U}_{l,z}(k-l)&:=\mathbb{P}(X_n(U)=k-l\mid N_n(U)=l-1,X_n(0)=z). \end{aligned} \tag{4.0.5}$$

The first term in (4.0.4) involves the difference arising when translating the Poisson distribution $\mathcal{L}(N_n(U))$; the second involves comparing the distributions of $X_n(U)$, given $X_n(0)$, conditional on two almost identical Poisson paths.

Step 4: Note that, when we multiply the first term in (4.0.4) by $\Pi_n(z)$ and sum over $z\in\mathbb{Z}$, we obtain the bound

$$\begin{aligned} &\sum_{z\in\mathbb{Z}}\Pi_n(z)\frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 0}|\mathbb{P}(N_n(U)=l)-\mathbb{P}(N_n(U)=l-1)|f^{U*}_{l,z}(k-l)\\ &\qquad\leq d_{TV}(\mathrm{Po}(n\lambda^0U),\ \mathrm{Po}(n\lambda^0U)*\delta_1)\leq\frac{1}{\sqrt{n\lambda^0U}}=O\Big(\frac{1}{\sqrt{n}}\Big), \end{aligned} \tag{4.0.6}$$

from Barbour, Holst and Janson (1992, Theorem 1.C).

Step 5: In view of (4.0.1), we now need to bound the second term in (4.0.4), multiplied by $\Pi_n(z)$ and summed over $z\in\mathbb{Z}$, as follows:

$$\frac{1}{2}\sum_{z\in\mathbb{Z}}\Pi_n(z)\sum_{l\geq 1}\mathbb{P}(N_n(U)=l-1)\sum_{k\in\mathbb{Z}}|f^{U*}_{l,z}(k-l)-f^{U}_{l,z}(k-l)|=O\Big(\frac{1}{\sqrt{n}}\Big). \tag{4.0.7}$$

Then, from (4.0.1), (4.0.4) and (4.0.6), the desired bound on $d_{TV}\{\Pi_n,\Pi_n*\delta_1\}$ is obtained. We proceed in several sub-steps.

First, in order to be able to perform calculations involving the probability densities (4.0.5), we note that the rates of the process X_n, conditional on a given path of the Poisson process N_n, become time-dependent; their detailed form is given in **Step 5.1**. Then fixing a path $n(\cdot)$ of the process N_n, we are able to give a more detailed description of the densities (4.0.5), see **Step 5.2**. We are actually interested in computing

$$\sum_{k\in\mathbb{Z}}|f^{U*}_{l,z}(k-l)-f^{U}_{l,z}(k-l)|,$$

where each element in the sum is itself an integral of probabilities conditional on a whole path of N_n. To handle this, we switch to considering ratios of conditional densities f^*/f, where each of f^* and f is the probability density of a particular path of X_n, given $N_n(\cdot) = n(\cdot)$ and $N_n(\cdot) = n(\cdot) + \mathbb{1}_{[s^*,\infty)}(\cdot)$, in **Step 5.3**. When well defined, f^*/f can be expressed as the exponential of a sum of terms, see **Step 5.4**. We use a trick which helps us get rid of the exponential and only keep the exponent, so that the sum $\sum_{k\in\mathbb{Z}} |f^{U*}_{l,z}(k-l) - f^{U}_{l,z}(k-l)|$ can now be bounded by the sum of two terms, one being precisely the expectation of the exponent, on some set B and with respect to a rather complicated probability measure, and the other being the probability of the complement of the set B, see **Step 5.5**. We are allowed to choose any set B we like, and this is very useful, since there exists a set B of paths, which has a very large probability and moreover the property that, on B, the transition rates of Z_n and therefore X_n are very well behaved, see **Step 5.6**. This choice of B will allow us to prove that the two terms that sum up at **Step 5.5** are indeed very small. To do so, we split the exponent into a sum of what we call a "martingale term", actually a sum of Itô integrals, and an "error term", so that now we have three terms to bound. At the end of **Step 5.7**, we state three lemmas, giving bounds of order at most $O(1/\sqrt{n})$ on the expectations of the martingale term and the error term, and on the probability of the complement of set B; proving these lemmas finally leads to (4.0.7).

Step 5.1: Conditional on a given path of the Poisson process N_n, the Markov process X_n is time inhomogeneous, with jump-rates from state x at time t given by

$$\begin{aligned} n\lambda_1^N\Big(\frac{x}{n},t\Big) &:= n\Big[\lambda_1\Big(\frac{x+N_n(t)}{n}\Big) - \lambda^0\Big] \quad \text{and} \\ n\lambda_j^N\Big(\frac{x}{n},t\Big) &:= n\lambda_j\Big(\frac{x+N_n(t)}{n}\Big) \quad \text{for all } \ j\in\mathbb{Z}, j\neq 0,1. \end{aligned} \tag{4.0.8}$$

Consequently, the rate with which the process X_n leaves state x at time t is given by

$$n\lambda^N\Big(\frac{x}{n},t\Big) := n\Big[\lambda\Big(\frac{x+N_n(t)}{n}\Big) - \lambda^0\Big]. \tag{4.0.9}$$

Moreover note that, by **Assumption 2**, the functions λ_j^N are of class $C^2(\mathbb{R})$ in their first argument. If $\partial_x\lambda_j^N$ denotes the first-order partial derivative of the function λ_j^N with respect to its first argument, then by differentiating we obtain that

$$\partial_x\lambda_j^N(z,t) = \lambda_j'\Big(z+\frac{N(t)}{n}\Big), \quad \text{for any } j\in\mathbb{Z}\setminus\{0\}, \tag{4.0.10}$$

and for any $t\geq 0$. If $\partial_x^2\lambda_j^N$ denotes the second-order partial derivative of the function λ_j^N with respect to its first argument, then by differentiating twice we obtain that

$$\partial_x^2\lambda_j^N(z,t) = \lambda_j''\Big(z+\frac{N(t)}{n}\Big), \quad \text{for any } j\in\mathbb{Z}\setminus\{0\}, \tag{4.0.11}$$

and for any $t\geq 0$.

Step 5.2: Now we fix a path of the Poisson process N_n, so that we can describe the densities (4.0.5) in more detail. Note that $f^{U*}_{l,z}(k-l)$ and $f^{U}_{l,z}(k-l)$ are both conditional probabilities of the paths of X_n that start at time 0 in z and are at time U in $k-l$, but the difference between these two densities is in the conditioning event: $f^{U*}_{l,z}(k-l)$ is conditioned on the fact that the Poisson process N_n makes l jumps up to time U, while $f^{U}_{l,z}(k-l)$ is conditioned on the fact that the Poisson process N_n makes only $l-1$ jumps up to time U.

For any positive integer k and any unordered, pairwise distinct points $\tau_1,\dots,\tau_k \in [0,U]$, we define on $\mathbb{R}_+$ the following function:

$$n^k(t;\tau_1,\dots,\tau_k) := \sum_{i=1}^{k} \mathbb{1}_{[\tau_i,U]}(t)\ .$$

Then, if $N_n(U)=l$, the whole path of N_n on $[0,U]$ can be described using the function n^l by

$$N_n := n^l(\cdot\,; s_1,\dots,s_{l-1},s^*), \tag{4.0.12}$$

where $\{s_1,s_2,\dots,s_{l-1},s^*\}$ is a set of realizations of $S_1,S_2,\dots,S_{l-1}$ and S^* respectively, independent random variables uniformly distributed on $[0,U]$, and represents the unordered set of the observed l jump-times of N_n in $[0,U]$.

Similarly, if $N_n(U)=l-1$, for any realizations $s_1,\dots,s_{l-1}$ of $S_1,\dots,S_{l-1}$, we take $\{s_1,s_2\dots,s_{l-1}\}$ to be the unordered set of the $l-1$ jump-times of N_n in $[0,U]$, and then the whole path of N_n on $[0,U]$ can be described by

$$N_n := n^{l-1}(\cdot\,; s_1,\dots,s_{l-1}). \tag{4.0.13}$$

In view of (4.0.8) and (4.0.12) − (4.0.13) let

$$\begin{aligned} n\lambda_1^{N*}\Big(\frac{x}{n},t\Big) &:= n\Big[\lambda_1\Big(\frac{x+n^l(t;s_1,\dots,s_{l-1},s^*)}{n}\Big)-\lambda^0\Big] \quad \text{and}\\ n\lambda_j^{N*}\Big(\frac{x}{n},t\Big) &:= n\lambda_j\Big(\frac{x+n^l(t;s_1,\dots,s_{l-1},s^*)}{n}\Big), \quad j\in\mathbb{Z}, j\neq 0,1 \end{aligned} \tag{4.0.14}$$

represent the jump-rates at time t from state x of the process X_n conditional on the event $N_n = n^l(\cdot\,;s_1,\dots,s_{l-1},s^*)$, while

$$\begin{aligned} n\lambda_1^{N}\Big(\frac{x}{n},t\Big) &:= n\Big[\lambda_1\Big(\frac{x+n^{l-1}(t;s_1,\dots,s_{l-1})}{n}\Big)-\lambda^0\Big] \quad \text{and}\\ n\lambda_j^{N}\Big(\frac{x}{n},t\Big) &:= n\lambda_j\Big(\frac{x+n^{l-1}(t;s_1,\dots,s_{l-1})}{n}\Big), \quad j\in\mathbb{Z}, j\neq 0,1 \end{aligned} \tag{4.0.15}$$

represent the jump-rates at time t from the state x of the Markov process X_n conditional on the event $N_n = n^{l-1}(\cdot\,;s_1,\dots,s_{l-1})$.

Remark 3. *Note that, for $t\in[0,s^*)$,*

$$n\lambda_j^{N*}\Big(\frac{x}{n},t\Big) = n\lambda_j^{N}\Big(\frac{x}{n},t\Big)$$

whereas, for $t\in[s^*,U]$, we have that

$$n\lambda_j^{N*}\Big(\frac{x}{n},t\Big) = n\lambda_j^{N}\Big(\frac{x+1}{n},t\Big),$$

for any $j \in \mathbb{Z} \setminus \{0\}$, *since* $n^l(\cdot\,; s_1, \dots, s_{l-1}, s^*) = n^{l-1}(\cdot\,; s_1, \dots, s_{l-1}) + \mathbb{1}_{[s^*,\infty)}(\cdot)$.

Write $s_{[1,l-1]}$ for $s_1, \dots, s_{l-1}$. Now use (4.0.12) and (4.0.13), as well as the total-probability formula, to rewrite (4.0.5) as follows:

$$\begin{aligned} f^{U*}_{l,z}(k-l) &= \frac{1}{U^l} \int_{[0,U]^l} ds_1 ... ds_{l-1} ds^* \\ &\quad \cdot \mathbb{P}(X_n(U) = k-l \mid N_n = n^l(\cdot\,; s_{[1,l-1]}, s^*), X_n(0) = z), \quad \text{and} \\ f^{U}_{l,z}(k-l) &= \frac{1}{U^l} \int_{[0,U]^l} ds_1 ... ds_{l-1} ds^* \\ &\quad \cdot \mathbb{P}(X_n(U) = k-l \mid N_n = n^{l-1}(\cdot\,; s_{[1,l-1]}), X_n(0) = z). \end{aligned} \tag{4.0.16}$$

Step 5.3: We wish to describe in even more detail the conditional probabilities that appear in (4.0.16). We do so using a new probability measure $\mathbb{Q}_U$ defined on a new probability space $(\Omega_U, \mathcal{K}_U)$. What we obtain in the end is a more explicit form for the term of interest $\sum_{k\in\mathbb{Z}} |f^{U*}_{l,z}(k-l) - f^{U}_{l,z}(k-l)|$, in terms of the ratio of two probability densities. Of course, we shall need to make sure that this ratio is well defined (i.e. we do not divide positive terms by 0.)

For each $j \in \mathbb{Z} \setminus \{0\}$, let $\xi^{(n)}_j$ denote the point process of the times of the jumps of size j of X_n in $[0, U]$. We also write $\xi^{(n)}_j(t) := \xi^{(n)}_j\{[0, t]\}$, for any $j \in \mathbb{Z} \setminus \{0\}$, and $R_n(t) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \xi^{(n)}_j(t)$, for any $t \le U$.

The paths of the process X_n on $[0, U]$ can then be encoded by the random vector $(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})$ defined on $(\Omega, \mathcal{K}, \mathbb{P})$, where $R_n(t)$ represents the number of jumps of X_n that happen in the time interval $[0, t]$, and, for any $i = 1, \dots, R_n(U)$, J_i represents the size of the $i-$th jump, and T_i the time of the $i-$th jump.

Thus, we can take $\Omega_U := \bigcup_{r\in\mathbb{Z}_+} \{r\} \times (\mathbb{Z} \setminus \{0\})^r \times [0, U]^r$ as the sample space of the paths of the process X_n on the interval $[0, U]$, and we write $\mathcal{K}_U$ for the set of all measurable subsets of Ω_U.

Write $p_r(t) := P\{R_n(t) = r\}$, for any $t \le U$ and any $r \in \mathbb{Z}_+$, and write also

$$\Omega_U(k) := \{(r, j_{[1,r]}, t_{[1,r]}) \in \Omega_U,\ j_1 + j_2 + \ldots + j_r = k\}$$

for any $k \in \mathbb{Z}$. Obviously, we have that $\Omega_U = \bigcup_{k\in\mathbb{Z}} \Omega_U(k)$.

Then, for each $l \in \mathbb{N}$, $s_1, \dots, s_{l-1}, s^* \in [0, U]$ and $z \in \mathbb{Z}$, we define a probability measure $\mathbb{Q}_U(l, s_{[1,l-1]}, s^*, z)$ as follows: for any $B \in \mathcal{K}_U$, and if

$$S := (R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})^{-1}(B) \in \mathcal{K},$$

we set

$$\mathbb{Q}_U(l, s_{[1,l-1]}, s^*, z)\{B\} := \mathbb{P}\{S \mid N_n = n^l(\cdot\,; s_{[1,l-1]}, s^*), X_n(0) = z\}, \tag{4.0.17}$$

which we can further express as

$$\begin{aligned} &\mathbb{P}\{S \mid N_n = n^l(\cdot\,; s_{[1,l-1]}, s^*), X_n(0) = z\} \\ &\quad = \sum_{r\in\mathbb{Z}_+} p_r(U) \sum_{j_1\in\mathbb{Z}\setminus\{0\}} \dots \sum_{j_r\in\mathbb{Z}\setminus\{0\}} \mathbb{1}_B(r, j_{[1,r]}, t_{[1,r]}) \end{aligned}$$

$$\int_{[0,U]^r} \mathbb{1}_{\{0<t_1<\ldots<t_r<U\}}(t_1,\ldots,t_r) \quad f^{U*}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z)\, dt_1 \ldots dt_r. \tag{4.0.18}$$

Here, $t_0 = 0$ and $\mathbb{1}_{\{0<t_1<\ldots<t_r<U\}}$ is an indicator function on $[0,U]^r$, and we write

$$\begin{aligned} &f^{U*}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z) \\ &:= \prod_{i=1}^{a-1} e^{-\int_{t_{i-1}}^{t_i} n\lambda^N\left(\frac{x_{i-1}}{n},u\right)du} \cdot n\lambda^N_{j_i}\left(\frac{x_{i-1}}{n}, t_i\right) \\ &e^{-\int_{t_{a-1}}^{s^*} n\lambda^N\left(\frac{x_{a-1}}{n},u\right)du} \cdot e^{-\int_{s^*}^{t_a} n\lambda^N\left(\frac{x_{a-1}+1}{n},u\right)du} \cdot n\lambda^N_{j_a}\left(\frac{x_{a-1}+1}{n}, t_a\right) \\ &\prod_{i=a+1}^{r} e^{-\int_{t_{i-1}}^{t_i} n\lambda^N\left(\frac{x_{i-1}+1}{n},u\right)du} \cdot n\lambda^N_{j_i}\left(\frac{x_{i-1}+1}{n}, t_i\right) \cdot e^{-\int_{t_r}^{U} n\lambda^N\left(\frac{x_r+1}{n},u\right)du}, \end{aligned}$$

with notation $a := \#\{i \mid t_i \le s^*\} + 1$, $x_0 := z$ and $x_i := x_0 + j_1 + \ldots + j_i$ for each i. The $(l-1)$ values $s_1, s_2, \ldots, s_{l-1}$ are implicit in the rates $n\lambda^N_j$, defined by (4.0.8), through (4.0.15) together with Remark 3; for convenience in the argument that follows, we distinguish the role of s^*.

Similarly, for any $B \in \mathcal{K}_U$ and if

$$S := (R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})^{-1}(B),$$

we write

$$\mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{B\} := \mathbb{P}\{S \mid N_n = n^{l-1}(\cdot\,; s_{[1,l-1]}), X_n(0) = z\}, \tag{4.0.19}$$

which we now express as

$$\begin{aligned} &\mathbb{P}\{S \mid N_n = n^{l-1}(\cdot\,; s_{[1,l-1]}), X_n(0) = z\} \\ &= \sum_{r\in\mathbb{Z}_+} p_r(U) \sum_{j_1\in\mathbb{Z}\setminus\{0\}} \cdots \sum_{j_r\in\mathbb{Z}\setminus\{0\}} \mathbb{1}_B(r, j_{[1,r]}, t_{[1,r]}) \\ &\int_{[0,U]^r} \mathbb{1}_{\{0<t_1<\ldots<t_r<U\}}(t_1,\ldots,t_r) \\ &f^{U}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z)\, dt_1 \ldots dt_r, \end{aligned} \tag{4.0.20}$$

with the same notation as before, and with

$$\begin{aligned} &f^{U}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z) \\ &:= \prod_{i=1}^{r} \left\{ e^{-\int_{t_{i-1}}^{t_i} n\lambda^N\left(\frac{x_{i-1}}{n},u\right)du} \cdot n\lambda^N_{j_i}\left(\frac{x_{i-1}}{n}, t_i\right)\right\} \cdot e^{-\int_{t_r}^{U} n\lambda^N\left(\frac{x_r}{n},u\right)du}, \end{aligned}$$

where the $(l-1)$ values $s_1, s_2, \ldots, s_{l-1}$ are again implicit in the rates $n\lambda^N_j$ through (4.0.15).

In view of (4.0.17) and (4.0.19), we now rewrite (4.0.16) as

$$f^{U*}_{l,z}(k-l) = \frac{1}{U^l}\int_{[0,U]^l} ds_1 \ldots ds_{l-1} ds^* \mathbb{Q}_U(l, s_{[1,l-1]}, s^*, z)\{\Omega_U(k-l-z)\},$$

and

$$f_{l,z}^U(k-l) = \frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^*\mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{\Omega_U(k-l-z)\},$$

respectively, for any $k \in \mathbb{Z}$.

Now let

$$\Gamma_U(l, s_{[1,l-1]}, z) := \{(r, j_{[1,r]}, t_{[1,r]}) \in \Omega_U,\ f^U(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z) > 0\},$$

and write

$$\Gamma_U(k, l, s_{[1,l-1]}, z) := \Gamma_U(l, s_{[1,l-1]}, z) \cap \Omega_U(k-l-z),$$

as well as

$$\Gamma_U^*(k, l, s_{[1,l-1]}, z) := \Omega_U(k-l-z) \setminus \Gamma_U(k, l, s_{[1,l-1]}, z).$$

By taking now $B = \Gamma_U^*(k, l, s_{[1,l-1]}, z)$ in (4.0.19) and (4.0.20), we have

$$\mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{\Gamma_U^*(k, l, s_{[1,l-1]}, z)\} = 0,$$

for any $k \in \mathbb{Z}$.

This suggests the following split for $\sum_{k\in\mathbb{Z}} |f_{l,z}^{U*}(k-l) - f_{l,z}^U(k-l)|$:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |f_{l,z}^{U*}(k-l) - f_{l,z}^U(k-l)| \\
&\le \frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^* \sum_{k\in\mathbb{Z}} \mathbb{Q}_U(l, s_{[1,l-1]}, s^*, z)\{\Gamma_U^*(k, l, s_{[1,l-1]}, z)\} \\
&+\frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^* \sum_{r\in\mathbb{Z}_+} p_r(U) \sum_{j_1\in\mathbb{Z}\setminus\{0\}} \cdots \sum_{j_r\in\mathbb{Z}\setminus\{0\}} \int_{[0,U]^r} \\
&\mathbb{1}_{\{0<t_1<\ldots<t_r<U\}}(t_1,\ldots,t_r) \cdot \left|\frac{f^{U*}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z)}{f^U(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z)} - 1\right| \\
&\mathbb{1}_{\Gamma_U(l, s_{[1,l-1]}, z)}(r, j_{[1,r]}, t_{[1,r]}) \cdot f^U(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z)\, dt_1 \ldots dt_r,
\end{aligned} \tag{4.0.21}$$

since $\mathbb{1}_{\Gamma_U(l,s_{[1,l-1]},z)}(r, j_{[1,r]}, t_{[1,r]}) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{\Gamma_U(k,l,s_{[1,l-1]},z)}(r, j_{[1,r]}, t_{[1,r]})$.

Step 5.4: The ratio of densities in (4.0.21) has an exponential form, the exponent being a sum of terms.

Remark 4. Let us note that, for any $(r, j_{[1,r]}, t_{[1,r]}) \in \Gamma_U(l, s_{[1,l-1]}, z)$, the ratio

$$\frac{f^{U*}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z)}{f^U(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, z)}$$

can be written in the form

$$e^{-\sigma_U^{(n)}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z)},$$

where

$$\sigma_U^{(n)}(r, j_{[1,r]}, t_{[1,r]} \mid l, s_{[1,l-1]}, s^*, z)$$

$$:= \int_{s^*}^{t_a} n\Big[\lambda^N\Big(\frac{x_{a-1}+1}{n},u\Big) - \lambda^N\Big(\frac{x_{a-1}}{n},u\Big)\Big]du - \ln\frac{\lambda^N_{j_a}\Big(\frac{x_{a-1}+1}{n},t_a\Big)}{\lambda^N_{j_a}\Big(\frac{x_{a-1}}{n},t_a\Big)}$$

$$+\sum_{i=a+1}^{r}\Big\{\int_{t_{i-1}}^{t_i} n\Big[\lambda^N\Big(\frac{x_{i-1}+1}{n},u\Big) - \lambda^N\Big(\frac{x_{i-1}}{n},u\Big)\Big]du - \ln\frac{\lambda^N_{j_i}\Big(\frac{x_{i-1}+1}{n},t_i\Big)}{\lambda^N_{j_i}\Big(\frac{x_{i-1}}{n},t_i\Big)}\Big\}$$

$$+\int_{t_r}^{U} n\Big[\lambda^N\Big(\frac{x_r+1}{n},u\Big) - \lambda^N\Big(\frac{x_r}{n},u\Big)\Big]du, \tag{4.0.22}$$

and where the value of $a = \#\{i \mid t_i \le s^*\}+1$ depends on s^* and on all the observed jump-times $t_{[1,r]}$ of X_n.

Remark 5. *The function $\Sigma_U^{(n)} := \sigma_U^{(n)}(\cdot \mid l, s_{[1,l-1]}, s^*, z)$ can be considered to be a random variable on the probability sub-space*

$$\big(\Gamma_U(l,s_{[1,l-1]},z), \mathcal{K}_U^\Gamma, \mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)\big)$$

of $\big(\Omega_U,\mathcal{K}_U,\mathbb{Q}_U(l-1,s_{[1,l-1]},z)\big)$, where $\mathcal{K}_U^\Gamma \subset \mathcal{K}_U$ is the set of all measurable subsets of $\Gamma_U(l,s_{[1,l-1]},z)$, and the measure $\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)$ is the restriction of $\mathbb{Q}_U(l-1,s_{[1,l-1]},z)$ to the set $\Gamma_U(l,s_{[1,l-1]},z)$. We have that

$$\mathbb{Q}_U(l-1,s_{[1,l-1]},z)\{\Gamma_U^C(l,s_{[1,l-1]},z)\} = 0, \tag{4.0.23}$$

since

$$\Gamma_U^C(l,s_{[1,l-1]},z) = \bigcup_{k\in\mathbb{Z}} \Gamma_U^*(k,l,s_{[1,l-1]},z),$$

so it is clear that $\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)$ is itself a probability measure.

In view of Remarks 4 and 5, we rewrite (4.0.21) as

$$\begin{aligned}\sum_{k\in\mathbb{Z}} &|f_{l,z}^{U*}(k-l) - f_{l,z}^{U}(k-l)| \\ &\le \frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^*\mathbb{Q}_U(l,s_{[1,l-1]},s^*,z)\{\Gamma_U^C(l,s_{[1,l-1]},z)\} \\ &+\frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^*\ \mathbb{E}_{\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)}\Big|e^{-\Sigma_U^{(n)}}-1\Big|.\end{aligned} \tag{4.0.24}$$

Step 5.5: Let us make another useful remark.

Remark 6. *For any random variable X defined on a probability space $(\Omega,\mathcal{K},\mathbb{P})$, we have*

$$\mathbb{E}|X-1| = \mathbb{E}X - 1 + 2\mathbb{E}(1-X)_+ \ .$$

If moreover there exists a random variable Y so that $X = e^Y$, then for any $B\in\mathcal{K}$ we also have

$$2\mathbb{E}(1-X)_+ \le 2\mathbb{E}|Y\cdot \mathbb{1}_B| + 2\mathbb{P}(B^C).$$

Based on Remark 4, we observe that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)}\, e^{-\Sigma_U^{(n)}} &= \mathbb{Q}_U(l,s_{[1,l-1]},s^*,z)\{\Gamma_U(l,s_{[1,l-1]},z)\} \\ &= 1-\mathbb{Q}_U(l,s_{[1,l-1]},s^*,z)\{\Gamma_U^C(l,s_{[1,l-1]},z)\}. \end{aligned} \tag{4.0.25}$$

In view of Remark 6 and (4.0.25), (4.0.24) now becomes

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} |f_{l,z}^{U*}(k-l)-f_{l,z}^{U}(k-l)| \\ &\le \frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^*\ 2\mathbb{E}_{\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)}|\Sigma_U^{(n)}\cdot \mathbb{1}_B| \\ &+\frac{1}{U^l}\int_{[0,U]^l} ds_1...ds_{l-1}ds^*\ 2\mathbb{Q}_U^\Gamma(l-1,s_{[1,l-1]},z)\{B^C\}, \end{aligned} \tag{4.0.26}$$

for any set $B\subset \Gamma_U(l,s_{[1,l-1]},z)$.

Step 5.6: Choice of the appropriate set B.
Let

$$S_{U,\delta} := \{\omega\in\Omega,\ \sup_{t\in[0,U]} |X_n(\omega,t)+N_n(\omega,t)-nc|\le n\delta\}$$

for the δ in **Assumption 2**, and

$$S_{U,\delta}(l,s_{[1,l-1]},z) := S_{U,\delta}\cap\{\omega\in\Omega, N_n(\omega,\cdot)=n^{l-1}(\cdot\,;s_{[1,l-1]}), X_n(\omega,0)=z\}.$$

Then define

$$\begin{aligned} B &:= B_{U,\delta}(l,s_{[1,l-1]},z) \\ &:= \Gamma_U(l,s_{[1,l-1]},z)\cap (R_n(U),J_{[1,R_n(U)]},T_{[1,R_n(U)]})\big(S_{U,\delta}(l,s_{[1,l-1]},z)\big). \end{aligned} \tag{4.0.27}$$

Step 5.7: Regard $\Sigma_U^{(n)}$ as a random variable on $(\Omega,\mathcal{K})$ rather than $(\Omega_U,\mathcal{K}_U)$, as we have done so far, and split it into what shall prove to be, in view of the lemmas that we shall give after this theorem, a martingale term and an error term, both on $(\Omega,\mathcal{K})$. Based on the results of the mentioned lemmas, we are now able to conclude the desired result, a bound of size $O(1/\sqrt{n})$ on (4.0.7).

Note that, using (4.0.22), we can define the measurable function

$$\begin{aligned} \Sigma_U^{(n)} &:= \sigma_U^{(n)}(R_n(U),J_{[1,R_n(U)]},T_{[1,R_n(U)]}\mid l,s_{[1,l-1]},s^*,z) \\ &= \int_{s^*}^{T_A} n\Big[\lambda^N\Big(\frac{X_{A-1}+1}{n},u\Big)-\lambda^N\Big(\frac{X_{A-1}}{n},u\Big)\Big]du - \ln\frac{\lambda_{J_A}^N\Big(\frac{X_{A-1}+1}{n},T_A\Big)}{\lambda_{J_A}^N\Big(\frac{X_{A-1}}{n},T_A\Big)} \\ &+\sum_{i=A+1}^{R_n(U)}\Big\{\int_{T_{i-1}}^{T_i} n\Big[\lambda^N\Big(\frac{X_{i-1}+1}{n},u\Big)-\lambda^N\Big(\frac{X_{i-1}}{n},u\Big)\Big]du - \ln\frac{\lambda_{J_i}^N\Big(\frac{X_{i-1}+1}{n},T_i\Big)}{\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n},T_i\Big)}\Big\} \\ &+\int_{T_{R_n(U)}}^{U} n\Big[\lambda^N\Big(\frac{X_{R_n(U)}+1}{n},u\Big)-\lambda^N\Big(\frac{X_{R_n(U)}}{n},u\Big)\Big]du, \end{aligned} \tag{4.0.28}$$

now defined on $(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})^{-1}(\Gamma_U(l, s_{[1,l-1]}, z)) \subset \Omega$; here, $A := \#\{i \mid T_i \leq s^*\} + 1$ and $X_i := z + J_1 + J_2 + \ldots + J_i$ for every i. Moreover, we can split $\Sigma_U^{(n)}$ into the sum of two random variables:

$$\Sigma_U^{(n)} = M_U^{(n)} + \epsilon_U^{(n)}, \tag{4.0.29}$$

where

$$\begin{aligned} M_U^{(n)} &:= M_U(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]} \mid l, s_{[1,l-1]}, s^*, z) \\ &= \int_{s^*}^{T_A} \partial_x \lambda^N\Big(\frac{X_{A-1}}{n}, u\Big) du - \frac{\partial_x \lambda^N_{J_A}\Big(\frac{X_{A-1}}{n}, T_A\Big)}{n\lambda^N_{J_A}\Big(\frac{X_{A-1}}{n}, T_A\Big)} \\ &+ \sum_{i=A+1}^{R_n(U)} \Big\{ \int_{T_{i-1}}^{T_i} \partial_x \lambda^N\Big(\frac{X_{i-1}}{n}, u\Big) du - \frac{\partial_x \lambda^N_{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)}{n\lambda^N_{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)} \Big\} \\ &+ \int_{T_{R_n(U)}}^{U} \partial_x \lambda^N\Big(\frac{X_{R_n(U)}}{n}, u\Big) du \end{aligned} \tag{4.0.30}$$

and

$$\begin{aligned} \epsilon_U^{(n)} &:= \epsilon_U(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]} \mid l, s_{[1,l-1]}, s^*, z) \\ &= \int_{s^*}^{T_A} nR_1\Big(\frac{X_{A-1}}{n}, u\Big) du + \frac{\partial_x \lambda^N_{J_A}\Big(\frac{X_{A-1}}{n}, T_A\Big)}{n\lambda^N_{J_A}\Big(\frac{X_{A-1}}{n}, T_A\Big)} - \ln \frac{\lambda^N_{J_A}\Big(\frac{X_{A-1}+1}{n}, T_A\Big)}{\lambda^N_{J_A}\Big(\frac{X_{A-1}}{n}, T_A\Big)} \\ &+ \sum_{i=A+1}^{R_n(U)} \Big\{ \int_{T_{i-1}}^{T_i} nR_1\Big(\frac{X_{i-1}}{n}, u\Big) du + \frac{\partial_x \lambda^N_{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)}{n\lambda^N_{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)} - \ln \frac{\lambda^N_{J_i}\Big(\frac{X_{i-1}+1}{n}, T_i\Big)}{\lambda^N_{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)} \Big\} \\ &+ \int_{T_{R_n(U)}}^{U} nR_1\Big(\frac{X_{R_n(U)}}{n}, u\Big) du, \end{aligned} \tag{4.0.31}$$

where, under **Assumptions 2 to 5**, we used Taylor's expansion and the following notation for the Lagrange remainder:

$$R_1\Big(\frac{x}{n}, t\Big) := \lambda^N\Big(\frac{x+1}{n}, t\Big) - \lambda^N\Big(\frac{x}{n}, t\Big) - \frac{1}{n}\partial_x \lambda^N\Big(\frac{x}{n}, t\Big).$$

Also note that, in view of (4.0.19), (4.0.27) and (4.0.28),

$$\mathbb{E}_{\mathbb{Q}_U^\Gamma(l-1, s_{[1,l-1]}, z)} |\Sigma_U^{(n)} \cdot \mathbb{1}_{B_{U,\delta}(l, s_{[1,l-1]}, z)}| = \mathbb{E}_{\mathbb{P}\{\cdot | N_n = n^{l-1}(\cdot\, ; s_{[1,l-1]}), X_n(0)=z\}} |\Sigma_U^{(n)} \cdot \mathbb{1}_{S_{U,\delta}^\Gamma(l, s_{[1,l-1]}, z)}|,$$

where

$$S_{U,\delta}^\Gamma(l, s_{[1,l-1]}, z) := S_{U,\delta}(l, s_{[1,l-1]}, z) \cap (R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})^{-1}\big(\Gamma_U(l, s_{[1,l-1]}, z)\big).$$

For the ease of notation, we shall write from now on

$$\mathbb{E}_{l,s,z} := \mathbb{E}_{\mathbb{P}\{\cdot | N_n = n^{l-1}(\cdot\, ; s_{[1,l-1]}), X_n(0)=z\}}.$$

Now replace B in (4.0.26) with $B_{U,\delta}(l, s_{[1,l-1]}, z)$, and note that if we can show that

$$\mathbb{E}_{l,s,z}|M_U^{(n)} \cdot \mathbb{1}_{S^{\Gamma}_{U,\delta}(l,s_{[1,l-1]},z)}| \leq K^{(1)}_{U,\delta}\, n^{-1/2} \tag{4.0.32}$$

and

$$\mathbb{E}_{l,s,z}|\epsilon_U^{(n)} \cdot \mathbb{1}_{S^{\Gamma}_{U,\delta}(l,s_{[1,l-1]},z)}| \leq K^{(2)}_{U,\delta}\, n^{-1/2}, \tag{4.0.33}$$

uniformly for all $(l, s_{[1,l-1]}, z) \in \bigcup_{l\in\mathbb{Z}}\{l\} \times [0,U]^{l-1} \times \mathbb{Z}$, and also that

$$\begin{aligned} &\sum_{z\in\mathbb{Z}} \Pi_n(z) \sum_{l\geq 1} \mathbb{P}(N_n(U) = l-1) \frac{1}{U^l} \int_{[0,U]^l} ds_1 ... ds_{l-1} ds^* \\ &\cdot \mathbb{Q}^{\Gamma}_U(l-1, s_{[1,l-1]}, z)\{B^C_{U,\tilde{\delta}}(l, s_{[1,l-1]}, z)\} \leq K_{U,\tilde{\delta}}\, n^{-1}, \end{aligned} \tag{4.0.34}$$

for any $\tilde{\delta} > 0$, then we have obtained the desired bound on (4.0.7). And this concludes the proof.

□

We give three lemmas to prove each of (4.0.32), (4.0.33) and (4.0.34). We start by proving (4.0.34) and note that, in view of (4.0.19), (4.0.23) and (4.0.27), we can write

$$\begin{aligned} &\mathbb{Q}^{\Gamma}_U(l-1, s_{[1,l-1]}, z)\{B^C_{U,\delta}(l, s_{[1,l-1]}, z)\} \\ &\quad = \mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{\Gamma^C_U(l, s_{[1,l-1]}, z) \cup \\ &\quad \big[(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})\big(S_{U,\delta}(l, s_{[1,l-1]}, z)\big)\big]^C\} \\ &\quad = \mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{\big[(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})\big(S_{U,\delta}(l, s_{[1,l-1]}, z)\big)\big]^C\} \\ &\quad = 1 - \mathbb{Q}_U(l-1, s_{[1,l-1]}, z)\{(R_n(U), J_{[1,R_n(U)]}, T_{[1,R_n(U)]})\big(S_{U,\delta}(l, s_{[1,l-1]}, z)\big)\} \\ &\quad = 1 - \mathbb{P}\{S_{U,\delta}(l, s_{[1,l-1]}, z) \mid N_n = n^{l-1}(\cdot\,; s_{[1,l-1]}), X_n(0) = z\} \\ &\quad = \mathbb{P}\{S^C_{U,\delta} \mid N_n = n^{l-1}(\cdot\,; s_{[1,l-1]}), X_n(0) = z\}, \end{aligned} \tag{4.0.35}$$

and therefore (4.0.34) is smaller then $\mathbb{P}\{S^C_{U,\delta}\}$. On the other hand, we have that

$$\begin{aligned} \mathbb{P}_{\Pi_n}\{S^C_{U,\tilde{\delta}}\} &= \mathbb{P}_{\Pi_n}\{\sup_{t\in[0,U]} |X_n(t) + N_n(t) - nc| > n\tilde{\delta}\} \\ &= \mathbb{P}_{\Pi_n}\{\sup_{t\in[0,U]} |Z_n(t) - nc| > n\tilde{\delta}\}, \end{aligned}$$

for any $\tilde{\delta} > 0$.

LEMMA 4.2. *Under* **Assumptions 1 to 4**, *for any $\tilde{\delta}$ so that $0 < \tilde{\delta} \leq \delta$, there exists a constant $K_{U,\tilde{\delta}} < \infty$ so that*

$$\mathbb{P}_{\Pi_n}\{\sup_{t\in[0,U]} |Z_n(t) - nc| > n\tilde{\delta}\} \leq K_{U,\tilde{\delta}}\, n^{-1}.$$

PROOF. Note that, under **Assumptions 2 and 3**, the process Z_n fulfils the regularity condition given by Proposition 2.2. Indeed, in view of (2.0.3),

$$\sum_{j\in\mathbb{Z}\backslash\{0\}} |j| n\lambda_j\Big(\frac{i}{n}\Big) \le \sum_{j\in\mathbb{Z}\backslash\{0\}} |j| c_j (n+|i-nc|) = O(|i|), \tag{4.0.36}$$

as $|i|\to\infty$. By Theorem 2.3, and since, by Corollary 3.4, $Z_n(0)$ is integrable with respect to its equilibrium distribution Π_n, it follows that

$$Z_n(t) - Z_n(0) - \int_0^t nF(z_n(s))ds,$$

is a martingale with expectation 0, when the initial distribution is Π_n, and where as usual $z_n = \frac{1}{n}Z_n$. Let $\tau_\delta := \inf\{t\ge 0 \mid \sup_{s\in[0,t]}|z_n(s)-c| > \delta\}$, for the δ in **Assumption 2**. Then,

$$\mathcal{M}_n(t) := Z_n(t\wedge\tau_\delta) - Z_n(0) - \int_0^{t\wedge\tau_\delta} nF(z_n(s))ds$$

is also a martingale with expectation 0 with respect to the initial distribution Π_n.

We further have that

$$|z_n(t\wedge\tau_\delta) - c| \le \frac{1}{n}\sup_{s\in[0,U]}|\mathcal{M}_n(s)| + |z_n(0)-c| + \int_0^{t\wedge\tau_\delta}|F(z_n(s))|ds,$$

for any $0\le t\le U$. From **Assumptions 1 to 4** we have that

$$\int_0^{t\wedge\tau_\delta}|F(z_n(s))|ds \le \sup_{|z-c|\le\delta}|F'(z)|\int_0^t |z_n(s\wedge\tau_\delta)-c|ds.$$

Write $\mathcal{C}_n := \frac{1}{n}\sup_{s\in[0,U]}|\mathcal{M}_n(s)| + |z_n(0)-c|$ and $\|F'\|_\delta := \sup_{|z-c|\le\delta}|F'(z)|$, and apply Gronwall's inequality to the positive function $|z_n(t\wedge\tau_\delta)-c|$ to obtain that

$$|z_n(t\wedge\tau_\delta)-c| \le \mathcal{C}_n e^{t\|F'\|_\delta}, \quad \text{for any } \ 0\le t\le U.$$

This further implies that

$$\sup_{t\in[0,U\wedge\tau_\delta]}|z_n(t)-c| \le \mathcal{C}_n e^{U\|F'\|_\delta}.$$

Note that, if $U\le\tau_\delta$, then

$$\sup_{t\in[0,U\wedge\tau_\delta]}|z_n(t)-c| = \sup_{t\in[0,U]}|z_n(t)-c|.$$

If $U\ge\tau_\delta$, then the events

$$\{\sup_{t\in[0,U\wedge\tau_\delta]}|z_n(t)-c| > \tilde\delta\} \ \text{ and } \ \{\sup_{t\in[0,U]}|z_n(t)-c|>\tilde\delta\}$$

are equivalent because

$$\inf\{t\ge 0 \mid \sup_{s\in[0,t]}|z_n(s)-c|>\tilde\delta\} \le \tau_\delta \le U,$$

for any $0<\tilde\delta\le\delta$. From the above, we now deduce that

$$\mathbb{P}_{\pi_n}\{\sup_{t\in[0,U]}|z_n(t)-c|>\tilde\delta\} \le \mathbb{P}_{\pi_n}\{\mathcal{C}_n e^{U\|F'\|_\delta} > \tilde\delta\}, \tag{4.0.37}$$

for any $0 < \tilde{\delta} \le \delta$. We further write

$$\mathbb{P}_{\pi_n}\{\mathcal{C}_n e^{U\|F'\|_\delta} > \tilde{\delta}\} = \mathbb{P}_{\Pi_n}\Big\{\sup_{s\in[0,U]} |\mathcal{M}_n(s)| + |Z_n(0) - nc| > \frac{n\tilde{\delta}}{e^{U\|F'\|_\delta}}\Big\}, \tag{4.0.38}$$

and once we show that both

$$\mathbb{P}_{\Pi_n}\Big\{\sup_{s\in[0,U]} |\mathcal{M}_n(s)| > \frac{n\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\} \tag{4.0.39}$$

and

$$\mathbb{P}_{\Pi_n}\Big\{|Z_n(0) - nc| > \frac{n\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\}$$

are at most of order $O(1/n)$, then it follows by (4.0.37) and (4.0.38) that also

$$\mathbb{P}_{\Pi_n}\{\sup_{t\in[0,U]} |Z_n(t) - nc| > n\tilde{\delta}\} = O\Big(\frac{1}{n}\Big), \quad \text{for any } 0 < \tilde{\delta} \le \delta.$$

By Corollary 3.5, since $\frac{\tilde{\delta}}{2e^{U\|F'\|_\delta}} \le \delta$, there exists a constant $K_{1,\tilde{\delta}} > 0$ so that the following holds:

$$\mathbb{P}_{\Pi_n}\Big\{|Z_n(0) - nc| > \frac{n\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\} = \mathbb{P}_{\pi_n}\Big\{|z_n - c| > \frac{\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\} \le \frac{K_{1,\tilde{\delta}}\ e^{2U\|F'\|_\delta}}{n}.$$

By Kolmogorov's inequality, (4.0.39) can be bounded by

$$\mathbb{P}_{\Pi_n}\Big\{\sup_{s\in[0,U]} |\mathcal{M}_n(s)| > \frac{n\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\} \le \frac{4e^{2U\|F'\|_\delta}\mathbb{E}_{\Pi_n}\{\mathcal{M}_n(U)^2\}}{n^2\tilde{\delta}^2}. \tag{4.0.40}$$

Again by Theorem 2.3 for $h(i) = i^2$, since

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} |(i+j)^2 - i^2| n\lambda_j\Big(\frac{i}{n}\Big) \le \sum_{j\in\mathbb{Z}\setminus\{0\}} (2|ij| + j^2)c_j(n + |i - nc|) = O(|i|^2),$$

for $|i| \to \infty$, it follows that $\mathcal{M}_n(s)$ is square integrable under the probability distribution δ_l for any $l \in \mathbb{Z}$, and that

$$[\mathcal{M}_n]_s - \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 n\Lambda_j(s\wedge\tau_\delta)$$

is a martingale with expectation 0, where $n\Lambda_j(s\wedge\tau_\delta) := \int_0^s n\lambda_j(z_n(u\wedge\tau_\delta))du$. Since

$$\mathbb{E}\mathcal{M}_n(U)^2 = \mathbb{E}\{\mathbb{E}\{[\mathcal{M}_n]_U \mid Z_n(0)\}\} = \mathbb{E}\{\mathbb{E}\{\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 n\Lambda_j(U\wedge\tau_\delta) \mid Z_n(0)\}\},$$

and since

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 n\Lambda_j(U\wedge\tau_\delta) \ \le\ nU \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j(1+\delta),$$

the inequality (4.0.40) now becomes

$$\mathbb{P}_{\Pi_n}\Big\{\sup_{s\in[0,U]} |\mathcal{M}_n(s)| > \frac{n\tilde{\delta}}{2e^{U\|F'\|_\delta}}\Big\} \le \frac{4nUe^{2U\|F'\|_\delta}\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2c_j(1+\delta)}{n^2\tilde{\delta}^2} \le \frac{K_{2,\tilde{\delta}}\ Ue^{2U\|F'\|_\delta}}{n}.$$

□

LEMMA 4.3. *There exists, under* **Assumptions 1 to 3, 4s and 5**, *a constant* $K_\delta^{(1)} < \infty$ *so that*

$$\mathbb{E}_{l,s,z}|M_U^{(n)} \cdot \mathbb{1}_{S_{U,\delta}^{\Gamma}(l,s_{[1,l-1]},z)}| \leq K_\delta^{(1)} \sqrt{U}\ n^{-1/2},$$

uniformly for all $(l, s_{[1,l-1]}, z) \in \bigcup_{l\in\mathbb{Z}} \{l\} \times [0,U]^{l-1} \times \mathbb{Z}$.

PROOF. Fix $(l, s_{[1,l-1]}, z) \in \bigcup_{l\in\mathbb{Z}} \{l\} \times [0,U]^{l-1} \times \mathbb{Z}$, and let τ_δ^Γ denote the first time to leave the set $S_{U,\delta}^{\Gamma}(l, s_{[1,l-1]}, z)$.

Remember that the processes X_n and N_n never jump simultaneously, a.s. Hence, the following holds

$$\lambda_j^N\Big(\frac{x}{n}, \tau\Big) = \lambda_j^N\Big(\frac{x}{n}, \tau-\Big), \quad \text{a.s.} \tag{4.0.41}$$

for any x and j, and for any jump-time τ of the process X_n, since it is clear that $N_n(\tau) = N_n(\tau-)$, a.s.

Note that, for any i,

$$X_{i-1} := X_n(t-) \ \text{ whenever } \ t \in (T_{i-1}, T_i]. \tag{4.0.42}$$

Also note that, for any $j \in \mathbb{Z} \setminus \{0\}$, the process $\xi_j^{(n)}(t)$ is right-continuous $(\mathcal{F}_t)$-adapted. The continuous increasing process defined as follows:

$$n\Lambda_j^*(t) := \int_0^t n\lambda_j^*(s)ds, \quad \text{for any } \ j \in \mathbb{Z} \setminus \{0\},$$

is the compensator of the point process $\xi_j^{(n)}$, where the function $n\lambda_j^*(\cdot)$ is the conditional intensity function for $\xi_j^{(n)}$, defined on $[0,U]$ by

$$\begin{aligned} \lambda_j^*(t) &:= \lambda_j^N\Big(\frac{X_n(t-)}{n}, t-\Big) = \lambda_j\Big(\frac{X_n(t-)+N_n(t-)}{n}\Big), \quad j \in \mathbb{Z}, j \neq 0, 1 \\ \lambda_1^*(t) &:= \lambda_1^N\Big(\frac{X_n(t-)}{n}, t-\Big) = \Big[\lambda_1\Big(\frac{X_n(t-)+N_n(t-)}{n}\Big) - \lambda^0\Big] \end{aligned} \tag{4.0.43}$$

as given by (4.0.8). Note that, for any j, the intensity function $n\lambda_j^*$ is a left-continuous function of time.

As an application of the Doob-Meyer decomposition theorem, it now follows by Daley and Vere-Jones (2003, Lemma 7.2.V.) that the process $(n\Lambda_j^* - \xi_j^{(n)})_t$ is, for any $j \in \mathbb{Z} \setminus \{0\}$, an $(\mathcal{F}_t)$-martingale with expectation 0, since, by definition, $n\Lambda_j^*(0) = \xi_j^{(n)}(0) = 0$. In particular,

$$\mathbb{E}_{l,s,z} n\Lambda_j^*(t) = \mathbb{E}_{l,s,z}\xi_j^{(n)}(t), \quad \text{for any } \ j \in \mathbb{Z} \setminus \{0\} \ \text{ and } \ t \geq 0. \tag{4.0.44}$$

For any $j \in \mathbb{Z} \setminus \{0\}$, we define the stochastic process $H_j^{(n)}(t)$ as follows:

$$H_j^{(n)}(t) := \begin{cases} \dfrac{\partial_x \lambda_j^N\Big(\frac{X_n(t-)}{n}, t-\Big)}{n\lambda_j^N\Big(\frac{X_n(t-)}{n}, t-\Big)}, & \text{if } \ t \leq \tau_\delta^\Gamma \\ 0, & \text{if } \ t > \tau_\delta^\Gamma, \end{cases}$$

and note that it is an $(\mathcal{F}_t)$-adapted, left-continuous, and therefore predictable process. The process $H_j^{(n)}$ is well defined, since by **Assumption 2**, for any $t \leq \tau_\delta^\Gamma$, the rate

$\lambda_j\Big(\frac{X_n(t-)+N_n(t-)}{n}\Big) \geq \varepsilon\lambda_j(c) > 0$, for any $j \in \mathbb{Z}, j \neq 0, 1$. For $j = 1$, we have that $\lambda_1^N\Big(\frac{X_n(t-)}{n}, t-\Big) \geq \lambda^0 > 0$, for any t.

Note that, in view of (4.0.41) and (4.0.42), under **Assumption 4**, the following two equalities hold

$$\int_{T_{i-1}}^{T_i} \partial_x\lambda^N\Big(\frac{X_{i-1}}{n}, s\Big)ds = \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_{(T_{i-1},T_i]} \partial_x\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)ds, \quad \text{a.e.} \tag{4.0.45}$$

for every $i = A, \ldots, R_n(U)+1$, if we take T_{A-1} to mean s^* and $T_{R_n(U)+1}$ to mean U, as well as

$$\sum_{i=A}^{R_n(U)} \frac{\partial_x\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n}, T_i\Big)}{n\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n}, T_i\Big)} = \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_{s^*}^{U} \frac{\partial_x\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)}{n\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)} d\xi_j^{(n)}(s). \tag{4.0.46}$$

With (4.0.46) and (4.0.45), we deduce that the term $M_U^{(n)}$ as given by (4.0.30) is equal to

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \Big\{ \int_{s^*}^{U} \frac{\partial_x\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)}{n\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)} n\lambda_j^*(s)ds - \int_{s^*}^{U} \frac{\partial_x\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)}{n\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)} d\xi_j^{(n)}(s)\Big\}, \quad \text{a.s.}$$

Hence, on the set $S_{U,\delta}^{\Gamma}(l, s_{[1,l-1]}, z)$, $M_U^{(n)}$ is actually $M_{U\wedge\tau_\delta^\Gamma}^{(n)}$ and equal to

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \{H_j^{(n)} \cdot (n\Lambda_j^* - \xi_j^{(n)})\}_U := \sum_{j\in\mathbb{Z}\setminus\{0\}} \int_{s^*}^{U} H_j^{(n)}(s)d(n\Lambda_j^* - \xi_j^{(n)})(s),$$

which is the sum over $j \in \mathbb{Z} \setminus \{0\}$ of the Itô integrals of the predictable processes $H_j^{(n)}$ with respect to the martingales $(n\Lambda_j^* - \xi_j^{(n)})$, up to time $U \wedge \tau_\delta^\Gamma$. We shall denote each of these Itô integrals by $M_j^{(n)}(t) := \{H_j^{(n)} \cdot (n\Lambda_j^* - \xi_j^{(n)})\}_t$.

Note that by **Assumption 2**, on the set $S_{U,\delta}^{\Gamma}(l, s_{[1,l-1]}, z)$, the intensity functions $n\lambda_j^*(t)$ have the property that

$$n\Lambda_j^*(t) \leq nUc_j(1+\delta) < \infty, \tag{4.0.47}$$

uniformly for all $0 \leq t \leq U$. By (4.0.10), we have that

$$\int_0^t |H_j^{(n)}(s)|\lambda_j^*(s)ds = \int_0^{t\wedge\tau_\delta^\Gamma} \Big|\partial_x\lambda_j^N\Big(\frac{X_n(s-)}{n}, s-\Big)\Big|ds \leq U\|\lambda_j'\|_\delta$$

for $0 \leq t \leq U$ and $j \in \mathbb{Z} \setminus \{0\}$, and hence

$$\mathbb{E}_{l,s,z} \int_0^t |H_j^{(n)}(s)|\lambda_j^*(s)ds \leq U\|\lambda_j'\|_\delta < \infty$$

in the same range of t. Under these conditions, by Brémaud (1981, Th. II.8 Integration Theorem (β)), it follows that $M_j^{(n)}(t)$ is an $(\mathcal{F}_t)$-martingale with expectation 0, for any $j \in \mathbb{Z} \setminus \{0\}$.

Now, we shall need the argument that, for any $j \in \mathbb{Z} \setminus \{0\}$, the process $\xi_j^{(n)}$ stopped at $U \wedge \tau_\delta^\Gamma$ is actually the quadratic variation process of the martingale $(n\Lambda_j^* - \xi_j^{(n)})$ stopped at $U \wedge \tau_\delta^\Gamma$. We therefore need to prove that the stopped martingale $(n\Lambda_j^* - \xi_j^{(n)})_u$, with $0 \le u \le U \wedge \tau_\delta^\Gamma$, is square-integrable. Note that (4.0.47) implies that the compensator of the counting process $\xi_j^{(n)}$ is bounded on $S_{U,\delta}^\Gamma(l, s_{[1,l-1]}, z)$, and by Fleming and Harrington (1991, Theorem 2.3.1 (2)) it follows that $(n\Lambda_j^* - \xi_j^{(n)})_u$ is a square-integrable martingale, for $0 \le u \le U \wedge \tau_\delta^\Gamma$. Since $(n\Lambda_j^* - \xi_j^{(n)})$ stopped at $U \wedge \tau_\delta^\Gamma$ is square-integrable, it follows by Meyer's Theorem, see Rogers and Williams (1987, (26)Theorem(ii)), that the quadratic variation process $[n\Lambda_j^* - \xi_j^{(n)}]_u$ exists for any $0 \le u \le U \wedge \tau_\delta^\Gamma$, and

$$\begin{aligned}\Delta[n\Lambda_j^* - \xi_j^{(n)}]_u &= \{\Delta(n\Lambda_j^* - \xi_j^{(n)})_u\}^2 \\ &= \{(n\Lambda_j^*(u) - n\Lambda_j^*(u-)) - (\xi_j^{(n)}(u) - \xi_j^{(n)}(u-))\}^2 = \Delta\xi_j^{(n)}(u),\end{aligned}$$

where $\Delta f(u) := f(u) - f(u-)$ for any right-continuous function on $[0, \infty)$. We used the fact that $n\Lambda_j^*$ is a continuous function and $\{\Delta\xi_j^{(n)}(u)\}^2 = \Delta\xi_j^{(n)}(u)$, since the values of this difference can only be equal to 0 or 1. It is now verified that the process $\xi_j^{(n)}$ stopped at $U \wedge \tau_\delta^\Gamma$ is the quadratic variation process of the martingale $(n\Lambda_j^* - \xi_j^{(n)})$ stopped at $U \wedge \tau_\delta^\Gamma$, and this is true for any $j \in \mathbb{Z} \setminus \{0\}$.

Note that, under **Assumptions 2, 3 and 4s**, the predictable process $H_j^{(n)}$ fulfils the following condition:

$$\mathbb{E}_{l,s,z} \int_0^U H_j^{(n)}(s)^2 d[n\Lambda_j^* - \xi_j^{(n)}]_s < \infty,$$

since $[n\Lambda_j^* - \xi_j^{(n)}]_t = \xi_j^{(n)}(t)$ for any $t \le U \wedge \tau_\delta^\Gamma$, and by (4.0.10), (4.0.44) and (4.0.47), we may write

$$\begin{aligned}&\mathbb{E}_{l,s,z} \int_0^U H_j^{(n)}(s)^2 d\xi_j^{(n)}(s) \\ &\le \mathbb{E}_{l,s,z} \int_0^{U \wedge \tau_\delta^\Gamma} \Big[\frac{\partial_x \lambda_j^N\left(\frac{X_n(s-)}{n}, s-\right)}{n\lambda_j^N\left(\frac{X_n(s-)}{n}, s-\right)}\Big]^2 d\xi_j^{(n)}(s) \\ &\le \frac{\|\lambda_j'\|_\delta^2}{n^2\varepsilon^2\lambda_j(c)^2}\mathbb{E}_{l,s,z}\xi_j^{(n)}(U \wedge \tau_\delta^\Gamma) = \frac{\|\lambda_j'\|_\delta^2}{n^2\varepsilon^2\lambda_j(c)^2}\mathbb{E}_{l,s,z} n\Lambda_j^*(U \wedge \tau_\delta^\Gamma) \\ &\le \frac{nUc_j(1+\delta)\|\lambda_j'\|_\delta^2}{n^2\varepsilon^2\lambda_j(c)^2} < \infty, \qquad \text{for any } \ j \in \mathbb{Z} \setminus \{0\}.\end{aligned} \tag{4.0.48}$$

Under the above condition, Rogers and Williams (1987, (27.6)Theorem (ii)) states that

$$\{H_j^{(n)} \cdot (n\Lambda_j^* - \xi_j^{(n)})\}_t^2 - \int_{s^*}^t H_j^{(n)}(s)^2 d[n\Lambda_j^* - \xi_j^{(n)}]_s$$

is a uniformly integrable martingale, stopped at τ_δ^Γ, having expectation 0. Now stop once more this martingale, but at time U, and evaluate its expectation, in order to obtain that

$$\mathbb{E}_{l,s,z}\Big\{M_j^{(n)}(U)^2 - \int_{s^*}^U H_j^{(n)}(s)^2 d\xi_j^{(n)}(s)\Big\} = 0, \quad \text{for all} \;\; j \in \mathbb{Z}\setminus\{0\}. \tag{4.0.49}$$

Now use (4.0.48) in order to deduce from (4.0.49) that

$$\mathbb{E}_{l,s,z} M_j^{(n)}(U)^2 \le \frac{nUc_j(1+\delta)\|\lambda_j'\|_\delta^2}{n^2\varepsilon^2\lambda_j(c)^2}, \;\; \text{for all} \;\; j \in \mathbb{Z}\setminus\{0\}.$$

Since $\mathbb{E}_{l,s,z}|M_j^{(n)}(U)| \le \sqrt{\mathbb{E}_{l,s,z}M_j^{(n)}(U)^2}$, for any j, this further implies that, for a constant $K_\delta^{(1)} < \infty$,

(4.0.50)

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \mathbb{E}_{l,s,z}|M_j^{(n)}(U)| \le \frac{1}{\varepsilon}\sqrt{\frac{U(1+\delta)}{n}} \sup_{j\in\mathbb{Z}\setminus\{0\}} \frac{\|\lambda_j'\|_\delta}{\lambda_j(c)} \sum_{j\in\mathbb{Z}\setminus\{0\}} \sqrt{c_j} \le K_\delta^{(1)}\sqrt{\frac{U}{n}},$$

under **Assumptions 2 to 4s**, since the fact that $\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2+\alpha}c_j < \infty$ implies, in particular, that there exists $j_1 \ge 1$ so that, for any j with $|j| > j_1$, $\sqrt{c_j} < |j|^{-1-\alpha/2}$, and the series $\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{-1-\alpha/2}$ converges, because $\alpha > 0$.

Since $\sum_{j\in\mathbb{Z}\setminus\{0\}} \mathbb{E}_{l,s,z}|M_j^{(n)}(U)| < \infty$, it follows by Fubini's theorem that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \mathbb{E}_{l,s,z}|M_j^{(n)}(U)| = \mathbb{E}_{l,s,z} \sum_{j\in\mathbb{Z}\setminus\{0\}} |M_j^{(n)}(U)|,$$

and so the desired bound is obtained:

$$\mathbb{E}_{l,s,z}|M_U^{(n)} \cdot \mathbb{1}_{S_{U,\delta}^\Gamma(l,s_{[1,l-1]},z)}| \le \mathbb{E}_{l,s,z} \sum_{j\in\mathbb{Z}\setminus\{0\}} |M_j^{(n)}(U)| \le K_\delta^{(1)}\sqrt{\frac{U}{n}}. \tag{4.0.51}$$

□

LEMMA 4.4. *There exists, under* **Assumptions 1 to 3, 4s and 5**, *a constant* $K_\delta^{(2)} < \infty$ *so that*

$$\mathbb{E}_{l,s,z}|\epsilon_U^{(n)} \cdot \mathbb{1}_{S_{U,\delta}^\Gamma(l,s_{[1,l-1]},z)}| \le K_\delta^{(2)} U\, n^{-1},$$

uniformly for all $(l, s_{[1,l-1]}, z) \in \bigcup_{l\in\mathbb{Z}}\{l\} \times [0,U]^{l-1} \times \mathbb{Z}$.

PROOF. Fix $(l, s_{[1,l-1]}, z) \in \bigcup_{l\in\mathbb{Z}}\{l\} \times [0,U]^{l-1} \times \mathbb{Z}$, and let τ_δ^Γ still denote the first time to leave the set $S_{U,\delta}^\Gamma(l, s_{[1,l-1]}, z)$.

By (4.0.42), we can rewrite (4.0.31) as follows:

$$\begin{aligned}\epsilon_U^{(n)} \quad &= \int_{s^*}^U nR_1\Big(\frac{X_n(s-)}{n}, s\Big)ds - \sum_{i=A}^{R_n(U)} \frac{R_1^{J_i}\Big(\frac{X_{i-1}}{n}, T_i\Big)}{\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n}, T_i\Big)} \\ &- \sum_{i=A}^{R_n(U)} \Big[\ln \frac{\lambda_{J_i}^N\Big(\frac{X_{i-1}+1}{n}, T_i\Big)}{\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n}, T_i\Big)} + 1 - \frac{\lambda_{J_i}^N\Big(\frac{X_{i-1}+1}{n}, T_i\Big)}{\lambda_{J_i}^N\Big(\frac{X_{i-1}}{n}, T_i\Big)}\Big],\end{aligned}$$

where we used the notation for the Lagrange remainder

$$R_1^j\Big(\frac{x}{n},t\Big) := \lambda_j^N\Big(\frac{x+1}{n},t\Big) - \lambda_j^N\Big(\frac{x}{n},t\Big) - \frac{1}{n}\partial_x\lambda_j^N\Big(\frac{x}{n},t\Big),$$

for any $j \in \mathbb{Z}\setminus\{0\}$. This further implies that

$$\begin{aligned}\epsilon_U^{(n)} \quad &= \int_{s^*}^{U} nR_1\Big(\frac{X_n(s-)}{n},s\Big)ds - \sum_{j\in\mathbb{Z}\setminus\{0\}}\int_{s^*}^{U}\frac{R_1^j\Big(\frac{X_n(s-)}{n},s\Big)}{\lambda_j^N\Big(\frac{X_n(s-)}{n},s\Big)}d\xi_j^{(n)}(s)\\ &+\sum_{j\in\mathbb{Z}\setminus\{0\}}\int_{s^*}^{U}\Big[\frac{\lambda_j^N\Big(\frac{X_n(s-)+1}{n},s\Big)}{\lambda_j^N\Big(\frac{X_n(s-)}{n},s\Big)} - 1 - \ln\frac{\lambda_j^N\Big(\frac{X_n(s-)+1}{n},s\Big)}{\lambda_j^N\Big(\frac{X_n(s-)}{n},s\Big)}\Big]d\xi_j^{(n)}(s).\end{aligned}$$

Note that, by **Assumptions 2, 3 and 4s**, for any $s\geq 0$ and any $\omega\in S_{U,\delta}^{\Gamma}(l,s_{[1,l-1]},z)$,

$$\Big|\frac{\lambda_j^N\Big(\frac{X_n(\omega,s-)+1}{n},s\Big)}{\lambda_j^N\Big(\frac{X_n(\omega,s-)}{n},s\Big)} - 1\Big| \leq \frac{\|\lambda_j'\|_\delta}{n\varepsilon\lambda_j(c)} = O\Big(\frac{1}{n}\Big) < \infty.$$

Hence, since $|\log(1+x)-x|\leq 2x^2$ if $|x|\leq\frac{1}{2}$, we can use Taylor's approximation to find, for $n\geq 2\max_j\frac{\|\lambda_j'\|_\delta}{\varepsilon\lambda_j(c)}$ and under **Assumptions 2, 3, 4s and 5**, a constant $K_\delta^{(2)}<\infty$ so that

$$\begin{aligned}\mathbb{E}_{l,s,z}|\epsilon_U^{(n)}\cdot\mathbb{1}_{S_{U,\delta}^{\Gamma}(l,s_{[1,l-1]},z)}| &\leq \frac{nU\|\lambda''\|_\delta}{2n^2} + \sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{\|\lambda_j''\|_\delta}{2n^2\varepsilon\lambda_j(c)}\,\mathbb{E}_{l,s,z}n\Lambda_j^*(U\wedge\tau_\delta^\Gamma)\\ &+\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{2\|\lambda_j'\|_\delta^2}{n^2\varepsilon^2\lambda_j(c)^2}\,\mathbb{E}_{l,s,z}n\Lambda_j^*(U\wedge\tau_\delta^\Gamma)\\ &\leq \frac{1}{n}\cdot\frac{U\|\lambda''\|_\delta}{2} + \frac{1}{n}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{Uc_j(1+\delta)\|\lambda_j''\|_\delta}{2\varepsilon\lambda_j(c)} + \frac{1}{n}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{2Uc_j(1+\delta)\|\lambda_j'\|_\delta^2}{\varepsilon^2\lambda_j(c)^2}\\ &= \frac{2U(1+\delta)}{n}\Big[\frac{\|\lambda''\|_\delta}{4(1+\delta)} + \sup_j\frac{\|\lambda_j''\|_\delta}{j^2\lambda_j(c)}\cdot\frac{\sum_{j\in\mathbb{Z}\setminus\{0\}}j^2c_j}{4\varepsilon} + \sup_j\frac{\|\lambda_j'\|_\delta^2}{\lambda_j(c)^2}\cdot\frac{\sum_{j\in\mathbb{Z}\setminus\{0\}}c_j}{\varepsilon^2}\Big]\\ &\leq K_\delta^{(2)}\frac{U}{n},\end{aligned}\tag{4.0.52}$$

since, under **Assumption 5**, $\|\lambda''\|_\delta := \sup_{|z-c|\leq\delta}|\lambda''(z)|<\infty$. We have again used Fubini's theorem, (4.0.44) and (4.0.47), which allow us to commute the expectation with the sum upon $j\in\mathbb{Z}\setminus\{0\}$, and we have also used (4.0.11) together with the fact that, on the set $S_{U,\delta}^{\Gamma}$,

$$|R_1(z,t)| \leq \frac{1}{2n^2}\sup_{(z,t)\in[c-\delta,c+\delta]\times[0,U]}\big|\partial_x^2\lambda^N(z,t)\big| \leq \frac{1}{2n^2}\|\lambda''\|_\delta$$

and, for any $j \in \mathbb{Z} \setminus \{0\}$,

$$|R_1^j(z,t)| \leq \frac{1}{2n^2} \sup_{(z,t)\in[c-\delta,c+\delta]\times[0,U]} \left|\partial_x^2 \lambda_j^N(z,t)\right| \leq \frac{1}{2n^2} \|\lambda_j''\|_\delta.$$

□

CHAPTER 5

Translated Poisson approximation to the equilibrium distribution

We prove, under **Assumptions 1 to 3, 4s and 5s**, that the distance in total variation between the translated distribution $\Pi_n - \lfloor nc \rfloor$ on $\mathbb{Z}$ and the translated Poisson distribution

$$\widehat{\mathrm{Po}}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big) := \mathrm{Po}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big) - \Big\lfloor \frac{n\sigma^2(c)}{-2F'(c)} \Big\rfloor$$

(also on $\mathbb{Z}$) is of order $O(n^{-\alpha/2})$, for the α in **Assumption 3**.

For the example we gave in the Preliminaries section, we have $c = \frac{a}{d - \sum_{j\ge 1} jb_j}$, $-2F'(c) = 2(d - \sum_{j\ge1} jb_j)$ and $\sigma^2(c) = a + \frac{a}{d-\sum_{j\ge1} jb_j}(d + \sum_{j\ge1} j^2 b_j)$, so that the shifted equilibrium distribution

$$\Pi_n - \Big\lfloor \frac{na}{d - \sum_{j\ge1} jb_j} \Big\rfloor$$

can be approximated in total variation by the shifted Poisson distribution

$$\widehat{\mathrm{Po}}\Big(\frac{na(2d + \sum_{j\ge1} j(j-1)b_j)}{2(d - \sum_{j\ge1} jb_j)^2}\Big),$$

to order $O(n^{-\alpha/2})$, for the α in **Assumption 3**. Note that if $b_j = 0$, for any $j \ge 1$, than the process becomes a simple death with immigration process, whose equilibrium distribution is precisely the Poisson distribution $\mathrm{Po}\big(\frac{na}{d}\big) = \mathrm{Po}(nc)$. In this case, the approximation is in fact exact.

1. Strategy of the proof

The idea of the proof is as follows. We use the generator $\mathcal{A}_n$ of the Markov process Z_n in its particular form given by Lemma 2.1 and apply Dynkin's formula to the process Z_n under equilibrium. This yields that the expectation of the Stein operator for translated Poisson approximation, acting on the first difference of a suitable function h in the domain of $\mathcal{A}_n$, is equal to the expectation of a sum of certain remaining terms. By taking absolute values on both sides of this equality, and then a supremum over all subsets of $\mathbb{Z}_+$, we are able to bound the desired distance in total variation on $\mathbb{Z}$ by a sum of terms on which we can prove a bound of size $O(n^{-\alpha/2})$.

Let A be a subset of $\mathbb{Z}_+$ and let $n\lambda = \frac{n\sigma^2(c)}{-2F'(c)}$. Solve the Stein equation (2.2.1) for A and this particular $n\lambda$, and denote the solution by $g_{n\lambda,A}$. We take $g_{n\lambda,A}(0) = 0$.

We extend the function $g_{n\lambda,A}$ to the set of all integers and redefine it as follows:

$$g_{n\lambda,A}(i) := \begin{cases} 0, & \text{if} \quad i < 0 \\ g_{n\lambda,A}(i), & \text{if} \quad i \geq 0. \end{cases} \tag{5.1.1}$$

When doing so, we implicitly extend the definitions of the functions $\tilde{g}_{n\lambda,B}$, with $B = A - \lfloor n\lambda \rfloor$, to the whole $\mathbb{Z}$, too:

$$\tilde{g}_{n\lambda,B}(l) = \tilde{g}_{n\lambda,A-\lfloor n\lambda \rfloor}(l) := \begin{cases} 0, & \text{if} \quad l < -\lfloor n\lambda \rfloor \\ g_{n\lambda,A}(l + \lfloor n\lambda \rfloor), & \text{if} \quad l \geq -\lfloor n\lambda \rfloor. \end{cases} \tag{5.1.2}$$

Now consider a function $h\colon \mathbb{Z} \to \mathbb{R}$ whose first difference $\triangle h\colon \mathbb{Z} \to \mathbb{R}$ is given by:

$$\triangle h(i) := \tilde{g}_{n\lambda,A-\lfloor n\lambda \rfloor}(i - \lfloor nc \rfloor), \quad \text{for any} \quad i \in \mathbb{Z}.$$

We shall denote this function by $h_{n\lambda,A}$, and we note that it is constant on the set $\{i \in \mathbb{Z},\ i < \lfloor nc \rfloor - \lfloor n\lambda \rfloor\}$.

Lemma 5.1. *Under* **Assumptions 1 to 3**, *the function $h_{n\lambda,A}$ fulfills the conditions of Theorem 2.3 with respect to the initial distribution Π_n.*

Proof. We already know, in view of (4.0.36), that under the above assumptions the process Z_n fulfills the regularity condition given by Proposition 2.2. We also have that

$$\begin{aligned}
(|\mathcal{A}_n| h_{n\lambda,A})(i) \quad &:= \sum_{j \in \mathbb{Z}\setminus\{0\}} n\lambda_j\Big(\frac{i}{n}\Big) |h_{n\lambda,A}(i+j) - h_{n\lambda,A}(i)| \\
&\leq \sum_{j \in \mathbb{Z}\setminus\{0\}} c_j (n + |i - nc|) \, \Big| \sum_{k=1}^{j} \tilde{g}_{n\lambda,B}(i + j - k - \lfloor nc \rfloor) \Big| \\
&\leq (n + n|c|) \, \sup_z |\tilde{g}_{n\lambda,B}(z)| \sum_{j \in \mathbb{Z}\setminus\{0\}} |j| c_j \\
&\quad + \sum_{j \in \mathbb{Z}\setminus\{0\}} c_j \, \Big| \sum_{k=1}^{j} i \, g_{n\lambda,A}(i + j - k - \lfloor nc \rfloor + \lfloor n\lambda \rfloor) \Big|.
\end{aligned} \tag{5.1.3}$$

In view of the equality (2.2.1), we have

$$\Big| \sum_{k=1}^{j} i \, g_{n\lambda,A}(i) \Big| \leq 2|j| + n\lambda |j| \sup_z |g_{n\lambda,A}(z)|, \quad \text{for any} \quad i, j \in \mathbb{Z}, \quad j \neq 0,$$

and it therefore follows that

$$\begin{aligned}
&\Big| \sum_{k=1}^{j} i \, g_{n\lambda,A}(i + j - k - \lfloor nc \rfloor + \lfloor n\lambda \rfloor) \Big| \\
&= \Big| \sum_{k=1}^{j} (i + j - k - \lfloor nc \rfloor + \lfloor n\lambda \rfloor) \, g_{n\lambda,A}(i + j - k - \lfloor nc \rfloor + \lfloor n\lambda \rfloor) \\
&- \sum_{k=1}^{j} (j - k) \, g_{n\lambda,A}(i + j - k - \lfloor nc \rfloor + \lfloor n\lambda \rfloor)
\end{aligned}$$

$$(5.1.4) \qquad +(\lfloor nc\rfloor - \lfloor n\lambda\rfloor)\sum_{k=1}^{j} g_{n\lambda,A}(i+j-k-\lfloor nc\rfloor+\lfloor n\lambda\rfloor)\big|$$
$$\leq 2|j| + \Big(n\lambda + \frac{|j-1|}{2} + |\lfloor nc\rfloor - \lfloor n\lambda\rfloor|\Big)|j|\sup_z |g_{n\lambda,A}(z)|.$$

We introduce (5.1.4) in (5.1.3) to conclude, under **Assumption 3** and in view of (2.2.2), that $(|\mathcal{A}_n|h_{n\lambda,A})$ is uniformly bounded on $\mathbb{Z}$.

□

Write $Y_n := Z_n - \lfloor nc\rfloor$. Under **Assumptions 1 to 3**, with $B = A - \lfloor n\lambda\rfloor$, use the result of Lemma 2.1 for the generator $\mathcal{A}_n$ acting on the function $h_{n\lambda,A}$, to obtain that

$$(\mathcal{A}_n h_{n\lambda,A})(Y_n+\lfloor nc\rfloor) = \frac{n}{2}\sigma^2\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big) \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)$$
$$+nF\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)\tilde{g}_{n\lambda,B}(Y_n) - \frac{n}{2}F\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big) \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)$$
$$+\sum_{j=2}^{\sqrt{n}}\Big[\sum_{k=2}^{j}\binom{k}{2} \bigtriangledown^2 \tilde{g}_{n\lambda,B}(Y_n+j-k+1)\Big]n\lambda_j\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)$$
$$+\sum_{j>\sqrt{n}}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n+j-k) - \bigtriangledown\tilde{g}_{n\lambda,B}(Y_n)\Big)\Big]n\lambda_j\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)$$
$$-\sum_{j=2}^{\sqrt{n}}\Big[\sum_{k=2}^{j}\binom{k}{2} \bigtriangledown^2 \tilde{g}_{n\lambda,B}(Y_n-j+k)\Big]n\lambda_{-j}\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)$$
$$-\sum_{j>\sqrt{n}}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - \bigtriangledown\tilde{g}_{n\lambda,B}(Y_n-j+k)\Big)\Big]n\lambda_{-j}\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big).$$

We reformulate the above expression of the generator $\mathcal{A}_n$ as follows:

$$(5.1.5) \qquad (\mathcal{A}_n h_{n\lambda,A})(Z_n) = \mathrm{SO}_{n\lambda,B} + \mathrm{RT}_{n\lambda,B},$$

in that we add the following "Stein Operator" term to $(\mathcal{A}_n h_{n\lambda,A})(Z_n)$

$$(5.1.6) \qquad \mathrm{SO}_{n\lambda,B} := \frac{n}{2}\sigma^2(c) \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)$$
$$+ F'(c)Y_n\tilde{g}_{n\lambda,B}(Y_n) - F'(c)\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle\tilde{g}_{n\lambda,B}(Y_n)$$

and then, remembering that $F(c) = 0$ by **Assumption 1**, we subtract the term $\mathrm{SO}_{n\lambda,B}$ again from $(\mathcal{A}_n h_{n\lambda,A})(Z_n)$ in order to obtain the following "Remaining Terms", that we shall be able to handle using Taylor expansion and the mean value theorem,

$$\mathrm{RT}_{n\lambda,B} \quad := \frac{n}{2}\Big[\sigma^2\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big) - \sigma^2(c)\Big] \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)$$

$$
\begin{aligned}
&+n\Big[F\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-F(c)-\frac{Y_n}{n}F'(c)\Big]\tilde g_{n\lambda,B}(Y_n)\\
&+F'(c)\Big\langle\frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle\tilde g_{n\lambda,B}(Y_n)-\frac{n}{2}\Big[F\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-F(c)\Big]\bigtriangledown\tilde g_{n\lambda,B}(Y_n)\\
&+\sum_{j=2}^{\sqrt n}\Big[\sum_{k=2}^{j}\binom{k}{2}\bigtriangledown^2\tilde g_{n\lambda,B}(Y_n+j-k+1)\Big]n\lambda_j(c)\\
&+\sum_{j=2}^{\sqrt n}\Big[\sum_{k=2}^{j}\binom{k}{2}\bigtriangledown^2\tilde g_{n\lambda,B}(Y_n+j-k+1)\Big]\Big(n\lambda_j\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-n\lambda_j(c)\Big)\\
&+\sum_{j>\sqrt n}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown\tilde g_{n\lambda,B}(Y_n+j-k)-\bigtriangledown\tilde g_{n\lambda,B}(Y_n)\Big)\Big]n\lambda_j\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)\\
&-\sum_{j=2}^{\sqrt n}\Big[\sum_{k=2}^{j}\binom{k}{2}\bigtriangledown^2\tilde g_{n\lambda,B}(Y_n-j+k)\Big]n\lambda_{-j}(c)\\
&-\sum_{j=2}^{\sqrt n}\Big[\sum_{k=2}^{j}\binom{k}{2}\bigtriangledown^2\tilde g_{n\lambda,B}(Y_n-j+k)\Big]\Big(n\lambda_{-j}\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-n\lambda_{-j}(c)\Big)\\
&-\sum_{j>\sqrt n}\Big[\sum_{k=1}^{j-1}k\Big(\bigtriangledown\tilde g_{n\lambda,B}(Y_n)-\bigtriangledown\tilde g_{n\lambda,B}(Y_n-j+k)\Big)\Big]n\lambda_{-j}\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big).
\end{aligned}
\tag{5.1.7}
$$

Note that, using the mean-value theorem, we can write

$$
\begin{aligned}
\frac{n}{2}\Big|\sigma^2\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-\sigma^2(c)\Big| &\le \frac12\|(\sigma^2)'\|_\delta\cdot|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|\le n\delta)\\
&+\frac12\sum_{j\in\mathbb{Z}\setminus\{0\}}j^2c_j\cdot|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)\\
&+\frac{n}{2}\Big[\sigma^2(c)+\sum_{j\in\mathbb{Z}\setminus\{0\}}j^2c_j\Big]\cdot\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)
\end{aligned}
\tag{5.1.8}
$$

as well as, with $F(c)=0$,

$$
\begin{aligned}
\frac{n}{2}\Big|F\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-F(c)\Big| &\le \frac12\|F'\|_\delta\cdot|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|\le n\delta)\\
&+\frac12\sum_{j\in\mathbb{Z}\setminus\{0\}}|j|c_j\cdot|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)\\
&+\frac{n}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}}|j|c_j\cdot\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta),
\end{aligned}
\tag{5.1.9}
$$

and, for any $j\in\mathbb{Z}$ with $|j|\ge 2$,

$$
n\Big|\lambda_j\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-\lambda_j(c)\Big| \le \|\lambda_j'\|_\delta\cdot|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|\le n\delta)
$$

$$+c_j \cdot |Y_n - \langle nc\rangle| \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta)$$
$$+n[\lambda_j(c) + c_j] \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta), \tag{5.1.10}$$

where $\|(\sigma^2)'\|_\delta := \sup_{|z-c|\le\delta} |(\sigma^2)'(z)|$ and $\|F'\|_\delta := \sup_{|z-c|\le\delta} |F'(z)|$.

We apply the Dynkin formula (2.1.1) for the process $Z_n = Y_n + \lfloor nc\rfloor$ in equilibrium, when $Z_n \sim \Pi_n$. That is to say, we replace (5.1.5) in (2.1.1), and take absolute values to obtain the equality

$$|\mathbb{E}\{\mathrm{SO}_{n\lambda,B}\}| = |\mathbb{E}\{\mathrm{RT}_{n\lambda,B}\}|. \tag{5.1.11}$$

One of our purposes will be to show that $\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} |\mathbb{E}\{\mathrm{SO}_{n\lambda,B}\}|$ is as small as order $O(n^{-\alpha/2})$. By (5.1.6), the equality (5.1.11) leads to the following inequality

$$|\mathbb{E}\{\mathrm{SO}_{n\lambda,B}\}| =$$
$$\Big|\mathbb{E}\{\frac{n}{2}\sigma^2(c) \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) + F'(c)Y_n\tilde{g}_{n\lambda,B}(Y_n) - F'(c)\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle\tilde{g}_{n\lambda,B}(Y_n)\}\Big|$$
$$= |\mathbb{E}\{\mathrm{RT}_{n\lambda,B}\}| \le \sum_{i=1}^{7} \mathrm{E^{(i)}}_{n\lambda,B}, \tag{5.1.12}$$

where, under **Assumption 3** and in view of (5.1.7) and (5.1.8), we write

$$\begin{aligned}
\mathrm{E}^{(1)}{}_{n\lambda,B} \quad &:= \frac{1}{2}\|(\sigma^2)'\|_\delta\ \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\} \\
&+ \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\ \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\} \\
&+ \frac{n}{2}\big[\sigma^2(c) + \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\big]\ \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\},
\end{aligned} \tag{5.1.13}$$

$$\begin{aligned}
\mathrm{E}^{(2)}{}_{n\lambda,B} \quad &:= n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \frac{Y_n}{n}F'(c)\Big| \cdot |\tilde{g}_{n\lambda,B}(Y_n)|\Big\} \\
&+ |F'(c)|\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle\ \mathbb{E}\{|\tilde{g}_{n\lambda,B}(Y_n)|\},
\end{aligned} \tag{5.1.14}$$

and by (5.1.9)

$$\begin{aligned}
\mathrm{E}^{(3)}{}_{n\lambda,B} \quad &:= \frac{1}{2}\|F'\|_\delta\ \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\} \\
&+ \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|\ c_j\ \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\} \\
&+ \frac{n}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|\ c_j\ \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot | \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)|\},
\end{aligned} \tag{5.1.15}$$

and by (5.1.10)

$$
\begin{aligned}
\mathrm{E}^{(4)}{}_{n\lambda,B} \quad &:= \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, n\lambda_j(c) \, |\mathbb{E}\{\triangledown^2 \tilde{g}_{n\lambda,B}(Y_n + j - k + 1)\}| \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, \|\lambda'_j\|_\delta \, \mathbb{E}\{|Y_n - \langle nc \rangle| \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n + j - k + 1)|\} \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, c_j \, \mathbb{E}\{|Y_n - \langle nc \rangle| \cdot \mathbb{1}(|Y_n - \langle nc \rangle| > n\delta) \\
&\qquad\qquad \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n + j - k + 1)|\} \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, n[\lambda_j(c) + c_j] \, \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc \rangle| > n\delta) \\
&\qquad\qquad \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n + j - k + 1)|\},
\end{aligned} \tag{5.1.16}
$$

and

$$
\begin{aligned}
\mathrm{E}^{(5)}{}_{n\lambda,B} \quad &:= \sum_{j > \sqrt{n}} nc_j \sum_{k=1}^{j-1} k \, \mathbb{E}\{| \triangledown \tilde{g}_{n\lambda,B}(Y_n)| + | \triangledown \tilde{g}_{n\lambda,B}(Y_n + j - k)|\} \\
&+ \sum_{j > \sqrt{n}} c_j \sum_{k=1}^{j-1} k \, \mathbb{E}\Big\{|Y_n - \langle nc \rangle| \cdot \Big(| \triangledown \tilde{g}_{n\lambda,B}(Y_n)| + | \triangledown \tilde{g}_{n\lambda,B}(Y_n + j - k)|\Big)\Big\},
\end{aligned} \tag{5.1.17}
$$

while, again, by (5.1.10)

$$
\begin{aligned}
\mathrm{E}^{(6)}{}_{n\lambda,B} \quad &:= \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, n\lambda_{-j}(c) \, |\mathbb{E}\{\triangledown^2 \tilde{g}_{n\lambda,B}(Y_n - j + k)\}| \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, \|\lambda'_{-j}\|_\delta \, \mathbb{E}\{|Y_n - \langle nc \rangle| \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n - j + k)|\} \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, c_{-j} \, \mathbb{E}\{|Y_n - \langle nc \rangle| \cdot \mathbb{1}(|Y_n - \langle nc \rangle| > n\delta) \\
&\qquad\qquad \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n - j + k)|\} \\
&+ \sum_{j=2}^{\sqrt{n}} \sum_{k=2}^{j} \binom{k}{2} \, n[\lambda_{-j}(c) + c_{-j}] \, \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc \rangle| > n\delta) \\
&\qquad\qquad \cdot | \triangledown^2 \, \tilde{g}_{n\lambda,B}(Y_n - j + k)|\},
\end{aligned} \tag{5.1.18}
$$

and finally

$$\mathrm{E}^{(7)}{}_{n\lambda,B} := \sum_{j>\sqrt{n}} nc_{-j} \sum_{k=1}^{j-1} k\, \mathbb{E}\{|\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)| + |\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n - j + k)|\}$$

$$+ \sum_{j>\sqrt{n}} c_{-j} \sum_{k=1}^{j-1} k\, \mathbb{E}\Big\{|Y_n - \langle nc\rangle| \cdot \Big(|\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)| + |\bigtriangledown \tilde{g}_{n\lambda,B}(Y_n - j + k)|\Big)\Big\}. \tag{5.1.19}$$

In the next section, giving our main result, we shall be able to prove that, with the inequalities (2.2.4), a bound of size $O(n^{-\alpha/2})$ on $\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} \sum_{i=1}^{7} \mathrm{E}^{(i)}{}_{n\lambda,B}$ can be obtained, provided that

$$\mathbb{E}\{|Y_n - \langle nc\rangle|\} = O(\sqrt{n}) \quad\text{and}\quad n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \frac{Y_n}{n}F'(c)\Big|\Big\} = O(1), \tag{5.1.20}$$

and also that

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} |\mathbb{E}\{\bigtriangledown^2 \tilde{g}_{n\lambda,B}(Y_n + C)\}| = O\Big(\frac{1}{n\sqrt{n}}\Big), \text{ for any integer constant } C. \tag{5.1.21}$$

2. The main result

First we give two lemmas to prove the bounds given in (5.1.20) and (5.1.21), and it is now that we will need the result of Theorem 4.1. After the lemmas, we give the main result, the translated Poisson approximation to the equilibrium distribution of Z_n.

Lemma 5.2. *For $Y_n \sim \Pi_n - \lfloor nc\rfloor$ in equilibrium, under* **Assumptions 1 to 4 and 5s**,

$$\mathbb{E}\{|Y_n - \langle nc\rangle|\} = O(\sqrt{n}),$$

and

$$n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \frac{Y_n}{n}F'(c)\Big|\Big\} = O(1).$$

Proof. At equilibrium, $Z_n \sim \Pi_n$, and therefore $z_n \sim \pi_n$, and we write

$$\mathbb{E}\{|Y_n - \langle nc\rangle|\} = \mathbb{E}\{|Z_n - \lfloor nc\rfloor - \langle nc\rangle|\} = n\mathbb{E}\{|z_n - c|\} = O(\sqrt{n}),$$

by Corollary 3.4, proving the first part of the lemma.

Note that one can write

$$\begin{aligned} &n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \frac{Y_n}{n}F'(c)\Big|\Big\} \\ &\le n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \Big(\frac{Y_n}{n} - \frac{\langle nc\rangle}{n}\Big)F'(c)\Big|\Big\} \\ &+ \langle nc\rangle|F'(c)|, \end{aligned} \tag{5.2.1}$$

where obviously $\langle nc\rangle|F'(c)| = O(1)$.

Note that

$$\begin{aligned} & n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n}{n}+\frac{\lfloor nc\rfloor}{n}\Big)-F(c)-\Big(\frac{Y_n}{n}-\frac{\langle nc\rangle}{n}\Big)F'(c)\Big|\Big\} \\ & = n\mathbb{E}\Big\{\Big|F\Big(\frac{Z_n}{n}\Big)-F(c)-\Big(\frac{Z_n}{n}-c\Big)F'(c)\Big|\Big\} \\ & = n\mathbb{E}\{|F(z_n)-F(c)-(z_n-c)F'(c)|\} \end{aligned}$$

and by truncation, since $F(c)=0$, it follows that

$$\begin{aligned} & n\mathbb{E}\{|F(z_n)-F(c)-(z_n-c)F'(c)|\} \\ & = n\mathbb{E}\{|F(z_n)-F(c)-(z_n-c)F'(c)|\cdot \mathbb{1}(|z_n-c|<\delta)\} \\ & + n\mathbb{E}\{|F(z_n)-(z_n-c)F'(c)|\cdot \mathbb{1}(|z_n-c|\geq\delta)\}. \end{aligned}$$

Using Taylor's theorem, we deduce that

$$\begin{aligned} & n\mathbb{E}\{|F(z_n)-F(c)-(z_n-c)F'(c)|\cdot \mathbb{1}(|z_n-c|<\delta)\} \\ & \leq \frac{n}{2}\mathbb{E}\{|z_n-c|^2\cdot \mathbb{1}(|z_n-c|<\delta)\} \sup_{|\epsilon|<\delta}|F''(c+\epsilon)| \end{aligned} \tag{5.2.2}$$

while, noting that $|F(z)|\leq(1+1/\delta)\sum_{j\in\mathbb{Z}}|j|c_j|z-c|$ whenever $|z-c|\geq\delta$,

$$\begin{aligned} & n\mathbb{E}\{|F(z_n)-(z_n-c)F'(c)|\cdot \mathbb{1}(|z_n-c|\geq\delta)\} \\ & \leq n\mathbb{E}\{|z_n-c|\cdot \mathbb{1}(|z_n-c|\geq\delta)\}\big((1+1/\delta)\sum_{j\in\mathbb{Z}}|j|c_j+F'(c)\big). \end{aligned} \tag{5.2.3}$$

Now use (3.0.1) in (5.2.2) and (5.2.3) and introduce the result in (5.2.1) to finally obtain, under the stronger **Assumption 5s**, that

$$n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n+\lfloor nc\rfloor}{n}\Big)-F(c)-\frac{Y_n}{n}F'(c)\Big|\Big\}=O(1).$$

□

LEMMA 5.3. *Under* **Assumptions 1 to 3, 4s and 5**, *if $Y_n\sim\Pi_n-\lfloor nc\rfloor$ is in equilibrium, then for any integer constant C,*

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor}|\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+C)\}|=O\Big(\frac{1}{n\sqrt{n}}\Big).$$

PROOF. Note that, if $Z_n\sim\Pi_n$, then

$$\begin{aligned} & \mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+C)\} \\ & = \mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Z_n-\lfloor nc\rfloor+C)\} \\ & = \sum_{i\in\mathbb{Z}}\{\bigtriangledown\tilde{g}_{n\lambda,B}(i-C')-\bigtriangledown\tilde{g}_{n\lambda,B}(i-C'-1)\}\cdot\Pi_n(i) \\ & = \sum_{i\in\mathbb{Z}}\bigtriangledown\tilde{g}_{n\lambda,B}(i-C')\Pi_n(i)-\sum_{i\in\mathbb{Z}}\bigtriangledown\tilde{g}_{n\lambda,B}(i-C')\Pi_n(i+1) \end{aligned}$$

since the series involved are absolutely convergent, and where $C':=\lfloor nc\rfloor-C$. It follows that

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor}|\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+C)\}|$$

$$\le \sup_{B\subset \mathbb{Z}_+ - \lfloor n\lambda\rfloor} \sum_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,B}(i-C')| \cdot |\Pi_n(i) - \Pi_n(i+1)|$$

$$\le \sup_{B\subset \mathbb{Z}_+ - \lfloor n\lambda\rfloor} \| \bigtriangledown \tilde{g}_{n\lambda,B}\| \cdot 2d_{TV}\{\Pi_n, \Pi_n * \delta_1\}, \tag{5.2.4}$$

where $\Pi_n * \delta_1$ denotes the distribution Π_n shifted by 1.

Use (2.2.4) and the result of Theorem 4.1 to obtain the desired bound.

□

THEOREM 5.4. *There exists, under* **Assumptions 1 to 3, 4s and 5s**, *a constant* $C > 0$ *so that*

$$d_{TV}\Big\{(\Pi_n - \lfloor nc\rfloor), \widehat{\text{Po}}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big)\Big\} \le Cn^{-\alpha/2},$$

for the α *in* **Assumption 3**.

PROOF. With notation $n\lambda = \frac{n\sigma^2(c)}{-2F'(c)}$, we have

$$d_{TV}\{(\Pi_n - \lfloor nc\rfloor), \widehat{\text{Po}}(n\lambda)\} = \sup_{S\subset\mathbb{Z}} \Big|(\Pi_n - \lfloor nc\rfloor)(S) - \widehat{\text{Po}}(n\lambda)(S)\Big|$$

$$= \sup_{B\subset \mathbb{Z}_+ - \lfloor n\lambda\rfloor} \Big|(\Pi_n - \lfloor nc\rfloor)(B) - \widehat{\text{Po}}(n\lambda)(B)\Big|,$$

since the distribution $\widehat{\text{Po}}(n\lambda)$ is only defined on $\mathbb{Z}_+ - \lfloor n\lambda\rfloor$, and one of the two subsets of $\mathbb{Z}$ on which the above supremum is attained is

$$\{i \in \mathbb{Z},\ \widehat{\text{Po}}(n\lambda)(i) > (\Pi_n - \lfloor nc\rfloor)(i)\} \subset \mathbb{Z}_+ - \lfloor n\lambda\rfloor.$$

If $Y_n \sim \Pi_n - \lfloor nc\rfloor$, then for any $B \subset \mathbb{Z}_+ - \lfloor n\lambda\rfloor$,

$$\begin{aligned}(\Pi_n - \lfloor nc\rfloor)(B) - \widehat{\text{Po}}(n\lambda)(B) &= \mathbb{E}\{\mathbb{1}_B(Y_n) - \widehat{\text{Po}}(n\lambda)(B)\}\\ &= \mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\text{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n \ge -\lfloor n\lambda\rfloor)\}\\ &\quad + \mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\text{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n < -\lfloor n\lambda\rfloor)\}.\end{aligned} \tag{5.2.5}$$

We use now the Stein Equation (2.2.3) for $l = Y_n \ge -\lfloor n\lambda\rfloor$, and write

$$\begin{aligned}&\mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\text{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n \ge -\lfloor n\lambda\rfloor)\}\\ &= \mathbb{E}\Big\{\Big[n\lambda \bigtriangleup \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\Big] \cdot \mathbb{1}(Y_n \ge -\lfloor n\lambda\rfloor)\Big\},\end{aligned} \tag{5.2.6}$$

where, with the notation of Section 2, $\tilde{g}_{n\lambda,B}(Y_n) = g_{n\lambda,A}(Y_n + \lfloor n\lambda\rfloor)$ as given by (5.1.2) and $g_{n\lambda,A}$ is the solution to the Stein Equation (2.2.1) extended to the set of all integers, as given by (5.1.1). Note that, since

$$\begin{aligned}&\mathbb{E}\{n\lambda \bigtriangleup \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\}\\ &= \mathbb{E}\{n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\}\\ &+ \mathbb{E}\{n\lambda \bigtriangleup \tilde{g}_{n\lambda,B}(Y_n) - n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n)\Big\},\end{aligned}$$

(5.2.6) now becomes

$$\mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\mathrm{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\}$$
$$= \mathbb{E}\Big\{\Big[n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\Big] \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\Big\}$$
$$+ n\lambda\, \mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1) \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\}.$$

This can be further written as

$$\begin{aligned}
&\mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\mathrm{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\}\\
&= \mathbb{E}\{n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\}\\
&- \mathbb{E}\Big\{\Big[n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\Big] \cdot \mathbb{1}(Y_n < -\lfloor n\lambda \rfloor)\Big\}\\
&+ n\lambda\, \mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1) \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\}.
\end{aligned} \tag{5.2.7}$$

Quite obviously, the term

$$\mathbb{E}\Big\{\Big[n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\Big] \cdot \mathbb{1}(Y_n < -\lfloor n\lambda \rfloor)\Big\} = 0$$

vanishes from the sum above, since the function $\tilde{g}_{n\lambda,B}$ was defined in (5.1.2) in such a way that $\tilde{g}_{n\lambda,B}(Y_n) = 0$ whenever $Y_n < -\lfloor n\lambda \rfloor$. The same observation allows us to write

$$\begin{aligned}
\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1) \cdot \mathbb{1}(Y_n < -\lfloor n\lambda \rfloor)\} &= \tilde{g}_{n\lambda,B}(-\lfloor n\lambda \rfloor) \cdot \mathbb{P}(Y_n = -\lfloor n\lambda \rfloor - 1)\\
&= g_{n\lambda,A}(0) \cdot \mathbb{P}(Y_n = -\lfloor n\lambda \rfloor - 1) = 0,
\end{aligned}$$

so that indeed

$$\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1) \cdot \mathbb{1}(Y_n \geq -\lfloor n\lambda \rfloor)\} = \mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1)\}.$$

Note also that, for the last term in (5.2.5), one can write

$$\begin{aligned}
&\mathbb{E}\{(\mathbb{1}_B(Y_n) - \widehat{\mathrm{Po}}(n\lambda)(B)) \cdot \mathbb{1}(Y_n < -\lfloor n\lambda \rfloor)\}\\
&\qquad = -\widehat{\mathrm{Po}}(n\lambda)(B) \cdot \mathbb{P}(Y_n < -\lfloor n\lambda \rfloor),
\end{aligned} \tag{5.2.8}$$

for any $B \subset \mathbb{Z}_+ - \lfloor n\lambda \rfloor$.

With the above remarks, introduce now (5.2.7) and (5.2.8) into (5.2.5) to obtain, in absolute value, that

(5.2.9)
$$\begin{aligned}
&|(\Pi_n - \lfloor nc \rfloor)(B) - \widehat{\mathrm{Po}}(n\lambda)(B)|\\
&\qquad \leq |\mathbb{E}\{n\lambda \bigtriangledown \tilde{g}_{n\lambda,B}(Y_n) - Y_n\tilde{g}_{n\lambda,B}(Y_n) + \langle n\lambda\rangle\tilde{g}_{n\lambda,B}(Y_n)\}|\\
&\qquad + n\lambda\, |\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1)\}| + \widehat{\mathrm{Po}}(n\lambda)(B) \cdot \mathbb{P}(Y_n < -\lfloor n\lambda \rfloor),
\end{aligned}$$

which, by (5.1.12), now becomes

(5.2.10)
$$\begin{aligned}
|(\Pi_n - \lfloor nc \rfloor)(B) - \widehat{\mathrm{Po}}(n\lambda)(B)| \leq \frac{1}{-F'(c)} \sum_{i=1}^{7} \mathrm{E^{(i)}}_{n\lambda,B} + n\lambda\, |\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1)\}|\\
+ \widehat{\mathrm{Po}}(n\lambda)(B) \cdot \mathbb{P}(Y_n < -\lfloor n\lambda \rfloor).
\end{aligned}$$

For the first element of (5.2.10), $\sum_{i=1}^{7} \mathrm{E}^{(i)}{}_{n\lambda,B}$, we proceed as follows. By (3.0.1) and the bounds (2.2.4), for any $B \subset \mathbb{Z}_+ - \lfloor n\lambda \rfloor$, and by the results of Lemmas 5.2 and 5.3, we can bound (5.1.13) by

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} \mathrm{E}^{(1)}{}_{n\lambda,B} \le \frac{1}{2}\|(\sigma^2)'\|_\delta \cdot O(\sqrt{n})\cdot O\Big(\frac{1}{n}\Big) + \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j \cdot O(1)\cdot O\Big(\frac{1}{n}\Big)$$
$$+\frac{n}{2}\big[\sigma^2(c) + \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\big]\cdot O\Big(\frac{1}{n}\Big)\cdot O\Big(\frac{1}{n}\Big), \tag{5.2.11}$$

and (5.1.14) by

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} \mathrm{E}^{(2)}{}_{n\lambda,B} \le O(1)\cdot O\Big(\frac{1}{\sqrt{n}}\Big)$$
$$+|F'(c)|\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle \cdot O\Big(\frac{1}{\sqrt{n}}\Big), \tag{5.2.12}$$

(5.1.15) by

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} \mathrm{E}^{(3)}{}_{n\lambda,B} \le \frac{1}{2}\|F'\|_\delta \cdot O(\sqrt{n})\cdot O\Big(\frac{1}{n}\Big) + \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|\; c_j \cdot O(1)\cdot O\Big(\frac{1}{n}\Big)$$
$$+\frac{n}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j|\; c_j\cdot O\Big(\frac{1}{n}\Big)\cdot O\Big(\frac{1}{n}\Big), \tag{5.2.13}$$

and, given that

$$|j|^{1-\alpha} \le (\sqrt{n})^{1-\alpha} \quad \text{whenever} \quad |j| \le \sqrt{n}$$

and that

$$|j|^{-\alpha} < (\sqrt{n})^{-\alpha} \quad \text{whenever} \quad |j| > \sqrt{n},$$

for any integer j and for $\alpha \in (0,1]$, we can bound the sum of (5.1.16) and (5.1.18) by

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} (\mathrm{E}^{(4)}{}_{n\lambda,B} + \mathrm{E}^{(6)}{}_{n\lambda,B}) \le \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{1-\alpha}\; nc_j\cdot O\Big(\frac{1}{n\sqrt{n}}\Big)$$
$$+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{1-\alpha}\; \|\lambda_j'\|_\delta\cdot O(\sqrt{n})\cdot O\Big(\frac{1}{n}\Big)$$
$$+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{1-\alpha}\; c_j\cdot O(1)\cdot O\Big(\frac{1}{n}\Big)$$
$$+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{1-\alpha}\; 2nc_j\cdot O\Big(\frac{1}{n}\Big)\cdot O\Big(\frac{1}{n}\Big), \tag{5.2.14}$$

and also we can bound the sum of (5.1.17) and (5.1.19) by

$$\begin{aligned}\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} (\mathrm{E}^{(5)}{}_{n\lambda,B}+\mathrm{E}^{(7)}{}_{n\lambda,B}) &\le \sum_{|j|>\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{-\alpha}\, nc_j\cdot O\Big(\frac{1}{n}\Big)\\ &+ \sum_{|j|>\sqrt{n}} |j|^{2+\alpha}(\sqrt{n})^{-\alpha}\, c_j\cdot O(\sqrt{n})\cdot O\Big(\frac{1}{n}\Big),\end{aligned} \tag{5.2.15}$$

where both $\sum_{|j|=2}^{\sqrt{n}}|j|^{2+\alpha}c_j$ and $\sum_{|j|>\sqrt{n}}|j|^{2+\alpha}c_j$ are smaller than $\sum_{j\in\mathbb{Z}\setminus\{0\}}|j|^{2+\alpha}c_j$ so that, under **Assumption 3**, they are bounded, and where

$$\sum_{|j|=2}^{\sqrt{n}}|j|^{2+\alpha}\|\lambda_j'\|_\delta \le \sup_{j\in\mathbb{Z}\setminus\{0\}}\frac{\|\lambda_j'\|_\delta}{\lambda_j(c)}\cdot\sum_{j\in\mathbb{Z}\setminus\{0\}}|j|^{2+\alpha}c_j<\infty,$$

under **Assumptions 3 and 4s**. We now have that $\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor}\sum_{i=1}^{7}\mathrm{E}^{(\mathrm{i})}{}_{n\lambda,B} = O(n^{-\alpha/2})$.

Again by Lemma 5.3, with $C=1$, it follows that

$$\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor} n\lambda\,|\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,B}(Y_n+1)\}|\le n\lambda\cdot O\Big(\frac{1}{n\sqrt{n}}\Big)=O\Big(\frac{1}{\sqrt{n}}\Big), \tag{5.2.16}$$

which deals with the second term in (5.2.10). For the final term, since $n\lambda-1\le\lfloor n\lambda\rfloor\le n\lambda$, it follows that

$$\mathbb{P}(Y_n<-\lfloor n\lambda\rfloor)\le\mathbb{P}(Y_n<-n\lambda+1).$$

We may then write

$$\begin{aligned}\mathbb{P}(Y_n<-\lfloor n\lambda\rfloor)&\le\mathbb{P}(Y_n-\langle nc\rangle<-n\lambda+1-\langle nc\rangle)\\&\le\mathbb{P}(|Y_n-\langle nc\rangle|>n\lambda-1+\langle nc\rangle).\end{aligned}$$

Using Markov's inequality, we obtain that

$$\begin{aligned}\mathbb{P}(Y_n<-\lfloor n\lambda\rfloor)&\le\mathbb{P}(|Y_n-\langle nc\rangle|>n\lambda-1+\langle nc\rangle)\\&\le\frac{\mathbb{E}\{|Y_n-\langle nc\rangle|\}}{n\lambda-1+\langle nc\rangle}=O\Big(\frac{1}{\sqrt{n}}\Big),\end{aligned} \tag{5.2.17}$$

since $\mathbb{E}\{|Y_n-\langle nc\rangle|\}=O(\sqrt{n})$ by Lemma 5.2.

In view of (5.2.11) − (5.2.17), it now follows from (5.2.10) that

$$\begin{aligned}&d_{TV}\Big\{(\Pi_n-\lfloor nc\rfloor),\widehat{\mathrm{Po}}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big)\Big\}=\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor}|(\Pi_n-\lfloor nc\rfloor)(B)-\widehat{\mathrm{Po}}(n\lambda)(B)|\\&\le\frac{1}{-F'(c)}\cdot O\Big(\frac{1}{\sqrt{n^\alpha}}\Big)+O\Big(\frac{1}{\sqrt{n}}\Big)+\sup_{B\subset\mathbb{Z}_+-\lfloor n\lambda\rfloor}\widehat{\mathrm{Po}}(n\lambda)(B)\cdot O\Big(\frac{1}{\sqrt{n}}\Big)=O\Big(\frac{1}{\sqrt{n^\alpha}}\Big),\end{aligned}$$

as required. □

CHAPTER 6

Local limit approximation for the equilibrium distribution

While in the previous chapter we were interested in finding a bound of order $O(n^{-\alpha/2})$, for the α in **Assumption 3**, on the total variation distance between the translated equilibrium distribution $\Pi_n - \lfloor nc \rfloor$ and the translated Poisson distribution

$$\widehat{\mathrm{Po}}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big) := \mathrm{Po}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big) - \Big\lfloor \frac{n\sigma^2(c)}{-2F'(c)} \Big\rfloor,$$

in the present chapter we are concerned with precisely how close the point probabilities of these two distributions are, and give the following result:

Theorem 6.1. *There exists, under* **Assumptions 1 to 3, 4s and 5s**, *a constant* $C > 0$ *so that*

$$\sup_{k\in\mathbb{Z}} \Big|(\Pi_n - \lfloor nc \rfloor)(k) - \widehat{\mathrm{Po}}\Big(\frac{n\sigma^2(c)}{-2F'(c)}\Big)(k)\Big| \le C n^{-\frac{\alpha}{2}-\frac{1}{4}},$$

for the α *in* **Assumption 3**.

Proof. The proof will be very much based on the previous results and their proofs, for which reason we keep all the notation that we have used so far. Note that, in view of (5.2.9),

$$\begin{aligned} &\sup_{k\in\mathbb{Z}} |(\Pi_n - \lfloor nc \rfloor)(\{k\}) - \widehat{\mathrm{Po}}(n\lambda)(\{k\})| \\ &\le \frac{1}{-F'(c)} \sup_{k\in\mathbb{Z}} |\mathbb{E}\{\mathrm{RT}_{n\lambda,k}\}| + \sup_{k\in\mathbb{Z}} n\lambda\, |\mathbb{E}\{\bigtriangledown^2 \tilde{g}_{n\lambda,k}(Y_n+1)\}| \\ &+ \sup_{k\in\mathbb{Z}} \widehat{\mathrm{Po}}(n\lambda)(\{k\}) \cdot \mathbb{P}(Y_n < -\lfloor n\lambda \rfloor) \\ &:= R_1 + R_2 + R_3, \end{aligned} \tag{6.0.1}$$

where $\mathrm{RT}_{n\lambda,k}$ is used to denote $\mathrm{RT}_{n\lambda,B}$, see formula (5.1.7), and $\tilde{g}_{n\lambda,k}$ to denote $\tilde{g}_{n\lambda,B}$, for $B = \{k\}$.

Now we need to show that each of R_1, R_2 and R_3 is of order $O(n^{-\frac{\alpha}{2}-\frac{1}{4}})$. To start with the easier part, we show first that R_3 is as small as $O(n^{-1})$.

From Barbour and Jensen (1989, Remark to Lemma 2.1) we note that, for any random variable X which has a Poisson distribution with parameter μ, it is true that

$$\sup_{k\in\mathbb{Z}} \mathbb{P}(X = k) \le \frac{1}{2\sqrt{\mu}},$$

and this fact obviously does not change if we shift the Poisson distribution (and X, for that matter) to the left or right. Therefore, in particular it is true that

$$\sup_{k\in\mathbb{Z}} \widehat{\mathrm{Po}}(n\lambda)(\{k\}) \le \frac{1}{2\sqrt{n\lambda}}.$$

This fact, together with (5.2.17), implies that

$$R_3 = \sup_{k\in\mathbb{Z}} \widehat{\mathrm{Po}}(n\lambda)(\{k\}) \cdot \mathbb{P}(Y_n < -\lfloor n\lambda\rfloor) = O\Big(\frac{1}{n}\Big). \tag{6.0.2}$$

In order to bound the other two terms in (6.0.1), R_1 and R_2, we shall first give two lemmas. The first one of them can be compared with the unconditional version given in Lemma 4.2.

LEMMA 6.2. *Under* **Assumptions 1 to 4**, *there exists, for any* $0 < \tilde\delta \le \delta$, *a constant* $C_{\tilde\delta} > 0$ *so that*

$$\sup_{|z-nc|<\frac{n\tilde\delta}{2e^{\|F'\|_\delta}}} \mathbb{P}\{\sup_{t\in[0,1]} |Z_n(t)-nc| > n\tilde\delta \mid Z_n(0)=z\} \le C_{\tilde\delta}\, n^{-1}.$$

PROOF. From (4.0.37) and (4.0.38) we deduce that

$$\sup_{|z-nc|<\frac{n\tilde\delta}{2e^{\|F'\|_\delta}}} \mathbb{P}\{\sup_{t\in[0,1]} |Z_n(t)-nc| > n\tilde\delta \mid Z_n(0)=z\}$$

$$\le \sup_{|z-nc|<\frac{n\tilde\delta}{2e^{\|F'\|_\delta}}} \mathbb{P}\Big\{\sup_{s\in[0,1]} |\mathcal{M}_n(s)| > \frac{n\tilde\delta}{e^{\|F'\|_\delta}} - |z-nc| \mid Z_n(0)=z\Big\}.$$

By Kolmogorov's inequality and the fact that the quadratic variation process of $\mathcal{M}_n(t)$ is $\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 n\Lambda_j(t\wedge\tau_\delta)$, where $\Lambda_j(t) := \int_0^t \lambda_j(z_n(u))du$, we now have that

$$\sup_{|z-nc|<\frac{n\tilde\delta}{2e^{\|F'\|_\delta}}} \mathbb{P}\{\sup_{t\in[0,1]} |Z_n(t)-nc| > n\tilde\delta \mid Z_n(0)=z\}$$

$$\le \sup_{|z-nc|<\frac{n\tilde\delta}{2e^{\|F'\|_\delta}}} \frac{n\mathbb{E}\{\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j \int_0^1 (1+|z_n(u\wedge\tau_\delta)-c|)du \mid Z_n(0)=z\}}{\big[\frac{n\tilde\delta}{e^{\|F'\|_\delta}} - |z-nc|\big]^2}$$

$$\le \frac{n\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j(1+\delta)}{n^2\frac{\tilde\delta^2}{4e^{2\|F'\|_\delta}}} \le \frac{K_{2,\tilde\delta}\, e^{2\|F'\|_\delta}}{n} := C_{\tilde\delta}\, n^{-1}.$$

□

LEMMA 6.3. *There exists, under* **Assumptions 1 to 3, 4s and 5s**, *a constant* $C > 0$ *so that*

$$\sup_{k\in\mathbb{Z}} |\Pi_n(k) - \Pi_n(k+1)| \le C\, n^{-3/4}.$$

PROOF. Let $t_n := n^{-1/2}$, for any fixed n so that $n > 1/\delta_2$. Since Π_n is the equilibrium distribution of Z_n, it is in particular true that

$$|\Pi_n(k) - \Pi_n(k+1)|$$

$$
\begin{aligned}
&= \Big|\sum_{z\in\mathbb{Z}} \Pi_n(z)\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z) \\
&\qquad - \sum_{z\in\mathbb{Z}} \Pi_n(z)\mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)\Big| \\
&= \Big|\sum_{z\in\mathbb{Z}} \Pi_n(z-1)\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z-1) \\
&\qquad - \sum_{z\in\mathbb{Z}} \Pi_n(z)\mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)\Big| \\
&\le \sum_{z\in\mathbb{Z}} \Pi_n(z-1)|\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z-1) - \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)| \\
&\qquad + \sum_{z\in\mathbb{Z}} |\Pi_n(z-1)-\Pi_n(z)|\mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z).
\end{aligned}
$$

Let us write $\delta_1 := \frac{\delta}{2e^{\|F'\|_\delta}}$ and $\delta_2 := \frac{\delta}{4e^{2\|F'\|_\delta}}$, for the δ in **Assumption 2.** Note that one can write

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}} \Pi_n(z-1)|\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z-1) - \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)| \\
&\le \Pi_n\Big\{|Z_n+1-nc| \ge n\delta_2\Big\} \\
&+ \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z-1) - \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)|,
\end{aligned}
$$

and also one can write

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}} |\Pi_n(z-1)-\Pi_n(z)|\mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z) \\
&\le \Pi_n\Big\{|Z_n+1-nc| \ge n\delta_1\Big\} + \Pi_n\Big\{|Z_n-nc| \ge n\delta_1\Big\} \\
&+ \sup_{|z-nc|<n\delta_1} \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)\cdot 2d_{TV}\{\Pi_n, \Pi_n * \delta_1\}.
\end{aligned}
$$

By applying the result of Corollary 3.5 with $a=\delta_2-1/n$, $a=\delta_1-1/n$ and then with $a=\delta_1$, we obtain that

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}} |\Pi_n(k)-\Pi_n(k+1)| \le O(n^{-1}) \\
&\quad + \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n(t_n)=k \mid Z_n(0)=z-1) \\
&\qquad\qquad - \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)| \\
&\quad + \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_1} \mathbb{P}(Z_n(t_n)=k+1 \mid Z_n(0)=z)\cdot 2d_{TV}\{\Pi_n, \Pi_n * \delta_1\} \\
&\quad := O(n^{-1}) + S_1 + S_2.
\end{aligned}
\tag{6.0.3}
$$

We shall now need two further lemmas.

LEMMA 6.4. *There exists, under* **Assumptions 1 to 3, 4s and 5s**, *a constant* $C' > 0$ *so that*

$$\sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_1} \mathbb{P}(Z_n(t_n) = k \mid Z_n(0) = z) \leq C'/\sqrt{nt_n} = C'\, n^{-1/4}.$$

PROOF. We make the following remark:

Remark 7. *Let* X *be an integer-valued random variable on a probability space* $(\Omega, \mathcal{K}, \mathbb{P})$. *Then, it is true, for any* $k \in \mathbb{Z}$, *that*

$$\begin{aligned}\mathbb{P}(X = k) &= \mathbb{P}(X \leq k) - \mathbb{P}(X < k) = \mathbb{P}(X \leq k) - \mathbb{P}(X \leq k-1)\\ &\leq \sup_{A\in\mathbb{Z}}(\mathbb{P}\{X \in A\} - \mathbb{P}\{X+1 \in A\})\\ &= d_{TV}\{\mathcal{L}(X), \mathcal{L}(X) * \delta_1\},\end{aligned}$$

where $\mathcal{L}(X)$ *denotes the distribution law of* X. *It follows that*

$$\sup_{k\in\mathbb{Z}} \mathbb{P}(X = k) \leq d_{TV}\{\mathcal{L}(X), \mathcal{L}(X) * \delta_1\}.$$

From the above remark, and in view of (4.0.4), we deduce that

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_1} \mathbb{P}(Z_n(t_n) = k \mid Z_n(0) = z)\\ &\leq \sup_{|z-nc|<n\delta_1} d_{TV}\{\mathcal{L}(Z_n(t_n) \mid Z_n(0) = z), \mathcal{L}(Z_n(t_n) + 1 \mid Z_n(0) = z)\}\\ &\leq \sup_{|z-nc|<n\delta_1} \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 0} |\mathbb{P}(N_n(t_n) = l) - \mathbb{P}(N_n(t_n) = l-1)| f_{l,z}^{t_n *}(k-l)\\ &+ \sup_{|z-nc|<n\delta_1} \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 1} \mathbb{P}(N_n(t_n) = l-1)|f_{l,z}^{t_n *}(k-l) - f_{l,z}^{t_n}(k-l)|\\ &:= T_1 + T_2.\end{aligned} \tag{6.0.4}$$

with the usual notations of Chapter 4. Again, as in (4.0.6), we have that

$$\begin{aligned}T_1 &= \sup_{|z-nc|<n\delta_1} \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{l\geq 0} |\mathbb{P}(N_n(t_n) = l) - \mathbb{P}(N_n(t_n) = l-1)| f_{l,z}^{t_n *}(k-l)\\ &\leq d_{TV}(\mathrm{Po}(n\lambda^0 t_n),\ \mathrm{Po}(n\lambda^0 t_n) * \delta_1) = O\Big(\frac{1}{\sqrt{nt_n}}\Big),\end{aligned}$$

from Barbour, Holst and Janson (1992, Theorem 1.C).

On the other hand, because $t_n < 1$, we shall be able to conclude that T_2, the second term in (6.0.4), is of smaller order than T_1, the first term in (6.0.4). This implies that their sum is also of order $O(1/\sqrt{nt_n})$. To reach this conclusion, we return to Step 5 in the proof of Theorem 4.1. We note that, in view of (4.0.26) and (4.0.29), we need to bound quantities similar to those in (4.0.32), (4.0.33) and (4.0.34), but with t_n for U. If we have

$$\sup_{|z-nc|<n\delta_1} \mathbb{E}_{l,s,z}|M_{t_n}^{(n)} \cdot \mathbb{1}_{S^{\Gamma}_{t_n,\delta}(l,s_{[1,l-1]},z)}| \leq \eta_1(n, t_n) \tag{6.0.5}$$

and

$$\sup_{|z-nc|<n\delta_1} \mathbb{E}_{l,s,z}|\epsilon_{t_n}^{(n)} \cdot \mathbb{1}_{S^{\Gamma}_{t_n,\delta}(l,s_{[1,l-1]},z)}| \leq \eta_2(n,t_n), \tag{6.0.6}$$

uniformly for all $(l, s_{[1,l-1]}) \in \bigcup_{l\in\mathbb{Z}}\{l\} \times [0,t_n]^{l-1}$, and also

$$\begin{aligned} &\sup_{|z-nc|<n\delta_1} \sum_{l\geq 1} \mathbb{P}(N_n(t_n) = l-1)\frac{1}{t_n^l}\int_{[0,t_n]^l} ds_1...ds_{l-1}ds^* \\ &\cdot \mathbb{Q}^{\Gamma}_{t_n}(l-1, s_{[1,l-1]}, z)\{B^C_{t_n,\delta}(l, s_{[1,l-1]}, z)\} \leq \eta_3(n,t_n), \end{aligned} \tag{6.0.7}$$

then we can bound T_2 as follows:

$$T_2 \leq \eta_1(n,t_n) + \eta_2(n,t_n) + \eta_3(n,t_n).$$

Remark 8. In view of (4.0.35), *where we replace* U *by* t_n, *we have that* (6.0.7) *is smaller than*

$$\sup_{|z-nc|<n\delta_1} \mathbb{P}\{S^C_{t_n,\delta} \mid X_n(0) = z\} = \sup_{|z-nc|<n\delta_1} \mathbb{P}\{\sup_{t\in[0,t_n]} |Z_n(t) - nc| > n\delta \mid Z_n(0) = z\},$$

since $\mathcal{L}(X_n + N_n \mid X_n(0) = z) = \mathcal{L}(Z_n \mid Z_n(0) = z)$. *Also note that, since* $t_n < 1$,

$$\mathbb{P}\{\sup_{t\in[0,t_n]} |Z_n(t) - nc| > n\delta \mid Z_n(0) = z\} \leq \mathbb{P}\{\sup_{t\in[0,1]} |Z_n(t) - nc| > n\delta \mid Z_n(0) = z\}.$$

The term (6.0.5) can be bounded, in view of (4.0.50) and (4.0.51) with $U = t_n = n^{-1/2}$, by

$$\eta_1(n,t_n) := K^{(1)}_\delta \sqrt{\frac{t_n}{n}} = O(n^{-3/4}),$$

while (6.0.6) can be bounded, in view of (4.0.52) and also with $U = t_n = n^{-1/2}$, by

$$\eta_2(n,t_n) := K^{(2)}_\delta \frac{t_n}{n} = O(n^{-3/2}).$$

In view of Remark 8 and the result of Lemma 6.2 with $\tilde{\delta} = \delta$, (6.0.7) can be bounded by

$$\eta_3(n,t_n) := \frac{K_{2,\delta}\, e^{2\|F'\|_\delta}}{n} = O(n^{-1}).$$

Now, since

$$T_2 \leq \eta_1(n,t_n) + \eta_2(n,t_n) + \eta_3(n,t_n) = O(n^{-3/4}),$$

we conclude that the dominant term in (6.0.4) is T_1, which is of order

$$O(1/\sqrt{nt_n}) = O(n^{-1/4}).$$

□

By the result of Theorem 4.1, it is true that $d_{TV}\{\Pi_n, \Pi_n * \delta_1\} = O(1/\sqrt{n})$. In view of this fact, and also of the result of Lemma 6.4, we now have that

$$\begin{aligned} S_2 &= \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_1} \mathbb{P}(Z_n(t_n) = k+1 \mid Z_n(0) = z) \cdot 2d_{TV}\{\Pi_n, \Pi_n * \delta_1\} \\ &= O(n^{-1/4}) \cdot O(n^{-1/2}) = O(n^{-3/4}). \end{aligned} \tag{6.0.8}$$

The result of Lemma 6.4 we also be used further on, for bounding S_1.

LEMMA 6.5. *There exists, under* **Assumptions 1 to 3, 4s and 5s**, *a constant* $C'' > 0$ *so that*

$$S_1 = \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n(t_n) = k \mid Z_n(0) = z-1) - \mathbb{P}(Z_n(t_n) = k+1 \mid Z_n(0) = z)| \le C''\, n^{-3/4}.$$

PROOF. We shall use a coupling argument. We take two copies of the process Z_n, denote them by $Z_n^{(1)}$ and $Z_n^{(2)}$, and try to couple their paths in such a way that, when they start at distance 1 apart at time 0, then with a very high probability they stay at distance 1 apart, making simultaneously jumps of the same size. More precisely, we assume that

$$Z_n^{(1)}(0) = z \quad \text{and} \quad Z_n^{(2)}(0) = z-1,$$

and that the process $\big(Z_n^{(1)}, Z_n^{(2)}\big)$ moves at rates

$$\begin{aligned}
(i, i-1) &\to (i+j, i-1+j) && \text{at rate } \min\{n\lambda_j\big(\tfrac{i}{n}\big), n\lambda_j\big(\tfrac{i-1}{n}\big)\}\\
(i, i-1) &\to (i+j, i-1) && \text{at rate } \Big(n\lambda_j\big(\tfrac{i}{n}\big) - n\lambda_j\big(\tfrac{i-1}{n}\big)\Big)_+\\
(i, i-1) &\to (i, i-1+j) && \text{at rate } \Big(n\lambda_j\big(\tfrac{i-1}{n}\big) - n\lambda_j\big(\tfrac{i}{n}\big)\Big)_+
\end{aligned}$$

for any $j \in \mathbb{Z}\backslash\{0\}$. Then, under **Assumption 2**, the rate at which the exact coupling fails when leaving the state $(i, i-1)$ is

$$\sum_{j\in\mathbb{Z}\backslash\{0\}} \Big|n\lambda_j\Big(\frac{i}{n}\Big) - n\lambda_j\Big(\frac{i-1}{n}\Big)\Big| \le \sum_{j\in\mathbb{Z}\backslash\{0\}} \sup_{\frac{i-1}{n}\le x\le\frac{i}{n}} |\lambda_j'(x)|,$$

for any $i \in \mathbb{Z}$.

Remark 9. *In particular, if* Z_n *is so that* $|Z_n(u) - nc| \le n\delta$, *for the* δ *in* **Assumption 2** *and for any time* $u > 0$, *then, under* **Assumptions 3 and 4**, *there exists a constant* $F > 0$ *so that*

$$\sum_{j\in\mathbb{Z}\backslash\{0\}} \Big|n\lambda_j\Big(\frac{Z_n(u)}{n}\Big) - n\lambda_j\Big(\frac{Z_n(u)-1}{n}\Big)\Big| \le \sup_{j\in\mathbb{Z}\backslash\{0\}} \frac{\|\lambda_j'\|_\delta}{|j|\lambda_j(c)} \sum_{j\in\mathbb{Z}\backslash\{0\}} |j|\lambda_j(c) < F.$$

Let τ denote the first time when the exact coupling fails. We have that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n(t_n) = k \mid Z_n(0) = z-1)\\
&\qquad\qquad -\mathbb{P}(Z_n(t_n) = k+1 \mid Z_n(0) = z)|\\
&= \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n^{(2)}(t_n) = k \mid Z_n^{(2)}(0) = z-1)\\
&\qquad\qquad -\mathbb{P}(Z_n^{(1)}(t_n) = k+1 \mid Z_n^{(1)}(0) = z)|\\
&\le \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n^{(2)}(t_n) = k, \tau \le t_n \mid Z_n^{(2)}(0) = z-1)
\end{aligned}$$

$$-\mathbb{P}(Z_n^{(1)}(t_n) = k+1, \tau \le t_n \mid Z_n^{(1)}(0) = z)| \tag{6.0.9}$$
$$\le 2 \sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} \mathbb{P}(Z_n(t_n) = k, \tau \le t_n \mid Z_n(0) = z).$$

For any $k \in \mathbb{Z}$ and z in $|z - nc| < n\delta_2$, we have that

$$\begin{aligned}
&\mathbb{P}(Z_n(t_n) = k, \tau \le t_n \mid Z_n(0) = z) \\
&= \int_0^{t_n} \mathbb{P}(Z_n(t_n) = k, \tau \in [u, u+du) \mid Z_n(0) = z) \\
&= \int_0^{t_n} \mathbb{P}(\tau \in [u, u+du) \mid Z_n(0) = z) \cdot \mathbb{P}(Z_n(t_n) = k \mid \tau = u, Z_n(0) = z).
\end{aligned}$$

By the strong Markov property for the process Z_n, this further implies that

$$\begin{aligned}
&\mathbb{P}(Z_n(t_n) = k, \tau \le t_n \mid Z_n(0) = z) \\
&= \int_0^{t_n} \mathbb{P}(\tau \in [u, u+du) \mid Z_n(0) = z) \sum_{l\in\mathbb{Z}} \mathbb{P}(Z_n(u) = l \mid \tau = u, Z_n(0) = z) \\
&\qquad \cdot \mathbb{P}(Z_n(t_n) = k \mid \tau = u, Z_n(u) = l, Z_n(0) = z) \\
&= \int_0^{t_n} \sum_{l\in\mathbb{Z}} \mathbb{P}(Z_n(u) = l, \tau \in [u, u+du) \mid Z_n(0) = z) \cdot \mathbb{P}(Z_n(t_n) = k \mid Z_n(u) = l) \\
&= \int_0^{t_n} \sum_{|l-nc|\ge n\delta_1} \mathbb{P}(Z_n(u) = l, \tau \in [u, u+du) \mid Z_n(0) = z) \\
&\qquad \cdot \mathbb{P}(Z_n(t_n) = k \mid Z_n(u) = l) \qquad (6.0.10) \\
&+ \int_0^{t_n} \sum_{|l-nc|<n\delta_1} \mathbb{P}(Z_n(u) = l, \tau \in [u, u+du) \mid Z_n(0) = z) \\
&\qquad \cdot \mathbb{P}(Z_n(t_n) = k \mid Z_n(u) = l).
\end{aligned}$$

Note that, in view of the result of Lemma 6.2 with $\tilde{\delta} = \delta_1 \le \delta$, it is true that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} \int_0^{t_n} \sum_{|l-nc|\ge n\delta_1} \mathbb{P}(Z_n(u) = l, \tau \in [u, u+du) \mid Z_n(0) = z) \\
&\qquad \cdot \mathbb{P}(Z_n(t_n) = k \mid Z_n(u) = l) \\
&\le \sup_{|z-nc|<n\delta_2} \int_0^{t_n} \mathbb{P}\Big(|Z_n(u) - nc| \ge n\delta_1, \tau \in [u, u+du) \mid Z_n(0) = z\Big) \\
&\le \sup_{|z-nc|<n\delta_2} \mathbb{P}\Big(\sup_{u\in[0,t_n]} |Z_n(u) - nc| \ge n\delta_1, \tau \le t_n \mid Z_n(0) = z\Big) \\
&\le \sup_{|z-nc|<n\delta_2} \mathbb{P}\Big(\sup_{u\in[0,1]} |Z_n(u) - nc| \ge n\delta_1 \mid Z_n(0) = z\Big) \le C_{\delta_1}\, n^{-1}. \qquad (6.0.11)
\end{aligned}$$

Also, it is true that

$$\sup_{k\in\mathbb{Z}}\sup_{|z-nc|<n\delta_2}\int_0^{t_n}\sum_{|l-nc|<n\delta_1}\mathbb{P}(Z_n(u)=l,\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\cdot\mathbb{P}(Z_n(t_n)=k\mid Z_n(u)=l)$$
$$\leq\sup_{|z-nc|<n\delta_2}\int_0^{t_n}\mathbb{P}(|Z_n(u)-nc|<n\delta_1,\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\cdot\sup_{k\in\mathbb{Z}}\sup_{|l-nc|<n\delta_1}\mathbb{P}(Z_n(t_n)=k\mid Z_n(u)=l)$$
$$\leq\sup_{|z-nc|<n\delta_2}\int_0^{t_n}\mathbb{P}(\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\cdot\sup_{k\in\mathbb{Z}}\sup_{|l-nc|<n\delta_1}\mathbb{P}(Z_n(t_n)=k\mid Z_n(u)=l).\tag{6.0.12}$$

In view of the result of Lemma 6.4 and of Remark 9, and since $\delta_1<\delta$ and $t_n<1$, we have that

$$\int_0^{t_n}\mathbb{P}(\tau\in[u,u+du)\mid Z_n(0)=z)\cdot\sup_{k\in\mathbb{Z}}\sup_{|l-nc|<n\delta_1}\mathbb{P}(Z_n(t_n)=k\mid Z_n(u)=l)$$
$$\leq\int_0^{t_n}\mathbb{P}(|Z_n(u-)-nc|<n\delta_1,\tau\in[u,u+du)\mid Z_n(0)=z)\cdot\frac{C'}{\sqrt{n(t_n-u)}}$$
$$+\int_0^{t_n}\mathbb{P}(|Z_n(u-)-nc|\geq n\delta_1,\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\leq FC'\int_0^{t_n}\frac{du}{\sqrt{n(t_n-u)}}+\int_0^{t_n}\mathbb{P}(|Z_n(u-)-nc|\geq n\delta_1,\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\leq 2FC'\sqrt{\frac{t_n}{n}}+\mathbb{P}(\sup_{u\in[0,t_n]}|Z_n(u-)-nc|\geq n\delta_1,\tau\leq t_n\mid Z_n(0)=z)$$
$$\leq 2FC'\sqrt{\frac{t_n}{n}}+\mathbb{P}(\sup_{u\in[0,1]}|Z_n(u)-nc|\geq n\delta_1\mid Z_n(0)=z).\tag{6.0.13}$$

Hence, in view of (6.0.12) and (6.0.13), and also of the result of Lemma 6.2 with $\tilde{\delta}=\delta_1$, we further have that

$$\sup_{k\in\mathbb{Z}}\sup_{|z-nc|<n\delta_2}\int_0^{t_n}\sum_{|l-nc|<n\delta_1}\mathbb{P}(Z_n(u)=l,\tau\in[u,u+du)\mid Z_n(0)=z)$$
$$\cdot\mathbb{P}(Z_n(t_n)=k\mid Z_n(u)=l)$$
$$\leq 2FC'\sqrt{\frac{t_n}{n}}+\sup_{|z-nc|<n\delta_2}\mathbb{P}(\sup_{u\in[0,1]}|Z_n(u)-nc|\geq n\delta_1\mid Z_n(0)=z)$$
$$\leq 2FC'\sqrt{\frac{t_n}{n}}+C_{\delta_1}\,n^{-1}.\tag{6.0.14}$$

In view of (6.0.9) to (6.0.11) and of (6.0.14), we now conclude that

$$\sup_{k\in\mathbb{Z}} \sup_{|z-nc|<n\delta_2} |\mathbb{P}(Z_n(t_n) = k \mid Z_n(0) = z-1) - \mathbb{P}(Z_n(t_n) = k+1 \mid Z_n(0) = z)|$$

$$\leq 4FC'\sqrt{\frac{t_n}{n}} + 4C_{\delta_1}\, n^{-1} = O(n^{-3/4}).$$

□

In view of the result of Lemma 6.5, S_1 is of order $O(n^{-3/4})$. Since also S_2 is of order $O(n^{-3/4})$, in view of (6.0.8), it follows from (6.0.3) that

$$\sup_{k\in\mathbb{Z}} |\Pi_n(k) - \Pi_n(k+1)| = O(n^{-3/4}).$$

□

For obtaining the desired bounds on R_1 and R_2, namely the first and the second term in (6.0.1), it is also very important to note that the function $\tilde{g}_{n\lambda,k}$, which is defined in (5.1.2) as the solution to the Stein Equation (2.2.1), for $A = \{k\} + \lfloor n\lambda \rfloor$, has very nice properties indeed, now that we reduced the set B to the one point set $\{k\} \in \mathbb{Z}$. The following remark makes these properties clear:

Remark 10. *By Barbour, Holst and Janson (1992, Lemma 1.1.1) we have that*

$$\sup_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,k}(i)| \leq \bigtriangledown\tilde{g}_{n\lambda,k}(k+1) \leq \min\{k^{-1}, (n\lambda)^{-1}(1-e^{-n\lambda})\};$$

furthermore, the function $\tilde{g}_{n\lambda,k}(i+1)$ is negative and strictly decreasing in $i < k$, and positive and strictly decreasing in $i \geq k$. Hence,

$$\sup_{i\in\mathbb{Z}} |\tilde{g}_{n\lambda,k}(i)| \leq \bigtriangledown\tilde{g}_{n\lambda,k}(k+1) \leq (n\lambda)^{-1}, \tag{6.0.15}$$

$$\sum_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,k}(i)| \leq 2 \bigtriangledown \tilde{g}_{n\lambda,k}(k+1) \leq 2(n\lambda)^{-1} \tag{6.0.16}$$

and therefore also

$$\sum_{i\in\mathbb{Z}} |\bigtriangledown^2 \tilde{g}_{n\lambda,k}(i)| \leq 4 \bigtriangledown \tilde{g}_{n\lambda,k}(k+1) \leq 4(n\lambda)^{-1}. \tag{6.0.17}$$

In view of (6.0.16) and the result of Lemma 6.3, we can bound R_2 as follows:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} n\lambda\, |\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,k}(Y_n+1)\}| \\
&= \sup_{k\in\mathbb{Z}} n\lambda\, \Big|\sum_{i\in\mathbb{Z}} \{\bigtriangledown\tilde{g}_{n\lambda,k}(i-\lfloor nc\rfloor+1) - \bigtriangledown\tilde{g}_{n\lambda,k}(i-\lfloor nc\rfloor)\}\Pi_n(i)\Big| \\
&= \sup_{k\in\mathbb{Z}} n\lambda\, \Big|\sum_{i\in\mathbb{Z}} \bigtriangledown\tilde{g}_{n\lambda,k}(i-\lfloor nc\rfloor)(\Pi_n(i-1)-\Pi_n(i))\Big| \\
&\leq \sup_{k\in\mathbb{Z}} n\lambda \cdot \sup_{i\in\mathbb{Z}} \big|\Pi_n(i-1)-\Pi_n(i)\big| \cdot \sum_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,k}(i-\lfloor nc\rfloor)| \qquad (6.0.18) \\
&\leq n\lambda \cdot \sup_{i\in\mathbb{Z}} \big|\Pi_n(i-1)-\Pi_n(i)\big| \cdot 2(n\lambda)^{-1}
\end{aligned}$$

$$= O(n^{-3/4}).$$

In order to bound R_1, we first choose ϵ_n depending on n, in such a way that $\frac{1}{n} < \epsilon_n < 1$. In view of (5.1.8), (5.1.9) and (5.1.10), we now can bound $|\mathbb{E}\{\mathrm{RT}_{n\lambda,k}\}|$ by the sum of the following terms:

$$\begin{aligned}
\mathrm{E}^{(1)}{}_{n\lambda,k} \quad &:= \frac{1}{2}\|(\sigma^2)'\|_\delta \; n\epsilon_n \; \mathbb{E}\{|\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{1}{2}\|(\sigma^2)'\|_\delta \; \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j \; \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{n}{2}\big[\sigma^2(c) + \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\big] \; \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\},
\end{aligned} \tag{6.0.19}$$

$$\begin{aligned}
\mathrm{E}^{(2)}{}_{n\lambda,k} \quad &:= n\mathbb{E}\Big\{\Big|F\Big(\frac{Y_n + \lfloor nc\rfloor}{n}\Big) - F(c) - \frac{Y_n}{n}F'(c)\Big| \cdot |\tilde{g}_{n\lambda,k}(Y_n)|\Big\} \\
&+ |F'(c)|\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle \; \mathbb{E}\{|\tilde{g}_{n\lambda,k}(Y_n)|\},
\end{aligned} \tag{6.0.20}$$

$$\begin{aligned}
\mathrm{E}^{(3)}{}_{n\lambda,k} \quad &:= \frac{1}{2}\|F'\|_\delta \; n\epsilon_n \; \mathbb{E}\{|\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{1}{2}\|F'\|_\delta \; \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{1}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j| \; c_j \; \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} \\
&+ \frac{n}{2}\sum_{j\in\mathbb{Z}\setminus\{0\}} |j| \; c_j \; \mathbb{E}\{\mathbb{1}(|Y_n - \langle nc\rangle| > n\delta) \cdot |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\},
\end{aligned} \tag{6.0.21}$$

$$\begin{aligned}
\mathrm{E}^{(4)}{}_{n\lambda,k} \quad &:= \sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2} \; n\lambda_j(c) \; |\mathbb{E}\{\bigtriangledown^2 \tilde{g}_{n\lambda,k}(Y_n + j - l + 1)\}| \\
&+ \sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2} \; \|\lambda_j'\|_\delta \; n\epsilon_n \; \mathbb{E}\{|\bigtriangledown^2 \tilde{g}_{n\lambda,k}(Y_n + j - l + 1)|\} \\
&+ \sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2} \; \|\lambda_j'\|_\delta \; \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta) \\
&\qquad \cdot |\bigtriangledown^2 \tilde{g}_{n\lambda,k}(Y_n + j - l + 1)|\}
\end{aligned}$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, c_j\, \mathbb{E}\{|Y_n-\langle nc\rangle|\cdot \mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)$$

$$\cdot|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n+j-l+1)|\}$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, n[\lambda_j(c)+c_j]\, \mathbb{E}\{\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)$$

$$\cdot|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n+j-l+1)|\}, \tag{6.0.22}$$

$$\mathrm{E}^{(5)}{}_{n\lambda,k} := \sum_{j>\sqrt{n}} nc_j \sum_{l=1}^{j-1} l\, \mathbb{E}\{|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n)|+|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n+j-l)|\}$$

$$+\sum_{j>\sqrt{n}} c_j \sum_{l=1}^{j-1} l\, \mathbb{E}\Big\{|Y_n-\langle nc\rangle|\cdot\Big(|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n)|+|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n+j-l)|\Big)\Big\}, \tag{6.0.23}$$

$$\mathrm{E}^{(6)}{}_{n\lambda,k} := \sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, n\lambda_{-j}(c)\, |\mathbb{E}\{\bigtriangledown^2\tilde{g}_{n\lambda,k}(Y_n-j+l)\}|$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, \|\lambda'_{-j}\|_\delta\, n\epsilon_n\, \mathbb{E}\{|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n-j+l)|\}$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, \|\lambda'_{-j}\|_\delta\, \mathbb{E}\{|Y_n-\langle nc\rangle|\cdot\mathbb{1}(n\epsilon_n<|Y_n-\langle nc\rangle|\le n\delta)$$

$$\cdot|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n-j+l)|\}$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, c_{-j}\, \mathbb{E}\{|Y_n-\langle nc\rangle|\cdot\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)$$

$$\cdot|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n-j+l)|\}$$

$$+\sum_{j=2}^{\sqrt{n}}\sum_{l=2}^{j}\binom{l}{2}\, n[\lambda_{-j}(c)+c_{-j}]\, \mathbb{E}\{\mathbb{1}(|Y_n-\langle nc\rangle|>n\delta)$$

$$\cdot|\bigtriangledown^2\,\tilde{g}_{n\lambda,k}(Y_n-j+l)|\}, \tag{6.0.24}$$

and

$$\mathrm{E}^{(7)}{}_{n\lambda,k} := \sum_{j>\sqrt{n}} nc_{-j} \sum_{l=1}^{j-1} l\, \mathbb{E}\{|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n)|+|\bigtriangledown\tilde{g}_{n\lambda,k}(Y_n-j+l)|\}$$

$$+\sum_{j>\sqrt{n}} c_{-j} \sum_{l=1}^{j-1} l\, \mathbb{E}\Big\{|Y_n - \langle nc\rangle| \cdot \Big(|\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)| + |\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n - j + l)|\Big)\Big\}. \tag{6.0.25}$$

Note that, in view of the results of Theorem 3.1, we have that

$$\begin{aligned}
&n\epsilon_n \mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta)\}\\
&\le \mathbb{E}\{(Y_n - \langle nc\rangle)^2 \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta)\}\\
&\le \mathbb{E}\{(Y_n - \langle nc\rangle)^2 \cdot \mathbb{1}(|Y_n - \langle nc\rangle| \le n\delta)\}\\
&= n^2\mathbb{E}\{(z_n - c)^2 \cdot \mathbb{1}(|z_n - c| \le \delta)\} = O(n).
\end{aligned}$$

It follows that

$$\mathbb{E}\{|Y_n - \langle nc\rangle| \cdot \mathbb{1}(n\epsilon_n < |Y_n - \langle nc\rangle| \le n\delta)\} = O\Big(\frac{1}{\epsilon_n}\Big). \tag{6.0.26}$$

Also note that

$$\begin{aligned}
\mathbb{E}\{|\bigtriangledown \tilde{g}_{n\lambda,k}(Y_n)|\} &= \sum_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,k}(i - \lfloor nc\rfloor)|\Pi_n(i)\\
&\le \sup_i \Pi_n(i) \cdot \sum_{i\in\mathbb{Z}} |\bigtriangledown \tilde{g}_{n\lambda,k}(i - \lfloor nc\rfloor)| = O\Big(\frac{1}{n^{3/2}}\Big),
\end{aligned}$$

in view of Remark 7 and (6.0.16), and similarly that

$$\begin{aligned}
\mathbb{E}\{|\bigtriangledown^2 \tilde{g}_{n\lambda,k}(Y_n + C)|\} &= \sum_{i\in\mathbb{Z}} |\bigtriangledown^2 \tilde{g}_{n\lambda,k}(i - \lfloor nc\rfloor + C)|\Pi_n(i)\\
&\le \sup_i \Pi_n(i) \cdot \sum_{i\in\mathbb{Z}} |\bigtriangledown^2 \tilde{g}_{n\lambda,k}(i - \lfloor nc\rfloor + C)| = O\Big(\frac{1}{n^{3/2}}\Big),
\end{aligned}$$

in view of Remark 7 and (6.0.17), for any integer constant C.

In view of the above, of (6.0.26) and (3.0.6), we now succeed in bounding $\sup_{k\in\mathbb{Z}} |\mathbb{E}\{\mathrm{RT}_{n\lambda,k}\}|$, based on (6.0.19) $-$ (6.0.25), by the sum of the following bounds:

$$\begin{aligned}
\sup_{k\in\mathbb{Z}} \mathrm{E}^{(1)}{}_{n\lambda,k} \quad &:= \frac{1}{2}\|(\sigma^2)'\|_\delta \Big[n\epsilon_n \cdot O\Big(\frac{1}{n^{3/2}}\Big) + O\Big(\frac{1}{\epsilon_n}\Big) \cdot O\Big(\frac{1}{n}\Big)\Big]\\
&+ \frac{1}{2} \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\, O(1) \cdot O\Big(\frac{1}{n}\Big)\\
&+ \frac{1}{2}\big[\sigma^2(c) + \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 c_j\big]\; n \cdot O\Big(\frac{1}{n}\Big) \cdot O\Big(\frac{1}{n}\Big),
\end{aligned}$$

$$\sup_{k\in\mathbb{Z}} \mathrm{E}^{(2)}{}_{n\lambda,k} := O(1) \cdot O\Big(\frac{1}{n}\Big) + |F'(c)|\Big\langle \frac{n\sigma^2(c)}{-2F'(c)}\Big\rangle \cdot O\Big(\frac{1}{n}\Big),$$

$$\sup_{k\in\mathbb{Z}} \mathrm{E}^{(3)}{}_{n\lambda,k} \quad := \frac{1}{2}\|F'\|_\delta \Big[n\epsilon_n \cdot O\Big(\frac{1}{n^{3/2}}\Big) + O\Big(\frac{1}{\epsilon_n}\Big) \cdot O\Big(\frac{1}{n}\Big)\Big]$$

$$+ \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \{0\}} |j| \; c_j \; O(1) \cdot O\Big(\frac{1}{n}\Big)$$
$$+ \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \{0\}} |j| \; c_j \; n \cdot O\Big(\frac{1}{n}\Big) \cdot O\Big(\frac{1}{n}\Big),$$

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \left(\mathrm{E}^{(4)}{}_{n\lambda,k} + \mathrm{E}^{(6)}{}_{n\lambda,k}\right) \quad &:= \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{1-\alpha} \; c_j \cdot O\Big(\frac{1}{n^{3/4}}\Big) \\
&+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{1-\alpha} \; \|\lambda_j'\|_\delta \; \Big[n\epsilon_n \cdot O\Big(\frac{1}{n^{3/2}}\Big) + O\Big(\frac{1}{\epsilon_n}\Big) \cdot O\Big(\frac{1}{n}\Big) \Big] \\
&+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{1-\alpha} \; c_j \cdot O(1) \cdot O\Big(\frac{1}{n}\Big) \\
&+ \sum_{|j|=2}^{\sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{1-\alpha} \; 2nc_j \cdot O\Big(\frac{1}{n}\Big) \cdot O\Big(\frac{1}{n}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \left(\mathrm{E}^{(5)}{}_{n\lambda,k} + \mathrm{E}^{(7)}{}_{n\lambda,k}\right) \quad &:= \sum_{|j| > \sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{-\alpha} n c_j \cdot O\Big(\frac{1}{n^{3/2}}\Big) \\
&+ \sum_{|j| > \sqrt{n}} |j|^{2+\alpha} (\sqrt{n})^{-\alpha} c_j \cdot O(\sqrt{n}) \cdot O\Big(\frac{1}{n}\Big).
\end{aligned}$$

After a careful look at each of the bounds above, we realize that the largest size for their sum, which is not larger than $n^{-\frac{\alpha}{2}}$, is attained for $\epsilon_n = n^{-\frac{1}{4}}$. Hence, we have obtained a bound of size $O(n^{-\frac{\alpha}{2}-\frac{1}{4}})$ on R_1, the first term in (6.0.1), which now becomes, in view of (6.0.2) and (6.0.18), the dominant term in (6.0.1), and this proves the desired result. □

CHAPTER 7

Appendix

1. Proof of Lemma 2.1

Proof of Lemma 2.1. By **Assumption 3**, the sums upon $j \in \mathbb{Z} \setminus \{0\}$ that appear in this proof are absolutely convergent. Also note that g is bounded. We write $|j| \geq 1$ instead of $j \in \mathbb{Z} \setminus \{0\}$, and then rewrite $(\mathcal{A}_n h)(i)$ as follows:

$$
\begin{aligned}
(\mathcal{A}_n h)(i) &= \sum_{|j|\geq 1} n\lambda_j\Big(\frac{i}{n}\Big)[h(i+j)-h(i)] \\
&= -\sum_{j\leq -1} n\lambda_j\Big(\frac{i}{n}\Big) \sum_{k=1}^{-j} g(i-k) \\
&\quad +\sum_{j\geq 1} n\lambda_j\Big(\frac{i}{n}\Big) \sum_{k=1}^{j} g(i+k-1), \qquad (7.1.1)
\end{aligned}
$$

for any $i \in \mathbb{Z}$. We rearrange, obtaining:

$$
\begin{aligned}
(\mathcal{A}_n h)(i) &= -n\sum_{k\geq 1}\sum_{j\geq k} \lambda_{-j}\Big(\frac{i}{n}\Big)g(i-k) \\
&+ n\sum_{k\geq 1}\sum_{j\geq k} \lambda_{j}\Big(\frac{i}{n}\Big)g(i+k-1) \\
&=: \mathcal{D}.
\end{aligned}
$$

We further write $\triangledown g(i) := g(i)-g(i-1)$ and by adding and subtracting to $(\mathcal{A}_n h)(i)$ the term $\frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) \triangledown g(i) + nF\Big(\frac{i}{n}\Big)g(i)$, we obtain

$$
\begin{aligned}
(\mathcal{A}_n h)(i) &= \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) \triangledown g(i) + nF\Big(\frac{i}{n}\Big)g(i) \\
&- g(i)\Big\{\frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) + nF\Big(\frac{i}{n}\Big)\Big\} \\
&+ g(i-1)\frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) \\
&+ \mathcal{D},
\end{aligned}
$$

which leads to the more explicit formula

$$
(\mathcal{A}_n h)(i) = \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big) \triangledown g(i) + nF\Big(\frac{i}{n}\Big)g(i)
$$

$$
\begin{aligned}
& - \quad g(i)\sum_{j\geq 1}\Big\{\Big(\frac{j^2}{2}-j\Big)n\lambda_{-j}\Big(\frac{i}{n}\Big)+\Big(\frac{j^2}{2}+j\Big)n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad g(i-1)\sum_{j\geq 1}\Big\{\frac{j^2}{2}n\lambda_{-j}\Big(\frac{i}{n}\Big)+\frac{j^2}{2}n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad \mathcal{D}.
\end{aligned}
$$

Since $\mathcal{D}$ equals

$$
\begin{aligned}
\mathcal{D} \;=\; & g(i)\sum_{j\geq 1}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-1)\sum_{j\geq 1}n\lambda_{-j}\Big(\frac{i}{n}\Big)\\
& +\sum_{k\geq 2}\Big\{g(i+k-1)\sum_{j\geq k}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-k)\sum_{j\geq k}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\},
\end{aligned}
$$

we may then write

$$
\begin{aligned}
(\mathcal{A}_n h)(i) \;=\; & \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big)\bigtriangledown g(i)+nF\Big(\frac{i}{n}\Big)g(i)\\
& - \quad g(i)\sum_{j\geq 1}\Big\{\Big(\frac{j^2-j}{2}-\frac{j}{2}\Big)n\lambda_{-j}\Big(\frac{i}{n}\Big)+\Big(\frac{j^2+j-2}{2}+\frac{j}{2}\Big)n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad g(i-1)\sum_{j\geq 1}\Big\{\Big(\frac{j^2+j-2}{2}-\frac{j}{2}\Big)n\lambda_{-j}\Big(\frac{i}{n}\Big)+\Big(\frac{j^2-j}{2}+\frac{j}{2}\Big)n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad \sum_{k\geq 2}\Big\{g(i+k-1)\sum_{j\geq k}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-k)\sum_{j\geq k}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\}.
\end{aligned}
$$

We notice that one can extract the term $\frac{n}{2}F\Big(\frac{i}{n}\Big)\bigtriangledown g(i)$ from the first 2 sums:

$$
\begin{aligned}
(\mathcal{A}_n h)(i) \;=\; & \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big)\bigtriangledown g(i)+nF\Big(\frac{i}{n}\Big)g(i)-\frac{n}{2}F\Big(\frac{i}{n}\Big)\bigtriangledown g(i)\\
& - \quad g(i)\sum_{j\geq 1}\Big\{\frac{j^2-j}{2}\cdot n\lambda_{-j}\Big(\frac{i}{n}\Big)+\frac{j^2+j-2}{2}\cdot n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad g(i-1)\sum_{j\geq 1}\Big\{\frac{j^2+j-2}{2}\cdot n\lambda_{-j}\Big(\frac{i}{n}\Big)+\frac{j^2-j}{2}\cdot n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
& + \quad \sum_{k\geq 2}\Big\{g(i+k-1)\sum_{j\geq k}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-k)\sum_{j\geq k}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\}.
\end{aligned}
$$

Next we shall again notice that the coefficients of the terms $g(i)$ and $g(i-1)$ are both, for $j=1$, equal to 0. Therefore we can also replace, for every $j>1$, the term $\frac{j^2-j}{2}$ by $\binom{j}{2}$ and the term $\frac{j^2+j-2}{2}$ by $[\binom{j}{2}+\binom{j-1}{1}]$:

$$
(\mathcal{A}_n h)(i) \;=\; \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big)\bigtriangledown g(i)+nF\Big(\frac{i}{n}\Big)g(i)-\frac{n}{2}F\Big(\frac{i}{n}\Big)\bigtriangledown g(i)
$$

$$
\begin{aligned}
- &\quad g(i)\sum_{j\geq 2}\Big\{\binom{j}{2}n\lambda_{-j}\Big(\frac{i}{n}\Big)+\Big[\binom{j}{2}+\binom{j-1}{1}\Big]n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
+ &\quad g(i-1)\sum_{j\geq 2}\Big\{\Big[\binom{j}{2}+\binom{j-1}{1}\Big]n\lambda_{-j}\Big(\frac{i}{n}\Big)+\binom{j}{2}n\lambda_j\Big(\frac{i}{n}\Big)\Big\}\\
+ &\quad \sum_{k\geq 2}\Big\{g(i+k-1)\sum_{j\geq k}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-k)\sum_{j\geq k}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\}.
\end{aligned}
$$

We rewrite the above in such a way to obtain an expression in which the transition rates $n\lambda_{-j}\Big(\frac{i}{n}\Big)$ and $n\lambda_j\Big(\frac{i}{n}\Big)$ are separately set:

$$
\begin{aligned}
(\mathcal{A}_n h)(i) &= \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big)\bigtriangledown g(i)+nF\Big(\frac{i}{n}\Big)g(i)-\frac{n}{2}F\Big(\frac{i}{n}\Big)\bigtriangledown g(i)\\
&\quad +\sum_{j\geq 2}n\lambda_j\Big(\frac{i}{n}\Big)\Big\{-\Big[\binom{j}{2}+\binom{j-1}{1}\Big]g(i)+\binom{j}{2}g(i-1)\Big\}\\
&\quad -\sum_{j\geq 2}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\{\binom{j}{2}g(i)-\Big[\binom{j}{2}+\binom{j-1}{1}\Big]g(i-1)\Big\}\\
&\quad +\sum_{k\geq 2}\Big\{g(i+k-1)\sum_{j\geq k}n\lambda_j\Big(\frac{i}{n}\Big)-g(i-k)\sum_{j\geq k}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\}.
\end{aligned}
$$

In view of (7.1.1), we have that

$$
\begin{aligned}
(\mathcal{A}_n h)(i) &= \frac{n}{2}\sigma^2\Big(\frac{i}{n}\Big)\bigtriangledown g(i)+nF\Big(\frac{i}{n}\Big)g(i)-\frac{n}{2}F\Big(\frac{i}{n}\Big)\bigtriangledown g(i)\\
&\quad +\sum_{j\geq 2}n\lambda_j\Big(\frac{i}{n}\Big)\Big\{\binom{j}{2}g(i-1)-\Big[\binom{j}{2}+\binom{j-1}{1}\Big]g(i)+\sum_{k=1}^{j-1}g(i+k)\Big\}\\
&\quad -\sum_{j\geq 2}n\lambda_{-j}\Big(\frac{i}{n}\Big)\Big\{\binom{j}{2}g(i)-\Big[\binom{j}{2}+\binom{j-1}{1}\Big]g(i-1)+\sum_{k=2}^{j}g(i-k)\Big\}.
\end{aligned}
\tag{7.1.2}
$$

It is easy to check the fact that, for any fixed integer $j>1$ and for any i,

$$
\begin{aligned}
&\sum_{k=2}^{j}\binom{k}{2}\bigtriangledown^2 g(i+j-k+1)\\
&\qquad =\binom{j}{2}g(i-1)-\Big[\binom{j}{2}+\binom{j-1}{1}\Big]g(i)+\sum_{k=1}^{j-1}g(i+k)
\end{aligned}
\tag{7.1.3}
$$

while it is also true that

$$
\sum_{k=1}^{j-1}k\Big(\bigtriangledown g(i+j-k)-\bigtriangledown g(i)\Big)
\tag{7.1.4}
$$

$$= \binom{j}{2} g(i-1) - \Big[\binom{j}{2} + \binom{j-1}{1}\Big] g(i) + \sum_{k=1}^{j-1} g(i+k).$$

One also has that

$$\sum_{k=2}^{j} \binom{k}{2} \bigtriangledown^2 g(i-j+k) \tag{7.1.5}$$
$$= \binom{j}{2} g(i) - \Big[\binom{j}{2} + \binom{j-1}{1}\Big] g(i-1) + \sum_{k=2}^{j} g(i-k),$$

while on the other hand it is true that

$$\sum_{k=1}^{j-1} k\Big(\bigtriangledown g(i) - \bigtriangledown g(i-j+k)\Big) \tag{7.1.6}$$
$$= \binom{j}{2} g(i) - \Big[\binom{j}{2} + \binom{j-1}{1}\Big] g(i-1) + \sum_{k=2}^{j} g(i-k).$$

Under **Assumption 3**, we now use (7.1.3) to (7.1.6) in order to obtain from (7.1.2) the desired form of the generator $\mathcal{A}_n$.

□

Bibliography

[1] S.E. Alm (1978) On the rate of convergence in diffusion approximation of jump Markov processes. Technical Report No. 1978:8, Uppsala University, Sweden.

[2] N.T.J. Bailey (1967) *The mathematical approach to biology and medicine.* John Wiley & Sons Ltd.

[3] A.D. Barbour (1980) Equilibrium distributions Markov population processes. *Adv. Appl. Prob.* **12**, 591-614.

[4] A.D. Barbour (2001) Topics in Poisson Approximation. *Handbook of Statistics, Vol. 19*, Elsevier Science B.V., 79-115.

[5] A.D. Barbour and V. Čekanavičius (2002) Total variation asymptotics for sums of independent integer random variables. *Ann. Prob.* **30**, 509-545.

[6] A.D. Barbour and G.K. Eagleson (1983) Poisson approximation for some statistics based on exchangeable trials. *Adv. Appl. Prob.* **15**, 585-600.

[7] A.D. Barbour, L. Holst and S. Janson (1992) *Poisson Approximation.* Oxford Univ. Press.

[8] A.D. Barbour and J.L. Jensen (1989) Local and tail approximations near the Poisson limit. *Scand. J. Statist.* **16**, 75-87.

[9] A.D. Barbour and A. Xia (1999) Poisson perturbations. *ESAIM, P&S* **3**, 131-150.

[10] E. Bolthausen (1984) An estimate of the remainder in a combinatorial central limit theorem. *Z. Wahrsch. verw. Gebiete* **66**, 379-386.

[11] P. Brémaud (1981) *Point processes and queues, martingale dynamics.* Springer-Verlag, New York.

[12] L.H.Y. Chen (1975) Poisson approximation for dependent trials. *Ann. Probab.* **3**, 534-545.

[13] L.H.Y. Chen, Q.-M. Shao (2003) Stein's method and normal approximation. *Tutorial notes for the Workshop on Stein's method and applications*, IMS National University of Singapore.

[14] J.E. Cohen (2005) Leben mit neun Milliarden Menschen. *Spektrum der Wissenschaft*, Ausgabe Oktober 2005, Heidelberg.

[15] D.J. Daley and D. Vere-Jones (2003) *An introduction to the theory of point processes, Vol. 1.* Springer-Verlag, New York.

[16] S.N. Ethier and T.G. Kurtz (1986) *Markov processes: Characterization and convergence.* John Wiley & Sons, Inc.

[17] W. Feller (1939) Die Grundlagen der volterraschen Theorie des Kampfes ums Dasein in wahrscheinlichkeitstheoretischer Behandlung. *Acta Biotheoretica* **5**, 11-40.

[18] T.R. Fleming and D.P. Harrington (1991) *Counting processes and survival analysis.* John Wiley & Sons Ltd.

[19] K. Hamza and F.C. Klebaner (1995) Conditions for integrability of Markov chains. *J. Appl. Prob.* **32**, 541-547.

[20] T.G. Kurtz (1970) Solutions of ordinary differential equations as limits of pure jump Markov processes. *J. Appl. Prob.* **7**, 49-58.

[21] T.G. Kurtz (1971) Limit theorems for sequences of jump Markov processes approximating ordinary differential processes. *J. Appl. Prob.* **8**, 344-356.

[22] T.G. Kurtz (1981) *Approximation of population processes. CBMS-NSF Regional Conf. Series in Appl. Math.* **36**, SIAM, Philadelphia.

[23] M.F. Norman (l972) *Markov processes and learning models.* Academic Press, New York.

[24] M.F. Norman (l974) A central limit theorem for Markov processes that move by small steps. *Annals of Probability* **2**, l065-l074.

[25] R. Pearl and L. Reed (1920) On the rate of growth of population of the United States since 1790 and its mathematical representation. *Proc. Nat. Academy Sci.* **6**, 275-288.

[26] R. Pearl (1925) *The biology of population growth.* Alfred A. Knopf, New York.

[27] R. Pearl (1927) The growth of populations. *Quart. Rev. Biol.* **2**, 532-548.

[28] L.C.G. Rogers and D. Williams (1987) *Diffusions, Markov processes, and martingales, Vol. 2.* John Wiley & Sons Ltd.

[29] C. Stein (1986) *Approximate Computation of Expectations.* Institute of Mathematical Statistics Lecture Notes - Monograph Series Vol. **7**, Hayward, California.

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