A Motivic Version of the Theorem of Fontaine and Wintenberger

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Zusammenfassung

Ein Satz von Fontaine und Wintenberger [**17**] besagt, dass es einen Isomorphismus zwischen den absoluten Galoisgruppen der Vervollständigung K des bewerteten Körpers $\mathbb{Q}_p(p^{1/p^\infty})$ und der Vervollständigung $K^\flat$ des bewerteten Körpers $\mathbb{F}_p((t))(t^{1/p^\infty})$ gibt. Nach der Definition von Scholze [**42**] bemerken wir, dass diese Körper beide *perfektoid* sind, d.h. vollständige nicht-archimedische, nicht-diskret bewertete Körper mit einem Restklassenkörper der Charakteristik p und so, dass die Frobenius-Abbildung auf O_K/p surjektiv ist. In loc. cit. zeigt der Autor, dass man für einen solchen Körper K das multiplikative Monoid $K^\flat = \varprojlim_{x\mapsto x^p} K$ mit einer Körperstruktur versehen kann, die perfektoid der Charakteristik p ist. Dieser Körper wird der *Tilt* von K genannt. Darüber hinaus induziert dieser "tilting"-Funktor eine Äquivalenz zwischen endlichen étalen Algebren über K und über $K^\flat$ und bietet damit eine Verallgemeinerung des Satzes von Fontaine und Wintenberger.

In einer Motivsprache kann das obige Ergebnis mit den Worten beschrieben werden, dass die Kategorien von Artin-Motiven über den beiden Körpern äquivalent sind. Das Ziel der vorliegenden Arbeit ist, diese Äquivalenz auf die gesamte Kategorie der (triangulierten) Motive analytischer Varietäten über K und über $K^\flat$ zu verallgemeinern.

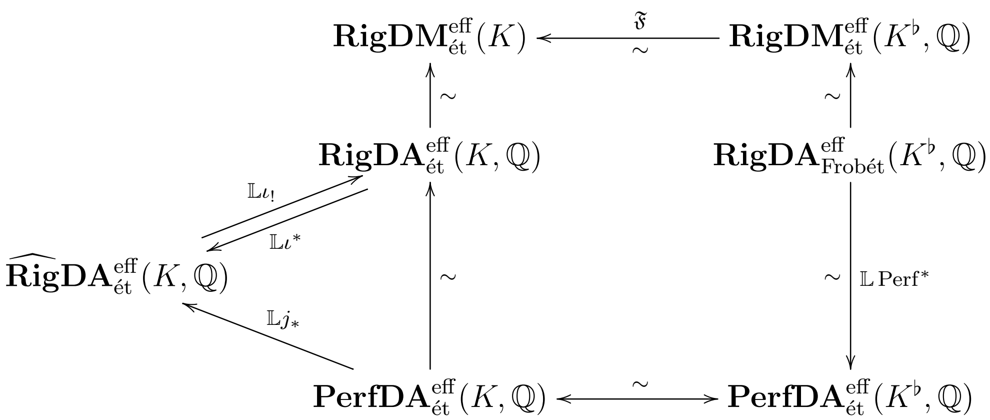
Das natürliche höher-dimensionale Analogon zur Kategorie von Artin-Motiven über einem lokalen Körper ist die Kategorie der analytischen Motive $\mathbf{RigDM}$, welche von Ayoub eingeführt und analysiert wurde [**5**]. In diesem Zusammenhang werden die Grundkörper mit ihren nicht-archimedischen Strukturen betrachtet, und nicht nur als abstrakte Körper wie in der Definition der Motive $\mathbf{DM}$ auf glatten Varietäten.

Das wichtigste Ergebnis unserer Arbeit ist der folgende Satz :

THEOREM. *Sei K ein perfektoider Körper mit Tilt $K^\flat$. Es gibt eine Äquivalenz von triangulierten monoidalen Kategorien:*

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\mathbb{Q}) \to \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K,\mathbb{Q}).$$

Die Situation kann mit dem folgenden Diagramm zusammengefasst werden. Die Schreibweise wird in der Dissertation beschrieben und erläutert.



Introduction

A theorem of Fontaine and Wintenberger [**17**], later expanded by Scholze [**42**], states that there is an isomorphism between the Galois groups of a perfectoid field K and the associated (tilted) perfect field $K^\flat$ of positive characteristic. The standard example of such a pair is formed by the completions of the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))(t^{1/p^\infty})$.

In a motivic language, the previous result can be rephrased by saying that the categories of Artin motives over the two fields are equivalent. The goal of this paper is to extend this equivalence to the whole category of (mixed derived) motives of rigid analytic varieties $\mathbf{RigDM}$ over K and over $K^\flat$ with $\mathbb{Q}$-coefficients. As a matter of fact, the natural analogue in higher dimension of the category of (derived) Artin motives over a local field is the category of rigid motives, introduced and analyzed by Ayoub [**5**], where the base field is considered as a non-archimedean valued field and not just as an abstract field as in the case of the category of algebraic motives $\mathbf{DM}$.

In this thesis, we prove the following (Theorem 1.7.8):

THEOREM. *Let K be a perfectoid field with tilt $K^\flat$ and let Λ be a $\mathbb{Q}$-algebra. There is a monoidal triangulated equivalence of categories*

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\Lambda)\to\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda).$$

We remark that the construction of the functor $\mathfrak{F}$ requires a lot of machinery and uses Scholze's tilting functor in a crucial way. We can roughly sketch the construction of this functor as follows. We start from a smooth rigid variety X over the perfectoid field of positive characteristic $K^\flat$ and we associate to it a perfectoid space $\widehat{X}$ obtained by taking the perfection of X. This operation can be performed canonically since $K^\flat$ has characteristic p. We then use Scholze's theorem to tilt $\widehat{X}$ obtaining a perfectoid space $\widehat{Y}$ in mixed characteristic. Suppose now that $\widehat{Y}$ is the limit of a tower of rigid analytic varieties

$$\ldots\to Y_{h+1}\to Y_h\to\ldots\to Y_1\to Y_0$$

such that Y_0 is étale over the Tate ball $\mathbb{B}^n=\operatorname{Spa}K\langle\upsilon_1,\ldots,\upsilon_n\rangle$ and each Y_h is obtained as the pullback of Y_0 by the map $\mathbb{B}^n\to\mathbb{B}^n$ defined by taking the p^h-powers of the coordinates $\upsilon_i\mapsto\upsilon_i^{p^h}$. Under such hypothesis (we will actually need slightly stronger conditions on the tower above) we then "de-perfectoidify" $\widehat{Y}$ by associating to it $Y_{\bar h}$ for a sufficiently big index $\bar h$.

The main technical problem of this construction is to show that it is independent of the choice of the tower, and on the index $\bar h$. It is also by definition a local procedure, which is not canonically extendable to arbitrary varieties by gluing. In order to overcome these obstacles, we use in a crucial way some techniques of approximating maps between spaces up to homotopy which are obtained by a generalization of the implicit function theorem in the non-archimedean setting. We also need to introduce a subcategory of adic spaces (in the sense of Huber [**26**]) $\widehat{\mathrm{RigSm}}$ where to embed both rigid analytic and perfectoid spaces, and adapt the motivic tools to develop homotopy theory on it.

Generalizing the results of [**3**, Appendix B], we also prove that the natural functor a_{tr} of adding transfers induces and equivalence between the category of motives without transfers $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda)$ and the category of motives with transfers $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda)$ in characteristic zero. In positive characteristic, it induces an equivalence between $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat,\Lambda)$ and $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\Lambda)$ where the former category is obtained as a localization of $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\Lambda)$ with respect to the set of relative Frobenius maps $X\to X\times_{K^\flat,\Phi}K^\flat$ for all rigid varieties X

over $K^\flat$. Our main theorem can therefore be stated as an equivalence between $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$ and $\mathbf{RigDA}^{\mathrm{eff}}_{\text{Frobét}}(K^\flat, \Lambda)$ for any perfectoid field K of characteristic 0.

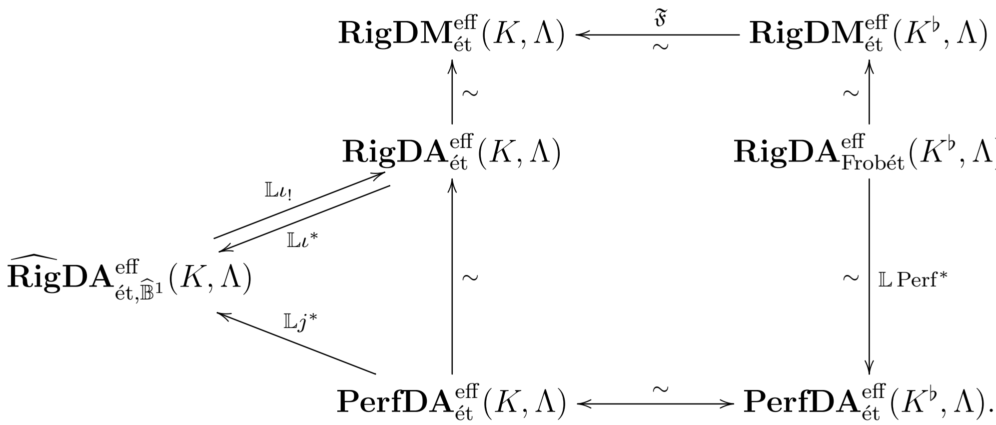
The statements above involve only rigid analytic varieties and their proofs use Scholze's theory of perfectoid spaces only in an auxiliary way. Nonetheless, we can restate our main result highlighting the role of perfectoid spaces as follows:

THEOREM. *Let K be a perfectoid field and let Λ be a $\mathbb{Q}$-algebra. There is a monoidal triangulated equivalence of categories*

$$\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda) \stackrel{\sim}{\to} \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$$

The category $\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$ is built in the same way as $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$ using as a starting point the big étale site of perfectoid spaces which are locally étale over some perfectoid ball $\widehat{\mathbb{B}}^n$.

The following diagram of categories of motives summarizes the situation. The equivalence in the bottom line follows easily from the "tilting equivalence" of Scholze, see [**42**, Proposition 6.17]. The notation introduced in the theorems and in the diagram will be described in later sections.



The thesis is organized as follows. The first chapter is devoted to the proof of the main theorem. In Section 1.1 we recall the basic definitions and the language of adic spaces while in Section 1.2 we define the environment in which we will perform our construction, namely the category of semi-perfectoid spaces $\widehat{\mathrm{Rig}}\mathrm{Sm}$ and we define the étale topology on it. In Section 1.3 we define the categories of motives for RigSm, $\widehat{\mathrm{Rig}}\mathrm{Sm}$ and PerfSm adapting the constructions of Voevodsky's and Ayoub's. Thanks to the general model-categorical tools introduced in this section, we give in Section 1.4 a motivic interpretation of some approximation results of maps valid for non-archimedean Banach algebras. In Sections 1.5 and 1.6 we prove the existence of the de-perfectoidification functor from perfectoid motives to rigid motives in zero and positive characteristics, respectively. Finally, we give in Section 1.7 the proof of our main result.

In the second chapter we prove the equivalence between rigid motives with and without transfers. In Section 2.1 we introduce the Frob-topology and the $\mathrm{Frob\acute{e}t}$-topology, which plays the same role of the étale topology in positive characteristic for our purposes. In Section 2.2 we define the categories of motives associated to these sites, as well as the auxiliary categories of motives of normal varieties and their relative counterpart. Finally in Section 2.3 we prove the desired equivalence $\mathbf{RigDA}^{\mathrm{eff}}_{\text{Frobét}}(K, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$.

In the appendix, we collect some technical theorems that are used along the proofs of the first chapter. Specifically, we first present a generalization of the implicit function theorem in the rigid setting, and conclude a result about the approximation of maps modulo homotopy as well as its geometric counterpart. We also prove the existence of compatible approximations of a collection of maps $\{f_1, \ldots, f_N\}$ from a variety in $\widehat{\mathrm{RigSm}}$ of the form $X \times \mathbb{B}^n$ to a rigid variety Y such that the compatibility conditions among the maps f_i on the faces $X \times \mathbb{B}^{n-1}$ are preserved. This fact has important consequences for computing maps to $\mathbb{B}^1$-local complexes of presheaves in the motivic setting.

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Contents

Zusammenfassung iii
Introduction iv
Acknowledgments vii

Chapter 1. The tilting equivalence of motives of rigid analytic varieties 1
1.1. Generalities on adic spaces 1
1.2. Semi-perfectoid spaces 9
1.3. Categories of adic motives 12
1.4. Motivic interpretation of approximation results 21
1.5. The de-perfectoidification functor in characteristic 0 23
1.6. The de-perfectoidification functor in characteristic p 28
1.7. The main theorem 31

Chapter 2. Rigid motives with and without transfers 37
2.1. The Frob-topology 37
2.2. Rigid motives and Frob-motives 42
2.3. The equivalence between motives with and without transfers 48

Appendix A. An implicit function theorem and approximation results 55
A.1. A non-archimedean implicit function theorem 55
A.2. Approximation of maps of adic spaces 59

Bibliography 64

CHAPTER 1

The tilting equivalence of motives of rigid analytic varieties

The purpose of this chapter is to construct the equivalence of categories between motives of analytic varieties over a perfectoid field K and its tilt $K^\flat$. To this aim, we first recall the theory of perfectoid spaces and we introduce the categories of adic spaces that we will be interested in.

1.1. Generalities on adic spaces

We start by recollecting the language of adic spaces, as introduced by Huber [**26**] and generalized by Scholze-Weinstein [**45**] including some terminology of Buzzard-Verberkmoes [**12**] and Wedhorn [**52**]. We will always work with adic spaces over a non-archimedean valued field.

1.1.1. DEFINITION. A *non-archimedean field* is a toplological field K whose topology is induced by a non-trivial valuation of rank one. The associated norm is a multiplicative map that we denote by $|\cdot|\colon K\to\mathbb{R}_{\geq 0}$ and its valuation ring is denoted by K°. A pair (K,K^+) is called a *valuation field* if K is a non-archimedean field and $K^+\subset K^\circ$ an open bounded valuation subring. We say it is *complete* if K (and hence also K^+) is complete. A map $(K,K^+)\to(L,L^+)$ of valuation fields is *local* if the inverse image of L^+ in K coincides with K^+.

1.1.2. REMARK. A map $(K,K^+)\to(L,L^+)$ is local if an only if the map $K^+\to L^+$ is a local map between local rings. In that case, the two valuations on K induced by K^+ and L^+ coincide. The valuation on K induced by K^+ has rank 1 precisely when K^+ coincides with K°.

From now on, we fix a complete non-archimedean field K and we pick a non-zero element $\pi\in K$ with $|\pi|<1$.

We now recall some definitions given in [**25**]. We also introduce the notion of a bounded affinoid K-algebra.

1.1.3. DEFINITION. A *Tate K-algebra* is a topological K-algebra R for which there exists a subring R_0 such that the set $\{\pi^k R_0\}$ forms a basis of neighborhoods of 0. A subring R_0 with the above property is called a *ring of definition*.

1.1.4. DEFINITION. Let R be a Tate K-algebra.

- A subset S of R is *bounded* if it is contained in the set $\pi^{-N}R_0$ for some integer N. An element x of R is *power-bounded* if the set $\{x^n\}_{n\in\mathbb{N}}$ is bounded. The set of power-bounded elements is a subring of R that we denote by R°. We say that R is *uniform* if R° is bounded.
- An element x of R is *topologically nilpotent* if $\lim_{n\to+\infty}x^n=0$. The set of topologically nilpotent elements is an ideal of R° that we denote by $R^{\circ\circ}$.

1.1.5. REMARK. Suppose that R is a Tate K-algebra. The definition of a bounded set does not depend on the choice of the ring of definition R_0. A subring of R is a ring of definition if and only if it is bounded and open. By [**25**, Corollary 1.3] the ring R° is the filtered union of all rings of definitions of R. In particular if $x\in R$ is algebraic over R° then it is algebraic over a ring of definition, and so it is power-bounded proving that R° is integrally closed in R. Moreover, since for any $x \in R$ the sequence $x\pi^n$ tends to zero, we conclude that $x\pi^n$ is contained in a ring of definition R_0 for a sufficiently big index n and hence $R_0[\pi^{-1}] = R$.

1.1.6. DEFINITION. Let K be a complete non-archimedean field.

- An *affinoid K-algebra* is a pair (R, R^+) where R is a Tate K-algebra and R^+ is an open and integrally closed K°-subalgebra of R°. A morphism $(R, R^+) \to (S, S^+)$ of affinoid K-algebras is a pair of compatible K°-linear continuous maps of rings (f, f^+).
- A *bounded affinoid K-algebra* is an affinoid K-algebra (R, R^+) such that R^+ is a ring of definition.
- An affinoid K-algebra (R, R^+) is called *complete* if R (and hence also R^+) is complete.

1.1.7. REMARK. If (R, R^+) is an affinoid K-algebra and R is uniform then (R, R^+) is bounded.

1.1.8. REMARK. If (R, R^+) is an affinoid K-algebra and x is topologically nilpotent, then there exists an integer N such that $x^N \in R^+$ and hence $x \in R^+$ since R^+ is integrally closed. We then deduce that R^+ contains the set $R^{\circ\circ}$. The restricted topology on a ring of definition R_0 coincides with the π-adic topology. In particular, the topology of a bounded affinoid K-algebra (R, R^+) is uniquely determined by the K°-algebra R^+.

1.1.9. EXAMPLE. By Remark 1.1.5, if R is a Tate K-algebra, then (R, R°) is an affinoid K-algebra.

Any affinoid K-algebra (R, R^+) is endowed with a universal map $(R, R^+) \to (\widehat{R}, \widehat{R}^+)$ to a complete affinoid K-algebra that we call the *completion* of (R, R^+) (see [**25**, Lemma 1.6]). In case (R, R^+) is bounded, then $\widehat{R}^+$ is the π-adic completion of R^+ and $\widehat{R}$ is $\widehat{R}^+[\pi^{-1}]$. More generally, for any affinoid K-algebra (R, R^+) we can define the *π-adic completion* to be the complete affinoid K-algebra (S, S^+) where S^+ is the π-adic completion of R^+ and S is $S^+[\pi^{-1}]$ endowed with the topology generated by the sets $\{\pi^k S^+\}$.

Let $\{(R_i, R_i^+), f_i\}$ be a direct system of maps of affinoid K-algebras. As remarked in [**44**], it is not true in general that the direct limit $(\varinjlim R_i, \varinjlim R_i^+)$ endowed with the direct limit topology is an affinoid K-algebra. In the bounded context, however, this nuisance can be easily solved.

1.1.10. LEMMA. *Let $\{(R_i, R_i^+), f_i\}$ be a direct system of maps of bounded affinoid K-algebras. Endow the ring $\varinjlim_i R_i$ with the topology for which $\varinjlim_i R_i^+$ is a ring of definition. The pair $(R, R^+) := (\varinjlim_i R_i, \varinjlim_i R_i^+)$ is a bounded affinoid K-algebra and one has $\operatorname{Hom}((R, R^+), (S, S^+)) \cong \varprojlim_i((R_i, R_i^+), (S, S^+))$ for any bounded affinoid K-algebra (S, S^+).*

PROOF. A map from (R, R^+) to (S, S^+) is uniquely determined by a K°-linear map from $R^+ = \varinjlim R_i^+$ to S^+. Similarly, a map from (R_i, R_i^+) to (S, S^+) is uniquely determined by a K°-linear map from R_i^+ to S^+. From the isomorphism $\operatorname{Hom}(\varinjlim R_i^+, S^+) \cong \varprojlim_i \operatorname{Hom}(R_i^+, S^+)$ we then deduce the claim. $\square$

From the lemma above, we conclude that the category of bounded affinoid K-algebras has direct limits.

We now examine some examples.

1.1.11. EXAMPLE. Let $\underline{\upsilon} = (\upsilon_1, \dots, \upsilon_N)$ be an N-tuple of coordinates. If (R, R^+) is a bounded affinoid K-algebra, then also the pair $(R\langle\underline{\upsilon}\rangle, R^+\langle\underline{\upsilon}\rangle)$ is, where $R\langle\underline{\upsilon}\rangle$ is the ring of strictly convergent power series in N variables with coefficients in R:

$$R\langle \underline{v}\rangle := \left\{ \sum_I a_I v^I \in R[[\underline{v}]] \colon \forall k \in \mathbb{N}, a_I \in \pi^k R^+ \text{ for almost all } I \right\}$$

with the topology having $\pi^k R^+\langle \underline{v}\rangle$ as a basis of neighborhoods of 0. In case R is normed, then also $R\langle \underline{v}\rangle$ is normed, with respect to the Gauss norm $|\sum_I a_I v^I| := \max_I\{|a_I|\}$ and is complete whenever R is (see [**9**, Section 1.4.1]).

1.1.12. EXAMPLE. If R is any normed K-algebra, then (R, R°) is an affinoid K-algebra. The classical definition of Tate gives therefore examples of affinoid K-algebras.

1.1.13. DEFINITION. A *topologically of finite type Tate algebra* (or simply *tft Tate algebra*) is a Banach K-algebra R isomorphic to a quotient of the normed K-algebra $K\langle v_1, \dots, v_n\rangle$ for some n.

If R is a tft Tate algebra, the pair (R, R°) is an affinoid K-algebra, which is bounded whenever R is reduced (see [**9**, Theorem 6.2.4/1]).

We now recall the definition of perfectoid pairs, introduced in [**42**]:

1.1.14. DEFINITION. A *perfectoid field* K is a complete non-archimedean field whose rank one valuation is non-discrete, whose residue characteristic is p and such that the Frobenius is surjective on K°/p. In case $\operatorname{char} K = p$ this last condition amounts to saying that K is perfect.

1.1.15. DEFINITION. Let K be a perfectoid field.

- A *perfectoid algebra* is a Banach K-algebra R such that R° is bounded and the Frobenius map is surjective on R°/p.
- A *perfectoid affinoid K-algebra* is an affinoid K-algebra (R, R^+) over a perfectoid field K such that R is perfectoid.

1.1.16. REMARK. Any perfectoid affinoid K-algebra is bounded. If R is a perfectoid algebra, then (R, R°) is a perfectoid affinoid K-algebra.

1.1.17. EXAMPLE. Suppose that K is a perfectoid field. A basic example of a perfectoid algebra is the following: let $\underline{v} = (v_1, \dots, v_N)$ be a N-tuple of coordinates and $K^\circ[\underline{v}^{1/p^\infty}]$ be the ring $\varinjlim_h K^\circ[\underline{v}^{1/p^h}]$ endowed with the sup-norm induced by the norm on K. We also denote by $K^\circ\langle \underline{v}^{1/p^\infty}\rangle$ its π-adic completion. By [**42**, Proposition 5.20], the ring $K^\circ\langle \underline{v}^{1/p^\infty}\rangle[\pi^{-1}]$ is a perfectoid K-algebra which we will denote by $K\langle \underline{v}^{1/p^\infty}\rangle$. The pair $(K\langle \underline{v}^{1/p^\infty}\rangle, K^\circ\langle \underline{v}^{1/p^\infty}\rangle)$ is a perfectoid affinoid K-algebra. We also define in the same way the perfectoid affinoid K-algebra $(K\langle \underline{v}^{\pm 1/p^\infty}\rangle, K^\circ\langle \underline{v}^{\pm 1/p^\infty}\rangle)$ (see [**43**, Example 4.4]).

1.1.18. REMARK. $K\langle \underline{v}^{1/p^\infty}\rangle$ is isomorphic as a $K\langle \underline{v}\rangle$-topological module to the completion $\widehat{\bigoplus} K\langle \underline{v}\rangle$ of the free module $\bigoplus K\langle \underline{v}\rangle$ with basis indexed by the set $(\mathbb{Z}[1/p] \cap [0, 1))^N$. By [**9**, Proposition 2.1.5/7] there is an explicit description of this ring as a subring of $\prod K\langle \underline{v}\rangle$.

The following theorem summarizes some results of Scholze, including the *tilting equivalence* of perfectoid algebras which will play a crucial role in our construction.

1.1.19. THEOREM ([**42**]). *Let K be a perfectoid field.*

(1) ([**42**, Lemma 3.4]) *The multiplicative monoid $\varprojlim_{x \mapsto x^p} K$ can be given a structure $K^\flat$ of perfectoid field with the norm induced by the multiplicative map $\sharp \colon K^\flat \to K$. The field $K^\flat$ has characteristic p and coincides with K in case* $\operatorname{char} K = p$.

(2) ([**42**, Theorem 3.7]) *The functor* $L \mapsto L^\flat$ *for* L *finite étale over* K *induces an isomorphism* $\mathrm{Gal}(K) \cong \mathrm{Gal}(K^\flat)$.
(3) ([**42**, Lemma 6.2]) *There is an equivalence of categories, the* tilting equivalence*, from perfectoid affinoid* K*-algebras to perfectoid affinoid* $K^\flat$*-algebras denoted by* $(R, R^+) \mapsto (R^\flat, R^{\flat +})$ *such that* $R^\flat$ *is multiplicatively isomorphic to* $\varprojlim_{x \mapsto x^p} R$ *and* $R^{\flat +}$ *is multiplicatively isomorphic to* $\varprojlim_{x \mapsto x^p} R^+$.
(4) ([**42**, Proposition 5.20 and Corollary 6.8]) *The tilting equivalence associates* $(K\langle \underline{v}^{1/p^\infty}\rangle, K^\circ\langle \underline{v}^{1/p^\infty}\rangle)$ *to* $(K^\flat\langle \underline{v}^{1/p^\infty}\rangle, K^{\flat\circ}\langle \underline{v}^{1/p^\infty}\rangle)$ *and* $(K\langle \underline{v}^{\pm 1/p^\infty}\rangle, K^\circ\langle \underline{v}^{\pm 1/p^\infty}\rangle)$ *to* $(K^\flat\langle \underline{v}^{\pm 1/p^\infty}\rangle, K^{\flat\circ}\langle \underline{v}^{\pm 1/p^\infty}\rangle)$.

We now introduce a geometric category. We make use of a definition of Wedhorn [**52**, Remark and Definition 8.9].

1.1.20. DEFINITION. Let X be a topological space and let $\mathcal{B}$ be a basis of open subsets of X. A presheaf $\mathcal{F}$ on X with values in a category where projective limits exist is *adapted to* $\mathcal{B}$ if for every open subset U of X one has $\mathcal{F}(U) \cong \varprojlim_{\mathcal{B} \ni B \subset U} \mathcal{F}(B)$.

1.1.21. REMARK. If $\mathcal{F}$ is a sheaf, it is adapted to any basis of open subsets. Vice-versa, if $\mathcal{F}$ is a presheaf on X adapted to $\mathcal{B}$ and a sheaf on $\mathcal{B}$ then it is a sheaf on X.

1.1.22. DEFINITION. Let K be a complete non-archimedean field.

- We denote by $\mathbf{V}_{\mathrm{psh}}$ the following category: objects are triples $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ with the following properties:
  - X is a topological space.
  - $\mathcal{O}_X$ resp. $\mathcal{O}_X^+$ is a presheaf on X of complete topological algebras over K resp. over K° with $\mathcal{O}_X \supseteq \mathcal{O}_X^+$ and the stalk at each point x is a local ring $\mathcal{O}_{X,x}$ resp. $\mathcal{O}_{X,x}^+$. We denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$.
  - There is a basis of open subsets $\mathcal{B}$ such that the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are adapted to it and the pair $(\mathcal{O}_X, \mathcal{O}_X^+)$ defines a presheaf on $\mathcal{B}$ of complete affinoid K-algebras.
  - For each point x in X the π-adic completion of the pair $(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^+)$ is a valuation field $(\widehat{k}(x), \widehat{k}(x)^+)$ such that the map $\mathcal{O}_{X,x} \to \widehat{k}(x)$ factors over $\mathcal{O}_{X,x}/\mathfrak{m}_x$.

  A morphism $f\colon (X, \mathcal{O}_X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^+)$ is a pair formed by a map of topological spaces $f\colon X \to Y$ and a couple of maps of presheaves of topological K°-algebras $(f^\sharp, f^{+\sharp})\colon (\mathcal{O}_Y, \mathcal{O}_Y^+) \to f_*(\mathcal{O}_X, \mathcal{O}_X^+)$ inducing local maps of valuation fields at each point. For each $x \in X$ we denote the totally ordered topological abelian group $\widehat{k}(x)^*/\widehat{k}(x)^{+*}$ by $\Gamma(x)$.
- We denote by $\mathbf{V}$ the full subcategory of $\mathbf{V}_{\mathrm{psh}}$ formed by triples $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ such that $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves of topological rings.

1.1.23. LEMMA. *Let* $(X, \mathcal{O}_X^+, \mathcal{O}_X)$ *be an object of* $\mathbf{V}_{\mathrm{psh}}$ *and* x *be a point of* X.

(1) *The completion map* $\mathcal{O}_{X,x}^+ \to \widehat{k}(x)^+$ *and the map* $\mathcal{O}_{X,x} \to \widehat{k}(x)$ *are local.*
(2) *If* $(f, f^\sharp, f^{+\sharp})\colon (X, \mathcal{O}_X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y, \mathcal{O}_Y^+)$ *is a morphism of* $\mathbf{V}$ *then the pairs* $(f, f^\sharp)$ *and* $(f, f^{+\sharp})$ *are morphisms of locally ringed spaces.*
(3) *The map* $\mathcal{O}_{X,x} \to \widehat{k}(x)$ *induces a continuous valuation* $|\cdot(x)|\colon \mathcal{O}_{X,x} \to \Gamma(x) \cup \{0\}$ *and morphism of* $\mathbf{V}_{\mathrm{psh}}$ *are compatible with these valuations.*
(4) *The ring* $\mathcal{O}_{X,x}^+$ *coincides with the subring of elements* f *with* $|f(x)| \leq 1$ *and its maximal ideal coincides with the set of elements* f *such that* $|f(x)| < 1$.

(5) *The maximal ideal* $\mathfrak{m}_x$ *of* $\mathcal{O}_{X,x}$ *coincides with the set of elements* f *such that* $|f(x)| = 0$.
(6) *If* $(X, \mathcal{O}_X^+, \mathcal{O}_X)$ *lies in* $\mathbf{V}$ *then* $\mathcal{O}^+(X)$ *coincides with the ring* $\{f \in \mathcal{O}(X) \colon |f(x)| \leq 1$ *for all* $x \in X\}$.
(7) *For any* $a, b \in \mathcal{O}_X(X)$ *the sets* $\{x \colon |a(x)| \neq 0\}$ *and* $\{x \colon |a(x)| \leq |b(x)| \neq 0\}$ *are open.*

PROOF. We start by proving the first claim. The local map $\mathcal{O}_{X,x}^+ \to \mathcal{O}_{X,x}^+/\pi$ factors by the completion map $\mathcal{O}_{X,x}^+ \to k(x)^+$ which is then also local. The claim about $\mathcal{O}_{X,x} \to k(x)$ follows from the very definition of the category $\mathbf{V}_{\mathrm{psh}}$.

For the second claim, we only need to prove that the induced maps $\mathcal{O}_{Y,f(x)}^+ \to \mathcal{O}_{X,x}^+$ and $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are local. This follows from the first claim and the fact that a local map of valuation fields $(\widehat{k}(y), \widehat{k}(y)^+) \to (\widehat{k}(x), \widehat{k}(x)^+)$ induces a local map $\widehat{k}(y)^+ \to \widehat{k}(x)^+$. This also proves the third claim.

If an element a in $\mathcal{O}_{X,x}$ satisfies $|a(x)| \leq 1$ then a lies in $\widehat{k}(x)^+$ which is the π-adic completion of $\mathcal{O}_{X,x}^+$. In particular, there exist elements $a', c \in \mathcal{O}_{X,x}^+$ such that $a - c\pi = a'$. We then deduce $a \in \mathcal{O}_{X,x}^+$ and hence the third claim. The fourth and fifth claims are easy consequences of the previous ones.

If $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves, then also the subsheaf $\mathcal{F}$ of $\mathcal{O}_X$ defined as $\mathcal{F}(U) = \{f \in \mathcal{O}_X(U) \colon |f(x)| \leq 1 \text{ for all } x \in U\}$ is a sheaf and by what proved above has the same stalks of $\mathcal{O}_X^+$. They therefore coincide and this shows the sixth claim.

We now move to the last claim. Fix now $a, b \in \mathcal{O}_X(X)$. From the previous results, we deduce that $|a(x)| \neq 0$ is equivalent to $a \in \mathcal{O}_{X,x}^*$ which is an open condition. In order to prove that the second set is also open it therefore suffices to show that the condition $|a(x)| \leq 1$ is open. From the third claim, this amounts to saying that $a \in \mathcal{O}_{X,x}^+$ which is again an open condition, as wanted. □

By the above result, each object $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ of $\mathbf{V}$ defines a triple $(X, \mathcal{O}_X, \{v_x\}_{x \in X})$ where v_x is a multiplicative valuation defined on the stalk $\mathcal{O}_{X,x}$ and the maps of $\mathbf{V}$ are compatible with them. The category $\mathbf{V}$ is then a subcategory of $\mathcal{V}$ as defined by Huber in [**27**, Section 2]. Our definition is more restrictive, as we assume that the valuation ring at each point coincides with the π-adic completion of the stalk of $\mathcal{O}_X^+$. On the other hand, valuations at each point are automatically induced by the properties of the stalks of $(\mathcal{O}_X, \mathcal{O}_X^+)$.

We now recall Huber's construction of the spectrum of a valuation ring (see [**26**]).

1.1.24. CONSTRUCTION. Let (R, R^+) be an affinoid K-algebra. The set $\mathrm{Spa}(R, R^+)$ is the set of equivalence classes of continuous multiplicative valuations, i.e. multiplicative maps $|\cdot| \colon R \to \Gamma \cup \{0\}$ where $(\Gamma, \cdot)$ is a totally ordered abelian group, such that $|0| = 0$, $|1| = 1$, $|x + y| \leq \max\{|x|, |y|\}$ and $|R^+| \leq 1$. It is endowed with the topology generated by *rational subsets* $\{U(f_1, \ldots, f_n \mid g)\}$ by letting $f_1, \ldots, f_n, g$ vary among elements in R such that $f_1, \ldots, f_n$ generate R as an ideal and where the set $U(f_1, \ldots, f_n \mid g)$ is the set of those valuations $|\cdot|$ satisfying $|f_i| \leq |g|$ for all i. Rational subsets form a basis of quasi-compact sets of the (quasi-compact) space $\mathrm{Spa}(R, R^+)$ ([**25**, Theorem 3.5]).

Alternatively, $\mathrm{Spa}(R, R^+)$ is the set $\varinjlim \mathrm{Hom}((R, R^+), (L, L^+))$ by letting (L, L^+) vary in the category of valuation fields over K and local maps. Its topology can be defined by declaring the sets $\{\phi \colon 0 \neq |\phi(f)| \leq |\phi(g)|\}$ to be open, for all pairs of elements f, g in R.

Let (R, R^+) be an affinoid K-algebra, let $f_1, \ldots, f_n$ be elements in R that generate R as an ideal and g be in R. We can endow the ring $R[1/g]$ with the topology generated by $\pi^k R_0[f_1/g, \ldots, f_n/g]$ where R_0 is a ring of definition of R. If we let B be the integral closure of $R^+[f_1/g, \dots, f_n/g]$ in $R[1/g]$ the pair $(R[1/g], B)$ is an affinoid algebra, and its completion will be denoted by $(R\langle f_1/g, \dots, f_n/g\rangle, R\langle f_1/g, \dots, f_n/g\rangle^+)$. If (R, R^+) is bounded (or even if R is uniform) the pair $(R\langle f_1/g, \dots, f_n/g\rangle, R\langle f_1/g, \dots, f_n/g\rangle^+)$ may not be bounded (see the proof of [**12**, Proposition 17]).

We associate to $U(f_1, \dots, f_n \mid g)$ the affinoid K-algebra

$$(\mathcal{O}(U), \mathcal{O}^+(U)) = (R\langle f_1/g, \dots, f_n/g\rangle, R\langle f_1/g, \dots, f_n/g\rangle^+)$$

introduced above. Whenever a rational subspace U is contained in another one U' there are canonical maps $\rho_U^{U'}\colon (\mathcal{O}(U'), \mathcal{O}^+(U')) \to (\mathcal{O}(U), \mathcal{O}^+(U))$ (see [**26**, Lemma 1.5]). For an arbitrary open V we can then define

$$\mathcal{O}(V) = \varprojlim_{V \supset U \text{ rational}} \mathcal{O}(U)$$

and similarly for $\mathcal{O}^+$. This way, we define a pair of presheaves of complete topological K-algebras $(\mathcal{O}, \mathcal{O}^+)$ on $Spa(R, R^+)$ adapted to rational subsets. By [**26**, Lemma 1.5, Proposition 1.6] we have $U \cong \mathrm{Spa}(\mathcal{O}(U), \mathcal{O}^+(U))$ which is called a *rational subspace* of $X = \mathrm{Spa}(R, R^+)$ and for any $x \in X$ the valuation at x extends to a valuation on $\mathcal{O}_{X,x}$ such that the stalk $\mathcal{O}_{X,x}^+$ is local and corresponds to $\{f \in \mathcal{O}_{X,x}\colon |f(x)| \leq 1\}$. The triple $(\mathrm{Spa}(R, R^+), \mathcal{O}_X, \mathcal{O}_X^+)$ defines an object of $\mathbf{V}_{\mathrm{psh}}$. The property at stalks is a consequence of [**42**, Proposition 2.25]. We point out that $(\mathcal{O}(X), \mathcal{O}^+(X)) \cong (\widehat{R}, \widehat{R}^+)$ and that $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(\widehat{R}, \widehat{R}^+)$ as remarked in [**25**, Proposition 3.9].

By [**26**, Proposition 1.6] there holds $\mathcal{O}^+(U) = \{f \in \mathcal{O}(U)\colon |f(x)| \leq 1 \text{ for all } x \in U\}$ for any rational open U of $\mathrm{Spa}(R, R^+)$ so that $\mathcal{O}^+$ is a sheaf if $\mathcal{O}$ is a sheaf. By Tate's acyclicity theorem [**9**, Theorem 8.2.1/1] and Scholze's acyclicity theorem [**42**, Theorem 6.3], if (R, R^+) is a tft Tate algebra or a perfectoid affinoid K-algebra, then $\mathcal{O}$, $\mathcal{O}^+$ are sheaves. Sadly enough, this does not hold in general as shown at the end of [**26**, Section 1].

1.1.25. REMARK. By [**12**, Theorem 7] if (R, R^+) is an affinoid K-algebra such that $\mathcal{O}(U)$ is uniform for all rational subspaces U of $\mathrm{Spa}(R, R^+)$ (i.e. it is *stably uniform* following [**12**]) then the presheaf $\mathcal{O}$ on $\mathrm{Spa}(R, R^+)$ is a sheaf.

1.1.26. REMARK. By abuse of notation, whenever R is a reduced tft Tate algebra we will sometimes denote by $\mathrm{Spa}\, R$ the object $\mathrm{Spa}(R, R^\circ)$ of $\mathbf{V}$.

The category $\mathbf{V}$ must be thought of as the analogue of the category of locally ringed spaces, and allows to have a completely abstract definition of the affinoid spectrum $\mathrm{Spa}(A, A^+)$ akin to the case of schemes (see [**14**, I.1.2.1]) as the following fact shows. It is a slight generalization of [**26**, Proposition 2.1(ii)].

1.1.27. PROPOSITION. *Let (R, R^+) be an affinoid K-algebra and X be an object of $\mathbf{V}$. The global section functor induces a bijection*

$$\mathrm{Hom}_{\mathbf{V}_{\mathrm{psh}}}(X, \mathrm{Spa}(R, R^+)) \cong \mathrm{Hom}_{K-\mathrm{cont}}((\widehat{R}, \widehat{R}^+), (\mathcal{O}(X), \mathcal{O}^+(X)))$$

where the set on the right is the set of continuous K°-linear maps of pairs of complete topological rings.

PROOF. We can assume that (R, R^+) is a complete affinoid K-algebra. There is a canonical map

$$\Gamma\colon \mathrm{Hom}_{\mathbf{V}_{\mathrm{psh}}}(X, \mathrm{Spa}(R, R^+)) \to \mathrm{Hom}_{K-\mathrm{cont}}((R, R^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$$

induced by the global section functor. We now define a map

$$\phi\colon \operatorname{Hom}_{K-\text{cont}}((R,R^+),(\mathcal{O}_X(X),\mathcal{O}^+_X(X)) \to \operatorname{Hom}_{\mathbf{V}_{\text{psh}}}(X,\operatorname{Spa}(R,R^+))).$$

Suppose we have a map $a\colon (R,R^+)\to(\mathcal{O}_X(X),\mathcal{O}^+_X(X))$. We associate to each $x\in X$ the point $\phi_a(x)$ of $\operatorname{Spa}(R,R^+)$ correponding to the composite map

$$(R,R^+)\to(\mathcal{O}_X(X),\mathcal{O}^+_X(X))\to(k(x),k(x)^+).$$

The map $x\mapsto\phi_a(x)$ from X to $\operatorname{Spa}(R,R^+)$ is continuous, since the condition $|a(f)(x)|\leq|a(g)(x)|\neq 0$ is open in X by Lemma 1.1.23. For each $f_1,\dots,f_n\in R$ generating R and any $g\in R$ let V be the subset $\{x\in X\colon |a(f_i)(x)|\leq|a(g)(x)|\neq 0 \text{ for all } i\}$. It is open by Lemma 1.1.23. For any subset B of V in the basis $\mathcal{B}$ there exists an induced map of affinoid K-algebras

$$(\phi^\sharp_a,\phi^{+\sharp}_a)(V)\colon (R\langle f_1/g,\dots,f_n/g\rangle, R\langle f_1/g,\dots,f_n/g\rangle^+)\to(\mathcal{O}_X(B),\mathcal{O}^+_X(B))$$

deduced by the universal property of $(R\langle f_i/g\rangle,R\langle f_i/g\rangle^+)$ [**26**, Proposition 1.3]. Since $\mathcal{O}_X$ and $\mathcal{O}^+_X$ are sheaves and since $\mathcal{O}$ and $\mathcal{O}^+$ are adapted to rational subsets the mapping above also defines

$$(\phi^\sharp_a,\phi^{+\sharp}_a)(U)\colon(\mathcal{O}(U),\mathcal{O}^+(U))\to(\mathcal{O}_X(\phi^{-1}(U)),\mathcal{O}^+_X(\phi^{-1}(U)))$$

for an arbitrary open subset U of $\operatorname{Spa}(R,R^+)$. Therefore the triple $(\phi_a,\phi^\sharp_a,\phi^{+\sharp}_a)$ defines an element of $\operatorname{Hom}_{\mathbf{V}_{\text{psh}}}(X,\operatorname{Spa}(R,R^+))$ as wanted.

The composition $\Gamma\circ\phi$ is the identity by definition. We are left to check that $\phi\circ\Gamma$ is the identity. Fix a map $(f,f^\sharp,f^{+\sharp})$ in $\operatorname{Hom}_{\mathbf{V}_{\text{psh}}}(X,\operatorname{Spa}(R,R^+))$ and let a be the associated map in $\operatorname{Hom}_{K-\text{cont}}((R,R^+),(\mathcal{O}_X(X),\mathcal{O}^+_X(X)))$. For each $x\in X$ we deduce the following commutative diagram:

$$\begin{array}{ccc}
(R,R^+) & \xrightarrow{\quad a\quad} & (\mathcal{O}_X(X),\mathcal{O}^+_X(X))\\
\downarrow & & \downarrow\\
(k(f(x)),k^+(f(x))) & \xrightarrow{(f^\sharp_x,f^{+\sharp}_x)} & (k(x),k^+(x))
\end{array}$$

where $(f^\sharp_x,f^{+\sharp}_x)$ is a local map of valuation fields. Since the composite map

$$(R,R^+)\to(\mathcal{O}_X(X),\mathcal{O}^+_X(X))\to(k(x),k^+(x))$$

coincides with ϕ_x we deduce that ϕ_x is equivalent to the valuation induced by the map $(R,R^+)\to(k(f(x)),k^+(f(x)))$ hence $f(x)=\phi_x$. Fix now a rational subset U of $\operatorname{Spa}(R,R^+)$ and let V be $f^{-1}(U)$. By covering it with open sets of $\mathcal{B}$ we conclude that the map a factors over

$$(f^\sharp,f^{+\sharp})(V)\colon(\mathcal{O}(U),\mathcal{O}^+(U))\to(\mathcal{O}_X(V),\mathcal{O}^+_X(V))$$

and it coincides with $(\phi^\sharp_a,\phi^{+\sharp}_a)(V)$ by the universal property of $(\mathcal{O}(U),\mathcal{O}^+(U))$. This proves the claim. □

1.1.28. REMARK. The functor Spa induces an adjunction between $\mathbf{V}$ and the category of affinoid K-algebras such that the presheaf $\mathcal{O}$ on $\operatorname{Spa}(R,R^+)$ is a sheaf, also known as *sheafy* affinoid K-algebras using the language of [**45**], and these include reduced tft Tate algebras and perfectoid affinoid K-algebras.

1.1.29. REMARK. Proposition 1.1.27 slightly differs from [**26**, Proposition 2.1(ii)] since we do not assume that X is locally affinoid and that $\operatorname{Spa}(R,R^+)$ is in $\mathbf{V}$.

1.1.30. DEFINITION. Let X be an object of $\mathbf{V}$.

- We say that X is an *affinoid adic space* if it is isomorphic to $\mathrm{Spa}(R, R^+)$ for some affinoid K-algebra (R, R^+). It is called *bounded* if (R, R^+) is bounded.
- We say that X is an *affinoid rigid variety* if it is isomorphic to $\mathrm{Spa}(R, R^\circ)$ for some tft Tate algebra R and it is called *reduced* if R is reduced.
- We say that X is a *perfectoid affinoid space* if it is isomorphic to $\mathrm{Spa}(R, R^+)$ for some perfectoid affinoid K-algebra (R, R^+).
- We say that X is an *adic space* if it is locally isomorphic to an affinoid adic space.
- We say that X is a *rigid variety* if it is locally isomorphic to an affinoid rigid variety. It is called *reduced* if it is locally isomorphic to a reduced affinoid rigid variety.
- We say that X is a *perfectoid space* if it is locally isomorphic to a perfectoid affinoid space.

In this work we will always be dealing with stably uniform affinoid K-algebras. For this reason, the adjectives "bounded" and "reduced" will sometimes be omitted.

There is an apparent clash of definitions between rigid varieties as presented above, and as defined by Tate [**49**]. In fact, the two categories are canonically equivalent. We refer to [**26**, Section 4] and [**42**, Section 2] for a more detailed collection of results on the comparison between these theories.

1.1.31. ASSUMPTION. From now on, we will always assume that K is a perfectoid field. We also make the extra assumption that the invertible element π of K satisfies $|p| \leq |\pi| < 1$ and coincides with $(\pi^\flat)^\sharp$ for a chosen $\pi^\flat$ in $K^\flat$. In particular, π is equipped with a compatible system of p-power roots π^{1/p^h} (see [**42**, Remark 3.5]).

We now consider some basic examples and fix some notation. Let $\underline{\upsilon} = (\upsilon_1, \dots, \upsilon_N)$ be a N-tuple of coordinates. The Tate N-ball $\mathrm{Spa}(K\langle\underline{\upsilon}\rangle, K^\circ\langle\underline{\upsilon}\rangle)$ will be denoted by $\mathbb{B}^N$ and the N-torus $\mathrm{Spa}(K\langle\underline{\upsilon}^{\pm 1}\rangle, K^\circ\langle\underline{\upsilon}^{\pm 1}\rangle)$ by $\mathbb{T}^N$. It is the rational open subset $U(1 \mid \upsilon_1 \dots \upsilon_N)$ of $\mathbb{B}^N$. The map of spaces induced by the inclusion $(K\langle\underline{\upsilon}\rangle, K^\circ\langle\underline{\upsilon}\rangle) \to (K\langle\underline{\upsilon}^{1/p^h}\rangle, K^\circ\langle\underline{\upsilon}^{1/p^h}\rangle)$ will be denoted by $\mathbb{B}^N\langle\underline{\upsilon}^{1/p^h}\rangle \to \mathbb{B}^N$. We use the analogous notation $\mathbb{T}^N\langle\underline{\upsilon}^{1/p^h}\rangle \to \mathbb{T}^N$ for the torus. These maps are clearly isomorphic to the endomorphism of $\mathbb{B}^N$ resp. $\mathbb{T}^N$ induced by $\upsilon_i \mapsto \upsilon_i^{p^h}$.

The space defined by the perfectoid affinoid K-algebra $(K\langle\underline{\upsilon}^{1/p^\infty}\rangle, K^\circ\langle\underline{\upsilon}^{1/p^\infty}\rangle)$ will be denoted by $\widehat{\mathbb{B}}^N$ and referred to as the *perfectoid N-ball*. The space defined by the perfectoid affinoid K-algebra $(K\langle\underline{\upsilon}^{\pm 1/p^\infty}\rangle, K^\circ\langle\underline{\upsilon}^{\pm 1/p^\infty}\rangle)$ coincides with the rational subspace $U(1 \mid \upsilon_1 \dots \upsilon_N)$ of $\widehat{\mathbb{B}}^N$ will be denoted by $\widehat{\mathbb{T}}^N$ and will be referred to as the *perfectoid N-torus*.

We now recall the definition of étale maps on the category of adic spaces, taken from [**42**, Section 7].

1.1.32. DEFINITION. A map of affinoid adic spaces $f\colon \mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ is *finite étale* if the associated map $R \to S$ is a finite étale map of rings, and if S^+ is the integral closure of R^+ in S. A map of adic spaces $f\colon X \to Y$ is *étale* if for any point $x \in X$ there exists an open neighborhood U of x and an affinoid open subset V of Y containing $f(U)$ such that $f|_U\colon U \to V$ factors as an open embedding $U \to W$ and a finite étale map $W \to V$ for some affinoid adic space W.

The previous definitions, when restricted to the case of tft Tate varieties, coincide with the usual ones, as proved in [**18**, Proposition 8.1.2].

1.1.33. REMARK. Suppose we are given a diagram of affinoid K-algebras

$$\begin{array}{ccc} (R, R^+) & \longrightarrow & (S, S^+) \\ \downarrow & & \\ (T, T^+) & & \end{array}$$

In general, it is not possible to define a push-out in the category of affinoid K-algebras. Nonetheless, this can be performed under some hypotheses. For example, if the affinoid K-algebras are tft Tate algebras then the push-out exists and it is the tft Tate algebra associated to the completion $S\widehat{\otimes}_R T$ of $S \otimes_R T$ endowed with the norm of the tensor product (see [**9**, Section 3.1.1]). In case the affinoid K-algebras are perfectoid affinoid, then the push-out exists and is also perfectoid affinoid. It coincides with the completion of (L, L^+) where L is the ring $S\otimes_R T$ endowed with the norm of the tensor product and L^+ is the algebraic closure of $S^+ \otimes_{R^+} T^+$ in L (see [**42**, Proposition 6.18]). The same construction holds in case the map $(R, R^+) \to (S, S^+)$ is finite étale and (T, T^+) is a perfectoid affinoid (see [**42**, Lemma 7.3]). By Proposition 1.1.27, the constructions above give rise to fiber products in the category $\mathbf{V}$.

1.2. Semi-perfectoid spaces

We can now introduce a convenient generalization of both smooth rigid varieties and smooth perfectoid spaces. We recall that our base field K is a perfectoid field.

1.2.1. PROPOSITION. *Let $\underline{\upsilon} = \upsilon_1, \ldots, \upsilon_N$ and $\underline{\nu} = \nu_1, \ldots, \nu_M$ be two systems of coordinates. Let (R_0, R_0°) be a tft Tate algebra and let*

$$f\colon \operatorname{Spa}(R_0, R_0^\circ) \to \mathbb{T}^N \times \mathbb{T}^M = \operatorname{Spa} K\langle \underline{\upsilon}^{\pm 1}, \underline{\nu}^{\pm 1}\rangle$$

be a map which is a composition of finite étale maps and rational embeddings. Let also $\operatorname{Spa}(R_h, R_h^\circ)$ be the affinoid rigid variety $\operatorname{Spa}(R_0, R_0^\circ) \times_{\mathbb{T}^N} \mathbb{T}^N\langle \underline{\upsilon}^{1/p^h}\rangle$. The π-adic completion (T, T^+) of $(\varinjlim_i R_i, \varinjlim_i R_i^\circ)$ represents the fiber product $\operatorname{Spa}(R_0, R_0^\circ) \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ and defines a bounded affinoid adic space. Moreover, (T, T^+) is isomorphic to the completion of (L, L^+) where L is the ring $R_0 \otimes_{K\langle \underline{\upsilon}\rangle} K\langle \underline{\upsilon}^{1/p^\infty}\rangle$ endowed with the norm of the tensor product and L^+ is the integral closure of R_0° in L.

PROOF. Let (T, T^+) be as in the last claim. We need to prove that $W := \operatorname{Spa}(T, T^+)$ is an adic space, i.e. that $\mathcal{O}$ is a sheaf on it. We let W' be the fiber product of $\operatorname{Spa}(R_0, R_0^\circ)$ and $\widehat{\mathbb{T}}^N \times \widehat{\mathbb{T}}^M$ over $\mathbb{T}^N \times \mathbb{T}^M$. If $\operatorname{char} K = 0$ by [**42**, Proposition 6.3(iii), Lemma 7.3 and Proposition 7.10] and the proof of [**43**, Lemma 4.5] it exists, is affinoid perfectoid represented by (T', T'^+) where T' is $R_0 \widehat{\otimes}_{K\langle \underline{\upsilon}, \underline{\nu}\rangle} K\langle \underline{\upsilon}^{1/p^\infty}, \underline{\nu}^{1/p^\infty}\rangle$ and where T'^+ is bounded in T' and corresponds to the completion of the algebraic closure of $R_0^\circ \otimes_{K^\circ\langle \underline{\upsilon}, \underline{\nu}\rangle} K^\circ\langle \underline{\upsilon}^{1/p^\infty}, \underline{\nu}^{1/p^\infty}\rangle$ in $R_0 \otimes_{K\langle \underline{\upsilon}, \underline{\nu}\rangle} K\langle \underline{\upsilon}^{1/p^\infty}, \underline{\nu}^{1/p^\infty}\rangle$. The same is true if $\operatorname{char} K = p$ as in this case it coincides with the completed perfection of X_0 (see [**19**, Theorem 3.5.13]).

Let $\{U_i\}$ be a finite rational covering of W and let $\{U_i'\}$ be the rational covering of W' obtained by pullback. We first prove that the pullback of $\mathcal{O}(W')$ and $\mathcal{O}(U_i)$ over $\mathcal{O}(U_i')$ coincides with $\mathcal{O}(W)$. Since as pointed out in Remark the ring $K\langle \underline{\nu}^{1/p^\infty}\rangle$ is isomorphic to $\widehat{\bigoplus} K\langle \underline{\nu}\rangle$ also $\mathcal{O}(W')$ is isomorphic to $\widehat{\bigoplus} \mathcal{O}(W)$ and $\mathcal{O}(U_i')$ is isomorphic to $\widehat{\bigoplus} \mathcal{O}(U_i)$ using [**9**, Proposition 2.1.7/8]. By the explicit description of this set as a subset of $\prod \mathcal{O}(U_i)$ given in [**9**, Proposition 2.1.5/7] we conclude that $\widehat{\bigoplus} \mathcal{O}(W) \times_{\widehat{\bigoplus} \mathcal{O}(U_i)} \mathcal{O}(U_i) = \mathcal{O}(W)$ as claimed. We then conclude that the equalizer of the diagram

$$\prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_i \cap U_j)$$

is obtained by pullback from equalizer of the diagram

$$\prod_i \mathcal{O}(U'_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U'_i \cap U'_j).$$

Since the latter coincides with $\mathcal{O}(W')$ we deduce that the former coincides wih $\mathcal{O}(W)$ as wanted.

Moreover, since the map $R_0 \to R_h$ is finite, R_h° is the algebraic closure in R_h of R_0° by [**9**, Theorem 6.3.5/1]. Passing to the direct limit, one finds that T^+ is the completion of $\varinjlim_h R_h^\circ$. We are left to prove that T^+ is bounded, and this follows as it strictly embeds in T'^+ which is bounded in T'. □

1.2.2. COROLLARY. *Let X be a reduced rigid variety with an étale map*

$$f\colon X \to \mathbb{T}^N \times \mathbb{T}^M = \operatorname{Spa} K\langle \underline{\upsilon}^{\pm 1}, \underline{\nu}^{\pm 1}\rangle.$$

Then the fiber product $X \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ exists.

PROOF. This follows from Proposition 1.2.1 and the fact that every étale map is locally (on the source) a composition of rational embeddings and finite étale maps. □

1.2.3. DEFINITION. We denote by $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$ the full subcategory of adic spaces whose objects are isomorphic to spaces $X = X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ with respect to a map of affinoid rigid varieties $f\colon X_0 \to \mathbb{T}^N \times \mathbb{T}^M$ that is a composition of rational embeddings and finite étale maps. Because of Proposition 1.2.1, such fiber products $X = X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ exist and are affinoid. Whenever $N = 0$ these varieties are rigid analytic varieties and the full subcategory they form will be denoted by $\mathrm{RigSm}^{\mathrm{gc}}/K$ and referred to as *smooth affinoid rigid varieties with good coordinates*. Whenever $M = 0$ these varieties are perfectoid affinoid spaces and the full subcategory they form will be denoted by $\mathrm{PerfSm}^{\mathrm{gc}}/K$ and referred to as *smooth affinoid perfectoids with good coordinates*. A perfectoid space X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$ will be sometimes denoted with $\widehat{X}$.

When $X = X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ is in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$ we denote by X_h the fiber product $X_0 \times_{\mathbb{T}^N} \mathbb{T}^N\langle \underline{\upsilon}^{1/p^h}\rangle$ and we will write $X = \varprojlim_h X_h$. We say that a presentation $X = \varprojlim_h X_h$ of an object X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$ has *good reduction* if the map $X_0 \to \mathbb{T}^n \times \mathbb{T}^m$ has an étale formal model $\mathfrak{X} \to \operatorname{Spf}(K^\circ\langle \underline{\upsilon}^{\pm 1}, \underline{\nu}^{\pm 1}\rangle)$. We say that a presentation $X = \varprojlim_h X_h$ of an object X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$ has *potentially good reduction* if there exists a finite separable field extension L/K such that $X_L = \varprojlim_h (X_h)_L$ has good reduction in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/L$. We warn the reader that the association $X \mapsto X_0$ is not functorial and the varieties X_h are not uniquely determined by X in general.

We denote by $\widehat{\mathrm{RigSm}}/K$ the full subcategory of adic spaces which are locally isomorphic to objects in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}/K$; we denote by RigSm/K the full subcategory of adic spaces which are locally isomorphic to objects in $\mathrm{RigSm}^{\mathrm{gc}}/K$ and by PerfSm/K the one of adic spaces which are locally isomorphic to objects in $\mathrm{PerfSm}^{\mathrm{gc}}/K$. Whenever the context allows it, we omit K from the notation.

1.2.4. REMARK. Any smooth rigid variety (see for example [**5**, Definition 1.1.39]) has locally good coordinates over $\mathbb{T}^N$ by [**5**, Corollary 1.1.49]. Hence RigSm coincides with the category of smooth rigid varieties.

We remark that the presentations of good reduction defined above are a special case of the objects considered in [**2**].

The notation $X = \varprojlim_h X_h$ is justified by the following corollary, which is inspired by [**45**, Proposition 2.4.5].

1.2.5. COROLLARY. *Let Y be a bounded affinoid adic space and let X be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ with $X = \varprojlim_h X_h$. Then $\mathrm{Hom}(Y, X) \cong \varprojlim_h \mathrm{Hom}(Y, X_h)$.*

PROOF. This follows from Lemma 1.1.10 and Proposition 1.2.1. □

Let $\{X_h, f_h\}_{h\in I}$ be a cofiltered diagram of rigid varieties and let $\{X \to X_h\}_{h\in I}$ be a collection of compatible maps of adic spaces. We recall that, according to [**27**, Remark 2.4.5], one writes $X \sim \varprojlim_h X_h$ when the following two conditions are satisfied:

(1) The induced map on topological spaces $|X| \to \varprojlim_h |X_h|$ is a homeomorphism.
(2) For any $x \in X$ with images $x_h \in X_h$ the map of residue fields $\varinjlim_h k(x_h) \to k(x)$ has dense image.

The apparent clash of notations is solved by the following fact.

1.2.6. PROPOSITION. *Let $X = \varprojlim_h X_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. Then $X \sim \varprojlim_h X_h$.*

PROOF. This follows from $\widehat{\mathbb{T}}^N \sim \varprojlim_h K\langle \underline{\upsilon}^{\pm 1/p^h}\rangle$ and from [**42**, Proposition 7.16]. □

Étale maps define a topology on $\widehat{\mathrm{RigSm}}$ in the following way.

1.2.7. DEFINITION. A collection of étale maps of adic spaces $\{U_i \to X\}_{i\in I}$ is an *étale cover* if the induced map $\bigsqcup_{i\in I} U_i \to X$ is surjective. These covers define a Grothendieck topology on $\widehat{\mathrm{RigSm}}$ called the *étale topology*.

The following facts are shown in the proof of [**42**, Theorem 7.17] and of [**27**, Proposition 2.4.4].

1.2.8. PROPOSITION. *Let $X = \varprojlim_h X_h$ be an object of $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$.*

(1) Any finite étale map $U \to X$ is isomorphic to $U_{\bar{h}} \times_{X_{\bar{h}}} X$ for some integer $\bar{h}$ and some finite étale map $U_{\bar{h}} \to X_{\bar{h}}$.
(2) Any rational subspace $U \subset X$ is isomorphic to $U_{\bar{h}} \times_{X_{\bar{h}}} X$ for some integer H and some rational subspace $U_{\bar{h}} \subset X_{\bar{h}}$.

PROOF. The first statement follows from [**42**, Lemma 7.5]. The second statement follows from [**25**, Lemma 3.10] and the fact that $\varinjlim_h \mathcal{O}(X_h)$ is dense in $\mathcal{O}(X)$. □

1.2.9. COROLLARY. *Let $X = \varprojlim_h X_h$ be an object of $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ and let $\mathcal{U} := \{f_i \colon U_i \to X\}$ be an étale covering of adic spaces. There exists an integer $\bar{h}$ and a finite affine refinement $\{V_j \to X\}$ of $\mathcal{U}$ which is obtained by pullback of an étale covering $\{V_{\bar{h}j} \to X_{\bar{h}}\}$ of $X_{\bar{h}}$ and such that $V = \varprojlim_h V_{hj}$ lies in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ by letting V_{hj} be $V_{\bar{h}j} \times_{X_{\bar{h}}} X_h$ for all $h \geq \bar{h}$.*

PROOF. Any étale map of adic spaces is locally a composition of rational embeddings and finite étale maps and they descend because of Proposition 1.2.8. □

1.2.10. COROLLARY. *A perfectoid space X lies in PerfSm if and only if it is locally étale over $\widehat{\mathbb{T}}^N$.*

PROOF. Let X be locally étale over $\widehat{\mathbb{T}}^N$. Then it is locally open in a finite étale space over a rational subaffinoid of $\widehat{\mathbb{T}}^N = \varprojlim_h \mathbb{T}^N\langle \underline{\upsilon}^{\pm 1/p^h}\rangle$. By Proposition 1.2.8, we conclude it is locally of the form $X_0 \times_{\mathbb{T}^N} \widehat{\mathbb{T}}^N$ for some étale map $X_0 \to \mathbb{T}^N = \operatorname{Spa}(K\langle \underline{\upsilon}^{\pm 1}\rangle, K^\circ\langle \underline{\upsilon}^{\pm 1}\rangle)$ which is the composition of rational embeddings and finite étale maps. □

1.2.11. REMARK. If X is a smooth affinoid perfectoid space, then it has a finite number of connected components. Indeed, it is quasi-compact and locally isomorphic to a rational domain of a perfectoid which is finite étale over a rational domain of $\widehat{\mathbb{T}}^N$.

For later use, we record the following simple example of a space $X = \varprojlim_h X_h$ for which the varieties X_h are easy to understand.

1.2.12. PROPOSITION. *Consider the smooth variety with good coordinates*

$$X_0 = U\left(\upsilon - 1 \mid \pi\right) \hookrightarrow \mathbb{T}^1 = \operatorname{Spa}(K\langle \upsilon^{\pm 1}\rangle).$$

One has $X_h \cong \mathbb{B}^1$ for all h and $\widehat{X} = \varprojlim_h X_h \cong \widehat{\mathbb{B}}^1$.

PROOF. By direct computation, the variety X_h is isomorphic to $\operatorname{Spa}(K\langle \upsilon, \omega\rangle/(\omega^{p^h} - (\pi\upsilon + 1)))$. Since $|p| \leq |\pi|$ we deduce that $|\binom{p^h}{i}| \leq |\pi|$ for all $0 < i < p^h$. In particular, in the ring $K\langle \upsilon, \omega\rangle/(\omega^{p^h} - (\pi\upsilon + 1))$ one has

$$|(\omega - 1)^{p^h}| = |\pi\upsilon + \sum_{i=1}^{p^h-1}\binom{p^h}{i}\omega^i| = |\pi|.$$

Analogously, in the ring $K\langle \chi\rangle$ one has

$$|(\chi + \pi^{-1/p^h})^{p^h} - \pi^{-1}| = |\chi^{p^h} + \sum_{i=1}^{p^h-1}\binom{p^h}{i}\chi^{p^h-1}\pi^{-i/p^h}| = 1.$$

The following maps are therefore well defined and clearly mutually inverse:

$$\begin{aligned} X_h = \operatorname{Spa}(K\langle \upsilon, \omega\rangle/(\omega^{p^h} - (\pi\upsilon + 1)) &\leftrightarrows \operatorname{Spa}(K\langle \chi\rangle) = \mathbb{B}^1 \\ (\upsilon, \omega) &\mapsto ((\chi + \pi^{-1/p^h})^{p^h} - \pi^{-1}, \pi^{1/p^h}\chi + 1) \\ \pi^{-1/p^h}(\omega - 1) &\leftarrow\!\shortmid \chi. \end{aligned}$$

Consider the multiplicative map $\sharp\colon K^\flat\langle \upsilon^{1/p^\infty}\rangle = (K\langle \upsilon^{1/p^\infty}\rangle)^\flat \to K\langle \upsilon^{1/p^\infty}\rangle$ defined in [**42**, Proposition 5.17]. By our assumptions on π the element $(\upsilon - 1)^\sharp - (\upsilon - 1)$ is divisible by π in $K^\circ\langle \upsilon^{1/p^\infty}\rangle$ and therefore the rational set $\widehat{X} \cong U\left(\upsilon - 1 \mid \pi\right)$ of $\widehat{\mathbb{T}}^1$ coincides with $U\left((\upsilon - 1)^\sharp \mid \pi^{\flat\sharp}\right)$. From [**42**, Theorem 6.3] we conclude $\widehat{X}^\flat \cong U\left(\upsilon - 1 \mid \pi^\flat\right) \hookrightarrow \widehat{\mathbb{T}}^{\flat 1}$ which is isomorphic to $\widehat{\mathbb{B}}^{\flat 1}$ hence the claim. □

From the previous proposition we conclude in particular that the perfectoid space $\widehat{\mathbb{B}}^1$ lies in $\mathrm{PerfSm}^{\mathrm{gc}}$.

1.3. Categories of adic motives

From now on, we fix a commutative ring Λ and work with Λ-enriched categories. In particular, the term "presheaf" should be understood as "presheaf of Λ-modules" and similarly for the tem "sheaf". The presheaf $\Lambda(X)$ represented by an object X of a category $\mathbf{C}$ sends an object Y of $\mathbf{C}$ to the free Λ-module $\Lambda\operatorname{Hom}(Y, X)$.

1.3.1. ASSUMPTION. Unless otherwise stated, we assume from now on that Λ is a $\mathbb{Q}$-algebra and we omit it from the notations.

We make extensive use of the theory of model categories and localization, following the approach of Ayoub in [**5**] and [**6**]. Fix a site $(\mathbf{C}, \tau)$. In our situation, this will be the étale site of RigSm or $\widehat{\mathrm{RigSm}}$. The category of complexes of presheaves $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ can be endowed with the *projective model structure* for which weak equivalences are quasi-isomorphisms and fibrations are maps $\mathcal{F} \to \mathcal{F}'$ such that $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a surjection for all X in $\mathbf{C}$ (cfr [**23**, Section 2.3] and [**6**, Proposition 4.4.16]).

Also the category of complexes of sheaves $\mathbf{Ch}(\mathbf{Sh}_\tau(\mathbf{C}))$ can be endowed with the *projective model structure* defined in [**6**, Proposition 4.4.41]. In this structure, weak equivalences are quasi-isomorphisms of complexes of sheaves.

1.3.2. REMARK. Let $\mathbf{C}$ be a category. As shown in [**16**] any projectively cofibrant complex $\mathcal{F}$ in $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ is a retract of a complex that is the filtered colimit of bounded above complexes, each constituted by presheaves that are direct sums of representable ones.

Just like in [**29**], [**36**], [**37**] or [**41**], we consider the left Bousfield localization of $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ with respect to the topology we select, and a chosen "contractible object". We recall that left Bousfield localizations with respect to a class of maps S (see [**22**, Chapter 3]) is the universal model categories in which the maps in S become weak equivalences. The existence of such structures is granted only under some technical hypothesis, as shown in [**22**, Theorem 4.1.1] and [**6**, Theorem 4.2.71].

1.3.3. PROPOSITION. *Let $(\mathbf{C}, \tau)$ be a site with finite direct products and let $\mathbf{C}'$ be a full subcategory of $\mathbf{C}$ such that every object of $\mathbf{C}$ has a covering by objects of $\mathbf{C}'$. Let also I be an object of $\mathbf{C}'$.*

(1) The projective model category $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ admits a left Bousfield localization $\mathbf{Ch}_I\,\mathbf{Psh}(\mathbf{C})$ with respect to the set S_I of all maps $\Lambda(I \times X)[i] \to \Lambda(X)[i]$ as X varies in $\mathbf{C}$ and i varies in $\mathbb{Z}$.

(2) The projective model categories $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C}')$ admit left Bousfield localizations $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C}')$ with respect to the class S_τ of maps $\mathcal{F} \to \mathcal{F}'$ inducing isomorphisms on the $\mathrm{\acute{e}t}$-sheaves associated to $H_i(\mathcal{F})$ and $H_i(\mathcal{F}')$ for all $i \in \mathbb{Z}$. Moreover, the two localized model categories are Quillen equivalent and the sheafification functor induces a Quillen equivalence to the projective model category $\mathbf{Ch}\,\mathbf{Sh}_\tau(\mathbf{C})$.

(3) The model categories $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C}')$ admit left Bousfield localizations $\mathbf{Ch}_{\tau,I}\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_{\tau,I}\,\mathbf{Psh}(\mathbf{C}')$ with respect to the set S_I defined above. Moreover, the two localized model categories are Quillen equivalent.

PROOF. The model structure on complexes is left proper and cellular. It follows that the projective model structures in the statement are also left proper and cellular. Any such model category admits a left Bousfield localization with respect to a set of maps ([**22**, Theorem 4.1.1]) hence the first claim.

For the first part of second claim, it suffices to apply [**6**, Proposition 4.4.32, Lemma 4.4.35] showing that the localization over S_τ is equivalent to a localization over a set of maps. The second part is a restatement of [**6**, Corollary 4.4.43, Proposition 4.4.56].

Since by [**6**, Proposition 4.4.32] the τ-localization coincides with the Bousfield localization with respect to a set, we conclude by [**6**, Theorem 4.2.71] that the model category $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$

is still left proper and cellular. The last statement then follows from [**22**, Theorem 4.1.1] and the second claim. ☐

In the situation above, we will denote by $S_{(\tau,I)}$ the union of the class S_τ and the set S_I.

1.3.4. REMARK. A geometrically relevant situation is induced when I is endowed with a multiplication map $\mu\colon I\times I\to I$ and maps i_0 and i_1 from the terminal object to I satisfying the relations of a monoidal object with 0 as in the definition of an interval object (see [**37**, Section 2.3]). Under these hypotheses, we say that the triple $(\mathbf{C},\tau,I)$ is a *site with an interval.*

1.3.5. EXAMPLE. The affinoid rigid variety with good coordinates $\mathbb{B}^1=\operatorname{Spa}K\langle\chi\rangle$ is an interval object with respect to the natural multiplication μ and maps i_0 and i_1 induced by the substitution $\chi\mapsto 0$ and $\chi\mapsto 1$ respectively.

We now apply the constructions above to the sites introduced in the previous sections. We recall that we consider adic spaces defined over a perfectoid field K.

1.3.6. COROLLARY. *The following pairs of model categories are Quillen equivalent.*

- $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\mathrm{RigSm})$ *and* $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}})$.
- $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm})$ *and* $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}})$.
- $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ *and* $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$.
- $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ *and* $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$.

PROOF. It suffices to apply Proposition 1.3.3 to the sites with interval $(\mathrm{RigSm},\text{ét},\mathbb{B}^1)$ and $(\widehat{\mathrm{RigSm}},\text{ét},\mathbb{B}^1)$ where $\mathbf{C}'$ is in both cases the subcategory of varieties with good coordinates. ☐

1.3.7. DEFINITION. For $\eta\in\{\text{ét},\mathbb{B}^1,(\text{ét},\mathbb{B}^1)\}$ we say that a map in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm})$ [resp. $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$] is a η*-weak equivalence* if it is a weak equivalence in the model structure $\mathbf{Ch}_\eta\,\mathbf{Psh}(\mathrm{RigSm})$ [resp. $\mathbf{Ch}_\eta\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$]. The triangulated homotopy category associated to the localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm})$ [resp. $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$] will be denoted by $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda)$ [resp. $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K,\Lambda)$]. We will omit Λ from the notation whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda(X)$. We say that an object $\mathcal{F}$ of the derived category $\mathbf{D}=\mathbf{D}(\mathbf{Psh}(\mathrm{RigSm}))$ [resp. $\mathbf{D}=\mathbf{D}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}))$] is η*-local* if the functor $\operatorname{Hom}_{\mathbf{D}}(\cdot,\mathcal{F})$ sends maps in S_η to isomorphisms. This amounts to say that $\mathcal{F}$ is quasi-isomorphic to a η-fibrant object.

We need to keep track of $\mathbb{B}^1$ in the notation of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K,\Lambda)$ since later we will perform a localization on $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ with respect to a different interval object.

1.3.8. REMARK. Using the language of [**8**], the localizations defined above induce endofunctors C^η of the derived categories $\mathbf{D}(\mathbf{Psh}(\mathrm{RigSm}))$, $\mathbf{D}(\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}}))$, $\mathbf{D}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}))$ and $\mathbf{D}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}}))$ such that $C^\eta\mathcal{F}$ is η-local for all $\mathcal{F}$ and there is a natural transformation $C^\eta\to\mathrm{id}$ which is a pointwise η-weak equivalencce. The functor C^η restricts to a triangulated equivalence on the objects $\mathcal{F}$ that are η-local and one can compute the Hom set $\operatorname{Hom}(\mathcal{F},\mathcal{F}')$ in the the homotopy category of the η-localization as $\mathbf{D}(\mathcal{F},C^\eta\mathcal{F}')$.

1.3.9. REMARK. By means of [**6**, Proposition 4.4.59] the complex $C^{\text{ét}}\mathcal{F}$ is such that $\mathbf{D}(\Lambda(X)[-i],C^{\text{ét}}\mathcal{F})=\mathbb{H}^i_{\text{ét}}(X,\mathcal{F})$ for all X in $\widehat{\mathrm{RigSm}}$ and all integers i. This property characterizes $C^{\text{ét}}\mathcal{F}$ up to quasi-isomorphisms.

We now show that the étale localization can alternatively be described in terms of étale hypercoverings $\mathcal{U}_\bullet \to X$ (see for example [**15**]). Any such datum defines a simplicial presheaf $n \mapsto \bigoplus_i \Lambda(U_{ni})$ whenever $\mathcal{U}_n = \bigsqcup_i h_{U_{ni}}$ is the sum of the presheaves of sets $h_{U_{ni}}$ represented by U_{ni}. This simplicial presheaf can be associated to a normalized chain complex, that we denote by $\Lambda(\mathcal{U}_\bullet)$. It is is endowed with a map to $\Lambda(X)$.

1.3.10. PROPOSITION. *The localization over $S_{\text{ét}}$ on $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}})$ [resp. on $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$] coincides with the localization over the set $\Lambda(\mathcal{U}_\bullet)[i] \to \Lambda(X)[i]$ as $\mathcal{U}_\bullet \to X$ varies among bounded étale hypercoverings of the objects X of $\mathrm{RigSm}^{\mathrm{gc}}$ [resp. $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$] and i varies in $\mathbb{Z}$.*

PROOF. Any ét-local object $\mathcal{F}$ is also local with respect to the maps of the statement. We are left to prove that a complex $\mathcal{F}$ which is local with respect to the maps of the statement is also ét-local.

Since Λ contains $\mathbb{Q}$ the étale cohomology of an étale sheaf $\mathcal{F}$ coincides with the Nisnevich cohomology (the same proof of [**36**, Proposition 14.23] holds also here). By means of [**5**, 1.2.19] we conclude that any rigid variety X has a finite cohomological dimension. By [**1**, Theorem V.7.4.1] and [**48**, Theorem 0.3], we obtain for any rigid variety X and any complex of presheaves $\mathcal{F}$ an isomorphism

$$\mathbb{H}^n_{\text{ét}}(X,\mathcal{F}) \cong \varinjlim_{\mathcal{U}_\bullet \in HR_\infty(X)} H_{-n}\,\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_\bullet),\mathcal{F})$$

where $HR_\infty(X)$ is the category of bounded étale hypercoverings of X (see [**1**, V.7.3]) and $\mathrm{Hom}_\bullet$ is the Hom-complex computed in the unbounded derived category of presheaves. Suppose now $\mathcal{F}$ is local with respect to the maps of the statement. Then $\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_\bullet),\mathcal{F})$ is quasi-isomorphic to $\mathrm{Hom}_\bullet(X,\mathcal{F})$ for every bounded hypercovering $\mathcal{U}_\bullet$ hence $H_{-n}\mathcal{F}(X) \cong \mathbb{H}^n_{\text{ét}}(X,\mathcal{F})$ by the formula above. We then conclude that the map $\mathcal{F} \to C^{\text{ét}}\mathcal{F}$ is a quasi-isomorphism, proving the proposition. □

As the following proposition shows, there are also alternative presentations of the homotopy categories introduced so far, which we will later use.

1.3.11. PROPOSITION. *Let Λ be a $\mathbb{Q}$-algebra. The natural inclusion induces Quillen equivalences $L_S\,\mathbf{Ch}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})) \leftrightarrows \mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ where L_S denotes the Bousfield localization with respect to the set S of shifts of the maps of complexes induced by étale Cech hypercoverings $\mathcal{U}_\bullet \to X$ of objects X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ such that for some presentation $X = \varprojlim_h X_h$ the covering $\mathcal{U}_0 \to X$ descends to a covering of X_0.*

PROOF. Using Proposition 1.3.10, it suffices to prove that the map $\Lambda(\mathcal{U}_\bullet) \to \Lambda(X)$ is an isomorphism in the homotopy category $L_S\,\mathbf{Ch}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}}))$ for a fixed bounded étale hypercovering $\mathcal{U}_\bullet$ of an object X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$.

Since the inclusion functor $\mathbf{Ch}_{\geq 0} \to \mathbf{Ch}$ is a Quillen functor, it suffices to prove that $\Lambda(\mathcal{U}_\bullet) \to \Lambda(X)$ is a weak equivalence in $L_T\,\mathbf{Ch}_{\geq 0}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}}))$ where T is the set of shifts of the maps of complexes induced by étale Cech hypercoverings descending at finite level. Let $L_{\tilde{T}}\,\mathbf{sPsh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ be the Bousfield localization of the projective model structure on simplicial presheaves of sets with respect to the set $\tilde{T}$ formed by maps induced by étale Cech hypercoverings $\mathcal{U}_\bullet \to X$ descending at finite level. We remark that the Dold-Kan correspondence (see [**46**, Section 4.1]) and the Λ-enrichment also define a left Quillen functor from $L_{\tilde{T}}\,\mathbf{sPsh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ to the category $L_T\,\mathbf{Ch}_{\geq 0}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}}))$. It therefore suffices to prove that $\mathcal{U}_\bullet \to X$ is a weak equivalence in $L_{\tilde{T}}\,\mathbf{sPsh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ and this follows from the fact that bounded hypercovering define the same localization as Cech hypercoverings (see [**15**, Theorem A.6]) together with the fact that coverings descending to finite level define the same topology (Corollary 1.2.9) and hence the same localization ([**15**, Corollary A.8]). We remark that [**15**, Corollary A.8] applies in our case even if the coverings $\mathcal{U} \to X$ descending to the finite level do not form a basis of the topology, as their pullback via an arbitrary map $Y \to X$ may not have the same property. However, the proof of the statement relies on [**15**, Proposition A.2], where it is only used that the chosen family of coverings $\mathcal{U} \to X$ generates the topology and that the fiber product $\mathcal{U} \times_X \mathcal{U}$ is defined. $\square$

1.3.12. REMARK. It is shown in the proof that the statements of Propositions 1.3.10 and 1.3.11 hold true without any assumptions on Λ under the condition that all varieties X have finite cohomological dimension with respect to the étale topology.

As we pointed out in Remark 1.3.9, there is a characterization of $C^{\text{ét}}\mathcal{F}$ for any complex $\mathcal{F}$. This is also true for the $\mathbb{B}^1$-localization, described in the following part.

1.3.13. DEFINITION. We denote by $\square$ the Σ-enriched cocubical object (see [**3**, Appendix A]) defined by putting $\square^n = \mathbb{B}^n = \operatorname{Spa} K\langle \tau_1, \ldots, \tau_n \rangle$ and considering the morphisms $d_{r,\epsilon}$ induced by the maps $\mathbb{B}^n \to \mathbb{B}^{n+1}$ corresponding to the substitution $\tau_r = \epsilon$ for $\epsilon \in \{0,1\}$ and the morphisms p_r induced by the projections $\mathbb{B}^n \to \mathbb{B}^{n-1}$. For any variety X and any presheaf $\mathcal{F}$ with values in an abelian category, we can therefore consider the Σ-enriched cubical object $\mathcal{F}(X \times \square)$ (see [**3**, Appendix A]). Associated to any Σ-enriched cubical object $\mathcal{F}$ there are the following complexes: the complex $C^\sharp_\bullet\mathcal{F}$ defined as $C^\sharp_n\mathcal{F} = \mathcal{F}_n$ and with differential $\sum(-1)^r(d^*_{r,1} - d^*_{r,0})$; the *simple complex* $C_\bullet\mathcal{F}$ defined as $C_n\mathcal{F} = \bigcap_{r=1}^n \ker d^*_{r,0}$ and with differential $\sum(-1)^r d^*_{r,1}$; the *normalized complex* $N_\bullet\mathcal{F}$ defined as $N_n\mathcal{F} = C_n \cap \mathcal{F} \bigcap_{r=2}^n \ker d^*_{r,1}$ and with differential $-d^*_{1,1}$. By [**4**, Lemma A.3, Proposition A.8, Proposition A.11], the inclusion $N_\bullet\mathcal{F} \hookrightarrow C_\bullet\mathcal{F}$ is a quasi-isomorphism and both inclusions $C_\bullet\mathcal{F} \hookrightarrow C^\sharp_\bullet\mathcal{F}$ and $N_\bullet\mathcal{F} \hookrightarrow C_\bullet\mathcal{F}$ split. For any complex of presheaves $\mathcal{F}$ we let $\operatorname{Sing}^{\mathbb{B}^1}\mathcal{F}$ be the total complex of the simple complex associated to the $\underline{\mathrm{Hom}}(\Lambda(\square), \mathcal{F})$. It sends the object X to the total complex of the simple complex associated to $\mathcal{F}(X \times \square)$.

The following lemma is the cocubical version of [**36**, Lemma 2.18].

1.3.14. LEMMA. *For any presheaf $\mathcal{F}$ the two maps of cubical sets $i_0^*, i_1^* \colon \mathcal{F}(\square \times \mathbb{B}^1) \to \mathcal{F}(\square)$ induce chain homotopic maps on the associated simple and normalized complexes.*

PROOF. Consider now the isomorphism $s_n \colon \mathbb{B}^{n+1} \to \mathbb{B}^n \times \mathbb{B}^1$ defined on points by separating the last coordinate and let s_n^* be the induced map $\mathcal{F}(\square^n \times \mathbb{B}^1) \to \mathcal{F}(\square^{n+1})$. We have $s^*_{n-1} \circ d^*_{r,\epsilon} = d^*_{r,\epsilon} \circ s^*_n$ for all $1 \leq r \leq n$ and $\epsilon \in \{0,1\}$. We conclude that

$$s^*_{n-1} \circ \sum_{r=1}^{n}(-1)^r(d^*_{r,1} - d^*_{r,0}) + \sum_{r=1}^{n+1}(-1)^r(d^*_{r,1} - d^*_{r,0}) \circ (-s^*_n) =$$
$$= (-1)^n(d^*_{n+1,1} \circ s^*_n - d^*_{n+1,0} \circ s^*_n) = (-1)^n(i_1^* - i_0^*).$$

Therefore, the maps $\{(-1)^n s^*_n\}$ define a chain homotopy from i_0^* to i_1^* as maps of complexes $C^\sharp_\bullet\mathcal{F}(\square \times \mathbb{B}^1) \to C^\sharp_\bullet\mathcal{F}(\square)$.

We automatically deduce that if an inclusion $C'_\bullet\mathcal{F} \to C^\sharp_\bullet\mathcal{F}$ has a functorial retraction, then the maps $i_0^*, i_1^* \colon C'_\bullet\mathcal{F}(\square \times \mathbb{B}^1) \to C'_\bullet\mathcal{F}(\square)$ are also chain homotopic. $\square$

The following proposition is the rigid analytic analogue of [**3**, Theorem 2.23], or the cocubical analogue of [**5**, Lemma 2.5.31].

1.3.15. PROPOSITION. *Let $\mathcal{F}$ be a complex in* $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$*. Then* $\mathrm{Sing}^{\mathbb{B}^1}\mathcal{F}$ *is $\mathbb{B}^1$-local and $\mathbb{B}^1$-weak equivalent to $\mathcal{F}$ in* $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$.

PROOF. In order to prove that $\mathrm{Sing}^{\mathbb{B}^1}\mathcal{F}$ is $\mathbb{B}^1$-local in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ we need to check that each homology presheaf $H_n(\mathrm{Sing}^{\mathbb{B}^1}\mathcal{F})$ is homotopy-invariant. By means of [**5**, Proposition 2.2.37] it suffices to show that the maps $i_0^*, i_1^*\colon N_\bullet\mathcal{F}(\square\times\mathbb{B}^1)\to N_\bullet\mathcal{F}(\square)$ are chain homotopic, and this follows from Lemma 1.3.14.

We now prove that $\mathrm{Sing}^{\mathbb{B}^1}\mathcal{F}$ is $\mathbb{B}^1$-weak equivalent to $\mathcal{F}$. We first prove that the canonical map $a\colon\mathcal{F}\to\underline{\mathrm{Hom}}(\Lambda(\square^n),\mathcal{F})$ has an inverse up to homotopy for a fixed n. Consider the map $b\colon\underline{\mathrm{Hom}}(\Lambda(\square^n),\mathcal{F})\to\mathcal{F}$ induced by the zero section of $\square^n$. It holds that $b\circ a=\mathrm{id}$ and $a\circ b$ is homotopic to id via the map

$$H\colon \Lambda(\mathbb{B}^1)\otimes\underline{\mathrm{Hom}}(\Lambda(\square^n),\mathcal{F})\to\underline{\mathrm{Hom}}(\Lambda(\square^n),\mathcal{F})$$

which is deduced from the adjunction $(\Lambda(\mathbb{B}^1)\otimes\cdot,\underline{\mathrm{Hom}}(\Lambda(\mathbb{B}^1),\cdot))$ and the map

$$\underline{\mathrm{Hom}}(\Lambda(\square^n),\mathcal{F})\to\underline{\mathrm{Hom}}(\Lambda(\mathbb{B}^1\times\square^n),\mathcal{F})$$

defined via the homothety of $\mathbb{B}^1$ on $\square^n$. As $\mathbb{B}^1$-weak equivalences are stable under filtered colimits and cones, we also conclude that the total complex associated to the simple complex of $\underline{\mathrm{Hom}}(\Lambda(\square),\mathcal{F})$ is $\mathbb{B}^1$-equivalent to the one associated to the constant cubical object $\mathcal{F}$ (see for example the argument of [**5**, Corollary 2.5.36]) which is in turn quasi-isomorphic to $\mathcal{F}$. □

1.3.16. COROLLARY. *Let Λ be a $\mathbb{Q}$-algebra. For any $\mathcal{F}$ in* $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ *the localization $C^{\mathbb{B}^1}\mathcal{F}$ is quasi-isomorphic to* $\mathrm{Sing}^{\mathbb{B}^1}\mathcal{F}$ *and the localization $C^{\text{ét},\mathbb{B}^1}\mathcal{F}$ is quasi-isomorphic to* $\mathrm{Sing}^{\mathbb{B}^1}(C^{\text{ét}}\mathcal{F}^\bullet)$.

PROOF. The first claim follows from Proposition 1.3.15. We are left to prove that the complex $\mathrm{Sing}^{\mathbb{B}^1}(C^{\text{ét}}\mathcal{F}^\bullet)$ is ét-local. To this aim, we use the description given in Proposition 1.3.10 and we show that $\mathrm{Sing}^{\mathbb{B}^1}(C^{\text{ét}}\mathcal{F}^\bullet)$ is local with respect to shifts of maps $\Lambda(\mathcal{U}_\bullet)\to\Lambda(X)$ induced by bounded hypercoverings $\mathcal{U}_\bullet\to X$.

Fix a bounded hypercovering $\mathcal{U}_\bullet\to X$. From the isomorphisms $H_p\,\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_\bullet\times\square^q),C^{\text{ét}}\mathcal{F})\cong H_p\,\mathrm{Hom}_\bullet(\Lambda(X\times\square^q),C^{\text{ét}}\mathcal{F})$ valid for all p,q and a spectral sequence argument (see [**48**, Theorem 0.3]) we deduce $\mathbf{D}(\Lambda(X)[n],\mathrm{Sing}^{\mathbb{B}^1}C^{\text{ét}}\mathcal{F})\cong\mathbf{D}(\Lambda(\mathcal{U}_\bullet)[n],\mathrm{Sing}^{\mathbb{B}^1}C^{\text{ét}}\mathcal{F})$ for all n as wanted. □

We now investigate some of the natural Quillen functors which arise between the model categories introduced so far. We start by considering the natural inclusion of categories $\mathrm{RigSm}\to\widehat{\mathrm{RigSm}}$

1.3.17. PROPOSITION. *The inclusion* $\mathrm{RigSm}\hookrightarrow\widehat{\mathrm{RigSm}}$ *induces a Quillen adjunction*

$$\iota^*\colon \mathbf{Ch}_{\text{ét},\mathbb{B}^1}\mathbf{Psh}(\mathrm{RigSm})\rightleftarrows\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\mathbf{Psh}(\widehat{\mathrm{RigSm}})\colon\iota_*.$$

Moreover, the functor $\mathbb{L}\iota^*\colon\ \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)\to\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)$ *is fully faithful.*

PROOF. The first claim is a special instance of [**6**, Proposition 4.4.46].

We prove the second claim by showing that $\mathbb{R}\iota_*\mathbb{L}\iota^*$ is isomorphic to the identity. Let $\mathcal{F}$ be a cofibrant object in $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\mathbf{Psh}(\mathrm{RigSm})$. We need to prove that the map $\mathcal{F}\to\iota_*(\mathrm{Sing}^{\mathbb{B}^1}C^{\text{ét}}(\iota^*\mathcal{F}))$ is an $(\text{ét},\mathbb{B}^1)$-weak equivalence. Since ι_* commutes with $\mathrm{Sing}^{\mathbb{B}^1}$ we are left to prove that the map $\iota_*\iota^*\mathcal{F} = \mathcal{F} \to \iota_* C^{\text{ét}}(\iota^*\mathcal{F})$ is an ét-weak equivalence. This follows since ι_* preserves ét-weak equivalences, as it commutes with ét-sheafification. □

We are now interested in finding a convenient set of compact objects which generate the categories above, as triangulated categories with small sums. This will simplify many definitions and proofs in what follows.

1.3.18. PROPOSITION. *The category* $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(K)$ *[resp.* $\widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét},\mathbb{B}^1}(K)$*] is compactly generated (as a triangulated category with small sums) by motives* $\Lambda(X)$ *associated to rigid varieties* X *which are in* RigSm^{gc} *[resp.* $\widehat{\text{Rig}}\text{Sm}^{\text{gc}}$*].*

PROOF. The statements are analogous, and we only consider the case of the category $\widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét},\mathbb{B}^1}(K)$. It is clear that the set of functors $H_i \operatorname{Hom}_\bullet(\Lambda(X), \cdot)$ detect quasi-isomorphisms between étale local objects, by letting X vary in $\widehat{\text{Rig}}\text{Sm}^{\text{gc}}$ and i vary in $\mathbb{Z}$. We are left to prove that the motives $\Lambda(X)$ with X in $\widehat{\text{Rig}}\text{Sm}^{\text{gc}}$ are compact. Since $\Lambda(X)$ is compact in $\mathbf{D}(\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm}^{\text{gc}}))$ and $\text{Sing}^{\mathbb{B}^1}$ commutes with direct sums, it suffices to prove that if $\{\mathcal{F}_i\}_{i\in I}$ is a family of ét-local complexes, then also $\bigoplus_i \mathcal{F}_i$ is ét-local. If I is finite, the claim follows from the isomorphisms $H_{-n} \operatorname{Hom}_\bullet(X, \bigoplus_i \mathcal{F}_i) \cong \bigoplus_i \mathbb{H}^n(X, \mathcal{F}_i) \cong \mathbb{H}^n(X, \bigoplus_i \mathcal{F}_i)$. A coproduct over an arbitrary family is a filtered colimit of finite coproducts, hence the claim follows from the stability of ét-local complexes under filtered colimits [**6**, Proposition 4.5.62]. □

1.3.19. REMARK. The above proof shows that the statement of Proposition 1.3.18 holds true without any assumptions on Λ under the condition that all varieties X have finite cohomological dimension with respect to the étale topology.

We now introduce the category of motives associated to smooth perfectoids, using the same formalism as before. In this category, the canonical choice of the "interval object" for defining homotopies is the perfectoid ball $\widehat{\mathbb{B}}^1$.

1.3.20. EXAMPLE. The perfectoid ball $\widehat{\mathbb{B}}^1 = \operatorname{Spa}(K\langle\chi^{1/p^\infty}\rangle, K^\circ\langle\chi^{1/p^\infty}\rangle)$ is an interval object with respect to the natural multiplication μ and maps i_0 and i_1 induced by the substitution $\chi^{1/p^h} \mapsto 0$ and $\chi^{1/p^h} \mapsto 1$ respectively.

The perfectoid variety $\widehat{\mathbb{B}}^1$ naturally lives in $\widehat{\text{Rig}}\text{Sm}$ and has good coordinates by Proposition 1.2.12. It can therefore be used to define another homotopy category out of $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm})$ and $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm}^{\text{gc}})$.

1.3.21. COROLLARY. *The following pairs of model categories are Quillen equivalent.*

- $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\text{PerfSm})$ *and* $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\text{PerfSm}^{\text{gc}})$.
- $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1}\,\mathbf{Psh}(\text{PerfSm})$ *and* $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1}\,\mathbf{Psh}(\text{PerfSm}^{\text{gc}})$.
- $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm})$ *and* $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm}^{\text{gc}})$.
- $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm})$ *and* $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm}^{\text{gc}})$.

PROOF. It suffices to apply Proposition 1.3.3 to the sites with interval $(\text{PerfSm}, \text{ét}, \widehat{\mathbb{B}}^1)$ and $(\widehat{\text{Rig}}\text{Sm}, \text{ét}, \widehat{\mathbb{B}}^1)$ where $\mathbf{C}'$ is in both cases the subcategory of affinoid rigid varieties with good coordinates. □

1.3.22. DEFINITION. For $\eta \in \{\text{ét}, \widehat{\mathbb{B}}^1, (\text{ét}, \widehat{\mathbb{B}}^1)\}$ we say that a map in $\mathbf{Ch}\,\mathbf{Psh}(\text{PerfSm})$ [resp. $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\text{Rig}}\text{Sm})$] is a η*-weak equivalence* if it is a weak equivalence in the model structure $\mathbf{Ch}_\eta \mathbf{Psh}(\mathrm{PerfSm})$ [resp. $\mathbf{Ch}_\eta \mathbf{Psh}(\widehat{\mathrm{RigSm}})$]. We say that an object $\mathcal{F}$ of the derived category $\mathbf{D} = \mathbf{D}(\mathbf{Psh}(\mathrm{PerfSm}))$ [resp. $\mathbf{D} = \mathbf{D}(\mathbf{Psh}(\widehat{\mathrm{RigSm}}))$] is *$\eta$-local* if the functor $\mathrm{Hom}_{\mathbf{D}}(\cdot, \mathcal{F})$ sends maps in S_η to isomorphisms. This amounts to say that $\mathcal{F}$ is quasi-isomorphic to a η-fibrant object. The triangulated homotopy category associated to the localization $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1} \mathbf{Psh}(\mathrm{PerfSm})$ [resp. $\mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1} \mathbf{Psh}(\widehat{\mathrm{RigSm}})$] will be denoted by $\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$ [resp. $\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K, \Lambda)$]. We will omit Λ whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda(X)$.

We recall one of the main results of Scholze [**42**], reshaped in our derived homotopical setting. It will constitute the bridge to pass from characteristic p to characteristic 0. We recall that as summarized in Theorem 1.1.19 there is an equivalence of categories between perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras, extending to an equivalence between the categories of perfectoid spaces over K and over $K^\flat$ (see [**42**, Proposition 6.17]). We refer to this equivalence as the *tilting equivalence*.

1.3.23. PROPOSITION. *There exists an equivalence of triangulated categories*

$$(-)^\sharp \colon \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat) \rightleftarrows \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K) :(-)^\flat$$

induced by the tilting equivalence.

PROOF. The tilting equivalence induces an equivalence of the étale sites on perfectoid spaces over K and over $K^\flat$ (see [**42**, Theorem 7.12]). Moreover $(\widehat{\mathbb{T}}^n)^\flat = \widehat{\mathbb{T}}^n$ and $(\widehat{\mathbb{B}}^n)^\flat = \widehat{\mathbb{B}}^n$. It therefore induces an equivalence of sites with interval $(\mathrm{PerfSm}/K, \text{ét}, \widehat{\mathbb{B}}^1) \cong (\mathrm{PerfSm}/K^\flat, \text{ét}, \widehat{\mathbb{B}}^1)$ hence the claim. □

We now investigate the triangulated functor between the categories of motives induced by the natural embedding $\mathrm{PerfSm} \to \widehat{\mathrm{RigSm}}$ in the same spirit of what we did previously in Proposition 1.3.17.

1.3.24. PROPOSITION. *The inclusion* $\mathrm{PerfSm} \hookrightarrow \widehat{\mathrm{RigSm}}$ *induces a Quillen adjunction*

$$j^* \colon \mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1} \mathbf{Psh}(\mathrm{PerfSm}) \rightleftarrows \mathbf{Ch}_{\text{ét},\widehat{\mathbb{B}}^1} \mathbf{Psh}(\widehat{\mathrm{RigSm}}) :j_*.$$

Moreover, the functor $\mathbb{L}j^* \colon \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K) \to \widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$ *is fully faithful.*

PROOF. The result follows in the same way as Proposition 1.3.17. □

Also in this framework, the $\widehat{\mathbb{B}}^1$-localization has a very explicit construction. Most proofs are straightforward analogues of those relative to the $\mathbb{B}^1$-localizations, and will therefore be omitted.

1.3.25. DEFINITION. We denote by $\widehat{\square}$ the Σ-enriched cocubical object (see [**4**, Appendix A]) defined by putting $\widehat{\square}^n = \widehat{\mathbb{B}}^n = \mathrm{Spa}\, K\langle \tau_1^{1/p^\infty}, \ldots, \tau_n^{1/p^{1/\infty}} \rangle$ and considering the morphisms $d_{r,\epsilon}$ induced by the maps $\widehat{\mathbb{B}}^n \to \widehat{\mathbb{B}}^{n+1}$ corresponding to the substitution $\tau_r^{1/p^h} = \epsilon$ for $\epsilon \in \{0, 1\}$ and the morphisms p_r induced by the projections $\widehat{\mathbb{B}}^n \to \widehat{\mathbb{B}}^{n-1}$. For any complex of presheaves $\mathcal{F}$ we let $\mathrm{Sing}^{\widehat{\mathbb{B}}^1} \mathcal{F}$ be the total complex of the simple complex associated to $\underline{\mathrm{Hom}}(\widehat{\square}, \mathcal{F})$. It sends the object X to the total complex of the simple complex associated to $\mathcal{F}(X \times \widehat{\square})$.

1.3.26. PROPOSITION. *Let $\mathcal{F}$ be a complex in* $\mathbf{Ch}\, \mathbf{Psh}(\mathrm{PerfSm})$ *[resp.* $\mathbf{Ch}\, \mathbf{Psh}(\widehat{\mathrm{RigSm}})$*]. Then* $\mathrm{Sing}^{\widehat{\mathbb{B}}^1} \mathcal{F}$ *is $\widehat{\mathbb{B}}^1$-local and $\widehat{\mathbb{B}}^1$-weak equivalent to $\mathcal{F}$.*

PROOF. The fact that $\mathrm{Sing}^{\widehat{\mathbb{B}}^1}\mathcal{F}$ is $\widehat{\mathbb{B}}^1$-local in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ can be deduced by Lemma 1.3.27 and Lemma 1.3.28. We are left to prove that $\mathrm{Sing}^{\widehat{\mathbb{B}}^1}\mathcal{F}$ is $\widehat{\mathbb{B}}^1$-weak equivalent to $\mathcal{F}$ and this follows in the same way as in the proof of Proposition 1.3.15. □

The following lemmas are used in the previous proof.

1.3.27. LEMMA. *A presheaf $\mathcal{F}$ in $\mathbf{Psh}(\mathrm{Sm}\,\mathrm{Perf})$ [resp. in $\mathbf{Psh}(\widehat{\mathrm{RigSm}})$] is $\widehat{\mathbb{B}}^1$-invariant if and only if $i_0^* = i_1^* \colon \mathcal{F}(X \times \widehat{\mathbb{B}}^1) \to \mathcal{F}(X)$ for all X in $\mathrm{Sm}\,\mathrm{Perf}$ [resp. in $\widehat{\mathrm{RigSm}}$].*

PROOF. This follows in the same way as [**36**, Lemma 2.16]. □

1.3.28. LEMMA. *For any presheaf $\mathcal{F}$ the two maps of cubical sets $i_0^*, i_1^* \colon \mathcal{F}(\widehat{\square}\times\widehat{\mathbb{B}}^1) \to \mathcal{F}(\widehat{\square})$ induce chain homotopic maps on the associated simple and normalized complexes.*

PROOF. This follows in the same way as Lemma 1.3.14. □

1.3.29. COROLLARY. *Let $\mathcal{F}$ be in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{PerfSm})$ [resp. in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$] the $(\text{ét}, \widehat{\mathbb{B}}^1)$-localization $C^{\text{ét},\widehat{\mathbb{B}}^1}\mathcal{F}$ is quasi-isomorphic to $\mathrm{Sing}^{\widehat{\mathbb{B}}^1}(C^{\text{ét}}\mathcal{F})$.*

PROOF. This follows in the same way as Corollary 1.3.16. □

1.3.30. PROPOSITION. *The category $\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K)$ [resp. $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$] is compactly generated (as a triangulated category with small sums) by motives $\Lambda(X)$ associated to rigid varieties X which are in $\mathrm{PerfSm}^{\mathrm{gc}}$ [resp. $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$].*

PROOF. This follows in the same way as Proposition 1.3.18. □

1.3.31. REMARK. The above proof shows that the statement of Proposition 1.3.30 holds true without any assumptions on Λ under the condition that all varieties X have finite cohomological dimension with respect to the étale topology.

So far, we have defined two different Bousfield localizations on complexes of presheaves on $\widehat{\mathrm{RigSm}}$ according to two different choices of intervals: $\mathbb{B}^1$ and $\widehat{\mathbb{B}}^1$. We remark that the second constitutes a further localization of the first, in the following sense.

1.3.32. PROPOSITION. *$\mathbb{B}^1$-weak equivalences in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}})$ are $\widehat{\mathbb{B}}^1$-weak equivalences.*

PROOF. It suffices to prove that $X \times \mathbb{B}^1 \to X$ induces a $\widehat{\mathbb{B}}^1$-weak equivalence, for any variety X in $\widehat{\mathrm{RigSm}}$. This follows as the multiplicative homothety $\widehat{\mathbb{B}}^1 \times \mathbb{B}^1 \to \mathbb{B}^1$ induces a homotopy between the zero map and the identity on $\mathbb{B}^1$. □

1.3.33. COROLLARY. *The category $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$ is equivalent to the full triangulated subcategory of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)$ formed by $\widehat{\mathbb{B}}^1$-local objects.*

PROOF. Because of Proposition 1.3.32, the triangulated category $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$ coincides with the localization of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)$ with respect to the set generated by the maps $\Lambda(\widehat{\mathbb{B}}^1_X)[n] \to \Lambda(X)[n]$ as X varies in $\widehat{\mathrm{RigSm}}$ and n in $\mathbb{Z}$. □

We end this section by recalling the definition of rigid motives with transfers. The notion of finite correspondence plays an important role in Voevodsky's theory of motives. In the case of rigid varieties over a field K correspondences give rise to the category $\mathrm{RigCor}(K)$ as defined in [**5**, Definition 2.2.27]. For further details, we refer to Definition 2.2.3.

1.3.34. DEFINITION. Additive presheaves over $\mathrm{RigCor}(K)$ are called *presheaves with transfers*, and the category they form is denoted by $\mathbf{PST}(\mathrm{RigSm}/K, \Lambda)$ or simply by $\mathbf{PST}(\mathrm{RigSm})$ when the context allows it.

By [**5**, Definition 2.5.15], the projective model category $\mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm})$ admits a Bousfield localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm})$ with respect to the union of the class of maps $\mathcal{F} \to \mathcal{F}'$ inducing isomorphisms on the ét-sheaves associated to $H_i(\mathcal{F})$ and $H_i(\mathcal{F}')$ for all $i \in \mathbb{Z}$ and the set of all maps $\Lambda(\mathbb{B}^1_X)[i] \to \Lambda(X)[i]$ as X varies in RigSm and i varies in $\mathbb{Z}$.

1.3.35. DEFINITION. The triangulated homotopy category associated to the localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm})$ will be denoted by $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$. We will omit Λ from the notation whenever the context allows it. The image of a variety X in will be denoted by $\Lambda_{\mathrm{tr}}(X)$.

1.3.36. REMARK. Since Λ is a $\mathbb{Q}$-algebra, one can equivalently consider the Nisnevich topology in the definition above and obtain a homotopy category $\mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{Nis}}(K, \Lambda)$ which is equivalent to $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K, \Lambda)$.

1.3.37. REMARK. The faithful embedding of categories $\mathrm{RigSm} \to \mathrm{RigCor}$ induces a Quillen adjunction (see [**5**, Lemma 2.5.18]):

$$a_{tr}\colon \mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}) \rightleftarrows \mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}) :o_{tr}$$

such that $a_{tr}\Lambda(X) = \Lambda_{tr}(X)$ for any $X \in \mathrm{RigSm}$ and o_{tr} is the functor of forgetting transfers. These functors induce an adjoint pair:

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) \rightleftarrows \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K) :\mathbb{R}o_{tr}.$$

1.4. Motivic interpretation of approximation results

In all this section, K is a perfectoid field of arbitrary characteristic. We begin by presenting an approximation result whose proof is differed to Appendix A.

1.4.1. PROPOSITION. *Let $X = \varprojlim_h X_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. Let also Y be an affinoid rigid variety endowed with an étale map $Y \to \mathbb{B}^m$. For a given finite set of maps $\{f_1, \ldots, f_N\}$ in $\mathrm{Hom}(X \times \mathbb{B}^n, Y)$ we can find corresponding maps $\{H_1, \ldots, H_N\}$ in $\mathrm{Hom}(X \times \mathbb{B}^n \times \mathbb{B}^1, Y)$ and an integer $\bar{h}$ such that:*

(1) For all $1 \leq k \leq N$ it holds $i_0^* H_k = f_k$ and $i_1^* H_k$ factors over the canonical map $X \to X_{\bar{h}}$.
(2) If $f_k \circ d_{r,\epsilon} = f_{k'} \circ d_{r,\epsilon}$ for some $1 \leq k, k' \leq N$ and some $(r, \epsilon) \in \{1, \ldots, n\} \times \{0, 1\}$ then $H_k \circ d_{r,\epsilon} = H_{k'} \circ d_{r,\epsilon}$.
(3) If for some $1 \leq k \leq N$ and some $h \in \mathbb{N}$ the map $f_k \circ d_{1,1} \in \mathrm{Hom}(X \times \mathbb{B}^{n-1}, Y)$ lies in $\mathrm{Hom}(X_h \times \mathbb{B}^{n-1}, Y)$ then the element $H_k \circ d_{1,1}$ of $\mathrm{Hom}(X \times \mathbb{B}^{n-1} \times \mathbb{B}^1, Y)$ is constant on $\mathbb{B}^1$ equal to $f_k \circ d_{1,1}$.

The statement above has the following interpretation in terms of complexes.

1.4.2. PROPOSITION. *Let $X = \varprojlim_h X_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ and let Y be in $\mathrm{RigSm}^{\mathrm{gc}}$. The natural map*

$$\phi\colon \varinjlim_h (\mathrm{Sing}^{\mathbb{B}^1} \Lambda(Y))(X_h) \to (\mathrm{Sing}^{\mathbb{B}^1} \Lambda(Y))(X)$$

is a quasi-isomorphism.

PROOF. We need to prove that the natural map

$$\phi\colon \varinjlim_h C_\bullet \Lambda \operatorname{Hom}(X_h \times \square, Y) \to C_\bullet \Lambda \operatorname{Hom}(X \times \square, Y)$$

defines bijections on homology groups.

We start by proving surjectivity. As $\square$ is a Σ-enriched cocubical object, the complexes above are quasi-isomorphic to the associated normalized complexes $N_\bullet$ which will then be considered instead. Suppose that $\beta \in \Lambda \operatorname{Hom}(X \times \square^n, Y)$ defines a cycle in N_n i.e. $\beta \circ d_{r,\epsilon} = 0$ for $1 \leq r \leq n$ and $\epsilon \in \{0,1\}$. This means that $\beta = \sum \lambda_k f_k$ with $\lambda_k \in \Lambda$, $f_k \in \operatorname{Hom}(X \times \square^n, Y)$ and $\sum \lambda_k f_k \circ d_{r,\epsilon} = 0$. This amounts to say that for every k, r, ϵ the sum $\sum \lambda_{k'}$ over the indices k' such that $f_{k'} \circ d_{r,\epsilon} = f_k \circ d_{r,\epsilon}$ is zero. By Proposition 1.4.1, we can find an integer h and maps $H_k \in \operatorname{Hom}(X \times \square^n \times \mathbb{B}^1, Y)$ such that $i_0^* H = f_k$, $i_1^* H = \phi(\tilde{f}_k)$ with $\tilde{f}_k \in \operatorname{Hom}(X_h \times \square^n, Y)$ and $H_k \circ d_{r,\epsilon} = H_{k'} \circ d_{r,\epsilon}$ whenever $f_k \circ d_{r,\epsilon} = f_{k'} \circ d_{r,\epsilon}$. If we denote by H the cycle $\sum \lambda_k H_k \in \Lambda \operatorname{Hom}(X \times \square^n \times \mathbb{B}^1, Y)$ we therefore have $d^*_{r,\epsilon} H = 0$ for all r, ϵ.

By Lemma 1.3.14, we conclude that $i_1^* H$ and $i_0^* H$ define the same homology class, and therefore β defines the same class as $i_1^* H$ which is the image of a class in $\Lambda \operatorname{Hom}(X_h \times \square^n, Y)$ as wanted.

We now turn to injectivity. Consider an element $\alpha \in \Lambda \operatorname{Hom}(X_0 \times \square^n, Y)$ such that $\alpha \circ d_{r,\epsilon} = 0$ for all r, ϵ and suppose there exists an element $\beta = \sum \lambda_i f_i \in \Lambda \operatorname{Hom}(X \times \square^{n+1}, Y)$ such that $\beta \circ d_{r,0} = 0$ for $1 \leq r \leq n+1$, $\beta \circ d_{r,1} = 0$ for $2 \leq r \leq n+1$ and $\beta \circ d_{1,1} = \phi(\alpha)$. Again, by Proposition 1.4.1, we can find an integer $\bar{h}$ and maps $H_k \in \operatorname{Hom}(X \times \square^{n+1} \times \mathbb{B}^1, Y)$ such that $H := \sum \lambda_k H_k$ satisfies $i_1^* H = \phi(\gamma)$ for some $\gamma \in \Lambda \operatorname{Hom}(X_{\bar{h}} \times \square^{n+1}, Y)$, $H \circ d_{r,0} = 0$ for $1 \leq r \leq n+1$, $H \circ d_{r,1} = 0$ for $2 \leq r \leq n+1$ and $H \circ d_{1,1}$ is constant on $\mathbb{B}^1$ and coincides with $\phi(\alpha)$. We conclude that $\gamma \in N_n$ and $d\gamma = \alpha$. In particular, $\alpha = 0$ in the homology group, as wanted. $\square$

1.4.3. COROLLARY. *Let $\mathcal{F}$ be a projectively cofibrant complex in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}})$. For any $X = \varprojlim_h X_h$ in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ the natural map*

$$\phi\colon \varinjlim_h (\operatorname{Sing}^{\mathbb{B}^1} \mathcal{F})(X_h) \to (\operatorname{Sing}^{\mathbb{B}^1} \iota^* \mathcal{F})(X)$$

is a quasi-isomorphism.

PROOF. As homology commutes with filtered colimits, by means of Remark 1.3.2 we can assume that $\mathcal{F}$ is a bounded above complex formed by sums of representable presheaves. For any X in $\widehat{\mathrm{RigSm}}$ the cohomology of $\operatorname{Sing}^{\mathbb{B}^1} \mathcal{F}(X)$ coincides with the cohomology of the total complex associated to $C_\bullet(\mathcal{F}(X \times \square))$. The result then follows from Proposition 1.4.2 and the convergence of the spectral sequence associated to the double complex above, which is concentrated in one quadrant. $\square$

The following technical proposition is actually a crucial point of our proof, as it allows some explicit computations of morphisms in the category $\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét}}(K)$.

1.4.4. PROPOSITION. *Let $\mathcal{F}$ be a cofibrant $(\mathbb{B}^1, \text{ét})$-fibrant complex in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}^{\mathrm{gc}})$. Then $\operatorname{Sing}^{\mathbb{B}^1}(\iota^* \mathcal{F})$ is $(\mathbb{B}^1, \text{ét})$-local in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$.*

PROOF. The difficulty lies in showing that the object $\operatorname{Sing}^{\mathbb{B}^1}(\iota^* \mathcal{F})$ is ét-local. By Propositions 1.3.11 and 1.3.15, it suffices to prove that $\operatorname{Sing}^{\mathbb{B}^1}(\iota^* \mathcal{F})$ is local with respect to the étale-Cech hypercoverings $\mathcal{U}_\bullet \to X$ in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ of $X = \varprojlim_h X_h$ descending at finite level. Let $\mathcal{U}_\bullet \to X$ be one of them. Without loss of generality, we assume that it descends to an étale covering of X_0. In particular we conclude that $\mathcal{U}_n = \varprojlim_h \mathcal{U}_{nh}$ is a disjoint union of objects in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$.

We need to show that $\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_\bullet), \mathrm{Sing}^{\mathbb{B}^1}(\iota^*\mathcal{F}))$ is quasi-isomorphic to $\mathrm{Sing}^{\mathbb{B}^1}(\iota^*\mathcal{F})(X)$. Using Corollary 1.4.3, we conclude that the complex $(\mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})(\mathcal{U}_n)$ is quasi-isomorphic to $\varinjlim_h(\mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})(\mathcal{U}_{nh})$ for each $n \in \mathbb{N}$. Passing to the homotopy limit on n on both sides, we get that $\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_\bullet), \mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})$ is quasi-isomorphic to $\varinjlim_h \mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_{\bullet h}), \mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})$. Using again Corollary 1.4.3, we also obtain that the complex $(\mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})(X)$ is quasi-isomorphic to $\varinjlim_h(\mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F})(X_h)$.

From the exactness of $\varinjlim$ it suffices then to prove that the maps

$$\mathrm{Hom}_\bullet(\Lambda(\mathcal{U}_{\bullet h}), \mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}) \to \mathrm{Hom}_\bullet(\Lambda(X_h), \mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F})$$

are quasi-isomorphisms. This follows once we show that the complex $\mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}$ is ét-local.

We point out that since $\mathcal{F}$ is $\mathbb{B}^1$-local, then the canonical map $\mathcal{F} \to \mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}$ is a quasi-isomorphism. As $\mathcal{F}$ is ét-local we conclude that $\mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}$ also is, hence the claim. □

We are finally ready to state the main result of this section.

1.4.5. PROPOSITION. *Let $X = \varprojlim_h X_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. For any complex of presheaves $\mathcal{F}$ on $\mathrm{RigSm}^{\mathrm{gc}}$ the natural map*

$$\varinjlim_h \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)(\Lambda(X_h), \mathcal{F}) \to \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)(\Lambda(X), \mathbb{L}\iota^*\mathcal{F})$$

is an isomorphism.

PROOF. Since any complex $\mathcal{F}$ has a fibrant-cofibrant replacement in a model category, we can assume that $\mathcal{F}$ is cofibrant and $(\text{ét}, \mathbb{B}^1)$-fibrant. Since it is $\mathbb{B}^1$-local, it is quasi-isomorphic to $\mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}$. By Corollary 1.4.3, for any integer i one has

$$\varinjlim_h \mathrm{Hom}(\Lambda(X_h)[i], \mathrm{Sing}^{\mathbb{B}^1}\,\mathcal{F}) \cong \mathrm{Hom}(\Lambda(X)[i], \mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F}).$$

As $\Lambda(X)$ is a cofibrant object in $\mathbf{Ch}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ and $\mathrm{Sing}^{\mathbb{B}^1}\,\iota^*\mathcal{F}$ is a $(\mathbb{B}^1, \text{ét})$-local replacement of $\mathcal{F}$ in $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\widehat{\mathrm{RigSm}}^{\mathrm{gc}})$ by Proposition 1.4.4, we conclude that the previous isomorphism can be rephrased in the following way:

$$\varinjlim_h \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)(\Lambda(X_h)[i], \mathcal{F}) \cong \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)(\Lambda(X)[i], \mathbb{L}\iota^*\mathcal{F})$$

proving the claim. □

1.5. The de-perfectoidification functor in characteristic 0

The results proved in Section 1.4 are valid both for $\mathrm{char}\,K = 0$ and $\mathrm{char}\,K = p$. On the contrary, the results of this section require that $\mathrm{char}\,K = 0$. We will present later their variant for the case $\mathrm{char}\,K = p$.

We start by considering the adjunction between motives with and without transfers (see Remark 1.3.37). Thanks to the following crucial theorem, we are allowed to add or ignore transfers according to the situation.

1.5.1. THEOREM. *Suppose that $\mathrm{char}\,K = 0$. The functors (a_{tr}, o_{tr}) induce an equivalence:*

$$\mathbb{L}a_{tr} \colon \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) \rightleftarrows \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K) : \mathbb{R}o_{tr}.$$

We postpone the proof of this fact to the second chapter, see Theorem 2.3.3.

1.5.2. REMARK. The proof of the statement above uses in a crucial way the fact that the ring of coefficients Λ is a $\mathbb{Q}$-algebra. This is the main reason of our assumption on Λ.

1.5.3. PROPOSITION. *Suppose* $\operatorname{char} K = 0$. *Let* $X = \varprojlim_h X_h$ *be in* $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. *If h is big enough, then the map* $\Lambda(X_{h+1}) \to \Lambda(X_h)$ *is an isomorphism in* $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$.

PROOF. By means of Proposition 1.5.1, we can equally prove the statement in the category $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$. We claim that we can also make an arbitrary finite field extension L/K. Indeed the transpose of the natural map $Y_L \to Y$ is a correspondence from Y to Y_L. Since Λ is a $\mathbb{Q}$-algebra, we conclude that $\Lambda_{\mathrm{tr}}(Y)$ is a direct factor of $\Lambda_{\mathrm{tr}}(Y_L) = \mathbb{L}e_\sharp \Lambda_{\mathrm{tr}}(Y_L)$ for any variety Y where $\mathbb{L}e_\sharp$ is the functor $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(L) \to \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$ induced by restriction of scalars. In particular, if $\Lambda_{\mathrm{tr}}((X_{h+1})_L) \to \Lambda_{\mathrm{tr}}((X_h)_L)$ is an isomorphism in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(L)$ then $\Lambda_{\mathrm{tr}}((X_{h+1})_L) \to \Lambda_{\mathrm{tr}}((X_h)_L)$ is an isomorphism in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$ and therefore also $\Lambda_{\mathrm{tr}}(X_{h+1}) \to \Lambda_{\mathrm{tr}}(X_h)$ is.

By means of Lemma [**5**, 1.1.50], we can suppose that $X_0 = \mathrm{Spa}(R_0, R_0^\circ)$ with $R_0 = S\langle\sigma,\tau\rangle/(P(\sigma,\tau))$ where $S = \mathcal{O}(\mathbb{T}^M)$, $\sigma = (\sigma_1,\ldots,\sigma_N)$ is a N-tuple of coordinates, $\tau = (\tau_1,\ldots,\tau_m)$ is a m-tuple of coordinates and P is a set of m polynomials in $S[\sigma,\tau]$ with $\det(\frac{\partial P}{\partial \tau}) \in R_0^\times$. In particular $X_1 = \mathrm{Spa}(R_1, R_1^\circ)$ with $R_1 = S\langle\sigma,\tau\rangle/(P(\sigma^p,\tau))$ and the map $f\colon X_1 \to X_0$ is induced by $\sigma \mapsto \sigma^p$, $\tau \mapsto \tau$. Since the map f is finite and surjective, we can also consider the transpose correspondence $f^T \in \mathrm{RigCor}(X_0, X_1)$. The composition $f \circ f^T$ is associated to the correspondence $X_0 \stackrel{f}{\leftarrow} X_1 \stackrel{f}{\to} X_0$ which is the cycle $\deg(f)X_0 = p^N \cdot \mathrm{id}_{X_0}$. The composition $f^T \circ f$ is associated to the correspondence $X_1 \stackrel{p_1}{\leftarrow} X_1 \times_{X_0} X_1 \stackrel{p_2}{\to} X_1$. Since $\mathbb{T}^N\langle\sigma^{1/p}\rangle \times_{\mathbb{T}^N} \mathbb{T}^N\langle\sigma^{1/p}\rangle \cong \mathbb{T}^N\langle\sigma^{1/p}\rangle \times \mu_p^N$ we conclude that the above correspondence is $X_1 \stackrel{p_1}{\leftarrow} X_1 \times (\mu_p)^N \stackrel{\eta}{\to} X_1$ where η is induced by the multiplication map $\mathbb{T}^N \times \mu_p^N \to \mathbb{T}^N$. Up to a finite field extension, we can assume that K has the p-th roots of unity. The above correspondence is then equal to $\sum f_\zeta$ where each f_ζ is a map $X_1 \to X_1$ defined by $\sigma_i \mapsto \zeta_i\sigma_i$, $\tau \mapsto \tau$ for each N-tuple $\zeta = (\zeta_i)$ of p-th roots of unity. If we prove that each f_ζ is homotopically equivalent to id_{X_1} then we get $\frac{1}{p^N} f^T \circ f = \mathrm{id}$, $f \circ \frac{1}{p^N} f^T = \mathrm{id}$ in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}$ as wanted.

We are left to find a homotopy between id and f_ζ for a fixed $\zeta = (\zeta_1,\ldots,\zeta_n)$ up to considering higher indices h. For the sake of clarity, we consider them as maps $\mathrm{Spa}\,\bar{R}_1 \to \mathrm{Spa}\,R_1$ where we put $\bar{R}_h = S\langle\bar\sigma,\bar\tau\rangle/(P(\bar\sigma^{p^h},\bar\tau))$ for any integer h. The first map is induced by $\sigma \mapsto \bar\sigma$, $\tau \mapsto \bar\tau$ and the second induced by $\sigma \mapsto \zeta\bar\sigma$, $\tau \mapsto \bar\tau$. Let $F_h = \sum_n a_n(\sigma - \bar\sigma)^n$ be the unique array of formal power series in $\bar{R}_h[[\sigma - \bar\sigma]]$ centered in $\bar\sigma$ associated to the polynomials $P(\sigma^{p^h},\tau)$ in $\bar{R}_h[\sigma,\tau]$ via Corollary A.1.2. Let also ϕ_h be the map $\bar{R}_h \to \bar{R}_{h+1}$. From the formal equalities $P(\sigma^{p^{(h+1)}}, F_{h+1}(\sigma)) = 0$, $P(\sigma^{p^h}, \phi(F_h(\sigma))) = \phi_h(P(\sigma^{p^h}, F_h(\sigma))) = 0$ and the uniqueness of F_{h+1} we deduce $F_{h+1}(\sigma) = \phi_h(F_h(\sigma^p))$.

We therefore have

$$F_{h+1}(\sigma) = \sum_n \phi_h(a_n)(\sigma^p - \bar\sigma^p)^n =$$
$$= \sum_n \phi_h(a_n)\left((\sigma-\bar\sigma)^{p-1} + \sum_{j=1}^{p-1}\binom{p}{j}(\sigma-\bar\sigma)^{j-1}\bar\sigma^{p-j}\right)^n(\sigma-\bar\sigma)^n$$

The expression

$$Q(x) = x^p + \sum_{j=1}^{p-1} \binom{p}{j} x^j \bar{\sigma}^{p-j}$$

is a polynomial in x and it easy to show that the mapping $x \mapsto Q(x)$ extends to a map $\bar{R}_{h+1}\langle x\rangle \to \bar{R}_{h+1}\langle x\rangle$ since $\bar{R}_{h+1}$ is complete and p divides $\binom{p}{j}$ for $1 < j < p$. We deduce that we can read off the convergence in the circle of radius 1 around $\bar{\sigma}$ and the values of F_{h+1} on its expression given above.

We remark that the norm of $Q(\sigma - \bar{\sigma})$ in the circle of radius $\rho \leq 1$ around $\bar{\sigma}$ is bounded by $\max\{\rho^p, |p|\} \leq \max\{\rho, |p|\}$. Suppose that F_h converges in a circle of radius ρ with $0 < \rho \leq 1$ around $\bar{\sigma}$ and in there it takes values in power-bounded elements. By the expression above, the same holds true for F_{h+1} in the circle of radius $\min\{\rho|p|^{-1}, 1\}$ around $\bar{\sigma}$. By induction we conclude that for a sufficiently big h the power series F_h converges in a circle of radius $\delta > |p|^{1/p}$ around $\bar{\sigma}$ and its values in it are power bounded. Up to rescaling indices, we suppose that this holds for $h = 1$.

The value $|p|^{1/p}$ is larger than $|\zeta_i - 1|$ for all i since $(\zeta_i - 1)^p$ is divisible by p. Also, from the relation $F_{h+1}(\sigma) = \phi_h(F_h(\sigma^p))$ we conclude $F_1(\zeta\bar{\sigma}) = F_1(\bar{\sigma}) = \bar{\tau}$. Therefore, the map

$$\begin{aligned} X_1 = \operatorname{Spa}(S\langle\sigma,\tau\rangle/P(\sigma^p,\tau)) \leftarrow X_1 \times \mathbb{B}^1 &= \operatorname{Spa}(S\langle\bar{\sigma},\bar{\tau},\chi\rangle/(P(\bar{\sigma}^p,\bar{\tau})) \\ (\sigma_i,\tau_j) &\mapsto (\bar{\sigma}_i + (\zeta_i - 1)\bar{\sigma}_i\chi, F_1(\bar{\sigma} + (\zeta - 1)\bar{\sigma}\chi)) \end{aligned}$$

is a well defined map, inducing a homotopy between id_{X_1} and f_ζ as claimed. □

It cannot be expected that all maps $X_{h+1} \to X_h$ are isomorphisms in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$: consider for example $X_0 = \mathbb{T}^1\langle\tau^{1/p}\rangle \to \mathbb{T}^1$. Then X_0 is a connected variety, while X_1 is not. That said, there is a particular class of objects $X = \varprojlim_h X_h$ in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ for which this happens: this is the content of the following proposition which nevertheless will not be used in the following.

We recall that a presentation $X = \varprojlim_h X_h$ of an object in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ is of good reduction if the map $X_0 \to \mathbb{T}^N \times \mathbb{T}^M$ has a formal model which is an étale map over $\operatorname{Spf} K^\circ\langle\underline{\upsilon}^{\pm1}, \underline{\nu}^{\pm1}\rangle$ and is of potentially good reduction if this happens after base change by a separable finite field extension L/K.

1.5.4. PROPOSITION. *Let* $\operatorname{char} K = 0$ *and let* $X = \varprojlim_h X_h$ *be a presentation of a variety in* $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ *of potentially good reduction. The maps* $\Lambda(X_{h+1}) \to \Lambda(X_h)$ *are isomorphisms in* $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$ *for all h.*

PROOF. If the map $X_0 \to \mathbb{T}^N \times \mathbb{T}^M$ has an étale formal model, then also the map $X_h \to \mathbb{T}^N\langle\underline{\upsilon}^{1/p^h}\rangle \times \mathbb{T}^M$ does. It is then sufficient to consider only the case $h = 0$. Since L/K is finite and Λ is a $\mathbb{Q}$-algebra, by the same argument of the proof of Proposition 1.5.3 we can assume that $\varprojlim_h X_h$ has good reduction. Also, by means of Proposition 1.5.1 and the Cancellation theorem [**5**, Corollary 2.5.49], we can equally prove the statement in the stable category $\mathbf{RigDA}_{\text{ét}}(K)$ defined in [**5**, Definition 1.3.19].

Let $\mathfrak{X}_0 \to \operatorname{Spf} K^\circ\langle\underline{\upsilon}^{\pm1}, \underline{\nu}^{\pm1}\rangle$ be a formal model of the map $X_0 \to \mathbb{T}^n \times \mathbb{T}^m$. We let $\bar{X}_0$ be the special fiber over the residue field k of K. The variety X_1 has also a smooth formal model $\mathfrak{X}_1$ whose special fiber is $\bar{X}_1$. By definition, the natural map $\bar{X}_1 \to \bar{X}_0$ is the push-out of the (relative) Frobenius map $\mathbb{A}^{\dim X}_k \to \mathbb{A}^{\dim X}_k$ which is isomorphic to the relative Frobenius map and hence an isomorphism of correspondences as p is invertible in Λ. We conclude that $\Lambda_{\mathrm{tr}}(\bar{X}_1) \to \Lambda_{\mathrm{tr}}(\bar{X}_0)$ is an isomorphism in $\mathbf{DM}_{\text{ét}}(k)$.

Let $\mathbf{FormDA}_{\text{ét}}(K^\circ)$ be the stable category of motives of formal varieties $\mathbf{FSH}_{\mathfrak{M}}(K^\circ)$ defined in [**5**, Definition 1.4.15] associated to the projective model category $\mathfrak{M} = \mathbf{Ch}(\Lambda\text{-}\mathbf{Mod})$. Using the canonical equivalence $\mathbf{DM}_{\text{ét}}(k) \cong \mathbf{DM}_{\text{ét}}(k)$ [**7**, Theorem B.1] we deduce that the map $\Lambda(\bar{X}_1) \to \Lambda(\bar{X}_0)$ is an isomorphism in $\mathbf{DA}_{\text{ét}}(k)$ as is its image via the following functor (see [**5**, Remark 1.4.30]) induced by the special fiber functor and the generic fiber functor:

$$\mathbf{DA}_{\text{ét}}(k) \xleftarrow[\sim]{(-)_\sigma} \mathbf{FormDA}_{\text{ét}}(K^\circ) \xrightarrow{(-)_\eta} \mathbf{RigDA}_{\text{ét}}(K).$$

This morphism is precisely the map $\Lambda(X_1) \to \Lambda(X_0)$ proving the claim. □

We are now ready to present the main result of this section.

1.5.5. THEOREM. *Let* $\operatorname{char} K = 0$. *The functor* $\mathbb{L}\iota^*\colon \mathbf{RigDA}^{\text{eff}}_{\text{ét}}(K) \to \widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét}}(K)$ *has a left adjoint* $\mathbb{L}\iota_!$ *and the counit map* $\mathrm{id} \to \mathbb{L}\iota_!\mathbb{L}\iota^*$ *is invertible. Whenever* $X = \varprojlim_h X_h$ *is an object of* $\widehat{\mathrm{Rig}}\mathrm{Sm}^{\text{gc}}$ *then* $\mathbb{L}\iota_!\Lambda(X) \cong \Lambda(X_h)$ *for a sufficiently large index* h. *If moreover* $X = \varprojlim_h X_h$ *is of potentially good reduction, then* $\mathbb{L}\iota_!\Lambda(X) \cong \Lambda(X_0)$.

PROOF. We start by proving that the canonical map

$$\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(K)(\Lambda(X_{\bar{h}}), \mathcal{F}) \to \widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét}}(K)(\Lambda(X), \mathbb{L}\iota^*\mathcal{F})$$

is an isomorphism, for every $X = \varprojlim_h X_h$ and for $\bar{h}$ big enough. By Proposition 1.4.5, it suffices to prove that the natural map

$$\mathbf{RigDA}^{\text{eff}}(K)(\Lambda(X_{\bar{h}}), \mathbb{L}a_{tr}\mathcal{F}) \to \varinjlim_h \mathbf{RigDA}^{\text{eff}}(K)(\Lambda(X_h), \mathbb{L}a_{tr}\mathcal{F})$$

is an isomorphism for some $\bar{h}$. This follows from Proposition 1.5.3 since all maps of the directed diagram are isomorphisms for $h \geq \bar{h}$ for some $\bar{h}$ big enough. In case $\varprojlim_h X_h$ is of potentially good reduction, then Proposition 1.5.4 ensures that we can choose $\bar{h} = 0$.

We conclude that the subcategory $\mathbf{T}$ of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$ formed by the objects M such that the functor $N \mapsto \widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)(M, \mathbb{L}\iota^*N)$ is corepresentable contains all motives $\Lambda(X)$ with X any object of $\widehat{\mathrm{Rig}}\mathrm{Sm}^{\text{gc}}$. Since these objects form a set of compact generators of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\text{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K)$ by Proposition 1.3.18, we deduce the existence of the functor $\mathbb{L}\iota_!$ by Lemma 1.5.6.

The formula $\mathbb{L}\iota_!\mathbb{L}\iota^* \cong \mathrm{id}$ is a formal consequence of the fact that $\mathbb{L}\iota_!$ is the left adjoint of a fully faithful functor $\mathbb{L}\iota^*$. □

1.5.6. LEMMA. *Let* $\mathfrak{G}\colon \mathbf{T} \to \mathbf{T}'$ *be a triangulated functor of triangulated categories. The full subcategory* $\mathbf{C}$ *of* $\mathbf{T}'$ *of objects* M *such that the functor* $a_M\colon N \mapsto \mathrm{Hom}(M, \mathfrak{G}N)$ *is corepresentable is closed under cones and small direct sums.*

PROOF. For any object M in $\mathbf{C}$ we denote by $\mathfrak{F}M$ the object corepresenting the functor a_M. Let now $\{M_i\}_{i\in I}$ be a set of objects in $\mathbf{C}$. It is immediate to check that $\bigoplus_i \mathfrak{F}M_i$ corepresents the functor $a_{\bigoplus_i M_i}$.

Let now M_1, M_2 be two objects of $\mathbf{C}$ and $f\colon M_1 \to M_2$ be a map between them. There are canonical maps $\eta_i\colon M_i \to \mathfrak{G}\mathfrak{F}M_i$ induced by the identity $\mathfrak{F}M_i \to \mathfrak{F}M_i$ and the universal property of $\mathfrak{F}M_i$. By composing with η_2 we obtain a morphism $\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(M_1, \mathfrak{G}\mathfrak{F}M_2) \cong \mathrm{Hom}(\mathfrak{F}M_1, \mathfrak{F}M_2)$ sending f to a map $\mathfrak{F}f$. Let C be the cone of f and D

be the cone of $\mathfrak{F}f$. We claim that D represents a_C. From the triangulated structure we obtain a map of distinguished triangles

$$\begin{array}{ccccccc} M_1 & \xrightarrow{f} & M_2 & \longrightarrow & C & \longrightarrow & \\ \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow & & \\ \mathfrak{G}\mathfrak{F}M_1 & \xrightarrow{\mathfrak{G}\mathfrak{F}f} & \mathfrak{G}\mathfrak{F}M_2 & \longrightarrow & \mathfrak{G}D & \longrightarrow & \end{array}$$

inducing for any object N of $\mathbf{T}$ the following maps of long exact sequences

$$\begin{array}{ccccccc} \longleftarrow & \mathrm{Hom}(M_1, \mathfrak{G}N) & \longleftarrow & \mathrm{Hom}(M_2, \mathfrak{G}N) & \longleftarrow & \mathrm{Hom}(C, \mathfrak{G}N) & \longleftarrow \\ & \uparrow & & \uparrow & & \uparrow & \\ \longleftarrow & \mathrm{Hom}(\mathfrak{G}\mathfrak{F}M_1, \mathfrak{G}N) & \longleftarrow & \mathrm{Hom}(\mathfrak{G}\mathfrak{F}M_2, \mathfrak{G}N) & \longleftarrow & \mathrm{Hom}(\mathfrak{G}D, \mathfrak{G}N) & \longleftarrow \\ & \uparrow & & \uparrow & & \uparrow & \\ \longleftarrow & \mathrm{Hom}(\mathfrak{F}M_1, N) & \longleftarrow & \mathrm{Hom}(\mathfrak{F}M_2, N) & \longleftarrow & \mathrm{Hom}(D, N) & \longleftarrow \end{array}$$

Since the vertical compositions are isomorphisms for M_1 and M_2 we deduce that they all are, proving that D corepresents a_C as wanted. □

We remark that we used the fact that Λ is a $\mathbb{Q}$-algebra at least twice in the proof of Theorem 1.5.5: to allow for field extensions and correspondences using Theorem 1.5.1 as well as to invert the map defined by multiplication by p. Nonetheless, it is expected that after inverting the Tate twist, Theorem 1.5.1 also holds for $\mathbb{Z}[1/p]$-coefficients therefore providing a stable version of previous result with more general coefficients.

The following fact is a straightforward corollary of Theorem 1.5.5.

1.5.7. PROPOSITION. *Let* $\operatorname{char} K = 0$. *The motive* $\mathbb{L}\iota_!\Lambda(\widehat{\mathbb{B}}^1)$ *is isomorphic to* Λ.

PROOF. In order to prove the claim, it suffices to prove that $\mathbb{L}\iota_!\Lambda(\widehat{\mathbb{B}}^1) \cong \Lambda(\mathbb{B}^1)$. This follows from Proposition 1.2.12 and the description of $\mathbb{L}\iota_!$ given in Theorem 1.5.5. □

We recall that all the homotopy categories we consider are monoidal (see [**6**, Propositions 4.2.76 and 4.4.63]), and the tensor product $\Lambda(X) \otimes \Lambda(X')$ of two motives associated to varieties X and Y coincides with $\Lambda(X \times X')$. The unit object is obviously the motive Λ. Due to the explicit description of the functor $\mathbb{L}\iota_!$ we constructed above, it is easy to prove that it respects the monoidal structures.

1.5.8. PROPOSITION. *Let* $\operatorname{char} K = 0$. *The functor*

$$\mathbb{L}\iota_! \colon \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K) \to \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$$

is a monoidal functor.

PROOF. Since $\mathbb{L}\iota_!$ is the left adjoint of a monoidal functor $\mathbb{L}\iota^*$ there is a canonical natural tranformation of bifunctors $\mathbb{L}\iota_!(M \otimes M') \to \mathbb{L}\iota_! M \otimes \mathbb{L}\iota_! M'$. In order to prove it is an isomorphism, it suffices to check it on a set of generators of $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}$ such as motives of semi-perfectoid varieties $X = \varprojlim_h X_h$, $X' = \varprojlim_h X'_h$. Up to rescaling, we can suppose that $\mathbb{L}\iota_!\Lambda(X) = \Lambda(X_0)$ and $\mathbb{L}\iota_!\Lambda(X') = \Lambda(X'_0)$ by Theorem 1.5.5. In this case, by definition of the tensor product, we obtain the following isomorphisms

$$\mathbb{L}\iota_!(\Lambda(X)\otimes\Lambda(X')) \cong \mathbb{L}\iota_!\Lambda(X\times X') \cong \Lambda(X_0\times X'_0) \cong \Lambda(X_0)\otimes\Lambda(X'_0) \cong \mathbb{L}\iota_!\Lambda(X)\otimes\mathbb{L}\iota_!\Lambda(X')$$

proving our claim.

□

The following proposition can be considered to be a refinement of Theorem 1.5.5.

1.5.9. PROPOSITION. *Let* $\operatorname{char} K = 0$. *The functor* $\mathbb{L}\iota_!$ *factors through* $\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1} \to \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}$ *and the image of the functor* $\mathbb{L}\iota^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) \to \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)$ *lies in the subcategory of* $\widehat{\mathbb{B}}^1$*-local objects. In particular, the triangulated adjunction*

$$\mathbb{L}\iota_!\colon \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) :\mathbb{L}\iota^*$$

restricts to a triangulated adjunction

$$\mathbb{L}\iota_!\colon \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) :\mathbb{L}\iota^*.$$

PROOF. By Propositions 1.5.7 and 1.5.8, $\mathbb{L}\iota_!$ is a monoidal functor sending $\Lambda(\widehat{\mathbb{B}}^1)$ to Λ. This proves the first claim.

From the adjunction $(\mathbb{L}\iota_!, \mathbb{L}\iota^*)$ we then obtain the following isomorphisms, for any X in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ and any M in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$:

$$\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)(\Lambda(X \times \widehat{\mathbb{B}}^1), \mathbb{L}\iota^* M) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)(\mathbb{L}\iota_!\Lambda(X) \otimes \Lambda, M) \cong$$
$$\cong \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)(\mathbb{L}\iota_!\Lambda(X), M) \cong \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K)(\Lambda(X), \mathbb{L}\iota^* M)$$

proving the second claim. □

1.5.10. REMARK. In the statement of the proposition above, we make a slight abuse of notation when denoting with $(\mathbb{L}\iota_!, \mathbb{L}\iota^*)$ both adjoint pairs. It will be clear from the context which one we consider at each instance.

1.6. The de-perfectoidification functor in characteristic p

We now consider the case of a perfectoid field $K^\flat$ of characteristic p and try to generalize the results of Section 1.5. We will need to perform an extra localization on the model structure, and in return we will prove a stronger result. In this section, we always assume that the base perfectoid field has characteristic p. In order to emphasize this hypothesis, we will denote it with $K^\flat$.

In positive characteristic, we are not able to prove the equivalence of motives with and without transfers (Theorem 1.5.1) as it is stated, and it is therefore not clear that the maps $X_{h+1} \to X_h$ associated to an object $X = \varprojlim_h X_h$ of $\widehat{\mathrm{RigSm}}$ are isomorphisms in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$ for a sufficiently big h. In order to overcome this obstacle, we localize our model category further.

For any variety X over $K^\flat$ we denote by $X^{(1)}$ the pullback of X over the Frobenius map $\Phi\colon K^\flat \to K^\flat$, $x \mapsto x^p$. The absolute Frobenius morphism induces a $K^\flat$-linear map $X \to X^{(1)}$. Since $K^\flat$ is perfect, we can also denote by $X^{(-1)}$ the pullback of X over the inverse of the Frobenius map $\Phi^{-1}\colon K^\flat \to K^\flat$ and $X \cong (X^{(-1)})^{(1)}$. There is in particular a canonical map $X^{(-1)} \to X$ which is isomorphic to the map $X' \to X$ induced by the absolute Frobenius, where we denote by X' the same variety X endowed with the structure map $X \to \operatorname{Spa} K \xrightarrow{\Phi} \operatorname{Spa} K$.

1.6.1. PROPOSITION. *The model category* $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/K^\flat)$ *admits a left Bousfield localization* $\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/K^\flat)$ *with respect to the set* S_{Frob} *of relative Frobenius maps* $\Phi\colon \Lambda(X^{(-1)})[i] \to \Lambda(X)[i]$ *as* X *varies in* RigSm *and* i *varies in* $\mathbb{Z}$.

PROOF. Since by [**6**, Proposition 4.4.32] the τ-localization coincides with the Bousfield localization with respect to a set, we conclude by [**6**, Theorem 4.2.71] that the model category $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}\,/K^\flat)$ is still left proper and cellular. We can then apply [**22**, Theorem 4.1.1]. □

1.6.2. DEFINITION. We will denote by $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat,\Lambda)$ the homotopy category of $\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}\,/K^\flat)$. We will omit Λ whenever the context allows it. The image of a rigid variety X in this category will be denoted by $\Lambda(X)$.

The category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)$ is canonically isomorphic to the full triangulated subcategory of $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$ formed by Frob-local objects, i.e. objects that are local with respect to the maps in S_{Frob}. Modulo this identification, there is an obvious functor $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)\to\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)$ associating to $\mathcal{F}$ a Frob-local object $C^{\mathrm{Frob}}\mathcal{F}$.

Inverting Frobenius morphisms is enough to obtain an analogue of Theorem 1.5.1 in characteristic p.

1.6.3. THEOREM. *Let* $\operatorname{char}K^\flat=p$. *The functors* (a_{tr},o_{tr}) *induce an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)\cong\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat).$$

We postpone the proof of this fact to the second chapter, see Theorem 2.3.2.

1.6.4. REMARK. The proof of the statement above uses in a crucial way the fact that the ring of coefficients Λ is a $\mathbb{Q}$-algebra. This is the main reason of our assumption on Λ.

We now investigate the relations between the category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)$ we have just defined, and the other categories of motives introduced so far.

1.6.5. PROPOSITION. *Let* X_0 *be in* $\mathrm{RigSm}\,/K^\flat$ *endowed with an étale map* $X_0\to\mathbb{T}^N\times\mathbb{T}^M=\mathrm{Spa}(K^\flat\langle\underline{\upsilon}^{\pm1},\underline{\nu}^{\pm1}\rangle)$. *The map* $X_1=X_0\times_{\mathbb{T}^N}\mathbb{T}^N\langle\underline{\upsilon}^{\pm1/p}\rangle\to X_0$ *is invertible in* $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)$.

PROOF. The map of the claim is a factor of $X_0\times_{(\mathbb{B}^N\times\mathbb{B}^M)}(\mathbb{B}^N\langle\underline{\upsilon}^{1/p}\rangle\times\mathbb{B}^M\langle\underline{\nu}^{1/p}\rangle)\to X_0$ which is isomorphic to the relative Frobenius map $X_0^{(-1)}\to X_0$ (see for example [**19**, Theorem 3.5.13]). If we consider the diagram

$$X_1^{(-1)}\xrightarrow{a}X_0^{(-1)}\xrightarrow{b}X_1\xrightarrow{c}X_0$$

we conclude that the two compositions ba and cb are isomorphisms hence also c is an isomorphism, as claimed. □

1.6.6. PROPOSITION. *The image via* $\mathbb{L}\iota^*$ *of a* Frob*-local object of* $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$ *is* $\widehat{\mathbb{B}}^1$*-local. In particular, the functor* $\mathbb{L}\iota^*$ *restricts to a functor*

$$\mathbb{L}\iota^*\colon\ \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)\to\widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K^\flat).$$

PROOF. Let $X'=\varprojlim_h X'_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. We consider the object $X'\times\widehat{\mathbb{B}}^1=\varprojlim_h(X'_h\times X_h)$ where we use the description $\widehat{\mathbb{B}}^1=\varprojlim_h X_h$ of Proposition 1.2.12. Let M be a Frob-local object of $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$. From Propositions 1.4.5 and 1.6.5 we then deduce the following isomorphisms

$$
\begin{aligned}
&\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K^\flat)(X'\times\widehat{\mathbb{B}}^1,\mathbb{L}\iota^*M)\cong\varinjlim_h\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(X'_h\times X_h,M)\cong\\
&\cong\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(X'_0\times\mathbb{B}^1,M)\cong\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(X'_0,M)\cong\\
&\cong\varinjlim_h\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(X'_h,M)\cong\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K^\flat)(X',\mathbb{L}\iota^*M)
\end{aligned}
$$

proving the claim. □

We remark that in positive characteristic the perfection $\mathrm{Perf}\colon X\mapsto\varprojlim X^{(-i)}$ is functorial. This makes the description of various functors a lot easier. We recall that we denote by

$$\mathbb{L}j^*\colon\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)\rightleftarrows\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K):\mathbb{R}j_*$$

the adjoint pair induced by the inclusion of categories $j\colon\mathrm{PerfSm}\to\widehat{\mathrm{RigSm}}$.

1.6.7. PROPOSITION. *The perfection functor* $\mathrm{Perf}\colon\widehat{\mathrm{RigSm}}\to\mathrm{PerfSm}$ *induces an adjunction*

$$\mathbb{L}\,\mathrm{Perf}^*\colon\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K^\flat)\rightleftarrows\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat):\mathbb{R}\,\mathrm{Perf}_*$$

and $\mathbb{L}\,\mathrm{Perf}^*$ *factors through* $\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K^\flat)\to\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K^\flat)$. *Moreover, the functor* $\mathbb{L}\,\mathrm{Perf}^*$ *coincides with* $\mathbb{R}j_*$ *on* $\widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K^\flat)$.

PROOF. The perfection functor is continuous with respect to the étale topology and maps $\mathbb{B}^1$ and $\widehat{\mathbb{B}}^1$ to $\widehat{\mathbb{B}}^1$ hence the first claim.

We now consider the functors $j\colon\mathrm{PerfSm}\to\widehat{\mathrm{RigSm}}$ and $\mathrm{Perf}\colon\widehat{\mathrm{RigSm}}\to\mathrm{PerfSm}$. They induce two Quillen pairs (j^*,j_*) and $(\mathrm{Perf}^*,\mathrm{Perf}_*)$ on the associated $(\text{ét},\widehat{\mathbb{B}}^1)$-localized model categories of complexes. Since Perf is a right adjoint of j we deduce that Perf^* is a right adjoint of j^* and hence we obtain an isomorphism $j_*\cong\mathrm{Perf}^*$ which shows the second claim. □

1.6.8. PROPOSITION. *Let* Λ *be a* $\mathbb{Q}$*-algebra. The functor*

$$\mathbb{L}\,\mathrm{Perf}^*\,\mathbb{L}\iota^*\colon\ \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)\to\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$$

factors over $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)$ *and is isomorphic to* $\mathbb{R}j_*\mathbb{L}\iota^*C^{\mathrm{Frob}}$.

PROOF. The first claim follows as the perfection of $X^{(-1)}$ is canonically isomorphic to the perfection of X for any object X in RigSm.

The second part of the statement follows from the first claim and the commutativity of the following diagram, which is ensured by Propositions 1.6.6 and 1.6.7.

$$
\begin{array}{ccccc}
\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat) & \xrightarrow{\mathbb{L}\iota^*} & \widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\widehat{\mathbb{B}}^1}(K^\flat) & \xrightarrow{\mathbb{R}j_*} & \\
\downarrow & & \downarrow & & \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)\\
\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat) & \xrightarrow{\mathbb{L}\iota^*} & \widehat{\mathbf{RigDA}}^{\mathrm{eff}}_{\text{ét},\mathbb{B}^1}(K^\flat) & \xrightarrow{\mathbb{L}\,\mathrm{Perf}^*} &
\end{array}
$$

□

1.6.9. THEOREM. *Let Λ be a $\mathbb{Q}$-algebra. The functor* $\mathbb{L}\operatorname{Perf}^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat) \to \mathbf{PerfDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)$ *defines a monoidal, triangulated equivalence of categories.*

PROOF. Let X_0 and Y be objects of $\mathrm{RigSm}^{\mathrm{gc}}$. Suppose X_0 is endowed with an étale map over $\mathbb{T}^N$ which is a composition of finite étale maps and inclusions, and let $\widehat{X}$ be $\varprojlim_h X_h$. We can identify $\widehat{X}$ with $\operatorname{Perf} X_0$. Since $C^{\mathrm{Frob}}\Lambda(Y)$ is Frob-local, by Proposition 1.6.5 the maps

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\Lambda(X_h), C^{\mathrm{Frob}}\Lambda(Y)) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\Lambda(X_{h+1}), C^{\mathrm{Frob}}\Lambda(Y))$$

are isomorphisms for all h. Using Propositions 1.4.5, 1.6.6 and 1.6.8, we obtain the following sequence of isomorphisms for any $n \in \mathbb{Z}$:

$$\begin{aligned}
&\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat)(\Lambda(X_0), \Lambda(Y)[n]) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\Lambda(X_0), C^{\mathrm{Frob}}\Lambda(Y)[n])\\
&\cong \varinjlim_h \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\Lambda(X_h), C^{\mathrm{Frob}}\Lambda(Y)[n]) \cong \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(K^\flat)(\Lambda(\widehat{X}), \mathbb{L}\iota^* C^{\mathrm{Frob}}\Lambda(Y)[n])\\
&\cong \widehat{\mathbf{Rig}}\mathbf{DA}^{\mathrm{eff}}_{\mathrm{\acute{e}t},\widehat{\mathbb{B}}^1}(K^\flat)(\Lambda(\widehat{X}), \mathbb{L}\iota^* C^{\mathrm{Frob}}\Lambda(Y)[n])\\
&\cong \mathbf{PerfDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\Lambda(\widehat{X}), \mathbb{R}j_*\mathbb{L}\iota^* C^{\mathrm{Frob}}\Lambda(Y)[n])\\
&\cong \mathbf{PerfDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat)(\mathbb{L}\operatorname{Perf}^*(X_0), \mathbb{L}\operatorname{Perf}^*(Y)[n]).
\end{aligned}$$

In particular, we deduce that the triangulated functor $\mathbb{L}\operatorname{Perf}^*$ maps a set of compact generators to a set of compact generators (see Propositions 1.3.18 and 1.3.30) and on these objects it is fully faithful. By means of [**5**, Lemma 1.3.32], we then conclude it is a triangulated equivalence of categories, as claimed. □

1.6.10. REMARK. From the proof of the previous claim, we also deduce that the inverse $\mathbb{R}\operatorname{Perf}_*$ of $\mathbb{L}\operatorname{Perf}^*$ sends the motive associated to an object $X = \varprojlim_h X_h$ to the motive of X_0. This functor is then analogous to the de-perfectoidification functor $\mathbb{L}j^* \circ \mathbb{L}\iota_!$ of Theorem 1.5.5.

1.7. The main theorem

Thanks to the results of the previous sections, we can reformulate Theorem 1.5.5 in terms of motives of rigid varieties. We will always assume that $\operatorname{char} K = 0$ since the results of this section are tautological when $\operatorname{char} K = p$.

1.7.1. COROLLARY. *There exists a triangulated adjunction of categories*

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K^\flat) \rightleftarrows \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K) :\mathfrak{G}$$

such that $\mathfrak{F}$ is a monoidal functor.

PROOF. From Theorem 1.5.5 and Proposition 1.5.8, we can define an adjunction

$$\mathfrak{F}'\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K^\flat) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K) :\mathfrak{G}'$$

by putting $\mathfrak{F}' := \mathbb{L}\iota_! \circ \mathbb{L}j^* \circ (-)^\sharp \circ \mathbb{L}\operatorname{Perf}^*$. We remark that by Proposition 1.5.8, $\mathfrak{F}'$ is also monoidal. The claim then follows from the equivalence of motives with and without transfers (see Propositions 1.5.1 and 1.6.3). □

Our goal is to prove that the adjunction of Corollary 1.7.1 is an equivalence of categories. To this aim, we recall the construction of the stable versions of the rigid motivic categories given in [**5**, Definition 2.5.27].

1.7.2. DEFINITION. Let T be the cokernel in $\mathbf{PST}(\mathrm{RigSm}/K)$ of the unit map $\Lambda_{\mathrm{tr}}(K) \to \Lambda_{\mathrm{tr}}(\mathbb{T}^1)$. We denote by $\mathbf{RigDM}_{\text{ét}}(K, \Lambda)$ or simply by $\mathbf{RigDM}_{\text{ét}}(K)$ the homotopy category associated to the stable $(\text{ét}, \mathbb{B}^1)$-local model structure on the category of symmetric spectra $\mathbf{Spect}_T^{\Sigma}(\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\mathbf{PST}(\mathrm{RigSm}/K))$.

As explained in [**5**, Section 2.5], T is cofibrant and the cyclic permutation induces the identity on $T^{\otimes 3}$ in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}$. Moreover, by [**24**, Theorem 9.3], $T \otimes -$ is a Quillen equivalence in this category, which is actually the universal model category where this holds (in some weak sense made precise by [**24**, Theorem 5.1, Proposition 5.3 and Corollary 9.4]). We recall that the canonical functor $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K) \to \mathbf{RigDM}_{\text{ét}}(K)$ is fully faithful, as proved in [**5**, Corollary 2.5.49] as a corollary of the Cancellation Theorem [**5**, Theorem 2.5.38].

1.7.3. DEFINITION. We denote by $\Lambda(1)$ the motive $T[-1]$ in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$. For any positive integer d we let $\Lambda(d)$ be $\Lambda(1)^{\otimes d}$. The functor $(\cdot)(d) := (\cdot) \otimes \Lambda(d)$ is an auto-equivalence of $\mathbf{RigDM}_{\text{ét}}(K)$ and its inverse will be denoted with $(\cdot)(-d)$.

1.7.4. DEFINITION. We denote by $\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K, \Lambda)$ or simply by $\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K)$ the full triangulated subcategory of $\mathbf{RigDM}_{\text{ét}}(K, \Lambda)$ whose objects are the compact ones. They are of the form $M(d)$ for some compact object M in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$ and some d in $\mathbb{Z}$. This category is called the category of *constructible motives*.

We now present an important result that is a crucial step toward the proof of our main theorem. The motivic property it induces will be given right afterwards.

1.7.5. PROPOSITION. *Let $\widehat{X}$ be a smooth affinoid perfectoid. The natural map of complexes*

$$\mathrm{Sing}^{\widehat{\mathbb{B}}^1}(\Lambda(\widehat{\mathbb{T}}^d))(\widehat{X}) \to \mathrm{Sing}^{\widehat{\mathbb{B}}^1}(\Lambda(\mathbb{T}^d))(\widehat{X})$$

is a quasi-isomorphism.

PROOF. We let $\widehat{X}$ be $\mathrm{Spa}(R, R^+)$. A map f in $\mathrm{Hom}(\widehat{X} \times \widehat{\mathbb{B}}^n, \mathbb{T}^d)$ [resp. in $\mathrm{Hom}(\widehat{X} \times \widehat{\mathbb{B}}^n, \widehat{\mathbb{T}}^d)$] corresponds to d elements $f_1, \ldots, f_d$ in the group $(R^+\langle \tau_1^{1/p^\infty}, \ldots, \tau_n^{1/p^\infty}\rangle)^\times$ [resp. in the group $(R^{\flat+}\langle \tau_1^{1/p^\infty}, \ldots, \tau_n^{1/p^\infty}\rangle)^\times$] and the map between the two objects is induced by the multiplicative tilt map $R^{\flat+}\langle \tau_1^{1/p^\infty}, \ldots, \tau_n^{1/p^\infty}\rangle \to R^+\langle \tau_1^{1/p^\infty}, \ldots, \tau_n^{1/p^\infty}\rangle$.

We now present some facts about homotopy theory for cubical objects, which mirror classical results for simplicial objects (see for example [**35**, Chapter IV]). We remark that the map of the statement is induced by a map of enriched cubical Λ-vector spaces (see [**3**, Definition A.6]), which is obtained by adding Λ-coefficients to a map of enriched cubical sets

$$\mathrm{Hom}(\widehat{X} \times \widehat{\square}, \widehat{\mathbb{T}}^d) \to \mathrm{Hom}(\widehat{X} \times \widehat{\square}, \mathbb{T}^d).$$

Any enriched cubical object has connections in the sense of [**10**, Section 1.2], induced by the maps m_i in [**3**, Definition A.6]. We recall that the category of cubical sets with connections can be endowed with a model structure by which all objects are cofibrant and weak equivalences are defined through the geometric realization (see [**30**]). Moreover, its homotopy category is canonically equivalent to the one of simplicial sets, as cubical sets with connections form a strict test category by [**34**].

The two cubical sets appearing above are abelian groups on each level and the maps defining their cubical structure are group homomorphisms. They therefore are cubical groups. By [**50**], they are fibrant objects and their homotopy groups π_i coincide with the homology H_iN of the associated normalized complexes of abelian groups (see Definition 1.3.13). The Λ-enrichment functor is tensorial with respect to the monoidal structure of cubical sets introduced in [**11**, Section 11.2] and the cubical Dold-Kan functor, associating to a cubical Λ-module with connection its normalized complex (see [**11**, Section 14.8]) is a left Quillen functor. We deduce that in order to prove the statement of the proposition it suffices to show that the two normalized complexes of abelian groups are quasi-isomorphic. We also remark that it suffices to consider the case $d = 1$.

We prove the following claim: the n-th homology of the complex $N((R\widehat{\otimes}\mathcal{O}(\widehat{\square}))^{+\times})$ is 0 for $n > 0$. Let f be invertible in $R^+\langle\tau_1^{1/p^\infty},\dots,\tau_n^{1/p^\infty}\rangle$ with $d_{r,\epsilon}f = 1$ for all (r,ϵ). We claim that $f-1$ is topologically nilpotent. Up to adding a topological nilpotent element, we can assume that $f \in R^+[\underline{\tau}]$. Since f is invertible, its image in $(R^+/R^{\circ\circ})[\underline{\tau}^{1/p^\infty}]$ is invertible as well. Invertible elements in this ring are just the invertible constants. We deduce that all coefficients of $f - f(0) = f - 1$ are topologically nilpotent and hence $f-1$ is topologically nilpotent. In particular, the element $H = f + \tau_{n+1}(1-f)$ in $R^+\langle\underline{\tau}^{1/p^\infty},\tau_{n+1}^{1/p^\infty}\rangle$ is invertible, satisfies $d_{r,\epsilon}H = 1$ for all ϵ and all $1 \leq r \leq n$ and determines a homotopy between f and 1. This proves the claim.

We can also prove that the 0-th homology of the complex $N((R\widehat{\otimes}\mathcal{O}(\widehat{\square}))^{+\times})$ coincides with $R^{+\times}/(1+R^{\circ\circ})$. This amounts to showing that the image of the ring map

$$\begin{aligned}\{f \in R^+\langle\tau^{1/p^\infty}\rangle^\times\colon f(0) = 1\} &\to R^{+\times}\\ f &\mapsto f(1)\end{aligned}$$

coincides with $1+R^{\circ\circ}$. Let f be invertible in $R^+\langle\tau^{1/p^\infty}\rangle$ with $f(0) = 1$. As proved above, $f-1$ is topologically nilpotent so that also $f(1)-1$ is. Vice-versa if $a \in R$ is topologically nilpotent then the element $1 + a\tau \in R^+\langle\tau^{1/p^\infty}\rangle$ is invertible, satisfies $f(0) = 1$ and $f(1) = 1+a$ proving the claim.

We are left to prove that the multiplicative map $\sharp$ induces an isomorphism $(R^{\flat+})^\times/(1+R^{\flat\circ\circ}) \to (R^+)^\times/(1+R^{\circ\circ})$. We start by proving it is injective. Let $a \in R^{\flat+}$ such that $(a^\sharp - 1)$ is topologically nilpotent. Since $(a^\sharp-1) = (a-1)^\sharp$ in R^+/π we deduce that the element $(a-1)^\sharp - (a^\sharp-1)$ is also topologically nilpotent. We conclude that $(a-1)^\sharp$ as well as $(a-1)$ are topologically nilpotent, as wanted.

We now prove surjectivity. Let a be invertible in R^+. In particular both a and a^{-1} are power-bounded. From the isomorphism $R^{\flat+}/\pi^\flat \cong R^+/\pi$ we deduce that there exists an element $b \in R^{\flat+}$ such that $b^\sharp = a + \pi\alpha = a(1+\pi\alpha a^{-1})$ for some (power bounded) element $\alpha \in R^+$. We deduce that $(1+\pi\alpha a^{-1})$ lies in $1+R^{\circ\circ}$ and that $b^\sharp$ is invertible. Since the multiplicative structure of $R^\flat$ is isomorphic to $\varprojlim_{x\mapsto x^p} R$ and $\sharp$ is given by the projection to the last component, we deduce that as $b^\sharp$ is invertible, then also b is. In particular, the image of $b \in (R^{\flat+})^\times$ in $(R^+)^\times/(1+R^{\circ\circ})$ is equal to a as wanted. $\square$

We recall that by Corollary 1.7.1 there is an adjunction

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat) \rightleftarrows \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K) \colon\! \mathfrak{G}$$

and our goal is to prove it is an equivalence.

1.7.6. PROPOSITION. *The motive $\mathfrak{G}\Lambda(d)$ is isomorphic to $\Lambda(d)$ for any positive integer d.*

PROOF. The natural map $\Lambda(d) \to \mathfrak{G}\Lambda(d)$ is induced by the isomorphism $\mathfrak{F}\Lambda(d) \cong \Lambda(d)$. We need to prove it is an isomorphism. The motive $\Lambda(d)$ is a direct factor of the motive $\Lambda(\mathbb{T}^d)[-d]$ and the map above is induced by $\Lambda(\mathbb{T}^d) \to \mathfrak{G}\Lambda(\mathbb{T}^d)$. It suffices then to prove that the map $\Lambda(\mathbb{T}^d) \to \mathfrak{G}\Lambda(\mathbb{T}^d)$ is an isomorphism.

By the definition of the adjoint pair $(\mathfrak{F},\mathfrak{G})$ given in Corollary 1.7.1, we can equivalently consider the adjunction

$$\mathbb{L}\iota_!\mathbb{L}j^*\colon \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K)\rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K) :\mathbb{R}j_*\mathbb{L}\iota^*$$

and prove that $\Lambda(\widehat{\mathbb{T}}^d)\to(\mathbb{R}j_*\circ\mathbb{L}\iota^*)\Lambda(\mathbb{T}^d)$ is an isomorphism in $\mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K)$.

From Proposition 1.7.5 we deduce that the complexes $\operatorname{Sing}^{\widehat{\mathbb{B}}^1}\Lambda(\widehat{\mathbb{T}}^d)$ and $j_*\operatorname{Sing}^{\widehat{\mathbb{B}}^1}\Lambda(\mathbb{T}^d)$ are quasi-isomorphic in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{PerfSm})$. By means of Remark 1.7.7 the quasi-isomorphism above can be restated as

$$\operatorname{Sing}^{\widehat{\mathbb{B}}^1}\Lambda(\widehat{\mathbb{T}}^d)\cong\mathbb{R}j_*\operatorname{Sing}^{\widehat{\mathbb{B}}^1}\Lambda(\mathbb{T}^d).$$

Due to Proposition 1.3.26 the complex $\operatorname{Sing}^{\widehat{\mathbb{B}}^1}\mathcal{F}$ is $\widehat{\mathbb{B}}^1$-equivalent to $\mathcal{F}$ for any complex $\mathcal{F}$. This fact, together with the isomorphism $\mathbb{L}\iota^*\Lambda(\mathbb{T}^d)\cong\Lambda(\mathbb{T}^d)$ implies $\Lambda(\widehat{\mathbb{T}}^d)\cong\mathbb{R}j_*\mathbb{L}\iota^*\Lambda(\mathbb{T}^d)$ as wanted.

□

1.7.7. REMARK. Since j_* commutes with ét-sheafification, it preserves ét-weak equivalences. It also commutes with $\operatorname{Sing}^{\widehat{\mathbb{B}}^1}$ and therefore preserves $\mathbb{B}^1$-weak equivalences. We conclude that $\mathbb{R}j_*=j_*$ and in particular $\mathbb{R}j_*$ commutes with small direct sums.

We are finally ready to present the proof of our main result.

1.7.8. THEOREM. *The adjunction*

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)\rightleftarrows\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K) :\mathfrak{G}$$

is a monoidal triangulated equivalence of categories.

PROOF. By Theorem 1.5.5 the functor $\mathbb{L}\iota_!\mathbb{L}j^*\colon \mathbf{PerfDA}^{\mathrm{eff}}_{\text{ét}}(K)\to\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$ sends the motive $\Lambda(\widehat{X})$ associated to a perfectoid $\widehat{X}=\varprojlim_h X_h$ to the motive $\Lambda(X_0)$ associated to X_0 up to rescaling indices. It is triangulated, commutes with sums, and its essential image contains motives $\Lambda(X_0)$ of varieties X_0 having good coordinates $X_0\to\mathbb{T}^N$ and such that $X_h=X_0\times_{\mathbb{T}^N}\mathbb{T}^N\langle\underline{\upsilon}^{1/p^h}\rangle\to X_0$ is an isomorphism in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$ for all h. We call these *rigid varieties with very good coordinates*. By Proposition 1.5.3, for every rigid variety with good coordinates $X_0\to\mathbb{T}^N$ there exists an index h such that $X_h=X_0\times_{\mathbb{T}^N}\mathbb{T}^N\langle\underline{\upsilon}^{\pm1/p^h}\rangle$ has very good coordinates. Since $\operatorname{char}K=0$ the map $\mathbb{T}^N\langle\underline{\upsilon}^{\pm1/p^h}\rangle\to\mathbb{T}^N$ is finite étale, and therefore also the map $X_h\to X_0$ is. We conclude that any rigid variety with good coordinates has a finite étale covering with very good coordinates, and hence the motives associated to varieties with very good coordinates generate the étale topos. In particular, the motives associated to them generate $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K)$ and hence the functor $\mathbb{L}\iota_!\circ\mathbb{L}j^*$ maps a set of compact generators to a set of compact generators.

Since $\mathfrak{F}$ is monoidal and $\mathfrak{F}(\Lambda(1))=\Lambda(1)$ it extends formally to a monoidal functor from the category $\mathbf{RigDA}^{\mathrm{ct}}_{\text{ét}}(K^\flat)$ to $\mathbf{RigDA}_{\text{ét}}(K)$ by putting $\mathfrak{F}(M(-d))=\mathfrak{F}(M)(-d)$. Let now M, N in $\mathbf{RigDM}_{\text{ét}}(K^\flat)$ be twists of the motives associated to the analytification of smooth projective varieties X resp. X'. They are strongly dualizable objects of $\mathbf{RigDM}_{\text{ét}}(K^\flat)$ since $\Lambda_{\mathrm{tr}}(X)$ and $\Lambda_{\mathrm{tr}}(X')$ are strongly dualizable in $\mathbf{DM}_{\text{ét}}(K^\flat)$. Fix an integer d such that $N^\vee(d)$ lies in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$. The objects M, N, $M^\vee$ and $N^\vee$ lie in $\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)$ and moreover $\mathfrak{F}(N^\vee)=\mathfrak{F}(N)^\vee$. From Lemma 1.7.9 we also deduce that the functor $\mathfrak{F}$ induces a bijection

$$\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)(M\otimes N^\vee,\Lambda)\cong\mathbf{RigDM}_{\text{ét}}(K)(\mathfrak{F}(M)\otimes\mathfrak{F}(N)^\vee,\Lambda).$$

By means of the Cancellation theorem [**5**, Corollary 2.5.49] the first set is isomorphic to the set $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(M,N)$ and the second is isomorphic to $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)(\mathfrak{F}(M),\mathfrak{F}(N))$. We then deduce that all motives M associated to the analytification of smooth projective varieties lie in the left orthogonal of the cone of the map $N \to \mathfrak{G}\mathfrak{F}N$ which is closed under direct sums and cones. Since Λ is a $\mathbb{Q}$-algebra, such motives generate $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$ by means of [**5**, Theorem 2.5.35]. We conclude that $N \cong \mathfrak{G}\mathfrak{F}N$. Therefore the category $\mathbf{T}$ of objects N such that $N \cong \mathfrak{G}\mathfrak{F}N$ contains all motives associated to the analytification of smooth projective varieties. It is clear that $\mathbf{T}$ is closed under cones. The functors $\mathfrak{F}$ and $\mathbb{L}\iota^*$ commute with direct sums as they are left adjoint functors. As pointed out in Remark 1.7.7 also the functor $\mathbb{R}j_*$ does. Since $\mathfrak{G}$ is a composite of $\mathbb{R}j_*\mathbb{L}\iota^*$ with equivalences of categories, it commutes with small sums as well. We conclude that $\mathbf{T}$ is closed under direct sums. Using again [**5**, Theorem 2.5.35] we deduce $\mathbf{T} = \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$ proving that $\mathfrak{F}$ is fully faithful. This is enough to prove it is an equivalence of categories, by applying [**5**, Lemma 1.3.32]. □

1.7.9. LEMMA. *Let M be an object of $\mathbf{RigDA}^{\mathrm{ct}}_{\text{ét}}(K^\flat)$. The functor $\mathfrak{F}$ induces an isomorphism*

$$\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)(M,\Lambda) \cong \mathbf{RigDM}_{\text{ét}}(K)(\mathfrak{F}(M),\Lambda).$$

PROOF. Suppose that d is an integer such that $M(d)$ lies in $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$. One has $\mathfrak{F}\Lambda(d) \cong \Lambda(d)$ and by Proposition 1.7.6 the unit map $\eta\colon \Lambda(d) \to \mathfrak{G}\mathfrak{F}\Lambda(d)$ is an isomorphism. In particular from the adjunction $(\mathfrak{F},\mathfrak{G})$ we obtain a commutative square

$$\begin{array}{ccc}
\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(M(d),\Lambda(d)) & \xrightarrow{\mathfrak{F}} & \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)(\mathfrak{F}M(d),\mathfrak{F}\Lambda(d)) \\
\big\downarrow{=} & & \big\downarrow{\sim} \\
\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(M(d),\Lambda(d)) & \underset{\sim}{\xrightarrow{\eta}} & \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)(M(d),(\mathfrak{G}\mathfrak{F})\Lambda(d))
\end{array}$$

in which the top arrow is then an isomorphism. By the Cancellation theorem [**5**, Corollary 2.5.49] we also obtain the following commutative square

$$\begin{array}{ccc}
\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)(M(d),\Lambda(d)) & \xrightarrow{\mathfrak{F}} & \mathbf{RigDM}_{\text{ét}}(K)(\mathfrak{F}M(d),\Lambda(d)) \\
\big\downarrow{\sim} & & \big\downarrow{\sim} \\
\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)(M(d),\Lambda(d)) & \underset{\sim}{\xrightarrow{\mathfrak{F}}} & \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)(\mathfrak{F}M(d),\Lambda(d))
\end{array}$$

and hence also the top arrow is an isomorphism. We conclude the claim from the following commutative square, whose vertical arrows are isomorphisms since the functor $(\cdot)(d)$ is invertible in $\mathbf{RigDM}_{\text{ét}}(K)$:

$$\begin{array}{ccc}
\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)(M,\Lambda) & \xrightarrow{\mathfrak{F}} & \mathbf{RigDM}_{\text{ét}}(K)(\mathfrak{F}M,\Lambda) \\
{\scriptstyle(\cdot)(d)}\big\downarrow{\sim} & & {\scriptstyle(\cdot)(d)}\big\downarrow{\sim} \\
\mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat)(M(d),\Lambda(d)) & \underset{\sim}{\xrightarrow{\mathfrak{F}}} & \mathbf{RigDM}_{\text{ét}}(K)(\mathfrak{F}M(d),\Lambda(d)).
\end{array}$$

□

1.7.10. REMARK. In the proof of Theorem 1.7.8 we again used the hypothesis that Λ is a $\mathbb{Q}$-algebra in order to apply [**5**, Theorem 2.5.35] which states that the motives associated to the analytification of smooth projective varieties generate $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat)$.

We remark that the proof above also induces the following statement.

1.7.11. COROLLARY. *The functor*

$$\mathfrak{F}\colon \mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K^\flat) \to \mathbf{RigDM}^{\mathrm{ct}}_{\text{ét}}(K)$$

is a monoidal equivalence of categories.

1.7.12. REMARK. The reader may wonder if the equivalence $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\Lambda)$ still holds true for an arbitrary ring of coefficients Λ such that $p \in \Lambda^\times$. With this respect, the case of rational coefficients that we tackled in this thesis is particularly meaningful. Indeed, it is expected that if l is coprime to p then the category $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K,\mathbb{Z}/l\mathbb{Z})$ coincides with the derived category of $\mathbb{Z}/l\mathbb{Z}$-Galois representations, in analogy to the case of $\mathbf{DM}^{\mathrm{eff}}_{\text{ét}}(K,\mathbb{Z}/l\mathbb{Z})$. It would then be equivalent to $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K^\flat,\mathbb{Z}/l\mathbb{Z})$ by the theorem of Fontaine and Wintenberger.

CHAPTER 2

Rigid motives with and without transfers

The purpose of this chapter is to prove an equivalence of categories $\mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(K,\Lambda)\cong\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(K,\Lambda)$ adapting the proof of [**3**, Theorem B.1] and [**7**, Theorem B.1] to the rigid analytic setting and to an arbitrary characteristic. To this aim, we first need to present a refinement of the étale topology.

2.1. The Frob-topology

In all this section, we assume that K is a perfect field which is complete with respect to a non-archimedean norm. Unless otherwise stated, we will use the term "variety" to indicate an affinoid rigid analytic variety over K.

2.1.1. DEFINITION. A map $f\colon Y\to X$ of varieties over K is called a Frob-*cover* if it is finite, surjective and for every affinoid U in X the affinoid inverse image $V=f^{-1}(U)$ is such that the induced map of rings $\mathcal{O}(U)\to\mathcal{O}(V)$ is radicial.

2.1.2. REMARK. By [**21**, Corollary IV.18.12.11] a morphism of schemes is finite, surjective and radicial if and only if it is a finite universal homeomorphism. The same holds true for rigid analytic varieties.

If $\operatorname{char}K=p$ and X is a variety over K then the absolute n-th Frobenius map $X\to X$ given by the elevation to the p^n-th power, factors over a map $X\to X^{(n)}$ where we denote by $X^{(n)}$ the base change of X by the absolute n-th Frobenius map $K\to K$. We denote by $\Phi^{(n)}$ the map $X\to X^{(n)}$ and we call it the *relative n-th Frobenius*. Since K is perfect, $X^{(n)}$ is isomorphic to X endowed with the structure map $X\to\operatorname{Spa}K\overset{\Phi^{-n}}{\to}\operatorname{Spa}K$ and the relative n-th Frobenius is isomorphic to the absolute n-th Frobenius of X over $\mathbb{F}_p$. We can also define $X^{(n)}$ for negative n to be the base change of X over the the map $\Phi^n\colon K\to K$ which is again isomorphic to X endowed with the structure map $X\to\operatorname{Spa}K\overset{\Phi^{-n}}{\to}\operatorname{Spa}K$. The Frobenius map induces a morphism $X^{(-1)}\to X$ and the collection of maps $\{X^{(-1)}\to X\}$ defines a coverage (see for example [**31**, Definition C.2.1.1]).

We also define $X^{(n)}$ to be X and the maps $\Phi\colon X^{(n-1)}\to X^{(n)}$ to be the identity maps for all $n\in\mathbb{Z}$ in case $\operatorname{char}K=0$.

2.1.3. PROPOSITION. *Let $Y\to X$ be a Frob-cover between normal varieties over K. There exists an integer n and a map $X^{(-n)}\to Y$ such that the composite map $X^{(-n)}\to Y\to X$ coincides with Φ^n and the composite map $Y\to X\to Y^{(n)}$ coincides with Φ^n.*

PROOF. We can equally prove the statement for affine schemes. Let $f\colon Y\to X$ a finite universal homeomorphism of affine normal schemes over K. By [**33**, Proposition 6.6] there exists an integer n and a map $h\colon X\to Y^{(n)}$ such that the composite map $Y\to X\to Y^{(n)}$ coincides with the relative n-th Frobenius. We remark that the map $Y\to X$ is an epimorphism (in the categorical sense) of normal varieties. From the equalities $fhf^{(n)}=\Phi^{(n)}_Y f^{(n)}=f\Phi^{(n)}_X$

we then conclude that the composite map $X \to Y^{(n)} \to X^{(n)}$ coincides with the n-relative Frobenius. This proves the claim. □

2.1.4. DEFINITION. Let B be a normal variety over K. We define RigSm/B to be the category of varieties which are smooth over B. We denote by $\tau_{\text{ét}}$ the étale topology.

2.1.5. DEFINITION. Let B be a normal variety over K. We define RigNor/B to be the category of normal varieties over B.

- We denote by τ_{Frob} the topology on RigNor/B induced by Frob-covers.
- We denote by $\tau_{\text{ét}}$ the étale topology.
- We denote by $\tau_{\text{Frobét}}$ the topology generated by τ_{Frob} and $\tau_{\text{ét}}$.
- We denote by τ_{fh} the topology generated by covering families $\{f_i \colon X_i \to X\}_{i\in I}$ such that I is finite, and the induced map $\sqcup f_i \colon \sqcup_{i\in I} X_i \to X$ is finite and surjective.
- We denote by $\tau_{\text{fhét}}$ the topology generated by τ_{fh} and $\tau_{\text{ét}}$.

2.1.6. REMARK. The fhét-topology is often denoted by qfh (see [**51**]). We stick to the notation fhét in order to be consistent with [**3**].

We are not imposing any additivity condition on the Frob-topology, i.e. the families $\{X_i \to \sqcup_{i\in I} X_i\}_{i\in I}$ are not Frob-covers. This does not interfere much with our theory since we will mostly be interested in the Frobét-topology, with respect to which such families are covering families.

2.1.7. REMARK. The fh-topology is obviously finer that the Frob-topology, which is the trivial topology in case $\operatorname{char} K = 0$.

2.1.8. REMARK. The category of normal affinoid is not closed under fiber products, and the fh-coverings do not define a Grothendieck pretopology. Nonetheless, they define a coverage which is enough to have a convenient description of the topology they generate (see for example [**31**, Section C.2.1]).

2.1.9. REMARK. A particular example of fh-covers is given by *pseudo-Galois covers* which are finite, surjective maps $f \colon Y \to X$ of normal integral affinoid varieties such that the field extension $K(Y) \to K(X)$ is obtained as a composition of a Galois extension and a finite, purely inseparable extension. The Galois group G associated to the extension coincides with $\operatorname{Aut}(Y/X)$. As shown in [**5**, Corollary 2.2.5], a presheaf $\mathcal{F}$ on RigNor/B with values in a complete and cocomplete category is an fh-sheaf if and only if the two following conditions are satisfied.

(1) For every finite set $\{X_i\}_{i\in I}$ of objects in RigNor/B it holds $\mathcal{F}(\sqcup_{i\in I} X_i) \cong \prod_{i\in I} \mathcal{F}(X_i)$.
(2) For every pseudo-Galois covering $Y \to X$ with associated Galois group G the map $\mathcal{F}(X) \to \mathcal{F}(Y)^G$ is invertible.

2.1.10. DEFINITION. Let B be a normal variety over K.

- We denote by $\operatorname{RigSm}/B^{\text{Perf}}$ the 2-limit category $2-\varinjlim_n \operatorname{RigSm}/B^{(-n)}$ with respect to the functors $\operatorname{RigSm}/B^{(-n-1)} \to \operatorname{RigSm}/B^{(-n)}$ induced by the pullback along the map $B^{(-n-1)} \to B^{(-n)}$. More explicitly, it is equivalent to the category $\mathbf{C}_B[S^{-1}]$ where $\mathbf{C}_B$ is the category whose objects are pairs $(X, -n)$ with $n \in \mathbb{N}$ and $X \in \operatorname{RigSm}/B^{(-n)}$ and morphisms $\mathbf{C}_B((X, -n), (X', -n'))$ are maps $f \colon X \to X'$ forming commutative squares

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ B^{(-n)} & \xrightarrow{\Phi} & B^{(-n')} \end{array}$$

and where S is the class of canonical maps $(X' \times_{B^{(-n')}} B^{(-n)}, -n) \to (X', -n')$ for each $X \in \mathrm{RigSm}\,/B^{(-n')}$ and $n \geq n'$ (see [**20**, Definition VI.6.3]).

- We say that a map $(X, -n) \to (X', -n')$ of $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ is a Frob-*cover* if the map $X \to X'$ is a Frob-cover. We denote by τ_{Frob} the topology on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ induced by Frob-covers.
- We say that a collection of maps $\{(X_i, -n_i) \to (X, -n)\}_{i \in I}$ is an *étale cover* if the induced collection $\{X_j \to X\}$ is. We denote by $\tau_{\text{ét}}$ the topology on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ generated by the étale coverings. It coincides with the one induced by putting the étale topology on each category $\mathrm{RigSm}\,/B^{(-n)}$ (see [**1**, Theorem VI.8.2.3]).
- We denote by $\tau_{\mathrm{Frob\acute{e}t}}$ the topology generated by τ_{Frob} and $\tau_{\text{ét}}$.

We now investigate some properties of the Frob-topology.

2.1.11. PROPOSITION. *Let B be a normal variety over K.*

- *A presheaf $\mathcal{F}$ on $\mathrm{RigNor}\,/B$ is a Frob-sheaf if and only if $\mathcal{F}(X^{(-1)}) \cong \mathcal{F}(X)$ for all objects X in $\mathrm{RigNor}\,/B$.*
- *A presheaf $\mathcal{F}$ on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ is a Frob-sheaf if and only if $\mathcal{F}(X^{(-1)}, -n-1) \cong \mathcal{F}(X, -n)$ for all objects $(X, -n)$ in $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$.*

PROOF. The two statements are analogous and we only prove the claim for $\mathrm{RigNor}\,/B$. By means of [**31**, Lemma C.2.1.6 and Lemma C.2.1.7] the topology generated by maps $f\colon Y \to X$ which factor a power of Frobenius $X^{(-n)} \to X$ is the same as the one generated by the coverage $X^{(-1)} \to X$. Using Proposition 2.1.3, we conclude that the Frob-topology coincides with the one generated by the coverage $\{X^{(-1)} \to X\}$. Since the Frobenius map is a monomorphism of normal varieties, the sheaf condition associated to the coverage $X^{(-1)} \to X$ is simply the one of the statement by [**31**, Lemma 2.1.3]. □

2.1.12. COROLLARY. *Let B be a normal variety over K.*

- *The class Φ of maps $\{X^{(-r)} \to X\}_{r \in \mathbb{N}, X \in \mathrm{RigNor}\,/B}$ admits calculus of fractions, and its saturation consists of Frob-covers. In particular, the continuous map*

$$(\mathrm{RigNor}\,/B, \mathrm{Frob}) \to \mathrm{RigNor}\,/B[\Phi^{-1}]$$

defines an equivalence of topoi.

- *The class Φ of maps $\{(X^{(-r)}, -n-r) \to (X, -n)\}_{r \in \mathbb{N}, (X,n) \in \mathrm{RigSm}\,/B^{\mathrm{Perf}}}$ admits calculus of fractions, and its saturation consists of Frob-covers. In particular, the continuous map*

$$(\mathrm{RigSm}\,/B^{\mathrm{Perf}}, \mathrm{Frob}) \to \mathrm{RigSm}\,/B^{\mathrm{Perf}}[\Phi^{-1}]$$

defines an equivalence of topoi.

PROOF. We only prove the first claim. The fact that Φ admits calculus of fractions is an easy check, and the characterization of its saturation follows from Proposition 2.1.3. The sheaf condition for a presheaf $\mathcal{F}$ with respect to the Frob-topology is simply $\mathcal{F}(X^{(-1)}) \cong \mathcal{F}(X)$ by Corollary 2.1.11 hence the last claim. □

2.1.13. REMARK. We follow the notations introduced in Definition 2.1.10. Any pullback of a finite, surjective radicial map between normal varieties is also finite, surjective and radicial. In particular, if B is a normal variety, the maps in the class S are invertible in $\mathrm{RigNor}/B[\Phi^{-1}]$. The functor $\mathbf{C}_B \to \mathrm{RigNor}/B[\Phi^{-1}]$ defined by mapping $(X, -n)$ to X factors through a functor $\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathrm{RigNor}/B[\Phi^{-1}]$. In particular, there is a functor $\mathrm{RigSm}/B^{\mathrm{Perf}}[\Phi^{-1}] \to \mathrm{RigNor}/B[\Phi^{-1}]$ defined by sending $(X, -n)$ to X hence, by Corollary 2.1.12, there is a functor $\mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B)$.

2.1.14. REMARK. If $e\colon B' \to B$ is a finite map of normal varieties, any étale hypercover $\mathcal{U} \to B'$ has a refinement by a hypercover $\mathcal{U}'$ obtained by pullback from an étale hypercover $\mathcal{V}$ of B (see for example [**47**, Section 44.45]). In particular, the functor $e_*\colon \mathbf{Psh}(\mathrm{RigSm}/B') \to \mathbf{Psh}(\mathrm{RigSm}/B)$ commutes with the functor $a_{\text{ét}}$ of ét-sheafification. The same holds true for the functor $e_*\colon \mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$.

From now on, we fix a commutative ring Λ and work with Λ-enriched categories. In particular, the term "presheaf" should be understood as "presheaf of Λ-modules" and similarly for the term "sheaf". It follows that the presheaf $\Lambda(X)$ represented by an object X of a category $\mathbf{C}$ sends an object Y of $\mathbf{C}$ to the free Λ-module $\Lambda\,\mathrm{Hom}(Y, X)$.

2.1.15. ASSUMPTION. Unless otherwise stated, we assume from now on that Λ is a $\mathbb{Q}$-algebra and we omit it from the notations.

The following facts are immediate, and will also be useful afterwards.

2.1.16. PROPOSITION. *Let B be a normal variety over K.*

- *If $\mathcal{F}$ is an étale sheaf on* $\mathrm{RigSm}/B^{\mathrm{Perf}}$ *[resp. on* RigNor/B*] then $a_{\mathrm{Frob}}\mathcal{F}$ is a* Frobét-*sheaf.*
- *If $\mathcal{F}$ is a* Frob-*sheaf on* $\mathrm{RigSm}/B^{\mathrm{Perf}}$ *[resp. on* RigNor/B*] then $a_{\text{ét}}\mathcal{F}$ is a* Frobét-*sheaf.*

PROOF. We only prove the claims for RigNor/B. First, suppose that $\mathcal{F}$ is an étale sheaf. By Proposition 2.1.3, we obtain that $a_{\mathrm{Frob}}\mathcal{F}(X) = \varinjlim_n \mathcal{F}(X^{(-n)})$. Whenever $U \to X$ is étale, then $U \times_X X^{(-n)} \cong U^{(-n)}$ and $U^{(-n)} \times_{X^{(-n)}} U^{(-n)} \cong (U \times_X U)^{(-n)}$ so that the following diagram is exact

$$0 \to \mathcal{F}(X^{(-n)}) \to \mathcal{F}(U^{(-n)}) \to \mathcal{F}((U \times_X U)^{(-n)}).$$

The first claim the follows by taking the limit over n.

We now prove the second claim. Suppose $\mathcal{F}$ is a Frob-sheaf. For any étale covering $\mathcal{U} \to X$ we indicate with $\mathcal{U}'$ the associated covering of $X^{(-1)}$ obtained by pullback. From Remark 2.1.14 one can compute the sections of $a_{\text{ét}}\mathcal{F}(X^{(-1)})$ with the formula

$$a_{\text{ét}}\mathcal{F}(X^{(-1)}) = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}'_0) \to \mathcal{F}(\mathcal{U}'_1)\right)$$

where $\mathcal{U} \to X$ varies among hypercovers of X. Since $\mathcal{F}$ is a Frob-sheaf, then $\mathcal{F}(U'_0) \cong \mathcal{F}(U_0)$ and $\mathcal{F}(U'_1) \cong \mathcal{F}(U_1)$. The formula above then implies

$$a_{\text{ét}}\mathcal{F}(X^{(-1)}) = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}_0) \to \mathcal{F}(\mathcal{U}_1)\right) = a_{\text{ét}}\mathcal{F}(X)$$

proving the claim. □

2.1.17. PROPOSITION. *Let B be a normal variety over K. If $\mathcal{F}$ is a* fh-*sheaf on* RigNor/B *then $a_{\text{ét}}\mathcal{F}$ is a* fhét-*sheaf.*

PROOF. Let $f\colon X'\to X$ be a pseudo-Galois cover with associated group G. In light of Remark 2.1.9, we need to show that $a_{\text{ét}}\mathcal{F}(X)\cong a_{\text{ét}}\mathcal{F}(X')^G$. For any étale covering $\mathcal{U}\to X$ we indicate with $\mathcal{U}'$ the associated covering of X' obtained by pullback. From Remark 2.1.14 one can compute the sections of $a_{\text{ét}}\mathcal{F}(X')$ with the formula

$$a_{\text{ét}}\mathcal{F}(X')=\varinjlim_{\mathcal{U}\to X}\ker\left(\mathcal{F}(\mathcal{U}'_0)\to\mathcal{F}(\mathcal{U}'_1)\right)$$

where $\mathcal{U}\to X$ varies among hypercovers of X. Taking the G-invariants is an exact functor as Λ is a $\mathbb{Q}$-algebra and when applied to the formula above it yields

$$a_{\text{ét}}\mathcal{F}(X')^G=\varinjlim_{\mathcal{U}\to X}\ker\left(\mathcal{F}(\mathcal{U}'_0)^G\to\mathcal{F}(\mathcal{U}'_1)^G\right)=\varinjlim_{\mathcal{U}\to X}\ker\left(\mathcal{F}(\mathcal{U}_0)\to\mathcal{F}(\mathcal{U}_1)\right)=a_{\text{ét}}\mathcal{F}(X)$$

as wanted. □

2.1.18. PROPOSITION. *Let B be a normal variety over K. The canonical inclusions*

$$o_{\text{Frob}}\colon \mathbf{Sh}_{\text{Frob}}(\operatorname{RigNor}/B)\to\mathbf{Psh}(\operatorname{RigNor}/B)$$
$$o_{\text{Frob}}\colon \mathbf{Sh}_{\text{Frob}}(\operatorname{RigSm}/B^{\text{Perf}})\to\mathbf{Psh}(\operatorname{RigSm}/B^{\text{Perf}})$$
$$o_{\text{fh}}\colon \mathbf{Sh}_{\text{fh}}(\operatorname{RigNor}/B)\to\mathbf{Psh}(\operatorname{RigNor}/B)$$

are exact.

PROOF. In light of Proposition 2.1.11 the statements about o_{Frob} are obvious. Since Λ is a $\mathbb{Q}$-algebra, the functor of G-invariants from $\Lambda[G]$-modules to Λ-modules is exact. The third claim then follows from Remark 2.1.9. □

We now investigate the functors of the topoi introduced above induced by a map of varieties $B'\to B$.

2.1.19. PROPOSITION. *Let $f\colon B'\to B$ be a map of normal varieties over K.*

- *Composition with f defines a functor $f_\sharp\colon \operatorname{RigNor}/B'\to\operatorname{RigNor}/B$ which induces the following adjoint pair*

$$f_\sharp\colon \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigNor}/B')\rightleftarrows\mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigNor}/B):f^*$$

- *The base change over f defines functors $f^{(-n)*}\colon \operatorname{RigSm}/B^{(-n)}\to\operatorname{RigSm}/B'^{(-n)}$ which induce the following adjoint pair*

$$f^*\colon \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigSm}/B^{\text{Perf}})\rightleftarrows\mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigSm}/B'^{\text{Perf}}):f_*$$

- *If f is a* Frob*-cover, the functors above are equivalences of categories.*
- *If f is smooth, the composition with f defines functors $f_\sharp^{(-n)}\colon \operatorname{RigSm}/B'^{(-n)}\to\operatorname{RigSm}/B^{(-n)}$ which induce the following adjoint pair*

$$f_\sharp\colon \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigSm}/B'^{\text{Perf}})\rightleftarrows\mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\operatorname{RigSm}/B^{\text{Perf}}):f^*$$

PROOF. We initially remark that the functors $f^{(-n)*}$ induce a functor $f^*\colon \mathbf{C}_B\to\mathbf{C}_{B'}$ where $\mathbf{C}_B$ is the fibered category introduced in Definition 2.1.10. As cartesian squares are mapped to cartesian squares, they also induce a functor $f^*\colon \operatorname{RigSm}/B^{\text{Perf}}\to\operatorname{RigSm}/B^{\text{Perf}}$.

The existence of the first two adjoint pairs is then a formal consequence of the continuity of the functors $f_\sharp$ and f^*.

Let now f be a Frob-cover. The functors $f^*\colon \operatorname{RigSm}/B^{\text{Perf}}[\Phi^{-1}]\to\operatorname{RigSm}/B'^{\text{Perf}}[\Phi^{-1}]$ and $f_\sharp\colon \operatorname{RigNor}/B'[\Phi^{-1}]\to\operatorname{RigNor}/B[\Phi^{-1}]$ are equivalences, and we conclude the third claim by what proved above and Corollary 2.1.12.

For the fourth claim, we use a different model for the $\mathrm{Frob\acute{e}t}$-topos on $\mathrm{RigSm}/B^{\mathrm{Perf}}$. The fibered category $\mathbf{C}_B$ can be endowed with the Frob-topology and the $\mathrm{Frob\acute{e}t}$-topology. Following the proof of Corollary 2.1.12, the map $(\mathbf{C}_B, \mathrm{Frob}) \to \mathbf{C}_B[\Phi^{-1}]$ induces an equivalence of topoi. Moreover, the canonical functor $\mathbf{C}_B[\Phi^{-1}] \to \mathrm{RigSm}/B^{\mathrm{Perf}}[\Phi^{-1}]$ induces an equivalence of categories.

The existence of the last Quillen functor is therefore a formal consequence of the continuity of the functor $f_\sharp\colon (\mathbf{C}_{B'}[\Phi^{-1}], \mathrm{\acute{e}t}) \to (\mathbf{C}_B[\Phi^{-1}], \mathrm{\acute{e}t})$. □

2.1.20. REMARK. Let $f\colon B' \to B$ be a map of normal varieties. The image via f^* of the presheaf represented by $(X, -n)$ is the the presheaf represented by $(X \times_B B'^{(-n)}, -n)$ and if f is smooth, the image via $f_\sharp$ of the presheaf represented by $(X', -n)$ is the sheaf represented by $(X', -n)$.

2.2. Rigid motives and Frob-motives

We apply the techniques and the terminology of Section 1.3 to the relative étale and Frob-étale site. We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra.

The category of complexes of presheaves $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ can be endowed with the *projective model structure* for which weak equivalences are quasi-isomorphisms and fibrations are maps $\mathcal{F} \to \mathcal{F}'$ such that $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a surjection for all X in $\mathbf{C}$ (cfr [**23**, Section 2.3] and [**6**, Proposition 4.4.16]).

We recall that from Proposition 1.3.3, whenever $(\mathbf{C}, \tau, I)$ is a site with an interval, the Bousfield localization over τ-local, I-local and (τ, I)-local maps is well defined. The induced model categories will be denoted by $\mathbf{Ch}_\tau \mathbf{Psh}(\mathbf{C})$, $\mathbf{Ch}_I \mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_{\tau,I} \mathbf{Psh}(\mathbf{C})$ respectively. The model category $\mathbf{Ch}_\tau \mathbf{Psh}(\mathbf{C})$ is canonically Quillen equivalent to the projective model structure on the category of complexes of sheaves $\mathbf{Ch}\,\mathbf{Sh}_\tau(\mathbf{C})$

2.2.1. DEFINITION. Let B be a normal variety over K.

- The triangulated homotopy category of the localization $\mathbf{Ch}_{\mathrm{\acute{e}t},\mathbb{B}^1} \mathbf{Psh}(\mathrm{RigSm}/B)$ will be denoted by $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B, \Lambda)$.
- The triangulated homotopy category of the localization $\mathbf{Ch}_{\mathrm{\acute{e}t},\mathbb{B}^1} \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ will be denoted by $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}, \Lambda)$ while the triangulated homotopy category of $\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1} \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ will be denoted by $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}, \Lambda)$.
- The triangulated homotopy category of the localization $\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1} \mathbf{Psh}(\mathrm{RigNor}/B)$ will be denoted by $\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B, \Lambda)$ while the triangulated homotopy category of $\mathbf{Ch}_{\mathrm{fh\acute{e}t},\mathbb{B}^1} \mathbf{Psh}(\mathrm{RigNor}/B)$ will be denoted by $\mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B, \Lambda)$.
- If $\mathbf{C}$ is one of the categories RigSm/B, $\mathrm{RigSm}/B^{\mathrm{Perf}}$ and RigNor/B and $\eta \in \{\mathrm{\acute{e}t}, \mathrm{Frob}, \mathrm{fh}, \mathrm{Frob\acute{e}t}, \mathrm{fh\acute{e}t}, \mathbb{B}^1, (\mathrm{\acute{e}t}, \mathbb{B}^1), (\mathrm{Frob\acute{e}t}, \mathbb{B}^1), (\mathrm{fh\acute{e}t}, \mathbb{B}^1)\}$ we say that a map in $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ is a *η-weak equivalence* if it is a weak equivalence in the model structure $\mathbf{Ch}_\eta \mathbf{Psh}(\mathbf{C})$ whenever this makes sense.
- We will omit Λ from the notation whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda(X)$.

We now want to introduce the analogue of the previous definitions for motives with transfers. By Remark 2.1.13 the map $(X, -n) \mapsto X$ induces a functor $\mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B)$. If we compose it with the Yoneda embedding and the functor a_{fh} of fh-sheafification we obtain a functor

$$\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B).$$

2.2.2. DEFINITION. Let B be a normal variety over K.

- We define the category RigCor/B as the category whose objects are those of RigSm/B and whose morphisms $\mathrm{Hom}(X,Y)$ are computed in $\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$. The category $\mathbf{Psh}(\mathrm{RigCor}/B)$ will be denoted by $\mathbf{PST}(\mathrm{RigSm}/B)$.
- We define the category $\mathrm{RigCor}/B^{\mathrm{Perf}}$ as the category whose objects are those of $\mathrm{RigSm}/B^{\mathrm{Perf}}$ and whose morphisms $\mathrm{Hom}(X,Y)$ are computed in $\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$. The category $\mathbf{Psh}(\mathrm{RigCor}/B^{\mathrm{Perf}})$ will be denoted by $\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$.

We remark that, as Λ is a $\mathbb{Q}$-algebra, our definition of RigCor/B is equivalent to the one given in [**5**, Definition 2.2.17]. We also remark that the inclusions of categories $\mathrm{RigSm}/B \to \mathrm{RigCor}/B$ and $\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathrm{RigCor}/B^{\mathrm{Perf}}$ induce the following adjunctions:

$$a_{\mathrm{tr}}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B) \rightleftarrows \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B) : o_{\mathrm{tr}}.$$

$$a_{\mathrm{tr}}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) : o_{\mathrm{tr}}.$$

We now define the category of motives with transfers.

2.2.3. PROPOSITION. *Let B be a normal variety over K and let $\mathbf{C}$ be either the category RigSm/B or the category $\mathrm{RigSm}/B^{\mathrm{Perf}}$. The projective model category $\mathbf{Ch}\,\mathbf{PST}(\mathbf{C})$ admits a left Bousfield localization $\mathbf{Ch}_{\text{ét}}\,\mathbf{PST}(\mathbf{C})$ with respect to $S_{\text{ét}}$, the class of of maps f such that $o_{\mathrm{tr}}(f)$ is a* ét-*weak equivalence. It also admits a further Bousfield localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathbf{C})$ with respect to the set formed by all maps $\Lambda(\mathbb{B}^1_X)[i] \to \Lambda(X)[i]$ by letting X vary in $\mathbf{C}$ and i vary in $\mathbb{Z}$.*

PROOF. The proof of [**5**, Theorem 2.5.7] also applies in our situation. For the second statement, it suffices to apply [**22**, Theorem 4.1.1]. □

2.2.4. REMARK. By means of an étale version of [**5**, Corollary 2.5.3], if $\mathcal{F}$ is a presheaf with transfers then the associated étale sheaf $a_{\text{ét}}\mathcal{F}$ can be endowed with a unique structure of presheaf with transfers such that $\mathcal{F} \to a_{\text{ét}}\mathcal{F}$ is a map of presheaves with transfers. The class $S_{\text{ét}}$ can then be defined intrinsecally, as the class of maps $\mathcal{F} \to \mathcal{F}'$ inducing isomorphisms of étale sheaves with transfers $a_{\text{ét}}H_i\mathcal{F} \to a_{\text{ét}}H_i\mathcal{F}'$.

2.2.5. DEFINITION. Let B be a normal variety over K.

- The triangulated homotopy category of the localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B)$ will be denoted by $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B,\Lambda)$.
- The triangulated homotopy category of the localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ will be denoted by $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}},\Lambda)$.
- We will omit Λ from the notation whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda_{\mathrm{tr}}(X)$.

We remark that if $\mathrm{char}\,K = 0$ the two definitions above coincide. Also, if B is the spectrum of the perfect field K the category $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}})$ coincides with $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(K)$. In this case, the definition of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ also coincides with the one of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K)$ given in Definition 1.6.2, as the following fact shows.

2.2.6. PROPOSITION. *Let B be a normal variety over K. There is a Quillen equivalence between the category $\mathbf{Ch}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ and the left Bousfield localization of $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ over the set of all shifts of maps $\Lambda(X^{(-1)},-n-1) \to \Lambda(X,-n)$ as $(X,-n)$ varies in $\mathrm{RigSm}/B^{\mathrm{Perf}}$.*

PROOF. From Lemmas 2.1.16, 2.1.18 and 2.2.7 we conclude that Frobét-local objects are those which are Frob-local and ét-local. We can then conclude using Lemma 2.2.8. ☐

2.2.7. LEMMA. *Let $\mathbf{C}$ be a category endowed with two Grothendieck topologies τ_1, τ_2 and let τ_3 be the topology generated by τ_1 and τ_2. We denote by a_{τ_i} the associated sheafification functor and with o_{τ_i} their right adjoint functors. If o_{τ_1} is exact and $a_{\tau_3} = a_{\tau_2}a_{\tau_1}$ then the following categories are canonically equivalent:*

(1) The homotopy category of $\mathbf{Ch}_{\tau_3}\,\mathbf{Psh}(\mathbf{C})$.
(2) The full triangulated subcategory of $\mathbf{D}(\mathbf{Psh}(\mathbf{C}))$ formed by objects which are τ_3-local.
(3) The full triangulated subcategory of $\mathbf{D}(\mathbf{Psh}(\mathbf{C}))$ formed by objects which are τ_1-local and τ_2-local.

PROOF. The equivalence between the first and the second category follows by definition of the Bousfield localization. We are left to prove the equivalence between the second and the third. We remark that τ_3-local objects are in particular (τ_1, τ_2)-local.

Since o_{τ_1} is exact, the category of τ_1-local objects coincides with the category of complexes quasi-isomorphic to complexes of τ_1-sheaves. Consider the model category $\mathbf{Ch}_{\tau_3}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ which is the Bousfield localization of $\mathbf{Ch}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ over the class of maps of complexes inducing isomorphisms on the τ_3-sheaves associated to the homology presheaves, that we will call τ_3-equivalences. From the assumption $a_{\tau_3} = a_{\tau_2}a_{\tau_1}$ the class of τ_3-equivalences coincides with the class of maps S_{τ_2} of complexes inducing isomorphisms on the τ_2-sheaves associated to the homology τ_1-sheaves. Hence $\mathbf{Ch}_{\tau_3}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ coincides with $\mathbf{Ch}_{\tau_2}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ and its derived category is equivalent to the category of (τ_1, τ_2)-local complexes.

Because of the following Quillen adjunction

$$\mathbb{L}a_{\tau_1} = a_{\tau_1} \colon \mathrm{Ho}(\mathbf{Ch}_{\tau_3}\,\mathbf{Psh}(\mathbf{C}) \rightleftarrows \mathrm{Ho}(\mathbf{Ch}_{\tau_3}\,\mathbf{Sh}_{\tau_1}(\mathbf{C})) : \mathbb{R}o_{\tau_1} = o_{\tau_1}.$$

we conclude that the image via o_{τ_1} of a τ_2-local complex of sheaves i.e. a $(\tau_1.\tau_2)$-local complex, is τ_3-local, as wanted. ☐

2.2.8. LEMMA. *Let B be a normal variety over K. A projectively fibrant object of $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ is* Frob*-local if and only if it is local with respect to the set of all shifts of maps $\Lambda(X^{(-1)}, -n-1) \to \Lambda(X, -n)$ as $(X, -n)$ varies in* $\mathrm{RigSm}/B^{\mathrm{Perf}}$.

PROOF. We initially remark that a fibrant complex $\mathcal{F}$ is local with respect to the set of maps in the claim if and only if $(H_i\mathcal{F})(X, -n) \cong (H_i\mathcal{F})(X^{(-1)}, -n-1)$ for all X and i. By Proposition 2.1.3, this amounts to say that $H_i\mathcal{F}$ is a Frob-sheaf for all i.

Suppose now that $\mathcal{F}$ is fibrant and Frob-local. Since the map of presheaves $\Lambda(X^{(-1)}, -n-1) \to \Lambda(X, -n)$ induces an isomorphism on the associated Frob-sheaves, we deduce that $(H_i\mathcal{F})(X^{(-1)}, -n-1) \cong (H_i\mathcal{F})(X, -n)$. This implies that $H_i\mathcal{F}$ is a Frob-sheaf and hence $\mathcal{F}$ is local with respect to the maps of the claim, as wanted.

Suppose now that $\mathcal{F}$ is fibrant and local with respect to the maps of the claim. Let $\mathcal{F} \to C^{\mathrm{Frob}}\mathcal{F}$ a Frob-weak equivalence to a fibrant Frob-local object. By definition, we deduce that the Frob-sheaves associated to $H_i\mathcal{F}$ and to $H_iC^{\mathrm{Frob}}\mathcal{F}$ are isomorphic. On the other hand, we know that these presheaves are already Frob-sheaves, and hence the map $\mathcal{F} \to C^{\mathrm{Frob}}\mathcal{F}$ is a quasi-isomorphism of presheaves and $\mathcal{F}$ is Frob-local. ☐

We now want to find another model for the category $\mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$. This is possible by means of the model-categorical machinery developed above.

By Remark 2.1.9 an object $\mathcal{F}$ in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigNor}/B)$ is fh-local if and only if it is additive and

$$\mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)(\Lambda(X),\mathcal{F}) \to \mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)(\Lambda(X'),\mathcal{F})^{\mathrm{Aut}(X'/X)}$$

is an isomorphism, for all pseudo-Galois coverings $X' \to X$. Therefore, if we consider $\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$ as the subcategory of $(\mathbb{B}^1, \mathrm{Frob\acute{e}t})$-local objects in $\mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)$ we say that an object $\mathcal{F}$ of $\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$ is fh-local if and only if

$$\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X),\mathcal{F}) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X'),\mathcal{F})^{\mathrm{Aut}(X'/X)}$$

is an isomorphism, for all pseudo-Galois coverings $X' \to X$.

2.2.9. PROPOSITION. *Let B be a normal variety over K. The category $\mathbf{D}^{\mathrm{fh}}_{\acute{\mathrm{e}}\mathrm{t},\mathbb{B}^1}(\mathrm{RigNor}/B)$ is canonically isomorphic to the category of* fh*-local objects in* $\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$.

PROOF. It suffices to prove the claim before performing the $\mathbb{B}^1$-localization on each category. The statement then follows from Propositions 2.1.16 and 2.1.17 together with Lemmas 2.1.18 and 2.2.7. □

We now study some functoriality properties of the categories just defined, and later prove a fundamental fact: the locality axiom (see [**37**, Theorem 3.2.21]).

2.2.10. PROPOSITION. *Let $f\colon B' \to B$ be a map of normal varieties over K. The first two adjoint pairs of Proposition 2.1.19 induce the following Quillen pairs:*

$$\mathbb{L}f_\sharp\colon \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B') \rightleftarrows \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B) :\mathbb{R}f^*$$
$$\mathbb{L}f^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B'^{\mathrm{Perf}}) :\mathbb{R}f_*$$

which are equivalences whenever f is a Frob*-covering. Moreover, if f is a smooth map, the third adjoint pair of Proposition 2.1.19 induces a Quillen pair:*

$$\mathbb{L}f_\sharp\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B'^{\mathrm{Perf}}) \rightleftarrows \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) :\mathbb{L}f^*$$

PROOF. The statement is a formal consequence of Proposition 2.1.19 and the formulas $f^*(\mathbb{B}^1_X) = \mathbb{B}^1_{f^*(X)}$ and $f_\sharp(\mathbb{B}^1_X) = \mathbb{B}^1_X$. □

2.2.11. PROPOSITION. *Let $e\colon B' \to B$ be a finite map of normal varieties over K. The functor*

$$e_*\colon\ \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$*-equivalences.*

PROOF. Let $e\colon B' \to B$ be a finite map of normal varieties. The functor e_* is induced by the map $\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathrm{RigSm}/B'^{\mathrm{Perf}}$ sending $(X,-n)$ to $(X \times_{B^{(-n)}} B'^{(-n)}, -n)$. From Remark 2.1.14 it commutes with ét-sheafification. As the image of $(X^{(-1)}, -n-1)$ is isomorphic to $((X \times_{B^{(-n)}} B'^{(-n)})^{(-1)}, -n-1)$ we deduce from Corollary 2.1.12 that e_* commutes with Frob-sheafification. Therefore by Proposition 2.1.16 we deduce that $e_*\colon\ \mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ commutes with the functor $a_{\mathrm{Frob\acute{e}t}}$ of Frobét-sheafification, hence it preserves Frobét-equivalences.

We now prove that it also preserves $\mathbb{B}^1$-equivalences. By [**6**, Proposition 4.2.74] it suffices to show that $e_*(\Lambda(\mathbb{B}^1_V) \to \Lambda(V))$ is a $\mathbb{B}^1$-weak equivalence for any V in $\mathrm{RigSm}/X'^{\mathrm{Perf}}$. This follows from the explicit homotopy between the identity and the zero map on $e_*(\Lambda(\mathbb{B}^1_V))$ (see the argument of [**5**, Theorem 2.5.24]). □

The following property is an extension of [**5**, Theorem 1.4.20] and referred to as the *locality axiom.*

2.2.12. THEOREM. *Let $i\colon Z \hookrightarrow B$ be a closed immersion of normal varieties over K and let $j\colon U \hookrightarrow B$ be the open complementary. For every object M in $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ there is a distinguished triangle*

$$\mathbb{L}j_{\sharp}\mathbb{L}j^{*}M \to M \to \mathbb{R}i_{*}\mathbb{L}i^{*}M \to$$

In particular, the pair $(\mathbb{L}j^{*}, \mathbb{L}i^{*})$ is conservative.

PROOF. First of all, we remark that by Proposition 2.2.11 one has $\mathbb{R}i_{*} = i_{*}$. In particular it suffices to prove the claim before performing the localization over the shifts of maps $\Lambda(X^{(-1)}, -n-1) \to \Lambda(X, -n)$ i.e. in the category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$.

The functors $\mathbb{L}j_{\sharp}$ $\mathbb{L}j^{*}$ and $\mathbb{L}i^{*}$ commute with small sums because they admit right adjoint functors. Also $\mathbb{R}i_{*}$ does, since it holds $\mathbb{R}i_{*} = i_{*}$. We conclude that the full subcategory of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ of objects M such that

$$\mathbb{L}j_{\sharp}\mathbb{L}j^{*}M \to M \to \mathbb{R}i_{*}\mathbb{L}i^{*}M \to$$

is a distinguished triangle is closed under cones, and under small sums. We can then equivalently prove the claim in the subcategory $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ of compact objects, since these motives generate $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ as a triangulated category with small sums.

Because of Lemma 2.2.13 and Proposition 2.2.11, we can prove the claim for each category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$. Therefore, it suffices to prove the claim for the categories $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Nis}}(B^{(-n)})$ as defined in [**5**], since the category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ is a further localization of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Nis}}(B^{(-n)})$. In this case, the statement is proved in [**5**, Theorem 1.4.20]. □

2.2.13. LEMMA. *Let B be a normal variety over K. The functors* $\mathrm{RigSm}/B^{(-n)} \to \mathrm{RigSm}/B^{\mathrm{Perf}}$ *induce a triangulated equivalence of categories*

$$\varinjlim_{n} \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)}) \cong \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$$

where we denote by $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ [resp. with $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$] the subcategory of compact objects of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ [resp. of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$].

PROOF. The functor $\varinjlim_{n} \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)}) \to \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ is triangulated and sends the objects $\Lambda(X)[i]$ which are compact generators of the first category, to a set of compact generators of the second. Up to shifting indeces, it therefore suffices to show that for X, Y in RigSm/B one has

$$\varinjlim_{n} \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})(\Lambda(X \times_{B} B^{(-n)}), \Lambda(Y \times_{B} B^{(-n)})) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})(\Lambda(\bar{X}), \Lambda(\bar{Y}))$$

where we denote by $\bar{X} = (X, 0)$ and $\bar{Y} = (Y, 0)$ the object of $\mathrm{RigSm}/B^{\mathrm{Perf}}$ associated to X resp. Y. To this aim, we simply follow the proof of [**5**, Proposition 1.A.1]. For the convenience of the reader, we reproduce it here.

Step 1: We consider the directed diagram $\mathcal{B}$ formed the maps $B^{(-n-1)} \to B^{(-n)}$ and we let $\mathrm{RigSm}/\mathcal{B}$ be the the category of rigid smooth varieties over it as defined in [**5**, Section 1.4.2]. We can endow the category $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B})$ with the $(\mathrm{\acute{e}t}, \mathbb{B}^{1})$-local model structure, and consider the Quillen adjunctions induced by the map of diagrams $\alpha_{n}\colon B^{(-n)} \to \mathcal{B}$, $f_{nm}\colon B^{(-n)} \to B^{(-m)}$:

$$\alpha_{n}^{*}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B}) \rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}) : \alpha_{n*}$$

$$\alpha_{n\sharp}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}) \rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B}) : \alpha_{n}^{*}$$

$$f_{nm}^{*}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-m)}) \rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}) : f_{nm*}$$

We also remark that the canonical map $\mathrm{RigSm}/B^{(-n)} \to \mathrm{RigSm}/B^{\mathrm{Perf}}$ induces a Quillen adjunction

$$f^*_{\infty n}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}) \rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) : f_{\infty n*}.$$

Consider a trivial cofibration $\alpha_{0*}\Lambda(Y) \to R$ with target R that is $(\text{ét}, \mathbb{B}^1)$-fibrant. Since α^*_n is a left and right Quillen functor and $\alpha^*_n\alpha_{0*} = f^*_{n0}$ we deduce that the map $\Lambda(Y \times_B B^{(-n)}) = f^*_{n0}\Lambda(Y) \to \alpha^*_n R$ is also an $(\text{ét}, \mathbb{B}^1)$-trivial cofibration with an $(\text{ét}, \mathbb{B}^1)$-fibrant target.

Step 2: By applying the left Quillen functors f^*_{nm} and $f^*_{\infty m}$ we also obtain that $f^*_{n0}\Lambda(Y) = f^*_{nm}f^*_{m0}\Lambda(Y) \to f^*_{nm}\alpha^*_m R$ and $f^*_{\infty 0}\Lambda(Y) = f^*_{\infty m}f^*_{m0}\Lambda(Y) \to f^*_{\infty m}\alpha^*_m R$ are $(\text{ét}, \mathbb{B}^1)$-trivial cofibrations. By the 2-out-of-3 property of weak equivalences applied to the composite map

$$f^*_{n0}\Lambda(Y) \to f^*_{nm}\alpha^*_m R \to \alpha^*_n R$$

we then deduce that the map $f^*_{nm}\alpha^*_m R \to \alpha^*_n R$ is an $(\text{ét}, \mathbb{B}^1)$-weak equivalence.

Step 3: We now claim that the natural map $\Lambda(\bar{Y}) \to \hat{R}$ with $\hat{R} := \mathrm{colim}_n\, f^*_{\infty n}\alpha^*_i R$ is an $(\text{ét}, \mathbb{B}^1)$-weak equivalence in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$. By what shown in Step 2, it suffices to prove that the functor

$$\mathrm{colim}\colon\ \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})^{\mathbb{N}} \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves $(\text{ét}, \mathbb{B}^1)$-weak equivalences. First of all, we remark that it is a Quillen left functor with respect to the projective model structure on the diagram category $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})^{\mathbb{N}}$ induced by the pointwise $(\text{ét}, \mathbb{B}^1)$-structure. Hence, it preserves $(\text{ét}, \mathbb{B}^1)$-weak equivalences between cofibrant objects. On the other hand, as directed colimits commute with homology, it also preserves weak equivalences of presheaves. Since any complex is quasi-isomorphic to a cofibrant one, we deduce the claim.

Step 4: We now prove that $\hat{R}$ is $\mathbb{B}^1$-local. Consider a variety U smooth over $B^{(-n)}$. From the formula

$$\hat{R}(\bar{U}) = \mathrm{colim}_{m\geq n}\, \alpha^*_m R(U \times_{B^{(-n)}} B^{(-m)})$$

and the fact that $\alpha^*_m R$ is $\mathbb{B}^1$-local, we deduce a quasi-isomorphism $\hat{R}(U) \cong \hat{R}(\mathbb{B}^1_U)$ as wanted.

Step 5: We now prove that $\hat{R}$ is ét-local. It suffices to show that for any U smooth over $B^{(-n)}$ one has $\mathbb{H}^k_{\text{ét}}(\bar{U}, \hat{R}) \cong H_{-k}\hat{R}(\bar{U})$. The topos associated to Et/U is equivalent to the one of $\varinjlim \mathrm{Et}/(U \times_{B^{(-n)}} B^{(-m)})$ and all these sites have a bounded cohomological dimension since Λ is a $\mathbb{Q}$-algebra. By applying [**1**, Theorem VI.8.7.3] together with a spectral sequence argument given by [**48**, Theorem 0.3], we then deduce the formula

$$\mathbb{H}^k_{\text{ét}}(\bar{U}, \hat{R}) \cong \mathrm{colim}_m\, \mathbb{H}^k_{\text{ét}}(U \times_{B^{(-n)}} B^{(-m)}, \alpha^*_m R).$$

On the other hand, as $\alpha^*_m R$ is ét-local, we conclude that

$$\mathrm{colim}_m\, \mathbb{H}^k_{\text{ét}}(U \times_{B^{(-n)}} B^{(-m)}, \alpha^*_i R) \cong \mathrm{colim}_m\, H_{-k}(\alpha^*_m R)(U \times_{B^{(-n)}} B^{(-m)}) \cong H_{-k}\hat{R}(\bar{U})$$

proving the claim.

Step 6: From Steps 3-5, we conclude that we can compute $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}})(\Lambda(\bar{X}), \Lambda(\bar{Y}))$ as $\hat{R}(\bar{X})$ which coincides with $\mathrm{colim}_n(\alpha^*_n R)(X \times_B B^{(-n)})$. By what is proved in Step 1, we also deduce that $\alpha^*_n R$ is a $(\text{ét}, \mathbb{B}^1)$-fibrant replacement of $\Lambda(Y \times_B B^{(-n)})$ and hence the last group coincides with $\mathrm{colim}_n\, \mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(B^{(-n)})(\Lambda(X \times_B B^{(-n)}), \Lambda(Y \times_B B^{(-n)}))$ proving the statement. □

2.3. The equivalence between motives with and without transfers

We can finally present the main result of this section. We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra.

2.3.1. THEOREM. *Let B be a normal variety over K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}).$$

As a corollary, we obtain the two following results, which are indeed our main motivation.

2.3.2. THEOREM. *The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(K) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(K).$$

2.3.3. THEOREM. *Let B be a normal variety over a field K of characteristic 0. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B).$$

2.3.4. REMARK. The statement of Theorem 2.3.1 in case B is a normal affinoid rigid analytic variety immediately implies the statement for the case of an arbitrary normal rigid analytic variety B. Therefore, we can suppose that B is affinoid, being consistent with our notations on the term “variety”.

The proof of Theorem 2.3.1 is divided into the following steps.

(1) We first produce a triangulated functor $\mathbb{L}a_{\mathrm{tr}}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ commuting with sums, sending a set of compact generators of the first category into a set of compact generators of the second.
(2) We define a fully faithful functor $\mathbb{L}i^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$.
(3) We define a fully faithful functor $\mathbb{L}j^*\colon \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$.
(4) We check that $\mathbb{L}j^* \circ \mathbb{L}a_{\mathrm{tr}}$ is isomorphic to $\mathbb{L}i^*$ proving that $\mathbb{L}a_{\mathrm{tr}}$ is also fully faithful.

We now prove the first step.

2.3.5. PROPOSITION. *Let B be a normal variety over K. The functor a_{tr} induces a triangulated functor*

$$\mathbb{L}a_{\mathrm{tr}}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$$

commuting with sums, sending a set of compact generators of the first category into a set of compact generators of the second.

PROOF. The functor a_{tr} induces a Quillen functor

$$\mathbb{L}a_{\mathrm{tr}}\colon\ \mathbf{Ch}_{\mathrm{\acute{e}t}}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Ch}_{\mathrm{\acute{e}t}}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

sending $\Lambda(X,-n)$ to $\Lambda_{\mathrm{tr}}(X)$. We are left to prove that it factors over the Frob-localization, i.e. that the map $\Lambda_{\mathrm{tr}}(X^{(-1)}) \to \Lambda_{\mathrm{tr}}(X)$ is an isomorphism in $\mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ for all $X \in \mathrm{RigSm}/B^{(-n)}$. Actually, since the map $X^{(-1)} \to X$ induces an isomorphism of fh-sheaves, we deduce that it is an isomorphism in the category $\mathrm{RigCor}/B^{\mathrm{Perf}}$ hence also in $\mathbf{RigDM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$. □

We are now ready to prove the second step.

2.3.6. PROPOSITION. *Let B be a normal variety over K. The functors* $\mathrm{RigSm}/B^{(-n)} \to \mathrm{RigNor}/B$ *induce a fully faithful functor*

$$\mathbb{L}i_B^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B).$$

PROOF. We let $\mathbf{C}_B$ be the category introduced in Definition 2.1.10. As already remarked in the proof of Proposition 2.1.19 we can endow it with the Frobét-topology and the topos associated to it is equivalent to the Frobét-topos on $\mathrm{RigSm}/B^{\mathrm{Perf}}$. In particular, the continuous functor $i_B\colon \mathbf{C}_B \to \mathrm{RigNor}/B$ induces an adjunction

$$\mathbb{L}i_B^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \rightleftarrows \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B) :\mathbb{R}i_{B*}.$$

As $i_{B*}i_B^*$ is isomorphic to the identity, it suffices to show that $\mathbb{R}i_{B*} = i_{B*}$ so that $\mathbb{R}i_{B*}\mathbb{L}i_B^*$ is isomorphic to the identity as well.

The functor i_{B*} commutes with Frobét-sheafification, and hence it preserves Frobét-weak equivalences, and since $i_{B*}(\Lambda(\mathbb{B}^1_V)) \cong \Lambda(\mathbb{B}^1_B) \otimes i_{B*}(\Lambda(V))$ is weakly equivalent to $i_{B*}(\Lambda(V))$ for every V in RigNor/B we also conclude that it preserves $\mathbb{B}^1$-weak equivalences, as wanted. □

2.3.7. REMARK. As a corollary of the proof of Proposition 2.3.6 we obtain that the functor i_{B*} preserves $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences.

We remark that the previous result does not yet prove our claim. This is reached by the following crucial fact.

2.3.8. PROPOSITION. *Let B be a normal variety over K. The image of $\mathbb{L}i_B^*$ is contained in the subcategory of* fh*-local objects.*

PROOF. Let M be an object of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ let $f\colon X \to B$ be a normal irreducible variety over B and let $r\colon X' \to X$ be a pseudo-Galois covering in RigNor/B with $G = \mathrm{Aut}(X'/X)$. We are left to prove that

$$\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X), \mathbb{L}i^*M) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X'), \mathbb{L}i^*M)^G$$

is an isomorphism. Using Lemma 2.3.9 we can equally prove that

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(\Lambda, \mathbb{L}f^*M) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X'^{\mathrm{Perf}})(\Lambda, \mathbb{L}r^*\mathbb{L}f^*M)^G$$

is an isomorphism. Using the notation of Lemma 2.3.12, it suffices to prove that the natural transformation $\mathrm{id} \to (\mathbb{R}r_*\mathbb{L}r^*)^G$ is invertible.

Using Lemma 2.3.13, we can define a stratification $(X_i)_{0\leq i\leq n}$ of X made of locally closed connected normal subvarieties of X such that $r_i\colon X_i' \to X_i$ is a composition of an étale cover and a Frob-cover of normal varieties, by letting X_i' be the reduction of the subvariety $X_i \times_X X' \subset X'$. Using the locality axiom (Theorem 2.2.12) for $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}$ applied to the inclusions $u_i\colon X_i \to X$ we can then restrict to proving that each transformation $\mathbb{L}u_i^* \to \mathbb{L}u_i^*(\mathbb{R}r_*\mathbb{L}r^*)^G \cong (\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G\mathbb{L}u_i^*$ is invertible, where the last isomorphism follows from Lemma 2.3.12. It suffices then to prove that $\mathrm{id} \to (\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G$ is invertible. If $s\colon Z \to T$ is a Frob-cover, the functors $(\mathbb{L}s^*, \mathbb{R}s_*)$ define an equivalence of categories $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(T^{\mathrm{Perf}}) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(Z^{\mathrm{Perf}})$ by Proposition 2.2.10 hence we can assume that the maps r_i are étale covers. Moreover, since $\mathbb{L}r_i^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X_i^{\mathrm{Perf}}) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X_i'^{\mathrm{Perf}})$ is conservative by Lemma 2.3.11, we can equivalently prove that $\mathbb{L}r_i^* \to \mathbb{L}r_i^*(\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G \cong (\mathbb{R}r'_{i*}\mathbb{L}r_i'^*)^G\mathbb{L}r_i^*$ is invertible, where r_i' is the base change of r_i over itself (see Lemma 2.3.12). By the assumptions on r_i we conclude that r_i' is a projection $\bigsqcup X_i' \to X_i'$ with G acting transitively on the fibers, so that the functor $(\mathbb{R}r'_{i*}\mathbb{L}r_i'^*)^G$ is the identity, proving the claim. □

The following lemmas were used in the proof of the previous proposition.

2.3.9. LEMMA. *Let* $f\colon B' \to B$ *be a map of normal rigid varieties over* K*. For any* $M \in \mathbf{RigDA}_{\text{Frobét}}(B)$ *there is a canonical isomorphism*

$$\mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B)(\Lambda(B'), \mathbb{L}i_B^*M) \cong \mathbf{RigDA}_{\text{Frobét}}(B')(\Lambda, \mathbb{L}f^*M).$$

PROOF. Consider the following diagram of functors:

$$\begin{array}{ccc} \mathbf{Psh}(\mathbf{C}_B[\Phi^{-1}]) & \xrightarrow{i_B^*} & \mathbf{Psh}(\text{RigNor}/B[\Phi^{-1}]) \\ \downarrow f^* & & \downarrow f^* \\ \mathbf{Psh}(\mathbf{C}_{B'}[\Phi^{-1}]) & \xrightarrow{i_{B'}^*} & \mathbf{Psh}(\text{RigNor}/B'[\Phi^{-1}]) \end{array}$$

Let $\mathcal{F}$ be in $\mathbf{Psh}(\mathbf{C}_B[\Phi^{-1}])$ and X' be in RigNor/B'. One has $(i_{B'}^* f^*)(\mathcal{F})(X') = \operatorname{colim}\mathcal{F}(V)$ where the colimit is taken over the maps $X' \to V \times_{B^{(-n)}} B'^{(-n)}$ in $\text{RigNor}/B'[\Phi^{-1}]$ by letting V vary among varieties which are smooth over some $B^{(-n)}$. On the other hand, one has $(f^* i_B^*)(\mathcal{F})(X') = \operatorname{colim}\mathcal{F}(V)$ where the colimit is taken over the maps $X' \to V$ in $\text{RigNor}/B[\Phi^{-1}]$ by letting V vary among varieties which are smooth over some $B^{(-n)}$. Since $V \times_{B^{(-n)}} B'^{(-n)} \cong (V \times_B B')^{\text{red}}$ in $\text{RigSm}/B'[\Phi^{-1}]$ we deduce that the indexing categories are equivalent, hence the diagram above is commutative and therefore by Corollary 2.1.12 and what shown in the proof of Proposition 2.1.19 also the following one is:

$$\begin{array}{ccc} \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\text{RigSm}/B^{\text{Perf}}) & \xrightarrow{i_B^*} & \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\text{RigNor}/B) \\ \downarrow f^* & & \downarrow f^* \\ \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\text{RigSm}/B'^{\text{Perf}}) & \xrightarrow{i_{B'}^*} & \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\text{RigNor}/B') \end{array}$$

This fact together with Lemma 2.3.10 implies $f^*\mathbb{L}i_B^* \cong \mathbb{L}i_{B'}^*\mathbb{L}f^*$. By Propositions 2.2.10 and 2.3.6 we then deduce

$$\begin{aligned} &\mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B)(\Lambda(B'), \mathbb{L}i_B^*M) = \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B)(\mathbb{L}f_\sharp(\Lambda), \mathbb{L}i_B^*M) \cong \\ &\cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B')(\Lambda, f^*\mathbb{L}i_B^*M) \cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B')(\Lambda, \mathbb{L}i_{B'}^*\mathbb{L}f^*M) \cong \\ &\cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B')(\mathbb{L}i_{B'}^*\Lambda, \mathbb{L}i_{B'}^*\mathbb{L}f^*M) \cong \mathbf{RigDA}_{\text{Frobét}}(B')(\Lambda, \mathbb{L}f^*M) \end{aligned}$$

as claimed. □

2.3.10. LEMMA. *Let* $f\colon B' \to B$ *be a map of normal varieties over* K*. The functor*

$$f^*\colon \mathbf{Ch}\,\mathbf{Psh}(\text{RigNor}/B) \to \mathbf{Ch}\,\mathbf{Psh}(\text{RigNor}/B')$$

preserves the $(\text{Frobét}, \mathbb{B}^1)$*-equivalences.*

PROOF. Since f^* commutes with Frobét-sheafification and with colimits, it preserves Frobét-equivalences. Since $f^*(\Lambda(\mathbb{B}^1_V)) \cong \mathbb{B}^1_B \otimes f^*(\Lambda(V))$ is weakly equivalent to $f^*(\Lambda(V))$ for every V in RigNor/B we also conclude that f^* preserves $\mathbb{B}^1$-weak equivalences, hence the claim. □

2.3.11. LEMMA. *Let* B *be a normal variety over* K *and let* $f\colon X \to Y$ *be a composition of* Frob*-coverings and* ét*-coverings in* RigNor/B*. The functor* $\mathbb{L}f^*\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(Y^{\text{Perf}}) \to \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(X^{\text{Perf}})$ *is conservative.*

PROOF. If f is a Frob-cover, then $\mathbb{L}f^*$ is an equivalence by Proposition 2.2.10. We are left to prove the claim in case f is an ét-covering. In this case, we can use the proof of the analogous statement in algebraic geometry [**7**, Lemma 3.4]. □

2.3.12. LEMMA. *Let $e\colon X' \to X$ be a finite morphism of normal varieties over K and let G be a finite group acting on $\mathbb{R}e_*\mathbb{L}e^*$. There exists a subfunctor $(\mathbb{R}e_*\mathbb{L}e^*)^G$ of $\mathbb{R}e_*\mathbb{L}e^*$ such that for all M, N in $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})$ one has*

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(M, (\mathbb{R}e_*\mathbb{L}e^*)^G N) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(M, \mathbb{R}e_*\mathbb{L}e^* N)^G.$$

Moreover for any map $f\colon Y \to X$ of normal rigid varieties factoring into a closed embedding followed by a smooth map, and any diagram of normal varieties

$$\begin{array}{ccc} (Y \times_X X')_{\mathrm{red}} & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ Y & \xrightarrow{f} & X \end{array}$$

there is an induced action of G on $\mathbb{R}e'_*\mathbb{L}e'^*$ and an invertible transformation $\mathbb{L}f^*(\mathbb{R}e_*\mathbb{L}e^*)^G \xrightarrow{\sim} (\mathbb{R}e'_*\mathbb{L}e'^*)^G\mathbb{L}f^*$.

PROOF. We define $(\mathbb{R}e_*\mathbb{L}e^*)^G$ to be subfunctor obtained as the image of the projector $\frac{1}{|G|}\sum g$ acting on $\mathbb{R}e_*\mathbb{L}e^*$.

In order to prove the second claim, it suffices to prove that $\mathbb{L}f^*\mathbb{R}e_*\mathbb{L}e^* \cong \mathbb{R}e'_*\mathbb{L}e'^*\mathbb{L}f^*$. As the latter term coincides with $\mathbb{R}e'_*\mathbb{L}(fe')^* = \mathbb{R}e'_*\mathbb{L}(ef')^* = \mathbb{R}e'_*\mathbb{L}f'^*\mathbb{L}e^*$ it suffices to show that the base change transformation $\mathbb{L}f^*\mathbb{R}e_* \to \mathbb{R}e'_*\mathbb{L}f'^*$ is invertible. We can consider individually the case in which f is smooth, and the case in which f is a closed embedding.

Step 1: Suppose that f is smooth. Then f^* has a left adjoint $f_\sharp$. We can equally prove that the natural tranformation $\mathbb{L}f'_\sharp\mathbb{L}e'^* \to \mathbb{L}e^*\mathbb{L}f_\sharp$ is invertible. This follows from the isomorphism between the functors $f'_\sharp e'^*$ and $e^*f_\sharp$ from $\mathbf{Psh}(\mathrm{RigSm}/X'^{\mathrm{Perf}})$ to $\mathbf{Psh}(\mathrm{RigSm}/Y^{\mathrm{Perf}})$ obtained by direct inspection.

Step 2: Suppose that f is a closed immersion. Let $j\colon U \to X$ be the open immersion complementary to f and j' be the open immersion complementary to f'. By the locality axiom (Theorem 2.2.12) we can equally prove that $\mathbb{L}j_\sharp\mathbb{R}e'_* \to \mathbb{R}e_*\mathbb{L}j'_\sharp$ is invertible.

Step 3: It is easy to prove that the transformation $\mathbb{L}j_\sharp\mathbb{R}e'_* \to \mathbb{R}e_*\mathbb{L}j'_\sharp$ is invertible once we know that e_*, e'_*, $j_\sharp$ and $j'_\sharp$ preserve the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences. Indeed, if this is the case, the functors derive trivially and it suffices to prove that for any $\mathrm{Frob\acute{e}t}$-sheaf $\mathcal{F}$ the map $(j_\sharp e'_*)(\mathcal{F}) \to (e_*j'_\sharp)(\mathcal{F})$ is invertible. This follows from the very definitions.

Step 4: The fact that $j_\sharp$ (and similarly $j'_\sharp$) preserves the $(\mathrm{Frob\acute{e}t})$-weak equivalences follows from the fact that it respects quasi-isomorphisms of complexes of $\mathrm{Frob\acute{e}t}$-sheaves, since it is the functor of extension by 0. In order to prove that it preserves the $\mathbb{B}^1$-equivalences, by [**6**, Proposition 4.2.74] we can prove that $j_\sharp(\Lambda(\mathbb{B}^1_V) \to \Lambda(V))$ is a $\mathbb{B}^1$-weak equivalence for all V in $\mathrm{RigSm}/U^{\mathrm{Perf}}$ and this is clear. The fact that e_* (and similarly e'_*) preserves the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences is proved in Proposition 2.2.11. We then conclude the claim in case f is a closed immersion. □

2.3.13. LEMMA. *Let $f\colon X' \to X$ be a pseudo-Galois map of normal varieties over K. There exists a finite stratification $(X_i)_{1\le i\le n}$ of locally closed normal subvarieties of X such that each induced map $f_i\colon (X' \times_X X_i)_{\mathrm{red}} \to X_i$ is a composition of an étale cover and a* Frob*-cover of normal rigid varieties.*

PROOF. For every affinoid rigid variety $\mathrm{Spa}(R, R^\circ)$ there is a map of ringed spaces $\mathrm{Spa}(R, R^\circ) \to \operatorname{Spec} R$ which is surjective on points, and such that the pullback of a finite étale map $\operatorname{Spec} S \to \operatorname{Spec} R$ [resp. of an open inclusion $U \to \operatorname{Spec} R$] over $\mathrm{Spa}(R, R^\circ) \to \operatorname{Spec} R$ exists (following the notation of [**26**, Lemma 3.8]) and is finite étale [resp. an open inclusion]. The claim then follows from the analogous statement valid for schemes over K. □

2.3.14. REMARK. In the proof of Proposition 2.3.8, we made use of the fact that Λ is a $\mathbb{Q}$-algebra in a crucial way, for instance, in order to define the functor $(\mathbb{R}e_*\mathbb{L}e^*)^G$.

The following result proves the second step.

2.3.15. COROLLARY. *Let B be a normal variety over K. The composite functor*

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$$

is fully faithful.

PROOF. This follows at once from Proposition 2.2.9 and Proposition 2.3.8. □

We now move to the third step. We recall that the category $\mathrm{RigCor}(B^{\mathrm{Perf}})$ is a subcategory of $\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$. We denote by j this inclusion of categories.

2.3.16. PROPOSITION. *Let B be a normal variety over K. The functor j induces a fully faithful functor $\mathbb{L}j^*\colon \mathbf{RigDM}^{\mathrm{eff}}(B^{\mathrm{Perf}}) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$.*

PROOF. The functor j extends to a functor $\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$ and induces a Quillen pair $j^*\colon \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B) :j_*$ with respect to the projective model structures. We prove that it is a Quillen adjunction also with respect to the $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-model structure on the two categories by showing that j_* preserves $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-local objects. From the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{RigSm}/B^{\mathrm{Perf}} & \longrightarrow & \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{i} & \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B) \\
\downarrow & & \downarrow{\scriptstyle a_{\mathrm{tr}}} & & \downarrow{\scriptstyle a_{\mathrm{fh}}} \\
\mathrm{RigCor}/B^{\mathrm{Perf}} & \longrightarrow & \mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{j} & \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)
\end{array}$$

we deduce that $o_{\mathrm{tr}}j_* = i_*o_{\mathrm{fh}}$ which is a right Quillen functor. It therefore suffices to show that if $o_{\mathrm{tr}}\mathcal{F}$ is $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-local then also $\mathcal{F}$ is, for every fibrant object $\mathcal{F}$. Let $\mathcal{F} \to \mathcal{F}'$ be a $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-weak equivalence to a $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-fibrant object of $\mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$. By Lemma 2.3.17, we deduce that $o_{\mathrm{tr}}\mathcal{F} \to o_{\mathrm{tr}}\mathcal{F}'$ is a $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-weak equivalence between $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-fibrant objects, hence it is a quasi-isomorphism. As o_{tr} reflects quasi-isomorphisms, we conclude that $\mathcal{F}$ is quasi-isomorphic to $\mathcal{F}'$ hence $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-local.

We now prove that $\mathbb{L}j^*$ is fully faithful by proving that $\mathbb{R}j_*\mathbb{L}j^*$ is isomorphic to the identity. As j_*j^* is isomorphic to the identity, it suffices to show that $\mathbb{R}j_* = j_*$. We start by proving that j_* preserves Frobét-weak equivalences. As shown in Remark 2.3.7, the functor i_* preserves Frobét-equivalences. It is also clear that o_{fh} does. Since o_{tr} reflects Frobét-weak equivalences, the claim follows from the equality $o_{\mathrm{tr}}j_* = i_*o_{\mathrm{fh}}$. Since $j_*(\Lambda(\mathbb{B}^1_V)) \cong \Lambda(\mathbb{B}^1_B) \otimes j_*(\Lambda(V))$ is weakly equivalent to $j_*(\Lambda(V))$ for every V in RigNor/B, we also conclude that j_* preserves $\mathbb{B}^1$-weak equivalences, hence the claim. □

2.3.17. LEMMA. *Let B be a normal variety over K. The functor*

$$o_{\mathrm{tr}}\colon \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves $(\mathrm{\acute{e}t}, \mathbb{B}^1)$-weak equivalences.

PROOF. The argument of [**3**, Lemma 2.111] easily generalizes to our context. □

The fourth step is just an easy check, as the next proposition shows.

2.3.18. PROPOSITION. *Let B be a normal variety over K. The composite functor $\mathbb{L}j^* \circ \mathbb{L}a_{\mathrm{tr}}$ is isomorphic to $\mathbb{L}i^*$. In particular $\mathbb{L}a_{\mathrm{tr}}$ is fully faithful.*

PROOF. It suffices to check that the following square is quasi-commutative.

$$\begin{array}{ccc} \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{a_{\mathrm{tr}}} & \mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \\ \downarrow i & & \downarrow j \\ \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B) & \xrightarrow{a_{\mathrm{fh}}} & \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B) \end{array}$$

This can be done by inspecting the two composite right adjoints, which are canonically isomorphic. □

This also ends the proof of Theorem 2.3.1.

We remark that in case K is endowed with the trivial norm, we obtain a result on the category of motives constructed from schemes over K. It is the natural generalization of [**3**, Theorem B.1] in positive characteristic. We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra.

2.3.19. THEOREM. *Let B be a normal algebraic variety over a perfect field K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \cong \mathbf{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}).$$

We now define the stable version of the categories of motives introduced so far, and remark that Theorem 2.3.3 extends formally to the stable case providing a generalization of the result [**13**, Theorem 15.2.16].

2.3.20. DEFINITION. We denote by $\mathbf{RigDA}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ [resp. by $\mathbf{RigDM}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$] the homotopy category associated to the model category of symmetric spectra (see [**6**, Section 4.3.2]) $\mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ [resp. $\mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$] where T is the cokernel of the unit map $\Lambda(B) \to \Lambda(\mathbb{T}^1_B)$ [resp $\Lambda_{\mathrm{tr}}(B) \to \Lambda_{\mathrm{tr}}(\mathbb{T}^1_B)$].

2.3.21. COROLLARY. *Let B be a normal variety over K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \cong \mathbf{RigDM}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}).$$

PROOF. Theorem 2.3.3 states that the adjunction

$$a_{tr}\colon \mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) : o_{\mathrm{tr}}$$

is a Quillen equivalence. It therefore induces a Quillen equivalence on the categories of symmetric spectra

$$a_{tr}\colon \mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) : o_{\mathrm{tr}}$$

by means of [**6**, Proposition 4.3.35]. □

We now assume that Λ equals $\mathbb{Z}$ if $\operatorname{char} K = 0$ and equals $\mathbb{Z}[1/p]$ if $\operatorname{char} K = p$. In analogy with the statement $\mathbf{DA}_{\mathrm{\acute{e}t}}(B, \Lambda) \cong \mathbf{DM}_{\mathrm{\acute{e}t}}(B, \Lambda)$ proved for motives associated to schemes (see [**7**, Appendix B]) it is expected that the following result also holds.

2.3.22. CONJECTURE. *Let B be a normal variety over K. The functors (a_{tr}, o_{tr}) induce an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}_{\text{ét}}(B,\Lambda) \cong \mathbf{RigDM}_{\text{ét}}(B,\Lambda).$$

We remark that in the above statement differs from Corollary 2.3.21 for two main reasons: the ring of coefficients is no longer assumed to be a $\mathbb{Q}$-algebra, and the class of maps with respect to which we localize are the ét-local maps and no longer the Frobét-local maps.

In order to reach this twofold generalization, using the techniques developed in [**7**], it would suffice to show the two following formal properties of the 2-functor $\mathbf{RigDA}_{\text{ét}}$:

- *Separateness*: for any Frob-cover $B' \to B$ the functor
$$\mathbf{RigDA}_{\text{ét}}(B,\Lambda) \to \mathbf{RigDA}_{\text{ét}}(B',\Lambda)$$
is an equivalence of categories.
- *Rigidity*: if $\operatorname{char} K \nmid N$ the functor
$$\mathbf{D}\,\mathbf{Sh}_{\text{ét}}(\operatorname{Et}/B, \mathbb{Z}/N\mathbb{Z}) \to \mathbf{RigDA}_{\text{ét}}(B,\mathbb{Z}/N\mathbb{Z})$$
is an equivalence of categories where Et/B is the small étale site over B.

APPENDIX A

An implicit function theorem and approximation results

The aim of this appendix is to prove approximation results for maps defined from objects $\varprojlim X_h$ in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ to rigid analytic varieties. We will show that, up to homotopy, any such map factors over an analytic space X_h in the direct system.

Along this chapter, we assume that K is a complete non-archimedean field.

A.1. A non-archimedean implicit function theorem

We begin our analysis with the analogue of the inverse mapping theorem, which is a variant of [**28**, Theorem 2.1.1].

A.1.1. PROPOSITION. *Let R be a K-algebra, let $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_m)$ be two systems of coordinates and let $P = (P_1, \dots, P_m)$ be a collection of polynomials in $R[\sigma, \tau]$ such that $P(\sigma = 0, \tau = 0) = 0$ and $\det(\frac{\partial P_i}{\partial \tau_j})(\sigma = 0, \tau = 0) \in R^\times$. There exists a unique collection $F = (F_1, \dots, F_m)$ of m formal power series in $R[[\sigma]]$ such that $F(\sigma = 0) = 0$ and $P(\sigma, F(\sigma)) = 0$ in $R[[\sigma]]$.*

Moreover, if R is a Banach K-algebra, then the polynomials $P_1, \dots, P_n$ have a positive radius of convergence.

PROOF. Let f be the polynomial $\det(\frac{\partial P_i}{\partial \tau_j})$ in $R[\sigma, \tau]$ and let S be the ring $R[\sigma, \tau]_f/(P)$. The induced map $R[\sigma] \to S$ is étale, and from the hypothesis $f(0,0) \in R^\times$ we conclude that the map $R[\sigma, \tau]/(P) \to R$, $(\sigma, \tau) \mapsto 0$ factors through S.

Suppose given a factorization as $R[\sigma]$-algebras $S \to R[\sigma]/(\sigma)^n \to R$ of the map $S \to R$. By the étale lifting property (see [**21**, Definition IV.17.1.1 and Corollary IV.17.6.2]) applied to the square

$$\begin{array}{ccc} R[\sigma] & \longrightarrow & R[\sigma]/(\sigma)^{n+1} \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow \\ S & \longrightarrow & R[\sigma]/(\sigma)^n \end{array}$$

we obtain a uniquely defined $R[\sigma]$-linear map $S \to R[\sigma]/(\sigma)^{n+1}$ factoring $S \to R$ and hence by induction a uniquely defined $R[\sigma]$-linear map $R[\sigma, \tau]/(P) \to R[[\sigma]]$ factoring $R[\sigma, \tau]/(P) \to R$ as wanted. The power series F_i is the image of τ_i via this map.

Assume now that R is a Banach K-algebra. We want to prove that the array $F = (F_1, \dots, F_m)$ of formal power series in $R[[\sigma]]$ constructed above is convergent around 0. As R is complete, this amounts to proving estimates on the valuation of the coefficients of F. To this aim, we now try to give an explicit description of them, depending on the coefficients of P. Whenever I is a n-multi-index $I = (i_1, \dots, i_n)$ we denote by σ^I the product $\sigma_1^{i_1} \cdot \ldots \cdot \sigma_n^{i_n}$ and we adopt the analogous notation for τ.

We remark that the claim is not affected by any invertible R-linear transformation of the polynomials P_i. Therefore, by multiplying the column vector P by the matrix $(\frac{\partial P_i}{\partial \tau_j})(0,0)^{-1}$

we reduce to the case in which $(\frac{\partial P_i}{\partial \tau_j})(0,0) = \delta_{ij}$. We can then write the polynomials P_i in the following form:

$$P_i(\sigma, \tau) = \tau_i - \sum_{|J|+|H|>0} c_{iJH}\sigma^J \tau^H$$

where J is an n-multi-index, H is an m-multi-index and the coefficients c_{iJH} equal 0 whenever $|J| = 0$ and $|H| = 1$.

We will determine the functions $F_i(\sigma)$ explicitly. We start by writing them as

$$F_i(\sigma) = \sum_{|I|>0} d_{iI}\sigma^I$$

with unknown coefficients d_{iI} for any n-multi-index I. We denote their q-homogeneous parts by

$$F_{iq}(\sigma) := \sum_{|I|=q} d_{iI}\sigma^I.$$

We need to solve the equation $P(\sigma, F(\sigma)) = 0$ which can be rewritten as

$$F_i(\sigma) = \sum_{J,H} c_{iJH}\sigma^J \left(\prod_{r=1}^{m} F_r(\sigma)^{h_r}\right)$$

where we denote by h_r the components of the m-multi-index H.

By comparing the q-homogeneous parts we get

$$F_{iq}(\sigma) = \sum_{(J,H,\Phi)\in\Sigma_{iq}} c_{iJH}\sigma^J \prod_{r=1}^{m}\prod_{s=1}^{h_r} F_{r,\Phi(r,s)}(\sigma)$$

where the set Σ_{iq} consists of triples (J, H, Φ) in which J is a n-multi-index, H is a m-multi-index and Φ is a function that associates to any element (r, s) of the set

$$\{(r, s) : r = 1, \ldots, m; s = 1, \ldots, h_r\}$$

a positive (non-zero!) integer $\Phi(r, s)$ such that $\sum \Phi(r, s) = q - |J|$.

If $\Phi(r, s) \geq q$ for some r we see by definition that $|J| = 0$, $|H| = 1$ and we know that in this case $c_{i0H} = 0$. In particular, we conclude that the right hand side of the formula above involves only $F_{rq'}$'s with $q' < q$. Hence, we can determine the coefficients d_{iI} by induction on $|I|$. Moreover, by construction, each coefficient d_{iI} can be expressed as

(1) $$d_{iI} = Q_{iI}(c_{iJH})$$

where each Q_{iI} is a polynomial in c_{iJH} for $|J| + |H| \leq |I|$ with coefficients in $\mathbb{N}$.

We can fix a non-zero topological nilpotent element π such that $||c_{iJK}|| \leq |\pi|^{-1}$ for all i, J, H. From the argument above, we deduce inductively that each coefficient d_{iI} is a finite sum of products of the form $\prod c_{kJH}$ with $\sum |J| \leq |I|$. In particular, each product has at most $|I|$ factors and hence $||d_{iI}|| \leq |\pi|^{-|I|}$. We conclude $||d_{iI}\pi^{2|I|}|| \leq |\pi|^{|I|}$ which tends to 0 as $|I| \to \infty$. □

The previous statement has an immediate generalization.

A.1.2. Corollary. *Let R be a non-archimedean Banach K-algebra, let $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots, \tau_m)$ be two systems of coordinates, let $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ and $\bar{\tau} = (\bar{\tau}_1, \ldots, \bar{\tau}_m)$ two sequences of elements of R and let $P = (P_1, \ldots, P_m)$ be a collection of polynomials in $R[\sigma, \tau]$ such that $P(\sigma = \bar{\sigma}, \tau = \bar{\tau}) = 0$ and $\det(\frac{\partial P_i}{\partial \tau_j})(\sigma = \bar{\sigma}, \tau = \bar{\tau}) \in R^\times$. There* *exists a unique collection $F = (F_1, \dots, F_m)$ of m formal power series in $R[[\sigma - \bar\sigma]]$ such that $F(\sigma = \bar\sigma) = \bar\tau$ and $P(\sigma, F(\sigma)) = 0$ in $R[[\sigma - \bar\sigma]]$ and they have a positive radius of convergence around $\bar\sigma$.*

PROOF. If we apply Proposition A.1.1 to the polynomials $P'_i := P(\bar\sigma + \eta, \bar\tau + \theta)$ we obtain an array of formal power series $F' = (F_1, \dots, F'_m)$ in $R[[\eta]]$ with positive radius of convergence such that $P'(\eta, F'(\eta)) = 0$. If we now put $\sigma := \bar\sigma + \eta$ and $F := \bar\tau + F'$ we get $P(\sigma, F(\sigma - \bar\sigma)) = 0$ in $R[[\sigma - \bar\sigma]]$ as wanted. □

We now assume that K is perfectoid and we come back to the category $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$ that we introduced above (see Definition 1.2.3). We recall that an object $X = \varprojlim_h X_h$ of this category is the pullback over $\widehat{\mathbb{T}}^N \to \mathbb{T}^N$ of a map $X_0 \to \mathbb{T}^N \times \mathbb{T}^M$ that is a composition of rational embeddings and finite étale maps from an affinoid tft adic space X_0 to a torus $\mathbb{T}^N \times \mathbb{T}^M = \operatorname{Spa} K\langle \underline{\upsilon}^{\pm1}, \underline{\nu}^{\pm1}\rangle$ and X_h denotes the pullback of X_0 by $\mathbb{T}^N\langle \underline{\upsilon}^{1/p^h}\rangle \to \mathbb{T}^N$.

A.1.3. PROPOSITION. *Let $X = \varprojlim_h X_h$ be an object of $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. If an element ξ of $\mathcal{O}^+(X)$ is algebraic and separable over each generic point of $\operatorname{Spec} \mathcal{O}(X_0)$ then it lies in $\mathcal{O}^+(X_{\bar h})$ for some $\bar h$.*

PROOF. Let X_0 be $\operatorname{Spa}(R_0, R_0^\circ)$ let X_h be $\operatorname{Spa}(R_h, R_h^\circ)$ and X be $\operatorname{Spa}(R, R^+)$. For any $h \in \mathbb{N}$ one has $R_h = R_0 \widehat\otimes_{K\langle \underline{\upsilon}^{\pm1}\rangle} K\langle \underline{\upsilon}^{\pm1/p^h}\rangle$ and R^+ coincides with the π-adic completion of $\varinjlim_h R_h^\circ$ by Proposition 1.2.1.The proof is divided in several steps.

Step 1: We can suppose that R is perfectoid. Indeed, we can consider the refined tower $X'_h = X_0 \times_{\mathbb{T}^N \times \mathbb{T}^M} (\mathbb{T}^N\langle \underline{\upsilon}^{1/p^h}\rangle \times \mathbb{T}^M\langle \underline{\nu}^{1/p^h}\rangle)$ whose limit $\widehat X$ is perfectoid. If the claim is true for this tower, we conclude that ξ lies in the intersection of $\mathcal{O}(X'_h)$ and $\mathcal{O}(X)$ inside $\mathcal{O}(\widehat X)$ for some h. By Remark 1.1.18 this is the intersection

$$\left(\widehat{\bigoplus}_{I \in (\mathbb{Z}[1/p] \cap [0,1))^N} R_0 \underline{\upsilon}^I\right) \cap \left(\widehat{\bigoplus}_{\substack{I \in \{a/p^h \colon 0 \le a < p^h\}^N \\ J \in \{a/p^h \colon 0 \le a < p^h\}^M}} R_0 \underline{\upsilon}^I \underline{\nu}^J\right)$$

which coincides with

$$\widehat{\bigoplus}_{I \in \{a/p^h \colon 0 \le a < p^h\}^N} R_0 \underline{\upsilon}^I = R_h.$$

Step 2: We can always assume that each R_h is an integral domain. Indeed, the number of connected components of $\operatorname{Spa} R_h$ may rise, but it is bounded by the number of connected components of the affinoid perfectoid X which is finite by Remark 1.2.11.

We deduce that the number of connected components of $\operatorname{Spa} R_h$ stabilizes for h large enough. Up to shifting indices, we can then suppose that $\operatorname{Spa} R_0$ is the finite disjoint union of irreducible rigid varieties $\operatorname{Spa} R_{i0}$ for $i = 1, \dots, k$ such that $R_{ih} = R_{i0} \widehat\otimes_{K\langle \underline{\upsilon}^{\pm1}\rangle} K\langle \underline{\upsilon}^{\pm1/p^h}\rangle$ is a domain for all h. We denote by R_i the ring $R_{i0} \widehat\otimes_{K\langle \underline{\upsilon}^{\pm1}\rangle} K\langle \underline{\upsilon}^{\pm1/p^\infty}\rangle$. Let now $\xi = (\xi_i)$ be an element in $R^+ = \prod R_i^+$ that is separable over $\prod \operatorname{Frac} R_i$ i.e. each ξ_i is separable over $\operatorname{Frac} R_i$. If the proposition holds for R_i we then conclude that ξ_i lies in R_{ih}° for some large enough h so that $\xi \in R_h^\circ$ as claimed.

Step 3: We prove that we can consider a non-empty rational subspace $U_0 = \operatorname{Spa} R_0\langle f_i/g\rangle$ of X_0 instead. Indeed, using Remark 1.1.18 if the result holds for U_0 assuming $\bar h = 0$ we deduce that ξ lies in the intersection of $R \cong \widehat\bigoplus R_0$ and of $R_0\langle f_i/g\rangle$ inside $R\langle f_i/g\rangle \cong \widehat\bigoplus R_0\langle f_i/g\rangle$ which coincides with R_0.

Step 4: We prove that we can assume ξ to be integral over R_0. Indeed, let P_ξ be its minimal polynomial over $\operatorname{Frac}(R_0)$. We can suppose there is a common denominator d such that P_ξ has coefficients in $R_0[1/d][x]$. By [**9**, Proposition 6.2.1/4(ii)] we can also assume that $|d| = 1$. In particular, by [**9**, Proposition 7.2.6/3], the rational subset associated to $R_0\langle 1/d\rangle$ is not empty. By Step 3, we can then restrict to it and assume ξ integral over R_0 and $R_0[\xi] \cong R_0[x]/P_\xi(x)$.

Step 5: We can suppose that $P_\xi(x)$ is the minimal polynomial of ξ with respect to all non-empty rational subspaces of X_h for all h. If it is not the case, from the previous steps we can rescale indices and restrict to a rational subspace with respect to which the degree of $P_\xi(x)$ is lower. Since the degree is bounded from below, we conclude the claim.

Step 6: We prove that we can assume that the sup-norm on R_h is multiplicative for all h. By [**9**, Proposition 6.2.3/5] this is equivalent to state that $\widetilde{R}_h := R_h^\circ/R_h^{\circ\circ}$ is a domain. The maps $R_h \to R_{h+1}$ induce inclusions $\tilde{R}_h \to \tilde{R}_{h+1}$ by [**9**, Lemma 3.8.1/6] and these rings are included in $\widetilde{R} := R^\circ/R^{\circ\circ}$ which is isomorphic to $\widetilde{R^\flat}$ by [**42**, Proposition 5.17]. Up to considering a rational subspace, we can assume that $R^\flat$ is the perfection of a smooth affinoid rigid variety $R_0^\flat$ and $\widetilde{R^\flat}$ is a domain if and only if $\widetilde{R_0^\flat}$ is. As this last ring is reduced, there is a Zariski open in which it is a domain, and hence by [**9**, Proposition 7.2.6/3] there is a non-empty rational subspace of $\mathrm{Spa}(R^\flat, R^{\flat+})$ and therefore of $\mathrm{Spa}(R, R^+)$ with the required property (the tilting equivalence preserves rational subspaces as proved in [**42**, Proposition 6.17]). We conclude the claim since rational subspaces of X descend to X_h for h big enough by Proposition 1.2.8. We can assume this happens at $h = 0$.

Step 7: Since R is the completion of $\varinjlim_h R_h$ with respect to the sup-norms, by the previous step we deduce that the norm $||\cdot||$ on R is multiplicative. Fix a separable closure L of the completion of $\mathrm{Frac}\, R$ with respect to $||\cdot||$. The element ξ and its conjugates $\xi_1, \ldots, \xi_n$ that are different from ξ all lie in the integral closure S of the ring $\varinjlim_h R_h$ in L which coincides with the integral closure of R_0 since all maps $R_0 \to R_h$ are integral. We can assume that for all i the minimal polynomial of $\xi - \xi_i$ over R_0 coincides with the one over all rings $R_h\langle 1/f\rangle$ with $|f| = 1$. Otherwise, restrict to some rational subspace $U(1 \mid \bar{f})$ of $X_{\bar{h}}$ with $|\bar{f}| = 1$ where this holds and rescale indices. By [**9**, Proposition 7.2.6/3] the hypotheses of the previous step are still preserved. Because R_0 is normal, by means of [**9**, Proposition 3.8.1/7] we can also endow S with the sup-norm $|\cdot|$. Let ϵ be the positive number $\min\{|\xi - \xi_i|\}$. By the density of $\varinjlim_h R_h$ in R we can find an element $\beta \in R_{\bar{h}}$ for some $\bar{h}$ such that $||\xi - \beta|| < \epsilon$. Up to rescaling indices, we can assume $\bar{h} = 0$.

Step 8: We prove that we can assume that the sup-norm on $R_0[\xi]$ is multiplicative. We remark that this ring is a tft Tate K-algebra by [**9**, Proposition 6.1.1/6]. The ring $\widetilde{R_0[\xi]}$ is reduced, contains the domain $\widetilde{R}_0$ and is finite over it (see [**9**, Proposition 1.2.5/7, Lemma 3.8.1/6, Theorem 6.3.1/6 and Theorem 6.3.5/1]). Up to considering an open of $\mathrm{Spec}\, \widetilde{R}_0$ and hence restricting to a non-empty rational subset $U(1 \mid f)$ of $\mathrm{Spa}\, R_0$ with $|f| = 1$ (see [**9**, Proposition 7.2.6/3]) we can then assume that the variety $\mathrm{Spec}\, \widetilde{R_0[\xi]}$ is a disjoint union of integral schemes. Since the spectrum of $R_0[\xi] \cong R_0[x]/P_\xi(x)$ is connected, we deduce that $\mathrm{Spec}\, \widetilde{R_0[\xi]}$ is also connected hence integral, and the sup norm on $R_0[\xi]$ is multiplicative by means of [**9**, Proposition 6.2.3/5]. We also remark that, by the construction of our restrictions, the rings $\tilde{R}_h$ are still domains hence the sup-norm is multiplicative on R_h. Moreover, the inequalities $||\xi - \beta|| < \epsilon$ and $|\xi - \xi_i| > \epsilon$ still hold since the maps $R_h \to R_h\langle 1/f\rangle$ are isometries with respect to the sup-norm (see [**9**, Lemma 6.3.1/6]) and because of our hypotheses from Step 7 together with the formulas computing the sup-norm on S (see [**9**, Proposition 3.8.1/7]).

Step 9: We prove that we the norm on $R_0[\xi]$ induced by R coincides with the sup-norm on this ring. By Step 7 and Step 8 the norm $||\cdot||$ on R and the sup-norm $|\cdot|_{\sup}$ on $R_0[\xi]$ are multiplicative, and both extend the sup-norm on R_0. Since the map $R_0[\xi] \to R$ is continuous, there is an integer n such that $|b|_{\sup} \leq |\pi|^n$ implies $||b|| \leq 1$ for all $b \in R_0[\xi]$. By Lemma A.1.4 we deduce that the two norms $|\cdot|_{\sup}$ and $||\cdot||$ on $R_0[\xi]$ coincide, as claimed.

Step 10: In this last step we argue as for Krasner's Lemma (see [**9**, Section 3.4.2]). The maps $R_h \to S$ and $R_0[\xi] \to S$ are all isometries with respect to sup-norms by [**9**, Lemma 3.8.1/6]. By the previous step, we deduce $|\xi - \beta| < \epsilon$ with respect to the sup-norm on $R_0[\xi]$. We now show that $n = 0$ i.e. that the degree of the separable polynomial $P_\xi(x)$ is 1 and therefore ξ lies in R_0. We argue by contradiction and we assume $n \geq 1$. Any choice of an element ξ_i induces a R_0-linear isomorphism $\tau_i\colon R_0[\xi] \cong R_0[\xi_i]$ which is an isometry with respect to the sup-norm. Therefore one has $|\xi - \xi_i| \leq \max\{|\xi - \beta|, |\xi_i - \beta|\} = \max\{|\xi - \beta|, |\tau_i(\xi - \beta)|\} = |\xi - \beta| < \epsilon$ leading to a contradiction. □

A.1.4. LEMMA. *Let $R \to S$ be an integral extension of integral domains over K. Let $|\cdot|$ be a multiplicative K-algebra norm on R and let $|\cdot|_1$ and $|\cdot|_2$ be two multiplicative norms on S extending the one of R such that $|b|_1 \leq \varepsilon$ implies $|b|_2 \leq 1$ for all $b \in S$ for a fixed $\varepsilon \in (0, 1] \subset \mathbb{R}$. Then $|\cdot|_1 = |\cdot|_2$.*

PROOF. We can suppose that $\varepsilon = |\alpha|$ for some $\alpha \in K^\times$. We first prove the inequality $\varepsilon|b|_2 \leq |b|_1$ for all $b \in S$. Fix an element $b \in S$ and a sequence of rational numbers in $\mathbb{Z}[1/p]$ such that $|\pi|^{m_i/n_i}$ converges to $|b|_1$ from above. From the inequality $|\pi^{-m_i/n_i}\alpha b|_1 \leq \varepsilon$ we deduce $\varepsilon|b|_2 \leq |\pi|^{m_i/n_i}$ and hence $\varepsilon|b|_2 \leq |b|_1$ as claimed.

We can endow the field $\operatorname{Frac} S$ with the extensions $|\cdot|_i$ of the norms of S by putting $|f/g|_i := |f|_i/|g|_i$. They are well defined and multiplicative. Since S is integral over R any element of $\operatorname{Frac} S$ is of the form f/g with $g \in R$. From what we proved above, one has $\varepsilon|b|_2 \leq |b|_1$ for all $b \in \operatorname{Frac} S$.

From standard valuation theory we then conclude that the two norms are equivalent on $\operatorname{Frac} S$ (for example, apply [**38**, Theorem II.3.4] with $a_1 = 0$ and $a_2 = 1$). Since they agree on K we conclude that they actually coincide on $\operatorname{Frac} S$ hence on S. □

A.2. Approximation of maps of adic spaces

We introduce now the geometric application of Propositions A.1.1 and A.1.3. It states that a map from $\varprojlim_h X_h \in \widehat{\mathrm{RigSm}}$ to a rigid variety factors, up to $\mathbb{B}^1$-homotopy, over one of the intermediate varieties X_h. Analogous statements are widely used in in [**5**] (see for example [**5**, Theorem 2.2.49]): there, these are obtained as corollaries of Popescu's theorem ([**39**] and [**40**]), which is not available in our non-noetherian setting.

A.2.1. PROPOSITION. *Let $X = \varprojlim_h X_h$ be in $\widehat{\mathrm{RigSm}}^{\mathrm{gc}}$. Let Y be an affinoid rigid variety endowed with an étale map $Y \to \mathbb{B}^n$ and let $f\colon X \to Y$ be a map of adic spaces.*

(1) There exist m polynomials $Q_1, \dots, Q_m$ in $K[\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_m]$ such that $Y \cong \operatorname{Spa} A$ with $A \cong K\langle\sigma, \tau\rangle/(Q)$ and $\det(\frac{\partial Q_i}{\partial \tau_j}) \in A^\times$.

(2) There exists a map $H\colon X \times \mathbb{B}^1 \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1$ factors over the canonical map $X \to X_h$ for some integer h.

Moreover, if f is induced by the map $K\langle\sigma, \tau\rangle \to \mathcal{O}(X)$, $\sigma \mapsto s, \tau \mapsto t$ the map H can be defined via

$$(\sigma, \tau) \mapsto (s + (\tilde{s} - s)\chi, F(s + (\tilde{s} - s)\chi))$$

where F is the unique array of formal power series in $\mathcal{O}(X)[[\sigma - s]]$ associated to the polynomials $P(\sigma, \tau)$ by Corollary A.1.2, and $\tilde{s}$ is any element in $\varinjlim_h \mathcal{O}^+(X_h)$ such that the radius of convergence of F is larger than $||\tilde{s} - s||$ and $F(\tilde{s})$ lies in $\mathcal{O}^+(X)$.

PROOF. The first claim follows from the proof of [**5**, Lemma 1.1.50]. We turn to the second claim. Let X_0 be $\operatorname{Spa}(R_0, R_0^\circ)$ and X be $\operatorname{Spa}(R, R^+)$. For any $h \in \mathbb{N}$ we denote $R_0 \widehat{\otimes}_{K\langle \underline{\upsilon}\rangle} K\langle \underline{\upsilon}^{\pm 1/p^h}\rangle$ with R_h so that R^+ coincides with the π-adic completion of $\varinjlim_h R_h^\circ$ by Proposition 1.2.1.

The map f is determined by the choice of n elements $s = (s_1, \ldots, s_n)$ and m elements $t = (t_1, \ldots, t_m)$ of R^+ such that $P(s,t) = 0$. We prove that the formula for H provided in the statement defines a map H with the required properties.

By Corollary A.1.2 there exists a collection $F = (F_1, \ldots, F_m)$ of m formal power series in $R[[\sigma - s]]$ with a positive radius of convergence such that $F(s) = t$ and $P(\sigma, F(\sigma)) = 0$. As $\varinjlim_h R_h^\circ$ is dense in R^+ we can find an integer $\bar{h}$ and elements $\tilde{s}_i \in R_{\bar{h}}^\circ$ such that $||\tilde{s} - s||$ is smaller than the convergence radius of F. By renaming the indices, we can assume that $\bar{h} = 0$. As F is continuous and R^+ is open, we can also assume that the elements $F_j(\tilde{s})$ lie in R^+. We are left to prove that they actually lie in $\varinjlim_h R_h^\circ$. Since the determinant of $(\frac{\partial P_i}{\partial \tau_j})(\tilde{s}, F(\tilde{s}))$ is invertible, the field $L := \operatorname{Frac}(R_0)(F_1(\tilde{s}), \ldots, F_m(\tilde{s}))$ is algebraic and separable over $\operatorname{Frac}(R_0)$. We can then apply Proposition A.1.3 to conclude that each element $F_j(\tilde{s})$ lies in R_h° for a sufficiently big integer h. □

The goal of the rest of this section is to prove Proposition 1.4.1. To this aim, we present a generalization of the results above for collections of maps. As before, we start with an algebraic statement and then translate it into a geometrical fact for our specific purposes.

A.2.2. PROPOSITION. *Let R be a Banach K-algebra and let $\{R_h\}_{h\in\mathbb{N}}$ be a collection of nested complete subrings of R such that $\varinjlim R_h$ is dense in R. Let $s_1, \ldots, s_N$ be elements of $R\langle\theta_1, \ldots, \theta_n\rangle$. For any $\varepsilon > 0$ there exists an integer h and elements $\tilde{s}_1, \ldots, \tilde{s}_N$ of $R_h\langle\theta_1, \ldots, \theta_n\rangle$ satisfying the following conditions.*

(1) $|s_\alpha - \tilde{s}_\alpha| < \varepsilon$ for each α.
(2) For any $\alpha, \beta \in \{1, \ldots, N\}$ and any $k \in \{1, \ldots, n\}$ such that $s_\alpha|_{\theta_k=0} = s_\beta|_{\theta_k=0}$ we also have $\tilde{s}_\alpha|_{\theta_k=0} = \tilde{s}_\beta|_{\theta_k=0}$.
(3) For any $\alpha, \beta \in \{1, \ldots, N\}$ and any $k \in \{1, \ldots, n\}$ such that $s_\alpha|_{\theta_k=1} = s_\beta|_{\theta_k=1}$ we also have $\tilde{s}_\alpha|_{\theta_k=1} = \tilde{s}_\beta|_{\theta_k=1}$.
(4) For any $\alpha \in \{1, \ldots, N\}$ if $s_\alpha|_{\theta_1=1} \in R_{h'}\langle\underline{\theta}\rangle$ for some h' then $\tilde{s}_\alpha|_{\theta_1=1} = s_\alpha|_{\theta_1=1}$.

PROOF. We will actually prove a stronger statement, namely that we can reinforce the previous conditions with the following:

(5) For any $\alpha, \beta \in \{1, \ldots, N\}$ any subset T of $\{1, \ldots, n\}$ and any map $\sigma\colon T \to \{0,1\}$ such that $s_\alpha|_\sigma = s_\beta|_\sigma$ then $\tilde{s}_\alpha|_\sigma = \tilde{s}_\beta|_\sigma$.
(6) For any $\alpha \in \{1, \ldots, N\}$ any subset T of $\{1, \ldots, n\}$ containing 1 and any map $\sigma\colon T \to \{0,1\}$ such that $s_\alpha|_\sigma \in R_h\langle\underline{\theta}\rangle$ for some h then $\tilde{s}_\alpha|_\sigma = s_\alpha|_\sigma$.

Above we denote by $s|_\sigma$ the image of s via the substitution $(\theta_t = \sigma(t))_{t\in T}$. We proceed by induction on N, the case $N = 0$ being trivial.

Consider the conditions we want to preserve that involve the index N. They are of the form

$$s_i|_\sigma = s_N|_\sigma$$

and are indexed by some pairs (σ, i) where i is an index and σ varies in a set of maps Σ. Our procedure consists in determining by induction the elements $\tilde{s}_1, \ldots, \tilde{s}_{N-1}$ first, and then deduce the existence of $\tilde{s}_N$ by means of Lemma A.2.5 by lifting the elements $\{\tilde{s}_i|_\sigma\}_{(\sigma,i)}$. Therefore, we first define $\varepsilon' := \frac{1}{C}\varepsilon$ where $C = C(\Sigma)$ is the constant introduced in Lemma A.2.5 and then apply the induction hypothesis to the first $N-1$ elements with respect to ε'.

By the induction hypothesis, the elements $\tilde{s}_i|_\sigma$ satisfy the compatibility condition of Lemma A.2.5 and lie in $R_h\langle\underline{\theta}\rangle$ for some integer h. Without loss of generality, we assume $h = 0$. By Lemma A.2.5 we can find an element $\tilde{s}_N$ of $R_h\langle\underline{\theta}\rangle$ lifting them such that $|\tilde{s}_N - s_N| < C\varepsilon' = \varepsilon$ as wanted. □

The following lemmas are used in the proof of the previous proposition.

A.2.3. LEMMA. *For any normed ring R and any map $\sigma\colon T_\sigma \to \{0,1\}$ defined on a subset T_σ of $\{1,\dots,n\}$ we denote by I_σ the ideal of $R\langle\underline{\theta}\rangle$ generated by $\theta_i - \sigma(i)$ as i varies in T_σ. For any finite set Σ of such maps and any such map η one has $\left(\bigcap_{\sigma\in\Sigma} I_\sigma\right) + I_\eta = \bigcap_{\sigma\in\Sigma}(I_\sigma + I_\eta)$.*

PROOF. We only need to prove the inclusion $\bigcap(I_\sigma + I_\eta) \subseteq (\bigcap I_\sigma) + I_\eta$. We can make induction on the cardinality of T_η and restrict to the case in which T_η is a singleton. By changing variables, we can suppose $T_\eta = \{1\}$ and $\eta(1) = 0$ so that $I_\eta = (\theta_1)$.

We first suppose that $1 \notin T_\sigma$ for all $\sigma \in \Sigma$. Let s be an element of $\bigcap(I_\sigma + (\theta_1))$. This means we can find elements $s_\sigma \in I_\sigma$ and polynomials $p_\sigma \in R\langle\underline{\theta}\rangle$ such that $s = s_\sigma + p_\sigma\theta_1$. Since I_σ is generated by polynomials of the form $\theta_i - \epsilon$ with $i \neq 1$ we can suppose that s_σ contains no θ_1 by eventually changing p_σ. Let now σ, σ' be in Σ. From the equality

$$s_\sigma = (s_\sigma + p_\sigma\theta_1)|_{\theta_1=0} = (s_{\sigma'} + p_{\sigma'}\theta_1)|_{\theta_1=0} = s_{\sigma'}$$

we conclude that $s_\sigma \in \bigcap I_\sigma$. Therefore $s \in \bigcap I_\sigma + (\theta_1)$ as claimed.

We now move to the general case. Suppose $\bar\sigma(1) = 1$ for some $\bar\sigma \in \Sigma$. Then $I_{\bar\sigma} + I_\eta = R\langle\underline{\theta}\rangle$ and if $f \in \bigcap_{\sigma\neq\bar\sigma} I_\sigma$ then $f = -f(\theta_1 - 1) + f\theta_1 \in \bigcap_\sigma I_\sigma + (\theta_1)$. Therefore, the contribution of $I_{\bar\sigma}$ is trivial on both sides and we can erase it from Σ. We can therefore suppose that $\sigma(1) = 0$ whenever $1 \in T_\sigma$.

For any $\sigma \in \Sigma$ let σ' be its restriction to $T_\sigma \setminus \{1\}$. We have $I_{\sigma'} \subseteq I_\sigma$ and $I_{\sigma'} + (\theta_1) = I_\sigma + (\theta_1)$ for all $\sigma \in \Sigma$. By what we already proved, the statement holds for the set $\Sigma' := \{\sigma' : \sigma \in \Sigma\}$. Therefore:

$$\bigcap_{\sigma\in\Sigma}(I_\sigma + (\theta_1)) = \bigcap_{\sigma'\in\Sigma'}(I_{\sigma'} + (\theta_1)) = \bigcap_{\sigma'\in\Sigma'} I_{\sigma'} + (\theta_1) \subseteq \bigcap_{\sigma\in\Sigma} I_\sigma + (\theta_1)$$

proving the claim. □

We recall (see [**9**, Definition 1.1.9/1]) that a morphism of normed groups $\phi\colon G \to H$ is *strict* if the homomorphism $G/\ker\phi \to \phi(G)$ is a homeomorphism, where the former group is endowed with the quotient topology and the latter with the topology inherited from H. In particular, we say that a sequence of normed K-vector spaces

$$R \xrightarrow{f} S \xrightarrow{g} T$$

is *strict and exact* at S if it exact at S and if f is strict i.e. the quotient norm and the norm induced by S on $R/\ker(f) \cong \ker(g)$ are equivalent.

A.2.4. LEMMA. *For any map $\sigma\colon T_\sigma \to \{0,1\}$ defined on a subset T_σ of $\{1,\dots,n\}$ we denote by I_σ the ideal of $R\langle\underline{\theta}\rangle = R\langle\theta_1\dots,\theta_n\rangle$ generated by $\theta_i - \sigma(i)$ as i varies in T_σ. For any finite set Σ of such maps and any complete normed K-algebra R the following sequence of Banach K-algebras is strict and exact*

$$0 \to R\langle\underline{\theta}\rangle/\bigcap_{\sigma\in\Sigma} I_\sigma \to \prod_{\sigma\in\Sigma} R\langle\underline{\theta}\rangle/I_\sigma \to \prod_{\sigma,\sigma'\in\Sigma} R\langle\underline{\theta}\rangle/(I_\sigma + I_{\sigma'})$$

and the ideal $\bigcap_{\sigma\in\Sigma} I_\sigma$ is generated by a finite set of polynomials with coefficients in $\mathbb{Z}$.

PROOF. We follow the notation and the proof of [**32**]. For a collection of ideals $\mathcal{I} = \{I_\sigma\}$ we let $A(\mathcal{I})$ be the kernel of the map $\prod_\sigma R\langle\underline{\theta}\rangle/I_\sigma \to \prod_{\sigma,\sigma'} R\langle\underline{\theta}\rangle/(I_\sigma + I_{\sigma'})$ and $O(\mathcal{I})$ be the cokernel of $R\langle\underline{\theta}\rangle/\bigcap_\sigma I_\sigma \to A(\mathcal{I})$. We make induction on the cardinality m of $\mathcal{I}$. The case $m = 1$ is obvious.

Let $\mathcal{I}'$ be $\mathcal{I} \cup \{I_\eta\}$. From the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & R\langle\underline{\theta}\rangle & \xrightarrow{id} & R\langle\underline{\theta}\rangle & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & A(\mathcal{I}') & \longrightarrow & A(\mathcal{I}) & & \end{array}$$

we obtain by the snake lemma the exact sequence

$$0 \to I_\eta \cap \bigcap I_\sigma \to \bigcap I_\sigma \to W \to O(\mathcal{I}') \to O(\mathcal{I}).$$

By direct computation, it holds $W = \bigcap(I_\sigma + I_\eta)/I_\eta$. By the induction hypothesis, we obtain $O(\mathcal{I}) = 0$. Moreover, since $\bigcap I_\sigma + I_\eta = \bigcap(I_\sigma + I_\eta)$ by Lemma A.2.3, we conclude that the map $\bigcap I_\sigma \to W$ is surjective and hence $O(\mathcal{I}') = 0$ proving the main claim.

The ideals I_σ are defined over $\mathbb{Z}$. In order to prove that the ideal $\bigcap I_\sigma$ is also defined over $\mathbb{Z}$ and that the sequence is strict, by means of [**9**, Proposition 2.1.8/6] it suffices to consider the cases $R = K = \mathbb{Q}_p$ or $R = K = \mathbb{F}_p((t))$ for which the statement is clear. □

Let σ and σ' be maps defined from two subsets T_σ resp. $T_{\sigma'}$ of $\{1, \dots, n\}$ to $\{0, 1\}$. We say that they are *compatible* if $\sigma(i) = \sigma'(i)$ for all $i \in T_\sigma \cap T_{\sigma'}$ and in this case we denote by (σ, σ') the map from $T_\sigma \cup T_{\sigma'}$ extending them.

A.2.5. LEMMA. *Let $X = \varprojlim_h X_h$ be an object in $\widehat{\mathrm{RigSm}}$ and Σ a set as in Lemma A.2.4. We denote $\mathcal{O}(X)$ by R and $\mathcal{O}(X_h)$ by R_h. For any $\sigma \in \Sigma$ let $\bar{f}_\sigma$ be an element of $R\langle\underline{\theta}\rangle/I_\sigma$ such that $\bar{f}_\sigma|_{(\sigma,\sigma')} = \bar{f}_{\sigma'}|_{(\sigma,\sigma')}$ for any couple $\sigma, \sigma' \in \Sigma$ of compatible maps.*

1. *There exists an element $f \in R\langle\underline{\theta}\rangle$ such that $f|_\sigma = \bar{f}_\sigma$.*
2. *There exists a constant $C = C(\Sigma)$ such that if for some $g \in R\langle\underline{\theta}\rangle$ one has $|\bar{f}_\sigma - g|_\sigma| < \varepsilon$ for all σ then the element f can be chosen so that $|f - g| < C\varepsilon$. Moreover, if $\bar{f}_\sigma \in R_0\langle\underline{\theta}\rangle/I_\sigma$ for all σ then the element f can be chosen inside $R_h\langle\underline{\theta}\rangle$ for some integer h.*

PROOF. The first claim and the first part of the second are simply a restatement of Lemma A.2.4, where $C = C(\Sigma)$ is the constant defining the compatibility $||\cdot||_1 \leq C||\cdot||_2$ between the norm $||\cdot||_1$ on $R\langle\underline{\theta}\rangle/\bigcap I_\sigma$ induced by the quotient and the norm $||\cdot||_2$ induced by the embedding in $\prod R\langle\underline{\theta}\rangle/I_\sigma$. We now turn to the last sentence of the second claim.

We apply Lemma A.2.4 to each R_h and to R. We then obtain exact sequences of Banach spaces:

$$0 \to R_h\langle\underline{\theta}\rangle/\bigcap_{\sigma\in\Sigma} I_\alpha \to \prod_{\sigma\in\Sigma} R_h\langle\underline{\theta}\rangle/I_\sigma \to \prod_{\sigma,\sigma'\in\Sigma} R_h\langle\underline{\theta}\rangle/(I_\sigma + I_{\sigma'})$$

$$0 \to R\langle\underline{\theta}\rangle/\bigcap_{\sigma\in\Sigma} I_\alpha \to \prod_{\sigma\in\Sigma} R\langle\underline{\theta}\rangle/I_\sigma \to \prod_{\sigma,\sigma'\in\Sigma} R\langle\underline{\theta}\rangle/(I_\sigma + I_{\sigma'})$$

where all ideals that appear are finitely generated by polynomials with $\mathbb{Z}$-coefficients, depending only on Σ.

In particular, there exist two lifts of $\{\bar{f}_\sigma\}$: an element f_1 of $R_0\langle\underline{\theta}\rangle$ and an element f_2 of $R\langle\underline{\theta}\rangle$ such that $|f_2 - g| < C\varepsilon$ and their difference lies in $\bigcap I_\sigma$. Hence, we can find coefficients

$\gamma_i \in R\langle\underline{\theta}\rangle$ such that $f_1 = f_2 + \sum_i \gamma_i p_i$ where $\{p_1, \dots, p_M\}$ are generators of $\bigcap I_\sigma$ which have coefficients in K. Let now $\tilde{\gamma}_i$ be elements of $R_h\langle\underline{\theta}\rangle$ with $|\tilde{\gamma}_i - \gamma_i| < C\varepsilon/M|p_i|$. The element $f_3 := f_1 - \sum_i \tilde{\gamma}_i p_i$ lies in $\varinjlim_h (R_h\langle\underline{\theta}\rangle)$ is another lift of $\{\bar{f}_\sigma\}$ and satisfies $|f_3 - g| \leq \max\{|f_2 - g|, |f_2 - f_3|\} < C\varepsilon$ proving the claim. $\square$

We can now finally prove the apporximation result that played a crucial role in Section 1.4.

PROOF OF PROPOSITION 1.4.1. For any $h \in \mathbb{Z}$ we will denote $\mathcal{O}(X_h)\langle\theta_1, \dots, \theta_n\rangle$ by R_h. We also denote the π-adic completion of $\varinjlim_h R_h^\circ$ by R^+ and $R^+[\pi^{-1}]$ by R.

By Proposition A.2.1 we conclude that there exist integers m and n and a m-tuple of polynomials $P = (P_1, \dots, P_m)$ in $K[\sigma, \tau]$ where $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1 \dots, \tau_m)$ are systems of variables such that $K\langle\sigma, \tau\rangle/(P) \cong \mathcal{O}(Y)$ and each f_k is induced by maps $(\sigma, \tau) \mapsto (s_k, t_k)$ from $K\langle\sigma, \tau\rangle/(P)$ to R for some m-tuples s_k and n-tuples t_k in R. Moreover, there exists a sequence of power series $F_k = (F_{k1}, \dots, F_{km})$ associated to each f_k such that

$$(\sigma, \tau) \mapsto (s_k + (\tilde{s}_k - s_k)\chi, F_k(s_k + (\tilde{s}_k - s_k)\chi) \in R\langle\chi\rangle \cong \mathcal{O}(X \times \mathbb{B}^n \times \mathbb{B}^1)$$

defines a map H_k satisfying the first claim, for any choice of $\tilde{s}_k \in \varinjlim_h R_h^\circ$ such that $\tilde{s}_k$ is in the convergence radius of F_k and $F_k(\tilde{s}_k)$ is in R^+.

Let now ε be a positive real number, smaller than all radii of convergence of the series F_{kj} and such that $F(a) \in R^+$ for all $|a - s| < \varepsilon$. Denote by $\tilde{s}_{ki}$ the elements associated to s_{ki} by applying Proposition A.2.2 with respect to the chosen ε. In particular, they induce a well defined map H_k and the elements $\tilde{s}_{ki}$ lie in $R_{\bar{h}}^\circ\langle\theta_1 \dots, \theta_n\rangle$ for some integer $\bar{h}$. We show that the maps H_k induced by this choice also satisfy the second and third claims of the proposition.

Suppose that $f_k \circ d_{r,\epsilon} = f_{k'} \circ d_{r,\epsilon}$ for some $r \in \{1, \dots, n\}$ and $\epsilon \in \{0, 1\}$. This means that $\bar{s} := s_k|_{\theta_r = \epsilon} = s_{k'}|_{\theta_r = \epsilon}$ and $\bar{t} := t_k|_{\theta_r = \epsilon} = t_{k'}|_{\theta_r = \epsilon}$. This implies that both $F_k|_{\theta_r = \epsilon}$ and $F_{k'}|_{\theta_r = \epsilon}$ are two m-tuples of formal power series $\bar{F}$ with coefficients in $\mathcal{O}(X \times \mathbb{B}^{n-1})$ converging around $\bar{s}$ and such that $P(\sigma, \bar{F}(\sigma)) = 0$, $\bar{F}(\bar{s}) = \bar{t}$. By the unicity of such power series stated in Corollary A.1.2, we conclude that they coincide.

Moreover, by our choice of the elements $\tilde{s}_k$ it follows that $\bar{\tilde{s}} := \tilde{s}_k|_{\theta_r = \epsilon} = \tilde{s}_{k'}|_{\theta_r = \epsilon}$. In particular one has

$$F_k((\tilde{s}_k - s_k)\chi)|_{\theta_r = \epsilon} = \bar{F}((\bar{\tilde{s}} - \bar{s})\chi) = F_{k'}((\tilde{s}_{k'} - s_{k'})\chi)|_{\theta_r = \epsilon}$$

and therefore $H_k \circ d_{r,\epsilon} = H_{k'} \circ d_{r,\epsilon}$ proving the second claim.

The third claim follows immediately since the elements $\tilde{s}_{ki}$ satisfy the condition (4) of Proposition A.2.2. $\square$

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