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Regular type distributions in mechanism design and ρ-concavity*

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Regular type distributions in mechanism design and ρ-concavity

Abstract. Some of the best-known results in mechanism design depend critically on R. Myerson's ("Optimal auction design," *Math. Oper. Res.* 6, 1981, pp. 58-73) regularity condition. E.g., the second-price auction with reserve price is revenue maximizing only if the type distribution is regular. This paper offers two main findings. First, a new interpretation of regularity is developed – similar to that of a monotone hazard rate – in terms of being *the next* to fail. Second, using expanded concepts of concavity, a tight sufficient condition is obtained for a density to define a regular distribution. New examples of regular distributions are identified. Applications are discussed.

Keywords. Virtual valuation, Regularity, Generalized concavity, Prékopa-Borell Theorem, Mechanism design.

JEL-Codes. D82 - Asymmetric and Private Information; D44 - Auctions; D86 - Economics of Contract: Theory; C16 - Specific Distributions.

1. Introduction

Some of the most celebrated results in the theory of mechanism design require the underlying type distribution to be *regular*. For example, the second-price auction with reserve price is revenue maximizing only under the condition of regularity.[1] Formally, regularity says that the *virtual valuation*,

$$J_f(x) = x - \frac{1 - F(x)}{f(x)}, \tag{1}$$

is strictly increasing in the type x, where f and F, respectively, denote the density and distribution function of the type distribution.[2]

A common way to ensure regularity is to impose that the inverse of the second term in (1), i.e., the *hazard rate* of the type distribution,

$$\lambda_f(x) = \frac{f(x)}{1 - F(x)}, \tag{2}$$

is monotone increasing. This approach has been found useful mainly for two reasons. First, the hazard rate allows an immediate interpretation as a conditional likelihood of failure.[3] Second, distributions with log-concave densities are known to possess a monotone hazard rate.[4]

[1] Cf. Myerson (1981). When the type distribution is not regular, the optimal mechanism will typically entail conditional minimum bids, as shown by Maskin and Riley (1989). In general, irregular type distributions necessitate the characterization of optimal bunches. See, e.g., Nöldeke and Samuelson (2007).

[2] Equivalently, the marginal revenue of a monopolist facing inverse demand $p = F^{-1}(1-q)$ is strictly declining in output. See Bulow and Roberts (1989).

[3] Indeed, if $F(x)$ is the probability that a machine will fail before time x, then the hazard rate is the instantaneous probability of failure, given that the machine has not failed before time x. See, e.g., Barlow and Proschan (1975).

[4] See, e.g., An (1998). This result can be used to identify many parameterized examples of regular type distributions. Specifically, as Bagnoli and Bergström (2005) show, regularity holds for the uniform, normal, exponential, logistic, extreme-value, Laplace, Maxwell, and Rayleigh distributions. With restrictions to parameters, this list extends to power, Weibull, Gamma, Chi-Squared, Chi, and beta distributions.

However, the hazard rate condition implies that virtual valuations increase with slope ≥ 1, which is overly restrictive. For example, as illustrated in Figure 1, the log-normal distribution does not possess a monotone hazard rate, but will still be regular unless the density is very flat.[5] However, specifications with precisely this shape have been found plausible as an empirical description of bidder valuations.[6] Thus, imposing the hazard rate condition or even a log-concave density not only impairs the power of theoretical findings, but also restricts in a substantial way the set of distributional specifications available for applied work.

This paper offers two main findings. The first is a statistical interpretation of regularity. As in the case of the hazard rate, the density function measures the instantaneous rate of failure. Regularity can then be captured in terms of the probability that a given machine will be *the next* to fail. This yields some intuition, e.g., regarding truncations of regular distributions. The second main result, which is central point of the paper, is a sufficient condition for a density to define a regular distribution. The condition, referred to as strong $(-\frac{1}{2})$-concavity, is much tighter than log-concavity. Numerous new examples of distributions can be shown to be regular. In particular, we establish the regularity of distributions of log-normal shape, for which existing criteria have no bite.

The rest of the paper is organized as follows. Section 2 reviews mathematical prerequisites. An interpretation of regularity is developed in Section 3. In Section 4, we prove a general characterization of distributions that

[5] Indeed, the log-normal distribution is regular provided its skewness is smaller than $(e^2+2)\sqrt{e^2-1} \approx 23.73$. See Table I and the Appendix.

[6] See, e.g., Baldwin et al. (1997), Guerre et al. (2000), and Laffont et al. (1995).

possess (weakly) increasing virtual valuations. Section 5 contains the key result of the paper, viz. that a strongly $(-\frac{1}{2})$-concave density defines a regular distribution. Illustrative applications are outlined in Section 6. Section 7 concludes. An Appendix provides background information on Tables I and II.

2. Mathematical tools

This section reviews some mathematical concepts and results that will be used in the analysis.

2.1. Generalized concavity

A function $g \geq 0$ on $\mathbb{R}^N$ is called *ρ-concave*, for $\rho \neq 0$, if the set $X_g = \{(x_1, ..., x_N) \in \mathbb{R}^N : g(x_1, ..., x_n) > 0\}$ is convex, and $(g(x_1, ..., x_N))^\rho/\rho$ is concave on X_g. For $\rho = 0$, the definition is extended by the requirement that g must be log-concave on X_g.[7]

Higher values of ρ correspond to more stringent variants of concavity. E.g., log-concavity is more stringent than ρ-concavity for any $\rho < 0$. We call a function g *strongly ρ-concave* if g is ρ'-concave for some $\rho' > \rho$. For a twice differentiable function $g : \mathbb{R} \to \mathbb{R}_{++}$ in a single variable x, the condition of ρ-concavity is equivalent to $g(x)g''(x) - (1-\rho)g'(x)^2 \leq 0$.

Among alternative notions of concavity, the definition above is highlighted by the fact that concavity properties are passed on from a density to the corresponding distribution.

Theorem 2.1. (Prékopa-Borell) *Let $g = g(x_1, ..., x_N) \geq 0$ be a density*

[7] Complemented by the two limit cases $\rho = \pm\infty$ (which are not needed here), this is the definition used in the economics literature since Caplin and Nalebuff (1991a, 1991b). Dierker (1991) is an early application of generalized concavity in the economics literature.

on $\mathbb{R}^N$. If g is ρ-concave for some $\rho > -\frac{1}{N}$, then

$$G(z) = \int_{\{x\in\mathbb{R}^N : x_N \leq z\}} g(x_1, ..., x_N) dx_1 ... dx_N \tag{3}$$

is $\widehat{\rho}$-concave with $\widehat{\rho} = \frac{\rho}{1+\rho N}$.

For a helpful discussion of this result, see Caplin and Nalebuff (1991a). In the simplest case ($N = 1$), the Prékopa-Borell Theorem says that if a density $g \geq 0$ on $\mathbb{R}$ is ρ-concave for some $\rho > -1$, then $G(z) = \int_{-\infty}^{z} g(x)dx$ is $\widehat{\rho}$-concave with $\widehat{\rho} = \frac{\rho}{1+\rho}$.[8]

2.2. Minimal conditions for monotonicity

A smooth function is monotone increasing provided that its first derivative is never negative. Here is a generalization to the non-differentiable case. For a given function g, denote by $\overline{g}^+(x) = \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}(g(x+\varepsilon) - g(x))$ the right-hand upper Dini derivative at x.

Theorem 2.2. *Assume that*

$$\limsup_{\varepsilon\to 0+} g(x-\varepsilon) \leq g(x) \leq \limsup_{\varepsilon\to 0+} g(x+\varepsilon) \tag{4}$$

at any x, that $\overline{g}^+(x) \geq 0$ a.e., and that $\overline{g}^+(x) > -\infty$ except possibly at a countable set. Then g is monotone increasing.

This result follows from Theorem 7.3 in Saks (1937).[9] Note that (4) holds if g is right-continuous and upper semi-continuous.

[8] Theorem 2.1 is best known in the special case where $\rho = 0$ (Prékopa 1973). E.g., if g is a log-concave density, then both G and $1 - G$ are log-concave, and hence, $\frac{g}{G}$ is monotone decreasing, and $\frac{g}{1-G}$ monotone increasing (An 1998; Bagnoli and Bergström 2005). Similarly, if $g(x)$ is (strictly) log-concave in $\log x$, then $\frac{xg(x)}{1-G(x)}$ is (strictly) increasing (van den Berg 2007; Zeng 2011). A multi-dimensional variant of Prékopa's theorem has been used, e.g., by Ivanov (2011).

[9] See also the discussion following the theorem.

3. An interpretation of regularity

McAfee and McMillan (1987) observed that in the smooth case, regularity is equivalent to the strict convexity of $1/(1-F(x))$. For a direct proof of this fact, assume that F is twice differentiable. Then

$$\frac{\partial J_f(x)}{\partial x} = \frac{\partial}{\partial x}(x - \frac{1-F(x)}{f(x)}) = 1 + \frac{f(x)^2 + (1-F(x))f'(x)}{f(x)^2}. \tag{5}$$

On the other hand,

$$\frac{\partial^2}{\partial x^2}\frac{1}{1-F(x)} = \frac{\partial}{\partial x}\frac{f(x)}{(1-F(x))^2} = \frac{(1-F(x))f'(x) + 2f(x)^2}{(1-F(x))^3}. \tag{6}$$

I.e., the respective signs of $J_f'(x)$ and $(1/(1-F(x)))''$ coincide.

It apparently went unnoticed that the above characterization implies the following statistical interpretation of regularity. Imagine a large number M of machines which fail one after another at rate $f(x)$. Pick one machine from the population, and assume it has been functional up to time x. By the law of large numbers, there are about $M(1-F(x))$ machines left. But all machines are ex-ante identical, so the uniform likelihood for the chosen machine to be *the next* to fail is $l(x) \approx 1/M(1-F(x))$. By the *zoom rate*, we mean the rate at which this likelihood grows over time (as a consequence of other machines failing). Regularity then requires the zoom rate to be increasing over time.[10]

[10] More formally, let $m \geq 1$ denote the exact number of machines that are still working at time x. Then, the likelihood for a given machine to be the next to fail is

$$\begin{aligned} l(x) &= \sum_{m=1}^{M} \frac{1}{m}\binom{M-1}{m-1}(1-F(x))^{m-1}F(x)^{M-m} \\ &= \frac{1}{M(1-F(x))}\sum_{m=1}^{M}\binom{M}{m}(1-F(x))^{m}F(x)^{M-m} \\ &= \frac{1-F(x)^M}{M(1-F(x))}. \end{aligned}$$

To see the interpretation at work, recall that any regular type distribution remains regular after arbitrary truncations.[11] This is quite obvious for truncations from below because dropping a subpopulation consisting of all machines that stop working before some time x_0 obviously does not affect the later development of the zoom rate. For truncations from above, the intuition is more involved. Note, however, that since the zoom rate is increasing in the initial distribution, the population must shrink sufficiently fast to compensate for any decline in the failure rate, i.e., $\frac{2f(x)^2}{1-F(x)} + f'(x) > 0$. In this situation, dropping a subpopulation consisting of all machines working beyond time x_1 accelerates the population effect, while there is no such impact on the failure rate. Formally, $\frac{2f(x)^2}{F(x_1)-F(x)} + f'(x) > 0$, and the zoom rate will be increasing also in the truncated distribution.[12]

4. A generalization

So far, we assumed that the density function is differentiable. However, this may be restrictive, e.g., when the distribution is a mixture or the result of endogenous decisions. To incorporate such possibilities, smoothness will be replaced by a somewhat weaker assumption.

Consider a density $f \geq 0$ on some interval X. Without loss of generality, we assume that f is strictly positive in the interior of X. Indeed, if $f(x) = 0$ at some interior point, then $J_f(x) = -\infty$, and J_f cannot be increasing. We will say that f satisfies the *Cantor-Lebesgue condition* (CL) if f is right-

Therefore, for x kept fixed, $l(x)$ is indeed asymptotically equivalent to $1/M(1 - F(x))$ as $M \to \infty$. Further, one can check that $\partial l/\partial x \approx f(x)/M(1 - F(x))^2$. This follows from differentiating the precise expression for $l(x)$ derived above.

[11] Cf. Hafalir and Krishna (2008), or Virág (2011).

[12] This zoom rate formulation of increasing virtual valuations has been taken up already by Szech (2011) to predict over- and underinvestment in attracting bidders to an auction.

continuous in the interior of X, upper semi-continuous, and satisfies $\overline{f}^+ > -\infty$ except possibly at a countable set. This condition is quite weak. For example, it is satisfied for right-continuous, piecewise differentiable densities that do not possess downward jumps.[13]

The following auxiliary result can be seen as a generalization of the smooth characterization of regularity. Note, however, that it concerns weakly increasing virtual valuations.

Lemma 4.1. *Let $f > 0$ be a density on some interval X, and assume that condition (CL) holds. Then, $J_f(x)$ is nondecreasing if and only if $1/(1 - F(x))$ is convex.*

Proof. The right-hand upper Dini derivative of $J_f(x) = x-(1-F(x))/f(x)$ is given by

$$\overline{J}_f^+(x) = 1 - \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\left\{\frac{1-F(x+\varepsilon)}{f(x+\varepsilon)} - \frac{1-F(x)}{f(x)}\right\} \tag{7}$$

$$= 1 + \limsup_{\varepsilon\to 0+} \frac{1}{f(x+\varepsilon)}\left\{\frac{F(x+\varepsilon)-F(x)}{\varepsilon} + \frac{1-F(x)}{f(x)}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right\}. \tag{8}$$

Clearly, the derivative of F is a.e. well-defined with $F' = f$. Hence, noting that f is right-continuous,

$$\overline{J}_f^+(x) = 2 + \frac{(1-F(x))\overline{f}^+(x)}{f(x)^2} \tag{9}$$

a.e. in X. Let $\vartheta_f(x) = f(x)/(1-F(x))^2$. Then the right-hand upper Dini

[13] Monteiro and Svaiter (2008) study optimal design for *arbitrary* distributions. For example, the support of the distribution may have gaps, and there may be mass points. Obviously, there is no role for regularity under such general conditions.

derivative of $\vartheta_f(x)$ is given by

$$\overline{\vartheta}_f^+(x) = \limsup_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left\{ \frac{f(x+\varepsilon)}{(1-F(x+\varepsilon))^2} - \frac{f(x)}{(1-F(x))^2} \right\} \tag{10}$$

$$= \frac{1}{(1-F(x))^2} \limsup_{\varepsilon \to 0+} \left\{ \frac{f(x+\varepsilon) - f(x)}{\varepsilon} + \frac{f(x)}{(1-F(x))^2} \frac{(1-F(x))^2 - (1-F(x+\varepsilon))^2}{\varepsilon} \right\}. \tag{11}$$

Hence,

$$\overline{\vartheta}_f^+(x) = \frac{\overline{f}^+(x)(1-F(x)) + 2f(x)^2}{(1-F(x))^3} \tag{12}$$

a.e. in X. Comparing (9) and (12) shows that $\overline{J}_f^+(x)$ and $\overline{\vartheta}_f^+(x)$ share the same sign a.e. in X.

"Only if." Assume that J_f is monotone. Then $\overline{J}_f^+ \geq 0$ on X, and therefore, $\overline{\vartheta}_f^+(x) \geq 0$ a.e. in X. An inspection of (11) shows that ϑ_f satisfies condition (CL). Hence, by Theorem 2.2, ϑ_f is nondecreasing. Thus, any integral of ϑ_f is convex, in particular $1/(1-F(x))$.

"If." Conversely, assume that $1/(1-F(x))$ is convex. Then the left derivative of $1/(1-F(x))$ is well-defined in the interior of X and monotone. But a.e. in X, the left derivative of $1/(1-F(x))$ is given by ϑ_f. Thus, $\overline{\vartheta}_f^+(x) \geq 0$ a.e. in X. As shown above, this implies $\overline{J}_f^+(x) \geq 0$ a.e. in X. One can check using (8) that J_f satisfies condition (CL). Therefore, by another application of Theorem 2.2, J_f is monotone increasing. □

5. A condition on the density

This section contains the key result of the paper. The Prékopa-Borell theorem is used to derive a tight criterion for regularity on the underlying density function. To deal with strict monotonicity, and to allow for modifications of the regularity assumption, we will write

$$J_f(x,a,b) = ax - \frac{b - F(x)}{f(x)}, \tag{13}$$

where $a, b \in \mathbb{R}$. Note that $J_f(x,1,1) \equiv J_f(x)$.

Theorem 5.1. *Let $f > 0$ be a density on some interval X, and $a > -1$. Then $J_f(x,a,b)$ is weakly increasing in x [strictly increasing in x] for any $b \in [0,1]$ if f is ρ-concave [strongly ρ-concave] for $\rho = -\frac{a}{1+a}$. In particular, $J_f(x)$ is strictly increasing if f is strongly $(-\frac{1}{2})$-concave.*

Proof. Assume that f is $(-\frac{a}{1+a})$-concave for some $a > -1$. Consider the mirror image density $g(y) = f(-y)$. Obviously, also g is $(-\frac{a}{1+a})$-concave. By Theorem 2.1, the integral $G(y) = 1 - F(-y)$ is $(-a)$-concave, and so is $1-F(x)$. Since f is continuous on X with finite right derivative in the interior, condition (CL) holds. Therefore, in straightforward extension of Lemma 4.1, $J_f(x,a,1)$ is nondecreasing. Similarly, $J_f(x,a,0)$ is nondecreasing since $F(x)$ is $(-a)$-concave. The unbracketed part of the theorem follows now from noting that $J_f(x,a,b)$ is linear in b. If f is even strongly $(-\frac{a}{1+a})$-concave for some $a > -1$ then, by the first part of the proof, $J_f(x,a',b)$ is weakly increasing in x for some $a' \in (-1,a)$. Hence, $J_f(x,a,b) = J_f(x,a',b) + (a - a')x$ is strictly increasing in x. $\square$

Thus, strong $(-\frac{1}{2})$-concavity of the density is sufficient for regularity. Since any log-concave function is strongly $(-\frac{1}{2})$-concave, Theorem 5.1 clearly implies the conventional log-concavity criterion.[14]

[14] Theorem 5.1 can also be applied if the density function has finitely many convex kinks and jump discontinuities. In such cases, one requires strong $(-\frac{1}{2})$-concavity of f in each smooth segment, and strong (-1)-concavity of F just left of critical points. For a proof, one constructs a strongly $(-\frac{1}{2})$-concave extension of the density right of the critical point. The details are omitted.

Many density functions are strongly $(-\frac{1}{2})$-concave without being log-concave. For example, as Table I shows, this is the case for the log-normal, Pareto, log-logistic, Student, Cauchy, F, beta prime, mirror-image Pareto, inverse gamma, inverse chi-squared, and Pearson distributions.[15] Thus, Theorem 5.1 finds new regular distributions and settles, in particular, the case of distributions with log-normal shape. Conversely, Table II lists various distributions that lack a strongly $(-\frac{1}{2})$-concave density function. We write $J_f' > 0$ if $J_f'(x) > 0$ holds for all $x \in X$ and for all (strictly positive) parameter values in the range indicated in the leftmost column. Similarly, we write $J_f' \ngeq 0$ if for all such parameter values, there is some $x \in X$ such that $J_f'(x) < 0$. The entry "mixed" is used when neither $J_f' > 0$ nor $J_f' \ngeq 0$ holds. It can be seen that most examples in Table II are never regular, regardless of parameter values. Overall, this clearly illustrates the tightness of Theorem 5.1.[16]

6. Applications

This section illustrates the use of Theorem 5.1 in specific settings. In all of these examples, the criterion opens the door for new classes of densities that hitherto have not been identified as regular.

6.1. Standard design problems. The revenue-maximizing mechanism in Myerson (1981) is a second-price auction with reserve price if the underlying type distribution is regular. While previous conditions on the density required log-concavity, the conclusion continues to hold if the density function is strongly $(-\frac{1}{2})$-concave only. Relatedly, Riley and Samuelson (1981) show

[15] Further details regarding Tables I and II can be found in the Appendix.

[16] In fact, strong $(-\frac{1}{2})$-concavity is the tightest condition possible in terms of generalized concavity. For example, the density function $f(x) = (1+x)^{-2}$ on $\mathbb{R}_+$ is $(-\frac{1}{2})$-concave, but not strictly so, and the corresponding distribution is irregular.

that the optimal reserve price in a broad class of auctions can be found by setting the virtual valuation equal to the seller's reservation value. The resulting equation has a unique solution provided the distribution of types is regular. The range of densities can be widened as before. For a somewhat richer example, recall that in Baron and Myerson (1982), optimal regulation of a monopoly discriminates between cost types provided that $\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}$ is increasing in θ, where $\alpha \in [0,1]$ is the policy weight of monopoly. Here, it suffices to assume that f is ρ-concave for $\rho > -\frac{1}{2-\alpha}$, which is less restrictive than log-concavity.

6.2. Two-sided markets and auctions with resale. An optimal trading mechanism exists in Myerson and Satterthwaite (1983) if the buyer's virtual valuation $J_f(x)$ and the seller's *virtual cost* $K_f(x) \equiv J_f(x,1,0) = x + \frac{F(\theta)}{f(\theta)}$ are both increasing in x. The new element here is the seller's virtual cost, which is increasing if and only if $1/F(x)$ is strict convex. The common condition on the density is log-concavity, but strong $(-\frac{1}{2})$-concavity is sufficient.[17] Similar conclusions can be drawn for auctions with resale where regularity conditions ensure that monopoly and monopsony prices are unique (see, e.g., Cheng and Tan 2011).

6.3. Distribution of bids. Guerre et al. (2000) prove that bids in a first-price auction can be rationalized as a Bayesian equilibrium under the independent private value paradigm whenever $x + \frac{1}{I}\frac{G(x)}{g(x)} = \frac{J_g(x,I,0)}{I}$ increases in x, where I is the number of bidders, while g and G, respectively, denote the predicted

[17] In fact, for unimodal densities, strong $(-\frac{1}{2})$-concavity is required only on the increasing tail of the density, which also explains the many positive findings in the rightmost column of Table II.

density and distribution of bids. While the common condition on the density would be log-concavity, any strongly ρ-concave density with $\rho = -\frac{I}{1+I}$ can be rationalized.

6.4. Affiliated types. Chung and Ely (2007) consider affiliated valuations $x_1, ..., x_N$, and show that a generalized hazard rate condition implies a single-crossing condition for virtual valuations. This allows them to provide a foundation for dominant-strategy mechanisms. We note that with continuous affiliated types, assuming that $f(x_1, ..., x_N)$ is strongly $(-\frac{1}{2})$-concave would be much less restrictive than the generalized hazard rate condition, but still ensure the single-crossing condition for virtual valuations.[18]

6.5. Multidimensional types with externalities. An object is sold to one of N buyers. With externalities, buyer i's type is a vector (s_i^i, s_{-i}^i) whose entries specify the respective payoff to i in case some buyer obtains the good. The distribution of buyer i's type follows some density $f_i(s_i^i, s_{-i}^i)$. Jehiel et al. (1999) show that the revenue-maximizing standard anonymous mechanism that always transfers the object is a second-price auction with entry fee if a modified regularity condition holds. Specifically, for

$$g(z) = \int_{\mathbb{R}^{N-1}} f_i(z + \frac{1}{N-1}\sum_{j\neq i} s_j^i, s_{-i}^i) ds_{-i}^i, \qquad (14)$$

the virtual valuation J_g needs to be increasing. By Theorem 5.1, it suffices that g is strongly $(-\frac{1}{2})$-concave. But from (14), the change of variables in the argument of f_i is an affine transformation of $\mathbb{R}^N$, which leaves generalized concavity unaffected. Therefore, the second-price auction with entry fee is

[18] Chung and Ely (2007) assume discrete type distributions, but their results could probably be extended to continuous distributions.

optimal if all f_i are strongly $(-\frac{1}{N+1})$-concave.

7. Concluding remarks

This paper has been devoted to Myerson's (1981) regularity condition. Methods introduced by Bagnoli and Bergström (2005) and An (1998) have been refined by focusing directly on the virtual valuation rather than on the hazard rate. This has led to a new interpretation of regularity. More importantly, we have derived strong $(-\frac{1}{2})$-concavity of the density as a new sufficient criterion for regularity. In particular, it has been shown that regularity is consistent with empirically important distributions such as the log-normal, whereas this is not the case for the hazard rate condition. Using various applications, it has been illustrated that replacing the commonly used log-concavity assumption on the density by the less restrictive conditions developed in this paper can significantly expand the scope of theoretical results.

Appendix. Parameterized distributions

This Appendix outlines the derivations underlying Tables I and II. The main tool is the following smooth criterion for ρ-concavity.

Lemma A.1. *Let $f > 0$ be twice continuously differentiable on some interval X, with a discrete set X_1 over which $f'(x) = 0$. Then, for finite ρ, the function f is ρ-concave if and only if $r_f(x) \equiv -(\ln f(x))''/(\ln f(x))'^2 \geq \rho$ for all $x \in X \setminus X_1$.*

Proof. A straightforward calculation shows that $r_f(x) = 1 - f(x)f''(x)/f'(x)^2$. Hence, $r_f(x) \geq \rho$ if and only if $f(x)f''(x) \leq (1-\rho)f'(x)^2$, provided $f'(x) \neq 0$. By continuity, this proves the assertion. $\square$

Lemma A.1 reduces the determination of the global concavity parameter to a straightforward minimization problem. More specifically, to find the tightest parameter ρ for which a given f is ρ-concave, one calculates the minimum (or infimum) of $r_f(x)$ on X. Table I shows the results for selected examples. A particular case is the Pearson distribution. Its density function solves the differential equation $f'(x) = f(x)(x - x^M)/\chi(x)$ for $x^M \in \mathbb{R}$ and $\chi(x) = b_0 + b_1 x + b_2 x^2$, where $b_0, b_1, b_2 \in \mathbb{R}$. We focus on distributions with unbounded support and such that $\chi(x^M) < 0$. Then, f is b_2-concave. Thus, both J_f and K_f are increasing provided that $b_2 > -\frac{1}{2}$.

Table II shows examples of distributions that do not allow a strongly $(-\frac{1}{2})$-concave density function. Unless noted otherwise, all parameter values are strictly positive. The only regular example is the mirror-image Pareto distribution. The entries of the form $J'_f \ngtr 0$, $K'_f \ngtr 0$, or "mixed" have been established by direct calculation at boundary values. In some cases, numerical calculations have been used. The entries with $K'_f > 0$ are typically straightforward, e.g., when the density function is everywhere declining. Some cases, however, need additional arguments. E.g., both the log-normal distribution and the F distribution (Finner and Roters 1997) possess a log-normal distribution function, even though the corresponding density functions are not log-normal. In other cases (inverse gamma distribution and inverse chi-squared), the density function is log-concave for low values and decreasing for high values, which again is sufficient for $K'_f > 0$.

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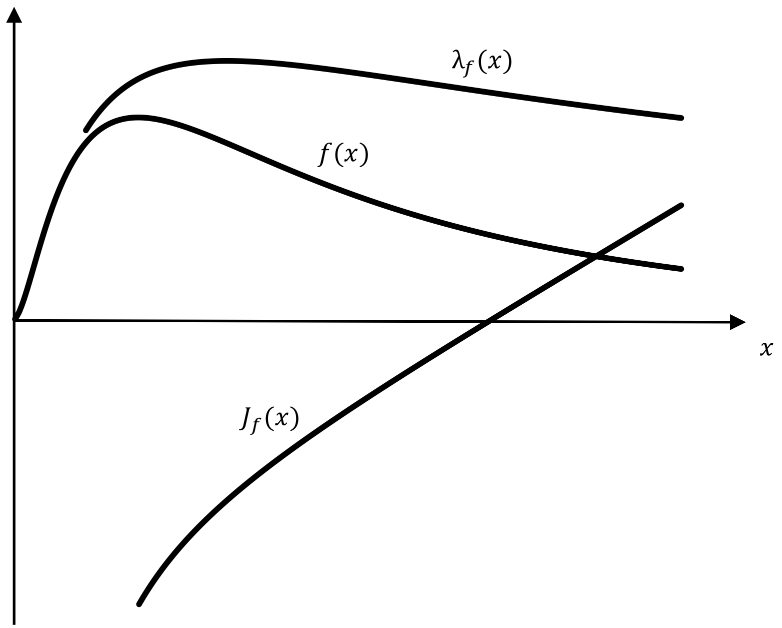
Figure 1. Regularity of the log-normal distribution.†



† The figure shows density, hazard rate, and virtual valuation of a random variable whose logarithm follows a standard normal distribution.

Table I. Distributions *with* strongly $\left(-\frac{1}{2}\right)$-concave density function.[a]

Name of distribution	**Interval** X	**P.d.f.** $f(x)$	**C.d.f.** $F(x)$	**Concavity** ρ
Any with log-concave density	- see Bagnoli and Bergström (2005, Table 1) -			≥ 0
Pareto ($\beta > 1$)	$[1; \infty)$	$\beta x^{-\beta-1}$	$1 - x^{-\beta}$	$-\frac{1}{\beta+1}$
Log-normal ($\sigma_L^2 < 2, \mu_L \in \mathbb{R}$)	$[0; \infty)$	$\propto \frac{1}{x}\exp\left(-\frac{(\ln x - \mu_L)^2}{2\sigma_L^2}\right)$	*	$-\frac{\sigma_L^2}{4}$
Student's t ($n > 1$)	$\mathbb{R}$	$\propto (1 + x^2/n)^{-\frac{n+1}{2}}$	*	$-\frac{1}{n+1}$
Cauchy[b]	$\mathbb{R}$	$\frac{1}{\pi(1+x^2)}$	$\frac{1}{2} + \frac{\arctan x}{\pi}$	$-\frac{1}{2}$
F distribution ($m_1 \geq 2, m_2 > 2$)	$[0; \infty)$	$\propto \frac{x^{\frac{m_1}{2}-1}}{(m_1 x + m_2)^{\frac{m_1+m_2}{2}}}$	*	$-\frac{2}{m_2+2}$
Mirror-image of Pareto ($\beta > 1$)	$(-\infty; -1]$	$\beta(-x)^{-\beta-1}$	$(-x)^{-\beta}$	$-\frac{1}{\beta+1}$
Log-logistic ($\beta > 1$)	$[0; \infty)$	$\frac{\beta x^{\beta-1}}{(1+x^\beta)^2}$	$\frac{x^\beta}{1+x^\beta}$	$-\frac{1}{\beta+1}$
Inverse gamma ($\alpha > 1$)	$[0; \infty)$	$\frac{\exp(-1/x)}{\Gamma(\alpha)x^{\alpha+1}}$	*	$-\frac{1}{\alpha+1}$
Inverse chi-squared ($\nu > 2$)	$[0; \infty)$	$\frac{x^{-(\nu/2)-1}}{2^{\nu/2}\Gamma(\nu/2)}\exp\left(-\frac{1}{2x}\right)$	*	$-\frac{2}{\nu+2}$
Beta prime ($\alpha \geq 1, \beta > 1$)	$[0; \infty)$	$\propto x^{\alpha-1}(1+x)^{-\alpha-\beta}$	*	$-\frac{1}{\beta+1}$
Pearson $\left(b_2 > -\frac{1}{2}\right)$	− see the Appendix −			b_2

[a] The symbol $\propto$ indicates that the density function, for fixed parameters, is proportional to the term given in the table; for cumulative distribution functions marked with *, there is no closed-form representation.
[b] The density function of the Cauchy distribution is strongly (-1/2)-concave on any compact interval.

Table II. Distributions *without* strongly $\left(-\frac{1}{2}\right)$-concave density function.[a]

Name of distribution	Interval X	P.d.f. $f(x)$	C.d.f. $F(x)$	Values J'_f	Costs K'_f
Power ($c < 1$)	$[0; 1]$	cx^{c-1}	x^c	$\ngeq 0$	> 0
Weibull ($c < 1$)	$[0; \infty)$	$cx^{c-1}\exp(-x^c)$	$1-\exp(-x^c)$	$\ngeq 0$	> 0
Gamma ($c < 1$)	$[0; \infty)$	$\frac{x^{c-1}\exp(-x)}{\Gamma(c)}$	*	$\ngeq 0$	> 0
Chi-squared ($c < 2$)	$[0; \infty)$	$\frac{x^{(c-2)/2}\exp(-x/2)}{2^{c/2}\Gamma(c/2)}$	*	$\ngeq 0$	> 0
Chi ($c < 1$)	$[0; \infty)$	$\frac{x^{c-1}\exp(-x^2/2)}{2^{(c-2)/2}\Gamma(c/2)}$	*	$\ngeq 0$	> 0
Beta ($\nu < 1$ or $\omega < 1$)	$[0; 1]$	$\propto x^{\nu-1}(1-x)^{\omega-1}$	*	mixed	mixed
Arc-sine	$[0; 1]$	$\frac{1}{\pi\sqrt{x(1-x)}}$	$\frac{2}{\pi}\arcsin(x)$	$\ngeq 0$	$\ngeq 0$
Pareto ($\beta < 1$)	$[1; \infty)$	$\beta x^{-\beta-1}$	$1-x^{-\beta}$	$\ngeq 0$	> 0
Log-normal ($\sigma_L^2 > 2$)	$[0; \infty)$	$\propto \frac{1}{x}\exp\left(-\frac{(\ln x-\mu_L)^2}{2\sigma_L^2}\right)$	*	mixed	> 0
Student's t ($n < 1$)	$\mathbb{R}$	$\propto \left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$	*	$\ngeq 0$	$\ngeq 0$
F distribution ($m_1 < 2$ or $m_2 \leq 2$)	$[0; \infty)$	$\propto \frac{x^{\frac{m_1}{2}-1}}{(m_1 x+m_2)^{\frac{m_1+m_2}{2}}}$	*	mixed	> 0
Mirror-image of Pareto ($\beta < 1$)	$(-\infty; -1]$	$\beta(-x)^{-\beta-1}$	$(-x)^{-\beta}$	> 0	$\ngeq 0$
Log-logistic ($\beta < 1$)	$[0; \infty)$	$\frac{\beta x^{\beta-1}}{(1+x^\beta)^2}$	$\frac{x^\beta}{1+x^\beta}$	$\ngeq 0$	> 0
Inverse gamma ($\alpha < 1$)	$[0; \infty)$	$\frac{\exp(-1/x)}{\Gamma(\alpha)x^{\alpha+1}}$	*	$\ngeq 0$	> 0
Inverse chi-squared ($\nu < 2$)	$[0; \infty)$	$\frac{x^{-(\nu/2)-1}}{\Gamma(\nu/2)}\exp\left(-\frac{1}{2x}\right)$	*	$\ngeq 0$	> 0
Beta prime ($\alpha, \beta < 1$)	$[0; \infty)$	$\propto x^{\alpha-1}(1+x)^{-\alpha-\beta}$	*	mixed	> 0
Pearson $\left(b_2 < -\frac{1}{2}\right)$		- see the Appendix -		mixed	mixed

[a] The symbol $\propto$ indicates that the density function, for fixed parameters, is proportional to the term given in the table; for cumulative distribution functions marked with *, there is no closed-form representation.