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Relaxation of variational problems in two-dimensional nonlinear elasticity

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Abstract Consider a plate occupying in a reference configuration a bounded open set $\Omega \subset \mathbb{R}^2$, and let $W : \mathbb{M}^{3\times 2} \to [0, +\infty]$ be its stored-energy function. In this paper we are concerned with relaxation of variational problems of type:

$$\inf\left\{\int_\Omega W\big(\nabla u(x)\big)dx - \int_\Omega \big\langle f(x), u(x)\big\rangle dx : u \in W_*^{1,p}(\Omega;\mathbb{R}^3)\right\},$$

where $W_*^{1,p}(\Omega;\mathbb{R}^3) := \{u \in W^{1,p}(\Omega;\mathbb{R}^3) : u(x) = (x,0) \text{ on } \partial\Omega\}$ with $p > 1$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^3$ and $f \in L^q(\Omega;\mathbb{R}^3)$, with $1/p + 1/q = 1$, is the external loading per unit surface. We take into account the fact that an infinite amount of energy is required to compress a finite surface of the plate into zero surface, i.e., $W\big(\xi_1 \mid \xi_2\big) \to +\infty$ as $\big|\xi_1 \wedge \xi_2\big| \to 0$.

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1 Introduction

Consider a plate occupying in a reference configuration a bounded open set $\Omega \subset \mathbb{R}^2$, and let $W : \mathbb{M}^{3\times 2} \to [0, +\infty]$ be its stored-energy function (assumed to be Borel measurable), where $\mathbb{M}^{3\times 2}$ denotes the space of real 3×2 matrices. Fix $p > 1$. In order to take into account the fact that an infinite amount of energy is required to compress a finite surface of the plate into zero surface, i.e.,

$$W(\xi_1 \mid \xi_2) \to +\infty \text{ as } |\xi_1 \wedge \xi_2| \to 0, \tag{1}$$

where $\xi_1 \wedge \xi_2$ denotes the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$, we assume that

(H_1) *there exist* $\alpha \in]0, 1]$ *and* $\beta > 0$ *such that for all* $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$, *if* $|\xi_1 \wedge \xi_2| \geq \alpha$ *then* $W(\xi) \leq \beta(1 + |\xi|^p)$.

In this paper we are concerned with relaxation of variational problems of type:

$$\inf\left\{\int_\Omega W(\nabla u(x))dx - \int_\Omega \langle f(x), u(x)\rangle dx : u \in W^{1,p}_*(\Omega; \mathbb{R}^3)\right\} \tag{P}$$

where $W^{1,p}_*(\Omega; \mathbb{R}^3) := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = (x, 0) \text{ on } \partial\Omega\}$, $\langle \cdot, \cdot\rangle$ is the scalar product in $\mathbb{R}^3$ and $f \in L^q(\Omega; \mathbb{R}^3)$, with $1/p + 1/q = 1$, is the external loading per unit surface. By relaxation of (P) we mean to find

$$\inf\left\{\int_\Omega \bar{W}(\nabla u(x))dx - \int_\Omega \langle f(x), u(x)\rangle dx : u \in W^{1,p}_*(\Omega; \mathbb{R}^3)\right\} \tag{$\overline{\mathrm{P}}$}$$

with $\bar{W} : \mathbb{M}^{3\times 2} \to [0, +\infty]$ related to W, such that

($\mathfrak{R}$) *for every minimizing sequence* $\{u_n\}_{n\geq 1}$ *of* (P), *i.e.,* $u_n \in W^{1,p}_*(\Omega; \mathbb{R}^3)$ *for all* $n \geq 1$ *and*

$$\lim_{n\to+\infty} \int_\Omega W(\nabla u_n)dx - \int_\Omega \langle f, u_n\rangle dx = \inf(\mathrm{P}),$$

there exists $\bar{u} \in W^{1,p}_*(\Omega; \mathbb{R}^3)$ *such that* (*up to a subsequence*) $u_n \rightharpoonup \bar{u}$ (*where* $\rightharpoonup$ *denotes the weak convergence in* $W^{1,p}$) *and*

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega W(\nabla u_n)dx - \int_\Omega \langle f, u_n\rangle dx &= \int_\Omega \bar{W}(\nabla \bar{u})dx - \int_\Omega \langle f, \bar{u}\rangle dx\\ &= \inf(\overline{\mathrm{P}}).\end{aligned}$$

Relaxation can be seen as a generalization of the so-called direct method in the Calculus of Variations. Indeed, under the following coercivity condition:

(H_2) $W(\xi) \geq C|\xi|^p$ *for all* $\xi \in \mathbb{M}^{3\times 2}$ *and some* $C > 0$

(which assures compactness of minimizing sequences), if

$$u \mapsto \int_\Omega W(\nabla u(x))dx \tag{2}$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^3)$, then $(\mathfrak{R})$ holds with $\bar{W} = W$, and so (P) has solutions. Usually, $(\overline{\mathrm{P}})$ is called the relaxed problem of (P), and solutions of $(\overline{\mathrm{P}})$ are interpreted as generalized solutions of (P).

In this paper we deal with the case where the functional in (2) fails to be lower semicontinuous. In such a situation, Dacorogna proved in [3] that $(\mathfrak{R})$ is satisfied with $\bar{W} = \mathcal{Q}W$ whenever W is continuous and of polynomial growth, (where $\mathcal{Q}W$ denotes the quasiconvex envelope of W). As condition (1) leads to a stored-energy function which is not of polynomial growth, Dacorogna's theorem cannot be applied in two-dimensional nonlinear elasticity. One object of this paper is to study relaxation of (P) when only (H_1) and (H_2) are satisfied.

The plan of the paper is as follows. In the next section we state our main results (cf. Theorems 1 and 2). Section 3 (resp. 4) is devoted to the proof of Theorem 1 (resp. 2).

2 Main results

Set $Y :=]0,1[^2$, denote by $\mathrm{Aff}(Y;\mathbb{R}^3)$ the space of all continuous piecewise affine functions from Y to $\mathbb{R}^3$ and define $\mathcal{Z}W : \mathbb{M}^{3\times 2} \to [0,+\infty]$ by

$$\mathcal{Z}W(\xi) := \inf\left\{\int_Y W(\xi + \nabla\phi(y))dy : \phi \in \mathrm{Aff}_0(Y;\mathbb{R}^3)\right\}$$

with $\mathrm{Aff}_0(Y;\mathbb{R}^3) := \{u \in \mathrm{Aff}(Y;\mathbb{R}^3) : u = 0 \text{ on } \partial Y\}$. Set $\mathcal{Z}_1 W := \mathcal{Z}W$ and $\mathcal{Z}_k W := \mathcal{Z}_1[\mathcal{Z}_{k-1}W]$ for all integers $k \geq 2$. Here are the two main results of the paper.

Theorem 1 *If* (H_1) *holds then* $\mathcal{Z}_2 W(\xi) \leq c(1+|\xi|^p)$ *for all* $\xi \in \mathbb{M}^{3\times 2}$ *and some* $c > 0$.

Let $m, N \geq 1$ be two integers. For $\Omega \subset \mathbb{R}^N$ and $W : \mathbb{M}^{m\times N} \to [0,+\infty]$ Borel measurable, we define $E : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ by

$$E(u) := \inf\left\{\liminf_{n\to+\infty}\int_\Omega W(\nabla u_n(x))dx : W^{1,p}_*(\Omega;\mathbb{R}^m) \ni u_n \rightharpoonup u\right\}$$

with $W^{1,p}_*(\Omega;\mathbb{R}^m) := \{u \in W^{1,p}(\Omega;\mathbb{R}^m) : u(x) = \mathrm{I}x \text{ on } \partial\Omega\}$, where $\mathrm{I} \in \mathbb{M}^{m\times N}$ is defined by $\mathrm{I}_{ij} := 1$ for $i = j$ and $\mathrm{I}_{ij} := 0$ for $i \neq j$.

Theorem 2 *If there exist* $k \geq 1$ *and* $c > 0$ *such that* $\mathcal{Z}_k W(\xi) \leq c(1+|\xi|^p)$ *for all* $\xi \in \mathbb{M}^{m\times N}$*, then*

$$E(u) = \begin{cases} \displaystyle\int_\Omega \mathcal{Q}W(\nabla u(x))dx & \text{if } u \in W^{1,p}_*(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

Then, a direct consequence of Theorems 1 and 2 is the following relaxation result.

Corollary 1 *Under* (H_1) *and* (H_2) *we have* ($\mathfrak{R}$) *with* $\bar{W} = \mathcal{Q}W$.

A Borel measurable function $g : \mathbb{M}^{m\times N} \to [0, +\infty]$ is quasiconvex (in the sense of Morrey [6]) if for every $\xi \in \mathbb{M}^{m\times N}$, every bounded open set $D \subset \mathbb{R}^N$ with $|\partial D| = 0$ and every $\phi \in W_0^{1,\infty}(D; \mathbb{R}^m)$,

$$g(\xi) \leq \frac{1}{|D|}\int_D g(\xi + \nabla\phi(x))dx,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$. By the quasiconvex envelope of g, that we denote by $\mathcal{Q}g$, we mean the greatest quasiconvex function which is less than or equal to g. Thus, g is quasiconvex if and only if $\mathcal{Q}g = g$. If g is continuous and of polynomial growth, from Dacorogna's quasiconvexification formula [4, Theorem 1.1 p. 201], we see that

$$\mathcal{Q}g(\xi) = \inf\left\{\int_Y g(\xi + \nabla\phi(y))dy : \phi \in \mathrm{Aff}_0(Y; \mathbb{R}^m)\right\}$$

for all $\xi \in \mathbb{M}^{m\times N}$. In Theorem 2, $\mathcal{Z}_k W$ is continuous by Proposition 2(iii) below, and so $\mathcal{Q}[\mathcal{Z}_k W] = \mathcal{Z}_{k+1}W$ (and $\mathcal{Z}_i W = \mathcal{Z}_{k+1}W$ for all integers $i \geq k+1$). Hence $\mathcal{Q}W = \mathcal{Z}_{k+1}W$ because it is easy to see that for every $i \geq 1$, $\mathcal{Q}[\mathcal{Z}_i W] = \mathcal{Q}W$. We thus have

Proposition 1 *If there exist* $k \geq 1$ *and* $c > 0$ *such that* $\mathcal{Z}_k W(\xi) \leq c(1 + |\xi|^p)$ *for all* $\xi \in \mathbb{M}^{m\times N}$, *then* $\mathcal{Q}W = \mathcal{Z}_{k+1}W$.

The following result gives three interesting properties of functions $\mathcal{Z}_k W$. This follows easily from [5, Lemma 2.16, Theorem 2.17 and Proposition 2.3] (the details are left to the reader). Recall first that (see [5, Definition 2.2]) a Borel function $g : \mathbb{M}^{m\times N} \to [0, +\infty]$ is rank-one convex at ξ in U, with $\xi \in \mathbb{M}^{m\times N}$ and U an open subset of $\mathbb{M}^{m\times N}$, if

$$g(\xi) \leq \lambda g(\xi + a\otimes b) + (1-\lambda)g\left(\xi - \frac{\lambda}{1-\lambda}a\otimes b\right) \tag{4}$$

for all $\lambda \in [0, 1[$ and all $a \in \mathbb{R}^m$, $b \in \mathbb{R}^N$ satisfying $\xi + \mu a\otimes b \in U$ for every $\mu \in [\lambda/(\lambda - 1), 1]$, where $a \otimes b$ denotes the tensor product of vectors a and b. We say that g is rank-one convex in U if g is rank-one convex at ξ in U for every $\xi \in U$. Also, g is said to be rank-one convex at ξ when (4) holds for all $\lambda \in [0, 1[$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^N$, and g is rank-one convex when it is rank-one convex at every $\xi \in \mathbb{M}^{m\times N}$.

Proposition 2 *Every function* $\mathcal{Z}_k W$ *with* $k \geq 1$ *satisfies the following three properties*:

(i) *for every bounded open set* $D \subset \mathbb{R}^N$ *with* $|\partial D| = 0$ *and every* $\xi \in \mathbb{M}^{m\times N}$,

$$\mathcal{Z}_k W(\xi) = \inf\left\{\frac{1}{|D|}\int_D \mathcal{Z}_{k-1}W(\xi + \nabla\phi(y))dy : \phi \in \mathrm{Aff}_0(D; \mathbb{R}^m)\right\}.$$

(ii) $\mathcal{Z}_k W$ *is rank-one convex in the interior of its effective domain*;
(iii) $\mathcal{Z}_k W$ *is continuous in the interior of its effective domain.*

Proposition 2(i) (resp. Proposition 2(iii)) is used in the proof of Theorem 1 (resp. Theorem 2). We need Proposition 2(ii) in the proof of Proposition 3 below.

An analogue result of Theorem 2 can be found in the paper of Ben Belgacem [1]. More precisely, denoting by $\mathcal{R}W$ the rank-one convex envelope of W (the greatest rank-one convex function which is less than or equal to W), Ben Belgacem states in [1, Theorem 3.1] that if $\mathcal{R}W$ is of polynomial growth and if two other technical assumptions hold, then

$$E(u) = \begin{cases} \displaystyle\int_\Omega \mathcal{Q}[\mathcal{R}W](\nabla u(x))dx & \text{if } u \in W^{1,p}_*(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases}$$

In the case $m = 3$ and $N = 2$, Ben Belgacem asserts in [1, Section 5.1] that if W satisfies (H_1), then $\mathcal{R}W$ is of polynomial growth, so that [1, Theorem 3.1] is applicable. In fact, for such stored-energy densities W, we have

$$\mathcal{Q}[\mathcal{R}W] = \mathcal{Q}W. \tag{5}$$

Indeed, Theorem 1 says that $\mathcal{Z}_2 W$ is of polynomial growth, and so $\mathcal{Q}W = \mathcal{Z}_3 W$ by Proposition 1. Thus $\mathcal{Q}W$ is continuous and of polynomial growth. Hence $\mathcal{Q}W$ is rank-one convex (see [4, Theorem 1.1 p. 102]), and (5) follows.

Generally speaking, as rank-one convexity and quasiconvexity do not coincide, Theorem 2 and [1, Theorem 3.1] are not identical. However, we have

Proposition 3 *If* $\mathcal{R}W(\xi) \leq c(1 + |\xi|^p)$ *for all* $\xi \in \mathbb{M}^{m\times N}$ *and some* $c > 0$ *and if there exists* $k \geq 1$ *such that* $\mathcal{Z}_k W(\xi) < +\infty$ *for every* $\xi \in \mathbb{M}^{m\times N}$, *then* (3) *holds.*

Proof. From Proposition 2(ii) we see that $\mathcal{Z}_k W$ is rank-one convex, so that $\mathcal{Z}_k W \leq \mathcal{R}W$, and Proposition 3 follows from Theorem 2. □

3 Proof of Theorem 1

We first prove the following lemma.

Lemma 1 *Under* (H_1) *there exists* $\gamma > 0$ *such that for all* $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$, *if* $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha$ *then* $\mathcal{Z}_1 W(\xi) \leq \gamma(1 + |\xi|^p)$.

Proof. Let $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$ be such that $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha$ (with $\alpha > 0$ given by (H_1)). Set

$$D := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 1 < x_2 < x_1 + 1 \quad \text{and} \quad -x_1 - 1 < x_2 < 1 - x_1\}.$$

For each $t \in \mathbb{R}$, define $\varphi_t \in \text{Aff}_0(D;\mathbb{R})$ by

$$\varphi_t(x_1, x_2) := \begin{cases} -tx_1 + t(x_2 + 1) & \text{if } (x_1, x_2) \in \Delta_1 \\ t(1 - x_1) - tx_2 & \text{if } (x_1, x_2) \in \Delta_2 \\ tx_1 + t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3 \\ t(x_1 + 1) + tx_2 & \text{if } (x_1, x_2) \in \Delta_4 \end{cases}$$

with

$\Delta_1 := \{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \leq 0\};$
$\Delta_2 := \{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \geq 0\};$
$\Delta_3 := \{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \geq 0\};$
$\Delta_4 := \{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \leq 0\}.$

Assume first $|\xi_1 \wedge \xi_2| \neq 0$. Define $\phi \in \mathrm{Aff}_0(D; \mathbb{R}^3)$ by

$$\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}) \text{ with } \nu := \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|},$$

(ν_1, ν_2, ν_3 are the components of the vector ν). Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 - \nu \mid \xi_2 + \nu) & \text{if } x \in \Delta_1 \\ (\xi_1 - \nu \mid \xi_2 - \nu) & \text{if } x \in \Delta_2 \\ (\xi_1 + \nu \mid \xi_2 - \nu) & \text{if } x \in \Delta_3 \\ (\xi_1 + \nu \mid \xi_2 + \nu) & \text{if } x \in \Delta_4. \end{cases}$$

Taking Proposition 2(i) into account, it follows that

$$\begin{aligned} \mathcal{Z}_1 W(\xi) \leq \frac{1}{4}(&W(\xi_1 - \nu \mid \xi_2 + \nu) + W(\xi_1 - \nu \mid \xi_2 - \nu) \\ &+ W(\xi_1 + \nu \mid \xi_2 - \nu) + W(\xi_1 + \nu \mid \xi_2 + \nu)). \end{aligned} \tag{6}$$

But

$$\begin{aligned} |(\xi_1 - \nu) \wedge (\xi_2 + \nu)|^2 &= |\xi_1 \wedge \xi_2 + (\xi_1 + \xi_2) \wedge \nu|^2 \\ &= |\xi_1 \wedge \xi_2|^2 + |(\xi_1 + \xi_2) \wedge \nu|^2 \\ &\geq |(\xi_1 + \xi_2) \wedge \nu|^2, \end{aligned}$$

and so

$$|(\xi_1 + \nu) \wedge (\xi_2 - \nu)| \geq |(\xi_1 + \xi_2) \wedge \nu| = |\xi_1 + \xi_2|.$$

Similarly, we obtain

$|(\xi_1 - \nu) \wedge (\xi_2 - \nu)| \geq |\xi_1 - \xi_2|;$
$|(\xi_1 + \nu) \wedge (\xi_2 - \nu)| \geq |\xi_1 + \xi_2|;$
$|(\xi_1 + \nu) \wedge (\xi_2 + \nu)| \geq |\xi_1 - \xi_2|.$

Thus, $|(\xi_1 - \nu) \wedge (\xi_2 + \nu)| \geq \alpha$, $|(\xi_1 - \nu) \wedge (\xi_2 - \nu)| \geq \alpha$, $|(\xi_1 + \nu) \wedge (\xi_2 - \nu)| \geq \alpha$ and $|(\xi_1 + \nu) \wedge (\xi_2 + \nu)| \geq \alpha$, because $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha$. Using ($H_1$), it follows that

$$\begin{aligned} W(\xi_1 - \nu \mid \xi_2 + \nu) &\leq \beta(1 + |(\xi_1 - \nu \mid \xi_2 + \nu)|^p) \\ &\leq \beta 2^p (1 + |(\xi_1 \mid \xi_2)|^p + |(-\nu \mid \nu)|^p) \\ &\leq \beta 2^{2p+1}(1 + |\xi|^p). \end{aligned}$$

In the same manner, we have

$W(\xi_1 - \nu \mid \xi_2 - \nu) \leq \beta 2^{2p+1}(1 + |\xi|^p);$
$W(\xi_1 + \nu \mid \xi_2 - \nu) \leq \beta 2^{2p+1}(1 + |\xi|^p);$
$W(\xi_1 + \nu \mid \xi_2 + \nu) \leq \beta 2^{2p+1}(1 + |\xi|^p),$

and, from (6), we conclude that

$$\mathcal{Z}_1 W(\xi) \leq \beta 2^{2p+1}(1+|\xi|^p).$$

Assume now $|\xi_1 \wedge \xi_2| = 0$. Then, one the three following possibilities holds:

(i) $\xi_1 \neq 0$ and $\xi_2 = 0$;
(ii) $\xi_1 = 0$ and $\xi_2 \neq 0$;
(iii) there exists $\lambda \in \mathbb{R}$ such that $\xi_1 = \lambda \xi_2$ (with $\xi_1 \neq 0$ and $\xi_2 \neq 0$).

Let $\sigma \in \mathbb{R}^3$ be such that

$$|\sigma| = 1 \text{ and } \begin{cases} \langle \xi_1, \sigma \rangle = 0 & \text{if either (i) or (iii) is satisfied} \\ \langle \xi_2, \sigma \rangle = 0 & \text{if (ii) is satisfied.} \end{cases}$$

Defining $\psi \in \mathrm{Aff}_0(D; \mathbb{R}^3)$ by $\psi := (\varphi_{\sigma_1}, \varphi_{\sigma_2}, \varphi_{\sigma_3})$, and using Proposition 2(i) we see that

$$\begin{aligned} \mathcal{Z}_1 W(\xi) \leq \frac{1}{4}(&W(\xi_1 - \sigma \mid \xi_2 + \sigma) + W(\xi_1 - \sigma \mid \xi_2 - \sigma) \\ &+ W(\xi_1 + \sigma \mid \xi_2 - \sigma) + W(\xi_1 + \sigma \mid \xi_2 + \sigma)). \end{aligned} \tag{7}$$

Moreover, we have

$$\begin{aligned} &|(\xi_1 - \sigma) \wedge (\xi_2 + \sigma)| = |(\xi_1 + \xi_2) \wedge \sigma| = |\xi_1 + \xi_2| \geq \alpha; \\ &|(\xi_1 - \sigma) \wedge (\xi_2 - \sigma)| = |\sigma \wedge (\xi_1 - \xi_2)| = |\xi_1 - \xi_2| \geq \alpha; \\ &|(\xi_1 + \sigma) \wedge (\xi_2 - \sigma)| = |\sigma \wedge (\xi_1 + \xi_2)| = |\xi_1 + \xi_2| \geq \alpha; \\ &|(\xi_1 + \sigma) \wedge (\xi_2 + \sigma)| = |(\xi_1 - \xi_2) \wedge \sigma| = |\xi_1 - \xi_2| \geq \alpha, \end{aligned}$$

and, by (H_1), we obtain

$$\begin{aligned} &W(\xi_1 - \sigma \mid \xi_2 + \sigma) \leq \beta 2^{2p+1}(1+|\xi|^p); \\ &W(\xi_1 - \sigma \mid \xi_2 - \sigma) \leq \beta 2^{2p+1}(1+|\xi|^p); \\ &W(\xi_1 + \sigma \mid \xi_2 - \sigma) \leq \beta 2^{2p+1}(1+|\xi|^p); \\ &W(\xi_1 + \sigma \mid \xi_2 + \sigma) \leq \beta 2^{2p+1}(1+|\xi|^p). \end{aligned}$$

From (7) it follows that

$$\mathcal{Z}_1 W(\xi) \leq \beta 2^{2p}(1+|\xi|^p),$$

and the proof is complete. □

Proof of Theorem 1. Let $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}$. For each $t \in \mathbb{R}$, define $\varphi_t \in \mathrm{Aff}_0(Y; \mathbb{R})$ by

$$\varphi_t(x_1, x_2) := \begin{cases} t x_2 & \text{if } (x_1, x_2) \in \Delta_1 \\ t(1 - x_1) & \text{if } (x_1, x_2) \in \Delta_2 \\ t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3 \\ t x_1 & \text{if } (x_1, x_2) \in \Delta_4 \end{cases}$$

with

$$\begin{aligned} &\Delta_1 := \{(x_1, x_2) \in Y : x_2 \leq x_1 \leq -x_2 + 1\}; \\ &\Delta_2 := \{(x_1, x_2) \in Y : -x_1 + 1 \leq x_2 \leq x_1\}; \\ &\Delta_3 := \{(x_1, x_2) \in Y : -x_2 + 1 \leq x_1 \leq x_2\}; \\ &\Delta_4 := \{(x_1, x_2) \in Y : x_1 \leq x_2 \leq -x_1 + 1\}. \end{aligned}$$

Assume first $|\xi_1 \wedge \xi_2| \neq 0$. Define $\phi \in \mathrm{Aff}_0(Y; \mathbb{R}^3)$ by

$$\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}) \text{ with } \nu := \frac{\alpha(\xi_1 \wedge \xi_2)}{|\xi_1 \wedge \xi_2|},$$

(ν_1, ν_2, ν_3 are the components of the vector ν and $\alpha > 0$ is given by (H_1)). Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 \mid \xi_2 + \nu) & \text{if } x \in \Delta_1 \\ (\xi_1 - \nu \mid \xi_2) & \text{if } x \in \Delta_2 \\ (\xi_1 \mid \xi_2 - \nu) & \text{if } x \in \Delta_3 \\ (\xi_1 + \nu \mid \xi_2) & \text{if } x \in \Delta_4. \end{cases}$$

But

$$\begin{aligned} |\xi_1 + (\xi_2 + \nu)|^2 &= |(\xi_1 + \xi_2) + \nu|^2 \\ &= |\xi_1 + \xi_2|^2 + |\nu|^2 \\ &= |\xi_1 + \xi_2|^2 + \alpha^2 \\ &\geq \alpha^2, \end{aligned}$$

hence $|\xi_1 + (\xi_2 + \nu)| \geq \alpha$. Similarly, we obtain $|\xi_1 - (\xi_2 + \nu)| \geq \alpha$, and so

$$\min\{|\xi_1 + (\xi_2 + \nu)|, |\xi_1 - (\xi_2 + \nu)|\} \geq \alpha.$$

In the same manner, we have

$$\begin{aligned} &\min\{|(\xi_1 - \nu) + \xi_2|, |(\xi_1 - \nu) - \xi_2|\} \geq \alpha; \\ &\min\{|\xi_1 + (\xi_2 - \nu)|, |\xi_1 - (\xi_2 - \nu)|\} \geq \alpha; \\ &\min\{|(\xi_1 + \nu) + \xi_2|, |(\xi_1 + \nu) - \xi_2|\} \geq \alpha. \end{aligned}$$

Noticing that

$$\begin{aligned} \mathcal{Z}_2 W(\xi) \leq \frac{1}{4}(&\mathcal{Z}_1 W(\xi_1 \mid \xi_2 + \nu) + \mathcal{Z}_1 W(\xi_1 - \nu \mid \xi_2) \\ &+ \mathcal{Z}_1 W(\xi_1 \mid \xi_2 - \nu) + \mathcal{Z}_1 W(\xi_1 + \nu \mid \xi_2)), \end{aligned}$$

from Lemma 1 we deduce that

$$\begin{aligned} \mathcal{Z}_2 W(\xi) &\leq \frac{\gamma}{4}(4 + |(\xi_1 \mid \xi_2 + \nu)|^p + |(\xi_1 - \nu \mid \xi_2)|^p \\ &\quad + |(\xi_1 \mid \xi_2 - \nu)|^p + |(\xi_1 + \nu \mid \xi_2)|^p) \\ &\leq \frac{\gamma}{4}2^p(4 + 4|(\xi_1 \mid \xi_2)|^p \\ &\quad + |(0 \mid \nu)|^p + |(-\nu \mid 0)|^p + |(0 \mid -\nu)|^p + |(\nu \mid 0)|^p) \\ &\leq \gamma 2^{p+1}(1 + |\xi|^p). \end{aligned}$$

Assume now $|\xi_1 \wedge \xi_2| = 0$. Then, one the four following possibilities holds:

(i) $\xi_1 = 0$ and $\xi_2 = 0$;
(ii) $\xi_1 \neq 0$ and $\xi_2 = 0$;
(iii) $\xi_1 = 0$ and $\xi_2 \neq 0$;
(iv) there exists $\lambda \in \mathbb{R}$ such that $\xi_1 = \lambda\xi_2$ (with $\xi_1 \neq 0$ and $\xi_2 \neq 0$).

Let $\sigma \in \mathbb{R}^3$ be such that

$$\begin{cases} |\sigma| = \alpha & \text{if (i) is satisfied} \\ |\sigma| = \alpha \text{ and } \langle \xi_1, \sigma\rangle = 0 & \text{if either (ii) or (iv) is satisfied} \\ |\sigma| = \alpha \text{ and } \langle \xi_2, \sigma\rangle = 0 & \text{if (iii) is satisfied.} \end{cases}$$

Defining $\psi \in \mathrm{Aff}_0(Y;\mathbb{R}^3)$ by $\psi := (\varphi_{\sigma_1}, \varphi_{\sigma_2}, \varphi_{\sigma_3})$, we have

$$\begin{aligned} \mathcal{Z}_2 W(\xi) \le \frac{1}{4}(&\mathcal{Z}_1 W(\xi_1 \mid \xi_2 + \sigma) + \mathcal{Z}_1 W(\xi_1 - \sigma \mid \xi_2) \\ &+ \mathcal{Z}_1 W(\xi_1 \mid \xi_2 - \sigma) + \mathcal{Z}_1 W(\xi_1 + \sigma \mid \xi_2)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\min\{|\xi_1 + (\xi_2 + \sigma)|, |\xi_1 - (\xi_2 + \sigma)|\} \ge \alpha; \\ &\min\{|(\xi_1 - \sigma) + \xi_2|, |(\xi_1 - \sigma) - \xi_2|\} \ge \alpha; \\ &\min\{|\xi_1 + (\xi_2 - \sigma)|, |\xi_1 - (\xi_2 - \sigma)|\} \ge \alpha; \\ &\min\{|(\xi_1 + \sigma) + \xi_2|, |(\xi_1 + \sigma) - \xi_2|\} \ge \alpha, \end{aligned}$$

and, by Lemma 1, we obtain

$$\begin{aligned} &\mathcal{Z}_1 W(\xi_1 \mid \xi_2 + \sigma) \le \gamma(1 + |(\xi_1 \mid \xi_2 + \sigma)|^p); \\ &\mathcal{Z}_1 W(\xi_1 - \sigma \mid \xi_2) \le \gamma(1 + |(\xi_1 - \sigma \mid \xi_2)|^p); \\ &\mathcal{Z}_1 W(\xi_1 \mid \xi_2 - \sigma) \le \gamma(1 + |(\xi_1 \mid \xi_2 - \sigma)|^p); \\ &\mathcal{Z}_1 W(\xi_1 + \sigma \mid \xi_2) \le \gamma(1 + |(\xi_1 + \sigma \mid \xi_2)|^p). \end{aligned}$$

It follows that

$$\mathcal{Z}_2 W(\xi) \le \gamma 2^{p+1}(1 + |\xi|^p),$$

and the proof is complete. □

4 Proof of Theorem 2

We begin by proving Proposition 4 below which will play an essential role in the proof of Theorem 2. Set $\mathrm{Aff}_*(\Omega;\mathbb{R}^m) := \{u \in \mathrm{Aff}(\Omega;\mathbb{R}^m) : u(x) = \mathrm{I}x \text{ on } \partial\Omega\}$ and $\mathcal{Z}_0 W := W$. For each integer $k \ge 0$, define the functional $F_k : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ by

$$F_k(u) := \inf\left\{\liminf_{n\to+\infty} \int_\Omega \mathcal{Z}_k W(\nabla u_n(x))dx : \mathrm{Aff}_*(\Omega;\mathbb{R}^m) \ni u_n \rightharpoonup u\right\}.$$

Proposition 4 *$F_k = F_0$ for all integers $k \ge 1$.*

To prove Proposition 4 we need the following lemma.

Lemma 2 *For every integer $k \ge 0$ and every $u \in \mathrm{Aff}_*(\Omega;\mathbb{R}^m)$,*

$$F_k(u) \le \int_\Omega \mathcal{Z}_{k+1} W(\nabla u(x))dx. \tag{8}$$

Proof. Let $k \geq 0$ be an integer and let $u \in \mathrm{Aff}_*(\Omega; \mathbb{R}^m)$. By definition, there exists a finite family $(\Omega_i)_{i \in I}$ of open disjoint subsets of Ω such that $|\Omega \setminus \cup_{i \in I} \Omega_i| = 0$ and, for every $i \in I$, $\nabla u(x) = \xi_i$ in Ω_i with $\xi_i \in \mathbb{M}^{m \times N}$. Given any $\delta > 0$ and any $i \in I$, we consider $\phi_i \in \mathrm{Aff}_0(Y; \mathbb{R}^m)$ such that

$$\int_Y \mathcal{Z}_k W(\xi_i + \nabla \phi_i(y))dy \leq \mathcal{Z}_{k+1} W(\xi_i) + \delta |\Omega|^{-1}. \tag{9}$$

Fix any integer $n \geq 1$. By Vitali's covering theorem, there exists a finite or countable family $(a_{i,j} + \varepsilon_{i,j} Y)_{j \in J_i}$ of disjoint subsets of Ω_i, where $a_{i,j} \in \mathbb{R}^N$ and $0 < \varepsilon_{i,j} < n^{-1}$, such that $|\Omega_i \setminus \cup_{j \in J_i} (a_{i,j} + \varepsilon_{i,j} Y)| = 0$. Define $\psi_n : \Omega \to \mathbb{R}^m$ by

$$\psi_n(x) := \varepsilon_{i,j} \hat{\phi}_i \left(\frac{x - a_{i,j}}{\varepsilon_{i,j}} \right) \text{ if } x \in a_{i,j} + \varepsilon_{i,j} Y,$$

where $\hat{\phi}_i$ is the Y-periodic extension of ϕ_i to $\mathbb{R}^N$. We have

$$\begin{aligned} \int_\Omega |\psi_n(x)|^p dx &= \sum_{i \in I} \int_{\Omega_i} |\psi_n(x)|^p dx \\ &= \sum_{i \in I} \sum_{j \in J_i} (\varepsilon_{i,j})^N \int_Y |\varepsilon_{i,j} \phi_i \, (y) \, |^p dy \\ &= \sum_{i \in I} \sum_{j \in J_i} (\varepsilon_{i,j})^N |\varepsilon_{i,j}|^p \int_Y |\phi_i \, (y) \, |^p dy. \end{aligned}$$

But $|\varepsilon_{i,j}|^p < n^{-p}$ for all $i \in I$ and all $j \in J_i$, hence

$$\begin{aligned} \int_\Omega |\psi_n(x)|^p dx &\leq n^{-p} \sum_{i \in I} \sum_{j \in J_i} (\varepsilon_{i,j})^N \int_Y |\phi_i \, (y) \, |^p dy \\ &= n^{-p} \sum_{i \in I} |\Omega_i| \int_Y |\phi_i \, (y) \, |^p dy, \end{aligned}$$

and so $\psi_n \to 0$ in $L^p(\Omega; \mathbb{R}^m)$. Since $\phi_i \in \mathrm{Aff}_0(Y; \mathbb{R}^m)$, there exists a finite family $(Y_{i,l})_{l \in L_i}$ of open disjoint subsets of Y such that $|Y \setminus \cup_{l \in L_i} Y_{i,l}| = 0$ and, for every $l \in L_i$, $\nabla \phi_i(y) = \zeta_{i,l}$ in $Y_{i,l}$ with $\zeta_{i,l} \in \mathbb{M}^{m \times N}$. Set $U_{i,l,n} := \cup_{j \in J_i} a_{i,j} + \varepsilon_{i,j} Y_{i,l}$, then $|\Omega \setminus \cup_{i \in I} \cup_{l \in L_i} U_{i,l,n}| = 0$ and $\nabla \psi_n(x) = \zeta_{i,l}$ in $U_{i,l,n}$, and so $\psi_n \in \mathrm{Aff}_0(\Omega; \mathbb{R}^m)$ and $\{\nabla \psi_n\}_{n \geq 1}$ is bounded in $L^p(\Omega; \mathbb{R}^m)$. Consequently, (up to a subsequence) $\psi_n \rightharpoonup 0$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. Moreover,

$$\begin{aligned} \int_\Omega \mathcal{Z}_k W \, (\nabla u(x) + \nabla \psi_n(x)) \, dx &= \sum_{i \in I} \int_{\Omega_i} \mathcal{Z}_k W \, (\xi_i + \nabla \psi_n(x)) \, dx \\ &= \sum_{i \in I} \sum_{j \in J_i} (\varepsilon_{i,j})^N \int_Y \mathcal{Z}_k W \, (\xi_i + \nabla \phi_i(y)) \, dy \\ &= \sum_{i \in I} |\Omega_i| \int_Y \mathcal{Z}_k W \, (\xi_i + \nabla \phi_i(y)) \, dy. \end{aligned}$$

As $u+\psi_n \in \mathrm{Aff}_*(\Omega;\mathbb{R}^m)$ for all $n \geq 1$ and $u+\psi_n \rightharpoonup u$, from (9) we deduce that

$$F_k(u) \leq \liminf_{n\to+\infty} \int_\Omega \mathcal{Z}_k W\left(\nabla u(x)+\nabla\psi_n(x)\right)dx \leq \sum_{i\in I} |\Omega_i| \mathcal{Z}_{k+1} W(\xi_i) + \delta$$
$$= \int_\Omega \mathcal{Z}_{k+1} W(\nabla u(x))dx + \delta,$$

and (8) follows. □

Proof of Proposition 4. It is sufficient to prove that for every integer $k \geq 0$,

$$F_k \leq F_{k+1}. \tag{10}$$

Let $k \geq 0$ be an integer. Fix any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and any sequence $u_n \rightharpoonup u$ with $u_n \in \mathrm{Aff}_*(\Omega;\mathbb{R}^m)$. Using Lemma 2, we have

$$F_k(u_n) \leq \int_\Omega \mathcal{Z}_{k+1} W(\nabla u_n(x))dx$$

for all $n \geq 1$. Thus,

$$F_k(u) \leq \liminf_{n\to+\infty} F_k(u_n) \leq \liminf_{n\to+\infty} \int_\Omega \mathcal{Z}_{k+1} W(\nabla u_n(x))dx,$$

and (10) follows. □

Proof of Theorem 2. Let $k \geq 1$ be an integer such that $\mathcal{Z}_k W(\xi) \leq c(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{m\times N}$ and some $c > 0$. Then $\mathcal{Z}_k W$ is finite, and so $\mathcal{Z}_k W$ is continuous by Proposition 2(iii). Since $\mathcal{Q}W = \mathcal{Q}[\mathcal{Z}_k W]$, it is sufficient to prove that

$$E(u) = \begin{cases} \displaystyle\int_\Omega \mathcal{Q}[\mathcal{Z}_k W](\nabla u(x))dx & \text{if } u \in W_*^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

As $\mathrm{Aff}_*(\Omega;\mathbb{R}^m)$ is strongly dense in $W_*^{1,p}(\Omega;\mathbb{R}^m)$, for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$,

$$F_k(u) = \inf\left\{\liminf_{n\to+\infty} \int_\Omega \mathcal{Z}_k W(\nabla u_n(x))dx : W_*^{1,p}(\Omega;\mathbb{R}^m) \ni u_n \rightharpoonup u\right\}.$$

Fix any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and any $u_n \rightharpoonup u$ with $u_n \in W_*^{1,p}(\Omega;\mathbb{R}^m)$. As $\mathcal{Z}_k W \leq W$ we have

$$\int_\Omega \mathcal{Z}_k W(\nabla u_n(x))dx \leq \int_\Omega W(\nabla u_n(x))dx$$

for all $n \geq 1$. Then,

$$F_k(u) \leq \liminf_{n\to+\infty} \int_\Omega \mathcal{Z}_k W(\nabla u_n(x))dx \leq \liminf_{n\to+\infty} \int_\Omega W(\nabla u_n(x))dx,$$

and so $F_k \leq E$. But $E \leq F_0$ and $F_0 = F_k$ by Proposition 4, hence $E = F_k$, and (11) follows from the (classical) integral representation theorem below.

Theorem *Let* $g : \mathbb{M}^{m\times N} \to [0, +\infty]$ *be a Borel measurable function and let* $G : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ *be defined by*

$$G(u) := \inf\left\{\liminf_{n\to+\infty}\int_\Omega g(\nabla u_n(x))dx : W^{1,p}_*(\Omega; \mathbb{R}^m) \ni u_n \rightharpoonup u\right\}.$$

If g *is continuous and* $g(\xi) \le c(1+|\xi|^p)$ *for all* $\xi \in \mathbb{M}^{m\times N}$ *and some* $c > 0$*, then*

$$G(u) = \begin{cases}\displaystyle\int_\Omega \mathcal{Q}g(\nabla u(x))dx & \text{if } u \in W^{1,p}_*(\Omega; \mathbb{R}^m)\\ +\infty & \text{otherwise.}\end{cases}$$

For a proof we refer to [2, Proposition 11.7 and Corollary 12.7].

References

1. Ben Belgacem, H.: Relaxation of singular functionals defined on Sobolev spaces. ESAIM COCV **5**, 71–85 (2000)
2. Braides, A.: Semicontinuity, Γ-convergence and homogenization of nonlinear functionals. Lecture notes. Trieste: SISSA (1994)
3. Dacorogna, B.: Quasiconvexity and relaxation of nonconvex problems in the Calculus of Variations. J. Funct. Anal. **46**, 102–118 (1982)
4. Dacorogna, B.: Direct Methods in the Calculus of Variations. Berlin: Springer-Verlag (1989)
5. Fonseca, I.: The lower quasiconvex envelope of the stored energy function for an elastic crystal. J. Math. Pures et Appl. **67**, 175–195 (1988)
6. Morrey, C.B.: Quasiconvexity and lower semicontinuity of multiple integrals. Pacif. J. Math. **2**, 25–53 (1952)